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GRAPHS WITH K-BALANCED CLOSED
NEIGHBORHOOD MATRICES

by

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Abstract

An n-cycle matrix is an $n \times n$ matrix $(a_{ij})$ with $a_{ii} = a_{i,i+1} = 1$ for $i = 1, \ldots, n-1$, $a_{nn} = a_{nl} = 1$ and $a_{ij} = 0$ otherwise, or any matrix obtained by applying row and column permutations to it. Let $K$ be a set of positive integers all of which are at least 3. A 0-1 matrix is $K$-balanced if it does not contain an n-cycle submatrix for all $n \in K$. This paper gives a characterization of $K$-balanced matrices in terms of closed neighborhood matrices of graphs.
1. **Introduction**

An n-cycle matrix is an \( n \times n \) matrix \((a_{ij})\) with \( a_{ii} = a_{i,i+1} = 1 \) for \( i = 1, \ldots, n-1 \), \( a_{nn} = a_{n1} = 1 \) and \( a_{ij} = 0 \) otherwise, or any matrix obtained by applying row and column permutations to it. A 0-1 matrix is balanced if it does not contain an n-cycle submatrix for all odd \( n \geq 3 \). Balanced matrices are important in combinatorial optimization because of the following theorem.

**Theorem** (Berge [1] and Fulkerson et al. [8]). If \( M \) is a balanced matrix then all of the extreme points of the polyhedra \( P_1 = \{x: xM \geq 1, x \geq 0\} \) and \( P_2 = \{y: My \leq 1, y \geq 0\} \) are integral.

Consequently, the integer programs \( \text{min}\{xc: x \in P_1 \text{ and } x \text{ is integral}\} \) and \( \text{max}\{wy: y \in P_2 \text{ and } y \text{ is integral}\} \) can be solved as linear programs. In fact, they can be solved in polynomial time by an ellipsoid algorithm [10]. However, the problems of recognizing a balanced matrix in polynomial time and of solving the integer programs by a polynomial time combinatorial algorithm are unsolved.

A 0-1 matrix is totally balanced if it does not contain an n-cycle submatrix for all \( n \geq 3 \). Totally balanced matrices are balanced. For totally balanced matrices, the recognition problem and integer programming problems have been solved by polynomial-time combinatorial algorithms, see Farber [6], Hoffman et al. [9], Kolen [11] and Lubiw [12]. These problems also have been solved for another special case of balanced matrices, see Chang and Nemhauser [5].
Given a simple graph $G = (V,E)$, the **closed neighborhood** of $x \in V$ is the set $N(x) = \{ y \in V : y = x \text{ or } (x,y) \in E \}$. To indicate $y \in N(x)$, we use the abbreviated notation $x \sim y$. The **closed neighborhood matrix** $N(G)$ of $G$ is the $|V| \times |V|$ matrix $(b_{ij})$ with $b_{ij} = 1$ if $x_i \sim x_j$ and $b_{ij} = 0$ otherwise.

In this paper, we generalize the notion of balanced matrices and characterize this class of matrices in terms of closed neighborhood matrices of graphs. Our main result covers Farber's theorem [7] on totally balanced matrices and gives a new characterization of balanced matrices.

In particular, let $K$ be any set of integers such that each integer in the set is at least 3. A 0-1 matrix is $K$-balanced if it does not contain an $n$-cycle submatrix for all $n \in K$. Our main theorem characterizes $K$-balanced matrices in terms of closed neighborhood matrices of graphs.

Denote by $\mathcal{Z}_n$ (resp. $\mathcal{O}_n$) the set of all integers $\geq n$ (all odd integers $\geq n$). Then a matrix is balanced (resp. totally balanced) if it is $\mathcal{O}_3$-balanced (resp. $\mathcal{Z}_3$-balanced).

2. **$K$-sun-free chordal graphs and $K$-balanced matrices**

In a graph $G$, an **$n$-hole** is a cycle of $n$ edges without chords. A graph is **chordal** if it does not contain an $n$-hole for all $n \geq 4$.

An $n$-sun is a chordal graph $G = (V,E)$ whose vertex set $V$ can be partitioned into $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$ such that

(S1) $Y$ is a stable set (no two vertices in $Y$ are joined by an edge),
(S2) $(z_1, \ldots, z_n, z_1)$ is a cycle,
(S3) $(y_i, z_j) \in E$ if and only if $i \in \{j, j+1\}$.\*$

\* It is to be understood in the sequel that for all vertices of an $n$-cycle or an $n$-sun addition of indices is assumed to be modulo $n$. 
In the above definition, the z's are called inner vertices of the n-sun and the y's outer vertices. A 4-sun is shown in Figure 1.

[Figure 1 here]

A graph is K-sun-free chordal (K-SF-chordal) if it does not contain an n-sun as an induced subgraph for all n ∈ K. Z₃-SF-chordal and O₃-SF-chordal graphs are known as SF-chordal and OSF-chordal graphs respectively, see Chang and Nemhauser [4,5]. Farber [7] called Z₃-SF-chordal graphs strongly chordal and proved that G is Z₃-SF-chordal if and only if N(G) is Z₃-balanced. In this section, we will give a more general theorem on K-SF-chordal graphs and K-balanced matrices.

Proposition 1. Suppose C is a cycle of a chordal graph, then for every edge (u,v) of the cycle there exists a vertex w of the cycle that is adjacent to both u and v.

Lemma 2. The following two statements are equivalent.

1. G = (V,E) is an n-sun.

2. G = (V,E) is a chordal graph where V = W ∪ X, W = \{w₁, \ldots, wₙ\}, X = \{x₁, \ldots, xₙ\} and wᵢ ∨ xⱼ if and only if i ∈ \{j, j+1\}.

Proof. (1 ⇒ 2). Set wᵢ = yᵢ and xᵢ = zᵢ, i = 1, \ldots, n.

(2 ⇒ 1). First we claim that W ∩ X = ∅. Suppose there is an r and an s such that wᵣ = xₛ. Then r = s or r = s+1. Without loss of generality, assume that r = s, so wᵣ = xₛ = xᵣ.

Consider the sequence of vertices C = (w₁, x₁, w₂, x₂, \ldots, wₙ, xₙ, w₁). Delete all wᵦ identical to xᵦ or xᵦ₋₁ from C. Then the resulting sequence is a cycle C'. In C', by Proposition 1, there is a vertex adjacent to
$x_r$ and $w_r = x_r$, which implies that $C'$ has a $w$-$x$ chord. Hence $w_i \sim x_j$ for some $i$ and $j$ such that $i \neq \{j, j+1\}$, which is a contradiction.

In the cycle $C = (w_1, x_1, w_2, x_2, \ldots, w_n, x_n, w_1)$, by Proposition 1, there is a vertex $v$ adjacent to $x_1$ and $w_2$. Since $C$ has no $w$-$x$ chord, either $v = x_2$ or $v = w_1$. If $v = x_2$, let $y_i = w_1$ and $z_i = x_i$ for $i = 1, \ldots, n$; if $v = w_1$, let $y_i = x_i-1$ and $z_i = w_i$ for $i = 1, \ldots, n$. In either case $(z_1, z_2) \in E$. We will prove that $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$ satisfy (S1), (S2) and (S3). (S3) is an immediate consequence of $w_i \sim x_j \iff i \in \{j, j+1\}$.

Next we prove by induction that $(z_1, z_2, \ldots, z_n, z_1)$ is a cycle. $(z_1, z_2)$ is a path. The induction hypothesis is that $(z_1, z_2, \ldots, z_i)$ is a path, where $i > 2$. Consider the cycle $C_i = (z_1, z_2, \ldots, z_i, y_{i+1}, z_{i+1}, \ldots, y_n, z_n, y_1, z_1)$. By Proposition 1, there is a vertex of $C_i$ that is adjacent to both $z_i$ and $y_{i+1}$. Since $C$ has no $y$-$z$ chord, the only possible vertex is $z_{i+1}$ and hence $(z_1, z_2, \ldots, z_i, z_{i+1})$ is a path. Continuing this process shows that $(z_1, z_2, \ldots, z_n, z_1)$ is a cycle. This proves (S2).

Finally, we prove that $y_i$ is not adjacent to $y_j$ for all $i \neq j$.

Suppose $y_i$ is adjacent to $y_j$ for some $i \neq j$. Without loss of generality, we can assume that $i \neq j-1$. Consider the cycle $\hat{C} = (y_i, z_i, z_{i+1}, \ldots, z_{j-1}, y_j, y_i)$ which contains exactly two $y$'s and at least two $z$'s. In $\hat{C}$, by Proposition 1, there is a vertex $z_k$ adjacent to both $y_i$ and $y_j$. Thus $\hat{C}$ contains a $y$-$z$ chord, which is a contradiction. This proves (S1).

Chang [3] gives several other equivalent definitions of an $n$-sun.
Theorem 3. The following two statements are equivalent.

1. G is K-SF-chordal.
2. G is chordal and \( N(G) \) is K-balanced.

Proof. For each \( n \in K \), it suffices to prove that the chordal graph G contains an \( n \)-sun as an induced subgraph if and only if \( N(G) \) contains an \( n \)-cycle submatrix.

\((\forall 1 \Rightarrow \forall 2)\). Suppose G has an \( n \)-sun \( H \). Then the submatrix of \( N(G) \) whose rows correspond to inner vertices of \( H \) and whose columns correspond to outer vertices of \( H \) is an \( n \)-cycle submatrix.

\((\forall 2 \Rightarrow \forall 1)\). Suppose \( N(G) \) has an \( n \)-cycle submatrix whose rows and columns correspond to \( X = \{x_1, \ldots, x_n\} \) and \( W = \{w_1, \ldots, w_n\} \) respectively. Note that \( w_i \sim x_j \) if and only if \( i \in \{j, j+1\} \). Hence, Lemma 2 implies that G contains an \( n \)-sun. 

Without the assumption of chordality, \( 2 \Rightarrow 1 \) of Theorem 3 is false. For example, a 4-hole has a K-balanced closed neighborhood matrix where \( K = \{4\} \), but is not chordal. We now consider conditions on \( K \) so that \( N(G) \) is K-balanced implies that G is chordal.

Lemma 4. Suppose \( H = (x_1, \ldots, x_n, x_1) \) is an \( n \)-hole with \( n \geq 4 \). \( N(H) \) contains an \( m \)-cycle submatrix if and only if \( m \geq 3 \) and \( \frac{1}{2} n \leq m \leq \frac{3}{4} n \).

Proof. (\( \Rightarrow \)) Suppose \( N(H) \) contains an \( m \)-cycle submatrix \( M \) whose rows and columns correspond to \( X \) and \( W \) respectively.

For any set \( S \) of vertices of \( H \), a run of length \( k \) is a subset \( T = \{x_j, x_{j+1}, \ldots, x_{j+k-1}\} \) of \( S \) such that \( x_{j-1}, x_{j+k} \notin S \), where indices are assumed to be modulo \( n \).
First we prove that each run of $X$ is of length not greater than three. If not, then $X$ contains $x_j x_{j+1} x_{j+2} x_{j+3}$ for some $j$. Since $W$ contains $x_p$ and $x_s$ such that $x_p \sim x_{j+1}$ and $x_s \sim x_{j+1}$ at least one of these vertices, say $x_p$, is $x_{j+1}$ or $x_{j+2}$. So the column corresponding to $x_p$ in $M$ contains at least three ones, which is impossible.

Next we show that each run of $V(H)-X$ is of length one. If not, then $x_{j-1} x_j x_{j+k} \in X$, $x_j x_{j+1}, \ldots, x_{j+k-1} \notin X$ for some $k \geq 2$. Then $x_j x_{j+1}, \ldots, x_{j+k-1} \notin W$, otherwise $M$ has a column with at most one 1. Since $M$ is an $m$-cycle submatrix, there is some $x_p$ with $x_p \sim x_{j-1}$ and $x_p \sim x_{j+k}$, which is impossible since $k \geq 2$.

Let $a_i$ denote the number of runs of length $i$ in $X$ for $i = 1, 2, 3$. Then $X$ has exactly $a_1 + a_2 + a_3$ runs. Since $V(H) \setminus X$ has as many runs as $X$, $|V(H) - X| = a_1 + a_2 + a_3$. Consequently, $m = |X| = a_1 + 2a_2 + 3a_3$ and $n = |X| + |V(H) - X| = 2a_1 + 3a_2 + 4a_3$. So

$$\frac{1}{2} n = a_1 + \frac{3}{2} a_2 + 2a_3 \leq m \leq \frac{3}{2} a_1 + \frac{9}{4} a_2 + 3a_3 = \frac{3}{4} n.$$

($\Leftarrow$) Conversely, suppose $m \geq 3$ and $\frac{1}{2} n \leq m \leq \frac{3}{4} n$. It is possible to find a subset $X$ of $V(H)$ and $(a_1, a_2, a_3)$ such that $X$ has exactly $a_i$ runs of length $i$ for $i = 1, 2, 3$ with $m = a_1 + 2a_2 + 3a_3$, and $V(H) - X$ has exactly $n-m = a_1 + a_2 + a_3$ runs of length one. In particular, first let $X_1 = \{x_i : i \text{ is odd and } 1 \leq i \leq n\}$. Next let $X_2 = \{x_{4i} : i = 1, \ldots, p\}$ where $p = m - |X_1|$. Since $|X_1| \geq \left\lceil \frac{n}{2} \right\rceil$ and $\frac{n}{2} \leq m \leq \frac{3}{4} n$, we have $0 \leq p \leq \frac{3}{4} n - \left\lceil \frac{n}{2} \right\rceil \leq \frac{1}{4} n$.

Then $X = X_1 \cup X_2$ satisfies the above requirements. Let $W = (V(H) - X) \cup \{x_i : (x_i, x_{i+1}) \text{ is a run of } X\} \cup \{x_i, x_{i+2} : (x_i, x_{i+1}, x_{i+2}) \text{ is a run of } X\}$. It is straightforward to check that the submatrix of $N(H)$ whose rows and columns correspond to $X$ and $W$ respectively is an $m$-cycle submatrix. \(\square\)
Consider the following condition on $K$.

(*) $K \cap \{ m \in \mathbb{Z}_3 : \frac{1}{2} n \leq m \leq \frac{3}{4} n \} \neq \emptyset$ for all $n \in \mathbb{Z}_4$.

**Theorem 5.** Suppose $K$ satisfies (*). $G$ is $K$-SF-chordal if and only if $N(G)$ is $K$-balanced.

**Proof.** If $G$ is $K$-SF-chordal, then $N(G)$ is $K$-balanced by Theorem 3. If $N(G)$ is $K$-balanced, then $G$ is chordal by Lemma 4. So again, by Theorem 3, $G$ is $K$-SF-chordal. \qed

**Corollary 6 (Farber [7]).** $G$ is SF-chordal if and only if $N(G)$ is totally balanced.

**Proof.** Apply Theorem 5 with $K = \mathbb{Z}_3$. \qed

**Corollary 7.** $G$ is OSF-chordal if and only if $N(G)$ is balanced.

**Proof.** Apply Theorem 5 with $K = \emptyset$. \qed

Corollary 7 is used in [5] to prove that $G$ is OSF-chordal implies that $G^2$ is perfect.

For any $K$, if a matrix $M$ is $K$-balanced then so are its transpose $M^t$, $(I,M)$, $(M,M)$ and any submatrix of $M$.

**Theorem 8.** Suppose $M$ is an $r \times s$ 0-1 matrix. $M$ has an $n$-cycle submatrix if and only if $M^* = \begin{pmatrix} I & M^t \\ M & J \end{pmatrix}$ has an $n$-cycle submatrix, where $I$ is an $s \times s$ identity matrix and $J$ is an $r \times r$ matrix of all ones.
Proof: \((\Rightarrow)\). \(M\) is a submatrix of \(M^\#\).

\((\Leftarrow)\). Suppose \(M^\#\) has an \(n\)-cycle submatrix \(A\). Suppose \(A\) has \(r^\#\) rows from \((M J)\), \(n-r^\#\) rows from \((I M^\top)\), \(s^\#\) columns from \((M^\top J)\), and \(n-s^\#\) columns from \((I M)\).

If \(r^\# = 0\), then \(A\) is a submatrix of \((I M^\top)\) and hence \((I M^\top)\) has an \(n\)-cycle submatrix which implies \(M\) also has one. Similarly, the result follows if \(s^\# = 0\).

So assume that both \(r^\#\) and \(s^\#\) are positive. If \(s^\# \geq 3\) or \(r^\# \geq 3\), then \(A\) has the submatrix \((1 1 1)\) or its transpose, which is impossible.

So \(1 \leq s^\# \leq 2\) and \(1 \leq r^\# \leq 2\). Suppose \(r^\# = s^\# = 2\), then \((1 1 1)\) is a submatrix of \(A\), which is a contradiction. Suppose \(1 = r^\# \leq s^\# \leq 2\). Then \(A\) has at most \(n-s^\#\) ones in \(I\), exactly \(2-s^\#\) ones in \(M\), exactly \(s^\#\) ones in \(M^\top\), and exactly \(s^\#\) ones in \(J\). Hence \(A\) has at most \(n+2\) ones, which contradicts the assumption that \(A\) is an \(n\)-cycle matrix and \(n \geq 3\). A similar argument applies to the case \(s^\# = 1\) and \(r^\# = 2\). \(\square\)

**Corollary 9.** Suppose \(K\) satisfies (*)\). \(M\) is \(K\)-balanced if and only if \(M^\#\) is the closed neighborhood matrix of a \(K\)-SF-chordal graph.
References


Figure 1. A 4-sun.