A TWO-STAGE MINIMAX PROCEDURE WITH SCREENING
FOR SELECTING THE LARGEST NORMAL MEAN

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Abstract

1. Introduction and summary

2. Preliminaries
   2.1 Assumptions
   2.2 Goal and probability requirement

3. Background: Single-stage and sequential procedures

4. A two-stage procedure \( (P_2) \)
   4.1 R-minimax design criterion
   4.2 U-minimax design criterion

5. Probability of a correct selection for \( P_2 \)
   5.1 General expression for \( P_\mu \{CS|P_2\} \)
   5.2 LF-configuration for \( P_2 \)
   5.3 Lower bound for \( P_\mu \{CS|P_2\} \)

6. Expected total sample size for \( P_2 \)
   6.1 General expression for \( E_\mu \{T|P_2\} \)
   6.2 The supremum of \( E_\mu \{T|P_2\} \)

7. Optimization problems yielding conservative solutions
   7.1 Discrete optimization problems
      7.1.1 R-minimax design criterion
      7.1.2 U-minimax design criterion
   7.2 Continuous optimization problems

8. Constants to implement \( P_2 \)
   8.1 Constants to implement \( P_2(U_E) \) and \( P_2(R_E) \) for \( k = 2 \)
   8.2 Constants to implement \( P_2(U_C) \) for \( k \geq 3 \)

9. The performance of \( P_2 \) relative to \( P_1 \)
   9.1 \( P_2(U_E) \) and \( P_2(U_C) \) vs. \( P_1 \) for \( k = 2 \)
   9.2 \( P_2(U_C) \) vs. \( P_1 \) for \( k \geq 3 \)

10. Directions of future research

11. Acknowledgments

12. References
Abstract

The problem of selecting the normal population with the largest population mean when the populations have a common known variance is considered. A two-stage procedure is proposed which guarantees the same probability requirement using the indifference-zone approach as does the single-stage procedure of Bechhofer [1954]. The two-stage procedure has the highly desirable property that the expected total number of observations required by the procedure is always less than the total number of observations required by the corresponding single-stage procedure, regardless of the configuration of the population means. The saving in expected total number of observations can be substantial, particularly when the configuration of the population means is favorable to the experimenter. The saving is accomplished by screening out "non-contending" populations in the first stage, and concentrating sampling only on "contending" populations in the second stage. The two-stage procedure can be regarded as a composite one which uses a screening subset-type approach (Gupta [1956], [1965]) in the first stage, and an indifference-zone approach (Bechhofer [1954]) applied to all populations retained in the selected subset in the second stage. Constants to implement the procedure for various $k$ and $P^*$ are provided, as are calculations giving the saving in expected total sample size if the two-stage procedure is used in place of the corresponding single-stage procedure.
1. Introduction and Summary

In many practical situations a statistician is faced with the problem of designing an experiment to select one (or more) out of \( k \geq 2 \) possible competing categories. Typically the categories (populations) are characterized by a real-valued parameter, and the experimenter is interested in selecting the population having the largest (or smallest) parameter value. This population is referred to as the "best" population. Thus, for example, the medical research worker might be studying the response of patients to different types of analgesic drugs in which case his interest might lie in selecting that drug which produces, on the average, the largest period of time without pain, or the agronomist might be conducting field trials with different varieties of grain in which case his purpose might be to select that variety which produces, on the average, the largest yield per acre.

Procedures for achieving such objectives have received considerable attention in recent years. Various probability distributions have been postulated as being appropriate to model these and other real-life problems, and several statistical formulations of these problems have been proposed, and associated statistical selection procedures devised. Recent reviews of the literature with particular reference to the normal means problem appear in Wetherill and Ofosu [1974] and Bechhofer [1975]. The present paper continues the study of the normal means problem, and explores in depth a new approach which has highly desirable properties. This same approach is also applicable to the normal variances problem.

The statistical formulation of the problem is given in Section 2. In Section 3 we sketch the relevant history of the normal means problem, and indicate the virtues and drawbacks of the various procedures which have
been proposed to deal with the problem; the reader is thus enabled to understand the role that our proposed procedure plays. The procedure itself as well as the design criterion that we adopt are described in Section 4. The main analytical results are contained in Sections 5 and 6 which deal with the probability of a correct selection and the expected total sample size, respectively; Section 5.2 discusses a key unsolved problem (that of determining the so-called least favorable configuration of the population means) associated with the procedure, while Section 5.3 contains a strategem which permits us to bypass this difficulty (at the expense of some loss of efficiency of our procedure). In Section 7 we formulate the problem that we must solve to obtain design constants to implement our procedure; tables of these constants are provided in Section 8. The performance of our two-stage procedure relative to that of the best competing single stage procedure is studied in Section 9, and is shown to be highly satisfactory. We conclude in Section 10 with suggestions for future research in this area.

2. Preliminaries

2.1 Assumptions

Let \( \Pi_i \) \((1 \leq i \leq k)\) denote a normal population with unknown mean \( \mu_i \) and known variance \( \sigma^2 \), and let \( \Omega = \{\mu = (\mu_1, \ldots, \mu_k) | -\infty < \mu_i < \infty (1 \leq i \leq k)\} \) be the parameter space of the \( \mu_i \). Denote the ranked values of the \( \mu_i \) by \( \mu[1] \leq \cdots \leq \mu[k] \), and let \( \delta_{i,j} = \mu[i] - \mu[j] \). We assume that the experimenter has no prior knowledge concerning the pairing of the \( \Pi_i \) with the \( \mu[j] \) \((1 \leq i, j \leq k)\). Let \( \Pi(j) \) denote the population associated with \( \mu[j] \). Suppose that \( \mu[k-r] < \mu[k-r+1] = \mu[k] \) for some \( r \) \((1 \leq r \leq k)\) where we define \( \mu[0] = -\infty \); then any one of the \( r \) populations \( \Pi(k-r+j) \) \((1 \leq j \leq r)\) is regarded as "best."
2.2 Goal and probability requirement

The goal of the experimenter is to select a best population. This event is referred to as a correct selection (CS). The experimenter restricts consideration to procedures (P) which guarantee the probability requirement

\[
P_u \{ \text{CS} | P \} \geq P_u^* \quad \forall u \in \Omega(\delta^*)
\]  

(2.1)

where \( \{ \delta^*, P^* \} \) \( 0 < \delta^* < \infty, \frac{1}{k} \leq P_u^* < 1 \) are specified prior to the start of experimentation, and

\[
\Omega(\delta^*) = \{ u \in \Omega | \delta_{k,k-1}^* \geq \delta^* \}.
\]

(2.2)

We refer to \( \Omega(\delta^*) \) as the preference zone for a CS and to \( \Omega_0(\delta^*) = \Omega - \Omega(\delta^*) \) as the associated indifference zone. The formulation (2.1) is called the indifference-zone approach.

3. Background: Single-stage and sequential procedures

The indifference-zone approach as applied to the normal means (common known variance) problem has received considerable study. Bechhofer [1954] proposed a single-stage procedure which guarantees (2.1); Hall [1959] showed that among single-stage procedures this procedure is "most economical" and Eaton [1967] proved that it has additional desirable decision theoretic properties. Bechhofer, Kiefer, and Sobel [1968] (see also, Bechhofer and Sobel [1954]) proposed an open sequential procedure without elimination which guarantees (2.1). Paulson [1964] proposed a closed sequential procedure with permanent elimination which also guarantees (2.1); Fabian [1974a] (see also, Fabian [1974b] and Lawing and David [1966]) showed how Paulson's procedure could be modified to improve its performance characteristics.
The single-stage procedure \( (P_1) \) of Bechhofer requires a common sample size \( n \) per population which is chosen in such a way that (2.1) is guaranteed even when \( \mu_{[k]} - \mu_{[i]} = \delta^* \) \((1 \leq i \leq k-1)\), this being the so-called least favorable (LF) configuration of the population means. However, the procedure is conservative in that if, unknown to the experimenter, \( \mu_{[k]} - \mu_{[i]} > \delta^* \) \((1 \leq i \leq k-1)\) with strict inequality for at least one \( i \)-value—in particular if \( \mu_{[k]} - \mu_{[k-1]} > \delta^* \), then
\[
P_{\mu}(CS|P_1) > p^* \text{ for the actual } \mu \in O(\delta^*) \text{ which the experimenter has encountered. If this is the situation he may have greatly overprotected himself, the overprotection having been purchased by the use of a much larger } n \text{-value than would have been necessary had the true } \mu \text{-values been known.}
\]

Unlike single-stage procedures, multistage or sequential procedures provide information concerning the true but unknown \( \mu \)-values as sampling proceeds.

The sequential procedure \( (P_{S_1}) \) of Bechhofer, Kiefer, and Sobel takes a single vector of observations at each stage of experimentation. Here the number of stages \( (N_{S_1}) \) to terminate experimentation is an unbounded r.v. (For \( P_{S_1} \) a vector consists of one observation from each of the \( k \) populations.) In addition to guaranteeing (2.1) when the population means are in the LF-configuration, it also reacts to favorable configurations of the population means and thereby tends to terminate experimentation early resulting in \( E_{\mu}(N_{S_1}|P_{S_1}) \)-values which are smaller than \( n \) (c.f., B-K-S [1968], Section 12.8.1). (Throughout this paper \( n \) will denote the single-stage sample size for \( P_1 \).) However, if \( p^* \) is sufficiently close to unity and if \( \mu_{[k]} - \mu_{[k-1]} < \delta^* \)--in particular, if \( \mu_{[k]} = \mu_{[1]} \), then
\[
E_{\mu}(N_{S_1}|P_{S_1}) > n \text{ for the actual } \mu \in O_0(\delta^*) \text{ which the experimenter has}
encountered. However, $P_{S_1}$ does have a practical disadvantage: It is open-ended, i.e., although $N_{S_1}$ is finite w.p. 1, it is unbounded; this latter fact may inhibit or prevent the use of the procedure in certain situations.

The sequential procedure $(P_{S_2})$ of Paulson which takes a single vector of observations at each stage of experimentation, and for which the number of stages $(N_{S_2})$ to terminate experimentation is a bounded r.v. $(\leq M)$, is an adaptive procedure. (For $P_{S_2}$ a vector consists of one observation from each of the $k_j$ populations still retained "in contention" by the procedure at stage $j$ ($1 \leq j \leq M$, $k = k_1 \geq k_2 \geq \ldots \geq k_M \geq 2$), those not retained at stage $j$ being permanently eliminated; here the $k_j$ ($2 \leq j \leq M$) are r.v.'s.) In addition to guaranteeing (2.1) when the population means are in the LF-configuration, it also reacts to favorable configurations of the population means, eliminating from further sampling populations which are indicated as not being in contention, and in general terminating experimentation early resulting in $\mu_{S_2}^{-1}|P_{S_2}$-values which are less than $n$. (See Ramberg [1966].) Of considerable interest is the fact that if $\mu_{[1]} = \mu_{[k]}$ and $P^*$ is close to unity, then

$E_{\mu_{S_2}} \{N_{S_2} | P_{S_2}\} < E_{\mu_{S_1}} \{N_{S_1} | P_{S_1}\}$ for the actual $\mu \in \Omega(\delta^*)$ which the experimenter has encountered. The quantity $E_{\mu_{S_2}} \{T_{S_2} | P_{S_2}\}$, where $T =$ total number of observations to terminate experimentation, behaves similarly w.r.t.

$E_{\mu_{S_1}} \{T_{S_1} | P_{S_1}\}$. Since $N_{S_2} \leq M$ we have $E_{\mu_{S_2}} \{N_{S_2} | P_{S_2}\} < M$ and $E_{\mu_{S_2}} \{T_{S_2} | P_{S_2}\} < kM$ for all $\mu \in \Omega$; the bound $M$ is a function of $k$, $(\delta^*, P^*)$ and also of a design parameter $\lambda$ ($0 < \lambda \leq \delta^*/2$) which is fixed by the experimenter before the start of experimentation.

Even though $P_{S_1}$ and $P_{S_2}$ have certain highly desirable properties relative to $P_1$, both being adaptive and therefore being able to capitalize
on favorable configurations of the population means, both have the drawback that they may require many stages to terminate experimentation. Such procedures are often very costly to implement, and in some experimental situations may be completely impractical, e.g., in agriculture where only one stage, i.e., vector of observations, can be obtained each growing season, the number of stages (years) to terminate experimentation would be prohibitively large.

Thus, in this present paper we study a two-stage procedure which takes a fixed number of vectors of observations at each stage of experimentation. The procedure guarantees (2.1) when the population means are in the LF-configuration. It is adaptive, eliminating from further sampling in the second stage populations which are indicated as not being in contention after the first stage, and in general terminating experimentation after the first stage if the configuration of the population means is very favorable, e.g., \( \mu[k] - \mu[k-1] \gg \delta^* \). In addition this procedure is designed to be minimax within a certain class of two-stage procedures.

4. A two-stage procedure \((P_2)\)

We propose a two-stage procedure \( P_2 = P_2(n_1, n_2, h) \) which depends on non-negative integers \( n_1, n_2 \) and a real constant \( h \geq 0 \) which are determined prior to the start of experimentation. The constants \( (n_1, n_2, h) \) depend on \( k \) and \( \{\delta^*, p^*\} \), and are chosen so that \( P_2 \) guarantees (2.1) and possesses a certain minimax property.

Procedure \( P_2 \):

1. In the first stage take \( n_1 \) independent observations \( X_{ij}^{(1)} \)

\( (1 \leq j \leq n_1) \) from \( \Pi_i \) \( (1 \leq i \leq k) \), and compute the \( k \)
sample means $\bar{x}_{i}^{(1)} = \frac{n_1}{n_1} \sum_{j=1}^{n_1} x_{ij}^{(1)} / n_1$ (1 $\leq$ i $\leq$ k). Let

$$\bar{x}_{[k]} = \max_{1 \leq i \leq k} \bar{x}_{i}^{(1)}.$$ Determine the subset $I$ of 

$$\{1, 2, \ldots, k\}$$ where $I = \{i | \bar{x}_{i}^{(1)} \geq \bar{x}_{[k]}^{(1)} - h\}$, and 

let $\Pi_I$ denote the associated subset of \{\Pi_1, \Pi_2, \ldots, \Pi_k\}. 

a) If $\Pi_I$ consists of one population, stop sampling 

and assert that the population associated with 

$$\bar{x}_{[k]}^{(1)}$$ is best. 

b) If $\Pi_I$ consists of more than one population 

proceed to the second stage. 

2. In the second stage take $n_2$ additional independent 

observations $x_{ij}^{(2)}$ (1 $\leq$ j $\leq$ $n_2$) from each population 

in $\Pi_I$, and compute the cumulative sample means 

$$\bar{x}_i = \left( \frac{1}{n_1} \sum_{j=1}^{n_1} x_{ij}^{(1)} + \frac{1}{n_2} \sum_{j=1}^{n_2} x_{ij}^{(2)} \right) / (n_1 + n_2)$$ for $i \in I$. Assert 

that the population associated with \max_{i \in I} \bar{x}_i is best. 

Remark 4.1: This procedure had been proposed previously by Cohen [1959] 
and Alam [1970]. Due to analytical and computational difficulties, most 
of their work was limited to the special case $k = 2$. 

Remark 4.2: If $h = 0 \ (h = \infty)$ the two-stage procedure $P_2$ reduces to 
Bechhofer's [1954] single-stage procedure $P_1$ with single-stage sample size 
$n = n_1 (n_1 + n_2)$ per population. Also, the rule determining $I$ in (4.1a) 
is of the type proposed by Gupta [1956,1965] in his subset selection 
procedure. 

There are an infinite number of combinations of $(n_1, n_2, h)$ which for 
any $k$ and \{\delta, \rho\} will exactly guarantee (2.1), and different design 
criteria lead to different choices. In the next sections we consider two 
of these criteria.
Let $S'$ denote the cardinality of the set $I$ in (4.1a), and let

$$
S = \begin{cases} 
0 & \text{if } S' = 1 \\
S' & \text{if } S' > 1.
\end{cases}
$$

(4.2)

Then the total sample size required by $P_2(n_1, n_2, h)$ is

$$
T = kn_1 + Sn_2.
$$

(4.3)

Let $E_{\mu}\{T|P_2\}$ denote the expected total sample size for $P_2(n_1, n_2, h)$ under $\mu$.

4.1 R-minimax design criterion

The design criterion proposed by Alam [1970] is the following: For given $k$ and specified $\{\delta^*, P^*\}$ choose $(n_1, n_2, h)$ to

$$
\text{minimize } \sup_{\mu \in \Omega(\delta^*)} E_{\mu}\{T|P_2\}
$$

subject to $\inf_{\mu \in \Omega(\delta^*)} P_2\{CS|P_2\} \geq P^*$,

(4.4)

where $n_1, n_2$ are non-negative integers and $h \geq 0$.

We denote by $(n_1, n_2, h|R_E)$ the exact solution to (4.4), and by $P_2(R_E)$ the procedure using this solution. The R-minimax criterion in which minimization takes place over a restricted portion of $\Omega$ insures that $E_{\mu}\{T|P_2(R_E)\} \leq kn$ for any given $k$ and specified $\{\delta^*, P^*\}$. However, it ignores what can happen to $E_{\mu}\{T|P_2(R_E)\}$ if, unknown to the experimenter, $\mu \in \Omega_0(\delta^*)$. Indeed, for $\mu[1] = \mu[k]$ it is possible to have
E_{T}\{P_2(R_\mu)\} \gg kn \text{ for } P_\mu \text{ sufficiently close to unity (as happens when }\frac{E_{T}}{E_{T}}\text{ of the Wald-Girshick SPRT is compared to the total single-stage sample size which guarantees the same probability requirement. See Bechhofer [1960] and B-K-S [1969], Section 12.8.1). It is to guard against this latter undesirable possibility that we propose the design criterion described below.}

4.2 **U-minimax design criterion**

Our design criterion is the following: For given $k$ and specified
\{\delta^*\},\{P^*\} choose $(n_1,n_2,h)$
to

\[
\begin{align*}
\text{minimize } & \sup_{\mu \in \Omega} E_{\mu}\{T\} & \text{subject to } & \inf_{\mu \in \Omega(\delta^*)} P_{\mu}\{CS|P_2\} \geq P^*, \\
\text{where } & n_1,n_2 \text{ are non-negative integers and } h \geq 0.
\end{align*}
\]

We denote by $(n_1,n_2,h|U_E)$ the exact solution to (4.5) and by $P_2(U_E)$ the procedure using this solution. Our U-minimax criterion (4.5) in which minimization takes place over the unrestricted parameter space $\Omega$ insures that $E_{\mu}\{T\}P_2(U_E) \leq kn \forall \mu \in \Omega$ for any given $k$ and specified $\{\delta^*,P^*\}$; in this sense $P_2(U_E)$ is uniformly (in $\mu$) superior to $P_1$.

As the first step in determining $(n_1,n_2,h|R_E)$ or $(n_1,n_2,h|U_E)$ we find an exact analytical expression for $P_{\mu}\{CS|P_2\}$. 
5. Probability of a correct selection for $P_2$

5.1 General expression for $P_{\mu}^C$ 

Our result concerning a general expression for $P_{\mu}^C$ is summarized in the following theorem:

Theorem 5.1: For any $\mu \in \Omega$ we have

$$P_{\mu}^C = \sum_{\mathbf{S} \subseteq S} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{i \in \mathbf{S}} \left[ \int_{x-(\delta_{ki}-h)\sqrt{n_1}/\sigma}^{x+(\delta_{ki}-h)\sqrt{n_1}/\sigma} \phi(y+(x-z)(p/q)^{1/2} + \frac{\delta_{ki}}{\sigma} \left( \frac{m}{q} \right)^{1/2} \right] \phi(\sigma) \right] d\phi(z)$$

$$\times \prod_{i \in \mathbf{S}} \phi[x + (\delta_{ki}-h)\sqrt{n_1}/\sigma] d\phi(y) d\phi(x)$$

$$\times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j \in \mathbf{S}_j} \left[ \int_{x-(\delta_{kj}+h)\sqrt{n_1}/\sigma}^{x-(\delta_{kj}+h)\sqrt{n_1}/\sigma} \phi(y+(x-u)(p/q)^{1/2} - \delta_{kj}(m/q)^{1/2} \right] \phi(\sigma) \right)$$

$$\times \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{i \in \mathbf{S}} \left[ \int_{x-(\delta_{ij}+h)\sqrt{n_1}/\sigma}^{x-(\delta_{ij}+h)\sqrt{n_1}/\sigma} \phi(y+(x-z)(p/q)^{1/2} + \delta_{ki}(m/q)^{1/2} \right] \phi(\sigma) \right)$$

$$\times \left( \prod_{i \in \mathbf{S}} \phi[x - (\delta_{ij}+h)\sqrt{n_1}/\sigma] d\phi(v) d\phi(u) \right) d\phi(y) d\phi(x),$$

where $\phi(\cdot)$ is the standard univariate normal cdf, and

- $S$ is the collection of all subsets of $\{1,2,\ldots,k-1\}$,
- $S_j$ is the collection of all subsets of $\{1,2,\ldots,j-1,j+1,\ldots,k-1\}$,
- $m = n_1 + n_2$, $p = n_1/m$, $q = n_2/m$. 
Proof: Let \( \bar{X}_s^{(i)} = \frac{1}{m} \sum_{j=1}^{n_j} X_{i,a}^{(j)} \) and \( \overline{X}_s^{(i)} = \frac{1}{m} \sum_{j=1}^{n_j} X_{i,a}^{(j)} \) where \( X_{i,a}^{(j)} \) is the \( a \)th observation in the \( j \)th stage from \( \Pi_s(i) \), all \( X_{i,a}^{(j)} \) \((1 \leq i \leq k, 1 \leq a \leq n_j, j = 1, 2)\) being independent. Then

\[
P_{s,j}^{\Pi} \{S | P_2 \} = \sum_{s \in S} P_{s,j}^{\Pi} \{ \bar{X}_s^{(i)} = \overline{X}_s^{(i)}, \bar{X}_s^{(i)} \geq \overline{X}_s^{(i)} \} - h \quad \forall \ i \in S;
\]

\[
\bar{X}_s^{(i)} < \overline{X}_s^{(i)} - h \quad \forall \ i \notin S; \quad \overline{X}_s^{(i)} > \overline{X}_s^{(i)} \quad \forall \ i \in S.
\]

\[
= \sum_{s \in S} A_s + \sum_{j=1}^{k-1} \sum_{s \in S} B_{s,j}.
\]

Denoting \([\bar{X}_s^{(i)} - \mu_s^{(i)}]\sqrt{n_j}/\sigma, (\overline{X}_s^{(i)} - \mu_s^{(i)})\sqrt{m}/\sigma]\) by \([X_i, Y_i]\), we see that \([X_i, Y_i]\) has a standard bivariate normal distribution with correlation coefficient \(\sqrt{\rho}\) \((1 \leq i \leq k)\). In what follows we shall use the equality

\[
\phi_2[a,b|\rho] = \int_{-\infty}^{a} \phi \left( \frac{b - rz}{(1 - \rho^2)^{1/2}} \right) \, dz \quad (5.3)
\]

where \(\phi_2[a,b|\rho]\) is the standard bivariate normal cdf with correlation coefficient \(\rho\) \((-1 < \rho < 1)\).

We first consider
\[ A_s = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{i \in s} \left( \phi_2^{u + \delta_{ki} \sqrt{n_1}/\sigma, v + \delta_{ki} \sqrt{m}/\sigma, u, v|\sqrt{p}_i} \right) \]

\[ \times \prod_{i \in s} \phi^{u + (\delta_{ki} - h) \sqrt{n_1}/\sigma} \right] \phi \left( u, v \right) d\phi \left( z \right) \]

\[ \times \prod_{i \in s} \phi^{u + (\delta_{ki} - h) \sqrt{n_1}/\sigma} \right] \phi \left( u, v \right) d\phi \left( z \right) \]

The next to last equality was obtained using (5.3), and the last was obtained by making the transformation \( x = u, \ y = (v - u/\sqrt{p})/(1 - p)^{1/2} \).

We next consider...
\[ \beta_{s,j} = P_{u_{i}}[X_{j} - \delta_{kj}\sqrt{n}/\sigma > X_{k} = X_{j} - (\delta_{kj} + h)\sqrt{n}/\sigma, \]

\[ X_{j} - \delta_{ij}\sqrt{n}/\sigma > X_{i} = X_{j} - (\delta_{ij} + h)\sqrt{n}/\sigma \quad \forall \ i \in s; \]

\[ X_{j} - (\delta_{ij} + h)\sqrt{n}/\sigma > X_{i} \quad \forall \ i \not\in s; \]

\[ Y_{k} + \delta_{kj}\sqrt{m}/\sigma > Y_{j}, \quad Y_{k} + \delta_{ki}\sqrt{m}/\sigma > Y_{i} \quad \forall \ i \in s \]  

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{x-\delta_{ij}\sqrt{n}/\sigma} \int_{t-\delta_{ij}\sqrt{m}/\sigma}^{\infty} \left( \prod_{i \in s} \Phi_{2}[x - \delta_{ij}\sqrt{n}/\sigma, w + \delta_{ki}\sqrt{m}/\sigma|v_{p}]\right) \right. \]

\[ - \Phi_{2}[x - (\delta_{ij} + h)\sqrt{n}/\sigma, w + \delta_{ki}\sqrt{m}/\sigma|v_{p}] \}

\[ \times \prod_{i \in s} \Phi[x - (\delta_{ij} + h)\sqrt{n}/\sigma]d\Phi_{2}(u, w|v_{p}) \right) d\Phi_{2}(x, t|v_{p}). \]

Proceeding as with \( A_{s} \), we apply (5.3) to (5.5) and then make the transformations \( x = x, y = (t-x\sqrt{p}/(1-p))^{1/2} \) and \( u = u, v = (w-u\sqrt{p})/(1-p)^{1/2} \).

Substituting the resulting expression and (5.4) in (5.2) we obtain (5.1).

**Corollary 5.1:** Let \( \mu(\delta) \) denote any \( \mu \in \Omega \) such that \( \mu_{[1]} = \mu_{[k-1]} = \mu_{[k]} - \delta \)
where \( \delta > 0 \). (\( \mu(\delta) \) is known as a slippage configuration.) Then we have

\[ P_{\mu(\delta)}(CS|P_{2}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{x+(\delta-h)\sqrt{n}/\sigma}^{x+\delta\sqrt{n}/\sigma} \Phi[y + (x-z)(p/q)^{1/2} + \delta(m/q)^{1/2}/\sigma]d\Phi(z) \right. \]

\[ + \Phi[x + (\delta-h)\sqrt{n}/\sigma] \right) \Phi_{2}(y) d\Phi(x) \]

\[ + (k-1) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{x+(\delta-h)\sqrt{n}/\sigma}^{x+\delta\sqrt{n}/\sigma} \Phi[y + (x-u)(p/q)^{1/2} - \delta(m/q)^{1/2}/\sigma]d\Phi(z) \right. \]

\[ + \Phi[x - h\sqrt{n}/\sigma] \right) \Phi_{2}(v) d\Phi(u) \left) \right) d\Phi(y) d\Phi(x). \]
Proof: The proof is straightforward.

Remark 5.1: For \( k = 2 \), (5.6) simplifies considerably; the resulting expression is given by Alam [1970] as his equation (3.1). Alam also gives an expression for \( P_{\mu}(\delta)|P_2 \) \) (see his second equation (3.24)); however, we were not able to verify his expression.

5.2 LF-configuration for \( P_2 \)

In order to solve (4.4) or (4.5) it first is necessary to determine the LF-configuration of the \( \mu_i \) for \( P_2 \), i.e., to determine any \( \mu_0 \in \Omega(\delta^\#) \) such that

\[
P_{\mu_0}(\mu|P_2) = \inf_{\mu \in \Omega(\delta^\#)} P_{\mu}(\mu|P_2).
\]

(5.7)

As a first step toward determining \( \mu_0 \) we now study the monotonicity of \( P_{\mu}(\mu|P_2) \) w.r.t. the \( \mu_{[i]} \) \((1 \leq i \leq k)\).

Lemma 5.1: For fixed \( k \) and \( \mu_{[i]} \) \((1 \leq i \leq k-1)\) and fixed \( (n_1, n_2, h) \), \( P_{\mu}(\mu|P_2) \) is non-decreasing in \( \mu_{[k]} \).

Proof: From (5.4) and (5.5) we have

\[
P_{\mu}(\mu|P_2) = P_{\mu} \left( \bigcup_{\mu \in \Omega(\delta^\#)} \left\{ (X_k + \delta_{ki} \sqrt{n_1}/\sigma > X_i > X_k + (\delta_{ki} - h) \sqrt{n_1}/\sigma \ \forall i \in s, \right. \right.
\]

\[
X_k + (\delta_{ki} - h) \sqrt{n_1}/\sigma > X_i \ \forall i \notin s) \bigcup \left( \bigcup_{j \in s} (X_j - \delta_{kj} \sqrt{n_1}/\sigma > X_k > X_j - (\delta_{kj} + h) \sqrt{n_1}/\sigma, \right.
\]

\[
X_j - \delta_{ij} \sqrt{n_1}/\sigma > X_i > X_j - (\delta_{ij} + h) \sqrt{n_1}/\sigma \ \forall i \in s, i \neq j; \right.
\]

\[
X_j - (\delta_{ij} + h) \sqrt{n_1}/\sigma > X_i \ \forall i \notin s) \right\} \cap \{ Y_k + \delta_{ki} \sqrt{m}/\sigma > Y_i \ \forall i \in s \})
\]

\[
= P\{A(\mu)\}
\]
where \( A(\mu) \) is in the sigma algebra generated by the r.v.'s \( [X_i, Y_i] \) \( (1 \leq i \leq k) \).

Now consider a vector \( \mu' = (\mu'_1, \ldots, \mu'_k) \) where \( \mu'_i = \mu'_i \) \( (1 \leq i \leq k-1) \) and \( \mu'_k > \mu'_k \). Then \( P_{\mu'}(CS \mid P_2) = P(A(\mu')) \). We shall show that \( A(\mu') \supset A(\mu) \). We denote the value taken on by a r.v. \( X \) at a sample point \( \omega \) by \( X(\omega) \). Also let \( \delta'_{ij} = \mu'_{[i]} - \mu'_{[j]} \) \( (1 \leq i, j \leq k) \). Then \( \delta'_{ki} > \delta_{ki} \) \( (1 \leq i \leq k-1) \) and \( \delta'_{ij} = \delta_{ij} \) \( (1 \leq i, j \leq k-1) \).

Fix \( \omega \in A(\mu) \) which corresponds to some set \( s \in S \).

**Case 1:** Suppose that \( \omega \) belongs to the following event:

\[
\{X_k(\omega) + \delta_{ki} \sqrt{n} \sigma > X_i(\omega) \text{ or } X_i(\omega) + (\delta_{ki}^* - h) \sqrt{n} \sigma > X_k(\omega) \forall i \notin s, \text{ or } \}
\]

\[
X_i(\omega) + (\delta_{ki}^* - h) \sqrt{n} \sigma > X_k(\omega) \forall i \notin s}. \]

Then it also belongs to the event

\[
\{X_k(\omega) + \delta'_{ki} \sqrt{n} \sigma > X_i(\omega) \text{ or } X_i(\omega) + (\delta'_{ki} - h) \sqrt{n} \sigma > X_k(\omega) \forall i \notin s', \}
\]

\[
X_i(\omega) + (\delta'_{ki} - h) \sqrt{n} \sigma > X_k(\omega) \forall i \notin s'} \}
\]

for some set \( s' \in S, s' \subseteq s \).

**Case 2:** Suppose that \( s \) is non-empty and for some \( j \notin s \), \( \omega \) belongs to the following event:

\[
\{X_j(\omega) - \delta_{kj} \sqrt{n} \sigma > X_k(\omega) > X_j(\omega) - (\delta_{kj} + h) \sqrt{n} \sigma, (5.9)\}
\]

\[
X_j(\omega) - \delta_{ij} \sqrt{n} \sigma > X_i(\omega) > X_j(\omega) - (\delta_{ij} + h) \sqrt{n} \sigma \forall i \in s, i \neq j; \]

\[
X_j(\omega) - (\delta_{ij} + h) \sqrt{n} \sigma > X_i(\omega) \forall i \notin s}. \]

Now suppose that $\delta_{ij}$ in (5.9) are replaced by $\delta'_{ij}$ and that $X_j(\omega) - \delta'_{kj}\sqrt{n_1}/\sigma > X_k(\omega)$ is still satisfied. Then (5.9) holds with $\delta_{ij}$ replaced by $\delta'_{ij}$ $(1 \leq i, j \leq k)$ and $s = s'$. On the other hand, if $X_j(\omega) - \delta'_{kj}\sqrt{n_1}/\sigma > X_k(\omega)$ is violated, then $\omega$ must belong to the following event:

$$\{X_k(\omega) + \delta'_{kj}\sqrt{n_1}/\sigma > X_i(\omega) > X_k(\omega) + (\delta'_{ki} - h)\sqrt{n_1}/\sigma \quad \forall i \in s',$$

$$X_k(\omega) + (\delta'_{ki} - h)\sqrt{n_1}/\sigma > X_i(\omega) \quad \forall i \notin s'\}$$

for some $s' \subseteq s$.

From Cases 1 and 2 we have $s' \subseteq s$, and hence we obtain

$$\{Y_k(\omega) + \delta_{ki}\sqrt{n_1}/\sigma > Y_i(\omega) \quad \forall i \in s\}$$

$$\Rightarrow \{Y_k(\omega) + \delta'_{ki}\sqrt{n_1}/\sigma > Y_i(\omega) \quad \forall i \notin s'\}.$$

Therefore $\omega \in A(\mu) \Rightarrow \omega \in A(\mu')$ and $A(\mu) \subseteq A(\mu')$. Hence $P(A(\mu')) \geq P(A(\mu))$ and $P_2(CS|P_2) > P_2(CS|\mu)$ which completes the proof of the lemma.

**Corollary 5.2:** $P_2(\mu)(CS|P_2)$ is non-decreasing in $\delta \geq 0$ when $\mu[1] = \mu[k-1]$ is fixed. In particular, for $k = 2$ $P_2(\mu)(CS|P_2)$ achieves its infimum over $\Omega(\delta')$, i.e., satisfies (5.7), at any $\mu$ satisfying $\mu[2] - \mu[1] = \delta'$.

**Proof:** Since $P_2$ is translation invariant, $P_2(\mu)(CS|P_2)$ depends on $\mu$ only through the $\delta_{ki}$ $(1 \leq i \leq k-1)$. The result then follows from Lemma 5.1.

**Remark 5.2:** The method of proof used for Lemma 5.1 does not carry over to prove our intuitive conjecture that $P_2(\mu)(CS|P_2)$ is non-increasing in each $\mu[i]$ $(i \neq k)$ when the remaining $\mu$-values remain fixed; nor have we been successful.
in finding alternative methods of proof. However, Monte Carlo samplings that we have conducted have supported this conjecture. Nonetheless, the monotonicity of \( \mu \{ CS | P_2 \} \) in the \( \delta_{ki} \) (1 \( \leq \) i \( \leq \) k-1) for \( k > 2 \) still remains an open question. We believe that the following conjecture made by Alam [1970] is correct:

**Conjecture 5.1:** For fixed \( k > 2 \) and \((n_1, n_2, h)\), the slippage configuration \( \mu(\delta^*) \) is a LF-configuration for \( P_2 \).

### 5.3 Lower bound for \( \mu \{ CS | P_2 \} \)

In this section we derive a lower bound for \( \mu \{ CS | P_2 \} \). This lower bound will prove to be particularly useful for \( k > 2 \) since we will prove that it achieves its infimum over \( \mu(\delta^*) \) at \( \mu(\delta^*) \), the conjectured LF-configuration for \( P_2 \); this result will permit us to construct a conservative 2-stage procedure (for \( k > 2 \)) which will guarantee (2.1). The lower bound involves integrals the values of which can be easily calculated on a digital computer.

**Theorem 5.2:** For any \( \mu \in \Omega \) we have

\[
\mu \{ CS | P_2 \} \geq \int_{-\infty}^{\infty} \left( \prod_{i=1}^{k-1} \phi[\frac{x + (\delta_{ki} + h)\sqrt{n_1}/\sigma}{\sqrt{n_1}/\sigma}] + \prod_{i=1}^{k-1} \phi[\frac{x + \delta_{ki}\sqrt{m}/\sigma}{\sqrt{m}/\sigma}] \right) d\Phi(x) - 1. \tag{5.10}
\]

**Proof:**

\[
1 - \mu \{ CS | P_2 \} = \mu \{ \text{Incorrect selection} | P_2 \}
\]

\[
\leq \mu \{ X_{(1)}^{(k)} < X_{(1)}^{(i)} - h \text{ for some } i \neq k \} + \mu \{ X_{(k)}^{(i)} > X_{(i)}^{(i)} \text{ for some } i \neq k \}
\]

\[
= 1 - \mu \{ X_{(1)}^{(k)} \geq X_{(1)}^{(i)} - h \text{ for some } i \neq k \} + 1 - \mu \{ X_{(k)}^{(i)} \geq X_{(i)}^{(i)} \text{ for some } i \neq k \}
\]

\[
= 2 - \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \phi[\frac{x + (\delta_{ki} + h)\sqrt{n_1}/\sigma}{\sqrt{n_1}/\sigma}] d\Phi(x) - \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \phi[\frac{x + \delta_{ki}\sqrt{m}/\sigma}{\sqrt{m}/\sigma}] d\Phi(x).
\]
A rearrangement of the terms gives the desired lower bound.

**Corollary 5.3:** For all \( \mu \in A(\delta^k) \) we have

\[
P_\mu \{CS|P_2\} = \int_{-\infty}^{\infty} \left( \phi^{k-1}[x + (\delta^k + h)\sqrt{n_1}/\sigma] + \phi^{k-1}[x + \delta^k\sqrt{m}/\sigma]\right) d\phi(x) - 1. \tag{5.11}
\]

**Proof:** The proof follows immediately on noting that the r.h.s. of (5.10) is non-decreasing in each \( \delta_{ki} \) for \( 1 \leq i \leq k-1 \).

**Corollary 5.4:** Since the r.h.s. of (5.11) is strictly increasing in each of \( n_1, m, h, \) and \( \lambda^2 \) as \( n_1 \) or as \( n_2 \) and \( h \to \infty \), we see that (2.1) can be guaranteed if all are chosen sufficiently large.

As a consequence of Corollary 5.4 it is clear that a conservative two-stage procedure which guarantees (2.1) and which employs either the R-minimax or the U-minimax design criterion can be constructed and implemented using the lower bound given by the r.h.s. of (5.11). We shall denote such procedures which employ these criteria by \( P_2(R_C) \) and \( P_2(U_C) \), respectively. \( P_2(R_C) \) is conservative relative to \( P_2(R_E) \) (as is \( P_2(U_C) \) relative to \( P_2(U_E) \)) since it overprotects the experimenter with respect to (2.1), this overprotection being purchased at the expense of an increase in \( E_\mu \{T|P_2(R_C)\} \) and \( E_\mu \{T|P_2(U_C)\} \) at \( \mu(\delta^k) \) and \( \mu(0) \), respectively. We consider \( P_2(R_C) \) and \( P_2(U_C) \) in detail in Sections 7-9.

**Remark 5.3:** If we let \( h \to \infty \) on the r.h.s. of (5.10) we obtain

\[
\int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \phi[x + \delta_{ki}\sqrt{m}/\sigma] d\phi(x) \quad \text{which is an expression for} \quad P_\mu \{CS|P_1\} \text{ where } P_1 \text{ uses a common single-stage sample size } m \text{ per population. Thus } P_1 \text{ is a special case of any } P_2(\cdot) \text{ based on the conservative lower bound.} \]
6. Expected total sample size for $P_2$

In order to solve either (4.4) or (4.5) we first find an analytical expression for $E_i(T|P_2)$; this is done in Section 6.1. Then it is necessary to determine $\text{Sup}_\mu E_i(T|P_2)$ for $\mu \in \Omega(\delta^2)$ and for $\mu \in \Omega$ for (4.4) and (4.5), respectively; the sets of $\mu$-values at which these suprema occur are found in Section 6.2.

6.1 General expression for $E_i(T|P_2)$

Our result concerning a general expression for $E_i(T|P_2)$ is summarized in the following theorem:

Theorem 6.1: For any $\mu \in \Omega$ we have

$$E_i(T|P_2) = \frac{kn_1 + n_2}{k} \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} \left( \prod_{j=1 \atop j \neq i}^{k} \phi(x + (\delta_{ij} + h)\sqrt{n_1}/\sigma) - \prod_{j=1 \atop j \neq i}^{k} \phi(x + (\delta_{ij} - h)\sqrt{n_1}/\sigma) \right) d\phi(x). \tag{6.1}$$

Proof:

$$E_i(T|P_2) = kn_1 + n_2E_{i}(S'|P_2)$$

$$= kn_1 + n_2[E_{i}(S'|P_2) - P_{i}(S' = 1|P_2)] \tag{6.2}$$

$$= kn_1 + n_2 \left[ \sum_{i=1}^{k} P_{i}(\bar{X}(i) \geq \bar{X}(1)) - h \ \forall \ j \neq i \right]$$

$$- \left[ \sum_{i=1}^{k} P_{i}(\bar{X}(i) \geq \bar{X}(1) + h \ \forall \ j \neq i \right].$$

Theorem 6.1 follows immediately.
6.2 The supremum of $E_{\mu} \{ T | P_2 \}$

Our results concerning the supremum of $E_{\mu} \{ T | P_2 \}$ for $\mu \in \Omega (\delta^k)$ and $\mu \in \Omega$ are summarized in a) and b) of the following theorem.

**Theorem 6.2**: For fixed $k$ and $(n_1, n_2, h)$ we have that

\[
\text{a) } \sup_{\mu \in \Omega (\delta^k)} E_{\mu} \{ T | P_2 \} = kn_1 \\
+ n_2 \left[ \int_{-\infty}^{\infty} \phi^{k-1} [x + (\delta^k + h) \sqrt{n_1} / \sigma] - \phi^{k-1} [x + (\delta^k - h) \sqrt{n_1} / \sigma] \right] d\phi(x) \\
+ (k-1) \int_{-\infty}^{\infty} \phi^{k-2} [x + h \sqrt{n_1} / \sigma] \phi [x - (\delta^k - h) \sqrt{n_1} / \sigma] \\
- \phi^{k-2} [x - h \sqrt{n_1} / \sigma] \phi [x - (\delta^k + h) \sqrt{n_1} / \sigma] \right] d\phi(x) \\
\]

(6.3)

which occurs when $\mu[1] = \mu[k-1] = \mu[k] = \delta^k$.

\[
\text{b) } \sup_{\mu \in \Omega} E_{\mu} \{ T | P_2 \} = kn_1 + kn_2 \int_{-\infty}^{\infty} \phi^{k-1} [x + h \sqrt{n_1} / \sigma] - \phi^{k-1} [x - h \sqrt{n_1} / \sigma] \right] d\phi(x) \\
\]

(6.4)

which occurs when $\mu[1] = \mu[k]$.

We shall prove part b) of Theorem 6.2; the proof of part a) follows along the same lines.

**Proof**: Gupta [1965] has shown that $E_{\mu} \{ S' | P_2 \}$ achieves its supremum for $\mu \in \Omega$ when $\mu[1] = \mu[k]$. It only remains to show that $P_{\mu} \{ S' = 1 | P_2 \}$ achieves its infimum when $\mu[1] = \mu[k]$. We use the method of Gupta [1965].

Set $\mu[1] = \cdots = \mu[b] = \mu < \mu[b+1]$ for some $b$ ($1 \leq b \leq k-1$) and define $\delta_i = \mu[i] - \mu$ for $b+1 \leq i \leq k$. Define
\[ Q(\mu) = P_{\mu} \{ S' = 1 | P_2 \}; \quad \mu[1] = \ldots = \mu[b] = \mu \]

\[ = b \int_{-\infty}^{\infty} \phi^{b-1}[x - \frac{h}{\sqrt{n_1}}] \prod_{j=b+1}^{k} \phi[x - (\delta_j - h)\sqrt{n_1} / \sigma] \, d\phi(x) \]

\[ + \sum_{i=b+1}^{k} \int_{-\infty}^{\infty} \phi^{b-1}[x + (\delta_i - h)\sqrt{n_1} / \sigma] \prod_{j=b+1}^{k} \phi[x + (\delta_j - h)\sqrt{n_1} / \sigma] \, d\phi(x). \]

After differentiating w.r.t. \( \mu \), and then interchanging the order of integration and summation in the first term, and making appropriate substitutions, we obtain

\[ \frac{dQ}{d\mu} = \frac{b\sqrt{n_1}}{\sigma} \sum_{i=b+1}^{k} \int_{-\infty}^{\infty} \phi^{b-1}[x + (\delta_i - h)\sqrt{n_1} / \sigma] \prod_{j=b+1}^{k} \phi[x + (\delta_j - h)\sqrt{n_1} / \sigma] \, d\phi(x) \]

\[ \cdot \left( \phi[x - \frac{h}{\sqrt{n_1}} / \sigma] \phi[x + \delta_i \sqrt{n_1} / \sigma] - \phi[x + (\delta_i - h)\sqrt{n_1} / \sigma] \phi(x) \right) dx \leq 0. \] (6.5)

The last inequality is obtained by noting that the quantity inside \{ \} in (6.5) is non-positive for every \( x \) and \( i \) for \( b+1 \leq i \leq k \) due to the monotone likelihood ratio property of \( \phi \). It follows that \( Q \) is non-increasing in \( \mu \) and is in fact strictly decreasing if \( h, n_1 > 0 \). Thus subject to \( \mu[1] = \ldots = \mu[b] \), we see that \( P_{\mu} \{ S' = 1 | P_2 \} \) is minimized by increasing the common value \( \mu \) until \( \mu = \mu[b+1] \). Since this is true for each \( b \leq k-1 \), it follows that \( P_{\mu} \{ S' = 1 | P_2 \} \) is minimized and hence \( E_{\mu}[T | P_2] \) is maximized over \( \Omega \) when \( \mu[1] = \mu[k] \).

Using the results of Theorem 5.1 and Lemma 5.1 along with Theorem 6.2 we can now formulate our optimization problems (4.4) and (4.5) precisely.
7. Optimization problems yielding conservative solutions

In this section we consider the optimization problems (4.4) and (4.5) which one must solve in order to determine \((n_1,n_2,h|R_e)\) and \((n_1,n_2,h|U_e)\) which are necessary to implement \(P_2(R_e)\) and \(P_2(U_e)\). As noted in Section 5.2, we have not been successful in determining the LP-configuration of the \(w[i]\)\((1 \leq i \leq k)\), except for \(k = 2\). Thus for \(k > 2\) we replace the exact probability \(\inf_{\mu \in \Omega(\delta^*)} E\{CS|P_2\}\) by the conservative lower bound on that probability given by the r.h.s. of (5.11), and consider the following optimization problems:

7.1 Discrete optimization problems

7.1.1 R-minimax design criterion

For given \(k\) and specified \(\{\delta^*, P^*\}\) choose \((n_1,n_2,h)\) to

\[
\text{minimize} \quad \sup_{\mu \in \Omega(\delta^*)} \frac{E_1[T|P_2]}{\mu} \tag{7.1}
\]

subject to \(\int_{-\infty}^{\infty} [\phi^{k-1}[x + (\delta^* + h)\sqrt{n_1}/\sigma] + \phi^{k-1}[x + \delta^*/\sigma]]d\phi(x) - 1 \geq P^*\),

where \(n_1, n_2\) are non-negative integers and \(h \geq 0\).

In (7.1), \(\sup_{\mu \in \Omega(\delta^*)} E_1[T|P_2]\) is given by (6.3). We denote by \((n_1,n_2,h|R_C)\) the solution to (7.1), and regard it as a conservative solution to (4.4); we denote the corresponding procedure by \(P_2(R_C)\).

7.1.2 U-minimax design criterion

For given \(k\) and specified \(\{\delta^*, P^*\}\) choose \((n_1,n_2,h)\) to
minimize $\sup_{\mu \in \mathcal{P}_2} E_{\mu} T | P_2 \rangle$

subject to $\int_{-\infty}^{\infty} \{ k^{k-1} [x + (\delta^k + h)\overline{n}_1/\sigma] + k^{k-1} [x + \delta^k \overline{m}/\sigma] \} d\phi(x) - 1 \geq P^*$,

where $n_1, n_2$ are non-negative integers and $h \geq 0$.

In (7.2), $\sup_{\mu \in \mathcal{P}_2} E_{\mu} T | P_2 \rangle$ is given by (6.4). We denote by $(n_1, n_2, h | U_C)$ the solution to (7.2), and regard it as a conservative solution to (4.5); we denote the corresponding procedure by $P_2(U_C)$.

7.2 Continuous optimization problems

The problems (7.1) and (7.2) are extremely complicated integer programming problems with non-linear constraints and objective functions. Although these problems can be solved in principle by enumeration, the search is likely to be a costly one because of the numerical evaluation of the integrals involved. Additionally, since the solution depends on $\delta^k$, a separate solution is required not only for each $k$ and $P^*$-value, but also for each $\delta^k$. Hence we shall remove the restriction that $n_1, n_2$ must be integers; we reparametrize the problem and regard the new design constants (which are functions of $n_1, n_2$, and $h$) as continuous. We use this continuous version as a large sample approximation to the discrete version.

We define the new design constants

$$c_1 = \delta^k \overline{n}_1/\sigma, \quad c_2 = \delta^k \overline{n}_2/\sigma, \quad d = h \overline{n}_1/\sigma.$$  \hspace{1cm} (7.3)

We note that the exact expression for $P_{\mu}(CS | P_2)$ and the conservative lower bound on it, as well as $E_{\mu}(T | P_2)$ depend on $(n_1, n_2, h)$, $\delta^k$, $\sigma$ only through $(c_1, c_2, d)$ and $\delta_i^k/\delta^k$ ($1 \leq i \leq k-1$).
Thus, for example, for given $k$ and specified \{$\delta^*, p^*$\} we can approximate the design constants \((n_1, n_2, h | U_C)\) necessary to implement \(P_2(U_C)\) by solving the continuous optimization problem:

\[
\begin{align*}
\text{minimize} & \quad k c_1^2 + k c_2^2 \int_{-\infty}^{\infty} \{\Phi^{k-1}(x+d) - \Phi^{k-1}(x-d)\} d\Phi(x) \\
\text{subject to} & \quad \int_{-\infty}^{\infty} \{\Phi^{k-1}(x+c_1+d) + \Phi^{k-1}(x + (c_1^2 + c_2^2)^{1/2})\} d\Phi(x) - 1 \geq p^* \\
\end{align*}
\tag{7.4}
\]

where \(c_1, c_2, d \geq 0\).

We denote by \((\hat{c}_1, \hat{c}_2, \hat{d} | U_C)\) the solution to (7.4), and use the approximate design constants

\[
\hat{n}_1 = \left[\left(\frac{1}{\delta^*}\right)^2 + 1\right], \quad \hat{n}_2 = \left[\left(\frac{\hat{c}_2^2}{\delta^*}\right)^2 + 1\right], \quad \hat{h} = \frac{\hat{d} \delta^*}{c_1}
\tag{7.5}
\]

where \([z]\) denotes the greatest integer \(< z\), to implement \(P_2(U_C)\).

Similarly, for \(k = 2\) and specified \(\{\delta^*, p^*\}\) we can approximate the design constants \((n_1, n_2, h | U_E)\) necessary to implement \(P_2(U_E)\) by solving the continuous optimization problem:

\[
\begin{align*}
\text{minimize} & \quad 2c_1^2 + 2c_2^2 \{\Phi(d/\sqrt{2}) - \Phi(-d/\sqrt{2})\} \\
\text{subject to} & \quad \Phi[(c_1-d)/\sqrt{2}] + \int_{(c_1-d)/\sqrt{2}}^{(c_1+d)/\sqrt{2}} \Phi[-x\sqrt{p/q} + \sqrt{(c_1^2 + c_2^2)/2q}] d\Phi(x) \geq p^*, \\
\end{align*}
\tag{7.6}
\]

where \(c_1, c_2, d \geq 0\).

We denote the solution by \((\hat{c}_1, \hat{c}_2, \hat{d} | U_E)\). Analogous expressions can be written
in order to approximate the design constants \( (n_1, n_2, h | R_C) \) for \( k \geq 2 \), and 
\( (n_1, n_2, h | R_E) \) for \( k = 2 \).

8. Constants to implement \( P_2 \)

8.1 Constants to implement \( P_2(U_E) \) and \( P_2(R_E) \) for \( k = 2 \)

Table 1 contains constants necessary to approximate \( (n_1, n_2, h | U_E) \) and 
\( (n_1, n_2, h | R_E) \) for \( k = 2 \) and selected \( P^* \); although we are primarily
interested in the ones associated with \( P_2(U_E) \), we have computed those
associated with \( P_2(R_E) \) for comparative purposes. (See Section 9.) The
computations for \( P_2(U_E) \) are the solutions of (7.5), while those for \( P_2(R_E) \)
are the solutions of the analogous problem wherein \( \sup E \{ T | P_2 \} \) over \( \Omega(\delta^*) \)
is minimized. The constants given here for \( P_2(U_E) \) and \( P_2(R_E) \) are exact
since the LF-configuration for \( P \{ CS \} \) is known for \( k = 2 \).

8.2 Constants to implement \( P_2(U_C) \) for \( k \geq 3 \)

Table 2 contains constants necessary to approximate \( (n_1, n_2, h | U_C) \) for \( k = 3, 4, 5, 10, 15, 25 \) and selected \( P^* \); the computations for \( P_2(U_C) \) are the
solution of (7.4). The constants given here for \( P_2(U_C) \) are conservative
since the LF-configuration for \( P \{ CS | P_2 \} \) is unknown for \( k \geq 3 \). (We have
not attempted to compute the constants \( (\hat{c}_1, \hat{c}_2, \hat{d} | U_C) \) which would be used
if the conjectured LF-configuration for \( P \{ CS | P_2 \} \) were indeed \( \mu[k] - \mu[i] = \delta^* \)
\((1 \leq i < k - 1)\) for \( k \geq 3 \); such computations, although of interest, would
be difficult to carry out because it would be necessary to evaluate numerically
very complicated iterated integrals.

All of the computations for Tables 1 and 2 (as well as those described in
Section 9) were carried out in double precision arithmetic on either Cornell's
IBM 360/65 and IBM 370/168 or on Northwestern's CDC 6400. To solve the
Table 1

Constants to implement $p_2(U_E)$ and $p_2(R_E)$ for $k = 2$

| $p^*$ | $(\hat{c}_1, \hat{c}_2, \hat{d}| U_E)$ | $(\hat{c}_1, \hat{c}_2, \hat{d}| R_E)$ |
|-------|--------------------------------------|--------------------------------------|
|       | $\hat{c}_1$ | $\hat{c}_2$ | $\hat{d}$ | $\hat{c}_1$ | $\hat{c}_2$ | $\hat{d}$ |
| 0.9999 | 4.5397 | 2.9087 | 0.97215 | 3.4801 | 4.4120 | 1.9162 |
| 0.9995 | 3.9742 | 2.6708 | 0.95623 | 3.1239 | 3.9034 | 1.6992 |
| 0.999 | 3.7062 | 2.5712 | 0.94824 | 2.9566 | 3.6606 | 1.6026 |
| 0.99  | 2.7189 | 2.0906 | 0.91613 | 2.2931 | 2.7371 | 1.2803 |
| 0.95  | 1.8621 | 1.6155 | 0.88072 | 1.6583 | 1.9347 | 1.0574 |
| 0.90  | 1.4270 | 1.3132 | 0.85278 | 1.3224 | 1.4996 | 0.92974 |
| 0.85  | 1.1391 | 1.0930 | 0.84174 | 1.0789 | 1.1880 | 0.90951 |
| 0.80  | 0.91577 | 0.90970 | 0.82702 | 0.88255 | 0.96174 | 0.87132 |
| 0.75  | 0.72801 | 0.74161 | 0.81999 | 0.71036 | 0.77072 | 0.84413 |
| 0.70  | 0.56240 | 0.58661 | 0.80783 | 0.55468 | 0.59769 | 0.82505 |
| 0.65  | 0.41227 | 0.43299 | 0.80202 | 0.40845 | 0.43957 | 0.81002 |
| 0.60  | 0.26982 | 0.28775 | 0.79714 | 0.26907 | 0.28914 | 0.79865 |
| 0.55  | 0.13374 | 0.14322 | 0.79140 | 0.13378 | 0.14313 | 0.79053 |
\textbf{Table 2}

Constants to implement $P_{2}(U_{c})$ for $k \geq 3$

| $k$ | $p^k$ | $(\hat{c}_1, \hat{c}_2, \hat{d}|U_{c})$ |
|-----|-------|-------------------------------------|
|     |       | $\hat{c}_1$ | $\hat{c}_2$ | $\hat{d}$ |
| 3   | 0.99  | 2.9326      | 2.4083      | 1.2458    |
|     | 0.95  | 2.0893      | 1.8974      | 1.6303    |
|     | 0.90  | 1.6699      | 1.5491      | 2.1814    |
|     | 0.75  | 1.0492      | 0.97980     | 3.9335    |
| 4   | 0.99  | 3.0432      | 2.5805      | 1.2596    |
|     | 0.95  | 2.2400      | 2.1106      | 1.4806    |
|     | 0.90  | 1.8262      | 1.7859      | 1.8245    |
|     | 0.75  | 1.2203      | 1.1712      | 3.2365    |
| 5   | 0.99  | 3.1035      | 2.7301      | 1.2712    |
|     | 0.95  | 2.3184      | 2.2522      | 1.4604    |
|     | 0.90  | 1.9209      | 1.9786      | 1.6403    |
|     | 0.75  | 1.3191      | 1.3304      | 2.7280    |
|     | 0.60  | 0.96047     | 0.91856     | 4.3038    |
| 10  | 0.99  | 3.2364      | 3.1620      | 1.3453    |
|     | 0.95  | 2.5094      | 2.7750      | 1.3529    |
|     | 0.90  | 2.1466      | 2.5349      | 1.3830    |
|     | 0.75  | 1.5712      | 1.9725      | 1.6980    |
|     | 0.60  | 1.2248      | 1.3491      | 2.7648    |
|     | 0.45  | 0.95453     | 0.92770     | 4.3433    |
| 15  | 0.99  | 3.2983      | 3.3883      | 1.3999    |
|     | 0.95  | 2.5886      | 3.0259      | 1.3771    |
|     | 0.90  | 2.2344      | 2.8212      | 1.3676    |
|     | 0.75  | 1.6899      | 2.4268      | 1.3974    |
|     | 0.60  | 1.3404      | 1.7600      | 2.0210    |
|     | 0.45  | 1.0897      | 1.1270      | 3.5471    |
| 25  | 0.99  | 3.3634      | 3.6572      | 1.4783    |
|     | 0.95  | 2.6646      | 3.3204      | 1.4401    |
|     | 0.90  | 2.3270      | 3.1393      | 1.3972    |
|     | 0.75  | 1.8000      | 2.8539      | 1.3234    |
|     | 0.60  | 1.4909      | 2.7798      | 1.1769    |
|     | 0.45  | 1.3026      | 3.1516      | 0.76886   |
continuous optimization problems, first a "reasonably good" discrete optimal solution was found by a search method. This solution was used as an initial guess in the computer program using a modified version of the steepest descent method to solve the continuous non-linear programming problem; Algorithm 304 of Hill and Joyce [1967] was used to evaluate \( \phi(\cdot) \); the integrals were evaluated using the Romberg method of integration. We do not claim that our solutions represent the absolute optima, but they are reasonably close to the optima. (The \( E \{ T | P_2(U_c) \} \)-surface is very flat in the neighborhood of the maximum for \( P^* = 1/k \) since \( P_2(U_c) \rightarrow P_1 \).) The tabulated values should be correct to the number of significant figures given.

9. The performance of \( P_2 \) relative to \( P_1 \)

As a measure of the efficiency of \( P_1 \) (Bechhofer [1954]) relative to that of \( P_2 \) when both guarantee the same probability requirement (2.1), we consider the ratio (termed relative efficiency (RE)) \( \frac{E \{ T | P_2 \} }{E \{ T | P_1 \} } /k \) where \( n = [(\hat{c}\sigma/\delta^*)^2 + 1] \), and \( \hat{c} \) is the solution of

\[
\int_{-\infty}^{\infty} \phi^{-1}(x + \hat{c}) \phi(x) = P^*.
\] (9.1)

Clearly RE depends on \( \hat{c} \) and \( \{ \delta^*, P^* \} \); values of RE less than unity favor \( P_2 \) over \( P_1 \). For mathematical convenience we shall use the continuous approximations to \( E \{ T | P_2 \} \) and \( n \) (thereby ignoring the fact that the sample sizes must be integers). RE is then given by

\[
\left( \frac{k^2 c_1 + c_2}{k} \right) \sum_{i=1}^{k} \left( \prod_{j=1}^{k} \phi(x + d + \delta_i j c_1 / \delta^*) - \prod_{j=1}^{k} \phi(x - d + \delta_i j c_1 / \delta^*) \right) \phi(x) \right) / k c^2 \] (9.2)
where we employ in (9.2) either \((\hat{c}_1, \hat{c}_2, \hat{d}| \mathcal{P}_2(U_E))\) for \(k = 2\) or
\((\hat{c}_1, \hat{c}_2, \hat{d}| \mathcal{P}_2(U_C))\) for \(k \geq 3\). (In order to compare the performance of \(\mathcal{P}_2(U_C)\)
with that of \(\mathcal{P}_2(U_E)\) for \(k = 2\), we also will employ \((\hat{c}_1, \hat{c}_2, \hat{d}| \mathcal{P}_2(U_C))\) for
\(k = 2\). See Table 3.) The value of \(\hat{c}\) in (9.1) has been tabulated for
selected \(k\) and \(P^*\) by Bechhofer [1954], Gupta [1963], and Milton [1963]
(Bechhofer's \(\lambda = \hat{c}\), Gupta's and Milton's \(H = \hat{c}/\sqrt{2}\).

Remark 9.1: For the equal means (EM) configuration \(\mu[1] = \mu[k]\), and for
the \(\mu(\delta^*)\) configuration \(\mu[1] = \mu[k-1] = \mu[k] - \delta^*\) (known to be LF for
\(\mathcal{P}_2(U_E)\) for \(k = 2\) and for \(\mathcal{P}_2(U_C)\) for \(k \geq 2\), and conjectured to be
LF for \(\mathcal{P}_2(U_E)\) for \(k > 3\)), we note that RE depends only on \(k\) and \(P^*\)
given \(\mathcal{P}_2(\cdot)\) and \(\mu \in \Omega\).

Table 3 which concerns \(\mathcal{P}_2(U_E)\) and \(\mathcal{P}_2(U_C)\) for \(k = 2\), and Table 4
which concerns \(\mathcal{P}_2(U_C)\) for \(k \geq 3\), give computed RE-values for selected \(P^*\)
and \(\mu \in \Omega\) to indicate the magnitude of the saving in \(E_{\mu}(T|P)\) achieved
by the screening property of \(\mathcal{P}_2(U_C)\) \((\mathcal{P}_2(U_E))\) when \(\mathcal{P}_2(\cdot)\) is used in
place of \(\mathcal{P}_1\) for \(k \geq 2\) \((k = 2)\); the computations for Tables 3 and 4 are
based on \((\hat{c}_1, \hat{c}_2, \hat{d})\) listed in Tables 1 and 2, respectively.

9.1 \(\mathcal{P}_2(U_E)\) and \(\mathcal{P}_2(U_C)\) vs. \(P_1\) for \(k = 2\)

For all \(P^*\) we note that \(\text{RE}_{\mu(0)}\) is less for \(\mathcal{P}_2(U_E)\) than for \(\mathcal{P}_2(U_C)\),
but \(\text{RE}_{\mu(\infty)}\) is greater for \(\mathcal{P}_2(U_E)\) than for \(\mathcal{P}_2(U_C)\) (since \(\hat{n}_1\) for
\(\mathcal{P}_2(U_E)\) turns out to be greater than \(\hat{n}_1\) for \(\mathcal{P}_2(U_C)\)). The range of
\((\mu[2] - \mu[1])/\delta^* = \delta/\delta^*\) values over which \(\text{RE}_{\mu(\delta)}\) is less for \(\mathcal{P}_2(U_E)\)
than for \(\mathcal{P}_2(U_C)\) for given \(P^*\) appears to depend critically on \(P^*\) being
small for \(P^*\) close to unity and large for \(P^* = 1/2\) (since in this latter
situation \(\mathcal{P}_2(U_C) \rightarrow P_1\)). However, of greatest importance, is the fact that
for either \(\mathcal{P}_2(U_E)\) or \(\mathcal{P}_2(U_C)\) used at any \(P^*\) \((1/2 < P^* < 1)\) we have
Table 3

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Table 4

Efficiency of $P_1$ relative to $P_2(U_C)$ for $k \geq 3$

when the $u_{[i]}$ (1 ≤ $i$ ≤ $k$) are in various configurations

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<th>$P_1^n$</th>
<th>$u_{[k]} = u_{[1]}$</th>
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<th>$\delta/\delta^n = 0.5$</th>
<th>$\delta/\delta^n = 1.0$</th>
<th>$\delta/\delta^n = 2.0$</th>
<th>$\delta/\delta^n = 4.0$</th>
<th>$u_{[k]} - u_{[k-1]}$ = $\delta$</th>
<th>$u_{[i]} - u_{[i-1]} = \delta$ (2 ≤ $i$ ≤ $k-1$)</th>
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<td>0.5008</td>
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<td>0.7683</td>
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<td>0.4766</td>
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<td>0.6058</td>
<td>0.5951</td>
<td>0.5940</td>
<td>0.5249</td>
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</table>
\[ \text{RE}_\theta(\delta) < 1 \text{ for all } \theta \in \Omega \text{ and } \text{RE}_\theta(\infty) << 1. \text{ Thus both } P_2(U_E) \text{ and } P_2(U_C) \text{ are highly effective as screening procedures.} \]

9.2 \( P_2(U_C) \) vs. \( P_1 \) for \( k \geq 3 \)

For given \( k \geq 3 \) the performance of \( P_2(U_C) \) relative to that of \( P_1 \) as measured by RE for \( 1/k < P^* < 1 \) and \( \theta \in \Omega \) is similar to that noted for \( k = 2 \). In addition, if we regard RE as a function of \( k \) for fixed \( P^* \) and configuration of the \( \theta_i \) (\( 1 \leq i \leq k \)), specifically for the configurations \( \theta(0); \theta[k] - \theta[k-1] = \delta^k, \theta[i] - \theta[i-1] = \delta \) (\( 2 \leq i \leq k-1, 0 \leq \delta < \infty \)); \( \theta[k] - \theta[k-1] = \infty \), our computations indicate that RE is decreasing in \( k \) (although this has not been established analytically). Thus the effectiveness of \( P_2(U_C) \) as a screening procedure appears to be increasing with increasing \( k \).

10. Directions of future research

The most important unsolved problem associated with \( P_2 \) is that of determining the LF-configuration of the \( \theta_i \) (see (5.7)) for \( k > 2 \); as noted earlier, we conjecture the answer to be the slippage configuration \( \theta(\delta^k) \). If this conjecture can be shown to be true, it will be necessary to find efficient algorithms for evaluating \( P_\theta(\delta^k)\{CS|P_2\} \) (as given by (5.6)) numerically before the design constants \( (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}|U_E) \) for use with \( P_2(U_E) \) can be determined.

A two-stage minimax screening procedure, analogous to \( P_2 \), can be devised for selecting the smallest (say) normal variance; this was done in Tamhane [1975b]. However, design constants to implement the procedure must still be computed.
Procedure $P_2$ given by (4.1) permanently eliminates populations for which $\bar{X}_i^{(1)} < \bar{X}_k^{(1)}$ - h. However, $P_2$ can be modified in such a way that populations from which a total of only $n_1$ observations are taken, are eligible for selection as "best" along with those from which a total of $n_1 + n_2$ observations are taken; in this modification we assert that the population associated with $\text{max}\{\text{max}_{i \in I} \bar{X}_i, \text{max}_{i \in I} \bar{X}_i^{(1)}\}$ is best. Such a procedure was considered in Tamhane [1975a]; an exact analytical expression for the PCS was derived, and the PCS performance was studied by Monte Carlo sampling methods. Aside from the analytical and computational difficulties (as well as the problem of determining the LF-configuration of the $\mu_i$), procedures of this type would appear to represent a fruitful direction of generalization.

11. Acknowledgments

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12. References


