ASYMPTOTIC EFFICIENCY OF ONE MULTIFACTOR EXPERIMENT RELATIVE TO SEVERAL ONE-FACTOR EXPERIMENTS FOR SELECTING THE NORMAL POPULATION WITH THE LARGEST MEAN

by

V.S. Bawa

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ABSTRACT

In this paper, we consider two single-stage procedures
for ranking means of normal populations with a common known
variance, for problems involving an r-way \( r \geq 2 \) classification
without interaction. For the problem of selecting the best level
of each of the \( r \) factors, the asymptotic \( (P^* \to 1) \) efficiency
of one r-factor experiment relative to \( r \) one-factor experiments
is determined. Some numerical results are provided for \( r = 2 \).
Applications to some related problems are noted.

* Bell Telephone Laboratories, Incorporated
Holmdel, New Jersey
1. Introduction

Bechhofer [1] proposed two single-stage procedures for ranking means of normal populations with a common known variance, for problems involving an r-way \((r \geq 2)\) classification without interaction. For the problem of selecting the best level of each of \(r\) factors, one of the procedures involves conducting one r-factor experiment and using the observations from the single experiment to select the \(r\) best levels; the other procedure involves conducting \(r\) one-factor experiments and using the observations from each experiment to select the best level of each of the associated r-factors. When proposing the factorial design for the 2-factor problem, Bechhofer ([1], p. 28) noted that "the factorial design of the experiment makes the data work twice and is in this sense more efficient than two separate experiments." In this paper, we are concerned with determining the efficiency of the factorial experiment relative to several one-factor experiments for the ranking problem. Relative efficiency is defined here as the ratio of the minimum total number of observations needed by each procedure to guarantee a certain requirement on the specified probability of a correct selection (under the indifference zone approach to the ranking problem). Asymptotic relative efficiency is defined as the limit of this relative efficiency as the specified probability of a correct selection approaches one. In this paper, we determine the asymptotic relative efficiency of the two single-stage procedures proposed
in [1]. For the 2-way classification problem, a table of numerical results for large sample relative efficiency and asymptotic relative efficiency for certain cases is included.

Bechhofer [2], in a paper on ranking multiply-classified variances of normal populations, noted that for large sample sizes (using a suitable log transformation) the problem of ranking multiply-classified variances of normal populations reduces to the problem of ranking multiply-classified means of normal populations. The results obtained in this paper would thus give asymptotic relative efficiency for the corresponding problem of ranking multiply-classified variances of normal populations. Similarly, the results obtained herein would be applicable to a large class of ranking problems which reduce asymptotically to ranking means of normal populations.

2. Formulation of the Ranking Problem

Let \( X_{i_1 i_2 \ldots i_r} \), \( m = 1, 2, \ldots, N_{i_1 i_2 \ldots i_r} \) denote i.i.d. random variables each having a normal distribution, common known variance \( \sigma^2 \), and with

\[
E \left[ X_{i_1 i_2 \ldots i_r} \right] = \mu + \sum_{j=1}^{r} \alpha_{i_j}^{(j)}
\]

\((i_j = 1, 2, \ldots, k_j; j = 1, 2, \ldots, r; m = 1, 2, \ldots, N_{i_1 i_2 \ldots i_r})\),

where \( \mu \) and the \( \alpha_{i_j}^{(j)} \) are unknown with \( \sum_{i_j=1}^{k_j} \alpha_{i_j}^{(j)} = 0, j = 1, 2, \ldots, r \).

For each fixed \( j \), the ranked values of the \( \alpha_{i_j}^{(j)}, i_j = 1, 2, \ldots, k_j \), are denoted by
\[(2.2) \quad \alpha^{(j)}_{[1]} \leq \alpha^{(j)}_{[2]} \leq \ldots \leq \alpha^{(j)}_{[k_j]} \cdot \]

We assume that we have no prior knowledge about the true pairing of the level of any of the factors with the ranked levels of the factors. For \(j = 1, 2, \ldots, r\), let \(\delta_j\) denote \(\alpha^{(j)}_{[k_j]} - \alpha^{(j)}_{[k_j-1]}\) and \(\delta\) denotes \((\delta_1, \delta_2, \ldots, \delta_r)\).

The \(\Pi_{j=1}^{r} k_j\) populations are characterized by their population means, the best population being the factor-level combination associated with the largest mean. (Alternatively, we may define the best population as the one associated with the smallest mean; the analysis is the same in each case.) Clearly, in our setup wherein we are assuming no interaction among the factors, the best combination of factor-levels is the one associated with \(\alpha^{(j)}_{[k_j]}\) \((j = 1, 2, \ldots, r)\). For example, for \(r = 2\), the \(k_1 k_2\) populations may be the \(k_1 k_2\) combinations of \(k_1\) varieties of grain and \(k_2\) types of fertilizers. Then \(\mu + \alpha^{(1)}_{i_1} + \alpha^{(2)}_{i_2}\) will be the mean yield per acre of the \(i_1\)th variety of grain used with the \(i_2\)th type of fertilizer. The best variety-fertilizer combination is the one associated with \(\mu + \alpha^{(1)}_{[k_1]} + \alpha^{(2)}_{[k_2]}\).

Under the indifference zone approach to the problem of selecting the best level of each of the \(r\) factors, the experimenter specifies, prior to experimentation, \(r + 1\) scalar constants \(\{\delta^*, P^*\}\), where \(\delta^* = (\delta^*_1, \ldots, \delta^*_r)\), \(0 < \delta^*_j < \infty\) \((j = 1, 2, \ldots, r)\) and \(\frac{1}{k_1 k_2 \ldots k_r} < P^* < 1\). The probability requirement which the selection procedures must guarantee is
(2.3) \( P\{\text{Correct Selection} | \delta_j \geq \delta_{j}^*, \ j = 1,2,...,r\} \geq p^* \),

where "correct selection" is defined as the selection of the population associated with \( \mu + \sum_{j=1}^{r} a(j) \frac{1}{k_j} \).

In this paper, we consider two single-stage procedures proposed in [1]. The selection procedures, henceforth called SP1 and SP2, are summarized as follows:

Procedure SP1

Take \( N \) independent observations from each of the \( k_1k_2...k_r \) populations. Compute the \( \prod_{j=1}^{r} k_j \) sample means

\[
\overline{X}_{i_1i_2...i_r} = \frac{N}{\sum_{m=1}^{r} X_{i_1i_2...i_r}/N},
\]

and then the

\[
\overline{X}_{i_j}^{(j)} = \left\{ \left. \sum_{i_j=1}^{k_j} \overline{X}_{i_1i_2...i_r} \right| i_j \neq i_1, i_2, \ldots, i_r \right\} / \left( \prod_{\ell=1, \ell \neq j}^{r} k_j \right)
\]

\((i_j = 1,2,...,k_j; j = 1,2,...,r)\).

For \( j = 1,2,...,r \), denote the ranked value of the \( \overline{X}_{i_j}^{(j)} \) by

\[
\overline{x}_{[1]}^{(j)} \leq \overline{x}_{[2]}^{(j)} \leq \cdots \leq \overline{x}_{[k_j]}^{(j)}.
\]
Select the levels associated with $\bar{x}_{[k_1]}, \bar{x}_{[k_2]}, \ldots, \bar{x}_{[k_r]}$ as the best levels for factors 1, 2, ..., r, respectively. The total number of observations required for SP1 is $N \prod_{j=1}^{r} k_j$.

**Procedure SP2**

Fix the levels of (r-1) factors, say all except factor $j$, and take $N_j$ independent observations at each level of factor $j$. For factor $j$ compute the sample mean of observations at each level. Select the level associated with the largest sample mean as the best level of factor $j$. Repeat this procedure for each of the r factors, taking an independent set of observations each time, and select (as above) the best level of each factor.

The total number of observations required by SP2 is $\sum_{j=1}^{r} k_j N_j$.

The sample size (N for SP1 and $(N_1, N_2, \ldots, N_r)$ for SP2) needed to completely specify the selection procedures is determined so as to guarantee the probability requirement (2.3).

It follows from results in [1] (Equation (20), p. 22 and Equation (48), p. 28) that the probability of a correct selection for the so-called "least favorable" parameter configuration is $\prod_{j=1}^{r} P'_j$ for SP1 and $\prod_{j=1}^{r} P_j$ for SP2, where, for $j = 1, 2, \ldots, r$,

$$P'_j = \int_{-\infty}^{\infty} F_j^{k_j-1} (x + \lambda_j \sqrt{N_j}) f(x) dx$$

and

$$P_j = \int_{-\infty}^{\infty} F_j^{k_j-1} (x + \lambda_j \sqrt{N_j}) f(x) dx.$$
In (2.4) and (2.5), for \( j = 1, 2, \ldots, r \), we have let \( \lambda_j \) denote \( \frac{\delta_j^*}{\sigma} \) and 
\( N_j' \) denote 
\[
N \sum_{\substack{x=1 \\
\# j}}^{r} k_x \; f(x)
\]
respectively of a standard normal random variable. The so-called "least favorable" parameter configuration is, for \( j = 1, 2, \ldots, r \), \( \alpha_{[1]}^{(j)} = \alpha_{[k_j-1]}^{(j)} \) and 
\[
\alpha_{[k_j]}^{(j)} - \alpha_{[k_j-1]}^{(j)} = \delta_j^*.
\]

The minimum number of observations needed for procedure SP1 is given as a solution to the following problem:

**Problem I:**

\[
\begin{align*}
\min & \quad N \left( \prod_{j=1}^{r} k_j \right) \\
\text{s.t.} & \quad \prod_{j=1}^{r} P_j' \geq P^*. 
\end{align*}
\]

The minimum number of observations needed for procedure SP2 is given as a solution to the following problem:

**Problem II:**

\[
\begin{align*}
\min & \quad \sum_{j=1}^{r} k_j N_j \\
(\text{s.t.} & \quad \prod_{j=1}^{r} P_j \geq P^*). 
\end{align*}
\]
Let \( \hat{N} \) and \( \hat{N}_1, \hat{N}_2, \ldots, \hat{N}_r \) denote the solution to Problems I and II, respectively. The relative efficiency (RE) and asymptotic relative efficiency (ARE) for the two single-stage procedures considered are defined as

\[
RE(k_1, k_2, \ldots, k_r; \delta^*, P^*) = \frac{\hat{N} \left( \prod_{j=1}^{r} k_j \right)}{\sum_{j=1}^{r} k_j \hat{N}_j},
\]

and

\[
ARE(k_1, k_2, \ldots, k_r; \delta^*) = \lim_{p^* \to 1} RE(k_1, k_2, \ldots, k_r; \delta^*, P^*),
\]

respectively.

It should be noted that although in practice the sample sizes are integer valued, for computing the asymptotic relative efficiency we treat the sample sizes, \( N \) for SP1 and \( (N_1, \ldots, N_r) \) for SP2, as non-negative continuous variables.

3. Asymptotic Relative Efficiency

To determine the asymptotic relative efficiency, we use a \((P^* \to 1)\) large sample approximation to the sample size given in Dudewicz [3]. We obtain, from Equation (7) of [3], for \( j = 1, 2, \ldots, r, \)

\[
(3.1) \quad N_j \sim -\frac{4}{\lambda_j^*} \log(1-P_j)
\]

and

\[
(3.2) \quad N_j' \sim -\frac{4}{\lambda_j^*} \log(1-P_j')
\]
Using the above approximations to the sample size, the minimization problems (I and II) reduce to the following:

**Problem I'**: 

\[
\min_{N'} \begin{pmatrix} \frac{r}{N} k_j \\ j=1 \end{pmatrix} \\
\text{s.t.} \\
\sum_{j=1}^{r} \left( \frac{r}{1-e^{-a_jN'_j}} \right) \geq p^* ,
\]

**Problem II'**: 

\[
\min_{(N'_1,N'_2,\ldots,N'_r)} \sum_{j=1}^{r} k_j N'_j \\
\text{s.t.} \\
\sum_{j=1}^{r} \left( \frac{r}{1-e^{-a_jN'_j}} \right) \geq p^* ,
\]

where, for \( j = 1,2,\ldots,r \), \( a_j \) denotes \( \frac{\lambda^*_j}{2}/4 \).

For typographical simplicity, let \( \varepsilon \) denote \( 1 - p^* \) and \( n_1 \) and \( n_2 \) denote the minimum number of observation for SP1 and SP2, respectively.

Then \( \text{RE} = \frac{n_1}{n_2} \) and \( \text{ARE} = \lim_{\varepsilon \to 0} [n_1/n_2] \). The asymptotic relative efficiency is given by the following:

**Theorem 3.1**

\[
(3.3) \quad \text{ARE} = \frac{\max_{1 \leq j \leq r} \{ k_j/a_j \}}{\sum_{j=1}^{r} (k_j/a_j)} .
\]
Proof: The constraint set of Problem I' may be rewritten as

\[
\sum_{j=1}^{r} \log \left\{ 1 - e^{-\frac{a_j}{k_j} n_j} \right\} \geq \log(1-\epsilon).
\]  

(3.4)

Since we are dealing with a minimization problem with a single constraint, the inequality in (3.4) may be replaced by an equality. Also for small \( \epsilon \), the constraint set reduces to the form

\[
\sum_{j=1}^{r} e^{-\frac{a_j}{k_j} n_j} = \epsilon.
\]  

(3.5)

Let \( C \) denote \( \min_{1 \leq j \leq r} (a_j/k_j) \). Then, from (3.5), we obtain

\[
n_1 = -\frac{1}{C} \log \epsilon + o(\epsilon).
\]  

(3.6)

The constraint set of Problem II' reduces for small \( \epsilon \) to the form

\[
\sum_{j=1}^{r} e^{-a_j N_j} = \epsilon.
\]  

(3.7)

For \( j = 1, 2, \ldots, r-1 \) let \( \rho_j \) denote \( e^{-a_j N_j} \). Using (3.7), Problem I' reduces to the following

\[
\min_{(\rho_1, \rho_2, \ldots, \rho_{r-1})} -\sum_{j=1}^{r-1} \frac{k_j}{a_j} \log(\rho_j \epsilon) - \frac{k_r}{a_r} \log \left( \left[ 1 - \sum_{j=1}^{r-1} \rho_j \right] \epsilon \right).
\]
It can readily be seen by setting the first derivative equal to zero
(necessary condition for minimization) and checking the second derivative
that (sufficient condition for minimization).

\[ n_2 = - \sum_{j=1}^{r} \frac{k_j}{a_j} \log \varepsilon + B \]

where \( B \) is independent of \( \varepsilon \). Equation (3.3) follows directly from (3.6)
and (3.8).

Q.E.D.

4. Numerical Results and Concluding Remarks

It follows from Problems I and II that for any \( P^* \), \( \frac{1}{k^r} < P^* < 1 \),
the relative efficiency for the symmetric case \( (k_j = k, \delta_j^* = \delta^*, j = 1, 2, \ldots, r) \)
is \( 1/r \). Thus in the symmetric case, the \( r \)-factor experiment is \( r \) times
more efficient than \( r \) one-factor experiments. However, in the unsymmetric
case, the factorial experiment, although more efficient asymptotically than
\( r \) one-factor experiments (i.e., ARE < 1 always), is not as efficient as in
the symmetric case. For example, for a 2-way classification problem with
\( \delta_1^* = \delta_2^* \), if \( k_2 >> k_1 \), then the asymptotic relative efficiency may be quite
close to one. Thus, for such a situation, one would not be taking many less
observations in conducting one two-factor experiment than two one-factor
experiments.

From a practical viewpoint, since it sometimes is easier to implement
several one-factor experiments than a factorial experiment, we recommend
for $P^*$ close to one the use of several one-factor experiments over a factorial experiment whenever the asymptotic relative efficiency is close to one.

For a 2-way classification problem, some numerical results are provided in Table I. The values in each cell of Table I are a large sample approximation to the relative efficiencies for $P^* = 0.999, 0.9999, 0.99999$, and $0.999999$ obtained from the solutions to minimization Problems I' and II'. Also included is the asymptotic relative efficiency. The values in the table are symmetric about the main diagonal. The results show the slow approach of the relative efficiencies to the asymptotic value.
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**Table 1.** 2-way classification with no interaction \( \lambda_1^* = \lambda_2^* = 0.1 \)
5. Acknowledgments

I would like to thank Professor R.E. Bechhofer for suggesting the problem, helpful discussions, and suggestions towards clarity in presentation. I also acknowledge helpful discussions with Professor L. Weiss.
REFERENCES

