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QUEUING OF SIDE STREET TRAFFIC AT A PRIORITY TYPE VEHICLE-ACTUATED SIGNAL

by

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1. Introduction and Summary

Vehicle-actuated signals can be divided into two types, non-priority and priority. The non-priority signal, at which neither street is favored, seems to be the type considered in the bulk of the literature. The commonly studied model is a signal located at the intersection of two one-way streets which operates under the rule of switch when the favored queue empties. This model has been studied by Tanner [5], Darroch, Newell and Morris [1], Dunne [2] and others.

The priority vehicle-actuated traffic signal, at which the main street signal is green unless traffic arrives on the side street, has received much less attention. Garwood [3] studied the delay to a side street vehicle which arrives to an empty queue during a red period. Haight [4] extended his results on the overflow at a fixed cycle traffic light to the overflow of main street traffic at a priority vehicle-actuated signal. Thedeen [6] studied a push-button type pedestrian crossing and obtained the expected delay to pedestrians and the distribution of cycle lengths. His model corresponds to a priority vehicle-actuated signal with a detector on the side road only and with queuing on the side road ignored.

In this paper we consider the queuing process of side street traffic at a priority type vehicle-actuated traffic signal located at the intersection of two one-way streets. The signal shows green on the main (priority) street unless it is actuated to change by side-street traffic. There is no detector on the main street, so that the density of main street traffic has no effect on the operation of the signal other than by way of design considerations. Side street vehicles cause the signal to change by means of a detector which senses whether or not there is a vehicle at the intersection. After detection of a vehicle, an activation period (t_a) must elapse before the light changes to green, this period corresponding to the yellow (caution) period on the main road. After changing, the light remains green to side-road traffic for a fixed interval of length g, and each green period is followed by a red period of minimum length r. During this minimum red period, the length of which is a constant programmed into the signal mechanism, the light will not change to green for side-road traffic.

In section 2 we develop a mathematical model of this intersection and obtain the transition probabilities of the queue length at the end of each cycle (green + red). In section 3 we obtain the probability distribution of the queue length in steady state and the mean queue length. In section 4 we find the distribution of the length of the side street red period in steady state. In section 5 we present a procedure for obtaining numerical results and an example of its use.

2. The Model

In this analysis time will be measured in discrete units. We make the following assumptions:

a) The side street green period length is g time units. The signal mechanism does not respond to the presence of a side street vehicle during the first $r-1$ time units following the end of a side street green phase. The actuation time (t_a) of the signal is one time unit. Thus the minimum length of the side street red period is r units. These are effective lengths, inclusive of the side street yellow period. We call a side-street green period followed by a red period a cycle; the minimum cycle length is $g+r$ units. For convenience of analysis we will assume that the initial (1st) cycle begins at time $t = 0$. Observe that a cycle cannot begin with no cars present.

b) Cars on the side street arrive at the intersection in a Bernoulli process; that is, in each time interval $(t-1,t]$, with probability p there is one arrival, and with probability $q = 1 - p$ there is no arrival. An arrival during $(t-1,t]$ will be said to have occurred at t. If A_x is the number of arrivals in a period of length x, then our assumption implies that A_x has the binomial distribution

$$b(k; x,p) = \binom{x}{k} p^k q^{x-k} \qquad (k = 0,1,\ldots,x) .$$

c) Each side street car requires one time unit to cross the intersection. Hence, at most g cars can cross the intersection during a green period.

Let Q_n be the queue length at the end of the n^{th} cycle $(n = 1,2,\ldots)$. We note that $Q_n > 0$, since a cycle cannot end with no cars present. Also, let Q_n^* be the queue length at the end of the green period of the n^{th} cycle. Then, considering the possible changes in queue length during the green period and the red period which follows this green period, we find that

$$Q_{n+1}^* = \max(0, Q_n + A_g - g) \tag{2.1}$$

since g is the maximum number of departures possible during the green period, while

$$Q_{n+1} = Q_{n+1}^* + A_R , \tag{2.2}$$

where R is the length of the red period. Our assumptions imply that R is a random variable which depends on Q_{n+1}^*. We have in fact

$$R = \begin{cases} r & \text{if } Q_{n+1}^* > 0 \\ r & \text{if } Q_{n+1}^* = 0 , \ A_{r-1} \geq 1 \\ x > r & \text{if } Q_{n+1}^* = 0 , \ A_{x-2} = 0 \quad A_{x-1} = 1 . \end{cases}$$

Therefore,

$$\begin{aligned} \Pr(R = x \mid Q_{n+1}^* > 0) &= 1 \quad (x = r) \\ &= 0 \quad (x \neq r) , \end{aligned} \tag{2.3a}$$

and

$$\begin{aligned} \Pr(R = x \mid Q_{n+1}^* = 0) &= 1 - q^{r-1} \quad (x = r) \\ &= pq^{x-2} \quad (x > r) . \end{aligned} \tag{2.3b}$$

From (2.1) and (2.2) it follows that the process $\{Q_n, n = 1,2,\dots\}$ is a time-homogeneous Markov chain on the state space $(1,2,\dots)$. To obtain its transition probabilities we have to consider these equations in detail. Firstly, let $g_{ij} = \Pr\{Q^*_{n+1} = j \mid Q_n = i\}$; from (2.1) we find that

$$g_{ij} = b(g + j - i; g,p) \quad (j > 0, \quad i > 0)$$
$$g_{i0} = \sum_{k=0}^{g-i} b(k;g,p) \quad (g \geq i > 0) . \tag{2.4}$$

Secondly, let $r_{ij} = \Pr(Q_{n+1} = j \mid Q^*_{n+1} = i)$. From (2.2) and (2.3) we find that

$$r_{ij} = \Pr(A_R = j - i \mid Q^*_{n+1} = i) = b(j-i;r,p) \quad (i > 0)$$

$$r_{0j} = \Pr(A_R = j \mid Q^*_{n+1} = 0) = \Pr(A_{r-1} \geq 1, A_r = j)$$
$$+ \sum_{x=r+1}^{\infty} \Pr(A_{x-2} = 0,\ A_{x-1} = 1, A_x = j)$$

$$= \begin{cases} b(j;r,p) & (j \geq 3) \\ b(2;r,p) + \sum_{r+1}^{\infty} q^{x-2}p \cdot p = b(2;r,p) + pq^{r-1} & (j = 2) \\ (r-1)pq^{r-1} + \sum_{r+1}^{\infty} q^{x-2}pq = (r-1)pq^{r-1} + q^r & (j = 1) . \end{cases} \tag{2.5}$$

We now combine (2.4) and (2.5) to obtain the transition probabilities of the queue length process $\{Q_n\}$, namely

$$p_{ij} = \Pr(Q_{n+1} = j \mid Q_n = i) \quad .$$

We have

$$p_{ij} = \sum_{\nu=0}^{\infty} g_{i\nu} r_{\nu j}$$

$$= \begin{cases} b(g + j - i; r + g, p) & (i > g) \\ b(g + j - i; r + g, p) & (i \le g, \quad j > r) \\ g_{i0} r_{0j} + \sum_{1}^{\min(i,j)} b(k - i + g; g, p) b(j - k; r, p) & (i \le g, \ j \le r). \end{cases} \qquad (2.6)$$

In arriving at the simplified expressions above, we have used the identity

$$b(n; r + g, p) = \sum_{m=1}^{n} b(m - n; r, p) b(m; g, p) \quad .$$

All transitions other than those listed in (2.6) have probability zero. For convenience we introduce the notations

$$a_k = b(g+k;\ g+r, p)$$

$$b_{ij} = g_{i0} r_{0j} + \sum_{1}^{\min(i,j)} b(k - i + g; g, p) b(j - k; r, p) \quad .$$

We can rewrite the transition probabilities (2.6) as

$$p_{ij} = \begin{cases} a_{j-i} & (i > g, i + r \geq j > i - g) \\ a_{j-i} & (i \leq g, r < j \leq i + r) \\ b_{ij} & (i \leq g, j \leq r) \\ 0 & \text{otherwise} \end{cases} \quad . \tag{2.7}$$

3. Steady state probabilities of queue length

Let u_i be the steady state probability that $Q_n = i > 0$. We have

$$u_j = \sum_i u_i p_{ij} \quad .$$

Applying equations (2.7), we find that

$$u_j = \sum_{i=j-r}^{j+g} u_i a_{j-i} \quad (j > r)$$

$$u_j = \sum_{i=1}^{g} u_i b_{ij} + \sum_{i=g+1}^{j+g} u_i a_{j-i} \quad (1 \leq j \leq r) \quad . \tag{3.1}$$

We now define the probability generating function

$$U(z) = \sum_{i=1}^{\infty} u_i z^i \quad (|z| < 1) .$$

Multiplying both sides of equations (3.1) by z^j and summing over $j = 1,2,...$ we obtain, after some simplification,

$$U(z) = \sum_{j=r+1}^{\infty} z^j \sum_{i=j-r}^{j+g} u_i a_{j-i} + \sum_{j=1}^{r} z^j \left(\sum_{i=1}^{g} u_i b_{ij} + \sum_{i=g+1}^{j+g} u_i a_{j-i} \right)$$

$$= A(z)U(z) - \sum_{i=1}^{g} a_{-i} z^{-i} \sum_{j=1}^{i} u_j z^j + \sum_{i=1}^{g} u_i [b_{ij} - a_{j-i}] z^i$$

where $A(z) = \sum_{-g}^{r} a_i z^i$. This gives

$$U(z) = \frac{- \sum_{i=1}^{g} a_{-i} z^{g-i} \sum_{j=1}^{i} u_j z^j + \sum_{j=1}^{r} z^{j+g} \sum_{i=1}^{g} u_i [b_{ij} - a_{j-i}]}{z^g - \sum_{0}^{r+g} a_{i-g} z^i}$$

$$= \frac{\sum_{i=1}^{g} u_i [\sum_{j=1}^{r} z^{j+g} (b_{ij} - a_{j-i}) - z^{i+g} \sum_{j=i}^{g} a_{-j} z^{-j}]}{z^g - \sum_{0}^{r+g} a_{i-g} z^i}$$

In order to evaluate the g unknown constants $u_1,...,u_g$ we first consider the zeros of the function

$$z^g - \sum_0^{r+g} a_{i-g} z^i$$

in the region $|z| \leq 1$. We first note that for $|z| = 1$,

$|\sum_0^{r+g} a_{i-g} z^i| \leq \sum_0^{r+g} a_{i-g}|z^i| = \sum_0^{r+g} a_{i-g} = 1$. Hence for $0 < \theta < 1$,

$|z| = 1$, we have $|z^g| > \theta \ |\sum_0^{r+g} a_{i-g} z^i$. Therefore, by Rouche's

theorem, the function $z^g - \theta \sum_0^{r+g} a_{i-g} z^i$ has g zeros in $|z| < 1$.

Let these zeros be denoted by $\xi_1(\theta),\ldots,\xi_g(\theta)$; $|\xi_i(\theta)| < 1 \quad (1 \leq i \leq g)$.

The $\xi_i(\theta)$ are continuous functions of θ; as $\theta \to 1 - 0$; the $\xi_i(\theta) \to \xi_i$

with $\xi_g = 1$, $|\xi_i| < 1 \quad (1 \leq i \leq g-1)$. Therefore the function

$z^g - \sum_0^{r+g} a_{i-g} z^i$ has the zeros $\xi_1,\ldots,\xi_g$ such that $|\xi_i| \leq 1$.

In order that $U(z)$ be analytic, the numerator must have zeros at $\xi_1,\ldots,\xi_g$. Hence we can determine $u_1,\ldots,u_g$ from the following set of equations:

$$\sum_{i=1}^{g} u_i \left[\sum_{j=1}^{r} \xi_k^{g+j} (b_{ij} - a_{j-i}) - \xi_k^{i+g} \sum_{j=i}^{g} a_{-j} \xi_k^{-j} \right] \quad (1 \leq k \leq g) . \tag{3.3}$$

We can then write the numerator of $U(z)$ as

$$c \sum_{1}^{r+g} (z - \theta_k)$$

where the θ_k $(1 \leq k \leq r+g)$ are its zeros and c is a constant. Equations (3.3) require that

$$\theta_i = \xi_i \quad (1 \leq i \leq g-1)$$

$$\theta_g = \xi_g = 1 \ .$$

Therefore

$$c \sum_{1}^{r+g} (z-\theta_k) = c \prod_{1}^{g-1} (z-\xi_k)(1-z) \prod_{g+1}^{r+g} (z-\theta_k),$$

and

$$U(z) = \frac{c \prod_{1}^{g-1} (z-\xi_k)(z-1) \prod_{g+1}^{r+g} (z-\theta_k)}{z^g(1 - A(z))} \ . \qquad (3.4)$$

To evaluate the constant c in (3.4), we set $z = 1$ and apply L'Hospital's rule to obtain

$$1 = U(1) = \frac{c \prod_{1}^{g-1} (1-\xi_k) \prod_{g+1}^{r+g} (1-\theta_k)}{-A'(1)} \ .$$

For consistency, we must have $A'(1) < 0$. This requires that, for the existence of steady state, the mean net input per cycle be negative. We then have

$$c = \frac{-A'(1)}{\prod_{1}^{g-1} (1-\xi_k) \prod_{g+1}^{r+g} (1-\theta_k)}$$

and

$$U(z) = \frac{\alpha(1-z)}{z^g(1-A(z))} \prod_{1}^{g-1} \frac{z-\xi_k}{1-\xi_k} \prod_{g+1}^{r+g} \frac{z-\theta_k}{1-\theta_k} \qquad (3.5)$$

where $\alpha = A'(1)$.

To obtain the expected value of Q_n in steady state, we write (3.5) as

$$U(z) = \frac{\alpha}{c(z)} \prod_{1}^{g-1} \frac{z-\xi_k}{1-\xi_k} \prod_{g+1}^{r+g} \frac{z-\theta_k}{1-\theta_k}$$

where

$$c(z) = \frac{z^g(1-A(z))}{1-z}, \quad c(1) = \alpha, \quad c'(1) = \frac{1}{2}\beta + g\alpha$$

where $\beta = A''(1)$. Taking logarithmic derivatives of $U(z)$, we find

$$U'(1) = -\frac{\beta}{2\alpha} - g + \sum_{1}^{g-1} \frac{1}{1-\xi_k} + \sum_{g+1}^{r+g} \frac{1}{1-\theta_k} \qquad (3.6)$$

to be the mean queue length in steady state.

4. Steady State Red-Period Length Distribution

The probability that the queue length is zero at the end of a green phase in steady state is

$$P_0 = \sum_{1}^{g} u_i g_{i0} .$$

From equations (2.2), the distribution of side street red period length in steady state is given by

$$\Pr(R = x) = \begin{cases} 0 & (x < r) \\ 1 - P_0 q^{r-1} & (x = r) \\ P_0 q^{x-2} p & (x > r) \end{cases} \qquad (4.1)$$

We note that we can write (4.1) as the following modified geometric distribution:

$$\Pr(R = x) = \begin{cases} 1 - \gamma & (x = r) \\ \gamma p q^{x-r-1} & (x > r) \\ 0 & \text{elsewhere} \end{cases}$$

where $\gamma = P_0 q^{r-1}$.

5. Numerical results

In this section we consider the use of the results obtained in previous sections to evaluate the performance of a signal of the type modeled. The queuing of side street traffic at the signal can be evaluated through the steady state probabilities and the mean queue length. The effect of the signal on main street traffic can be analyzed through use of the side street red period distribution (which corresponds to the main street green period length distribution). The first step in applying the results to a given situation is to obtain the length of the time unit, which corresponds to the time taken by a side street car to cross the intersection. Following this the probability of a side street arrival during one time unit should be obtained. One then calculates numerical values, using the following procedures.

(a) Compute the coefficients involved in (3.2). We need

$$a_i \quad (-g \leq i \leq r)$$

and

$$b_{ij} = g_{i0} r_{0j} + \sum_{1}^{\min(i,j)} b(k-i+g;\ g,p) b(j-k;\ r,p) \qquad (1 \leq i \leq g, \quad 1 \leq j \leq r)$$

where g_{i0} and r_{0j} are defined in (2.4) and (2.5).

(b) Write down the denominator of (3.2); that is

$$g(z) = z^g(1 - A(z)) = z^g(1 - \sum_{-g}^{r} a_i z^i) .$$

We wish to find the roots of the equation $g(z) = 0$, and shall henceforth refer to these roots as the "zeros." Note that $g(z)$ is of order $r+g$. From previous comments we already know that $z = 1$ is a zero. We now obtain the zeros of $g(z)$ by an appropriate exact or approximation method. Some of the zeros may be complex. We next identify $\xi_1,\ldots,\xi_g$, the zeros of $g(z)$ which have absolute value less than or equal to 1.

(c) Insert the ξ_i, the a_i and the b_{ij} into (3.3), thus obtaining a set of g linear equations in $u_1,\ldots u_g$. We note that this set of linear equations is homogeneous, so that we may find $u_2,\ldots,u_g$ in terms of u_1. We now calculate u_1 by substituting the expressions in terms of u_1 for $u_2,\ldots,u_g$ into the numerator of (3.2) and solving the equation

$$1 = U'(1) = \frac{F'(1)}{-A'(1)} \tag{5.1}$$

for u_1. Equation (5.1) is equivalent to (3.4) when $F(z)$ is the numerator of (3.2).

(d) We now have a complete specification of $U(z)$. The steady state probabilities may be obtained from

$$u_n = \left.\frac{d^n U(z)}{n!dz^n}\right|_{z=0}$$

or in the case of $u_2,\ldots,u_q$ from the relationships of step (c). We find that the mean queue length is given by

$$U'(1) = \frac{F'(1)}{2F''(1)} - \frac{A'(1)}{2A''(1)} - g \qquad (5.2)$$

which is equivalent to (3.6).

(e) In order to find the red period length distribution, we use the values of g_{i0} calculated in step (a) and $u_1,\ldots,u_g$ in (4.1).

To illustrate the preceding ideas, we now carry out the calculation for the case $g = 2$, $r = 2$, $p = 1-q = 0.25$.

(a) We have

$g_{10} = q^2 + 2pq = .9375$

$g_{20} = q^2 = .5625$

$r_{01} = pq + q^2 = .75$

$r_{02} = p^2 + pq = .25$

$a_{-2} = q^4 = .3164062$

$a_{-1} = 4pq^3 = .421875$

$a_0 = 6p^2q^2 = .2109375$

$a_1 = 4p^3q = .046875$

$a_2 = p^4 = .00390625$

$b_{11} = 3pq^3 + q^4 + 3p^2q^2 = .7382811$

$b_{12} = 4p^3q + 3p^2q^2 + pq^3 = .2578024$

$b_{21} = 3pq^3 + q^4 = .6328124$

$b_{22} = 6p^2q^2 + pq^3 = .3164062$

(b) We find

$$g(z) = -(.00390625z^4 + .046875z^3 - .7890625z^2 + .421875z + .3164062) .$$

Knowing that $z = 1$ is a zero we divide by $z-1$ to obtain

$$\frac{g(z)}{z-1} = -.00390625z^3 - .05078125z^2 + .7382812z + .3164062 .$$

We next apply the well known procedure for solving cubic equations to $g(z)/(z-1)$ (see, e.g. Chemical Rubber Company Standard Mathematical Tables, 12th ed., p. 358) and find that the zeros of $g(z)$ and their absolute values are

Zero	Absolute Value
8.999997	8.999997
-21.58297	21.58297
-.4170036	.4170036
1.00	1.00

Note that in the special case under consideration, all of the zeros of $g(z)$ are real. We see that the required zeros are $\xi_1 = -.417$ and $\xi_2 = 1.00$ (since these are the only zeros with absolute value ≤ 1).

(c) We insert ξ_1 and ξ_2 into (3.3) and obtain

$$(\xi_1) \qquad .0267207u_1 - .0671271u_2 = 0.0$$

$$(\xi_2) \qquad 0.0u_1 + 0.0u_2 = 0.0$$

From this set of equations we have $u_2 = .3980613u_1$. We can now write $F(z)$ as

$$F(z) = u_1(.6113097z^4 + .2529204z^3 - .5479239z^2 - .3164062z) .$$

We also have $A(z) = .3164062z^{-2} + .421875z^{-1} + .2109375 + .046875z + .00390625z^2$

Hence

$$1 = U'(1) = \frac{u_1(1.722585)}{-(-1.0)}$$

and $u_1 = .5580523.$

(d) We have $u_2 = .3980613u_1$ or $u_2 = .2221390$ and

$$U(z) = \frac{.3411428z^4 + .1411428z^3 - .3057144z^2 - .1765712z}{-.00390625z^4 - .046875z^3 + .7890625z^2 - .421875z + .3164062}$$

We find from (5.2) that

$$U'(1) = 1.451284 .$$

(e) Applying (4.1), we find that

$$P_0 = \sum_1^g g_{io}u_i = .6481272$$

$$P(2) = .5139045$$

$$Pr(x > 2) = (.6481272)(.25)(.75)^{x-3} .$$

We have the following further results for the case $g = 2$, $r = 2$.

p	Zeros of g(z)	U'(1)	u_1	u_2	A'(1)	p_0	P(2)
.25	1.0,-.4170036	1.451284	.5580523	.222139	-1.0	.6481272	.5139045
.375	1.0,-.2731027	1.764572	.4472672	.3853778	-.50	.5349084	.6656822
.45	1.0,-.2082166	3.098022	.2629420	.3544846	-.20	.3169278	.8256897

The reader is reminded that for the very simple special case $g = 2$, $r = 2$ which we have just considered, the polynomial $g(z)/(z-1)$ is of degree 3, and hence the zeros can be obtained in closed form. In general, when $g+r > 4$, the root would have to be obtained by approximate methods. Here at Cornell, an appropriate method would be the use of the computer subroutine CGRT9 which uses Muller's method with deflation [7] for finding all of the zeros of a complex polynomial.

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