

## ON INEQUALITY COMPARISONS<sup>1</sup>

BY GARY S. FIELDS AND JOHN C. H. FEI

IS ONE DISTRIBUTION (of income, consumption, or some other economic variable) among families or individuals more or less equal in relative terms than another?<sup>2</sup> Despite the seeming straightforwardness of this question, there has been and continues to be considerable debate over how to go about finding the answer.

There are two points of contention. One is the issue of cardinality vs. ordinality. Practitioners of the cardinal approach compare distributions by means of summary measures such as a Gini coefficient, variance of logarithms, and the like. For purposes of ranking the relative inequality of two distributions, the cardinality of the usual measures is not only a source of controversy, but it is also redundant.<sup>3</sup> Accordingly, some researchers prefer an ordinal approach, adopting Lorenz domination as their criterion. The difficulty with the Lorenz criterion is its incompleteness, affording rankings of only some pairs of distributions but not others. Current practice in choosing between a cardinal or an ordinal approach is now roughly as follows: Check for Lorenz domination in the hope of making an unambiguous comparison; if Lorenz domination fails, calculate one or more cardinal measures.

This raises the second contentious issue: which of the many cardinal measures in existence should one adopt? The properties of existing measures have been discussed extensively in several recent papers.<sup>4</sup> Typically, these studies have started with the measures and then examined their properties.

In this paper, we reverse the direction of inquiry. Our approach is to start by specifying as axioms a relatively small number of properties which we believe a "good" index of inequality should have and then examining whether the Lorenz criterion and the various cardinal measures now in use satisfy those properties. The key issue is the reasonableness of the postulated properties. Work to date has shown the barrenness of the Pareto criterion.<sup>5</sup> Only recently have researchers begun to develop an alternative axiomatic structure.<sup>6</sup> The purpose of this paper is to contribute to such a development.

<sup>1</sup> Portions of this research were financed by the International Bank for Reconstruction and Development under RPO/284. However, the views expressed do not necessarily reflect those of IBRD.

<sup>2</sup> Throughout this paper, we shall talk in terms of income distributions among families. All results apply, however, without modification to comparisons of inequality in the distribution of *any* quantifiable economic magnitude.

<sup>3</sup> Cardinality of inequality is redundant and controversial for purposes of ranking of distributions in the same sense that cardinal utility is redundant and controversial in the analysis of consumer choices. See Hicks [5, p. 17].

<sup>4</sup> See Champernowne [2], Kondor [7], Sen [10, Chapter 2], and Szal and Robinson [12].

<sup>5</sup> For an axiomatic development of the Pareto criterion, see Sen [10].

<sup>6</sup> Noteworthy are the axiomatic approaches in the work by Kondor [7] on relative inequality indices and by Sen [11] on absolute poverty measures.

In Section 1, we shall postulate three axioms: scale irrelevance, symmetry, and desirability of rank-preserving equalization. Then, in Section 2, we will use these axioms to investigate and strengthen previous results by Rothschild and Stiglitz [9] and others regarding the consistency of alternative orderings in terms of Lorenz domination. The principal result of this paper is that the three axioms are sufficient to justify the Lorenz criterion for comparing the relative inequality of two distributions. Like the Lorenz criterion, the axiomatic system so constructed is incomplete. This incompleteness is intentional, for it allows us to ascertain which of the indices in current use do or do not satisfy our axioms (Section 3).<sup>7</sup> Those that do are the Gini coefficient, coefficient of variation, Atkinson index, and Theil index. This lends support to their reasonableness. However, they differ in ways which lie outside the scope of our axioms. Hopefully, future researchers will add to our axioms so as to narrow down this incompleteness.

### 1. THREE AXIOMS FOR INEQUALITY COMPARISONS

Suppose there are  $n$  families in an economy whose incomes may be represented by the non-negative row vector

$$(1.1) \quad X = (X_1 \quad X_2 \quad \dots \quad X_n) \geq 0$$

in the non-negative orthant of the  $n$ -dimension income distribution space  $\Omega^+$ . A point in  $\Omega^+$  is a pattern of income distribution. In this paper, we shall exclude the origin  $(0 \ 0 \ \dots \ 0)$  (i.e., when no family receives any income) from  $\Omega^+$ . The object of inequality comparisons between two such patterns is to be able to say that one is more or less equal than the other. More specifically, we wish to introduce a *complete pre-ordering*<sup>8</sup> of all points in  $\Omega^+$ , i.e., a binary relation " $R$ " defined on ordered pairs in  $\Omega^+$  satisfying the conditions of comparability and transitivity:

- (1.2a) Comparability. For any  $X$  and  $Y$  in  $\Omega^+$ , exactly one of the following is true: (i)  $XRY \dots$  in which case we write  $X > Y$  and read "X is more equal than Y;" (ii)  $YRX \dots$  in which case we write  $Y > X$  and read "Y is more equal than X;" (iii)  $XRY$  and  $YRX \dots$  in which case we write  $X = Y$  and read "X and Y are equally unequal."

- (1.2b) Transitivity.  $XRY$  and  $YRZ$  implies  $XRZ$ .

We now introduce three properties which we shall propose as axioms for inequality comparisons. Not only do these seem reasonable to us but in addition they have been used by previous writers on inequality.

<sup>7</sup> Indices of inequality, including those mentioned above, are cardinal measures which naturally introduce a pre-ordering. Thus, rigorously, it is the pre-ordering  $R$  induced by the index which satisfies our axioms.

<sup>8</sup> Intuitively, a complete pre-ordering has exactly the same meaning as the ranking of commodity bundles by ordinary (ordinal) indifference curves in consumer analysis.

First, suppose two distributions  $X$  and  $Y$  are scalar multiples of one another:

$$(1.3) \quad X = aY, \text{ i.e., } (X_1 \ X_2 \dots X_n) = (aY_1 \ aY_2 \dots aY_n), \quad a > 0.$$

Because *inequality* in the distribution of income and the *level* of income enter as separate arguments into judgments of social well-being, it would seem reasonable and desirable for the *measure* of inequality to be independent of the level of income.<sup>9</sup> For this reason, we require that the two distributions  $X$  and  $Y$  in (1.3) be ranked as equally unequal.<sup>10</sup> Hence, we postulate:

A1. AXIOM OF SCALE IRRELEVANCE:  $X = aY$  ( $a > 0$ ) implies  $X \simeq Y$ .

This axiom allows us to normalize all distributions  $X$  in  $\Omega^+$  according to the fraction of income received by each family:

$$(1.4) \quad [X = (X_1 \ X_2 \ \dots \ X_n)] \simeq [\theta = (\theta_1 \ \theta_2 \dots \ \theta_n)]$$

where

$$\theta_i = X_i / (X_1 + X_2 + \dots + X_n) \quad \text{for } i = 1, 2, \dots, n.$$

The totality of all such normalized patterns,  $\Omega^c$ , is the subset of points  $\theta = (\theta_1 \ \theta_2 \dots \ \theta_n)$  of  $\Omega^+$  satisfying the conditions

$$(1.5) \quad \theta_i \geq 0 \quad \text{and} \quad \sum_i \theta_i = 1.$$

Axiom 1 assures us:

LEMMA 1.1: *If a preordering  $R$  is first defined on  $\Omega^c$ , then it can be extended uniquely to  $\Omega^+$ .*

Next, suppose the elements in one vector  $X$  are a permutation of the elements of  $Y$ , i.e., the frequency distributions of income are the same but different individuals receive the income in the two cases. On the principle of treating all individuals or families as the same with regard to income distributions, these two patterns can be characterized by the same degree of inequality. Hence, we state:

A2. AXIOM OF SYMMETRY:<sup>11</sup> *If  $(i_1, i_2, \dots, i_n)$  is any permutation of  $(1, 2, \dots, n)$  then  $(X_1 \ X_2 \dots X_n) \simeq (X_{i_1} \ X_{i_2} \dots X_{i_n})$ .*

<sup>9</sup> Note that in positing that the *measure* of relative inequality is independent of the level of income, we do not wish to suggest that our *feelings* about inequality are invariant with income level. On this, see Hirschman and Rothschild [6].

<sup>10</sup> Following Atkinson [1], we would note that this condition is analogous to constant relative inequality aversion. For further applications of this notion to inequality comparisons, see also the papers by Rothschild and Stiglitz [9] and Dasgupta, Sen, and Starrett [4].

<sup>11</sup> A2 is sometimes referred to as the axiom of anonymity in the literature (see Sen [10]). Sen also includes an illuminating discussion highlighting the conflicts between A2 and a Benthamite utilitarian approach to social judgments (in which social welfare is taken as the sum of individual utilities).

Let  $(i_1^*, i_2^*, \dots, i_n^*)$  be a particular permutation of  $(1, 2, \dots, n)$ . Then those  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$  in  $\Omega^c$  which satisfy the condition

$$(1.6) \quad \theta_{i_1^*} \leq \theta_{i_2^*} \leq \dots \leq \theta_{i_n^*}$$

comprise a *rank-preserving subset* of  $\Omega^c$ . There are altogether  $n!$  such rank-preserving subsets. Suppose  $R$  is defined for *any one* of them. Then A2 allows us to extend it uniquely to the entire set  $\Omega^c$  and, by Lemma 1.1, to the full income distribution space  $\Omega^+$ . For convenience, we shall work with the permutation with the natural order  $(1, 2, \dots, n)$ . Denote the corresponding rank-preserving subset as  $\Omega_0$ , which includes all points satisfying the conditions

$$(1.7) \quad \theta_1 \leq \theta_2 \leq \dots \leq \theta_n; \quad \theta_i \geq 0; \quad \sum_{i=1}^n \theta_i = 1.$$

$\Omega_0$  will be referred to as the *monotonic rank-preserving set*. A1 and A2 allow us to state the following:

**LEMMA 1.2:** *Under A1 and A2, if R is first defined on the monotonic rank-preserving set  $\Omega_0$ , then it can be extended uniquely to  $\Omega^+$ .*

Notice from Lemma 1.2 that after postulating A1 and A2, we can restrict our search for "reasonable" properties to the space  $\Omega_0$ .

Next, let  $X$  and  $Y$  be two alternative distributions in  $\Omega_0$  such that  $X$  is obtained from  $Y$  by the transfer of a positive amount of income  $h$  from a relatively rich family  $j$  to a poorer family  $i$ ,  $i < j$ . We shall write  $X = E(Y)$  and say that  $X$  is obtained from  $Y$  by a *rank-preserving equalization*. For a particular pair  $i, j$  ( $i < j$ ), there is a maximum amount which can be transferred if the rank is to be preserved. Formally:

**DEFINITION: Rank-Preserving Equalization.**  $X = E(Y)$  if for some  $i, j$  ( $i < j$ ) and  $h > 0$ ,

$$(1.8a) \quad X_k = Y_k \quad \text{for } k \neq i, j,$$

$$X_i = Y_i + h,$$

$$X_j = Y_j - h, \quad \text{where:}$$

$$(1.8b) \quad \text{if } j = i + 1, \quad h \leq \frac{1}{2}(Y_j - Y_i);$$

$$\text{if } j > i + 1, \quad h \leq \min [(Y_{i+1} - Y_i), (Y_j - Y_{i-1})].$$

The next axiom which we shall introduce is:

**A3. AXIOM OF RANK-PRESERVING EQUALIZATION:** *In  $\Omega_0$ , if  $X = E(Y)$ , then  $X > Y$ .*<sup>12</sup>

The intuitive justification for this axiom is simply that it is reasonable to regard as more equal a distribution which can be derived from another by a richer person giving a part of his income to a poorer person. Defining the *perfect equality point* as  $\Phi = (1/n \ 1/n \ \dots \ 1/n)$ , any income distribution point  $X$  in  $\Omega_0$  can be transformed into  $\Phi$  by a finite sequence of rank-preserving equalizations.<sup>13</sup> Thus A3 and the transitivity of the ordering imply:

**LEMMA 1.3:**  $\Phi = (1/n \ 1/n \ \dots \ 1/n) > X$  for all  $X \neq \Phi \in \Omega_0$ .

The proof is immediate.

Notice that A3 has been introduced only on  $\Omega_0$ . Suppose now we introduce an  $R$  on  $\Omega_0$  satisfying all three axioms. By Lemma 1.2,  $R$  can be extended uniquely to the entire income distribution space  $\Omega^+$ . It is clear that A3 is automatically extended. Formally:

**DEFINITION:** Let  $X$  and  $Y$  be two patterns of income distribution in  $\Omega^+$ . We shall say that  $X$  is obtained from  $Y$  by a *rank preserving equalization*, in notation  $X = E(Y)$ , if (a)  $X$  and  $Y$  belong to the same rank preserving subset;<sup>14</sup> (b)  $X$  is obtained from  $Y$  by the transfer of a positive amount of income  $h$  from a relatively rich family (e.g.  $Y_q = X_q - h$ ) to a relatively poor family (e.g.  $Y_p = X_p + h$ ) for  $Y_q > Y_p$ .

Notice that  $X = E(Y)$  is now defined for the entire income distribution space  $\Omega^+$ . However, this definition coincides with the previous definition (1.8a, b) where both  $X$  and  $Y$  belong to  $\Omega_0$ . Thus:

**LEMMA 1.4:** *If  $R$  is first defined on the monotonic rank-preserving set  $\Omega_0$  satisfying A1–A3, the unique extension of  $R$  to  $\Omega^+$  also possesses the property of desirability of rank preserving equalization, i.e., if  $X = E(Y)$ , then  $X > Y$ .*

## 2. ORDINAL APPROACH TO INEQUALITY COMPARISONS: ZONES OF AMBIGUITY AND LORENZ DOMINATION

In the last section, we showed that if we postulate a set of “reasonable” axioms for  $R$  on  $\Omega_0$ , then  $R$  can be extended from  $\Omega_0$  to the entire income

<sup>12</sup> Precedent for this axiom dates back at least half a century to Dalton [3], who called this the “principle of transfers.”

<sup>13</sup> This assertion is easily proven by constructing a sequence of transfers from families above the mean to those below.

<sup>14</sup> For some permutation  $i_1 \ i_2 \ \dots \ i_n$ , if  $Y_{i_1} \leq Y_{i_2} \leq \dots \leq Y_{i_n}$  then  $X_{i_1} \leq X_{i_2} \leq \dots \leq X_{i_n}$ .

distribution space  $\Omega^+$ . We have not as yet considered whether the three axioms are *sufficient* to allow us to compare any two points  $X, Y$  in  $\Omega^+$  according to the comparability condition (1.2.a). In this section, we examine when inequality comparisons can or cannot be made using A1–A3.

### A. Zones of Ambiguity

We shall now show that there are well-defined ranges in which inequality comparisons can be made using A1–A3 alone and other well-defined zones of ambiguity where comparisons cannot be made without further specification of the rules of ordering. We shall also establish that there is a direct one-to-one correspondence between the zones of ambiguity and the more familiar concept of Lorenz domination, which we examine below.

The first concept we need to introduce is a sequence of equalizations from a given point  $Y \in \Omega_0$  according to the following definition:

**DEFINITION:**  $X$  is obtained from  $Y$  by a finite *sequence of equalizations*,  $X = T(Y)$ , when

$$(2.1) \quad X = E_k(\dots E_2(E_1(Y))\dots).$$

Starting from a given point  $Y$ , we can define three sets  $Y^*$ ,  $Y_*$ , and  $M$  as follows:

$$(2.2a) \quad Y^* = \{X | X = T(Y)\},$$

$$(2.2b) \quad Y_* = \{X | T(X) = Y\},$$

$$(2.2c) \quad M = \Omega_0 - Y^* \cup Y_*.$$

$Y^*$  is the set of all points in  $\Omega_0$  obtained from  $Y$  by a sequence of *equalizing transfers*, while  $Y_*$  includes those points in  $\Omega_0$  from which a sequence of equalizing transfers will lead to  $Y$ . We can also talk about *disequalizing transfers* as the transfer of income from a relatively poor to a relatively rich family, in which case  $Y_*$  is the set of all  $X$  which can be obtained from  $Y$  by a sequence of disequalizing transfers. The set  $M$  contains all other points of  $\Omega_0$ .

It follows from A3 that points in  $Y^*$ , obtained from  $Y$  by a sequence of equalizations, are more equal than  $Y$ , i.e.,  $X \in Y^*$  implies  $X > Y$ . Similarly,  $X \in Y_*$  implies  $X < Y$ . From (2.2c), it follows that the set  $M$  contains all points which are not unambiguously comparable with  $Y$  under A1–A3. A point  $Z$  in  $M$  can always be transformed into  $Y$  by a finite sequence of rank-preserving transfers. However, any such sequence necessarily involves at least one equalization and at least one disequalization—which is why  $Z$  cannot be compared with  $Y$ . The theorem we prove below, Theorem 2.1, implies that the Lorenz curves of  $Z$  and  $Y$  necessarily cross each other. We now consider Lorenz domination.

### B. Lorenz Domination

One distribution is said to Lorenz-dominate another if the Lorenz curve of the first distribution never lies below that of the second and lies above it at least one point. For two points  $X$  and  $Y$  in  $\Omega_0$ :

**DEFINITION:**  $X$  Lorenz-dominates  $Y$  (in notation,  $L_X \geq L_Y$ ) when

$$(2.3a) \quad X_1 + X_2 + \dots + X_i \geq Y_1 + Y_2 + \dots + Y_i \quad \text{for } j = 1, 2, \dots, n-1,$$

$$(2.3b) \quad X_1 + X_2 + \dots + X_j > Y_1 + Y_2 + \dots + Y_j \quad \text{for some } j < n.$$

Notice that

$$(2.4) \quad \sum_{i=1}^n X_i = \sum_{i=1}^n Y_i = 1 \quad \text{in } \Omega_0.$$

Our basic theorem is:

**THEOREM 2.1:**  $X \in Y^*$  if and only if  $L_X \geq L_Y$ .

Thus, the Lorenz Curve of  $Y$  is dominated by the Lorenz Curves of all  $X \in Y^*$ , dominates those of  $X \in Y_*$ , and crosses those of  $X \in M$ , i.e., neither dominates the other.

The necessary condition of the theorem (i.e.,  $X \in Y^*$  implies that the Lorenz curve of  $X$  dominates that of  $Y$ ) is a well-known result.<sup>15</sup> The sufficient condition of the theorem states that whenever the Lorenz curve of  $X$  dominates that of  $Y$  (i.e.,  $L_X \geq L_Y$ ),  $X$  can be obtained from  $Y$  by a sequence of equalizations which are rank-preserving.<sup>16</sup> This sufficient condition, when proved, along with (2.2.a), will allow us to conclude that Lorenz domination implies greater equality. This may be summarized as:

**COROLLARY 2.2.** Under A3, for  $X, Y$  in  $\Omega_0$ ,  $L_X \geq L_Y$  implies  $X > Y$ .

### C. Proof of the Sufficient Condition of Theorem 2.1

The sufficient condition of Theorem 2.1 holds that whenever  $X$  Lorenz-dominates  $Y$ , there exists a sequence of rank-preserving equalizations leading from  $Y$  to  $X$ . In order to prove the validity of this part of the theorem, we must produce a rule for finding the necessary sequence. The derivation of the rule follows.

<sup>15</sup> See Atkinson [1], Rothschild and Stiglitz [9], and Dasgupta, Sen, and Starrett [4].

<sup>16</sup> Rothschild and Stiglitz have proven that when the Lorenz Curve of  $X$  dominates that of  $Y$ , it is possible to construct a sequence of transfers which may or may not be rank-preserving, i.e., they may move out of and back into  $\Omega_0$ . The sufficient condition which we shall prove in the text is a stronger version requiring that such a sequence be rank-preserving and stay within  $\Omega_0$ .

Consider any two distributions  $X$  and  $Y$  in  $\Omega_0$ . Let their difference be denoted by

$$(2.5a) \quad d = (d_1 \ d_2 \ \dots \ d_n) = (X_1 - Y_1 \ X_2 - Y_2 \ \dots \ X_n - Y_n),$$

$$(2.5b) \quad \sum d_i = 0.$$

The  $n$  elements of  $d$  can be partitioned consecutively into  $r$  subsets  $(D_1 \ D_2 \ \dots \ D_r)$  according to the following rules:

- (2.6a) Every  $d_i$  belongs to one  $D_j$ .
- (2.6b) Every  $D_j$  contains at least one non-zero  $d_i$ .
- (2.6c) If  $d_i \in D_j$  and  $d_p \in D_q$ , then  $j < q$  implies  $i < p$ .
- (2.6d) The first element of  $D_2, D_3, \dots, D_r$  is non-zero.
- (2.6e) All  $d_i \in D_j$  are non-positive or non-negative [ $D_j$  is called positive (or negative) according to the signs of the  $d_i$  in  $D_j$ .]
- (2.6f) The  $D_j$  alternate in sign.

The partition determined by (2.6.a-f) is unique. Furthermore, if  $X \neq Y$ , then there is at least one strictly positive  $d_i$  and one strictly negative  $d_j$ . Thus,

$$(2.7) \quad \text{if } X \neq Y, \quad r \geq 2.^{17}$$

We can also define

$$(2.8) \quad S_j = \sum_{d_i \in D_j} d_i \quad \text{for } j = 1, \dots, r$$

with the properties

$$(2.9a) \quad S_j \neq 0, \quad j = 1, 2, \dots, r,$$

$$(2.9b) \quad S_1, S_2, \dots, S_r \text{ alternate in sign,}$$

$$(2.9c) \quad \sum_{j=1}^r S_j = \sum_{i=1}^n d_i = 0.$$

We may now state a *general rule for rank preserving equalizations* from  $X$  to  $Y$ : (a) Identify the groups according to (2.6). (b) With each transfer, eliminate the gap between  $X$  and  $Y$  of one family's income by (i) taking from the poorest family (the  $p$ th) with non-zero  $d$  in the richest group ( $S_r$ ), (ii) giving to the richest family (the  $q$ th) with non-zero  $d$  in the next lower group ( $S_{r-1}$ ), (iii) compute the amount of transfer as the smaller of  $d_p$  and  $-d_q$ . (c) Repeat these steps (a, b) again, each time eliminating the gap for another family's income.

To prove the validity of this rule, we need to draw on the Lorenz domination condition of Theorem 2.1 by the following lemma.

<sup>17</sup> The number  $(r-1)$  may be thought of as a *crossing index* since if we were to plot the two distributions  $X$  and  $Y$  with two curves they would cross  $(r-1)$  times.

LEMMA 2.3: When  $X \neq Y$ ,  $L_X \geq L_Y$  is equivalent to (2.10.a, b)

$$(2.10a) \quad \sum_{j=1}^i d_j \geq 0 \quad \text{for } i = 1, \dots, n,$$

$$(2.10b) \quad \sum_{j=1}^p S_j \geq 0 \quad \text{for } p = 1, \dots, r.$$

PROOF: (2.3.a, b) imply (2.10.a). Conversely (2.10.a) implies (2.3.a) and, since  $X \neq Y$ , it also implies (2.3.b). Thus, (2.3.a, b) and (2.10a) are equivalent when  $X \neq Y$ . It follows directly from (2.8) that (2.10.a) implies (2.10.b). Thus we need only prove the reverse implication. Suppose  $d_i \in D_q = (d_{a+1} \ d_{a+2} \ \dots \ d_{a+m_q})$ . Then define

$$V_i = \sum_{j=1}^i d_j = S_1 + S_2 + \dots + S_{q-1} + d_{a+1} + \dots + d_i.$$

We want to prove  $V_i \geq 0$ . In this expression,

$$(2.11a) \quad S = S_1 + S_2 + \dots + S_{q-1} \geq 0 \quad (\text{by (2.10.b)}),$$

$$(2.11b) \quad S_q = d_{a+1} + \dots + d_{a+m_q}, \quad \text{where all } d\text{'s have the same sign,}$$

$$(2.11c) \quad V_{a+m_q} = S + S_q \geq 0 \quad (\text{by 2.10.b}).$$

Thus,  $d_i$  is one member of a sequence  $(V_{a+1}, V_{a+2}, \dots, V_{a+m_q})$  which either (i) is monotonically increasing from  $S \geq 0$  if  $D_q$  is a positive set, or (ii) is monotonically decreasing to  $S + S_q \geq 0$  if  $D_q$  is a negative set. In either case,  $V_i \geq 0$ .

Q.E.D.

Notice that (2.9.a, c) and (2.10.b) imply  $S_1 > 0$  and  $S_r < 0$ . Thus (2.9.b) implies  $r$  is even. Hence,

LEMMA 2.4:  $L_X \geq L_Y$  implies  $r$  is even and the  $S_i$  can be grouped into  $r/2$  pairs with the indicated signs:

$$(2.12) \quad (S_1^+ S_2^-) \ (S_3^+ S_4^-) \dots (S_{r-1}^+ S_r^-)$$

Then when  $L_X \geq L_Y$ , families in the last group  $S_r^-$  of  $X$  must be relatively poorer than those in  $Y$ . The opposite is true for the group  $S_{r-1}^+$ .

Before we can prove the validity of the rule, we need one more lemma. In this lemma suppose  $Y' = (Y'_1 \ Y'_2 \ \dots \ Y'_n)$  is obtained from  $Y$  by a single rank-preserving equalization. Let

$$(2.13) \quad d' = (d'_1 \ \dots \ d'_n) = (X_1 - Y'_1 \quad X_2 - Y'_2 \quad \dots \quad X_n - Y'_n).$$

Then:

**LEMMA 2.5:** *If  $L_X \geq L_Y$ , there exists  $Y' \in \Omega_0$  such that*

$$(2.14a) \quad Y' = E(Y),$$

$$(2.14b) \quad L_X \geq L_{Y'},$$

$$(2.14c) \quad d_i = 0 \text{ implies } d'_i = 0,$$

$$(2.14d) \quad \text{there is at least one integer } j \text{ such that } d_j \neq 0 \text{ and } d'_j = 0.$$

**PROOF:** Suppose the last non-zero  $d_i$  in  $S_{r-1}^+$  in (2.12) is  $d_p$  and the first non-zero  $d_i$  in  $S_r^-$  is  $d_q$  ( $q > p$ ). Thus, by this choice, we have

$$(2.15) \quad d_{p+1} = d_{p+2} = \dots = d_{q-1} = 0.$$

Let

$$(2.16) \quad h = \min(d_p, -d_q) = \min(X_p - Y_p, Y_q - X_q) > 0.$$

When  $h$  is transferred from the  $q$ th family to the  $p$ th family of  $Y$ , let the result be denoted by  $Y'$ . Then obviously (2.14.a, c, d) are satisfied. To prove (b), we have

$$(2.17) \quad d'_1 + d'_2 + \dots + d'_i = \begin{cases} d_1 + d_2 + \dots + d_i & \text{for } i < p \text{ or } i \geq q, \\ d_1 + d_2 + \dots + d_i - h = d_1 + d_2 + \dots + d_{p-1} + d_p - h & \text{for } p \leq i < q. \end{cases}$$

The first sum  $d_1 + d_2 + \dots + d_i \geq 0$  by (2.10.a). In the second sum,  $d_1 + d_2 + \dots + d_{p-1}$  is non-negative by Lorenz-domination (2.10.a) and  $(d_p - h)$  is non-negative because  $h \leq d_p$ . Thus  $d'_1 + d'_2 + \dots + d'_i \geq 0$  and  $L_X \geq L'_{Y'}$  by (2.10.a). *Q.E.D.*

Lemma 2.5 assures us that we can repeat the same operation on  $Y'$  by reducing one additional non-zero entry of  $d'$ . Since there are only a finite number of non-zero  $d_i$  we have:

**LEMMA 2.6:** *If  $L_X \geq L_Y$ , then there exists a sequence of rank-preserving transfers  $T$  such that  $X = T(Y)$  and  $T$  involves at most  $m$  steps, where  $m$  is the number of non-zero  $d_i$  in  $d$  (as given by (2.5)).*

The proof of the sufficient condition of Theorem 2.1 follows directly from Lemma 2.6, as does the validity of the rule presented above.

#### D. Theorem 2.1 and the Lorenz Criterion

Theorem 2.1 has a ready application to zones of ambiguity and Lorenz-domination. When comparing two distributions  $X$  and  $Y$ , a simple rule for determining when  $L_X$  crosses  $L_Y$  (i.e., when  $X \in M$ ) is to examine the sign of the

first and last non-zero  $d_i$  and, if they have the same sign, the Lorenz curves must cross.<sup>18</sup>

From Theorem 2.1, it follows that the axiomatic system A1–A3 constitutes a rigorous justification for the Lorenz criterion for comparing the relative inequality of two income distributions. Of course, the Lorenz criterion is incomplete, and the three axioms are also. Completeness is customarily achieved via cardinality, but the cardinal indices in current use do not necessarily satisfy A1–A3. In what follows, we examine which of the usual indices satisfy our three axioms and which do not.

### 3. CARDINAL INDEX APPROACH TO INEQUALITY COMPARISONS

The traditional approach for comparing the inequality of two distributions is to compute a cardinal index of inequality  $I$  with domain  $\Omega^+$ :

$$(3.1) \quad I = f(X) = f(X_1 \ X_2 \ \dots \ X_n), \quad X_i \geq 0.$$

Examples are the Gini coefficient, coefficient of variation, range, and others which we shall consider below. Inequality comparisons are made according to the following definition:

**DEFINITION—Pre-Ordering Induced by an Index:** A real-valued index of inequality  $I = f(X)$  induces a complete pre-ordering  $R$  as follows: for all  $X, Y \in \Omega^+$ ,  $X R Y$  when  $f(X) \leq f(Y)$ .<sup>19</sup>

Notice that the *cardinality* of the index (3.1) is unnecessary for the question of determining which is the more equal of two distributions, since the essential information for this purpose is all contained in the pre-ordering  $R$  which  $f(X)$  induces.

It is the purpose of this section to explore whether  $R$ 's induced by many familiar inequality indices indeed satisfy the three axioms introduced in Section 1. When a particular index  $I = f(X)$  in (3.1) satisfies restrictions (3.2.a–c), the following theorem insures that  $R$  satisfies A1–A3:

**THEOREM 3.1:** *The pre-ordering  $R$  induced by an index  $I = f(X)$  satisfies A1–A3 when:*

$$(3.2a) \quad \text{homogeneous of degree zero: } f(X) = f(aX), \quad a > 0;$$

$$(3.2b) \quad \text{symmetry: } f(X_{i_1} \ X_{i_2} \ \dots \ X_{i_n}) = f(X_1 \ X_2 \ \dots \ X_n), \quad \text{where } (i_1, i_2, \dots, i_n) \text{ is a permutation of } (1, 2, \dots, n);$$

<sup>18</sup> However, if they have opposite signs, they may or may not cross and it is necessary to compare the full distributions.

<sup>19</sup> Notice that (3.1) measures *inequality* and therefore a more equal distribution has a lower index.

(3.2c) *monotonicity of partial derivative:*

$$\frac{\partial f}{\partial X_i} = f_i(X) < \frac{\partial f}{\partial X_j} = f_j(X) \quad \text{for } i < j \quad \text{and} \quad X \in \Omega_0.$$

**PROOF:** (3.2a) and (3.2b) respectively insure that the induced ordering satisfies A1 and A2. To show A3 holds on  $\Omega_0$ , suppose  $X$  is obtained from  $Y$  by a rank-preserving equalization. It is readily seen that the difference  $I(X) - I(Y)$  is negative by (2.2.c) for an equalization of any positive sum. Thus, A3 is satisfied.

Using Theorem 3.1, we may prove:

**THEOREM 3.2:** *The following inequality indices satisfy A1–A3:*

(3.3) *Coefficient of Variation:*  $C = \sigma/\bar{X}$  where  $\sigma = \sqrt{\sum_i (X_i - \bar{X})^2/n}$  and  $\bar{X} = \sum_i X_i/n$ .

(3.4) *Gini Coefficient:*  $G = -1 - \frac{1}{n} + \frac{2}{n^2 \bar{X}} [X_1 + 2X_2 + \dots + nX_n]$ .

(3.5) *Atkinson Index:*<sup>20</sup>

$$A = 1 - \left[ \left[ \left( \frac{X_1}{\bar{X}} \right)^{1-\varepsilon} + \left( \frac{X_2}{\bar{X}} \right)^{1-\varepsilon} + \dots + \left( \frac{X_n}{\bar{X}} \right)^{1-\varepsilon} \right] \frac{1}{n} \right]^{1/(1-\varepsilon)}, \quad \varepsilon > 1.$$

(3.6) *Theil Index:*<sup>21</sup>  $T = \sum_i X_i \log nX_i$ .

At the suggestion of the editor, the proof of Theorem 3.2 is omitted due to space limitations.

The fulfillment of A1–A3 by these measures strengthens both the axioms and the indices. The axioms are seen to be relevant to a number of measures with which we have considerable experience. In turn, the indices are seen to have several properties whose desirability is a matter of substantial agreement.

Despite the large number of indices which satisfy our three axioms, there are other indices in common use which violate them, particularly A1 and A3. The difficulty with those indices which violate A1 (e.g., *variance*) is that they are not independent of the level of income. Those indices which do not satisfy A3 are in some circumstances insensitive to certain rank-preserving equalizations. One example is the family of *fractile ranges* (e.g., *interquartile range*); any rank-preserving equalizations *within a segment* (e.g., within a quartile) leave the index unchanged, in violation of A3. Another example is the *Kuznets Ratio*:<sup>22</sup>

(3.7)  $K = \sum |\theta_i - 1/n|$ ,

<sup>20</sup> See Atkinson [1].

<sup>21</sup> See Theil [13].

<sup>22</sup> See Kuznets [8].

which is unchanged by any rank-preserving equalization or disequalization *on the same side of the mean*. To the extent that A1–A3 are reasonable, all indices which violate them are less than satisfactory; their popular use in empirical work cannot be defended by these axioms and must be justified on other grounds.

#### 4. CONCLUSION

In this paper, we have developed an approach to inequality comparisons which differs from the conventional one. Beginning by postulating three axioms, we showed that the axiomatic system so constructed is sufficient to justify the Lorenz criterion for inequality comparisons. However, like the Lorenz criterion, the axiomatic system is incomplete. Past researchers have achieved completeness by the use of cardinal inequality measures. We showed that many but by no means all of the commonly used indices satisfy our three axioms. The ones which do satisfy the axioms agree on the ranking of distributions whose Lorenz curves do not intersect. However, when Lorenz curves do intersect, the various measures partition the income distribution space differently. Since the three axioms are insufficient to determine the specific partition to use, the use of any of the conventional measures implicitly accepts the additional welfare judgments associated with that measure.

The key issue for inequality comparisons is the reasonableness of the ordering criterion, which in the case of cardinal measures is the index itself. An axiomatic approach is probably the ideal method for confronting this issue, because the reasonable properties (i.e., the axioms) are postulated explicitly. At minimum, this approach facilitates communication by enabling (and indeed requiring) one to set forth clearly his own viewpoints and value judgments for scrutiny by others. But in addition, to the extent that one person's judgments (such as those in our three axioms) are acceptable to others, controversies over inequality comparisons may be resolved. We have seen that our three axioms are incomplete insofar as they cannot determine the ordinal ranking uniquely. A feasible and desirable direction for future research is to investigate what further axioms could be introduced to complete the axiomatic system or at least to reduce further the zones of ambiguity.

It is conceivable that beyond some point the search for new axioms may turn out to be unrewarding. In that case, inequality comparisons will always be subject to arbitrary specifications of welfare weights. The selection of suitable weights by whatever reasonable criterion one cares to exercise is a less desirable but possibly more practical alternative than a strictly axiomatic approach.

Our research has hopefully made clear that inequality comparisons cannot be made without adopting value judgments, explicit or otherwise, about the desirability of incomes accruing to persons at different positions in the income distribution. Even the Lorenz criterion, which permits us to rank the relative inequality of different distributions in only a fraction of the cases, embodies such judgments. The traditional inequality indices such as those considered in Section 3, to the extent they complete the ordering, embody *some* value judgments

beyond our three axioms. The axiomatic bases for these judgments are at present vague, and it would be helpful if future researchers could state these implicit value judgments in axiomatic terms so that when a particular inequality index is used we will know exactly what judgments are being made.

*Yale University*

#### REFERENCES

- [1] ATKINSON A. B.: "On the Measurement of Inequality," *Journal of Economic Theory*, 5 (1970), 244-263.
- [2] CHAMPERNOWNE, D. G.: "A Comparison of Measures of Inequality of Income Distribution," *The Economic Journal*, 84 (1974), 787-816.
- [3] DALTON, H.: "The Measurement of the Inequality of Incomes," *Economic Journal*, 30 (1920), 348-361.
- [4] DASGUPTA, PARTHA, AMARTYA SEN, AND DAVID STARRETT: "Notes on the Measurement of Inequality," *Journal of Economic Theory*, 8 (1973), 180-187.
- [5] HICKS, J. R.: *Value and Capital*. Oxford: The Clarendon Press, 1939.
- [6] HIRSCHMAN, ALBERT: "The Changing Tolerance for Income Inequality in the Course of Economic Development," with a mathematical appendix by Michael Rothschild, *Quarterly Journal of Economics*, 87 (1973), 544-566.
- [7] KONDOR, YAAKOV: "Value Judgements Implied by the Use of Various Measures of Income Inequality," *Review of Income and Wealth*, 21 (1975), 309-321.
- [8] KUZNETS, SIMON: "Quantitative Aspects of the Economic Growth of Nations, II: Industrial Distribution of National Product and Labor Force," *Economic Development and Cultural Change*, 5 (1957) (Supplement), 3-111.
- [9] ROTHSCHILD, MICHAEL, AND JOSEPH E. STIGLITZ: "Some Further Results on the Measurement of Inequality," *Journal of Economic Theory*, 8 (1973), 188-204.
- [10] SEN, AMARTYA K.: *On Economic Inequality*. New York: Norton, 1973.
- [11] ———: "Poverty: An Ordinal Approach to Measurement," *Econometrica*, 44 (1976), 219-231.
- [12] SZAL, RICHARD, AND SHERMAN ROBINSON: "Measuring Income Inequality," Princeton University and Brookings Institution, mimeo, 1975.
- [13] THEIL, HENRI: *Economics and Information Theory*. Chicago: Rand McNally, 1967.