On the Proof Theory of the modal logic $G$. 

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1. An arithmetic interpretation $*$ of propositional formulas (fls) is determined by interpreting atoms $p_1$ by arithmetic sentences $p_1^*$. This may be extended to the language of modality by a suitable interpretation of the "necessity" connective $\Box$. Of special interest is the interpretation of $\Box$ as (arithmetized) provability in, say, Peano's arithmetic $PA$; i.e., one defines, given $[p_1]_i^*$, $\langle \Box \rangle^* := i$, $(\varphi \ast \psi)^* := \varphi^* \ast \psi^*$ for each binary connective $\ast$, and $(\Box \varphi)^* := \text{Pr}(\text{Pr}^* \varphi)$, where $\text{Pr}$ is a (canonical) provability predicate for $PA$.

Under any such interpretation, any instance of the following schemas becomes a theorem or rule of $PA$:

$$(A1) \quad \Box(\varphi \ast \psi) \rightarrow \Box \varphi \ast \Box \psi$$

$$(A2) \quad \Box \varphi \rightarrow \Box \Box \varphi$$

$$(R1) \quad \varphi \Rightarrow \Box \varphi$$

In fact, these are the derivability conditions used in the proof of Gödel's incompleteness theorems. While these schemas are valid also for trivial interpretations of $\Box$ (e.g., as a vacuous operator), the self-referential mechanism of $PA$ yields as a theorem of $PA$ also each $*$ interpretation of

$$(A3) \quad \Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi$$

(see [5]).
Let \( \mathcal{G} \) (for Gödel) be the extension of classical propositional logic \( \mathcal{G}_p \) with (A1)-(A3), (R1). We indicated that each \(*\)-interpretation of a theorem of \( \mathcal{G} \) is a theorem of \( \mathcal{G}_p \). Solovay [2] proved the converse: if \( \vdash_{\mathcal{G}_p} \varphi \) for all \(*\), then \( \vdash_\mathcal{G} \varphi \). The logic \( \mathcal{G}_n \) is discussed in extenso by Boolos [1] and Smoryński [6] (where it is denoted \( \mathcal{L} \)).

De Jongh, Sambin and Kripke have independently shown that (A2) is derived in \( \mathcal{G}^- := \mathcal{G} - (A2) \). (cf. [1], p. 30.)

2. An alternative axiomatization of \( \mathcal{G} \). Let \( \mathcal{G}' \) be like \( \mathcal{G} \), except that (A3) is replaced by the inference rule

\[
(R2) \quad \vdash \Box \varphi \rightarrow \varphi.
\]

We show that \( \mathcal{G}' \) is equivalent to \( \mathcal{G} \).

2.1. \textsc{Lemma.} \( \vdash_\mathcal{G} \Box \varphi \rightarrow \vdash_\mathcal{G} \varphi \).

\textbf{Proof:} Assume \( \vdash_\mathcal{G} \Box \varphi \); then \( \vdash_{\mathcal{G}_p} \text{Pr} \varphi \) for all \(*\), so \( \vdash_\mathcal{G} \varphi \) by the soundness of \( \mathcal{G}_p \), and hence \( \vdash_\mathcal{G} \varphi \) by Solovay's completeness theorem. \( \square \)

2.2. \textsc{Proposition.} \( \vdash_\mathcal{G} \varphi \rightarrow \vdash_\mathcal{G} \varphi \).

\textbf{Proof:} We only have to verify that \( \mathcal{G} \) is closed under \((\Box \cdot \cdot)\).

Assume \( \vdash_\mathcal{G} \Box \varphi \rightarrow \varphi \); then \( \vdash_\mathcal{G} \Box (\Box \varphi \rightarrow \varphi) \) by (R1), so \( \vdash_\mathcal{G} \Box \varphi \) by (A1), and \( \vdash_\mathcal{G} \varphi \) by 2.1. \( \square \)
2.3. INDUCTION. \( \varphi \rightarrow \varphi' \varphi \).

Proof: We only have to prove in \( \varphi' \) every instance of (A3), say

\[
\varphi' = \Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi.
\]

By (P2) it suffices to derive \( \Box \varphi' \rightarrow \varphi' \). Arguing in \( \varphi' \), assume

(1) \( \Box \varphi' \) and (2) \( \Box(\Box \varphi \rightarrow \varphi) \). Then (3) \( \Box \Box(\Box \varphi \rightarrow \varphi) \) by (2), (A2); also (4) \( \Box \Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \Box \varphi \) by (1), (Al); so (5) \( \Box \Box \varphi \)

by (3), (4); (6) \( \Box \Box \varphi \rightarrow \Box \varphi \) by (2), (Al), and \( \Box \varphi \) by (5),

(6). \( \Box \varphi \).

3. A SEQUENTIAL CALCULUS FOR \( \varphi \). Let \( \Gamma, \Delta \) stand for finite sets of

fns. A sequent is an ordered pair \( \Gamma; \Delta \). Write \( \Gamma, \Delta \) for \( \Gamma \cup \Delta \);

\( \Gamma; \Sigma \) for \( \Gamma \cup \{ \varphi \} \); \( \Box \Gamma \) for \( \{ \Box \varphi \mid \varphi \in \Gamma \} \). We define \( \varphi \o \) as the

sequential calculus built on the following inference rules.

\[
\begin{array}{c}
p; p \quad ; \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\Gamma; \Delta, \varphi \quad \Sigma, \varphi ; \Xi \\
\hline
\Gamma, \Delta, \varphi + \varphi ; \Sigma, \Xi
\end{array}
\]

\[
\begin{array}{c}
\Gamma; \varphi ; \varphi, \Delta, \Xi \\
\hline
\Gamma, \varphi + \varphi, \Delta, \Xi
\end{array}
\]

The usual rules for \( \land \) and \( \lor \) (if one wishes to refer to these

connectives)

\[
\begin{array}{c}
\Gamma; \varphi, \Delta \quad \Sigma, \psi, \Xi \\
\hline
\Gamma, \Sigma, \varphi, \psi, \Xi
\end{array}
\]

\[
\begin{array}{c}
\Gamma; \Delta \\
\hline
\Gamma, \Sigma, \Delta, \Xi
\end{array}
\]

\[
\begin{array}{c}
\Gamma; \Delta \\
\hline
\Gamma, \Sigma, \Delta, \Xi
\end{array}
\]
$\Gamma$: \[ \phi \]

Clearly, $G_0$ is the same as $G_1 + (A1) + (K1)$. Let $G_1 := G_0 + (A1)$, and $G_2 := G_1 + (L)$ where $L$ is the rule:

\[ L \quad \frac{L \psi \psi}{\phi} \]

Then $G_2$ is the same as $G'$, and hence the same as $G$.

3.1. **LEMMA**. (Cut-elimination) Every theorem of $G_0$ has a proof in $G_0$ without cut.

**Proof:** Same as the standard cut-elimination argument for $G_2$ (cf. e.g. [3] p. 454). Permutation of cut over $\Box I$ is never needed, since all $f$s are active in the conclusion of $\Box I$. When both active occurrences of a cut formula are derived by $\Box I$, we have

\[
\Gamma : \phi \\
\Sigma : \psi \\
\frac{\Box \Gamma : \Box \phi \\
\Box \Sigma : \Box \psi}{\Box \Gamma, \Box \Sigma : \Box \psi}
\]

This is reduced to

\[
\Gamma : \phi \\
\Sigma : \psi \\
\frac{\Box \Gamma, \Box \Sigma : \Box \psi}{\Box \Gamma, \Box \Sigma : \Box \psi}
\]

We do not have cut elimination for either $G_1$ or $G_2$. However, the simple axiomatization of $G_2$ over $G_0$ permits some interesting applications of 3.1. This is done via the following lemma.

3.2. **LEMMA**. (i) If $\Gamma \vdash_0 \Delta$, then $\Sigma \sqcup \Xi, \Gamma \vdash_0 \Delta$, where each $\sigma \in \Sigma$ is an instance of $(A2)$, and each $\xi \in \Xi$ is a theorem of $G$.
(ii) Let \( \mathcal{G}_2^- := \mathcal{G}_2 - (A2) \). If \( \Gamma \vdash \mathcal{G}_2^- \Delta \) then \( \Box \Xi, \Gamma \vdash \mathcal{G}_0 \Delta \)

where each \( \xi \in \Xi \) is a theorem of \( \mathcal{G}_2^- \).

Proof: (i) If \( \Gamma \vdash \mathcal{G}_0 \Delta \) then there is a proof \( \pi \) in \( \mathcal{G}_0 + (L) \) deriving \( \Gamma, \Sigma : \Delta \) for \( \Sigma \) as above. Skipping in \( \pi \) each instance

\[ \Box \eta : \xi \]

of \( (L) \) and collecting \( \Box \psi \) in all antecedents below such an instance we obtain the result. The proof of (ii) is the same. \( \Box \)

4. Closure under rules. Cut-free systems are useful in demonstrating closure under rules. We give two examples.

4.1. PROPOSITION. If \( \Box \Gamma \vdash \Box \Delta, \Box \Delta' \) then \( \Gamma, \Box \Gamma \vdash \Box \Delta, \Box \Delta' \).

Proof: Assume \( \Box \Gamma \vdash \Box \Delta, \Box \Delta' \). Then, by 3.2(i), 3.1, there is a cut free proof \( \pi \) of \( \mathcal{G}_0 \) deriving

\[ (\star) \]

\[ \Sigma, \Box \Xi, \Box \Gamma : \Box \Delta, \Box \Delta' \]

where \( \Sigma = (\Box t_1 + \Box t_1) \) and each \( \xi \in \Xi \) is a theorem of \( \mathcal{G} \). We show by induction on the height \( h \) of \( \pi \) that (**) \( \Gamma, \Box \Gamma \vdash \Delta, \Box \Delta' \).

Basis. \( h = 1 \). (\ast) has no premise. Case (a). \( \Delta S \in \Box \Xi \) for some \( S \in \Delta \cup \Delta' \); then \( \Gamma \vdash \Box S \) and \( \Gamma \vdash \Box S \). (b) \( \Delta S \in \Box \Gamma \); then \( \Gamma \vdash \Box S \), and \( \Gamma \vdash \Box \Box \Delta \).

Ind. Step. \( h > 1 \). Case 1. (\ast) is derived by \( \Box \):
By ind. hyp. applied to the premises,

\[ \Gamma \Box \Gamma \vdash \Delta, \Box \Psi \quad \text{and} \quad \Gamma, \Box \Gamma, \Box \top, \Box \top \vdash \Delta, \Box \Delta' \]

So \( \Gamma, \Box \Gamma \vdash \Delta, \Box \Delta' \)

**Case 2.** (*) is derived by thin; trivial.

**Case 3.** (*) is derived by \( \Box \land \): 

\[
\frac{\Xi, \Gamma; \Box \land \Delta}{\Box \Xi, \Box \Gamma; \Box \Delta}
\]

and \( \Box \Gamma \vdash \Box \Delta \).

then \( \Gamma \vdash \Box \Delta \). Since \( \pi \) is cut-free, these are the only possible cases. \( \Box \)

4.2. **COROLLARY.** If \( \Box \Box \Gamma \vdash \Box \Delta \) then \( \Box \Gamma \vdash \Box \Delta \). \( \Box \)

4.3. **PROPOSITION.** If \( \Box \Gamma \vdash \Box \Delta \) then \( \Gamma \vdash \Box \Delta \).

**Proof:** Similar to (and simpler than) 4.1., using 3.2(ii) in place of 3.2(i). \( \Box \)

4.4. **COROLLARY.** \( \Box p \vdash \Box \Box \top \).

This contrasts with the derivability of \( (\Lambda \top) \) in \( \Box \top = \Box \top \) - (\( \Lambda \top \)),

mentioned in §1.

We now give a second example of a rule under which \( \Box \top \) is closed.

4.5. **PROPOSITION.** If \( \Box \Gamma \vdash \Box \top \Lambda \), where each \( \lambda \in \Lambda \) is a propositional letter, then \( \Box \Gamma \vdash \Box \Omega \) for some \( \Omega \in (\Box \top, \Lambda) \).
Proof: Using the conventional notations of 4.1, it suffices to show that $\Box \Gamma \vdash p$ whenever there is a cut-free proof $\pi$ of $\Box_0$ deriving (4) $\Sigma b \vdash \Box \Delta, \Lambda$. We proceed, again, by induction on the height of $\pi$. The basis is trivial.

**Ind. Step. Case 1.** (4) is derived by $\rightarrow L$.

$$
\frac{
\Sigma b \vdash \Box \Delta, \Lambda,
\Sigma b \vdash \Box \Delta, \Lambda
}{
\Sigma b \vdash \Box \Delta, \Lambda
}
$$

By ind. hyp. applied to the left premise, if $\Box \Gamma \vdash p$ for $p \in (\Box \Delta, \Lambda)$, then $\Box \Gamma \vdash p$. And by ind. hyp. for the right premise, $\Box \Gamma, \Box \vdash p$ for some $p \in (\Box \Delta, \Lambda)$. Hence $\Box \Gamma \vdash p$.

**Case 2:** thin; trivial. **Case 3.** $\Box I$; then the succedent of (4) must consist of a single fl to start with. 

Some examples of application of 4.5: (1)

$$
\Box (p \rightarrow q) \vdash \Box p \rightarrow \Box q
$$

(2) $\Box (p \lor \Box p \lor \cdots \lor \Box^n p) \vdash \Box p \lor \Box p \lor \cdots \lor \Box^n p$.

Here $\Box_0 p := p$,

$$
\Box (p) := \Box \Box p
$$

5. The reflection principle. This is the schema $\Box \varphi \rightarrow \varphi$. By (A3), $\vdash \Box (p \rightarrow \varphi)$ iff $\vdash \Box \varphi$. The next result shows that the reflection principle is not finitely axiomatizable over $\Box$. This has been shown model-theoretically by Boolos [2].

**5.1. Proposition.** Assume (4) $(\Box \varphi_1 \rightarrow \varphi_1)_{i=1}^{k} \vdash \Box \Box^n p \rightarrow p$. Then $k \geq n$. 

Proof: By induction on \( n \). Basis \( n = 1 \); trivial. Ind. step.

\( n > 1 \). Using again the notational conventions of 4.1, if (*) holds, then there is a cut-free proof \( \kappa \) of \( \Box_0 \) deriving

\[(**) \quad \Sigma_\Box \Xi \Box_1, (\Box \varphi_i \rightarrow \varphi_i)_{i=1}^k, \Box^n p : p. \]

(*) must be derived by thin or \(-\bot\). The left premise have the form \( \Sigma_\Box \Xi \Box_1, (\Box \varphi_i \rightarrow \varphi_i)_{i=1}^k, \Box^n p : p, \Box \top \) and is derived again by thin or \(-\bot\). These inferences may be ordered at will, with a single instance of thin on the top; this is simply because such instances of \(-\bot\) may be permuted ([4]). To recall:

\[
\Gamma; \Delta, \alpha, \phi \quad \Gamma, \psi; \Delta, \alpha' \quad \Gamma, \psi \rightarrow \psi; \beta, \Delta
\]

\[
\Gamma, \alpha + \psi, \beta \rightarrow \psi; \Delta
\]

may be rearranged as

\[
\Gamma; \Delta, \alpha, \phi \quad \Gamma, \psi; \Delta, \beta' \quad \Gamma, \psi; \Delta, \beta \quad \Gamma, \psi; \Delta, \alpha'
\]

\[
\Gamma, \psi; \Delta, \beta \rightarrow \psi, \alpha \rightarrow \psi; \Delta
\]

We may assume, therefore, that instances of reflection are active below active occurrences of \( \Box \)s in \( \Sigma \); taking successively left premises of \(-\bot\), we then get in \( \pi \) a sequent \( \Sigma_\Box \Xi \Box_1, (\Box \varphi_i)_{i=1}^k, \Box^n p : p \). Since \( \Box^n p \vdash \Box p \), we get, by 4.5,

\( \Box^n p \vdash \Box \varphi_i \) for some \( i \), say \( i = k \). Since \( n > 1 \), \( \Box^{n-1} p \vdash \Box \varphi_k \) by 4.2. Hence \( \Box \varphi_i \rightarrow \varphi_i \) for \( i = 1 \). Since \( n > 1 \), \( \Box^{n-1} p \vdash \varphi_k \) by ind. hyp. \( k - 1 \geq n - 1 \), and so \( k \geq n \). \( \Box \)}
6. **Interpolation.** A system \( S \) satisfies (Craig's) interpolation if \( \varphi \models_S \psi \) implies that \( \varphi \models_{S_x} \chi \) and \( \chi \models_S \psi \) for some \( S_x \) with logical constants common to \( \varphi \) and \( \psi \).

6.1. **Proposition.** \( G_0 \) satisfies interpolation.

**Proof:** We apply cut-elimination (3.1) via Maehara's partition method (cf. [4] p. 35). The presence of the rule \( \Box \Gamma \) necessitates only two additional clauses, (1) Consider \( \Gamma, \Delta : \Box \psi \), and assume \( \chi \) is an interpolant for the premise: \( \Gamma, \chi \vdash \gamma \) and \( \Delta, \chi \vdash \phi \). Then

\( \Box \Gamma, \Box \neg \chi \vdash \Box \gamma \) and \( \Box \Delta, \Box \neg \chi \vdash \Box \phi \). So \( \Box \Gamma, \Box \neg \chi, \Box \gamma \) and

\( \Box \Delta, \Box \neg \chi, \Box \phi \), and \( \Box \neg \chi \) is an interpolant for the conclusion. (2)

Similarly, if \( \chi \) is an interpolant for \( \Gamma, \Delta : \psi \), then \( \Box \chi \) is for \( \Box \Gamma, \Box \Delta : \Box \psi \).

6.2. **Lemma.** Assume \( \Gamma, \Delta, \Gamma_1 [\bar{r}] \vdash_{G_2} \Delta, \Gamma_2 [\bar{r}] \), \( \bar{r} = (r_1, \ldots, r_n) \). Set \( \Sigma_1 = (\sigma(\delta_1, \ldots, \delta_n) \mid \sigma \in \Sigma_1, \delta_j = \tau \text{ or } 1, j = 1, \ldots, n) \). Then

\( \Gamma, \Delta, \Sigma_1 \vdash_{G_2} \Delta, \Sigma_2 \).

**Proof:** A straightforward and trivial induction on the length of the proof in \( G_2 \) for \( \Gamma, \Delta, \Sigma_1 \).

6.3. **Proposition.** If \( H = G_0 + S \), where \( S \) is a set of axioms (no rules!) closed under substitution, then \( H \) satisfies interpolation.

**Proof:** Assume \( \varphi(\bar{b}, \bar{q}) \vdash_H \psi(\bar{p}, \bar{b}) \); then \( \Sigma(\bar{b}, \bar{q}, \bar{p}) \), \( \varphi \vdash_{G_0} \psi \), with \( \Sigma \subset S \). So by 5.2. \( \Sigma, \bar{q} \vdash_{[\bar{p}]} \varphi \), \( \varphi \vdash_{G_0} \psi \). By 5.1 there is an interpolant \( \chi \) in \( G_0 \) for \( \Lambda(\bar{q}, \bar{b}) \Lambda \varphi \) and \( \psi \). Since \( \Sigma, \bar{q} \subset S \) by our assumption on \( S \), \( \varphi \vdash_H \chi \) and \( \chi \vdash_H \psi \).
6.4. **Corollary.** \( G_1 \) and \( G_2 = G \) satisfy interpolation. \( \Box \)

The interpolation theorem for \( G \) was proved independently by Boolos [1] and Smoryński [7], using Kripke models for \( G \). As usual, from the interpolation theorem Beth's definability theorem for \( G \) readily follows.
REFERENCES


