On Branching Numbers of Normal Manifolds*

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Abstract

Let $\mathcal{M}$ be a finite piecewise linear (pl) manifold of $\mathbb{R}^n$, and $P : \mathbb{R}^n \to \mathbb{R}^n$ be pl with respect to $\mathcal{M}$, i.e. P is affine on each set in $\mathcal{M}$. The *branching number* of $\mathcal{M}$, Kuhn and Löwen [8], is the maximum number of sets in $\mathcal{M}$ that can contain any common face of codimension 2. [8, Thm. 5.3] shows that if $\mathcal{M}$ has branching number less than or equal to 4, then P is a homeomorphism if and only if it is coherently oriented, i.e. the determinants of P on the sets in $\mathcal{M}$ have the same nonzero sign.

Let C be a nonempty polyhedral convex set in $\mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$. Robinson [14] defines a finite pl manifold $\mathcal{N}_C$ of $\mathbb{R}^n$ called the *normal manifold* of C; and the *normal map* $A_C : \mathbb{R}^n \to \mathbb{R}^n$ induced by (A, C), which is pl with respect to $\mathcal{N}_C$. [14, Thm. 4.3] shows that A_C is homeomorphic if and only if it is coherently oriented.

We show that $\mathcal{N}_C$ has branching number less than or equal to 4, hence Robinson's result is actually a corollary of Kuhn and Löwen's.

Key Words

Piecewise linear, piecewise affine, branching number, normal map, pl-normal, normal manifold, homeomorphism, coherently oriented

Abbreviated Title

On Branching Numbers of $\mathcal{N}_C$.

1 Introduction

Kuhn and Löwen [8] and Robinson [14] have provided deep and elegant results that characterize the homeomorphic members of certain classes of piecewise linear (pl) mappings. The purpose of this paper is to show that the class of mappings examined by Robinson is contained in the class of mappings examined by Kuhn and Löwen.

Let $\mathcal{M}$ be a finite piecewise linear (pl) manifold of $\mathbb{R}^n$, and $P : \mathbb{R}^n \to \mathbb{R}^n$ be pl with respect to $\mathcal{M}$, i.e. P is affine on each set in $\mathcal{M}$. The *branching number* of $\mathcal{M}$, Kuhn and Löwen [8], is the maximum number of sets in $\mathcal{M}$ that can contain any common face of codimension 2. [8, Thm. 5.3] shows that if $\mathcal{M}$ has branching number less than or equal to 4, then P is a homeomorphism if and only if it is coherently oriented, i.e. the determinants of P on each set in $\mathcal{M}$ have the same nonzero sign.

Let C be a nonempty polyhedral convex set in real n-dimensional space, $\mathbb{R}^n$, and A belong to the space of real $n \times n$ matrices, $\mathbb{R}^{n \times n}$. Robinson [14] defines a finite pl manifold $\mathcal{N}_C$ of $\mathbb{R}^n$ [14, Prop. 2.4] called the *normal manifold* of C; and the *normal map* $A_C : \mathbb{R}^n \to \mathbb{R}^n$ induced by (A, C), which is pl with respect to $\mathcal{N}_C$ [14, Prop. 2.5]. The main result [14, Thm. 4.3] shows that A_C is homeomorphic if and only if it is coherently oriented; see Ralph [10] for an alternative proof, and Robinson [15] for a much simpler proof in the special case that A is symmetric. It is pointed out in [14, §1] that if $\mathcal{N}_C$ could be shown to have branching number less than or equal to 4, then [14, Thm. 4.3] would follow from [8, Thm. 5.3]. Indeed, showing that normal manifolds have this branching number property is precisely our goal.

Throughout the paper, C and A will be as above. Let π_C be the Euclidean projection onto C, that is π_C maps each point of $\mathbb{R}^n$ to its nearest point in C with respect to the Euclidean norm. The identity in $\mathbb{R}^{n\times n}$ is I.

Definition 1 [14] *The* normal *mapping induced by (A,C) is the mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *given by*

$$A_C \stackrel{\text{def}}{=} A\pi_C + I - \pi_C.$$

Such normal mappings are called pl-normal *mappings.*

Pl-normal systems are important in many optimization and equilibrium problems. They arise directly from variational inequalities or, equivalently, generalized equations specified by linear mappings and polyhedral convex sets; and indirectly as approximations to such systems specified by smooth nonlinear functions over polyhedral convex sets. For example, see Robinson [16], Ralph [11]. Perhaps the best known of these problems is the linear complementarity problem [3, 9] which can be posed as a pl-normal equation $A_C(x) = a$, for appropriate $A \in \mathbb{R}^{n\times n}$ and $a \in \mathbb{R}^n$, by taking C to be the nonnegative orthant $\mathbb{R}^n_+$.

The question of necessary and sufficient conditions for pl mappings to be homeomorphic was answered for the linear complementarity problem by Samelson, Thrall and Wesler [18], and later by Murty [9]. Other work on the general question includes Fujisawa and Kuh [6], Rheinboldt and Vandergraft [12], Kojima and Saigal [7], Schramm [19], and Eaves and Rothblum [5].

The remainder of this section will be spent giving rigor to our introductory remarks on pl manifolds, branching numbers, coherent orientation, and homeomorphism results. §2 contains the main result on branching numbers, Proposition 12.

We recall the following terms.

- A set in $\mathbb{R}^n$ is ***polyhedral convex*** if it is the intersection of finitely many closed half-spaces.

- A *face* of a convex set $D \subset \mathbb{R}^n$ is a subset F of D such that for $x_1, x_2 \in D$ and $0 < t < 1$, if $tx_1 + (1-t)x_2$ belongs to F then x_1, x_2 belong to F.

Definition 2

1. *An* n-cell [4], *or* chamber [8], *(in* $\mathbb{R}^n$*) is a polyhedral convex set with nonempty interior in* $\mathbb{R}^n$.

2. *A* pl manifold, *or* chamber system [8], *of* $\mathbb{R}^n$ *is a locally finite family of* n*-cells covering* $\mathbb{R}^n$*, any two of which intersect in a (possibly empty) common face. A pl manifold of* $\mathbb{R}^n$ *is* finite *if it consists of finitely many* n*-cells.*

3. *Let* $\mathcal{M}$ *be a pl manifold of* $\mathbb{R}^n$*. The* branching number [8] *of* $\mathcal{M}$ *is the least* $b \in \{1, 2, \ldots, \infty\}$ *such that every face of an* n*-cell of* $\mathcal{M}$ *which has codimension 2 is contained in at most* b n*-cells.*

We note that, according to the proof of [14, Prop. 2.4], any pl manifold $\mathcal{M}$ of $\mathbb{R}^n$ is also a ***subdivided pl manifold of dimension*** n in the sense of Eaves [4]; namely, in addition to satisfying the requirements of Definition 2.2, it has the property that each face of dimension $n-1$ of an n-cell of $\mathcal{M}$ is contained in at most two n-cells of $\mathcal{M}$.

Definition 3 *Let $P : \mathbb{R}^n \to \mathbb{R}^n$, and $\mathcal{M}$ be a pl manifold of $\mathbb{R}^n$.*

1. *P is* pl with respect to $\mathcal{M}$ *if for each $\sigma \in \mathcal{M}$, there are $A_\sigma \in \mathbb{R}^{n\times n}$ and $b_\sigma \in \mathbb{R}^n$ such that*

$$P(x) = A_\sigma x + b_\sigma, \quad \forall x \in \sigma.$$

2. *If P pl with respect to $\mathcal{M}$ and the matrices A_σ ($\sigma \in \mathcal{M}$) are as above, the* determinants *of P are the determinants of the matrices A_σ.*

Suppose P, $\mathcal{M}$ and A_σ ($\sigma \in \mathcal{M}$) are as in the definition. Note that $\mathcal{M}$ is not uniquely determined by P. The set of matrices $\{A_\sigma \,|\, \sigma \in \mathcal{M}\}$ and the set of their determinants are, however, uniquely determined by P and are independent of $\mathcal{M}$. Furthermore it is easy to see, since $\mathcal{M}$ is locally finite, that P is not only continuous but locally Lipschitz; and that P is a Lipschitz mapping if $\mathcal{M}$ is finite.

We now present the characterization theorems of Kuhn and Lõwen [8], and Robinson [14], respectively.

Theorem 4 *Suppose $\mathcal{M}$ is a finite pl manifold of $\mathbb{R}^n$ with branching number less than or equal to 4. If $P : \mathbb{R}^n \to \mathbb{R}^n$ is pl with respect to $\mathcal{M}$, then it is a homeomorphism if and only if it is coherently oriented.*

Proof Let P be pl with respect to $\mathcal{M}$. [8, Thm. 5.3] says that if P is proper, i.e. $P^{-1}(D)$ is compact for any compact set $D \subset \mathbb{R}^n$, then it is bijective if and only if it is coherently oriented.

Suppose P is bijective. It is easy to see that $P(\mathcal{M}) \stackrel{\text{def}}{=} \{P(\sigma) \,|\, \sigma \in \mathcal{M}\}$ is a pl manifold of $\mathbb{R}^n$, and that P^{-1} is pl with respect to $P(\mathcal{M})$. Hence both

P and P^{-1} are continuous, and P is a homeomorphism. To conclude, we note that each of the conditions of bijectivity and coherent orientation imply that P is invertible on each of its n-cells, from which it follows, by finiteness of $\mathcal{M}$, that P is proper. $\square$

Theorem 5 *A_C is a homeomorphism if and only if A_C is coherently oriented.*

Proof According to [14, Thm. 4.3], A_C is a Lipschitzian homeomorphism if and only if A_C is coherently oriented. Now A_C is pl with respect to the finite pl manifold $\mathcal{N}_C$ [14, Thm. 2.5(i)] hence, as noted previously, A_C is Lipschitz, and the stated result follows. $\square$

Details of $\mathcal{N}_C$ will be given in §2. It follows from our main result, Proposition 12, that the branching number of $\mathcal{N}_C$ is less than or equal to 4, hence we can apply Theorem 4 directly to the pl-normal mapping A_C to obtain Theorem 5.

2 Normal manifolds, their quotients and local structure

Further notation:

- The relative interior of a set F in $\mathbb{R}^n$ is denoted $\operatorname{ri} F$. A basic fact [17, Thm. 6.1] is that each nonempty convex set in $\mathbb{R}^n$ has nonempty relative interior.
- The affine hull of a set $F \subset \mathbb{R}^n$ is denoted $\operatorname{aff} F$. The linear subspace parallel to $\operatorname{aff} F$ is denoted $\operatorname{par} F$. The linear subpace parallel to the largest affine subspace of F is denoted $\operatorname{lin} F$.

- A set K in $\mathbb{R}^n$ is a *cone* if for each $\alpha > 0$ and $k \in K$, $\alpha k \in K$. If K is nonempty cone, the *polar* cone of K is

$$K^\circ \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid \langle x, k \rangle \leq 0,\ \forall k \in K\}.$$

The polar cone of the empty set is defined to be the empty set.

- The *normal* cone to C at $\hat{c} \in C$ is

$$N_C(\hat{c}) \stackrel{\text{def}}{=} \{\nu \in \mathbb{R}^n \mid \langle \nu, c - \hat{c} \rangle \leq 0,\ \forall c \in C\}.$$

$N_C(\hat{c})$ is defined to be the empty set if $\hat{c} \notin C$.

The *tangent* cone to C at $\hat{c} \in \mathbb{R}^n$, is $T_C(\hat{c}) \stackrel{\text{def}}{=} N_C(\hat{c})^\circ$.

- If X is a nonempty set in $\mathbb{R}^n$, $X^\perp$ is the set of vectors $z \in \mathbb{R}^n$ such that $\langle z, x \rangle = 0$ for each $x \in X$.

- The *critical cone* [13] to C at $x \in \mathbb{R}^n$ is $T_C(\pi_C(x)) \cap [x - \pi_C(x)]^\perp$.

By convention, the sum or cartesian product of two sets in $\mathbb{R}^n$, one of which is empty, is also empty.

Proposition 6 [14, Prop. 2.1, 2.4, 2.5]

1. *For each face F of C and each point f in the relative interior of F, the normal cone to C at f is the same (nonempty) set. Denote this by N_F. Also, for any point $c \in F$,*

$$N_F = N_C(c) \cap (F - c)^\perp.$$

2. *The family of sets $F + N_F$, where F is any nonempty face of C, is a pl manifold of* $\mathbb{R}^n$.

3. *On each n-cell $F + N_F$, where F is a nonempty face of C, A_C acts as an affine mapping:*

$$A_C(f + \nu) = Af + \nu, \;\; \forall\, f \in F, \, \nu \in N_F.$$

The constancy of $N_C(x)$ for x in the relative interior of a face F of C is also shown by Burke and Moré [2, Thm. 2.3].

Definition 7

1. *Let F be a nonempty face of C and $f \in \operatorname{ri} F$. The* normal cone of C on F (in $\mathbb{R}^n$) *is* $N_F \stackrel{\text{def}}{=} N_C(f)$.

2. *The* normal manifold [14] of C (in $\mathbb{R}^n$), *denoted $\mathcal{N}_C$, is the family of sets $F + N_F$ where F is a nonempty face of C.*

Since C is polyhedral convex, it has finitely many nonempty faces F (see [17, Thm. 19.1]). So part 2 of Proposition 6 says that $\mathcal{N}_C$ is a finite pl manifold of $\mathbb{R}^n$. Part 3 says that A_C is pl with respect to $\mathcal{N}_C$.

Our first task is to show one-to-one correspondence between the nonempty faces of C and n-cells of $\mathcal{N}_C$.

Proposition 8

1. *Let F be a nonempty face of C.*

   The mapping $x \mapsto (\pi_C(x), x - \pi_C(x))$, where $x \in F + N_F$, is a bijection from $F + N_F$ to $F \times N_F$.

2. *The mapping $F \mapsto F + N_F$ is a bijection from the family of nonempty faces F of C to $\mathcal{N}_C$.*

Proof

1. Let $\mu(x) \stackrel{\text{def}}{=} (\pi_C(x), x - \pi_C(x))$. Clearly the addition mapping $+$ is the inverse of μ. Also $+ : F \times N_F \to F + N_F$. So we only have to show $\mu(F + N_F) \subset F \times N_F$.

   Let $f \in F$, $\nu \in N_F$. Then $\nu \in N_C(f)$ by Proposition 6.1. So $f + \nu \in (I + N_C)(f)$ hence, by [1, Ex. 2.8.2], $f = \pi_C(f + \nu)$. So $\mu(f + \nu) = (f, \nu) \in F \times N_F$.

2. Suppose F, G are nonempty faces of C such that $F + N_F = G + N_G$. Let $f \in \operatorname{ri} F$ and $\nu \in N_F$. Since $f + \nu \in F + N_F = G + N_G$, we have from part 1 that $f = \pi_C(f + \nu) \in G$. So $(\operatorname{ri} F) \cap G$ is nonempty and it follows, by [17, Thm. 18.1], that $F \subset G$. By symmetry, $G \subset F$, i.e. $F = G$.

$\square$

We now consider linear subspaces that are either contained in $\operatorname{lin} C$ or orthogonal to $\operatorname{par} C$, and show how they can be factored out of $\mathcal{N}_C$.

Proposition 9 *Let L, M be linear subspaces of* $\operatorname{lin} C$, $(\operatorname{par} C)^\perp$ *respectively.*

1. *The mapping $F \mapsto F \cap L^\perp$ is a bijection from the faces of C to the faces of $C \cap L^\perp$.*

2. Let $c_0 \in C \cap L^\perp$ and m be the dimension of $(L+M)^\perp$. The mapping $\sigma \mapsto (\sigma - c_0) \cap (L+M)^\perp$ is a bijection from the n-cells σ of $\mathcal{N}_C$ to the m-cells of the normal manifold of $(C - c_0) \cap L^\perp$ in $(L+M)^\perp$.

Proof

1. Note that, since L is parallel to an affine subspace of C,

$$C = (C \cap L^\perp) + L;$$

in particular $(C \cap L^\perp) \neq \emptyset$. Let F be a face of C. If $F \cap L^\perp$ is nonempty, let c_1 and c_2 belong to $C \cap L^\perp$, and $t \in (0,1)$ be such that $tc_1 + (1-t)c_2 \in F \cap L^\perp$. Then $c_1, c_2 \in F \cap L^\perp$ because F is a face and $L^\perp$ is convex, that is $F \cap L^\perp$ is a face of $C \cap L^\perp$. So the mapping $F \mapsto F \cap L^\perp$ from the faces F of C to the faces of $C \cap L^\perp$ is well defined. For injectivity, suppose F and F' are faces of C such that $F \cap L^\perp = F' \cap L^\perp$. Without loss of generality assume that F and F' are nonempty. Now $C = (C \cap L^\perp) + L$ hence for any $f \in F \subset C$, $f + L \subset C$. Let $l \in L$, then $(1/2)(f - l) + (1/2)(f + l) \in F$, so $f + l \in F$. Hence $f + L \subset F$, i.e. L is parallel to an affine subspace of F, and $F = (F \cap L^\perp) + L$. Likewise L is parallel to an affine subspace of F', hence

$$F' = (F' \cap L^\perp) + L = (F \cap L^\perp) + L = F.$$

For surjectivity, let G be a nonempty face of $C \cap L^\perp$. Since $G \subset L^\perp$ we have $(G + L) \cap L^\perp = G$. It is left to show that $G + L$ is a face of C.

Now $G+L \subset C$ because $G \subset C \cap L^{\perp}$ and $(C \cap L^{\perp}) + L = C$. Suppose $c_1, c_2 \in C$, $t \in (0,1)$ and $c \stackrel{\text{def}}{=} tc_1 + (1-t)c_2 \in G + L$. For $i = 1, 2$ we can decompose c_i as the sum $l_i + z_i$ where $l_i \in L$ and $z_i \in C \cap L^{\perp}$. As $c \in G + L$ and $G \subset L^{\perp}$, we must have $tz_1 + (1-t)z_2 \in G$. Thus $z_1, z_2 \in G$, G being a face of $C \cap L^{\perp}$, and $c_1, c_2 \in G + L$.

2. We use Proposition 8.2 and part 1 above. Recall that c_0 belongs to $C \cap L^{\perp}$, and let $C_0 \stackrel{\text{def}}{=} C - c_0$. Note that part 1 applies when C is replaced by C_0, and we have

$$\begin{aligned} C_0 \cap L^{\perp} &\subset \operatorname{par} C \cap L^{\perp} \\ &\subset M^{\perp} \cap L^{\perp} = (L+M)^{\perp}. \end{aligned}$$

Now consider the following mapping from $\mathcal{N}_C$ to the normal manifold of $C_0 \cap L^{\perp}$ in $(L+M)^{\perp}$: given $\sigma \in \mathcal{N}_C$, take the unique face F of C such that $\sigma = F + N_F$, map it to the face $G \stackrel{\text{def}}{=} F \cap L^{\perp}$ of $C \cap L^{\perp}$, then map G to the face $G_0 \stackrel{\text{def}}{=} G - c_0$ of $(F - c_0) \cap L^{\perp}$, and finally map G_0 to the sum of G_0 and the normal cone, in $(L+M)^{\perp}$, of $C_0 \cap L^{\perp}$ on G_0. This mapping is well defined and, as the composition of bijective mappings, is itself bijective. Therefore, any surjective mapping from (the finite set) $\mathcal{N}_C$ to the normal manifold of $C_0 \cap L^{\perp}$ in $(L+M)^{\perp}$ must be bijective.

Let F be a face of C, so $F_0 \stackrel{\text{def}}{=} F - c_0$ is a face of C_0 and $F_0 \cap L^{\perp}$ is a face of $C_0 \cap L^{\perp}$. Since $F = (F \cap L^{\perp}) + L$, [16, Thm. 6.7] yields

$$\operatorname{ri} F = \operatorname{ri}(F \cap L^{\perp}) + \operatorname{ri} L = \operatorname{ri}(F \cap L^{\perp}) + L,$$

and, in particular, $\operatorname{ri} F$ contains $\operatorname{ri}(F \cap L^{\perp})$. Choose $g_0 \in \operatorname{ri}(F_0 \cap L^{\perp})$, so $g_0 + c_0 \in \operatorname{ri}(F \cap L^{\perp})$ and, by Proposition 6.1,

$$N_F = N_C(g_0 + c_0) = N_{C \cap L^{\perp}}(g_0 + c_0) \cap L^{\perp} = N_{C_0 \cap L^{\perp}}(g_0) \cap L^{\perp},$$

where the second equality follows from the identity $C = (C \cap L^{\perp}) + L$. Also $F_0 = (F_0 \cap L^{\perp}) + L$, so we have

$$\begin{aligned}
&[F + N_F - c_0] \cap (L + M)^{\perp} \\
&= [F_0 \cap L^{\perp} + L + N_{C_0 \cap L^{\perp}}(g_0) \cap L^{\perp}] \cap (L^{\perp} \cap M^{\perp}) \\
&= [F_0 \cap L^{\perp} + N_{C_0 \cap L^{\perp}}(g_0) \cap L^{\perp}] \cap M^{\perp}, \\
&= F_0 \cap L^{\perp} + [N_{C_0 \cap L^{\perp}}(g_0) \cap L^{\perp} \cap M^{\perp}], \text{ since } F_0 \cap L^{\perp} \subset M^{\perp}, \\
&= F_0 \cap L^{\perp} + [N_{C_0 \cap L^{\perp}}(g_0) \cap (L + M)^{\perp}].
\end{aligned}$$

We already know that $F_0 \cap L^{\perp}$ is a face of $C_0 \cap L^{\perp}$. Moreover, by choice of the g_0, $N_{C_0 \cap L^{\perp}}(g_0) \cap (L + M)^{\perp}$ is the normal cone, in $(L + M)^{\perp}$, of $C_0 \cap L^{\perp}$ on $F_0 \cap L^{\perp}$.

We have shown that $\sigma \mapsto (\sigma - c_0) \cap (L + M)^{\perp}$ defines a mapping from the n-cells σ of $\mathcal{N}_C$ to the m-cells of the normal manifold of $C_0 \cap L^{\perp}$ in $(L + M)^{\perp}$; and that, given part 1, this mapping is surjective. The proof is complete.

□

When $0 \in C$, i.e. $\operatorname{lin} C$ is the largest subspace contained in C, part 2 of the last proposition suggests the idea of a quotient of a normal manifold.

Definition 10 *Let L, M be linear subspaces of C, $(\mathrm{par}\, C)^\perp$ respectively. The quotient of $\mathcal{N}_C$ over $L+M$, denoted $\mathcal{N}_C/(L+M)$, is the normal manifold of $C \cap L^\perp$ in $(L+M)^\perp$.*

This can easily be extended to the situation in which L is a subspace of $\mathrm{lin}\, C$ but not of C. We will not need such generality for our investigation of branching numbers, however.

Next we examine the local structure of $\mathcal{N}_C$.

Proposition 11 *Let $\hat{x} \in \mathbb{R}^n$, and K be the critical cone to C at $\hat{x}$.*

1. *The mapping $F \mapsto T_F(\pi_C(\hat{x}))$ is a bijection from the faces F of C such that $\hat{x} \in F + N_F$, to the nonempty faces of K.*

2. *The mapping $\sigma \mapsto T_\sigma(\hat{x})$ is bijection from the n-cells of $\mathcal{N}_C$ that contain $\hat{x}$ to the n-cells of $\mathcal{N}_K$.*

Proof

1. Let $\hat{c} \stackrel{\text{def}}{=} \pi_C(\hat{x})$ and $\hat{\nu} \stackrel{\text{def}}{=} \hat{x} - \pi_C(\hat{x})$. Let $\mathcal{F}(\hat{x})$ be the family of faces F of C such that $\hat{x} \in F + N_F$. Observe that for each face F of C we have $F = (\mathrm{aff}\, F) \cap C$: $F \subset (\mathrm{aff}\, F) \cap C$, and for any $x \in (\mathrm{aff}\, F) \cap C$, take $f \in \mathrm{ri}\, F$ and observe for sufficiently small $t > 0$ that $(1-t)f + tx \in F$, hence $x \in F$. Note that each $F \in \mathcal{F}(\hat{x})$ is polyhedral convex because C is polyhedral convex. We will also use the fact that for a polyhedral convex set D and each $d \in D$, $T_D(d) = \mathrm{cone}(D - d)$, where for any nonempty set $S \subset \mathbb{R}^n$, $\mathrm{cone}(S) \stackrel{\text{def}}{=} \{\alpha s \,|\, \alpha > 0,\ s \in S\}$.

Let $F \in \mathcal{F}(\hat{x})$; by Proposition 8.1, $\hat{c} \in F$ and $\hat{\nu} \in N_F$. We will show that $G \stackrel{\text{def}}{=} T_F(\hat{c})$ is a face of K. By Proposition 6.1, $N_F \subset (F - \hat{c})^{\perp}$, hence $F - \hat{c} \subset \hat{\nu}^{\perp}$. In particular,

$$G = \text{cone}(F - \hat{c}) \subset [\text{cone}(C - \hat{c})] \cap \hat{\nu}^{\perp} = K.$$

Next suppose $k_1, k_2 \in K$ and $t \in (0,1)$ are such that $k \stackrel{\text{def}}{=} tk_1 + (1 - t)k_2 \in G$. Choose $\alpha > 0$ small enough such that

$$\begin{aligned} \hat{c} + \alpha k_1, \hat{c} + \alpha k_2 &\in C, \\ \hat{c} + \alpha k &\in F. \end{aligned}$$

Since $\hat{c} + \alpha k$ equals $t(\hat{c} + \alpha k_1) + (1 - t)(\hat{c} + \alpha k_2)$ and F is a face of C, we see that $\alpha k_1, \alpha k_2 \in (F - \hat{c})$. Therefore $k_1, k_2 \in G$, and G is a face of K. We have shown that the mapping $F \mapsto T_F(\hat{c})$ from $\mathcal{F}(\hat{x})$ to the nonempty faces of K is well defined.

For injectivity, suppose $T_F(\hat{c}) = T_{F'}(\hat{c})$ where $F, F' \in \mathcal{F}(\hat{x})$. So for each $f \in F$ there is $\alpha \in (0,1)$ such that $\alpha(f - \hat{c}) \in F' - \hat{c}$, hence $(1 - \alpha)\hat{c} + \alpha f \in F'$. As F' is a face of C, $f \in F'$, i.e. $F \subset F'$. By symmetry, $F \supset F'$, and $F = F'$.

For surjectivity, let G be a nonempty face of K. Since K is a cone, G is a cone too. Let $F \stackrel{\text{def}}{=} (G + \hat{c}) \cap C$. As $G + \hat{c}$ and C are polyhedral convex sets containing $\hat{c}$, we can decompose the tangent cone of F at $\hat{c}$:

$$T_F(\hat{c}) = [\text{cone}\, G] \cap [\text{cone}(C - \hat{c})] = G \cap T_C(\hat{c}).$$

Therefore $T_F(\hat{c}) = G$, so it is only left to be shown that $F \in \mathcal{F}(\hat{x})$.

First we note that $K = T_{\hat{F}}(\hat{c})$, where $\hat{F} \stackrel{\text{def}}{=} C \cap (\hat{c} + \hat{\nu}^\perp)$: $\hat{c} \in \hat{F}$ and

$$T_{\hat{F}}(\hat{c}) = \text{cone}[(C - \hat{c}) \cap \hat{\nu}^\perp] = [\text{cone}(C - \hat{c})] \cap \hat{\nu}^\perp = K.$$

Second, as G is a face of $T_{\hat{F}}(\hat{c})$, $(G + \hat{c}) \cap \hat{F}$ is a face of $\hat{F}$: If $f_1, f_2 \in \hat{F}$ and $t \in (0,1)$ are such that $f \stackrel{\text{def}}{=} tf_1 + (1-t)f_2 \in (G + \hat{c}) \cap \hat{F}$, then $f_1 - \hat{c}, f_2 - \hat{c} \in T_{\hat{F}}(\hat{c})$, and $t(f_1 - \hat{c}) + (1-t)(f_2 - \hat{c})$ equals $f - \hat{c} \in G$. So $f_1, f_2 \in (G + \hat{c}) \cap \hat{F}$.

Third, $(G + \hat{c}) \cap \hat{F} = F$: since $G \cap (C - \hat{c}) \subset G \subset K \subset \hat{\nu}^\perp$, we get

$$F = (G + \hat{c}) \cap C = (G + \hat{c}) \cap C \cap (\hat{c} + \hat{\nu}^\perp) = (G + \hat{c}) \cap \hat{F}.$$

Fourth, we show $\hat{F}$ is a face of C, for then F, which we have shown is a face of $\hat{F}$, must also be a face of C. Recall that, by construction, $\hat{\nu} \in N_C(\hat{c})$. Let $c_1, c_2 \in C, t \in (0,1)$, $c \stackrel{\text{def}}{=} tc_1 + (1-t)c_2 \in \hat{F}$ and assume $c_1 \notin \hat{F}$. Then $\langle c_1 - \hat{c}, \hat{\nu} \rangle < 0$ and, as $\langle c - \hat{c}, \hat{\nu} \rangle = 0$, $\langle c_2 - \hat{c}, \hat{\nu} \rangle > 0$ which contradicts $\hat{\nu} \in N_C(\hat{c})$. So $\hat{F}$ is a face of C as needed.

It only remains to be seen that $\hat{x} \in F + N_F$. We already know that $\hat{x} = \hat{c} + \hat{\nu}$, where $\hat{c} \in F$ and $F - \hat{c} \subset \hat{F} - \hat{c} \subset \hat{\nu}^\perp$. Thus $\hat{\nu} \in (F - \hat{c})^\perp \cap N_C(\hat{c})$, and Proposition 6.1 assures us that $\hat{\nu} \in N_F$.

2. The proof proceeds in a fashion similar to the proof of Proposition 9.2. We will use the proof of part 1 without reference.

   Denote by $\mathcal{N}_C(\hat{x})$ the family of n-cells of $\mathcal{N}_C$ that contain $\hat{x}$. Let $\sigma \in \mathcal{N}_C(\hat{x})$ and, using Proposition 8.2, F be the unique face of C such that

$\sigma = F + N_F$. Then $\hat{c} \in F$ and $G \stackrel{\text{def}}{=} T_F(\hat{c})$ is a face of K. The mapping of σ to the sum of G and the normal cone to K on G therefore defines a function from $\mathcal{N}_C(\hat{x})$ to $\mathcal{N}_K$. This function is the composition of invertible mappings, hence is invertible. It follows that any surjective mapping from $\mathcal{N}_C(\hat{x})$ to $\mathcal{N}_K$ is bijective.

Let σ and F be as above, so $\hat{c} \in F$ and $\hat{\nu} \in N_F$. Consider

$$\begin{aligned} T_\sigma(\hat{x}) &= \text{cone}[(F - \hat{c}) + (N_F - \hat{\nu})] \\ &= \text{cone}(F - \hat{c}) + \text{cone}(N_F - \hat{\nu}) \end{aligned}$$

since $F - \hat{c}$, $N_F - \hat{\nu}$ are polyhedral convex and contain zero. Also, as N_F is a cone containing $\hat{\nu}$, $\text{cone}(N_F - \hat{\nu})$ equals $N_F + \text{span}(\hat{\nu})$, where $\text{span}(\hat{\nu}) \stackrel{\text{def}}{=} \{\alpha\hat{\nu} \mid \alpha \in \mathbb{R}\}$. So

$$T_\sigma(\hat{x}) = T_F(\hat{c}) + N_F + \text{span}(\hat{\nu}).$$

Recall that $T_F(\hat{c})$ is face of K, and, by definition, $K = T_C(\hat{c}) \cap \hat{\nu}^\perp$. Let N be the normal cone to K on $T_F(\hat{c})$. By Proposition 6.1,

$$\begin{aligned} N &= N_K(0) \cap [T_F(\hat{c}) - 0]^\perp \\ &= [T_C(\hat{c}) \cap \hat{\nu}^\perp]^\circ \cap [\text{cone}(F - \hat{c})]^\perp \\ &= [N_C(\hat{c}) + \text{span}(\hat{\nu})] \cap [F - \hat{c}]^\perp, \end{aligned}$$

since $T_C(\hat{c})$ and $\hat{\nu}^\perp$ are polyhedral convex cones, and $(\hat{\nu}^\perp)^\circ = \text{span}(\hat{\nu})$. Now $\hat{\nu} \in N_F \subset (F - \hat{c})^\perp$, so $\text{span}(\hat{\nu}) \subset (F - \hat{c})^\perp$ and we get

$$N = [N_C(\hat{c}) \cap (F - \hat{c})^\perp] + \text{span}(\hat{\nu}) = N_F + \text{span}(\hat{\nu}),$$

where the second equality follows from Proposition 6.1. We deduce that $\sigma \mapsto T_\sigma(\hat{x})$ defines a mapping from the n-cells σ of $\mathcal{N}_C(\hat{x})$ to the n-cells of $\mathcal{N}_K$; and, given part 1, that this mapping is a surjection. Hence the mapping is bijective.

$\square$

We have arrived at our main result.

Proposition 12 *Suppose G is a nonempty face of $F + N_F$ of codimension m, where F is a face of C. Let $\hat{x} \in \operatorname{ri} G$ and K be the critical cone to C at $\hat{x}$. Define $\hat{c} \stackrel{\text{def}}{=} \pi_C(\hat{x})$, $\hat{\nu} \stackrel{\text{def}}{=} \hat{x} - \hat{c}$,*

$$L \stackrel{\text{def}}{=} \operatorname{par}[(G - \hat{\nu}) \cap F],\ M \stackrel{\text{def}}{=} \operatorname{par}[(G - \hat{c}) \cap N_F].$$

Then $L \subset K$, $M \subset K^\perp$, $L+M = \operatorname{par} G$, and there is a one-to-one correspondence between the n-cells of $\mathcal{N}_C$ containing $\hat{x}$ and the m-cells of $\mathcal{N}_K/(L+M)$.

Proof If $L \subset K$, $M \subset K^\perp$, and $L + M = \operatorname{par} G$, then the dimension of $(L+M)^\perp$ is m, and the bijective correspondence between the n-cells of $\mathcal{N}_C$ containing $\hat{x}$ and the m-cells of $\mathcal{N}_K/(L+M)$ follows from Propositions 11.2 and 9.2.

We will show

$$G = [(G - \hat{\nu}) \cap F] + [(G - \hat{c}) \cap N_F]. \tag{1}$$

Let $c + \nu \in G$ where $c \in F$ and $\nu \in N_F$, then $c + \hat{\nu}, \hat{c} + \nu \in F + N_F$ and, by convexity of G,

$$(1/2)(c + \hat{\nu}) + (1/2)(\hat{c} + \nu) \ = \ (1/2)(c + \nu) + (1/2)(\hat{c} + \hat{\nu}) \in G.$$

So $c+\hat{\nu}$ and $\hat{c}+\nu$ belong to G, and it follows that $c+\nu$ lies in $[(G-\hat{\nu})\cap F]+[(G-\hat{c})\cap N_F]$. Conversely if $c\in(G-\hat{\nu})\cap F$ and $\nu\in(G-\hat{c})\cap N_F$, then both $c+\hat{\nu}$ and $\hat{c}+\nu$ belong to G, and an argument similar to the one above yields $c+\nu\in G$.

Recall (Proposition 6.1) that $N_F = N_C(\hat{c})\cap(F-\hat{c})^{\perp}$, so $\operatorname{par} F$ and $\operatorname{par} N_F$ are orthogonal. Hence the decomposition of $\hat{x}$ as the sum $c+\nu$, where $c\in(G-\hat{\nu})\cap F$ and $\nu\in(G-\hat{c})\cap N_F$, implies c and v are unique, thus $c=\hat{c}$ and $\nu=\hat{\nu}$. Using (1) and [17, Thm. 6.7], we can express $\operatorname{ri} G$ is the sum of $\operatorname{ri}[(G-\hat{\nu})\cap F]$ and $\operatorname{ri}[(G-\hat{c})\cap N_F]$. As $\hat{x}\in\operatorname{ri} G$ it now follows that $\hat{c}$ and $\hat{\nu}$ are relative interior points of $(G-\hat{\nu})\cap F$ and $(G-\hat{c})\cap N_F$, respectively.

Therefore we have, in the notation of the previous proof,

$$L=\operatorname{cone}([(G-\hat{\nu})\cap F]-\hat{c}), \tag{2}$$

so $L\subset\operatorname{cone}(C-\hat{c})=T_C(\hat{c})$. Also $\hat{\nu}$ belongs to N_F (Proposition 8.1) hence to $(F-\hat{c})^{\perp}$, therefore $\hat{\nu}^{\perp}\supset\operatorname{cone}(F-\hat{c})\supset L$. This shows that the critical cone $K\stackrel{\text{def}}{=}T_C(\hat{c})\cap\hat{\nu}^{\perp}$ contains L. Similarly,

$$M=\operatorname{cone}([(G-\hat{c})\cap N_F]-\hat{\nu}), \tag{3}$$

so $M\subset\operatorname{cone}(N_F-\hat{\nu})$. For $\nu\in N_F\subset N_C(\hat{c})$ and $d\in(C-\hat{c})\cap\hat{\nu}^{\perp}$, $\langle\nu-\hat{\nu},d\rangle=\langle\nu,d\rangle\leq 0$, hence

$$\operatorname{cone}(N_F-\hat{\nu})\subset[\operatorname{cone}(C-\hat{c})\cap\hat{\nu}^{\perp}]^{\circ}=K^{\circ}.$$

Thus the linear space M lies in K°, which shows it is a subspace of $K^{\perp}$.

It only remains to be seen that $\operatorname{par} G = L + M$. This follows from (1), (2), and (3), given that for any nonempty sets A and B in $\mathbb{R}^n$, $\operatorname{par}(A+B) = \operatorname{par} A + \operatorname{par} B$. The easy proof of the last fact is left to the reader. $\square$

Corollary 13 *$\mathcal{N}_C$ has branching number less than or equal to 4. In particular, A_C is pl with respect to a finite pl manifold of $\mathbb{R}^n$ that has branching number less than or equal to 4.*

Proof If $n = 1$ then, using the convention that the empty set has dimension -1, the only face of C of codimension 2 is the empty set. Trivially, $\mathcal{N}_C$ has cardinality of 3 or less, hence its branching number is less than or equal to 3. (Alternatively, one could ignore the empty set and argue that the idea of branching numbers is vacuous.)

Let $n \geq 2$, and suppose G is a (nonempty) face of $F + N_F$ of codimension 2, where F is a (nonempty) face of C. Let $\hat{x}$, K, L and M be as in Proposition 12. The proposition says that $L + M$ has codimension 2, and that the number of n-cells of $\mathcal{N}_C$ containing G (which contains $\hat{x}$) is not greater than the number of 2-cells of $\mathcal{N}_K/(L + M)$.

Consider $\mathcal{N}_K/(L+M)$ which is the normal manifold of a nonempty polyhedral convex cone in a 2-dimensional space. The maximum number of 2-cells of $\mathcal{N}_K/(L+M)$ is four: it consists of one 2-cell if $K \cap L^\perp$ is $\{0\}$ or $(L+M)^\perp$, two if $K \cap L^\perp$ is a ray or a halfspace, and four otherwise.

$\square$

Therefore Robinson's homeomorphism result, Theorem 5, is a corollary of Kuhn and Löwen's homeomorphism result, Theorem 4.

A minor point is that for a finite pl manifold $\mathcal{M}$ of $\mathbb{R}$ and a mapping $P : \mathbb{R} \rightarrow \mathbb{R}$ that is pl with respect to $\mathcal{M}$, it is almost trivial that P is a homeomorphism if and only if it is both proper and coherently oriented. In other words the branching number of $\mathcal{M}$, which equals the cardinality of $\mathcal{M}$ if the empty set is assigned codimension of 2, has no bearing on whether or not P is a homeomorphism. However, as Kuhn and Löwen [8, p. 122] note, in all higher dimensions the upper bound of 4 on the branching number is tight: there are coherently oriented maps that are pl with respect to finite pl manifolds of branching number 5, but that are not homeomorphic.

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