A HOPF ALGEBRA FROM PREPROJECTIVE MODULES

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Let Q be a finite type quiver i.e. ADE Dynkin quiver. Denote by Λ its preprojective algebra. It is known that there are finitely many indecomposable Λ-modules if and only if Q is of type A_1, A_2, A_3, A_4. Extending Lusztig's construction of $U\mathfrak{n}$, we study an algebra generated by these indecomposable submodules. It turns out that it forms the universal enveloping algebra of some nilpotent Lie algebra inside the function algebra on Lusztig's nilpotent scheme. The defining relations of the corresponding nilpotent Lie algebra for type A_1, A_2, A_3, A_4 are given here.

BIOGRAPHICAL SKETCH

Pak Hin Li was born in Hong Kong. He completed his Bachelor of Science at the Chinese University of Hong Kong. Then he came to Cornell University to pursue a Ph.D. in Mathematics in 2014. In 2020, he completed his thesis under the supervision of his advisor, Allen Knutson.

This thesis is dedicated to my parents.

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CHAPTER 1

INTRODUCTION

Preprojective algebra of a quiver Q is constructed by considering the path algebra of the double of Q and modding out the Gelfand and Ponomarev relations. The notion of preprojective algebra was first introduced by Gelfand and Ponomarev [8]. See also the paper by Ringel [17] for an introduction.

Later in [10], Lusztig constructed nilpotent variety which parametrizes modules of preprojective algebra and endowed an algebra structure on its constructible functions. Denote by $\widetilde{\mathcal{M}}$ the function algebra of nilpotent variety. The convolution multiplication of $\widetilde{\mathcal{M}}$ is similar to Ringel's multiplication for Hall algebras [16] and it was shown by Lusztig that certain subalgebra is isomorphic to the universal enveloping algebra $U\mathfrak{n}$ where $\mathfrak{n}$ is the positive part of Lie algebra associated with Q. This identification allows Lusztig to define a new basis of $U\mathfrak{n}$ which is called semicanonical basis. Semicanonical basis is compatible with various filtrations of $U\mathfrak{n}$ and it has a surprising property: when you apply the semicanonical basis to a lowest weight vector of an irreducible representation, the remaining nonzero vectors form a basis for that representation. We can see that nilpotent variety provides a bridge connecting semicanonical basis and preprojective algebra. See [11] and [12] for the construction and properties of semicanonical basis. See also [9] and [15] for a geometric realization of this surprising property by Nakajima. In Lusztig's construction, a certain subalgebra inside $\widetilde{\mathcal{M}}$ was shown to be a universal enveloping algebra but little is known about the structure of the entire $\widetilde{\mathcal{M}}$.

For a connected quiver Q, the preprojective algebra Λ_Q is finite representation type if and only if Q is of type A_n where $n \leq 4$. See [7] for further details.

$\widetilde{\mathcal{M}}$ is I-graded where I is the vertex set of Q. Then by the validity of the second Brauer-Thrall conjecture, it is possible that when Q is not of type A_1, A_2, A_3, A_4, $\widetilde{\mathcal{M}}_v$ is infinite dimensional for infinitely many $v \in \mathbb{N}^I$.

Hence we mainly focused on the case $Q = A_1, A_2, A_3, A_4$. In Chapter 3, we showed that $\widetilde{\mathcal{M}}$ is isomorphic to the universal enveloping algebra $U\tilde{\mathfrak{n}}$ where $\tilde{\mathfrak{n}}$ is the nilpotent Lie algebra generated freely by the indecomposable modules. In Chapter 4, the Lie bracket table, lower, upper central series and Lie algebra cohomology of $\tilde{\mathfrak{n}}$ are also given.

CHAPTER 2

BACKGROUND

In this thesis, we will mainly work on the function algebra of Lusztig's nilpotent scheme. We will list the definition and basic properties of preprojective algebra, nilpotent scheme and their relations to representation theory. We will follow the notation in [5] closely.

2.1 Quiver

A quiver is a simple mathematical object, and briefly speaking, is a directed graph. Given a quiver, we can talk about representations of it and it has interesting relations to representation theory. We will follow the notation in [3].

Definition 2.1.1. *A quiver is a finite directed graph and it can be described by* (I, Q_1, s, t) *where* I, Q_1 *are finite sets (*I *and* Q_1 *are the set of "vertices" and "arrows" respectively).* s, t *are maps from* Q_1 *to* I *and map an arrow to its source and target respectively.*

We can also define representation of a quiver.

Definition 2.1.2. *A representation* M *of a quiver* $Q = (I, Q_1, s, t)$ *is a collection of vector spaces* $(V_i)_{i\in I}$ *and a collection of linear maps* $\{f_a : V_{s(a)} \to V_{t(a)} : a \in Q_1\}$.

Definition 2.1.3. *Given a representation* M *of quiver* Q*, we define the dimension vector* $\dim(M)$ *of* M *as* $(\dim(V_i))_{i\in I} \in \mathbb{Z}^I$.

Given two representations M, N of Q, we can talk about morphisms between them.

Definition 2.1.4. *A morphism* $f = (f_i : M_i \to N_i)$ *between representations* $M = (M_i, u_a)$ *and* $N = (N_i, v_a)$ *is defined as a collection of linear maps which satisfies the following commuting diagram:*

$$\begin{array}{ccc} M_{s(a)} & \xrightarrow{u_a} & M_{t(a)} \\ {\scriptstyle f_{s(a)}}\downarrow & & \downarrow{\scriptstyle f_{t(a)}} \\ N_{s(a)} & \xrightarrow{v_a} & N_{t(a)} \end{array}$$

for all $a \in Q_1$.

Plainly, the composition of morphism is associative and there is an identity element. Hence, we can consider the category of representations of Q, denote it by $Rep(Q)$, and it is in fact an abelian category.

Definition 2.1.5. *Given a I-graded vector space* $V = (V_i)_{i \in I}$*, we define* $Rep(Q, V)$ *to be representations of* Q *with vector space* V*. More specifically,* $Rep(Q, V) \cong \prod_{\alpha \in Q_1} Hom(V_{s(\alpha)}, V_{t(\alpha)})$.

There is an interesting theorem by Pierre Gabriel which connects finite type quivers with Dynkin quivers .

Theorem 2.1.6. *[4](Gabriel) A quiver Q has only finitely many isomorphism classes of representations of any prescribed dimension vector if and only if each connected component of its underlying undirected graph is a simply-laced Dynkin diagram.*

Simply-laced Dynkin diagram consists of Dynkin diagram of type A, D, E and will be the only quivers that we consider when we define nilpotent scheme.

2.2 Path algebra

We can also define the category of representation from the path algebra perspective and it will be useful to define preprojective algebra in later section.

Definition 2.2.1. *The path algebra kQ of a quiver Q is the k-algebra generated by $e_i, i \in I$ and $\alpha \in Q_1$ with relations $e_i^2 = e_i$, $e_{t(\alpha)}\alpha = \alpha e_{s(\alpha)} = \alpha$ and $e_i e_j = 0$ if $i \neq j$.*

Given a kQ-module M, we can consider vector spaces $(e_i M)_{i \in I}$ and maps from $e_{s(\alpha)}M \to e_{t(\alpha)}M$ induced by $e_{t(\alpha)}\alpha = \alpha e_{s(\alpha)}$ and we can see that we have a representation of Q. In fact, we have the following:

Theorem 2.2.2. *[3] Given a quiver Q, the representation category $Rep(Q)$ is equivalent to the the category of left kQ-modules.*

Definition 2.2.3. *A relation of a quiver Q is a subspace of kQ spanned by linear combinations of paths having a common source and a common target, and of length at least 2.*

Let J be a two-sided ideal generated by relations in kQ. We can define quotient algebra kQ/J and $Rep(Q, J)$.

Definition 2.2.4. *Given a I-graded vector space V, we can consider representations which are inside $Rep(Q, V)$ and also satisfy relations J. Denote these representations by $Rep(Q, J, V)$ and we can see that it is an affine algebraic subvariety of $Rep(Q, V)$.*

2.3 Preprojective algebra

In this thesis, we only focus on simply laced A, D, E type quivers. Given a quiver $Q = (I, Q_1, s, t)$, we can consider a double of Q denoted by $\overline{Q}$ by replacing each arrow $\alpha \in Q_1$ by a pair (α, α^*) where α^* is the reverse of original arrow α. Notice that $\overline{Q}$ only depends on the underlying undirected graph of the diagram.

Definition 2.3.1. *Given a simply laced quiver $Q = (I, Q_1, s, t)$, we consider the double $\overline{Q} = (I, \overline{Q_1}, \bar{s}, \bar{t})$ and Gelfand-Ponomarev relations $R := \{\sum_{a \in Q_1 : s(a)=i} a^* a - \sum_{a \in Q_1 : t(a)=i} a a^* : i \in I\}$. Denote by J the two-sided ideal generated by R. Then the preprojective algebra Λ is defined as the quotient algebra $k\overline{Q}/J$.*

2.4 Lusztig's nilpotent scheme

Definition 2.4.1. *Given a finite dimensional I-graded vector space V with graded dimension $|V|$ (associate an integer at each vertex of the Dynkin quiver Q), we can construct Lusztig's nilpotent scheme $\Lambda_V := Rep(Q, J, V)$ where J is the two-sided ideal generated by Gelfand-Ponomarev relations.*

Remark 2.4.2. *The general definition of nilpotent variety for arbitrary quivers is sightly different from the one presented here and it is given in [10].*

Example 2.4.3. *Let Q be a type A_n quiver and $V = (V_i)_{i=1,2,..,n}$ be a vector space . Then $\Lambda_V \cong \{((f_i)_{i=1,2.,,n-1}, (q_i)_{i=1,2,...,n-1}) \in \prod_{i=1}^{n-1} Hom(V_i, V_{i+1}) \times \prod_{i=1}^{n-1} Hom(V_{i+1}, V_i) : q_i f_i = f_{i-1} q_{i-1}, i = 1, 2, ..., n\}$, where the maps are zero if the index is not in $\{1, 2..., n-1\}$.*

$$\left(V_1 \underset{q_1}{\overset{f_1}{\rightleftarrows}} V_2 \underset{q_2}{\overset{f_2}{\rightleftarrows}} \cdots \underset{q_{n-1}}{\overset{f_{n-1}}{\rightleftarrows}} V_n \right).$$

There are some interesting properties of nilpotent scheme proven by Lusztig:

Theorem 2.4.4. *[10, Theorem 12.3, 12.9],[12, Lemma 2.6] : Let V be a I-graded vector space. Denote $Rep(\overline{Q}, V)$ by E_V and we have $\Lambda_V \subset E_V$. Then Λ_V has the following properties:*

1. Λ_V is a closed subvariety of E_V of pure dimension $dim(E_V)/2$.

2. Λ_V is a Lagrangian subvariety of E_V if we consider $E_V \cong T^*(Rep(Q,V))$ which naturally has a symplectic structure.

3. The number of irreducible components of Λ_V is equal to $\dim((U\mathfrak{n}_+)_{|V|})$ where $|V| = \sum_{i\in I} dim(V_i)\alpha_i$, α_i is the simple root corresponding to i and $\mathfrak{n}_+$ corresponds to the positive part of Lie algebra associated with Dynkin graph Q.

2.5 Function algebra on Lusztig's nilpotent scheme

Given a I-graded vector space V, there is a group $G_V := \prod_{i\in I} GL(V_i)$ acting on Λ_V by change of basis as follows. Let $x = (x_\alpha) \in \Lambda_V$ and $g = (g_i)_{i\in I} \in G_V$. Then $g \cdot x = (y_\alpha)_{\alpha\in\overline{Q_1}}$ is given by: $y_\alpha = g_{t(\alpha)} x_\alpha g_{s(\alpha)}^{-1}$. This action is well-defined since it preserves the preprojective relations.

Definition 2.5.1. *Given an algebraic variety X over $\mathbb{C}$, a constructible set is defined to be a finite union of locally closed sets under the Zariski topology. A function $f : X \to \mathbb{C}$ is defined to be constructible if $f = \sum_A c_A 1_A$ is a linear combination of characteristic functions of constructible sets.*

Define $M(\Lambda_V)$ to be the space of all constructible functions on Λ_V and denote by $M(\Lambda_V)^{G_V}$ the G_V-invariant functions. Let $\widetilde{\mathcal{M}} = \oplus_{|V|\in\mathbb{N}^I} M(\Lambda_V)^{G_V}$, where $\mathbb{N}$ denotes the set of nonnegative integers. We will define an algebra structure $(\widetilde{\mathcal{M}}, *)$ on $\widetilde{\mathcal{M}}$. Since constructible functions are linear combinations of characteristic functions, we only need to define the product on G_V-invariant characteristic functions and extend it to all constructible functions.

Definition 2.5.2. *[12] If V_1, V_2, V_3 are I-graded vector spaces and $|V_1| + |V_2| = |V_3|$, we can define $* : M(\Lambda_{V_1})^{G_{V_1}} \times M(\Lambda_{V_2})^{G_{V_2}} \to M(\Lambda_{V_3})^{G_{V_3}}$ by: $1_{O_1} * 1_{O_2}(x) := \chi(\Phi_{O_1,O_2,x})$*

where $\Phi_{O_1,O_2,x} = \{U \in Gr(|V_2|, V_3) : U$ *is a* Λ*-submodule of* x *of type* O_2*,* x/U *has type* $O_1\}$ *and* χ *is compactly supported Euler characteristic.* $Gr(|V_2|, V_3)$ *is defined as the Grassmannian of dimension* $|V_2|$ *inside* V_3*.*

Remark 2.5.3. *Let* A, B *be two disjoint constructible sets in an algebraic variety* X *over* $\mathbb{C}$*. We can have* $\chi(A \cup B) = \chi(A) + \chi(B)$ *which will be very handy when we do calculations on the function algebra of the nilpotent scheme. This additivity property only works on constructible sets.*

Example 2.5.4. *Consider* A_2 *quiver and* I*-graded vector space* V_1, V_2, V_3 *with* $|V_1| = (1,0)$ *,* $|V_2| = (0,1)$ *and* $|V_3| = (1,1)$*. Let* $f_1 = 1_{V_1}, f_2 = 1_{V_2}$*, then* $f_i \in M(\Lambda_{V_i})^{G_{V_i}}$ *and* $x \in \Lambda_{V_3}$*. We want to find* $f_1 * f_2$ *and* $f_2 * f_1$*. There are in total 3* G_{V_3}*-orbits of* Λ_{V_3} *namely:* $x_1 = [1 \quad 1], x_2 = [1 \rightarrow 1], x_3 = [1 \leftarrow 1]$ *where an arrow indicates a rank* 1 *map.* $f_1 * f_2(x_1) = \chi(Gr((0,1),(1,1))) = 1$*,* $f_1 * f_2(x_2) = \{U \in Gr(Gr((0,1),(1,1)) : x_2(U) \subset U, V_3/U \cong V_1\} = 1$*,* $f_1 * f_2(x_3) = \{U \in Gr(Gr((0,1),(1,1)) : x_3(U) \subset U, V_3/U \cong V_1\} = 0$*. Hence,* $f_1 * f_2 = 1_{[1 \quad 1]} + 1_{[1 \rightarrow 1]}$*. Similarly,* $f_2 * f_1 = 1_{[1 \quad 1]} + 1_{[1 \leftarrow 1]}$*.*

2.6 The coproduct structure on the function algebra

In [5], given V', V'', V such that $|V| = |V'| + |V''|$, there is defined a map $Res^V_{V',V''} : M(\Lambda_V)^{G_V} \rightarrow M(\Lambda_{V'} \times \Lambda_{V''})^{G_{V'} \times G_{V''}}$ as follows:

$$Res^V_{V',V''}(f)(x', x'') = f(x' \oplus x''),$$

where $(x', x'') \in \Lambda_{V'} \times \Lambda_{V''}$. It is proven in [5] that the map $Res^V_{V',V''}$ descends to $M(\Lambda_{V'})^{G_{V'}} \otimes M(\Lambda_{V''})^{G_{V''}}$ when restricted to the algebra generated by $\{e_i = 1_{Z[i]} : i \in I\}$ (where $Z[i]$ is the Λ-module the dimension vector of which is Kronecker delta $(\delta_{ij})_{j \in I}$) and it agrees with the coproduct structure of $U\mathfrak{n}_+$ where $\mathfrak{n}_+$ is the positive nilpotent Lie algebra of $\mathfrak{g}$ (the Lie algebra constructed from Dynkin quiver Q).

Remark 2.6.1. *It is not clear that whether Res extends to a coproduct to the entire $\widetilde{\mathcal{M}}$. However, for $Q = A_1, A_2, A_3, A_4$, it will be proven that it does extend.*

Theorem 2.6.2. *[10] Given a Dynkin A, D, E type quiver Q, let $\widetilde{\mathcal{M}} = \oplus_{|V|\in\mathbb{N}^I} M(\Lambda_V)^{G_V}$. Let $\mathfrak{n}_+$ be the positive nilpotent Lie algebra corresponding to Q, then define $\phi : U\mathfrak{n}_+ \to \mathcal{M}$ by mapping $e_i \mapsto 1_{Z[i]}$, which is I-graded. ϕ is an embedding of algebras from the universal enveloping algebra $U\mathfrak{n}_+$ to $(\widetilde{\mathcal{M}}, *)$. Define $\mathcal{D}$ to be the subalgebra of $\widetilde{\mathcal{M}}$ generated by $1_{Z[i]}, i \in I$. Then Res extends to a coproduct on $\mathcal{D}$ and ϕ becomes an ismorphism between Hopf algerbras $U\mathfrak{n}_+$ and $\mathcal{D}$.*

2.7 Semicanonical basis

Lusztig used the isomorphism between $\mathcal{D}$ and $U\mathfrak{n}_+$ to define semicanonical basis associated to Q. He proved in [12, Theorem 2.7] that given any irreducible component Z of Λ_V, there is a unique function f_Z in $\mathcal{D}$ such that f_Z is 1 on an open dense set of Z and is zero on a dense open set of any other components of Λ_V. Then he defined $B_v := \phi^{-1}(\{f_Z; Z \in Irr(\Lambda_V)\})$ and $B := \cup_{v\in\mathbb{N}^I} B_v$ to be the semicanonical basis.

Theorem 2.7.1. *[12] Given Q, let $\mathfrak{g}$ be the Lie algebra asosciated with Q and $\mathfrak{n}_+$ be the positive part. Let λ be a highest weight of $\mathfrak{g}$ and V^λ be the highest weight representation. It has natural $U\mathfrak{n}_+$-mod structure and a canonical map $p : U\mathfrak{n}_+ \to V^\lambda$. Then $\{p(b) : p(b) \neq 0, b \in B\}$ forms a basis of V^λ.*

Remark 2.7.2. *This theorem shows that semicanonical basis is compatible with various filtrations of $U\mathfrak{n}_+$ and it is surprising that the nonzero projections of semicanonical basis still form a basis of any irreducible representation.*

CHAPTER 3

LIE ALGEBRA REALIZATION

Theorem 2.6.2 says that $\mathcal{D} \subset \widetilde{\mathcal{M}}$ is a Hopf algebra is isomorphic to $U\mathfrak{n}_+$. In this chapter, we will see that for $Q = A_1, A_2, A_3, A_4$, $\widetilde{\mathcal{M}}$ is also a Hopf algebra containing $\mathcal{D}$ as a subalgebra with compatible Hopf algebra structure and it is also the universal enveloping algebra of some nilpotent Lie algebra.

3.1 Algebra of indecomposable modules

Theorem 3.1.1. *[5, Section 20], [6, Section 8.1-8.3] : Let Q be a simply laced Dynkin quiver. Then the preprojective algebra Λ_Q has finitely many isomorphism types of indecomposable modules if and only if $Q = A_1, A_2, A_3, A_4$. There are $1, 4, 12, 40$ different isomorphism types of indecomposable modules for $Q = A_1, A_2, A_3, A_4$ respectively.*

Definition 3.1.2. *For $Q = A_1, A_2, A_3, A_4$, define $\mathcal{A}$ to be the subalgebra of $(\widetilde{\mathcal{M}}, *)$ generated by $\{1_K : K$ is an orbit of an indecomposable module$\}$.*

Remark 3.1.3. *We will later prove that $\mathcal{A} = \widetilde{\mathcal{M}}$.*

Lemma 3.1.4. *Let $f = 1_{K_1} * 1_{K_2} * ... * 1_{K_n}$ where K_i are G_{V_i}-invariant subsets of Λ_{V_i}. Then $f(x) = \chi(\{(V^0 \supseteq V^1 \supseteq ... \supseteq V^n = 0) : x(V^i) \subset V^i, [V^{i-1}/V^i] \in K_i, \text{for } i = 1, 2, ..., n\})$.*

Proof. This follows directly from the definition of product structure on $\widetilde{\mathcal{M}}$. □

Definition 3.1.5. *Let $U_1, ..., U_r$ be the different isomorphism classes of indecomposable Λ-module. Each function in $M(\Lambda_V)^{G_V}$ can be expressed as a linear combination of indicator functions $1_{a_1 U_1 \oplus ... \oplus a_r U_r}$. Let $Y \subset \Lambda_V$ be the G_V-orbit of $a_1 U_1 \oplus ... \oplus a_r U_r$. We define the indicator functions 1_Y and 1_{U_i} to be $I(a_1, ..., a_r)$ and I_i respectively. We also let $a = (a_1, ..., a_r)$ and denote $I(a_1, ..., a_r)$ by $I(a)$.*

3.2 Coproduct structure on $\mathcal{A}$

In this section, we prove that Res gives us a coproduct structure on $\mathcal{A}$ and the coproduct is multiplicative, so $\mathcal{A}$ is necessarily the universal enveloping algebra of its primitive elements.

Lemma 3.2.1. *For $Q = A_1, A_2, A_3, A_4$, the map $Res^V_{V',V''} : M(\Lambda_V)^{G_V} \to M(\Lambda_{V'} \times \Lambda_{V''})^{G_{V'} \times G_{V''}}$ factors through $M(\Lambda_{V'})^{G_{V'}} \otimes M(\Lambda_{V''})^{G_{V''}}$. Hence the map Res induces a map $\Delta : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}} \otimes \widetilde{\mathcal{M}}$.*

Proof. Each function in $M(\Lambda_V)^{G_V}$ is a linear combination of 1_K where the K are G_V-orbits of Λ-modules. Say that module is of the form $a_1V_1 \oplus a_2V_2 ... \oplus a_rV_r$ where V_i are different isomorphism types of indecomposable module. Then by the Krull-Schmidt theorem, every Λ-module has a unique decomposition into indecomposable modules, so we get:

$$Res^V_{V',V''}(1_K)(x' \oplus x'') = \sum_{\substack{(b_1,b_2,...,b_r): \\ b_1v_1+...+b_rv_r=|V'|, b_i \le a_i}} 1_{b_1V_1 \oplus ... \oplus b_rV_r}(x') 1_{(a_1-b_1)V_1 \oplus ... \oplus (a_r-b_r)V_r}(x'')$$

Hence, Res induces a map $\Delta : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}} \otimes \widetilde{\mathcal{M}}$. □

Lemma 3.2.2. *Let $X \subset Y$ be quasiprojective varieties and T be a torus acting on Y algebraically and preserving X where Y is a partial flag variety. Then the compactly supported Euler characteristic of X is equal to the compactly supported Euler characteristic of the T-fixed points in X.*

Proof. This standard fact follows from the Białynicki-Birula decomposition [2] and the additivity of compactly supported Euler characteristic on locally closed sets in an algebraic variety. □

We need a basic Lemma here.

Lemma 3.2.3. *Let $V = \bigoplus_{i=1}^{N} V_i$ be the isotopic decomposition of a T-representation where $T = (\mathbb{C}^*)^N$. If U is a T-invariant subspace of V, then $U = \oplus_{i=1}^{N}(U \cap V_i)$.*

Lemma 3.2.4. *Let $V = V_1 \oplus V_2 ... \oplus V_t$, where V_i are indecomposable Λ-modules (possibly with repetition). Let $T = (\mathbb{C}^*)^t$ be the torus acting on V by scaling each indecomposable summand. If U is an indecomposable submodule in V such that U is T-invariant, then U is a submodule of one of the V_i.*

Proof. Since U is T-invariant, then by Lemma 3.2.3, we must have $U = \oplus_{i=1}^{r}(U \cap V_i)$. U is a Λ-module so $U \cap V_i$ is also a Λ-module. As a result, U is a direct sum of Λ-modules. However, U is indecomposable so U is a submodule of one of the V_i. □

Definition 3.2.5. *Given a sequence of indecomposable modules $U_{j_1}, U_{j_2}, ..., U_{j_t}$, define $d_J := I_{j_1} * .... * I_{j_t}$ where $J = (j_1, j_2, ..., j_t)$. Given $c \in \{0,1\}^t$, we also define $d_{J,c}$ to be the product of I_{j_i} where $c_i = 1$.*

We are now going to show that $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is an algebra homomorphism.

Lemma 3.2.6. *Let $J = (j_1, ..., j_N)$ be a sequence of indices of indecomposable modules. Following section 2.6, let $V = V' \oplus V''$ and x', x'' be Λ-modules with dimension vectors $|V'|, |V''|$ respectively. We have $Res^V_{V',V''} d_J(x', x'') = \sum_{(c',c'')} d_{J,c'}(x') d_{J,c''}(x'')$, where (c', c'') runs through all pairs of sequences in $\{0,1\}^N \times \{0,1\}^N$ which satisfy $c'_i + c''_i = 1$ for all $i = 1, 2, ..., N$, $\sum_{i=1}^{N} c'_i |U_{j_i}| = |V'|$ and $\sum_{i=1}^{N} c''_i |U_{j_i}| = |V''|$.*

Proof. By the definition of $Res^V_{V',V''}$ and Lemma 3.1.4, we have $Res^V_{V',V''} d_J(x', x'') = d_J(x' \oplus x'') = \chi(\{V = V^0 \supseteq V^1 \supseteq .... \supseteq V^N = 0 : x(V^i) \subset V^i, [V^{i-1}/V^i] \in K_{j_i}\})$. Let $T = \mathbb{C}^* \times \mathbb{C}^*$ be a torus acting on $V = V' \oplus V''$ by scaling summand V' and V''. It induces an action on $\{V = V^0 \supseteq V^1 \supseteq .... \supseteq V^N = 0 : x(V^i) \subset V^i, [V^{i-1}/V^i] \in K_{j_i}\}$.

Let $X = \{V = V^0 \supseteq V^1 \supseteq .... \supseteq V^N = 0 : x(V^i) \subset V^i, [V^{i-1}/V^i] \in K_{j_i}\}$. Then by Lemma 3.2.2, we have $d_J(x' \oplus x'') = \chi(X) = \chi(X^T)$. By Lemma 3.2.4, for each flag $(V^i)_{i=0,1,...,N}$ in X^T, we have $V^i = V^i \cap V' \oplus V^i \cap V''$. Then $V^{i-1}/V^i \cong V^{i-1} \cap V''/V^i \cap V'' \oplus (V^{i-1}/V'')/(V^i/V'')$. As a result, $V^{i-1} \cap V''/V^i \cap V''$ is either trivial or isomorphic to V^{i-1}/V^i since V^{i-1}/V^i is indecomposable. Then following [5, Lemma 6.1], we have $Res^V_{V',V''} d_J(x', x'') = \sum_{(c',c'')} d_{J,c'}(x') d_{J,c''}(x'')$ where (c', c'') runs through all pairs of sequences in $\{0,1\}^N \times \{0,1\}^N$ which satisfy $c'_i + c''_i = 1$, $\sum_{i=1}^N c'_i |U_{j_i}| = |V'|$ and $\sum_{i=1}^N c''_i |U_{j_i}| = |V''|$. □

Corollary 3.2.7. *The map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is multiplicative i.e. $\Delta(fg) = \Delta(f)\Delta(g)$, coassociative and cocommutative . Hence, $(\mathcal{A}, *, u, \Delta, \varepsilon)$ is a bialgebra where $u : \mathbb{C} \to \mathcal{A}$ is defined by $k \mapsto k$ and $\varepsilon : \mathcal{A} \to \mathbb{C}$ is defined by $x \mapsto x_0$ which is projection to $(\mathcal{A})_{(0 \in \mathbb{N}^I)}$.*

Proof. It directly follows from Lemma 3.2.6 that $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is an algebra homomorphism and $(\mathcal{A}, *, u, \Delta, \varepsilon)$ is a bialgebra. □

It is proven by Takeuchi [18] that under certain conditions, a bialgebra has an antipode and hence a Hopf algebra. The theorem was summarized by Carolina Benedetti and Bruce E. Sagan [1].

Lemma 3.2.8. *[1, Theorem 1.1] :*

Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra over $\mathbb{C}$ which satisfies:

1. *H can be written as $\oplus_{n \geq 0} H_n$*
2. *$H_i H_j \subset H_{i+j}$ for all $i, j \geq 0$,*
3. *$\Delta H_n \subset \oplus_{i+j=n} H_i \otimes H_j$ for all $n \geq 0$,*
4. *$\varepsilon H_n = 0$ for all $n \geq 1$, and*
5. *$H_0 \cong \mathbb{C}$.*

Define a projection map $\pi : H \to H$ by $\pi|_{H_n} = Id|_{H_n}$ if $n > 0$ and $= 0$ if $n = 0$. Then H is *a Hopf algebra with antipode* $S = \sum_{k\geq 0}(-1)^k m^{k-1}\pi^{\otimes k}\Delta^{k-1}$ *where we define* $m^{-1} = u$ *and* $\Delta^{-1} = \varepsilon$.

Theorem 3.2.9. *The bialgebra* $(\mathcal{A}, *, u, \Delta, \varepsilon)$ *has an antipode map and hence it is a Hopf algebra.*

Proof. By lemma 3.2.7, $(\mathcal{A}, *, u, \Delta, \varepsilon)$ is a bialgebra. $\mathcal{A}$ is I-graded algebra and hence $\mathbb{N}$-graded by summing up the dimension vector and we have $\mathcal{A}_n = \oplus_{V:\Sigma|V_i|=n}\mathcal{A}_V$ and $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$. Besides, $\mathcal{A}_0 \cong \mathbb{C}$ and by lemma 3.2.1 we have $\Delta\mathcal{A}_n \subset \oplus_{i+j=n}\mathcal{A}_i \otimes \mathcal{A}_j$ and $\varepsilon(\mathcal{A}_n) = 0$ for $n > 0$. As a result, by lemma 3.2.8, $S = \sum_{k\geq 0}(-1)^k m^{k-1}\pi^{\otimes k}\Delta^{k-1}$ is an antipode and $(\mathcal{A}, *, u, \Delta, \varepsilon, S)$ is a Hopf algebra. $\square$

3.3 $\mathcal{A} = \widetilde{\mathcal{M}}$ for $Q = A_1, A_2, A_3, A_4$

Lemma 3.3.1. *Let* r *be the number of indecomposable modules for* Q $(1, 4, 12, 40$ *for* $A_{1,2,3,4})$. *Denote by* U_i *the i-th indecomposable* Λ*-module, ordered such that if* U_i *is a submodule of* U_j*, then* $i \leq j$ *by a refinement of total dimension. Following Definition 3.1.5, we have*

$$I(a_1, a_2, ..., a_r) * I_j = (a_j + 1)I(a + e_j) + \sum_{b\in S} c_b I(b),$$

where $e_j = (\delta_{ij})_{i=1}^r$ *and* $S = \{(b_1, ..., b_r) \in \mathbb{N}^r : b_{i_0} = a_{i_0} + 1$ *for some* $i_0 > j, \sum_i b_i|U_i| = |V''|\}$.

Proof. Let V, V' be the ambient vector spaces such that $I(a_1, a_2, ..., a_r) \in \Lambda_V$, $I_j \in \Lambda_{V'}$. Let $V'' = V \oplus V'$ and $x = b_1U_1 \oplus b_2U_2 \oplus .. \oplus b_rU_r \in \Lambda_{V''}$. Suppose $1_{b_1U_1\oplus b_2U_2\oplus ..\oplus b_rU_r}$ is a nonzero summand with coefficient c_b. Then $I(a_1, a_2, ..., a_r) * I_j(x) = c_b$.

By the definition of the product structure on $\tilde{\mathcal{M}}$, $c_b = \chi(\{U \in Gr(|V'|, V'') : U \cong U_j, x/U \cong a_1U_1 \oplus a_2U_2 \oplus ... \oplus a_rU_r\})$. Let $T = (\mathbb{C}^*)^{\sum_i b_i}$ act on x by scaling each indecomposable module independently. The action is well-defined since it preserves each preprojective relation. Then by Lemma 3.2.2, $c_b = \chi(\{U \in Gr(|V'|, V'') : U \cong U_j, x/U \cong \oplus_{i=1}^r a_iU_i\}^T)$. Let $U \in \{U \in Gr(|V'|, V'') : U \cong U_j, x/U \cong \oplus_{i=1}^r a_iU_i\}^T$. Since U is invariant under T and is isomorphic to an indecomposable module U_j, by Lemma 3.2.4, U is a Λ-submodule of one of the U_i summands of $x = \oplus_i b_iU_i$. Since $c_b \neq 0$, there is an integer i_0 between 1 and r such that U is a submodule of U_{i_0} as a summand of x. From the ordering of U_i, we have $i_0 \geq j$. Since $x/U \cong \oplus_{i=1}^r a_iU_i$, we have $b_1U_1 \oplus b_2U_2 \oplus ... \oplus (b_{i_0} - 1)U_{i_0} \oplus ... \oplus b_rU_r \oplus (U_{i_0}/U) \cong \oplus_{i=1}^r a_iU_i$. Then we have $b_{i_0} - 1 = a_{i_0}$ since U_{i_0}/U does not have summand isomorphic to U_{i_0}. If $i_0 = j$, $c_b = a_j + 1$. As a result, we have:

$$I(a_1, a_2, ..., a_r) * I_j = (a_j + 1)I(a + e_j) + \sum_{b \in S} c_b I(b),$$

where $S = \{(b_1, ..., b_r) \in \mathbb{N}^r : b_{i_0} = a_{i_0} + 1 \text{ for some } i_0 > j, \sum_i b_i|U_i| = |V''|\}$. □

Lemma 3.3.2. *$\mathcal{A}$ is equal to $\widetilde{\mathcal{M}}$.*

Proof. $\widetilde{\mathcal{M}}$ is I-graded. It suffices to show that for each dimension vector $v = (v_i)_{i \in I} \in \mathbb{N}^I$, $\mathcal{A}_v = \widetilde{\mathcal{M}}_v$. In the base case when the sum of v is equal to 1, $\mathcal{A}_v = \widetilde{\mathcal{M}}_v$ because both are 1-dimensional. We proceed to show $\mathcal{A}_v = \widetilde{\mathcal{M}}_v$ by induction. Suppose $\mathcal{A}_v = \widetilde{\mathcal{M}}_v$ for all v such that $\sum_{i \in I} v_i < N$. Now let $v = (v_i)_{i \in I} \in \mathbb{N}^I$ such that $\sum_{i \in I} v_i = N$. Using the notation in Definition 3.1.5, it suffices to show that for any $I(a_1, ..., a_r) \in \widetilde{\mathcal{M}}_v$, we have $I(a_1, ..., a_r) \in \mathcal{A}_v$. We can further order the indecomposable modules such that if U_i is a submodule of U_j, then $i \leq j$ by a refinement of total dimension. Now we fix N. Let m be the largest index such that a_m is not zero (i.e. $(a_1, a_2, ..., a_r) = (a_1, ..., a_m, 0, .., 0)$). We can further use induction on m to show $I(a_1, ..., a_r) \in \mathcal{A}_v$. For the base case, when $m = r$, by Lemma 3.3.1, $I(a_1, ..., a_r - 1) * I_r = a_r I(a_1, ..., a_r)$. By induction assumption $I(a_1, ..., a_r - 1), I_r \in \mathcal{A}$ so $I(a_1, ..., a_r) \in \mathcal{A}_v$. Now assume that $I(a_1, ..., a_r) \in \mathcal{A}_v$ whenever the largest index m such that $a_m > 0$ is greater than M. When $m = M$, we want to show that $I(a_1, ..., a_M, 0, ..., 0) \in \mathcal{A}_v$. We can consider $I(a_1, ..., a_M - 1, 0, ..., 0) * I_M$. By Lemma 3.3.1, $I(a_1, ..., a_M - 1, 0, ..., 0) * I_M = a_M I(a_1, a_2, ..., a_M, 0, ..., 0) + L$, where L is a linear combination of other $I(a'_1, ..., a'_r)$ whose highest index m such that $a'_m > 0$ is larger than M. By induction assumption, we have $I(a_1, ..., a_m, 0, ..., 0) \in \mathcal{A}_v$ if $m > M$ and $a_m > 0$. As a result, $L \in \mathcal{A}_v$, and $I(a_1, a_2, ..., a_M, 0, ..., 0) = (I(a_1, ..., a_M - 1, 0, ..., 0) * I_M - S)/a_M \in \mathcal{A}_v$ since by induction we also have $I(a_1, ..., a_M - 1, 0, ..., 0) \in \mathcal{A}$. □

Lemma 3.3.3 (Milnor-Moore theorem [14]). *Given a connected graded cocommutative Hopf algebra A over a field of characteristic zero with $\dim A_n < \infty$, the natural Hopf algebra homomorphism $U(P(A)) \to A$ from the universal enveloping algebra of the graded Lie algebra $P(A)$ of primitive elements of A is an isomorphism.*

Theorem 3.3.4. *For $Q = A_1, A_2, A_3, A_4$, let $\mathcal{A}$ be the subalgebra generated by the characteristic functions of orbits of indecomposable preprojective modules. Then $\mathcal{A}$ is isomorphic to $U\tilde{\mathfrak{n}}$ for some Lie algebra $\tilde{\mathfrak{n}}$ with a basis indexed by indecomposable modules.*

Proof. By Theorem 3.2.9, $\mathcal{A}$ is a Hopf algebra. From the definition of Δ, we can see that the primitive elements are of the form 1_K, where K is an orbit of an indecomposable Λ-module. Then by Lemma 3.3.3, $\mathcal{A}$ is isomorphic to the universal enveloping algebra $U\tilde{\mathfrak{n}}$ where $\tilde{\mathfrak{n}}$ is freely spanned by primitive elements which are indicator functions of orbits of indecomposable modules. □

Corollary 3.3.5. *The antipode S of $\mathcal{A}$ has this property: $S(u_1 * u_2 * .... * u_n) = (-u_n) * (-u_{n-1}) * ... * (-u_1)$ where each u_i is indicator function of an orbit of an indecomposable module.*

Proof. It follows directly from $\mathcal{A}$ being a universal enveloping algebra. □

Remark 3.3.6. *Since $\widetilde{\mathcal{M}} = \mathcal{A} = U\tilde{\mathfrak{n}}$ for $Q = A_1, A_2, A_3, A_4$, we can compute the dimension of $\widetilde{\mathcal{M}}_v$ for $v \in \mathbb{N}^I$ by using the fact that it is a universal enveloping algebra and the Poincaré–Birkhoff–Witt theorem.*

CHAPTER 4

PROPERTIES OF LIE ALGEBRA $\tilde{\mathfrak{n}}$

It is well-known that there are $1, 4, 12, 40$ indecomposable modules for $Q = A_1, A_2, A_3, A_4$ respectively. These indecomposable modules and the Auslander-Reiten quivers of preprojective algebra for $Q = A_2, A_3, A_4$ were described in [5, Section 20]. For the sake of completeness, I will also label all of the indecomposable modules and include the Auslander-Reiten quivers of preprojective algebra for $Q = A_2, A_3, A_4$ here. The dimension vectors are represented by the number at each vertex. The dimension at a vertex is the total sum of the numbers in the same column. If there is no arrow between two numbers, it means the map between the two vertices is a zero map.

From Theorem 3.3.4, $\widetilde{\mathcal{M}} = \mathcal{A} = U\tilde{\mathfrak{n}}$ for some Lie algebra $\tilde{\mathfrak{n}}$ which is spanned freely by characteristic functions of orbit of indecomposable modules for $Q = A_1, A_2, A_3, A_4$. In this chapter, we will prove that $\tilde{\mathfrak{n}}$ is nilpotent and its Lie algebra bracket tables, lower, upper central series will be given here.

4.1 Indecomposable modules for $Q = A_1, A_2, A_3, A_4$

4.1.1 For $Q = A_2$:

$$U_1 = [1 \quad 0] \quad U_2 = [0 \quad 1] \quad U_3 = [1 \rightarrow 1] \quad U_4 = [1 \leftarrow 1].$$

4.1.2 For $Q = A_3$:

$U_1 = [1 \quad 0 \quad 0]$ $U_2 = [0 \quad 1 \quad 0]$ $U_3 = [0 \quad 0 \quad 1]$ $U_4 = [1 \rightarrow 1 \quad 0]$

$U_5 = [0 \quad 1 \rightarrow 1]$ $U_6 = [1 \leftarrow 1 \quad 0]$ $U_7 = [0 \quad 1 \leftarrow 1]$ $U_8 = [1 \rightarrow 1 \rightarrow 1]$ $U_9 = [1 \leftarrow 1 \leftarrow 1]$

$$U_{10} = [1 \rightarrow 1 \leftarrow 1] \quad U_{11} = [1 \leftarrow 1 \rightarrow 1] \quad U_{12} = \begin{bmatrix} & & 1 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 1 \\ & \searrow & & \swarrow & \\ & & 1 & & \end{bmatrix}.$$

4.1.3 For $Q = A_4$:

$U_1 = [1 \quad 0 \quad 0 \quad 0]$ $U_2 = [0 \quad 1 \quad 0 \quad 0]$ $U_3 = [0 \quad 0 \quad 1 \quad 0]$ $U_4 = [0 \quad 0 \quad 0 \quad 1]$

$U_5 = [1 \rightarrow 1 \quad 0 \quad 0]$ $U_6 = [1 \leftarrow 1 \quad 0 \quad 0]$ $U_7 = [0 \quad 1 \rightarrow 1 \quad 0]$ $U_8 = [0 \quad 1 \leftarrow 1 \quad 0]$

$U_9 = [0 \quad 0 \quad 1 \rightarrow 1]$ $U_{10} = [0 \quad 0 \quad 1 \leftarrow 1]$ $U_{11} = [1 \rightarrow 1 \rightarrow 1 \quad 0]$ $U_{12} = [1 \leftarrow 1 \leftarrow 1 \quad 0]$

$U_{14} = [1 \leftarrow 1 \rightarrow 1 \quad 0]$ $U_{15} = [0 \quad 1 \rightarrow 1 \rightarrow 1]$ $U_{16} = [0 \quad 1 \leftarrow 1 \leftarrow 1]$ $U_{17} = [0 \quad 1 \rightarrow 1 \leftarrow 1]$

$$U_{18} = [0 \quad 1 \leftarrow 1 \rightarrow 1] \quad U_{19} = \begin{bmatrix} & & 1 & & & \\ & \swarrow & & \searrow & & \\ 1 & & & & 1 & 0 \\ & \searrow & & \swarrow & & \\ & & 1 & & & \end{bmatrix} \quad U_{20} = \begin{bmatrix} & & & 1 & & \\ & & \swarrow & & \searrow & \\ 0 & 1 & & & & 1 \\ & & \searrow & & \swarrow & \\ & & & 1 & & \end{bmatrix},$$

$U_{21} = [1 \rightarrow 1 \rightarrow 1 \rightarrow 1]$ $U_{22} = [1 \rightarrow 1 \rightarrow 1 \leftarrow 1]$ $U_{23} = [1 \rightarrow 1 \leftarrow 1 \rightarrow 1]$,

$U_{24} = [1 \rightarrow 1 \leftarrow 1 \leftarrow 1]$ $U_{25} = [1 \leftarrow 1 \rightarrow 1 \rightarrow 1]$ $U_{26} = [1 \leftarrow 1 \rightarrow 1 \leftarrow 1]$,

$$U_{27} = [1 \leftarrow 1 \leftarrow 1 \rightarrow 1] \quad U_{28} = [1 \leftarrow 1 \leftarrow 1 \leftarrow 1] \quad U_{29} = \begin{bmatrix} & & 1 & & & & \\ & \swarrow & & \searrow & & & \\ 1 & & & & 1 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & 1 & & & & 1 \end{bmatrix},$$

$$U_{30} = \begin{bmatrix} & & 1 & & & & 1 \\ & \swarrow & & \searrow & & \swarrow & \\ 1 & & & & 1 & & \\ & \searrow & & \swarrow & & & \\ & & 1 & & & & \end{bmatrix} \quad U_{31} = \begin{bmatrix} & & & & 1 & & \\ & & & \swarrow & & \searrow & \\ & & 1 & & & & 1 \\ & \swarrow & & \searrow & & \swarrow & \\ 1 & & & & 1 & & \end{bmatrix}$$

$$U_{32}=\begin{bmatrix}1&&&&1&&\\&\searrow&&\swarrow&&\searrow&\\&&1&&&&1\\&&&\searrow&&\swarrow&\\&&&&1&&\end{bmatrix}\quad U_{33}=\begin{bmatrix}&&1&&&&\\&\swarrow&&\searrow&&&\\1&&&&2&&\\&\searrow&&\swarrow&&\searrow&\\&&1&&&&1\end{bmatrix}$$

$$U_{34}=\begin{bmatrix}&&1&&&&1\\&\swarrow&&\searrow&&\swarrow&\\1&&&&2&&\\&\searrow&&\swarrow&&&\\&&1&&&&\end{bmatrix}\quad U_{35}=\begin{bmatrix}&&&&1&&\\&&&\swarrow&&\searrow&\\&&2&&&&1\\&\swarrow&&\searrow&&\swarrow&\\1&&&&1&&\end{bmatrix}$$

$$U_{36}=\begin{bmatrix}1&&&&1&&\\&\searrow&&\swarrow&&\searrow&\\&&2&&&&1\\&&&\searrow&&\swarrow&\\&&&&1&&\end{bmatrix}\quad U_{37}=\begin{bmatrix}&&1&&&&\\&\swarrow&&\searrow&&&\\1&&&&1&&\\&\searrow&&\swarrow&&\searrow&\\&&1&&&&1\\&&&\searrow&&\swarrow&\\&&&&1&&\end{bmatrix}$$

$$U_{38}=\begin{bmatrix}&&&&1&&\\&&&\swarrow&&\searrow&\\&&1&&&&1\\&\swarrow&&\searrow&&\swarrow&\\1&&&&1&&\\&\searrow&&\swarrow&&&\\&&1&&&&\end{bmatrix}\quad U_{39}=\begin{bmatrix}1&&&&1&&\\&\searrow&&\swarrow&&\searrow&\\&&2&&&&1\\&\swarrow&&\searrow&&\swarrow&\\1&&&&1&&\end{bmatrix}$$

$$U_{40}=\begin{bmatrix}&&1&&&&1\\&\swarrow&&\searrow&&\swarrow&\\1&&&&2&&\\&\searrow&&\swarrow&&\searrow&\\&&1&&&&1\end{bmatrix}.$$

The most nontrivial indecomposable modules in $Q = A_4$ are U_{39} and U_{40} so I give their explicit maps here.

$$U_{39} = k^2 \underset{\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}}{\rightleftarrows}} k^2 \underset{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{\overset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{\rightleftarrows}} k^2 \underset{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & 0 \end{pmatrix}}{\rightleftarrows}} k \qquad U_{40} = U_{39} \text{ reversed}$$

4.1.4 The Auslander-Reiten quivers of Λ for $Q = A_2, A_3, A_4$

The Auslander-Reiten quiver of an algebra provides a very efficient summary of the category over that algebra.

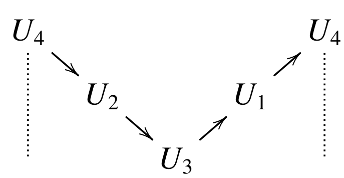

Table 4.1: The Auslander-Reiten quiver of Λ for $Q = A_2$

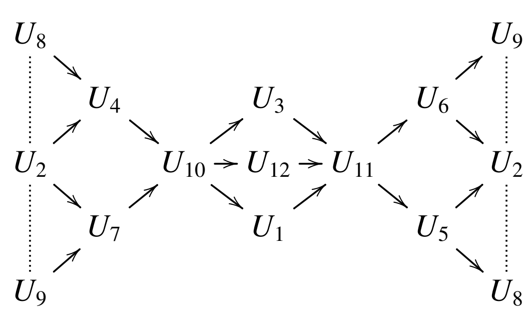

Table 4.2: The Auslander-Reiten quiver of Λ for $Q = A_3$

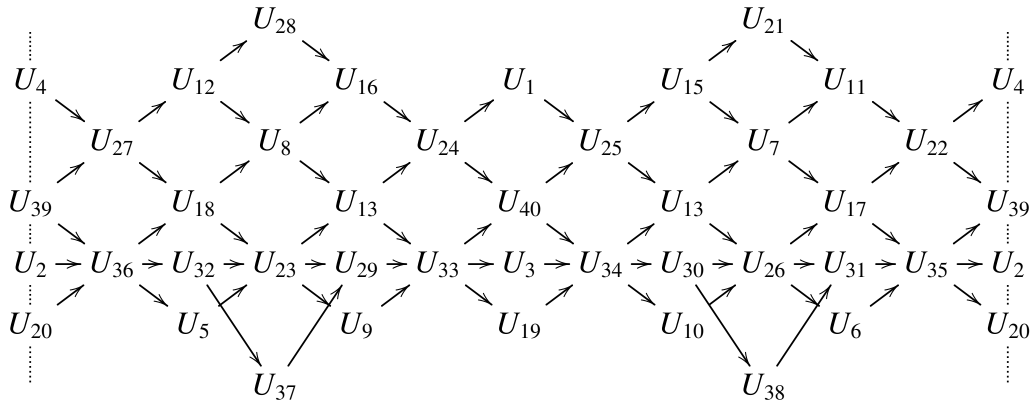

Table 4.3: The Auslander-Reiten quiver of Λ for $Q = A_4$

4.2 The Lie algebra $\tilde{\mathfrak{n}}$ with basis indexed by indecomposable modules

In this section, we will use the labels in section 4.1 and explicitly list all the Lie bracket relations among these generators.

Define $u_i = 1_{K_i}$ to be the constructible function where K_i is the orbit of indecomposable U_i for $i = 1, 2, ..., r$. Here r is the number of isomorphism types of indecomposable modules.
The (i, j)-entry of the matrix is the Lie bracket $[u_i, u_j] = u_i \cdot u_j - u_j \cdot u_i$. We only show the lower triangular part since the matrix satisfies $M^T = -M$. We only need to compute brackets when the sum of the two dimension vectors is again the dimension vector of an indecomposable module, since otherwise the bracket must be zero.

4.2.1 For $Q = A_1$:

There is only one indecomposable module so $\mathcal{A} \cong \mathbb{C}[x]$, where $x = 1_K$ and K is the orbit of the unique indecomposable module and the Lie algebra is just one dimensional.

4.2.2 For $Q = A_2$:

$[u_i, u_j]$	u_1	u_2	u_3	u_4
u_1				
u_2	$u_4 - u_3$			
u_3	0	0		
u_4	0	0	0	

Table 4.4: Lie bracket table for $Q = A_2$

4.2.3 For $Q = A_3$:

$[u_i, u_j]$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}
u_1												
u_2	$u_6 - u_4$											
u_3	0	$u_7 - u_5$										
u_4	0	0	$u_8 - u_{10}$									
u_5	$u_{11} - u_8$	0	0	u_{12}								
u_6	0	0	$u_{11} - u_9$	0	0							
u_7	$u_9 - u_{10}$	0	0	0	0	$-u_{12}$						
u_8	0	0	0	0	0	0	0					
u_9	0	0	0	0	0	0	0	0				
u_{10}	0	$-u_{12}$	0	0	0	0	0	0	0			
u_{11}	0	u_{12}	0	0	0	0	0	0	0	0		
u_{12}	0	0	0	0	0	0	0	0	0	0	0	

Table 4.5: Lie bracket table for $Q = A_3$

4.2.4 For $Q = A_4$:

For the dimension vector reason, $[u_i, u_j] = 0$ if $21 \leq i, j \leq 40$.

$[u_i, u_j]$	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
u_1										
u_2	$u_6 - u_5$									
u_3	0	$u_8 - u_7$								
u_4	0	0	$u_{10} - u_9$							
u_5	0	0	$u_{11} - u_{13}$	0						
u_6	0	0	$u_{14} - u_{12}$	0	0					
u_7	$u_{14} - u_{11}$	0	0	$u_{15} - u_{17}$	u_{19}	0				
u_8	$u_{12} - u_{13}$	0	0	$u_{18} - u_{16}$	0	$-u_{19}$	0			
u_9	0	$u_{18} - u_{15}$	0	0	$u_{23} - u_{21}$	$u_{27} - u_{25}$	u_{20}	0		
u_{10}	0	$u_{16} - u_{17}$	0	0	$u_{24} - u_{22}$	$u_{28} - u_{26}$	0	$-u_{20}$	0	
u_{11}	0	0	0	$u_{21} - u_{22}$	0	0	0	0	$-u_{32}$	0
u_{12}	0	0	0	$u_{27} - u_{28}$	0	0	0	0	0	u_{31}
u_{13}	0	$-u_{19}$	0	$u_{23} - u_{24}$	0	0	0	0	0	u_{32}
u_{14}	0	u_{19}	0	$u_{25} - u_{26}$	0	0	0	0	$-u_{31}$	0
u_{15}	$u_{25} - u_{21}$	0	0	0	u_{29}	0	0	0	0	0
u_{16}	$u_{28} - u_{24}$	0	0	0	0	$-u_{30}$	0	0	0	0
u_{17}	$u_{26} - u_{22}$	0	$-u_{20}$	0	u_{30}	0	0	0	0	0
u_{18}	$u_{27} - u_{23}$	0	u_{20}	0	0	$-u_{29}$	0	0	0	0
u_{19}	0	0	0	$u_{29} - u_{30}$	0	0	0	0	$u_{33} - u_{38}$	$u_{37} - u_{34}$
u_{20}	$u_{31} - u_{32}$	0	0	0	$u_{38} - u_{36}$	$u_{35} - u_{37}$	0	0	0	0
u_{21}	0	0	0	0	0	0	0	0	0	0
u_{22}	0	0	$-u_{32}$	0	0	0	$-u_{37}$	$-u_{36}$	0	0
u_{23}	0	$-u_{29}$	u_{32}	0	0	0	$u_{33} - u_{36}$	0	0	0
u_{24}	0	$-u_{30}$	0	0	0	0	$-u_{34}$	$-u_{38}$	0	0
u_{25}	0	u_{29}	0	0	0	0	u_{37}	u_{33}	0	0
u_{26}	0	u_{30}	$-u_{31}$	0	0	0	0	$u_{35} - u_{34}$	0	0
u_{27}	0	0	u_{31}	0	0	0	u_{35}	u_{38}	0	0
u_{28}	0	0	0	0	0	0	0	0	0	0
u_{29}	0	0	$u_{37} - u_{33}$	0	0	0	0	0	0	$-u_{40}$
u_{30}	0	0	$u_{34} - u_{38}$	0	0	0	0	0	u_{40}	0
u_{31}	0	$u_{38} - u_{35}$	0	0	$-u_{39}$	0	0	0	0	0
u_{32}	0	$u_{36} - u_{37}$	0	0	0	u_{39}	0	0	0	0
u_{33}	0	0	0	$-u_{40}$	0	0	0	0	0	0
u_{34}	0	0	0	u_{40}	0	0	0	0	0	0
u_{35}	$-u_{39}$	0	0	0	0	0	0	0	0	0
u_{36}	u_{39}	0	0	0	0	0	0	0	0	0
u_{37}	0	0	0	0	0	0	0	0	0	0
u_{38}	0	0	0	0	0	0	0	0	0	0
u_{39}	0	0	0	0	0	0	0	0	0	0
u_{40}	0	0	0	0	0	0	0	0	0	0

Figure 4.1: Lie bracket table for $Q = A_4$ (1)

$[u_i, u_j]$	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	u_{19}	u_{20}
u_{11}										
u_{12}	0									
u_{13}	0	0								
u_{14}	0	0	0							
u_{15}	u_{37}	0	u_{33}	0						
u_{16}	0	$-u_{38}$	0	$-u_{34}$	0					
u_{17}	0	$-u_{35}$	$-u_{34} + u_{36}$	$-u_{37}$	0	0				
u_{18}	u_{36}	0	u_{38}	$-u_{35} + u_{33}$	0	0	0			
u_{19}	0	0	0	0	0	0	0	0		
u_{20}	0	0	0	0	0	0	0	0	0	
u_{21}	0	0	0	0	0	0	0	0	0	0
u_{22}	0	$-u_{39}$	0	0	0	0	0	0	0	0
u_{23}	0	0	0	$-u_{39}$	0	0	u_{40}	0	0	0
u_{24}	0	0	0	0	$-u_{40}$	0	0	0	0	0
u_{25}	0	0	0	0	0	u_{40}	0	0	0	0
u_{26}	0	0	u_{39}	0	0	0	0	$-u_{40}$	0	0
u_{27}	u_{39}	0	0	0	0	0	0	0	0	0
u_{28}	0	0	0	0	0	0	0	0	0	0
u_{29}	0	0	0	0	0	0	0	0	0	0
u_{30}	0	0	0	0	0	0	0	0	0	0
u_{31}	0	0	0	0	0	0	0	0	0	0
u_{32}	0	0	0	0	0	0	0	0	0	0
u_{33}	0	0	0	0	0	0	0	0	0	0
u_{34}	0	0	0	0	0	0	0	0	0	0
u_{35}	0	0	0	0	0	0	0	0	0	0
u_{36}	0	0	0	0	0	0	0	0	0	0
u_{37}	0	0	0	0	0	0	0	0	0	0
u_{38}	0	0	0	0	0	0	0	0	0	0
u_{39}	0	0	0	0	0	0	0	0	0	0
u_{40}	0	0	0	0	0	0	0	0	0	0

Figure 4.2: Lie bracket table for $Q = A_4$ (2)

4.3 Lower and upper central series of the Lie algebra $\tilde{\mathfrak{n}}$

In this section we will show that $\tilde{\mathfrak{n}}$ is nilpotent by computing its lower central series. Since we already know all the Lie brackets of $\tilde{\mathfrak{n}}$ in terms of u_i, we can iteratively compute the series. Recall the definition of the **lower central series** of a Lie algebra $\mathfrak{n}$: $\mathfrak{n}_{i+1} = [\mathfrak{n}, \mathfrak{n}_i]$ and $\mathfrak{n}_0 = \mathfrak{n}$. Define generalized center of an ideal $\mathfrak{I} \subset \mathfrak{n}$ as $GC(\mathfrak{I}) := \{x \in \mathfrak{n} : [x, y] \in \mathfrak{I}, \forall y \in \mathfrak{n}\}$. Then we can define the **upper central series** as follows: $\mathfrak{n}^0 = 0$, $\mathfrak{n}^{i+1} = GC(\mathfrak{n}^i)$. A Lie algebra $\mathfrak{n}$ is **nilpotent** if there is an integer N such that $\mathfrak{n}_N = 0$. The positive part $(\mathfrak{sl}_{n+1})_+$ of $\mathfrak{sl}_{n+1}$ is naturally a Lie subalgebra of $\tilde{\mathfrak{n}}$ and the lower central series of $\mathfrak{sl}_{n+1}$ in terms of basis elements u_i is also computed. The calculation in this section was done by using the software Maple™[13].

4.3.1 For $Q = A_2$:

Lower central series:

$$\tilde{\mathfrak{n}}_1 = \mathrm{span}\{u_3 - u_4\},$$

$$\tilde{\mathfrak{n}}_2 = 0.$$

	$\tilde{\mathfrak{n}}_0$	$\tilde{\mathfrak{n}}_1$	$\tilde{\mathfrak{n}}_2$
dim	4	1	0

Table 4.6: Lower central series for $Q = A_2$

Upper central series:

$$\tilde{\mathfrak{n}}^1 = \text{span}\{u_3, u_4\},$$

$$\tilde{\mathfrak{n}}^2 = \tilde{\mathfrak{n}}.$$

	$\tilde{\mathfrak{n}}^0$	$\tilde{\mathfrak{n}}^1$	$\tilde{\mathfrak{n}}^2$
dim	0	2	4

Table 4.7: Upper central series for $Q = A_2$

$(\mathfrak{sl}_3)_+$ as a sub Lie algebra:

$$((\mathfrak{sl}_3)_+)_0 = \text{span}\{u_1, u_2\},$$

$$((\mathfrak{sl}_3)_+)_1 = \text{span}\{u_3 - u_4\},$$

$$((\mathfrak{sl}_3)_+)_2 = 0.$$

4.3.2 For $Q = A_3$:

Lower central series:

$$\tilde{\mathfrak{n}}_1 = \text{span}\{-u_4 + u_6, -u_8 + u_{11}, u_9 - u_{10}, -u_5 + u_7, -u_{12}, u_8 - u_{10}\},$$

$$\tilde{\mathfrak{n}}_2 = \text{span}\{u_8 + u_9 - u_{10} - u_{11}, u_{12}\},$$

$$\tilde{\mathfrak{n}}_3 = 0.$$

	$\tilde{\mathfrak{n}}_0$	$\tilde{\mathfrak{n}}_1$	$\tilde{\mathfrak{n}}_2$	$\tilde{\mathfrak{n}}_3$
dim	12	6	2	0

Table 4.8: Lower central series for $Q = A_3$

Upper central series:

$$\tilde{\mathfrak{n}}^1 = \text{span}\{u_8, u_9, u_{10} + u_{11}, u_{12}\},$$

$$\tilde{\mathfrak{n}}^2 = \text{span}\{-u_4 + u_6, -u_5 + u_7, u_8, u_9, u_{10}, u_{11}, u_{12}\},$$

$$\tilde{\mathfrak{n}}^3 = \tilde{\mathfrak{n}}.$$

	$\tilde{\mathfrak{n}}^0$	$\tilde{\mathfrak{n}}^1$	$\tilde{\mathfrak{n}}^2$	$\tilde{\mathfrak{n}}^3$
dim	0	4	7	12

Table 4.9: Upper central series for $Q = A_3$

$(\mathfrak{sl}_4)_+$ as a sub Lie algebra:

$$((\mathfrak{sl}_4)_+)_0 = \text{span}\{u_1, u_2, u_3\}$$

$$((\mathfrak{sl}_4)_+)_1 = \text{span}\{-u_4 + u_6, -u_5 + u_7\}$$

$$((\mathfrak{sl}_4)_+)_2 = \text{span}\{u_8 + u_9 - u_{10} - u_{11}\}$$

$$((\mathfrak{sl}_4)_+)_3 = 0$$

4.3.3 For $Q = A_4$:

Lower central series:

$$\tilde{\mathfrak{n}}_1 = \text{span}\{u_5 - u_6, u_{11} - u_{14}, u_{13} - u_{12}, u_{21} - u_{25}, u_{24} - u_{28}, u_{22} - u_{26}, u_{23} - u_{27}, u_{32} - u_{31}, u_{39}, u_7 - u_8, u_{15} - u_{18}, u_{17} - u_{16}, u_{19}, u_{29}, u_{30}, u_{35} - u_{38}, u_{36} - u_{37}, u_9 - u_{10}, u_{11} - u_{13}, u_{20}, u_{32}, u_{33} - u_{37}, u_{34} - u_{38}, u_{15} - u_{17}, u_{21} - u_{22}, u_{27} - u_{28}, u_{40}, u_{21} - u_{23}, u_{36} - u_{38}, u_{37}\}$$

$$\tilde{\mathfrak{n}}_2 = \text{span}\{u_{11} + u_{12} - u_{13} - u_{14}, u_{19}, u_{21} - u_{23} - u_{25} + u_{27}, u_{22} - u_{24} - u_{26} + u_{28}, u_{29}, u_{30}, u_{35} + u_{36} - u_{37} - u_{38}, u_{39}, u_{21} - u_{22} - u_{25} + u_{26}, u_{31} - u_{32}, u_{37}, u_{34}, u_{33} - u_{35} - u_{36}, u_{33}, u_{40}, u_{15} + u_{16} - u_{17} - u_{18}, u_{20}, u_{36} - u_{37}, -u_{21} + u_{22} + u_{23} - u_{24}, u_{32}\}$$

$$\tilde{\mathfrak{n}}_3 = \text{span}\{u_{21} - u_{22} - u_{23} + u_{23} - u_{25} + u_{26} - u_{27} + u_{28}, u_{31} - u_{32}, u_{33} - u_{37}, u_{34} - u_{38}, -u_{34} + u_{35} + u_{36} - u_{37}, u_{39}, u_{29} - u_{30}, u_{33} - u_{38}, u_{40}, u_{36} - u_{37}\}$$

$$\tilde{\mathfrak{n}}_4 = \text{span}\{-u_{33} - u_{34} + u_{35} + u_{36}, u_{39}, u_{40}, -u_{35} - u_{36} + u_{37} + u_{38}\}$$

$$\tilde{\mathfrak{n}}_5 = 0$$

	$\tilde{\mathfrak{n}}_0$	$\tilde{\mathfrak{n}}_1$	$\tilde{\mathfrak{n}}_2$	$\tilde{\mathfrak{n}}_3$	$\tilde{\mathfrak{n}}_4$	$\tilde{\mathfrak{n}}_5$
dim	40	30	20	10	4	0

Table 4.10: Lower central series for $Q = A_4$

Upper central series:

$$\tilde{\mathfrak{n}}^1 = \text{span}\{u_{40}, u_{39}, u_{38}, u_{37}, u_{35} + u_{36}, u_{33} + u_{34}, u_{28}, u_{21}\},$$

$$\tilde{\mathfrak{n}}^2 = \text{span}\{u_{40}, u_{39}, u_{38}, u_{37}, u_{36}, u_{35}, u_{34}, u_{33}, u_{31} - u_{32},$$
$$u_{29} - u_{30}, u_{28}, -u_{22} - u_{23} + u_{24} - u_{25} + u_{26} + u_{27}, u_{21}\},$$

$$\tilde{\mathfrak{n}}^3 = \text{span}\{u_{40}, u_{20}, u_{19}, -u_{15} - u_{16} + u_{17} + u_{18}, -u_{11} - u_{12} + u_{13} + u_{14}, u_{39}, u_{38}, u_{37}, u_{36}, u_{35}, u_{34},$$
$$u_{33}, u_{32}, u_{31}, u_{30}, u_{29}, u_{28}, u_{22} + u_{27}, u_{23} + u_{26}, u_{22} + u_{23} + u_{25}, -u_{22} - u_{23} + u_{24}, u_{21}\},$$

$$\tilde{\mathfrak{n}}^4 = \text{span}\{u_{19}, u_{20}, u_{21}, u_{22}, u_{23}, u_{24}, u_{25}, u_{26}, u_{27}, u_{28}, u_{29}, u_{30}, u_{31}, u_{32}, u_{33}, u_{34}, u_{35}, u_{36}, u_{37}, u_{38}, u_{39}, u_{40},$$
$$u_{15} - u_{17}, u_{15} - u_{16}, u_{11} - u_{14}, u_{11} - u_{13}, u_{11} - u_{12}, u_9 - u_{10}, u_7 - u_8, u_5 - u_6, u_{15} - u_{18}\},$$

$$\tilde{\mathfrak{n}}^5 = \tilde{\mathfrak{n}}$$

	$\tilde{\mathfrak{n}}^0$	$\tilde{\mathfrak{n}}^1$	$\tilde{\mathfrak{n}}^2$	$\tilde{\mathfrak{n}}^3$	$\tilde{\mathfrak{n}}^4$	$\tilde{\mathfrak{n}}^5$
dim	0	8	13	22	31	40

Table 4.11: Upper central series for $Q = A_4$

$(\mathfrak{sl}_5)_+$ as a sub Lie algebra:

$$((\mathfrak{sl}_5)_+)_0 = \text{span}\{u_1, u_2, u_3, u_4\}$$

$$((\mathfrak{sl}_5)_+)_1 = \text{span}\{-u_5 + u_6, -u_7 + u_8, -u_9 + u_{10}\}$$

$$((\mathfrak{sl}_5)_+)_2 = \text{span}\{u_{11} + u_{12} - u_{13} - u_{14}, u_{15} + u_{16} - u_{17} - u_{18}\}$$

$$((\mathfrak{sl}_5)_+)_3 = \text{span}\{-u_{21} + u_{22} + u_{23} - u_{24} + u_{25} - u_{26} - u_{27} + u_{28}\}$$

$$((\mathfrak{sl}_5)_+)_4 = 0$$

4.4 Lie algebra cohomology of $\tilde{\mathfrak{n}}$

In this section, we will compute the Lie algebra cohomology of $\tilde{\mathfrak{n}}$. The calculation in this section was done by using the software Maple™[13]. The Lie algebra cohomology is graded by cohomological dimension and the the dimension vector of Λ-module. The dimension of each graded piece will be given. Since $\tilde{\mathfrak{n}}$ is nilpotent, the Lie algebra cohomology satisfies Poincaré duality. The Lie algebra cohomology for the case $Q = A_4$ is too big to calculate and is not given here.

4.4.1 For $Q = A_2$:

	H^0	H^1	H^2	H^3	H^4
dim	1	3	4	3	1

Table 4.12: Dimension of Lie algebra cohomology for $Q = A_2$

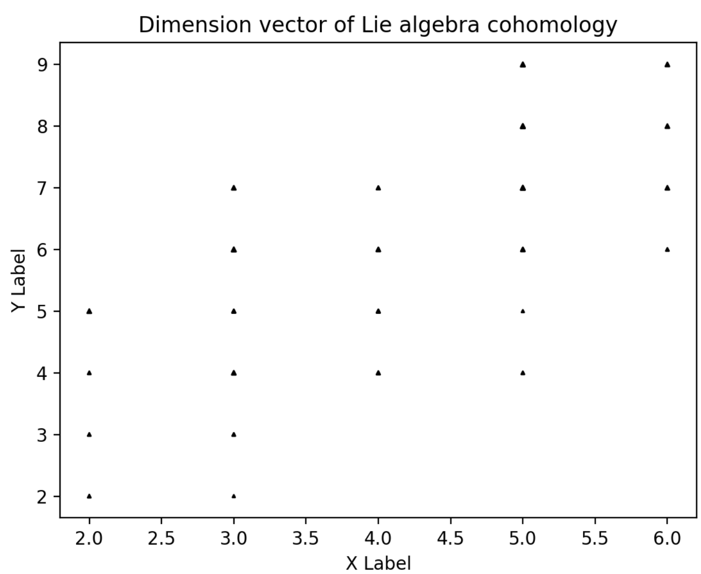

Figure 4.3: Multi-dimension vector of Lie algebra cohomology for $Q = A_2$

4.4.2 For $Q = A_3$:

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8	H^9	H^{10}	H^{11}	H^{12}
dim	1	6	20	47	85	121	136	121	85	47	20	6	1

Table 4.13: Dimension of Lie algebra cohomology for $Q = A_3$

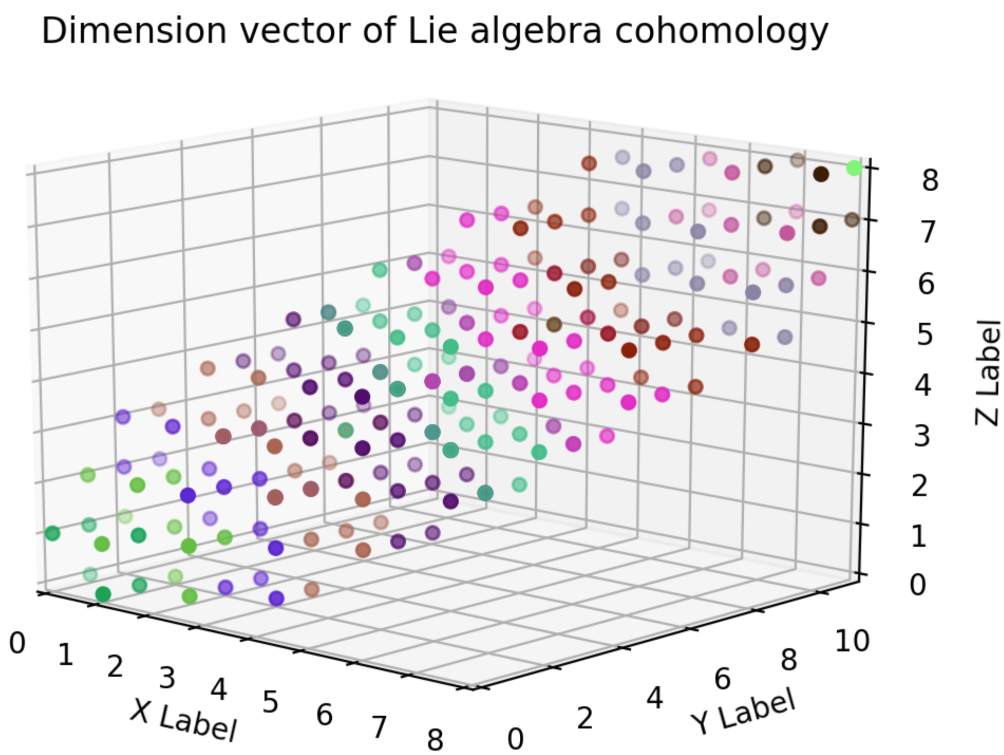

Figure 4.4: Multi-dimension vector of Lie algebra cohomology for $Q = A_3$

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