ON THE SEMIGROUP OF STRONGLY CONNECTED AUTOMATA

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* This work was supported by the Ministry of Higher Education, Science and Technics,Poland. Author's address: Institute of Control Engineering, Technical University of Poznań, Poland.

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In 1964 Weeg [6] has posed a question asking which semigroups admit strongly connected automata. The presumed answer was given in 1971 by Oehmke [5] through the following statement: A semigroup admits strongly connected automaton iff it has a homomorphic image either a nontrivial finite group or semigroup of the transformations of some right zero semigroup. This theorem can be reformulated as follows: A semigroup is a characteristic semigroup of strongly connected automaton iff it has a homomorphic image either a nontrivial finite group or a semigroup of transformations of some right zero semigroup. In this paper the "only if" part of this theorem has been overthrown. Some related problems concerning strongly connectedness of finite automata are also investigated.

Key Words and Phrases: semigroup, characteristic semigroup of an automaton, strongly connected automaton,

CR Categories: 5.22

1. Introduction

Let $A = (S,\Sigma,M)$ be a finite automaton, where S is a finite set of states, Σ is finite input alphabet and M is a transition function such that $M: S \times \Sigma \to S$. The set of all finite strings over Σ will be denoted I and called free semigroup. The elements of I are called words. If $x \in I$ and $x = \sigma_0 \sigma_1 \ldots \sigma_{k-1}$ then $|x|=k$ is the length of x.

Let us define a relation R on I as follows:

xRy iff for every $s \in S$ $M(s,x) = M(s,y)$.

R is of course an equivalence relation. A set of elements of I equivalent to x with respect to R will be denoted $\overline{x}$ and called the equivalence class of I. The set of such equivalence classes with respect to R for automaton A will be denoted $\overline{I(A)}$ and called the characteristic semigroup of A.

For automaton $A = (S,\Sigma,M)$ we define characteristic automaton $\overline{A}$ as an ordered triple $(S, \overline{I(A)}, \overline{M})$, where $M: S \times \overline{I(A)} \to S$ is defined as follows:

$$\overline{M}(s,\overline{x}) = M(s,x).$$

There exits also another interpretation of Σ, namely: for every $\sigma \in \Sigma$ we define a mapping f_σ of the set S into S as follows:

$$f_\sigma(s) = M(s,\sigma) \text{ for every } s \in S.$$

One says that f_σ is implied by T. The set of all mappings of S into S implied by all elements of Σ will be denoted J(A). J(A) together with an operation of composition of functions is a set of generators of some semigroup F(A) such that $(f_\sigma f_{\sigma'})(s)=f_\sigma(f_{\sigma'}(s$ for $f_\sigma, f_{\sigma'} \in J(A)$ and $s \in S$. Now an automaton can be defined as an

ordered pair (S,J(A)) and characteristic automaton of (S,J(A)) as an ordered pair (S,F(A)).

Let S_1 and S_1' be semigroups; a mapping ϕ: S_1 into S_1' is called a <u>semigroup homomorphism</u> iff

(1) $\phi(s_1 \circ s_1') = \phi(s_1)\phi(s_1')$ for every $s_1, s_1' \in S_1$ and $\phi(s_1)$, $\phi(s_1') \in S_1'$. If ϕ is one-to-one and onto then ϕ is called <u>semigroup isomorphism</u>.

If ϕ in (1) has a property $\phi(s_1 \circ s_1') = \phi(s_1')\phi(s_1)$ for every $s_1, s_1' \in S_1$ and $\phi(s_1), \phi(s_1') \in S_1'$, then ϕ is called <u>semigroup antihomomorphism</u> and <u>semigroup antiisomorphism</u>, respectively.

It is easily seen that F(A) and $\overline{I(A)}$ are semigroup anti-isomorphic and $\phi(f_x) = \overline{x}$, where $x \in I$.

An automaton A = (S,Σ,M) is called <u>strongly connected</u> iff for every $(s,s') \in SxS$ there exists word $x \in I$ such that $M(s,x) = s'$.

In the literature [6] the following question has been posed: which semigroups admit strongly connected automata. This question can be restated as follows: characterize characteristic semigroup of strongly connected automata or give necessary and sufficient conditions for characteristic semigroup to be a semigroup of strongly connected automaton.

In this paper we shall give partial answer to this question, first giving a counterexample to the theorem 1[5] and then proving a necessary and sufficient conditions of the strong connectedness for cyclic automata.

2. Characteristic semigroup of strongly connected automaton

First we quote some notion from [5]. Let $\mathscr{S}$ be an arbitrary semigroup, S a finite state set and M transition function on $S \times \mathscr{S} \to S$ such that $M(s,xx') = M(M(s,x),s')$ for $s \in S$ and $x,x' \in \mathscr{S}$. An ordered triple $(S,\mathscr{S},M)$ will be called an automaton with input semigroup $\mathscr{S}$. It is easily seen that this notion of an automaton is equivalent to our notion of characteristic automaton $\overline{A} = (S, \overline{I(A)}, \overline{M})$. It is obvious that $A = (S,\Sigma,M)$ is strongly connected iff $\overline{A} = (S, \overline{I(A)}, \overline{M})$ is strongly connected.

Let $A = (S,\Sigma,M)$ and $B = (U,\Sigma,N)$ be two automata. An automaton $C = (T,\Sigma,P)$ is called direct sum of A and B if $T = S \cup U$ and

$$P(t,\sigma) = \begin{cases} M(t,\sigma) & \text{if } t \in S \\ N(t,\sigma) & \text{if } t \in U \end{cases}$$

for every $t \in T$ and $\sigma \in \Sigma$.

Theorem 1. Let $A = (S,\Sigma,M)$ and $B = (U,\Sigma,N)$ be two finite automata and let $C = (T,\Sigma,P)$ be a direct sum of A and B, then characteristic semigroups $\overline{I(C)}$ of C is isomorphic to the direct product of characteristic semigroups $\overline{I(A)}$ and $\overline{I(B)}$ i.e. $\overline{I(A)} \times \overline{I(B)} \simeq \overline{I(C)}$.

Proof For the sake of convenience only we prove this result for characteristic semigroups of transformations on the sets of states, namely $F(C) \simeq F(A) \times F(B)$.

Define

$\phi: F(C) \to F(A) \times F(B)$ as follows $\phi(f_x^C) = (f_x^A, f_x^B)$, where $f_x^C = P(t,x)$ $f_x^A = M(s,x)$, $f_x^B = N(u,x)$ for $t \in T$, $s \in S$ and $u \in U$, respectively.

i) first we show that ϕ preserves operation

$\phi(f_x^C f_y^C) = \phi(f_{xy}^C) = (f_{xy}^A, f_{xy}^B) = (f_x^A f_y^A, f_x^B f_y^B) = (f_x^A, f_x^B)(f_y^A, f_y^B) = \phi(f_x^C)\phi(f_y^C)$.

ii) now we show that ϕ is "one-to-one",

$\phi(f_x^C) = \phi(f_y^C) \Rightarrow (f_x^A, f_x^B) = (f_y^A, f_y^B) \Rightarrow f_x^A \equiv f_y^A \;\&\; f_x^B \equiv f_y^B \Rightarrow f_x^C \equiv f_y^C$.

iii) now we prove that ϕ is "onto",

$\phi(f_x^C) = (f_x^A, f_x^B) \Rightarrow \phi^{-1}\phi(f_x^C) = \phi^{-1}(f_x^A, f_x^B) \Rightarrow f_x^C = \phi^{-1}(f_x^A, f_x^B)$

Q.E.D.

In [5] the following theorem was proven: a semigroup admits a strongly connected automaton iff it has a homomorphic image either a nontrivial finite group or a semigroup of mappings on some right zero semigroup. From our theorem 1 follows immediately a counterexample. Namely, it is not true that a semigroup which homomorphic image is a nontrivial finite group or a semigroup of transformations on right zero semigroup admits strongly connected automaton. In tab. 1 a semigroup $\mathcal{S}$ is given which is isomorphic to the characteristic semigroup of automaton A in Tab. 3. The isomorphism ψ is given below Tab. 3.

Tab. 1. Semigroup $\mathscr{S}$.

$\mathscr{S}$	a	b	c	d	e	f	g	h	i	j	k	l
a	a	b	c	d	e	f	g	h	i	j	k	l
b	c	d	a	b	k	l	i	j	g	h	e	f
c	c	d	a	b	k	l	i	j	g	h	e	f
d	a	b	c	d	e	f	g	h	i	j	k	l
e	e	f	g	h	a	b	c	d	k	l	i	j
f	g	h	e	f	i	j	k	l	c	d	a	b
g	g	h	e	f	i	j	k	l	c	d	a	b
h	e	f	g	h	a	b	c	d	k	l	i	j
i	i	j	k	l	g	h	e	f	a	b	c	d
j	k	l	i	j	c	d	a	b	e	f	g	h
k	k	l	i	j	c	d	a	b	e	f	g	h
l	i	j	k	l	g	h	e	f	a	b	c	d

Tab. 2. Group T.

T	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	A	D	C	F	E
C	C	E	A	F	B	D
D	D	F	B	E	A	C
E	E	C	F	A	D	B
F	F	D	E	B	C	A

Tab. 3. Automaton A

S \ Σ	0	1
1	2	1
2	2	1
3	4	3
4	3	5
5	5	4

Tab. 4 Semigroup T'

T'	K	L
K	K	L
L	K	L

$$\phi = \begin{pmatrix} a,b & c,d & e,f & g,h & i,j & k,l \\ C & A & E & F & D & B \end{pmatrix}$$

$$\psi = \begin{pmatrix} a & b & c & d & e & f & g & h & i & j & k & l \\ \overline{11} & \overline{0} & \overline{011} & \overline{00} & \overline{1} & \overline{10} & \overline{0101} & \overline{100} & \overline{101} & \overline{1010} & \overline{01} & \overline{010} \end{pmatrix}$$

In Tab. 2 a group T is given and T is a homomorphic image of $\mathscr{S}$ under ϕ. Note that automaton A in Tab. 3 is not strongly connected. In Tab. 4 right zero semigroup T' is given. Note that T' is homomorphic image of $\mathscr{S}$ under

$$\phi = \begin{pmatrix} \overline{1}, & \overline{11}, & \overline{011}, & \overline{0101}, & \overline{101}, & \overline{01} & \overline{0} & \overline{00} & \overline{10} & \overline{100} & \overline{1010} & \overline{010} \\ & & & K & & & & & & L & & \end{pmatrix}$$

From the fact that given semigroup can be a characteristic semigroup of strongly connected and unconnected automata follows immediately that in general case strong connectedness cannot be described with the help of characteristic semigroup. Oehmke's result is still valid for cyclic automata (automata with single state called generator from which all other states can be achieved) and for this case can be restated as follows:

Theorem 2. For cyclic automata characteristic semigroup $\mathcal{S}$ admits strongly connected automaton iff it has as a homomorphic image either a nontrivial finite group or a semigroup of mappings of some right zero semigroup.

Proof (i) => identical as a proof of theorem 1 in [5].

(ii) <= let us consider two cases:

a) homomorphic image T of $\mathcal{S}$ is nontrivial finite group; let $(S, \mathcal{S}, M)$ and (S,T,M^*) be corresponding automata and let $x,x' \in \mathcal{S}$ and $h(x)$, $h(x') \in T$ be such that $h(x)h(x') = id.$, and

(2) $M(s,x) = s'$ implies $M^*(s,h(x)) = s'$ for every $s \in S$ and $x \in \mathcal{S}$. (S,T,M^*) is cyclic automaton because $(S, \mathcal{S}, M)$ is cyclic. (S,T,M^*) is strongly connected because is cyclic and T is a group (finite and nontrivial). Let us assume now that $(S, \mathcal{S}, M)$ is not strongly connected. This means that for some $s_1, s_2 \in S$ do not exist such $x,x' \in \mathcal{S}$ that

$$M(s_1,x) = s_2 \quad \text{or} \quad M(s_2,x') = s_1.$$

but than also

$$M^*(s_1,h(x)) = s_2 \text{ or } M^*(s_2,h(x')) = s_1$$

fails because of (2). So we have a contradiction.

b) homomorphic image W of $\mathcal{S}$ is a set of transformations of some left zero semigroup, i.e. for every $w,w' \in W$ we have $ww' = w'$. Let $(\mathcal{S}/_\theta, W, M_\theta)$ be automaton, a homomorphic image of $(S, \mathcal{S}, M)$ under homomorphism h

(θ is two-sided congruence associated with h). Then for arbitrary $\theta(s_1)$ and $\theta(s_2)$ we have

$$M_\theta(\theta(s_1),s_2) = \theta(s_1s_2) = \theta(s_2)$$

$$M_\theta(\theta(s_2),s_1) = \theta(s_2s_1) = \theta(s_1)$$

because W is a right zero semigroup. This means that $(\mathcal{S}/_\theta, W, M_\theta)$ is strongly connected automaton. The proof of the second part is similar to that of a) and therefore omitted.

Q.E.D.

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