Global Convergence of Damped Newton's Method for Nonsmooth Equations, via the Path Search

Daniel Ralph

TR 90-1181
December 1990

Department of Computer Science
Cornell University
Ithaca, NY 14853-7501

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Cornell University

Computer Sciences Department

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Abstract. A natural damping of Newton's method for nonsmooth equations is presented. This damping, via the *path search* instead of the traditional line search, enlarges the domain of convergence of Newton's method and therefore is said to be globally convergent. Convergence behavior is like that of line search damped Newton's method for smooth equations, including Q-quadratic convergence rates under appropriate conditions.

Applications of the path search include damping Robinson-Newton's method for nonsmooth normal equations corresponding to nonlinear complementarity problems and variational inequalities, hence damping both Wilson's method (sequential quadratic programming) for nonlinear programming and Josephy-Newton's method for generalized equations.

Computational examples from nonlinear programming are given.

Key words. Damped Newton's method, global convergence, first-order approximation, line search, path search, nonsmooth equations, normal mappings, variational inequalities, generalized equations, complementarity problems, nonlinear programming

Abbreviated title. Global Convergence of Nonsmooth Newton's Method

1 Introduction

This paper presents a novel, natural damping of (local) Newton's method — essentially Robinson-Newton's method [Rob88] — for solving nonsmooth equations. This damping, via the so-called *path search* instead of the traditional line search, enlarges the domain of convergence of Newton's method and therefore is said to be globally convergent. The convergence behavior of the method is almost identical to that of the traditional line search damped Newton's method applied to smooth equations, which, under appropriate conditions, is roughly described as linear convergence far away from a solution and superlinear, possibly quadratic, convergence near a solution. We also investigate the *nonmonotone* path search (§3, §4) which is an easy extension of the nonmonotone line search [GLL].

An immediate application is damping (Robinson-)Newton's method for solving the nonsmooth *normal* equations [Rob90], yielding a damping procedure for both Wilson's method (or sequential quadratic programming) [Wil; Fle, Ch. 12 §4] for nonlinear programs, and Josephy-Newton's method for generalized equations [Jos]. Path search damped Newton's method applies equally to the normal equation formulation of variational inequalities and nonlinear complementarity problems, as described in §5.

We need some notation to aid further discussion. Suppose X and Y are both N dimensional Euclidean spaces (though Banach spaces can be dealt with) and $f : X \to Y$. We wish to solve the equation

$$f(x) = 0, \quad x \in X.$$

For the traditional damped Newton's method we assume f is a continuously differentiable function. Suppose x^k is the kth iterate of the algorithm, and $\hat{x}^{k+1}$ is the zero of the linearization

$$A_k(x) \stackrel{\text{def}}{=} f(x^k) + \nabla f(x^k)(x - x^k).$$

Since A_k approximates f, $\hat{x}^{k+1}$ at least formally approximates a zero of f. Newton's method defines $x^{k+1} \stackrel{\text{def}}{=} \hat{x}^{k+1}$, so $\hat{x}^{k+1}$ is called *Newton's iterate.* We may test the "accuracy" of this approximate solution by line searching along the interval from x^k to $\hat{x}^{k+1}$. For example, using the idea of the Armijo line search, we start at the point $x \stackrel{\text{def}}{=} \hat{x}^{k+1}$, test x for accuracy by comparing $\|f(x)\|$ against $\|A_k(x)\|$, and move x to the current midpoint of the interval $[x^k, x]$ if the two norms are not "close". (Alternatively, the point x is defined as $x^k + t(\hat{x}^{k+1} - x^k)$ starting with $t = 1$, then $t = 1/2$ if necessary, and so on.) The line search proceeds iteratively, halving the distance from x to x^k, till the decrease in $\|f(\cdot)\|$ in moving from x^k to x is close to the decrease predicted by $\|A_k(\cdot)\|$. The first point x at which sufficient accuracy is found is the next iterate: $x^{k+1} \stackrel{\text{def}}{=} x$. We refer to such methods as *line search damped.*

The computational success of line search damped Newton's method is due to its very nice convergence behavior (under appropriate conditions) described below. This convergence relies on two key properties of the linearizations (A_k): firstly A_k is a "good" approximation of f near x^k, *independent* of k; and, secondly, A_k goes to zero *rapidly* on the path $p^k(t) \stackrel{\text{def}}{=} x^k + t(\hat{x}^{k+1} - x^k)$ as t goes from 0 to 1. These properties yield that f moves rapidly

toward zero on the path p^k, at least for all sufficiently small t independent of k. Hence the (monotone) line search, which samples points on the path p^k, determines the next iterate x^{k+1} such that the residual $\|f(x^{k+1})\|$ is less than or equal to some fixed fraction (less than 1) of $\|f(x^k)\|$. After finitely many iterations, the residual will fall below a positive threshold after which the iterates (x^k) converge to a solution point at a superlinear, perhaps quadratic, rate.

We propose a damping of Newton's method suitable for a nonsmooth equation $f(x) = 0$, using approximations A_k and paths p^k with the properties summarized above. For example, suppose f is piecewise smooth and A_k piecewise linear, the nonsmoothness of A_k being necessary to maintain uniform accuracy of approximations for all k. Newton's method — essentially Robinson-Newton's method [Rob88] in this context — is to set $x^{k+1} \stackrel{\text{def}}{=} \hat{x}^{k+1}$ where $\hat{x}^{k+1} = A_k{}^{-1}(0)$ (as in the smooth case). A naive approach to damping this method is to line search along the interval $[\hat{x}^{k+1}, x^k]$. Instead, we follow a path p^k from x^k $(t = 0)$ to $\hat{x}^{k+1}$ $(t = 1)$ on which A_k goes to zero rapidly; hence we expect that, at least initially, f will move rapidly toward zero along this path. In rough terms, we increase t so long as the actual residual $\|f(p^k(t))\|$ is close to the approximate residual $\|A_k(p^k(t))\|$, the maximum such t being used to define the next iterate: $x^{k+1} \stackrel{\text{def}}{=} p^k(t)$. This procedure yields *path search damped* Newton's method.

Of course it is likely that p^k will not be affine. In this case, there is no basis for the line search on $[x^k, \hat{x}^{k+1}]$. It is even possible that $\|f(x^k + t(\hat{x}^{k+1} - x^k))\|$ initially increases as t increases from 0, causing the line search to fail altogether. The path search however may still be numerically and theoretically sound.

Another, and perhaps the simplest approach to (damping) Newton's method for nonsmooth equations is to set

$$A_k(x) \stackrel{\text{def}}{=} f(x^k) + f'(x^k; x - x^k)$$

where we assume the existence of the directional derivative $f'(x; d)$ at each x in each direction d. As before, suppose $\hat{x}^{k+1}$ solves $A_k(x) = 0$. Since $f'(x; \cdot)$ is positive homogeneous, A_k is affine on the interval $[x^k, \hat{x}^{k+1}]$, so damping by line searching makes sense. A major difficulty here is that the closer x^k is to a point of nondifferentiability f, the smaller the neighborhood of x^k in which A_k accurately approximates f. This difficulty is reflected in the dearth of results showing general global convergence of such a scheme. In J.-S. Pang's proposal of this method [Pan], it is necessary for convergence of the iterates to a solution x^* that f be strongly Frećhet differentiable at x^*. This requirement partly defeats the aim of solving nondifferentiable equations. Differentiability at solution points is not necessary in our approach. [HX] applies the theory of [Pan] to the nonsmooth normal equation associated with the nonlinear complementarity problem (cf. Theorem 10 and remarks following), and provides some computational experience of this damped Newton's method.

This alternative approach to damping nonsmooth Newton's method is less general than, and seems to lack the elegance of path search damping. These observations, however, may have little bearing on the ultimate usefulness of the two methods. For example we use a

modification of Lemke's algorithm to determine each path p^k in §5, hence this implementation might be prey to the same exponential behavior as the original Lemke's algorithm for certain pathological problems, a difficulty not observed when applying damped Newton's method using directional derivatives to such problems [HX].

The remainder of the paper is organized as follows:

§2 Notation and preliminary results.

§3 Motivation: line search damped Newton's method for smooth equations.

§4 Path search damped Newton's method for nonsmooth equations.

§5 Applications.

This report will also appear as a Working Paper in the Systems and Decision Sciences Program, International Institute of Applied Systems Analysis, Laxenburg, Austria.

Acknowledgements

This research was initiated under funding from the Air Force Office of Scientific Research, Air Force Systems Command, USAF, through Grants no. AFOSR-88-0090 and AFOSR-89-0058; and the National Science Foundation through Grant No. CCR-8801489. Further support was given by the National Science Foundation for the author's participation in the 1990 Young Scientists' Summer Program at the International Institute of Applied Systems Analysis, Laxenburg, Austria. Finally, this research was partially supported by the U.S. Army Research Office through the Mathematical Sciences Institute, Cornell University, and by the Computational Mathematics Program of the National Science Foundation under grant DMS-8706133. The US Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation thereon.

I would like to thank Professor Stephen M. Robinson for his support of this project, through both the initial funding and his continued interest. I also thank Laura M. Morley for her assistance in implementing path search damped Newton's method.

2 Notation and Preliminary Results

Most of the interest for us is in finite dimensions, when both X and Y are Euclidean N-space, $\mathbb{R}^N$. Our most fundamental result, however, Theorem 8, is valid for Banach spaces X, Y hence this generality of X and Y will be used throughout the paper. Also throughout, f is function mapping X to Y, and $\mathbb{B}_X$, $\mathbb{B}_Y$ denote the closed unit balls in X,Y respectively. The unit ball may be written $\mathbb{B}$ when the context is clear. A neighborhood in X is a subset of X with nonempty interior; a neighborhood of a point $x \in X$ is a subset of X containing x in its interior.

By $o(t)$ (as $t \downarrow 0$) we mean a scalar function of the scalar t such that $o(t)/t \to 0$ as $t \downarrow 0$. Likewise, by $O(t)$ (as $t \downarrow 0$) we mean $|O(t)/t|$ is bounded above as $t \downarrow 0$.

The function f is Lipschitz (of modulus $l \geq 0$) on a subset X_0 of X if $\|f(x) - f(x')\|$ is bounded above by a constant multiple (l) of $\|x - x'\|$, for any points x,x' in X_0. A function g from a subset X_0 of X to Y is said to be (continuously, or Lipschitz) invertible if it is bijective (and its inverse mapping is continuous, or Lipschitz respectively). Such a function g is (continuously, or Lipschitz) invertible *near* a point $x \in X_0$ if, for some neighborhoods U of x in X and V of $f(x)$ in Y, the restricted mapping $g|_{U \cap X_0} : U \cap X_0 \to V$ is (continuously, or Lipschitz) invertible. In defining this restricted mapping it is tacitly assumed that $g(U \cap X_0) \subset V$.

We are interested in approximating f when it is not necessarily differentiable.

Definition 1 *Let $X_0 \subset X$.*

1. *A* first-order approximation *of f at $x \in X$ is a mapping $\hat{f} : X \to Y$ such that*

$$\|\hat{f}(x') - f(x')\| = o(\|x - x'\|), \quad \text{as } x' \to x.$$

   A first-order approximation *of f on X_0 is a mapping A on X_0 such that for each $x \in X_0$, $A(x)$ is a first-order approximation of f at x.*

2. *Let A be a first-order approximation of f on X_0. A is a* uniform *first-order approximation (with respect to X_0) if there is a function $\Delta(t) = o(t)$ such that for any $x, x' \in X_0$,*

$$\|A(x)(x') - f(x')\| \leq \Delta(\|x - x'\|). \tag{1}$$

   A is a uniform first-order approximation near *$x^0 \in X_0$ if, for some $\Delta(t) = o(t)$, (1) holds for x, x' near x^0.*

It may be the case that we are only interested in (defining) a first-order approximation $\hat{f}$ of f on X_0 rather than on all X. The above definition is a matter of notational convenience.

The idea of a path will be needed to define the path search damping of Newton's method.

Definition 2 *A* path *(in X) is a continuous function $p : [0,T] \to X$ where $T \in [0,1]$. The* domain *of p is $[0,T]$, denoted* $\mathrm{dom}(p)$.

We note a trivial path lifting result.

Lemma 3 *Let $\Phi : X \to Y$, $x \in X$ and $\Phi(x) \neq 0$. If for some neighborhood U of x and a radius $\epsilon > 0$ the restricted mapping*

$$\hat{\Phi} \stackrel{\text{def}}{=} \Phi|_U : U \to \Phi(x) + \epsilon \mathbb{B}_Y$$

is continuously invertible, then for $0 \leq T \leq \min\{\epsilon/\|\Phi(x)\|, 1\}$, the unique path p of domain $[0,T]$ such that

$$p(0) = x$$
$$\Phi(p(t)) = (1-t)\Phi(x) \quad \forall t \in [0,T]$$

is given by

$$p(t) = \hat{\Phi}^{-1}((1-t)\Phi(x)) \quad \forall t \in [0,T].$$

For a nonempty, closed convex set C in $\mathbb{R}^N$ and each $x \in \mathbb{R}^N$, $\pi_C(x)$ denotes the nearest point in C to x. The existence and uniqueness of the projected point $\pi_C(x)$ is classical in the generality of Hilbert spaces. We refer the reader to [Bré, Ex. 2.8.2 and Prop. 2.6.1] where we also see that the projection operator π_C is Lipschitz of modulus 1. The *normal cone* to C at x is

$$N_C(x) \stackrel{\text{def}}{=} \begin{cases} \{y \in \mathbb{R}^N \mid \langle y, c-x\rangle \leq 0,\ \forall c \in C\} & \text{if } x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

Next we have the normal maps of [Rob90]. These will be our source of applications (§5).

Definition 4 *Let $F : \mathbb{R}^N \to \mathbb{R}^N$, and C be a closed, convex, nonempty set in $\mathbb{R}^N$. The* normal *function induced by F and C is*

$$F_C \stackrel{\text{def}}{=} F \circ \pi_C + I - \pi_C$$

where I is the identity operator on $\mathbb{R}^N$.

Our applications will concern finding a zero of a normal function induced by a continuously differentiable function F and a nonempty, convex polyhedral set C. We point out that the type of differentiability is not relevant: a consequence of the vector mean value theorem [OR, Thm. 3.2.3] is that F is continuously Fréchet differentiable iff it is continuously Gâteaux differentiable.

We relate the normal function F_C to the set mapping $F + N_C$.

Lemma 5 *Suppose F, C are as in the Definition 4, and F is locally Lipschitz.*

The mapping F_C is Lipschitz invertible near x iff for some neighborhoods U, V of $\pi_C(x)$, $F_C(x)$ respectively, the set-valued mapping $(F + N_C)^{-1} \cap U$ is a Lipschitz function when restricted to V.

Proof It is well known [Bré, Ex. 2.8.2] that $c = \pi_C(x)$ iff $c \in C$ and

$$\langle x - c, c' - c\rangle \leq 0,\ \forall c' \in C;$$

thus $c = \pi_C(\xi + c)$ iff $\xi \in N_C(c)$. It follows that

$$\xi = F_C(x),\ c = \pi_C(x) \iff \xi \in (F + N_C)(c),\ x = \xi + (I - F)(c). \tag{2}$$

Let $x^0 \in X$ and $\xi^0 \stackrel{\text{def}}{=} F_C(x^0)$.

Suppose F_C is Lipschitz invertible near x^0; so for some $\delta > 0$ and neighborhood V^0 of ξ^0, $(F_C)^{-1} \cap (x^0 + 2\delta\mathbb{B})$ is a Lipschitz function from V^0 onto $x^0 + 2\delta\mathbb{B}$. Let

$$\begin{aligned} U &\stackrel{\text{def}}{=} (I - F)^{-1}(x^0 - \xi^0 + \delta\mathbb{B}), \\ V &\stackrel{\text{def}}{=} (F + N_C)(U) \cap (\xi^0 + \epsilon\mathbb{B}), \end{aligned}$$

where $\epsilon \in (0, \delta)$ is chosen such that $\xi^0 + \epsilon\mathbb{B} \subset V^0$. Observe U is a neighborhood of $\pi_C(x^0)$ since $(I - F)(\pi_C(x^0)) = x^0 - \xi^0$. Using (2) we find that $(F + N_C)(U)$ equals $F_C\pi_C{}^{-1}(U)$, so for $\xi \in V$

$$\emptyset \neq (F + N_C)^{-1}(\xi) \cap U = [\pi_C(F_C)^{-1}(\xi)] \cap U. \tag{3}$$

We also see that V, the intersection of $F_C\pi_C{}^{-1}(U)$ and $\xi^0 + \epsilon\mathbb{B}$, is a neighborhood of ξ^0: $\pi_C{}^{-1}(U)$ is a neighborhood of x^0, so Lipschitz invertibility of F_C near x^0 yields $F_C\pi_C{}^{-1}(U)$ is a neighborhood of ξ^0. Moreover for $\xi \in V$ we have

$$\begin{aligned} & x \in (F_C)^{-1}(\xi),\ \pi_C(x) \in U \\ \Longrightarrow\quad & x = \xi + (I - F)(\pi_C(x)),\ (I - F)(\pi_C(x)) \in x^0 - \xi^0 + \delta\mathbb{B} \\ \Longrightarrow\quad & x \in x^0 + \xi - \xi^0 + \delta\mathbb{B} \subset x^0 + 2\delta\mathbb{B}. \end{aligned}$$

Thus

$$[\pi_C(F_C)^{-1}(\xi)] \cap U \subset \pi_C[(F_C)^{-1}(\xi) \cap (x^0 + 2\delta\mathbb{B})].$$

Since the set on the right is a singleton, this inclusion combined with (3) yields

$$(F + N_C)^{-1}(\xi) \cap U = \pi_C[(F_C)^{-1}(\xi) \cap (x^0 + 2\delta\mathbb{B})].$$

In particular $(F + N_C)^{-1} \cap U$, as a mapping on V, is a Lipschitz function.

Conversely suppose $(F + N_C)^{-1} \cap U$ is Lipschitz on V, where U and V are respective neighborhoods of $\pi_C(x^0)$ and ξ^0. Let U^0 be neighborhood of $\pi_C(x^0)$ contained in U, on which F is Lipschitz. So $(F+N_C)^{-1} \cap U^0$ is a Lipschitz function on $V^0 \stackrel{\text{def}}{=} (F+N_C)(U^0)$. In fact V^0 is a neighborhood of ξ^0 since U^0 is a neighborhood of $\pi_C(x^0) = (F + N_C)^{-1}(\xi^0) \cap U^0$, $(F+N_C)^{-1} \cap U^0$ a continuous map on V, and V is a neighborhood of ξ^0. Let $U^1 \stackrel{\text{def}}{=} \pi_C{}^{-1}(U^0)$, a neighborhood of x^0. Again using (2), we have

$$(F_C)^{-1} \cap U^1 = I + [I - F][(F + N_C)^{-1} \cap U^0].$$

So $(F_C)^{-1} \cap U^1$ is a Lipschitz function on V^0, and it follows that F_C is Lipschitz invertible near x^0. □

Finally, we have a generalization of the Banach perturbation lemma from [Rob88].

Lemma 6 *Let Ω be a set in X. Let g and g' be functions from X into Y such that $g|_\Omega : \Omega \to g(\Omega)$ has an inverse that is Lipschitz of modulus $L \geq 0$, and $g - g'$ is Lipschitz on Ω of modulus $\eta \geq 0$. Let $x_0 \in \Omega$. If*

a. $\Omega \supset x^0 + \delta\mathbb{B}_X$ for some $\delta > 0$,

b. $g(\Omega) \supset g(x^0) + \epsilon\mathbb{B}_Y$ for some $\epsilon > 0$, and

c. $\eta L < 1$,

then $g'|_\Omega : \Omega \to g'(\Omega)$ has an inverse that is Lipschitz of modulus $L/(1 - L\eta) > 0$, and

$$g'(\Omega) \supset g'(x^0) + (1 - \eta L)\epsilon\mathbb{B}_Y.$$

Proof Define the perturbation function, $h(\cdot) = g'(\cdot) - g(\cdot) - [g'(x^0) - g(x^0)]$. Let $\hat{g} \stackrel{\text{def}}{=} g|_\Omega$ and observe that the Lipschitz property of $\hat{g}^{-1}$ gives

$$\begin{aligned} 1/L &\leq 1/\sup\left\{\|\hat{g}^{-1}(y) - \hat{g}^{-1}(y')\|/\|y - y'\| \mid y, y' \in g(\Omega),\ y \neq y'\right\} \\ &= \inf\{\|g(x) - g(x')\|/\|x - x'\| \mid x, x' \in \Omega,\ x \neq x'\}. \end{aligned}$$

Thus according to [Rob88, Lemma 2.3], $g + h$ satisfies

$$(1/L) - \eta \leq \inf\{\|(g+h)(x) - (g+h)(x')\|/\|x - x'\| \mid x, x' \in \Omega,\ x \neq x'\}$$

and $(g+h)(\Omega)$ contains $g(x^0)+(1-\eta L)\epsilon\mathrm{IB}_Y$. Therefore $(g+h)|_\Omega : \Omega \to (g+h)(\Omega)$ is invertible and, similar to the above, its inverse is Lipschitz of modulus $1/[(1/L) - \eta] = L/(1 - L\eta)$. The claimed properties of g' hold because $g' = g + h + g'(x^0) - g(x^0)$.

□

3 Motivation: Line Search Damped Newton's Method for Smooth Equations

Let $f : X \to Y$ be smooth, that is continuously differentiable. We wish to solve the nonlinear equation

$$f(x) = 0, \quad x \in X.$$

Suppose $x^k \in X$ ($k \in \{0, 1, \ldots\}$) and there exists *Newton's iterate* $\hat{x}^{k+1}$, i.e. $\hat{x}^{k+1}$ solves the equation

$$f(x^k) + \nabla f(x^k)(x - x^k) = 0, \quad x \in X.$$

Newton's method is inductively given by setting $x^{k+1} \stackrel{\text{def}}{=} \hat{x}^{k+1}$.

The algorithm is also called *local* Newton's method because the Kantorovich-Newton theorem [OR, Thm. 12.6.2] — probably the best known convergence result for the Newton's method — shows convergence of Newton's iterates to a solution in a δ-ball ($\delta > 0$) of the starting point x^0. Assumptions include that $\nabla f(x^0)$ is boundedly invertible, and δ is small enough to ensure, by continuity of ∇f, that $\nabla f(x)$ is boundedly invertible at each $x \in x^0 + \delta\mathrm{IB}_X$. It is well known, however, that the domain of convergence of the algorithm can be substantially enlarged by the use of line search damping, described below, which preserves the asymptotic convergence properties of the local method.

We have, by choice of $\hat{x}^{k+1}$,

$$\begin{aligned} f(x^k + t(\hat{x}^{k+1} - x^k)) &= f(x^k) + t\nabla f(x^k)(\hat{x}^{k+1} - x^k) + o(t) \\ &= (1 - t)f(x^k) + o(t). \end{aligned}$$

The operand on the left side of the equation

$$x^k + t(\hat{x}^{k+1} - x^k)$$

which we denote by $p^k(t)$, is just a path from the last iterate x^k to a zero $\hat{x}^{k+1}$ of the approximating function $A_k : x \mapsto f(x^k) + \nabla f(x^k)(x - x^k)$ such that

$$f(p^k(t)) = (1-t)f(x^k) + o(t) \tag{4}$$

I.e. f moves quickly toward zero along the path p^k as t increases from 0, at least initially.

In the Armijo procedure for finding the step length t_k, familiar in optimization [McC, Ch 6 §1], we fix $\sigma, \tau \in (0,1)$ and, assuming $f(x^k) \neq 0$ (x^k is not a solution point), observe that for all sufficiently small positive t

$$\|o(t)\| < t(1-\sigma)\|f(x^k)\|.$$

With (4) we deduce for small positive t that we have Monotone Descent of the residual $\|f(p^k(t)\|$ from $\|f(x^k)\|$:

$$\|f(p^k(t))\| < (1-\sigma t)\|f(x^k)\| \tag{MD}$$

hence there is a least $l = l(k) \in \{0,1,\ldots\}$ such that (MD) holds for $t = \tau^l$. We take $t_k \stackrel{\text{def}}{=} \tau^l$ and *damped* Newton's iterate to be $x^{k+1} \stackrel{\text{def}}{=} p^k(t_k)$. This kind of procedure, which determines the step length t_k by checking function values on the line segment from x^k to $\hat{x}^{k+1}$, is a *line search damping* of Newton's method.

The Armijo procedure, like other standard line search methods, is effective because it finds $t_k \in [0,1]$ such that, firstly, progress toward a solution is made (eg. (MD) holds at $t = t_k$) and, secondly, given some other conditions, the progress is sufficient to prevent premature convergence to a nonsolution point. The Armijo procedure monotone in the sense that the sequence of residuals ($\|f(x^k)\|$) decreases monotonically. We abstract the general properties of an unspecified Monotone Linesearch procedure:

(MLs)

If (MD) holds at $t = 1$, let $t_k \stackrel{\text{def}}{=} 1$.

Otherwise, choose any $t_k \in [0,1]$ such that (MD) holds at $t = t_k$ and

$$t_k \geq \tau \sup\{T \in [0,1] \mid \text{(MD) holds } \forall t \in [0,T]\}$$

The conditions on t_k ensure that a Newton iterate ($t_k = 1$) is taken if possible, otherwise t_k is at least some constant fraction (τ) of the length of the largest "acceptable" interval containing $t = 0$. It is easy to see that the Armijo line search described above produces a step length that fulfills (MLs). The parameter τ need not be explicitly used in damped Newton's algorithm so other line search procedures in which τ is not specified may be valid; only the existence of τ, independent of the iterate, is needed.

More recently, in the context of unconstrained optimization, Grippo et al [GLL] have developed a line search using a Nonmonotone Descent condition that often gives better computational results than monotone damping. Let $M \in \mathbb{N}$, the *memory* length of the procedure, and relax the progress criterion (MD) to

$$\|f(p^k(t))\| < (1-\sigma t)\max\{\|f(x^{k+1-j})\| \mid j = 1,\ldots,M,\ j \leq k+1\} \tag{NmD}$$

The NonMonotone Line search is

(NmLs)

If (NmD) holds at $t = 1$, let $t_k \stackrel{\text{def}}{=} 1$.

Otherwise, choose any $t_k \in [0,1]$ such that (NmD) holds at $t = t_k$ and

$$t_k \geq \tau \sup\{T \in [0,1] \mid \text{(NmD) holds } \forall t \in [0,T]\}$$

Clearly (NmLs) is identical to (MLs) when $M = 1$.

The formal algorithm is given below.

Line search damped Newton's method. Given $x^0 \in X$, the sequence (x^k) is inductively defined for $k = 0, 1, \ldots$ as follows.

If $f(x^k) = 0$, **stop.**

Find $\hat{x}^{k+1} \stackrel{\text{def}}{=} x^k - \nabla f(x^k)^{-1} f(x^k)$.

Line search: Let $p^k(t) \stackrel{\text{def}}{=} x^k + t(\hat{x}^{k+1} - x^k)$ for $t \in [0,1]$. Find $t_k \in [0,1]$ satisfying (NmLs).

Define $x^{k+1} \stackrel{\text{def}}{=} p^k(t_k)$.

We present a basic convergence result for the line search damped, or so-called global Newton's method. It is a corollary of Proposition 9.

Proposition 7 *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be continuously differentiable, $\alpha_0 > 0$ and*

$$X_0 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \mid \|f(x)\| \leq \alpha_0\}.$$

Let $\sigma, \tau \in (0,1)$ and $M \in \mathbb{N}$ be the line search parameters governing the condition (NmLs).

Suppose X_0 is bounded and $\nabla f(x)$ is invertible for each $x \in X_0$. independent of x, Then for each x^0 in X_0, line search damped Newton's method is well-defined and the sequence (x^k) converges to a zero x^* of f.

The residual converges to zero at least at an R-linear rate: for some constant $\rho \in (0,1)$ and all k,

$$\|f(x^k)\| \leq \rho^k \|f(x^0)\|.$$

The rate of convergence of (x^k) to x^* is Q-superlinear. In particular, if ∇f is Lipschitz near x^* then (x^k) converges Q-quadratically to x^*:

$$\|x^{k+1} - x^*\| \leq d\|x^k - x^*\|^2$$

for some constant $d > 0$ and all sufficiently large k.

The convergence properties of the method depend both on the uniform accuracy of each of the approximations A_k of f at x^k for all k, i.e.

$$\|f(x) - A_k(x)\| / \|x - x^k\| \to 0 \quad \text{as } x \to x^k \ (x \neq x^k), \quad \text{uniformly } \forall k$$

and on the uniform invertibility of the approximations A_k:

$$\|\nabla f(x^k)^{-1}\| \text{ is bounded above, independent of } k.$$

These uniformness properties are disguised in the boundedness (hence compactness) hypothesis on X_0.

4 Path Search Damped Newton's Method for Nonsmooth Equations

We want to solve the nonlinear and, in general, nonsmooth equation

$$f(x) = 0, \quad x \in X$$

where $f : X \to Y$. We proceed as in the smooth case, the main difference being the use of first-order approximations of the function f instead linearizations.

Suppose $x^k \in X$ ($k \in \{0, 1, \ldots\}$) and A_k is a first-order approximation of f at x^k. Recall (Definition 2) a path is a continuous mapping of the form $p : [0, T] \to X$ where $T \in [0, 1]$. Assume there exists a path $p^k : [0, 1] \to X$ such that, for $t \in [0, 1]$,

$$A_k(p^k(t)) = (1 - t)f(x^k) \tag{5}$$

and $\|p^k(t) - x^k\| = O(t)$. Note that $\hat{x}^{k+1} \stackrel{\text{def}}{=} p^k(1)$ is a solution of the equation $A_k(x) = 0$.

In nonsmooth Newton's method, the next iterate is given by $x^{k+1} \stackrel{\text{def}}{=} \hat{x}^{k+1}$ just as in smooth Newton's method. For nonsmooth functions having a uniform first-order approximation we call this Robinson-Newton's method since the ideas behind convergence results are essentially the same as those employed in the seminal paper [Rob88], although a special uniform first-order approximation called the *point-based approximation* is required there (see discussion after Proposition 9). [Rob88] provides the corresponding version of the Kantorovich-Newton convergence theorem for nonsmooth equations. Applications of Robinson-Newton's method include sequential quadratic programming [Fle], or Wilson's method [Wil] for nonlinear programming; and Josephy-Newton's method [Jos] for generalized equations (see also §5). As in the smooth case, however, convergence of the method is shown within a ball of radius $\delta > 0$ about x^0. The point-based approximation $A(x^0)$ of f at x^0 is assumed to have a Lipschitz inverse such that, by the continuity properties of $A(\cdot)$, $A(x)$ is Lipschitz invertible near each $x \in x^0 + \delta \mathbb{B}_X$. We propose to enlarge the domain of convergence by means of a suitable damping procedure.

Now A_k is a first-order approximation of f at x^k, and $o(p^k(t)) = o(t)$ since $p^k(t) = O(t)$. With (5) we find that

$$f(p^k(t)) = (1 - t)f(x^k) + o(t) \tag{6}$$

i.e. f moves toward zero rapidly on the path p^k as t increases from 0, at least initially.

In the spirit of §3, we fix $\sigma, \tau \in (0,1)$. As before, assuming $f(x^k) \neq 0$, we have for all sufficiently small positive t

$$\|o(t)\| < t(1-\sigma)\|f(x^k)\|$$

hence with (6),

$$\|f(p^k(t))\| < (1-\sigma t)\|f(x^k)\|.$$

So the nonmonotone descent condition below, identical to that given in §3, is valid given any memory size $M \in \mathbb{N}$ and all small positive t:

$$\|f(p^k(t))\| < (1-\sigma t)\max\{\|f(x^{k+1-j})\| \mid j = 1,\ldots,M,\ j \leq k+1\} \qquad \text{(NmD)}$$

The path search is any procedure satisfying

If (NmD) holds at $t = 1$, let $t_k \stackrel{\text{def}}{=} 1$.

Otherwise, choose any $t_k \in [0,1]$ such that (NmD) holds at $t = t_k$ and

$$t_k \geq \tau \sup\{T \in [0,1] \mid \text{(NmD) holds } \forall t \in [0,T]\}$$

This path search takes $t_k = 1$, hence Newton's iterate $x^{k+1} = p^k(1)$, if possible, otherwise a path length t_k large enough to prevent premature convergence under further conditions. Also, as in §3, τ need not be used or specified explicitly.

However the path search given above is too restrictive in practise; in particular it assumes existence of Newton's iterate $\hat{x}^{k+1} \in A_k{}^{-1}(0)$. Motivated by computation (§5) we only assume the path $p^k : [0,T_k] \to X$ can be constructed for some $T_k \in (0,1]$ by iteratively extending its domain $[0,T_k]$ until either it cannot be extended further (eg. $T_k = 1$) or the progress criterion (NmD) is violated at $t = T_k$. The path length t_k is then chosen with reference to this upper bound T_k. Of course we are still assuming that the path p^k satisfies the Path conditions:

$$\begin{aligned} &p^k(0) = x^k \\ &A_k(p^k(t)) = (1-t)f(x^k), \quad \forall t \in \text{dom}(p^k) \end{aligned} \qquad \text{(P)}$$

where $\text{dom}(p^k) = [0,T_k]$. The idea of extending the path p^k is explained by Lemma 3, taking $\Phi = A_k$ and $x = p^k(T_k)$, which says that if A_k is continuously invertible near $p^k(T_k)$ and $T_k < 1$, then p^k can be defined over a larger domain (i.e. T_k is strictly increased) while still satisfying (P).

We use the following Nonmonotone Pathsearch in which $p^k : [0,T_k] \to X$ is supposed to satisfy (P).

(NmPs)

If (NmD) holds at $t = T_k$, let $t_k \stackrel{\text{def}}{=} T_k$.

Otherwise, choose any $t_k \in [0,T_k]$ such that (NmD) holds at $t = t_k$ and

$$t_k \geq \tau \sup\{T \in [0,T_k] \mid \text{(NmD) holds } \forall t \in [0,T]\}$$

Now we give the algorithm formally. In general (§5) the choice of t_k is partly determined during the construction of p^k, hence we have not separated the construction of p^k from the path search in the algorithm. On this point the smooth and nonsmooth damped Newton's methods differ.

Path search damped Newton's method. Given $x^0 \in X$, the sequence (x^k) is inductively defined for $k = 0, 1, \ldots$ as follows.

If $f(x^k) = 0$, **stop.**

Path search: Let $A_k \stackrel{\text{def}}{=} A(x^k)$. Construct a path $p^k : [0, T_k] \to X$ satisfying (P) such that if $T_k < 1$ then either A_k is not continuously invertible near $p^k(T_k)$, or (NmD) fails at $t = T_k$. Find $t_k \in [0, 1]$ satisfying (NmPs).

Define $x^{k+1} \stackrel{\text{def}}{=} p^k(t_k)$.

Our main result, showing the convergence properties of global Newton's method, is now given. The first two assumptions on the first-order approximation A correspond, in the smooth case, to uniform continuity of ∇f on X_0 and uniformly bounded invertibility of $\nabla f(x)$ for $x \in X_0$, respectively. The purpose of the third, technical assumption is to guarantee the existence of paths used by the algorithm (cf. the *continuation* property of [Rhe]). With the exception of dealing with the paths p^k, the proof uses techniques well developed in standard convergence theory of damped algorithms (eg. [OR, McC, Fle]).

Theorem 8 *Let $f : X \to Y$ be continuous, $\alpha_0 > 0$ and $X_0 \stackrel{\text{def}}{=} \{x \in X \mid \|f(x)\| \leq \alpha_0\}$. Let $\sigma, \tau \in (0,1)$ and $M \in \mathbb{N}$ be the parameters governing the path search condition condition (NmPs).*

Suppose

1. *A is a uniform first-order approximation of f on X_0.*
2. *$A(x)$ is uniformly Lipschitz invertible near each $x \in X_0$, meaning for some constants $\delta, \epsilon, L > 0$ and for each $x \in X_0$, there are sets U_x and V_x containing $x + \delta \mathbb{B}_X$ and $f(x) + \epsilon \mathbb{B}_Y$ respectively, such that $A(x)|_{U_x} : U_x \to V_x$ has an inverse that is Lipschitz of modulus L.*
3. *For each $x \in X_0$, if $p : [0, T) \to X$ ($T \in (0, 1]$) is continuous with $p(0) = x$ such that $A(x)(p(t)) = (1 - t)f(x)$ and $A(x)$ is continuously invertible near $p(t)$ for each $t \in [0, T)$, then there exists $p(T) \stackrel{\text{def}}{=} \lim_{t \uparrow T} p(t)$ with $A(x)(p(T)) = (1 - T)f(x)$.*

Then for any $x^0 \in X_0$, path search damped Newton's method is well defined such that the sequence (x^k) converges to a zero x^* of f.

The residual converges to zero at least at an R-linear rate: for some constant $\rho \in (0,1)$ and all k,

$$\|f(x^k)\| \leq \rho^k \|f(x^0)\|.$$

The rate of convergence of (x^k) to x^* is at least as high as the rate of convergence of the error $(A_k(x^*) - f(x^*))$ to 0, hence is Q-superlinear. In particular, if for $c > 0$ and all points x near x^* we have $\|A(x)(x^*) - f(x^*)\| \leq c\|x - x^*\|^2$, then convergence is Q-quadratic:

$$\|x^{k+1} - x^*\| \leq cL\|x^k - x^*\|^2$$

for sufficiently large k.

Proof We begin by showing the algorithm is well defined. Suppose $x \in X_0$. Lemma 3 can be used to show existence of a (unique) continuous function $p : I \to Y$ of largest domain I, with respect to the conditions

- $p(0) = x$;
- either $I = [0,1]$, or $I = [0,T)$ for some $T \in (0,1]$; and
- for each $t \in I$

$$A(x)(p(t)) = (1-t)f(x),$$
$$A(x) \text{ is continuously invertible near } p(t).$$

If $I = [0,1]$, $p^k = p$ is a path acceptable to the algorithm if $x^k = x$. If $I = [0,T)$ then, by assumption 3, we can extend p continuously to domain $[0,T]$ by $p(T) \stackrel{\text{def}}{=} \lim_{t \uparrow T} p(t)$, for which $A(x)(p(T)) = (1-T)f(x)$. In this case, by maximality of I, $A(x)$ is not continuously invertible at $p(T)$ so the extension $p : [0,T] \to Y$ is acceptable as p^k if $x^k = x$. So it is enough that each $x^k \in X_0$, which is easy to show by induction.

Assume, without of generality, that $f(x^k) \neq 0$ for each k. Let δ, ϵ, L be the constants given by hypothesis 2 of the theorem and, for each k, $\hat{A}_k$ be the Lipschitz invertible mapping $A(x^k)|_{U_{x^k}} : U_{x^k} \to V_{x^k}$ given there. Recall that $p^k : [0,T_k] \to Y$, $0 \leq T_k \leq 1$, is the path determined by the algorithm.

We aim to find a positive constant γ such that for each k,

$$T_k \geq S_k \stackrel{\text{def}}{=} \min\{\gamma/\|f(x^k)\|, 1\}$$

and

$$p^k(t) = \hat{A}_k^{-1}((1-t)f(x^k)) \quad \forall t \in [0, S_k] \tag{7}$$
$$\text{(NmD) holds} \quad \forall t \in [0, S_k] \tag{8}$$

To show these we need several other properties of the path search, the first of which is given by Lemma 3 when $\Phi \stackrel{\text{def}}{=} A_k$ and $U \stackrel{\text{def}}{=} U_{x^k}$:

$$p^k(t) = \hat{A}_k^{-1}((1-t)f(x^k)), \quad \forall 0 \leq t \leq \min\{\epsilon/\|f(x^k)\|, T_k\}. \tag{9}$$

Now if $T_k < 1$ and (NmD) holds at $t = T_k$ then, by choice of p^k, A_k is not continuously invertible at T_k; thus

$$T_k \geq \min\{\epsilon/\|f(x^k)\|, 1\}.$$

Another fact is that

$$\text{(NmD) holds for } 0 \le t \le \min\{\gamma/\|f(x^k)\|, T_k\} \tag{10}$$

where $\gamma \in (0, \epsilon]$ will be specified below. Let $\Delta(t) = o(t)$ be the uniform bound on the accuracy of $A(x)$, $x \in X_0$, as given by Definition 1.2. Recall the path search parameter $\sigma \in (0,1)$. We choose $\hat{\beta} > 0$ such that $\Delta(\beta) < \beta(1-\sigma)/L$ for $0 < \beta \le \hat{\beta}$. Then for

$$0 \le t \le \min\left\{\frac{\hat{\beta}}{L\|f(x^k)\|}, \frac{\epsilon}{\|f(x^k)\|}, T_k\right\}$$

we have

$$\begin{aligned} \|p^k(t) - x^k\| &= \|\hat{A}_k^{-1}((1-t)f(x^k)) - \hat{A}_k^{-1}(f(x^k))\| \quad \text{by (9)} \\ &\le L\|(1-t)f(x^k) - f(x^k)\| = tL\|f(x^k)\| \quad \text{by assumption 2.} \end{aligned}$$

So for such t, $\|p^k(t) - x^k\| \le \hat{\beta}$ hence

$$\Delta(\|p^k(t) - x^k\|) < \|p^k(t) - x^k\|(1-\sigma)/L \le t(1-\sigma)\|f(x^k)\| \tag{11}$$

and furthermore

$$\begin{aligned} \|f(p^k(t))\| &\le \|A_k(p^k(t))\| + \Delta(\|p^k(t) - x^k\|) \\ &\quad \text{by assumption 1} \\ &< (1-t)\|f(x^k)\| + t(1-\sigma)\|f(x^k)\| = (1-\sigma t)\|f(x^k)\| \\ &\quad \text{by choice of } p^k \text{ and (11)} \\ &\le (1-\sigma t)\max\{\|f(x^{k+1-j})\| \mid j = 1, \ldots, M,\ j \le k+1\}. \end{aligned}$$

Let $\gamma \stackrel{\text{def}}{=} \min\{\hat{\beta}/L, \epsilon\}$ $(\in (0, \epsilon])$. We have verified (10).

If $T_k < 1$ and (NmD) holds at $t = T_k$ we have already seen that T_k is greater than or equal to $\min\{\epsilon/\|f(x^k)\|, 1\}$, hence $T_k \ge S_k$. If (NmD) is violated at $t = T_k$ then, with the statement (10), we see that $T_k > \gamma/\|f(x^k)\|$. So we always have $T_k \ge S_k$. Statement (7) now immediately follows from (9) because $\epsilon \ge \gamma$ and $T_k \ge S_k$. Likewise, statement (8) immediately follows from (10).

Next we show that $t_k \ge \tau S_k$ for every k (recall $\tau \in (0,1)$ is another path search parameter). From the rules of (NmPs), if (NmD) holds at $t = T_k$ then $t_k \stackrel{\text{def}}{=} T_k \ge S_k \ge \tau S_k$; otherwise t_k is chosen to satisfy

$$\begin{aligned} t_k &\ge \tau \sup\{T \in [0, T_k] \mid \text{(NmD) holds } \forall t \in [0, T]\} \\ &\ge \tau S_k \quad \text{by (8).} \end{aligned}$$

Since each iterate x^k belongs to X_0, we get

$$t_k \ge \tau \min\{\gamma/\|f(x^k)\|, 1\} \ge \hat{t} \stackrel{\text{def}}{=} \tau \min\{\gamma/\alpha_0, 1\} > 0.$$

Therefore, by a short induction argument, for each k

$$\|f(x^k)\| \leq \rho^k \|f(x^0)\| \tag{12}$$

where $\rho \stackrel{\text{def}}{=} (1-\sigma\hat{t})^{1/M}$. This validates the claim of linear convergence of the residuals. As a result, for some $K_1 > 0$ and each $k \geq K_1$

$$\gamma/\|f(x^k)\| \geq 1$$

whence $S_k = 1$. So $p^k(1) = \hat{A}_k^{-1}(0)$ (by (7)) and (NmD) holds at $t = 1$ (by (8)). Also $T_k = 1$, since $1 = S_k \leq T_k \leq 1$, hence (NmPs) determines $t_k \stackrel{\text{def}}{=} 1$ and damped Newton's iterate *is* Newton's iterate: $x^{k+1} = \hat{A}_k^{-1}(0)$. For $k \geq K_1$

$$\begin{aligned} \|x^{k+1} - x^k\| &= \|\hat{A}_k^{-1}(0) - \hat{A}_k^{-1}(f(x^k))\| \\ &\leq L\|0 - f(x^k)\| && \text{by assumption 2} \\ &\leq L\rho^k\|f(x^0)\| && \text{by (12).} \end{aligned}$$

So if $K, K' \in \mathbb{N}$, $K_1 \leq K \leq K'$, then

$$\begin{aligned} \|x^K - x^{K'}\| &\leq \sum_{k=K}^{\infty} \|x^{k+1} - x^k\| \\ &\leq \sum_{k=K}^{\infty} L\rho^k\|f(x^0)\| \\ &= \frac{L\|f(x^0)\|}{1-\rho}\rho^K \\ &\to 0 \text{ as } K \to \infty, K' \geq K. \end{aligned}$$

This shows that (x^k) is a Cauchy sequence, hence convergent in the complete normed space X with limit, say, x^*. Since $\|f(x^k)\| \to 0$, continuity of f yields $f(x^*) = 0$.

Finally note that for all sufficiently large k, $x^* \in x^k + \delta\mathbb{B}_X \subset U_{x^k}$. For such $k \geq K_1$, assumption 2 yields

$$\begin{aligned} \|x^{k+1} - x^*\| &= \|\hat{A}_k^{-1}(f(x^*)) - \hat{A}_k^{-1}(A_k(x^*))\| \\ &\leq L\|f(x^*) - A_k(x^*)\| \\ &\leq L\Delta(\|x^k - x^*\|). \end{aligned}$$

The second inequality demonstrates Q-superlinear convergence. With the first inequality we see that if $\|A(x)(x^*) - f(x^*)\| \leq c\|x - x^*\|^2$ for some $c > 0$ and all x near x^*, then

$$\|x^{k+1} - x^*\| \leq cL\|x^k - x^*\|^2$$

for sufficiently large k.

□

We will find the following version of Theorem 8 useful in applications (§5).

Proposition 9 *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be continuous, $\alpha_0 > 0$ and*

$$X_0 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \mid \|f(x)\| \leq \alpha_0\}.$$

Let $\sigma, \tau \in (0,1)$ and $M \in \mathbb{N}$ be the parameters governing the path search condition condition (NmPs).

Suppose X_0 is bounded and for each $x \in X_0$ the following hold:

1. *A is a uniform first-order approximation of f near x.*
2. *$A(x)$ is Lipschitz invertible near x.*
3. *$A(x)$ is piecewise linear.*
4. *There exists $\eta_x(s) \geq 0$ for $s \geq 0$, such that $\lim_{s \downarrow 0} \eta_x(s) = 0$ and $A(x^1) - A(x^2)$ is Lipschitz of modulus $\eta_x(\|x^1 - x^2\|)$ near x, for x^1, x^2 near x.*

Then for $X = Y = \mathbb{R}^N$, the hypotheses (and conclusions) of Theorem 8 hold.

Proof We first strengthen hypothesis 2: each $x \in X_0$ has a neighborhood U_x such that

2′. For some scalars $\epsilon_x, L_x > 0$ and each $x' \in U_x$, the mapping

$$A(x')|_{U_x} : U_x \to A(x')(U_x)$$

has an inverse that is Lipschitz of modulus L_x, and $A(x')(U_x)$ contains $f(x') + \epsilon_x \mathbb{B}_Y$.

To see this appeal to hypotheses 2 and 4. There are neighborhoods U, V of x, $f(x)$ respectively and $L > 0$ for which $A|_U : U \to V$ has an inverse that is Lipschitz of modulus $L > 0$; and there is $\eta : [0,\infty) \to [0,\infty)$ such that $\lim_{s \downarrow 0} \eta(s) = 0$ and $A(x^1) - A(x^2)$ is Lipschitz of modulus $\eta(\|x^1 - x^2\|)$ for $x^1, x^2 \in U$. Choose $\hat{s} > 0$ such that $x + \hat{s}\mathbb{B}_X \subset U$ and

$$\eta(s) \leq 1/(2L), \ \ \forall s \in [0, \hat{s}].$$

Let $\epsilon > 0$ satisfy $f(x) + \epsilon \mathbb{B}_Y \subset V$. Then for $x' \in x + \hat{s}\mathbb{B}_X$, Lemma 6 says $A(x')|_U : U \to A(x')(U)$ is an invertible mapping, its inverse is Lipschitz of modulus $2L$, and $A(x')(U)$ contains $f(x') + (\epsilon/2)\mathbb{B}_Y$.

Let $U_x \stackrel{\text{def}}{=} x + \hat{s}\mathbb{B}_X$, $L_x \stackrel{\text{def}}{=} 2L$ and $\epsilon_x \stackrel{\text{def}}{=} \min\{\epsilon/2, \hat{s}/(2L)\}$. Then for each $x' \in U_x$, Lipschitz continuity of $(A(x')|_U)^{-1}$ gives

$$(A(x')|_U)^{-1}[A(x')(f(x') + \epsilon_x \mathbb{B}_Y)] \subset x' + 2L\epsilon_x \mathbb{B}_X \subset U_x.$$

As $f(x') + \epsilon_x \mathbb{B}_Y \subset A(x')(U)$ this yields $f(x') + \epsilon_x \mathbb{B}_Y \subset A(x')(U_x)$. 2′ is confirmed.

By 1 and the above, each $x \in X_0$ has a neighborhood U_x such that for some $\Delta_x(t) = o(t)$

$$\|A(x')(x'') - f(x'')\| \leq \Delta_x(\|x' - x''\|), \ \ \forall x', x'' \in U_x \tag{13}$$

and 2′ holds. Since X_0 is compact we may cover it by finitely many such neighborhoods (U_{x^i}) corresponding to a finite sequence $(x^i) \subset X_0$. For each i we have the $o(t)$ function $\Delta_{x^i}(t)$ satisfying (13), and the scalars ϵ_{x^i}, L_{x^i} of 2′. Now there exists $\delta > 0$ such that for each $x \in X_0$, $x + \delta \mathrm{IB}_X \subset U_{x^i}$ for some i; if not, sequential compactness of X_0 leads to an easy contradiction. Let

$$\Delta(t) \stackrel{\text{def}}{=} \begin{cases} \max_i \Delta_{x^i}(t) & \text{if } 0 \le t \le \delta, \\ \sup\{\|A(x')(x'') - f(x'')\| \mid x', x'' \in X_0,\ \|x' - x''\| \le t\} & \text{if } t > \delta. \end{cases}$$

Note Δ is finite valued because $\|A(x')(x'') - f(x'')\|$ is continuous in (x', x'') on the compact set $\{(x', x'') \in X_0 \times X_0 \mid \|x' - x''\| \le t\}$. Since any two points of X_0 at a distance from one another of less than δ lie in some U_{x^i}, $\Delta(t) = o(t)$ and

$$\|A(x')(x'') - f(x'')\| \le \Delta(\|x' - x''\|) \ \ \forall x', x'' \in X_0.$$

So hypothesis 1 of Theorem 8 holds.

Hypothesis 2 of Theorem 8 also holds, with δ as already defined, $\epsilon \stackrel{\text{def}}{=} \min_i \epsilon_{x^i}$, and $L \stackrel{\text{def}}{=} \max_i L_{x^i}$. For each $x \in X_0$ we may take $U_x \stackrel{\text{def}}{=} U_{x^i}$, where $x + \delta \mathrm{IB}_X \subset U_{x^i}$, and $V_x \stackrel{\text{def}}{=} A(x)(U_{x^i})$ $(\supset f(x) + \epsilon \mathrm{IB}_Y)$.

Hypothesis 3 of Theorem 8 follows from piecewise linearity of $A(x)$, and the piecewise linearity of

$$p(t) \stackrel{\text{def}}{=} A(x)^{-1}((1-t)f(x)), \ \ \forall t \in [0, T)$$

for suitable $T \in (0, 1]$.

□

Hypotheses 1 and 4 of the proposition specify a weak local version of the point-based approximation of f at x, used to define Robinson-Newton's method [Rob88]. A point-based approximation of f on $\Omega \subset X$ is a function A on Ω, each value $A(x)$ of which is itself a mapping from Ω to Y, such that for some $\kappa \ge 0$ and every $x^1, x^2 \in \Omega$,

a. $\|A(x^1)(x^2) - f(x^2)\| \le (1/2)\kappa \|x^1 - x^2\|^2$, and

b. $A(x^1) - A(x^2)$ is Lipschitz of modulus $\kappa \|x^1 - x^2\|$.

Suppose A is a point-based approximation of f on a neighborhood Ω of x, and for x^1 near x we extend the domain of each $A(x^1)$ to X by arbitrarily defining the values $A(x^1)(x^2)$, $x^2 \in X \backslash \Omega$. Then property a implies hypothesis 1, and property b implies hypothesis 4.

5 Applications

The Variational Inequality is the problem of finding a vector $z \in \mathrm{IR}^N$ such that

$$\begin{aligned} & z \in C \\ & \langle F(z), c - z \rangle \ge 0, \quad \forall c \in C \end{aligned} \tag{VI}$$

where F is a function from $\mathbb{R}^N$ to $\mathbb{R}^N$ and C is a nonempty convex set in $\mathbb{R}^N$. For the purpose of implementation, we will also assume F is continuously differentiable and C is polyhedral. Harker's and Pang's paper [HP] is recommended for survey of variational inequalities, their analysis, and algorithms for their solution.

Equivalently we can solve the Generalized Equation [Rob79]

$$0 \in F(z) + N_C(z) \tag{GE}$$

where N_C is the normal cone to C at z (see §2); or the Normal Equation [Rob90]:

$$0 = F(\pi_C(x)) + x - \pi_C(x) \tag{NE}$$

where $\pi_C(x)$ is the projection of x to its nearest point in C. So (NE) is just $0 = F_C(x)$. We will work with (NE) because it is a (nonsmooth) equation.

Note that (VI) and (GE) are, but for notation, identical, whereas the equivalence between each of these two problems and (NE) is indirect: if z solves (VI) or (GE) then $x = z - F(z)$ solves (NE), while if x solves (NE) then $z = \pi_C(x)$ solves (VI) and (GE). In fact we have a stronger result from Lemma 5, assuming F is locally Lipschitz: F_C is Lipschitz invertible near x iff for some neighborhoods U, V of $\pi_C(x)$, $F_C(x)$ respectively, $(F + N_C)^{-1} \cap U$ is a Lipschitz function on V. This result has links to *strongly regular* generalized equations (see [Rob80], and proof of Proposition 11).

A natural first-order approximation of $f \stackrel{\text{def}}{=} F_C$ at x is obtained by linearizing F about $c \stackrel{\text{def}}{=} \pi_C(x)$:

$$A(x)(x') \stackrel{\text{def}}{=} F(c) + \nabla F(c)(\pi_C(x') - c) + x' - \pi_C(x'), \quad \forall x' \in \mathbb{R}^N. \tag{14}$$

So $A(x) = F(c) - \nabla F(c)(c) + \nabla F(c)_C$, a piecewise linear normal function. Furthermore, for any $x^1, x^2 \in \mathbb{R}^N$,

$$\begin{aligned}
&\|A(x^1)(x^2) - f(x^2)\| \\
&\quad \leq \sup_{0 \leq s \leq 1} \|\nabla F(s\pi_C(x^1) + (1-s)\pi_C(x^2)) - \nabla F(\pi_C(x^1))\| \|\pi_C(x^1) - \pi_C(x^2)\| \\
&\qquad \text{by the vector mean value theorem [OR, Thm. 3.2.3]} \\
&\quad = o(\|x^1 - x^2\|) \text{ as } x^1, x^2 \to x \\
&\qquad \text{by continuity of } \nabla F \text{ and Lipschitz continuity of } \pi_C \text{ [Bré, Ch. II].}
\end{aligned}$$

A similar argument further ensures that, for small positive t and

$$\eta_x(t) \stackrel{\text{def}}{=} \sup\{\|\nabla F(\pi_C(\xi^1)) - \nabla F(\pi_C(\xi^2))\| \mid \xi^1, \xi^2 \in x + t\mathbb{B}\},$$

$A(x^1) - A(x^2)$ is Lipschitz in $x + \|x^1 - x^2\|\mathbb{B}$ of modulus $\eta_x(\|x^1 - x^2\|)$. Continuity of ∇F yields that $\eta_x(t) \downarrow 0$ as $t \downarrow 0$. We have verified the first, third and fourth assumptions of Proposition 9 for each $x \in \mathbb{R}^N$.

To apply Proposition 9 it is left to find a constant $\alpha_0 > 0$ such that the level set

$$X_0 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \mid \|f(x)\| \leq \alpha_0\}$$

is bounded and only contains points x at which $A(x)$ is locally Lipschitz invertible. As we saw above, the first-order approximation $A(x)$ is the normal function $\nabla F(\pi_C(x))_C$ plus a constant vector. Robinson's homeomorphism theorem [Rob90] says that such a piecewise linear mapping is homeomorphic iff its determinants in each of its full dimensional linear pieces have the same (nonzero) sign. More importantly for us, [Rob90] also provides homeomorphism results near points x, via the critical cone to C at x. See also [Ral, Ch. 4]. These results provide testable conditions for the second assumption of Proposition 9.

Josephy-Newton's method. In Josephy-Newton's method [Jos] for solving (GE), given the kth iterate c^k, the next iterate is defined to be a solution $\hat{c}^{k+1}$ of the linearized generalized equation

$$0 \in F(c^k) + \nabla F(c^k)(c - c^k) + N_C(c).$$

The equivalence between Josephy-Newton's method on (GE) and Robinson-Newton's method on the associated (NE) is well known: if x^k is such that $\pi_C(x^k) = c^k$ and

$$\hat{x}^{k+1} \stackrel{\text{def}}{=} \hat{c}^{k+1} - F(c^k) - \nabla F(c^k)(\hat{c}^{k+1} - c^k),$$

we get $\hat{x}^{k+1}$ is a zero of $A(x^k)(\cdot)$ and $\hat{c}^{k+1} = \pi_C(\hat{x}^{k+1})$. I.e. Josephy-Newton's iterate is the projection of Robinson-Newton's iterate. Path search damping of Robinson-Newton's method produces x^{k+1} on the path p^k from x^k to $\hat{x}^{k+1}$. The projection $\pi_C(x^{k+1})$, on the path from c^k to $\hat{c}^{k+1}$, is a damped Josephy-Newton iterate.

The normal function $f = F_C$ belongs to a more general class of nonsmooth equations which have natural first-order approximations, namely the class of functions of the form

$$f = H \circ h$$

where $H : \mathbb{R}^K \to \mathbb{R}^N$ is smooth, and $h : \mathbb{R}^N \to \mathbb{R}^K$ is locally Lipschitz. This class of functions was introduced in [Rob88] in the context of point-based approximations. Similar to above, it is easy to see that the mapping

$$A(x)(x') \stackrel{\text{def}}{=} H(h(x)) + \nabla H(h(x))(h(x') - h(x))$$

is a first-order approximation of f at x satisfying assumptions 1 and 4 of Proposition 9. The normal function F_C is given by setting $K \stackrel{\text{def}}{=} 2N$, $h(x) \stackrel{\text{def}}{=} (\pi_C(x), x - \pi_C(x))$ and $H(a,b) \stackrel{\text{def}}{=} F(a) + b$.

Before specializing, let us describe the construction of the path p^k given the kth damped Newton's iterate x^k. As A_k is piecewise linear, p^k is also piecewise linear. We construct it, piece by affine piece, using a pivotal method. Starting from $t = 0$ ($p^k(0) = x^k$), ignoring

degeneracy, each pivot increases t to the next breakpoint (in the derivative) of p^k while maintaining the equation

$$A_k(p^k(t)) = (1-t)f(x^k);$$

thereby extending the domain of p^k. We continue to pivot so long as pivoting is possible and our latest breakpoint t satisfies the nonmonotone descent condition (NmD). If, after a pivot, (NmD) fails, then we line search on the interval $[p^k(t_{old}), p^k(t)]$ to find x^{k+1}, where t_{old} is the value of t at the last breakpoint. The line search makes sense here because p^k is affine between successive breakpoints, hence affine on $[t_{old}, t]$. It is easy to see that the Armijo line search applied with parameters $\sigma, \tau \in (0,1)$ produces $t_k \in [t_{old}, t]$ that fulfills (NmPs). On the other hand, if (NmD) holds at every breakpoint t then we must eventually stop because $t = 1$ or further pivots are not possible (i.e. A_k is not continuously invertible at $p^k(t)$). In this case we take $x^{k+1} \stackrel{\text{def}}{=} p(t)$ (and $T_k \stackrel{\text{def}}{=} t$).

We now confine our attention to $C \stackrel{\text{def}}{=} \mathbb{R}^N_+$, the nonnegative orthant in $\mathbb{R}^N$. The problems (VI), (GE) and (NE) are equivalent to the NonLinear Complementarity Problem:

$$\begin{aligned} z, F(z) &\geq 0 \\ \langle z, F(z) \rangle &= 0 \end{aligned} \qquad \text{(NLCP)}$$

where vector inequalities are taken pointwise. The normal equation form of this problem is $F_{\mathbb{R}^N_+}(x) = 0$ or

$$F(x_+) + x - x_+ = 0$$

where x_+ denotes $\pi_{\mathbb{R}^N_+}(x)$. The associated first-order approximation (14) is

$$A(x)(x') \stackrel{\text{def}}{=} F(x_+) + \nabla F(x_+)(x'_+ - x_+) + x' - x'_+, \quad \forall x' \in \mathbb{R}^N. \tag{15}$$

(NLCP) has many applications, for example in nonlinear programming (below) and economic equilibria problems [HP, HX].

More notation is needed. Given a matrix $M \in \mathbb{R}^{N \times N}$ and index sets $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, N\}$, let $M_{\mathcal{I},\mathcal{J}}$ be the submatrix of M of elements $M_{i,j}$ where $(i,j) \in \mathcal{I} \times \mathcal{J}$. Also let $\backslash\mathcal{I}$ be the complement of $\mathcal{I}$, $\{1, \ldots, N\} \backslash \mathcal{I}$, and $M/M_{\mathcal{I},\mathcal{I}}$ be the Schur complement of M with respect to $M_{\mathcal{I},\mathcal{I}}$,

$$M/M_{\mathcal{I},\mathcal{I}} \stackrel{\text{def}}{=} \begin{cases} M_{\backslash\mathcal{I},\backslash\mathcal{I}} - M_{\backslash\mathcal{I},\mathcal{I}}[M_{\mathcal{I},\mathcal{I}}]^{-1} M_{\mathcal{I},\backslash\mathcal{I}} & \text{if } \mathcal{I} \neq \emptyset \\ M & \text{otherwise} \end{cases}$$

assuming $T_{\mathcal{I},\mathcal{I}}$ is invertible or vacuous.

Proposition 10 *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be continuously differentiable, and $\sigma, \tau \in (0,1)$, $M \in \mathbb{N}$ be the parameters governing the path search condition condition (NmPs). Suppose $\alpha_0 > 0$ and*

$$X_0 \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N \mid \|F_{\mathbb{R}^N_+}(x)\| \leq \alpha_0\}$$

is bounded. Suppose for each $x \in X_0$ the normal map $\nabla F(x_+)_{\mathbb{R}^N_+}$ is Lipschitz invertible near x or, equivalently, the following (possibly vacuous) conditions hold:

$\nabla F(x_+)_{\mathcal{I}\mathcal{I}}$ *is invertible, where* $\mathcal{I} \stackrel{\text{def}}{=} \{i \mid x_i > 0\}$
$\nabla F(x_+)_{\mathcal{J},\mathcal{J}} / \nabla F(x_+)_{\mathcal{I}\mathcal{I}}$ *is a P-matrix, where* $\mathcal{J} \stackrel{\text{def}}{=} \{i \mid x_i \geq 0\}$.

Let path search damped Newton's method for solving $F_{\mathbb{R}^N_+}(x) = 0$ *be defined using the first-order approximation (15). Then for any* $x^0 \in X_0$, *damped Newton's iterates* x^k *converge to a zero* x^* *of* $F_{\mathbb{R}^N_+}$.

Convergence of the residual $F_{\mathbb{R}^N_+}(x^k)$ *to zero is R-linear. Convergence of the iterates* x^k *to* x^* *is Q-superlinear; indeed convergence is Q-quadratic if* ∇F *is Lipschitz near* $x^*{}_+$.

Proof Given the equivalence between the above conditions on $\nabla F(x_+)$ and local Lipschitz invertibility of $F_{\mathbb{R}^N_+}$, the result is a corollary of Proposition 9. The claimed equivalence is well known, and follows from Robinson's homeomorphism theorem [Rob90] in any case. □

This result is similar in statement to [HX, Thm. 3], but the conclusions are stronger in two ways. Firstly, strict complementarity (i.e. $x_i \neq 0$, $\forall i$) is not required at a solution point to guarantee convergence; and, secondly, superlinear convergence is achieved (convergence rates are not mentioned in [HX]).

Our computational examples are all optimization problems in NonLinear Programming. A general form of the nonlinear programming problem is

$$\min \theta(z) \quad \text{subject to } z \in D,\ g(z) = 0$$

where $\theta : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions, and D is a nonempty polyhedral convex set in $\mathbb{R}^n$. Under a constraint qualification the standard first-order conditions necessary for optimality of this problem are of the form (GE), where $N \stackrel{\text{def}}{=} n + m$,

$$\begin{aligned} C &\stackrel{\text{def}}{=} D \times \mathbb{R}^m, \\ F(z,y) &\stackrel{\text{def}}{=} (\nabla\theta(z)^T + \nabla g(z)^T y,\ g(z)), \quad \forall (z,y) \in \mathbb{R}^{n+m}. \end{aligned}$$

(see, for example [Rob83, §1 and Thm. 3.2]). Point-multiplier pairs satisfying the first-order conditions can be rewritten as solutions of (NE) for these F and C ([Par, Ch. 3 §4; Rob90]). We will confine ourselves to a more restrictive class of nonlinear programs which contains our computational examples:

$$\min \theta(z) \quad \text{subject to } z \geq 0,\ g(z) \leq 0 \qquad \text{(NLP)}$$

Again under a constraint qualification such as Mangasarian-Fromowitz condition [Man, 11.3.5; McC, 10.2.16] the first-order conditions necessary for optimality of (NLP) can be written as (NLCP), or the corresponding normal equation, where $N \stackrel{\text{def}}{=} n + m$ and

$$F(z,y) \stackrel{\text{def}}{=} (\nabla\theta(z)^T + \nabla g(z)^T y,\ -g(z)), \quad \forall (z,y) \in \mathbb{R}^{n+m}. \qquad (16)$$

Kojima [Koj] introduced an equation formulation, similar to the normal equation, for programs with (nonlinear) inequality and equality constraints.

Wilson's method. In Wilson's method [Wil, Fle], also known as sequential quadratic programming (SQP), given the kth variable-multiplier pair $(a^k, b^k) \in \mathbb{R}^{n+m}_+$ the next iterate is defined as the optimal variable-multiplier pair $(\hat{a}^{k+1}, \hat{b}^{k+1})$ for the approximate Lagrangian quadratic program:

$$\begin{array}{ll} \min\limits_{a \in \mathbb{R}^n} & f(a^k) + \nabla f(a^k)(a - a^k) + (1/2)\left\langle a - a^k, \nabla^2(\theta + (b^k)^T g)(a^k)(a - a^k)\right\rangle \\ \text{subject to} & a \geq 0, \;\; g(a^k) + \nabla g(a^k)(a - a^k) \leq 0. \end{array}$$

Suppose $x^k = (z^k, y^k) \in \mathbb{R}^{n+m}$ satisfies $(z^k_+, y^k_+) = (a^k, b^k)$, F is given by (16), and $A(x^k)$ is given by (15). By definition, the SQP iterate $(\hat{a}^{k+1}, \hat{b}^{k+1})$ satisfies a the first-order condition for the quadratic program, which is is equivalent, by previous discussion, to saying the point

$$\hat{x}^{k+1} = (\hat{z}^{k+1}, \hat{y}^{k+1}) \stackrel{\text{def}}{=} (\hat{a}^{k+1}, \hat{b}^{k+1}) - F(a^k, b^k) - \nabla F(a^k, b^k)(\hat{a}^{k+1} - a^k, \hat{b}^{k+1} - b^k)$$

is a zero of $A(x^k)$. Also $(\hat{z}^{k+1}_+, \hat{y}^{k+1}_+) = (\hat{a}^{k+1}, \hat{b}^{k+1})$. The path search applied to Robinson-Newton's method for (NE) determines the next iterate $x^{k+1} = (z^{k+1}, y^{k+1})$ as a point on the path p^k from x^k to $\hat{x}^{k+1}$. The nonnegative part (z^{k+1}_+, y^{k+1}_+), a point on the path from (a^k, b^k) to $(\hat{a}^{k+1}, \hat{b}^{k+1})$, is a damped iterate for SQP.

There are standard conditions that jointly ensure local uniqueness of an optimal point-multiplier pair $(\hat{z}, \hat{y}) \geq 0$ of (NLP), hence of the solution (also $(\hat{z}, \hat{y})$) of the associated (NLCP), and of the solution $(z^*, y^*) = (\hat{z}, \hat{y}) - F(\hat{z}, \hat{y})$ of the associated (NE). At non-solution points (z, y) of the associated (NE), analogous conditions will guarantee local invertibility of $F_{\mathbb{R}^{n+m}_+}$ and its first-order approximation (15) at (z, y).

To specify these conditions at $(z, y) \in \mathbb{R}^{n+m}$, let

$$\begin{array}{ll} \mathcal{I}_z \stackrel{\text{def}}{=} \{i \mid z_i > 0\}, & \mathcal{J}_z \stackrel{\text{def}}{=} \{i \mid z_i \geq 0\} \\ \mathcal{I}_y \stackrel{\text{def}}{=} \{l \mid y_l > 0\}, & \mathcal{J}_y \stackrel{\text{def}}{=} \{l \mid y_l \geq 0\} \end{array} \tag{17}$$

and $|\mathcal{I}_z|$ be the cardinality of $\mathcal{I}_z$ etc. We present the conditions of Linear Independence of binding constraint gradients at (z, y):

$$\nabla g(z_+)_{\mathcal{J}_y \mathcal{I}_z} \text{ has linearly independent rows} \tag{LI}$$

and Strong Second-Order Sufficiency at (z, y):

$$\begin{array}{l} \text{if } \nabla g(z_+)_{\mathcal{I}_y, \mathcal{J}_z} \hat{d} = 0, \; 0 \neq \hat{d} \in \mathbb{R}^{|\mathcal{J}_z|}, \;\; \text{then} \\ \quad \hat{d}^T [\nabla^2 \theta(z_+) + y_+^T \nabla^2 g(z_+)]_{\mathcal{J}_z, \mathcal{J}_z} \hat{d} \; > \; 0 \end{array} \tag{SSOS}$$

We note that if (z, y) is a zero of $F_{\mathbb{R}^{n+m}_+}$ then (LI) and (SSOS) correspond to more familiar conditions (eg. [Rob80, §4]) defined with respect to (z_+, y_+) rather than (z, y). This connection is explored further in the proof of our next result.

Proposition 11 *Let $\theta : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ be twice continuously differentiable functions, and F be given by (16). Let $\sigma, \tau \in (0,1)$ and $M \in \mathbb{N}$ be the parameters governing the path search condition condition (NmPs). Suppose $\alpha_0 > 0$ and*

$$X_0 \stackrel{\text{def}}{=} \{(z,y) \in \mathbb{R}^{n+m} \mid \|F_{\mathbb{R}^{n+m}_+}(z,y)\| \leq \alpha_0\}.$$

is bounded. Suppose for each $(z,y) \in X_0$, $\nabla F(z_+,y_+)_{\mathbb{R}^{n+m}_+}$ is Lipschitz invertible near (z,y) or, sufficiently, the above (LI) and (SSOS) conditions hold at (z,y).

If path search damped Newton's method for solving $F_{\mathbb{R}^{n+m}_+}(z,y) = 0$ is defined using the first-order approximation (15), where $x \stackrel{\text{def}}{=} (z,y)$, then for any $(z^0,y^0) \in X_0$, the iterates (z^k,y^k) converge to a zero (z^*,y^*) of $F_{\mathbb{R}^{n+m}_+}$. In fact z^*_+ is a local minimizer of (NLP).

Convergence of the residual to zero is at least R-linear. Convergence of the iterates to (z^*,y^*) is Q-superlinear; indeed convergence is Q-quadratic if $\nabla^2\theta$ and $\nabla^2 g$ are Lipschitz near z^*_+.

Proof This is essentially a corollary of Proposition 10. Most of the proof is devoted to showing that the (LI) and (SSOS) conditions at a given point $(z^0,y^0) \in \mathbb{R}^n \times \mathbb{R}^m$ are sufficient for $\nabla F(z^0_+,y^0_+)_{\mathbb{R}^{n+m}_+}$ to be Lipschitz invertible near that point. Below, I denotes the n times n identity matrix; and the fact that

$$0 \in u + N_{\mathbb{R}^n_+}(z) \iff u,z \geq 0, \langle u,z\rangle = 0 \iff 0 \in z + N_{\mathbb{R}^n_+}(u).$$

will be used without reference. Also [Rob80] is needed, so generalized equations rather than normal equations are stressed.

Let $x^0 \stackrel{\text{def}}{=} (z^0,y^0)$ and $u^0 \stackrel{\text{def}}{=} -z^0$ (so that $u^0_+ = z^0_+ - z^0$). Define

$$\xi^0 = (\xi^0_z, \xi^0_y) \stackrel{\text{def}}{=} F_{\mathbb{R}^{n+m}_+}(x^0).$$

Consider the nonlinear program, a perturbed version of (NLP),

$$\min \tilde{\theta}(z) \text{ subject to } \tilde{g}(z) \leq 0 \qquad (\widetilde{\text{NLP}})$$

where $\tilde{\theta}(z) \stackrel{\text{def}}{=} \theta(z) - \langle \xi^0_z, z\rangle$ and $\tilde{g}(z) \stackrel{\text{def}}{=} (g(z) + \xi^0_y, -z)$. The first-order optimality condition [Rob80] for this problem is the perturbed generalized equation

$$0 \in (\tilde{F} + N_{\mathbb{R}^n \times \mathbb{R}^{n+m}_+})(z,y,u) \qquad (\widetilde{\text{GE}})$$

where

$$\tilde{F}(z,y,u) \stackrel{\text{def}}{=} (\nabla\tilde{\theta}(z)^T + \nabla\tilde{g}(z)^T(y,u), -\tilde{g}(z)) = (F(z,y) - \xi^0 - (u,0), z).$$

Now

$$\tilde{F}(z^0_+,y^0_+,u^0_+) = (0, y^0_+ - y^0, z^0_+) \in -N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+}(z^0_+,y^0_+,u^0_+)$$

i.e. (z_+^0, y_+^0, u_+^0) solves $(\widetilde{\mathrm{GE}})$. This generalized equation is said to be *strongly regular* [Rob80] at (z_+^0, y_+^0, u_+^0) if the linearized set mapping

$$T(z,y,u) \stackrel{\text{def}}{=} \tilde{F}(z_+^0, y_+^0, u_+^0) + \nabla \tilde{F}(z_+^0, y_+^0, u_+^0)(z - z_+^0, y - y_+^0, u - u_+^0) + N_{\mathbb{R}^n \times \mathbb{R}^{n+m}_+}(z,y,u)$$

is such that for some neighborhoods $\tilde{U}$ of (z_+^0, y_+^0, u_+^0) and $\tilde{V}$ of $0 \in \mathbb{R}^{2n+m}$, $T^{-1} \cap \tilde{U}$ is a Lipschitz mapping on $\tilde{V}$.

Given a subset $\mathcal{K}$ of the row indices of a matrix M, $M_{\mathcal{K}}$ denotes the submatrix consisting of the rows of M of indices in $\mathcal{K}$. According to [Rob80, Thm. 4.1], it is sufficient for strong regularity of $(\widetilde{\mathrm{GE}})$ at (z_+^0, y_+^0, u_+^0) that two conditions hold, where, for $\zeta^0 \stackrel{\text{def}}{=} (y_+^0, u_+^0)$,

$$\mathcal{K}_+ \stackrel{\text{def}}{=} \{i \,|\, \tilde{g}(z_+^0)_i = 0,\ \zeta_i^0 > 0\},\ \mathcal{K}_0 \stackrel{\text{def}}{=} \{i \,|\, \tilde{g}(z_+^0)_i = 0,\ \zeta_i^0 = 0\}.$$

The first is linear independence of the binding constraints:

$$\nabla \tilde{g}(z_+^0)_{\mathcal{K}_+ \cup \mathcal{K}_0} \text{ has full row rank;}$$

and the second, strong second-order sufficiency:

$$\text{if } z \in \mathbb{R}^n \backslash \{0\} \text{ and } \nabla \tilde{g}(z_+^0)_{\mathcal{K}_+} z = 0, \text{ then } z^T \tilde{\mathcal{L}}'' z > 0$$

where

$$\tilde{\mathcal{L}}'' \stackrel{\text{def}}{=} \nabla^2(\tilde{\theta} + (y_+^0, u_+^0)^T \tilde{g})(z_+^0) = \nabla^2(\theta + (y_+^0)^T g)(z_+^0).$$

In terms of g and the index sets (17), and $\backslash \mathcal{I}_z \stackrel{\text{def}}{=} \{1, \ldots, n\} \backslash \mathcal{I}_z$, the linear independence condition is

$$\begin{bmatrix} \nabla g(z_+^0)_{\mathcal{J}_y} \\ -I_{\backslash \mathcal{I}_z} \end{bmatrix} \text{ has linearly independent rows;}$$

hence is equivalent to (LI) at (z^0, y^0). Likewise, the strong second order sufficient condition may be rewritten

$$\text{if } z \in \mathbb{R}^n \backslash \{0\} \text{ and } \begin{bmatrix} \nabla g(z_+^0)_{\mathcal{I}_y} \\ -I_{\backslash \mathcal{J}_z} \end{bmatrix} z = 0, \text{ then } z^T \tilde{\mathcal{L}}'' z > 0$$

which clearly is equivalent to (SSOS) at (z^0, y^0). Therefore the conditions (LI) and (SSOS) at (z^0, y^0) guarantee strong regularity of $(\widetilde{\mathrm{GE}})$ at (z_+^0, y_+^0, u_+^0).

Furthermore if $(z^0, y^0) = (z^*, y^*)$ then $\xi^0 = (0,0)$ and $(\widetilde{\mathrm{NLP}})$ is just the original problem (NLP). So, as shown above, (LI) and (SSOS) at (z^*, y^*) are respectively equivalent to the usual conditions of linear independence and strong second order sufficiency for (NLP) with respect to the variable z_+^*, the multipliers y_+^* corresponding the the nonlinear constraints $g(z_+^*) \leq 0$, and the multipliers $z_+^* - z^*$ corresponding to the constraints $z_+^* \geq 0$. Applying [McC, 10.4.1] we find that z_+^* is a strict local minimizer of (NLP).

We now relate strong regularity of $(\widetilde{\mathrm{GE}})$ to local Lipschitz invertibility of $\nabla F(z_+^0, y_+^0)_{\mathbb{R}^{n+m}_+}$. Let $M \stackrel{\text{def}}{=} \nabla F(z_+^0, y_+^0)$, $\tilde{M} \stackrel{\text{def}}{=} \nabla \tilde{F}(z_+^0, y_+^0, u_+^0)$. Observe that

$$M = \begin{bmatrix} \mathcal{L}'' & G^T \\ -G & \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} \mathcal{L}'' & G^T & -I \\ -G & & \\ I & & \end{bmatrix}$$

where $\mathcal{L}'' \stackrel{\text{def}}{=} \nabla^2(\theta + (y_+^0)^T g)(z_+^0)$ $(= \tilde{\mathcal{L}}'')$, $G \stackrel{\text{def}}{=} \nabla g(z_+^0)$. Define

$$(\xi_z^{00}, \xi_y^{00}) \stackrel{\text{def}}{=} \xi^0 - (F - M)(z_+^0, y_+^0);$$

then

$$(\xi_z^{00}, \xi_y^{00}, 0) = -(\tilde{F} - \tilde{M})(z_+^0, y_+^0, u_+^0) \in (\tilde{M} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})(z_+^0, y_+^0, u_+^0).$$

Note $T(\cdot) + (\xi_z^{00}, \xi_y^{00}, 0) = (\tilde{M} + N_{\mathbb{R}^n \times \mathbb{R}^{n+m}_+})(\cdot)$.

For any $(z, y), (\xi_z, \xi_y) \in \mathbb{R}^n \times \mathbb{R}^m$

$$\begin{aligned} & & (\xi_z, \xi_y) \in (M + N_{\mathbb{R}^{n+m}_+})(z, y) \\ \iff & \exists u \in \mathbb{R}^n, & \xi_z = \mathcal{L}'' z + G^T y - u \\ & & \xi_y \in -Gz + N_{\mathbb{R}^m_+}(y) \\ & & 0 \in z + N_{\mathbb{R}^n_+}(u) \\ \iff & \exists u \in \mathbb{R}^n, & (\xi_z, \xi_y, 0) \in (\tilde{M} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})(z, y, u) \end{aligned}$$

Hence if $\tilde{U}$, $\tilde{V}$ are respective neighborhoods of (z_+^0, y_+^0, u_+^0), $(\xi_z^{00}, \xi_y^{00}, 0)$ such that $(\tilde{M} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})^{-1} \cap \tilde{U}$ is a Lipschitz map on $\tilde{V}$, then $(M + N_{\mathbb{R}^{n+m}_+})^{-1} \cap U$ is a Lipschitz map on V, where

$$\begin{aligned} V & \stackrel{\text{def}}{=} \{(\xi_z, \xi_y) \in \mathbb{R}^{n+m} \mid (\xi_z, \xi_y, 0) \in \tilde{V}\}, \\ U & \stackrel{\text{def}}{=} \{(z, y) \in \mathbb{R}^{n+m} \mid (z, y, u) \in \tilde{U} \text{ for some } u \in \mathbb{R}^n\}. \end{aligned}$$

In this case U and V are neighborhoods of (z_+^0, y_+^0) and (ξ_z^{00}, ξ_y^{00}) respectively, so Lemma 5 (and equation (2)) assures us that $M_{\mathbb{R}^{n+m}_+}$ is also Lipschitz invertible near

$$(\xi_z^{00}, \xi_y^{00}) + (I - M)(z_+^0, y_+^0) = \xi^0 + (I - F)(z_+^0, y_+^0) = (z^0, y^0).$$

To summarize: (z_+^0, y_+^0, u_+^0) solves $(\widetilde{\mathrm{GE}})$ and

$$
\begin{aligned}
& \text{(LI) and (SSOS) hold at } (z^0, y^0) \\
\Longrightarrow\quad & T^{-1} \cap \tilde{U} \text{ is Lipschitz on } \tilde{V}, \\
& \text{for some neighborhoods } \tilde{U}, \tilde{V} \text{ of } (z_+^0, y_+^0, u_+^0), 0 \in \mathbb{R}^{2n+m} \text{ respectively} \\
\Longleftrightarrow\quad & (\tilde{M} + N_{\mathbb{R}^n \times \mathbb{R}_+^{m+n}})^{-1} \cap \tilde{U} \text{ is Lipschitz on } \tilde{V}, \\
& \text{for some neighborhoods } \tilde{U}, \tilde{V} \text{ of } (z_+^0, y_+^0, u_+^0), (\xi_z^{00}, \xi_y^{00}, 0) \text{ respectively} \\
\Longrightarrow\quad & (M + N_{\mathbb{R}_+^{n+m}})^{-1} \cap U \text{ is Lipschitz on } V, \\
& \text{for some neighborhoods } U, V \text{ of } (z_+^0, y_+^0), (\xi_z^{00}, \xi_y^{00}) \text{ respectively} \\
\Longleftrightarrow\quad & M_{\mathbb{R}_+^{n+m}} \text{ is Lipschitz invertible near } (z^0, y^0).
\end{aligned}
$$

The proof is complete.

□

We show how to apply Proposition 11 to convex programs.

Lemma 12 *Let $\theta, g, F, \tau, \sigma, M$ be as in Proposition 11. Let θ and each component function g_l $(l = 1, \ldots, m)$ of g be convex. Suppose there is $z > 0$ with $g(z) < 0$ (Slater constraint qualification) and (NLP) has a (global) minimum at $\hat{z} \in \mathbb{R}_+^n$. Then there exists $\hat{y} \in \mathbb{R}_+^m$ such that $(z^*, y^*) = (\hat{z}, \hat{y}) - F(\hat{z}, \hat{y})$ is a zero of $F_{\mathbb{R}_+^{n+m}}$.*

If (LI) and (SSOS) hold at (z^*, y^*) then there is $\alpha_0 > 0$ such that

$$X_0 \stackrel{\text{def}}{=} \{(z, y) \in \mathbb{R}^{n+m} \mid \|F_{\mathbb{R}_+^{n+m}}(z, y)\| \leq \alpha_0\}$$

is bounded, and $\nabla F(z_+, y_+)_{\mathbb{R}_+^{n+m}}$ is Lipschitz invertible near each $(z, y) \in X_0$.

Proof It is a classical result in optimization [Man, 7.3.7; McC, 10.2.16] that there exists a Lagrange multiplier $(\hat{y}, \hat{u}) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

$$
\begin{aligned}
0 &= \nabla\theta(\hat{z}) + \hat{y}^T \nabla g(\hat{z}) - \hat{u} \\
0 &\leq \hat{u},\ \langle \hat{u}, \hat{z} \rangle = 0 \\
0 &\leq \hat{y},\ \langle \hat{y}, g(\hat{z}) \rangle = 0
\end{aligned}
$$

Since $\hat{z}$ is also feasible ($\hat{z} \geq 0$ and $g(\hat{z}) \leq 0$), $(z, y, u) = (\hat{z}, \hat{y}, \hat{u})$ is a solution of

$$0 \in (\tilde{F}^* + N_{\mathbb{R}^n \times \mathbb{R}_+^{m+n}})(z, y, u) \qquad (\widetilde{\mathrm{GE}}^*)$$

where $\tilde{F}^*(z, y, u) \stackrel{\text{def}}{=} (F(z, y) - (u, 0), z)$. $(\widetilde{\mathrm{GE}}^*)$ is just the generalized equation $(\widetilde{\mathrm{GE}})$ from the proof of Proposition 11 when ξ^0 is $0 \in \mathbb{R}^{n+m}$. It is easy to see that $(\hat{z}, \hat{y})$ solves (GE),

$$0 \in (F + N_{\mathbb{R}_+^{n+m}})(\hat{z}, \hat{y})$$

hence $(z^*, y^*) \stackrel{\text{def}}{=} (\hat{z}, \hat{y}) - F(\hat{z}, \hat{y})$ is a zero of $F_{\mathbb{R}^{n+m}_+}$; and $(z^*_+, y^*_+, z^*_+ - z^*)$ is $(\hat{z}, \hat{y}, \hat{u})$, a solution of $(\widetilde{\mathrm{GE}}^*)$.

Since (LI) and (SSOS) hold at (z^*, y^*), the proof of Proposition 11 demonstrates that $\nabla F(z^*_+, y^*_+)_{\mathbb{R}^{n+m}_+}$ is Lipschitz invertible near (z^*, y^*). Lemma 6 with $g \stackrel{\text{def}}{=} \nabla F(z^*_+, y^*_+)_{\mathbb{R}^{n+m}_+}$ and $g' \stackrel{\text{def}}{=} F_{\mathbb{R}^{n+m}_+}$ can be used to show that $F_{\mathbb{R}^{n+m}_+}$ is Lipschitz invertible near (z^*, y^*). Let U^* be a neighborhood of (z^*, y^*) and $\alpha_* > 0$ be such that

$$F_{\mathbb{R}^{n+m}_+}|_{U^*} : U^* \to \alpha_* \mathbb{B}^\circ$$

is Lipschitz invertible, where $\mathbb{B}^\circ$ is the open unit ball in $\mathbb{R}^{n+m}$. So U^* is open and bounded. For any element (z^0, y^0) of U^*, another application of Lemma 6, this time with $g \stackrel{\text{def}}{=} F_{\mathbb{R}^{n+m}_+}$ and $g' \stackrel{\text{def}}{=} \nabla F(z^0, y^0)_{\mathbb{R}^{n+m}_+}$, shows that $\nabla F(z^0, y^0)_{\mathbb{R}^{n+m}_+}$ is Lipschitz invertible near (z^0, y^0). It only remains to be seen that for some $\alpha_0 > 0$,

$$X_0 \stackrel{\text{def}}{=} (F_{\mathbb{R}^{n+m}_+})^{-1}(\alpha_0 \mathbb{B}) \subset U^*,$$

because then X_0 is bounded and, for each $(z^0, y^0) \in X_0$, $\nabla F(z^0, y^0)_{\mathbb{R}^{n+m}_+}$ is Lipschitz invertible near (z^0, y^0).

Let $u^* \stackrel{\text{def}}{=} -z^*$, so (z^*_+, y^*_+, u^*_+) solves $(\widetilde{\mathrm{GE}}^*)$. Recall from the proof of Proposition 11 the strong regularity of $(\widetilde{\mathrm{GE}}^*)$ at (z^*_+, y^*_+, u^*_+). By [Rob80, Thm. 2.1], there exist a neighborhood $\tilde{U}$ of (z^*_+, y^*_+, u^*_+) and $\alpha_0 \in (0, \alpha_*)$ such that $(\tilde{F}^* + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})^{-1} \cap \tilde{U}$ is a Lipschitz when restricted to $\alpha_0 \mathbb{B} \subset \mathbb{R}^{2n+m}$. Assume without loss of generality that $\alpha_0 > 0$ has been chosen small enough so that $(\tilde{F}^* + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})^{-1}(\alpha_0 \mathbb{B}) \cap \tilde{U}$ is a subset of the interior of $\tilde{U}$. Suppose $\xi = (\xi_z, \xi_y) \in \mathbb{R}^{n+m}$ has norm not greater than α_0. Then the generalized equation

$$(\xi_z, \xi_y, 0) \in (\tilde{F}^* + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})(z, y, u) \tag{18}$$

has a solution (z, y, u) in the interior of $\tilde{U}$, which is the only solution in $\tilde{U}$. In fact a solution of (18) satisfies the first-order optimality condition for the (convex) nonlinear program $(\widetilde{\mathrm{NLP}})$ of the previous proof, when ξ^0 is chosen to be (ξ_z, ξ_y), i.e.

$$(\tilde{F}^* + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})^{-1}(\xi^0_z, \xi^0_y, 0) = (\tilde{F} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})^{-1}(0)$$

where $\tilde{F}(z, y, u) \stackrel{\text{def}}{=} (F(z, y) - (\xi_z, \xi_y) - (u, 0), z)$. Denote $(\tilde{F} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})^{-1}(0)$ by S. It is well known that S is a convex set, being the cartesian product of the convex set of minimizers of $(\widetilde{\mathrm{NLP}})$ and the convex set of optimal multipliers of $(\widetilde{\mathrm{NLP}})$ (i.e. the pairs (y, u) such that $(\tilde{F} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+})(\cdot, y, u)^{-1}(0)$ is nonempty); for example, this follows from the saddle point conditions for convex programming [Man, 5.4.7]. As (z, y, u) is an interior point of $\tilde{U}$ and $\tilde{U} \cap S = \{(z, y, u)\}$, convexity of S yields $S = \{(z, y, u)\}$. So there is a unique solution in $\mathbb{R}^{2n+m}$ to (18).

It is now easy to check that for each (ξ_z, ξ_y) of norm not greater than α_0 there is a unique solution to

$$(\xi_z, \xi_y) \in (F + N_{\mathbb{R}^{n+m}_+})(z, y)$$

(cf. the relationship between $M + N_{\mathbb{R}^{n+m}_+}$ and $\tilde{M} + N_{\mathbb{R}^n \times \mathbb{R}^{m+n}_+}$ in the previous proof). Hence by (2) in the proof of Lemma 5, for each such (ξ_z, ξ_y)

$$(F_{\mathbb{R}^{m+n}_+})^{-1}(\xi_z, \xi_y) = (\xi_z, \xi_y) + (I - F)(F + N_{\mathbb{R}^{n+m}_+})^{-1}(\xi_z, \xi_y),$$

a singleton. This single point set must be contained in U^* since $\|(\xi_z, \xi_y)\| < \alpha_*$ and $F_{\mathbb{R}^{n+m}_+}(U^*) = \alpha_* \mathbb{B}^\circ$. So

$$(F_{\mathbb{R}^{m+n}_+})^{-1}(\alpha_0 \mathbb{B}) \subset U^*$$

and we are done.

□

Implementation of the path search

We will outline how the path p^k is obtained in computation at iteration k, for (NE) when $C \stackrel{\text{def}}{=} \mathbb{R}^N_+$. We are thinking of solving (NLP), that is letting $N \stackrel{\text{def}}{=} n + m$ and F be given by (16). First Lemke's algorithm [CD], a standard pivotal method for solving linear complementarity problems, is reviewed.

The Linear Complementarity Problem is a special case of the nonlinear complementarity problem when the function F defining (NLCP) is affine: find $v, w \in \mathbb{R}^N$ such that

$$\begin{aligned} w &= Mv + q \\ 0 &\leq v, w \\ 0 &= \langle v, w \rangle \end{aligned} \tag{LCP}$$

where $M \in \mathbb{R}^{N \times N}$ and $q \in \mathbb{R}^N$. In Lemke's algorithm an artificial variable $v_0 \in \mathbb{R}$ is introduced. Let $e \stackrel{\text{def}}{=} (1, \ldots, 1)^T \in \mathbb{R}^N$. At each iteration of the algorithm we have a basic feasible solution or BFS [Chv], (v, w, v_0), of the system

$$\begin{aligned} w &= Mv + q + ev_0 \\ 0 &\leq (v, w, v_0) \end{aligned} \tag{19}$$

which is almost complementary, i.e. for each basic variable v_i (respectively w_i), $1 \leq i \leq N$, its *complement* w_i (respectively v_i) is nonbasic. In particular $\langle v, w \rangle = 0$, hence if $v_0 = 0$ then (v, w) solves (LCP). Given v_0, (v, w) solves the parametric linear complementarity problem

$$\begin{aligned} w &= Mv + q + ev_0 \\ 0 &\leq v, w \\ 0 &= \langle v, w \rangle \end{aligned}$$

The initial almost complementary BFS is given by taking v_0 large and positive, and $(v, w) = (0, q + ev_0)$. So Lemke's algorithm may be viewed as a method of constructing a path of solutions $(v(v_0), w(v_0))$ to the parametric problem as v_0 moves down toward zero[1]. We use this idea, but with a different parametric problem, to construct the path p^k we need for iteration $k+1$ of damped Newton's method.

Let $N \stackrel{\text{def}}{=} n+m$, F be given by (16) and $x^k \stackrel{\text{def}}{=} (z^k, y^k)$. Let the first-order approximation A to F be given by (15); and denote $A(x^k)$ by A_k. Now $p^k(0) \stackrel{\text{def}}{=} x^k$ and, for each $t \in \text{dom}(p^k)$, $p^k(t)$ is the solution x to

$$\begin{aligned}(1-t)F_{\mathbb{R}^N_+}(x^k) &= A_k(x) \\ &= F(x^k_+) + \nabla F(x^k_+)\left(x_+ - x^k_+\right) + x - x_+ \\ &= (F(x^k_+) - \nabla F(x^k_+)x^k_+) + \nabla F(x^k_+)_{\mathbb{R}^N_+}(x).\end{aligned}$$

Equivalently, $(v, w) = (p^k(t)_+, p^k(t)_+ - p^k(t))$ solves the parametric linear complementarity problem:

$$\begin{aligned} w &= M^k v + q^k - (1-t)r^k \\ 0 &\leq v, w \\ 0 &= \langle v, w \rangle \end{aligned} \qquad \text{(LCP)(t)}$$

where

$$M^k \stackrel{\text{def}}{=} \nabla F(x^k_+) = \begin{bmatrix} \mathcal{L}''_k & \nabla g(z^k_+)^T \\ -\nabla g(z^k_+) & \end{bmatrix}$$

$$q^k \stackrel{\text{def}}{=} F(x^k_+) - \nabla F(x^k_+)x^k_+ = \begin{bmatrix} \nabla\theta(z^k_+) - \mathcal{L}''_k z^k_+ \\ -g(z^k_+) + \nabla g(z^k_+)z^k_+ \end{bmatrix}$$

$$r^k \stackrel{\text{def}}{=} F_{\mathbb{R}^N_+}(x^k) = M^k v^k + q^k - w^k$$

and

$$\begin{aligned} \mathcal{L}''_k &\stackrel{\text{def}}{=} \nabla^2(\theta + (y^k_+)^T g)(z^k_+) \\ (v^k, w^k) &\stackrel{\text{def}}{=} (x^k_+, x^k_+ - x^k). \end{aligned}$$

Clearly (v^k, w^k) solves (LCP)(0), and a solution (v, w) of (LCP)(1) yields a zero $v - w$ of A_k i.e. Newton's iterate.

[1]Actually v_0 may not strictly decrease during some iterations of Lemke's algorithm, in which case the corresponding solution (v, w) of the parametric problem is not a *function* of v_0. In any case (v, w) traces out a path.

To construct p^k we use a modification of Lemke's algorithm in which the rôle of the auxiliary variable v_0 is played by t. Assume (v,w,t) is an almost complementary BFS of

$$\begin{aligned} w &= M^k v + q^k + (t-1)r^k \\ 0 &\leq v, w \\ 0 &\leq t \leq 1 \end{aligned} \tag{20}$$

such that t is basic. This defines the current point on p^k: $p^k(t) \stackrel{\text{def}}{=} v - w$. As there are exactly N basic variables, there is some i for which both v_i and w_i are nonbasic. Let either of these nonbasics be the *entering* variable. The modified method iterates likes Lemke's algorithm, in the following way.

Increase the entering variable — altering the basic variables as needed to maintain the equation $w = M^k v + q^k + (t-1)r^k$ — until (at least) one of the basic variables is forced to a bound. We choose one of these basic variables to be the *leaving* variable, that is the variable to be replaced in the basis by the entering variable. This transformation of the entering and basic variables is called a *pivot* operation, and corresponds to moving along an affine piece of the path p^k from the value of t before the pivot, t_{old}, to the current parameter value. After a pivot, (v,w,t) is still an almost complementary BFS for (20); the leaving variable is nonbasic, and, assuming it is not t, its complement is nonbasic too. The next entering variable is defined as the complement of the leaving variable. This completes an iteration of the modified algorithm, and we are ready to begin the next iteration.

The modified algorithm is initialized at $(v,w,t) = (v^k, w^k, 0)$ with t as the nonbasic entering variable (so $p^k(0) = v^k - w^k$ as needed).

Now the algorithm cannot continue if either t becomes the leaving variable (because no entering variable is defined) or no leaving variable can be found (i.e. increasing the entering variable causes no decrease in any basic variable). In the latter case we have detected a *ray*. Other stopping criteria are needed to reflect the aim of the exercise, namely to path search. We stop iterating if $t = 1$ (the entire path has been traversed), or t, with $p^k(t) = v - w$, does not satisfy the descent condition (NmD), or t strictly decreases as a result of the last pivot.

If the method halts because a ray is detected or t decreases we know A_k is not invertible at $p^k(t_{old})$, for invertibility implies t would have strictly increased from $t_{old}(< 1)$ as a result of the final pivot. In either case we take $x^{k+1} \stackrel{\text{def}}{=} p^k(t_{old})$. Otherwise the method halts because (NmD) fails at t, so we use the Armijo line search on $[p(t_{old}), p(t)]$ to determine a path length $t_k \in [t_{old}, t]$ satisfying (NmPs).

It is a subtle point that, when solving (NLP), the first-order approximation at $x^k = (z^k, y^k) \in \mathbb{R}^{n+m}$ used to define the path p^k is *exact* for points $x = (z,y)$ such that $z_+ = z_+^k$: $A_k(x) = F(x_+) + x - x_+$. Using this idea, we save some work by only checking (NmD) if the positive part of the variables z have changed after a pivot. This corresponds in the modified Lemke's algorithm to only checking (NmD) if the subvector of the first n components of v has changed as a result of a pivot.

One difficulty we have not discussed is the possibility that (v^k, w^k) is not a basic solution of

$$w = M^k v + q^k - r^k,\ 0 \leq v, w$$

i.e. $(v^k, w^k, 0)$ with t nonbasic cannot be used as a starting point of the modified Lemke's algorithm. To overcome this, we performed a modified Cholesky factorization [GMW] of the Hessian of the Lagrangian, $\mathcal{L}''_k$ above, changing it if necessary into a positive definite matrix. This operation — somewhat crude because it attacks the entire Hessian of the Lagrangian, rather than just the "basis" columns — may not always be sufficient to correct the problem since columns of M^k not corresponding to columns of $\mathcal{L}''_k$ are generally specified in the initial basis. However it was sufficient in the problems we tried, below. It has the added advantage of possibly preventing convergence to a Karush-Kuhn-Tucker point of the nonlinear program which corresponds to a local maximum or saddle point instead of a local minimum of the program.

Computational Examples

Our computational examples are of global, or path search damped Newton's method applied to the nonlinear programs of the form (NLP). The computer program was written in C and the computation carried out on a Sun 4 (Sparcstation). Most problems are taken from [Fer] with the starting points used there[2], which are close to solutions, and setting the initial multipliers y^0 to zero. Other problems, showing convergence of the damped method in spite of the cycling (nonconvergence) that occurs without damping, are tested.

For comparison we have also tested local (Robinson- or Josephy-)Newton's method on the same problems, using Lemke's algorithm to find successive Newton's iterates. This method was found to be rather fragile with respect to starting points: quite often the method failed because a ray was detected by Lemke's algorithm. To make this more robust we modified the method to continue, even if a ray was detected in trying to find a zero of A_k, by setting $x^{k+1} = v - w$ where (v, w) was the last complementary BFS before the ray was found. In most cases this allowed the method to solve the problem at hand.

Our conclusion with regard to the problems from [Fer] (Table 1) is that the benefit of path searching is only clear on problems when Newton's method fails because, when it works, modified Newton's method works rather well. This suggests further computational study is needed to determine how best to trade off the extra work of damping against the need for robustness, an obvious strategy being to test the descent condition (NmD) only every 'few' pivots and back track if necessary. Also we note that the problems all have small dimension: no more than 15 variables and 10 nonlinear constraints. In higher dimensions we expect the value of path searching to be even greater, a view partly supported by the following observations.

[2]Starting points and Fortran subroutines were supplied in a personal communication by Professor Michael C. Ferris, Computer Sciences Department, University of Wisconsin–Madison.

One peculiarity of Newton's method is the possibility of cycling, depending on the starting point, as in the smooth case. This is observable for the modified Newton's algorithm when testing the Colville 2 problem with a feasible starting point (Table 1). It is easy to find a real function of one variable, which, when minimized over $[0, \infty)$ by Newton's method, demonstrates cycling of (unmodified) Newton's method (Example 13 below). It turns out that for any problem (NLP), the new problem formed by adding such a function in an $(n+1)$th variable to the objective function θ *cannot* be solved by Newton's method for some starting points (Example 14): at best it will converge in the first n variables and cycle in the $(n+1)$th variable. So, in some sense, the likelihood of cycling in Newton's method increases as the dimension of the problem increases. Global Newton's method converges for otherwise well behaved problems.

Table 1 summarizes our results on the problems from [Fer]. The following parameters were set: In the path search, $\sigma \stackrel{\text{def}}{=} 0.1$, $\tau \stackrel{\text{def}}{=} 0.5$, $M \stackrel{\text{def}}{=} 4$. The problem was considered solved when the norm of the residual satisfied

$$\|F(x_+^k) + x^k - x_+^k\| \leq 10^{-5}.$$

Starting points $x^0 \stackrel{\text{def}}{=} (z^0, y^0) \in \mathbb{R}^{n+m}$ were taken with the variables z^0 used in [Fer] and the multipliers $y^0 \stackrel{\text{def}}{=} 0$. In the first column the name of the problem is given, and the second column lists the number of constraints $g_l(x) \leq 0$ times the number of variables. The remaining columns contain pairs of numbers, at left the number used by global Newton's method to solve the problem, and at right in parentheses, the number required by modified local Newton's method. An asterisk * indicates failure to solve the problem. The 'Pivots' column lists the total number of pivots required by the two methods to solve a given problem. The 'Evaluations' column lists the total number of evaluations of the objective function θ and nonlinear constraints g, and their derivatives, used by the two methods. The final column 'Iterations' shows the number of iterations needed by the methods to solve the problems.

Remarks on Table 1.

1. The most important feature of the results is the number of iterations needed, i.e. the number of evaluations of the Hessian of the Lagrangian, $\mathcal{L}''$. In this regard the advantages of the global method over the local method, and vice versa, are not clear. Three problems where the number of iterations differ widely are discussed further.

   The objective function of the Rosenbrock problem is highly nonlinear causing any damping of Newton's method (even in the unconstrained case) to take very short steps, hence many iterations, unless the current iterate is very close to a minimizer. The nonmonotone line search alleviates this problem: with memory length of 4, 9 iterations are needed, while more than 500 iterations were needed in an unrecorded test using the monotone path search (memory length of 1). In other tests using the monotone path search, the remaining problems required the same or slightly fewer iterations than listed in Table 1.

Problem	Size $m \times n$	Global (Local) Pivots		Evaluations		Iterations	
Rosenbrock	4×2	19	(20)	17	(7)	9	(6)
Himmelblau	3×4	7	(25)	6	(6)	5	(5)
Wright	3×5	31	(99)	28	(8)	27	(7)
Colville 1	10×5	41	(31)	5	(4)	3	(3)
Colville 2 (feas.)	5×15	23	(*)	21	(*)	8	(*)
Colville 2 (infeas.)	5×15	40	(113)	23	(8)	7	(7)

Table 1: Testing standard (NLP)'s from local starting points.

In the Wright problem, the local method does much better (7 iterations) than the global method (27 iterations) because it uses the actual Hessian of the Lagrangian whereas the global method introduces error by using a modified Cholesky version of this. In an unrecorded test, 27 iterations were also required by the local method using the modified Cholesky version of the Hessian of the Lagrangian. This highlights a difficulty of the damped method: care is needed to ensure that the path is well defined near current iterate. On the other hand, this has advantages (other than damping) as we see in 3 below.

For the Colville 2 problem with a feasible starting point, modified local Newton's method cycled while global Newton's method solved the problem easily. In the local method, after the first iteration the pair (v^k, w^k) alternated between two different vector pairs for which the associated residual $\|F(v^k) - w^k\|$ took values 98 and 5 respectively, the former corresponding to an unbounded ray.

2. As expected, the number of evaluations of the objective function θ and constraint function g, and their derivatives, is much higher for the damped method than the undamped method. The damped method compares $\|F(v) - w\|$ to $(1-t)\|F(v^k) - w^k\|$ at almost every pivot during iteration k, whereas the local method only checks the norm of the residual at the start of the iteration.

   For the damped method, the difference between the number in the 'Pivots' column and the number in the 'Evaluations' column is the savings in function/derivative evaluations obtained by only checking (NmD) when there is a change in the first n components of v during a pivot of modified Lemke's method.

3. The global method usually requires less pivots per iteration than does the local iteration. This is not surprising:

   The damped method requires the current iterate x^k to correspond to a basis of (LCP)(0). We initiate the modified Lemke's algorithm at this basis by explicitly factorizing the corresponding matrix. By starting with this 'warm' basis, the damped method generally take many fewer pivots than Lemke's algorithm to find Newton's iterate, the solution of (LCP)(1).

Example 13 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a differentiable function whose gradient is*

$$\nabla\phi(z) \stackrel{\text{def}}{=} \arctan(z - 10),$$

the shifted inverse trigonometric tangent function. Integrating and setting the constant of integration to be zero we obtain

$$\phi(z) = (z - 10)\arctan(z - 10) - (1/2)\log_e(1 + (z - 10)^2).$$

Now consider the simplest problem of the form (NLP):

$$\min_{z \in \mathbb{R}} \phi(z) \text{ subject to } z \geq 0.$$

The unique solution is $z = 10$.

It can be easily shown that for any starting point $|z^0 - 10| \geq 2$ Newton's method cycles infinitely. This has been computationally verified for several starting points. Damped Newton's method, with the same parameters as used to obtain the results of Table 1, converged in 4-33 iterations over a variety of starting points $2 \leq |z^0 - 10| \leq 100$.

Example 14 *Derive a problem from (NLP) by including an extra variable z_{n+1}. The new objective function is*

$$\hat{\theta}(z_1, \ldots, z_{n+1}) \stackrel{\text{def}}{=} \theta(z_1, \ldots, z_n) + \phi(z_{n+1}),$$

where ϕ is defined in the last example. The constraints are $g(z_1, \ldots, z_n) \leq 0$ as before, and $z \geq 0$. It is not hard to see that for any starting point $(z^0, y^0) \in \mathbb{R}^{n+1} \times \mathbb{R}^m$ in which $|z^0_{n+1} - 10| \geq 2$, Newton's method cannot converge in z_{n+1}.

We have tested this for the Himmelblau (NLP), where $n \stackrel{\text{def}}{=} 4$, $(z^0_1, \ldots, z^0_4)$ are the initial variables used in [Fer], $z^0_5 = 0$, and all multipliers are initially zero. Newton's method cycles infinitely; damped Newton's method, with the same parameters as used for Table 1, converges in 33 iterations.

In other tests using the the monotone path search, damped Newton's method only required 4-7 iterations to solve the problem in Example 13 from various starting points; and 9 iterations for the augmented problem of Example 14 using the starting point specified there. In both cases this is an improvement on nonmonotone damping. It is interesting that the same reason nonmonotone damping helps on some problems, namely by accepting iterates that need not decrease the size of the residual, hurts here because it perpetuates *almost* cyclic behavior of the iterates.

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