A Note on Wavelet Bases for Two-Dimensional Surfaces

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Abstract

Recent work by Beylkin, Coifman and Rokhlin has demonstrated that integral equations for functions on $\mathbb{R}$ can be solved rapidly by expressing the integrands in a wavelet basis. Boundary element methods for solving partial differential equations in three dimension rely on integral equations for functions defined on surfaces embedded in $\mathbb{R}^3$. Accordingly, it is of interest to extend the wavelet work to functions defined on surfaces.

In this report we define a basis of piecewise constant functions on surfaces in $\mathbb{R}^3$ with properties akin to a wavelet basis. The basis we define is not useful for numerical computation because piecewise constant functions have poor approximation properties, but this work suggests an approach to define smoother wavelet bases for surfaces.

1 Wavelets and integral equations

A *wavelet* basis is a basis for $L^2(\mathbb{R})$ with the following properties:

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- All the functions in the basis take the form

$$W(2^j x + k)$$

where j, k range over all integers, and W is a fixed function in $L^2(\mathbb{R})$. Thus, the set of basis functions is closed under dilation by powers of 2 and under translations by multiples of 2^{-j}.

- The basis functions are pairwise orthogonal.

In addition, the following properties are important for numerical applications.

- The function $W(x)$ (which is sometimes called "the" wavelet) is a compactly supported function.

- The identity

$$\int_{\mathbf{R}} x^m W(x)\,dx = 0 \tag{1}$$

holds for $m = 0, \ldots, p-1$, where p measures the smoothness of the wavelet basis.

Wavelets are the work of many authors; works of S. Mallat and Y. Meyer have been especially influential. The first family of wavelets (parameterized by p, the smoothness factor) with all of these properties is due to I. Daubechies [1988]. We refer the reader to Strang's [1989] excellent review article for further explanation and a bibliography.

The simplest wavelet basis with $p = 1$ is called the Haar basis. The function $W(x)$ for the Haar basis is

$$W(x) = \begin{cases} -1 & \text{for } x \in [0, 1/2] \\ 1 & \text{for } x \in [1/2, 1] \\ 0 & \text{elsewhere.} \end{cases}$$

This function is plotted in Figure 1. Beylkin, Coifman and Rokhlin [1989] (referred to as the BCR) have observed that a wavelet basis is a powerful tool for evaluating integrals of the form:

$$f(x) = \int_{\mathbf{R}} K(x,y) g(y)\,dy \tag{2}$$

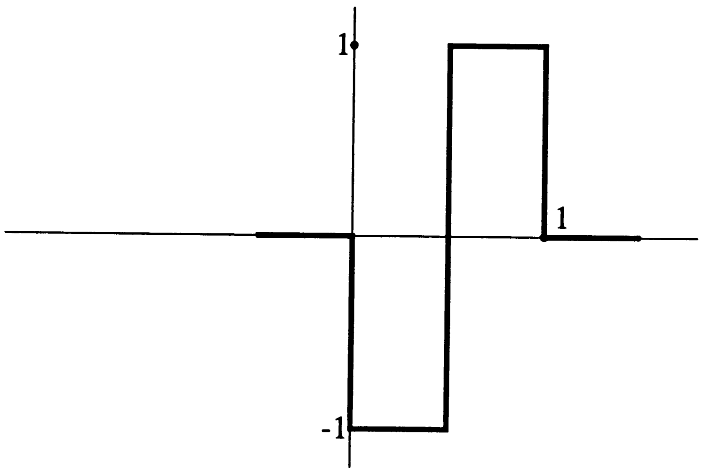

Figure 1: The function $W(x)$ for the Haar wavelet basis

rapidly for N values of x (rapidly means in time proportional to N or $N \log N$), where $g(y)$ is the "input data." Function $g(y)$ can be expressed in terms of a wavelet basis; this conversion is rapid because of the orthogonality of the basis. If $K(x, y)$ is also expressed in terms of a conforming wavelet basis, then most of the coefficients in the discretized version of $K(x, y)$ will be nearly equal to zero because of (1). Accordingly, an accurate approximation to the preceding integral is obtained by setting to zero the coefficients in the discretized version of $K(x, y)$ that are close to zero. The reader is referred to the BCR paper for further details.

The BCR approach can be applied to the problem of solving an integral equation of the form (2) in which the input data is $f(x)$ and the problem is to compute $g(y)$. In this case an iterative method for linear systems can be applied. The main step of an iterative method is to evaluate (2) for the current guess for $g(y)$, which can be done efficiently as described in the previous paragraph.

An important application of solving integral equations is the boundary element method for partial differential equations. For more information about the method, we refer the reader to the series edited by Brebbia [1987].

Recently, iterative methods for boundary integral equations have received attention. A very attractive approach to boundary integral equations would be the BCR wavelet techniques. A wavelet-like technique was proposed Canning [1990] for a boundary element method.

The BCR results do not apply directly to functions of surfaces embedded in $\mathbb{R}^3$. One approach is to parametrize the surface via coordinates in $\mathbb{R}^2$, and then use the product wavelet basis for $\mathbb{R}^2$ proposed by BCR. The difficulty is that a parametric representation of the surface could be very complicated to formulate, particularly when the surface moves during the course of the problem. This is the situation for the application described by Vavasis [1990].

In the next section we propose a basis for functions on a surface based on a triangulation of the surface. This approach is free of parametrization. In addition, triangulations of the surface are likely to be available since they are needed for traditional boundary element methods. The kind of triangulation needed for our approach is a special hierarchical triangulation.

Our basis has many of the properties of the wavelet basis, including hierarchical elements, local support (compact support in this case is meaningless since the surface itself is compact) and orthogonality.

Our wavelet basis is not useful for integral equations because it is not able to efficiently approximate functions—it can approximate smooth functions with accuracy of only $O(h)$. In the third section we indicate an approach to defining smoother wavelet functions.

2 The basis

The surface S under consideration should be compact, without boundary and (infinitely) smooth. The results of this section can be generalized to surfaces with boundaries or to surfaces with certain breaks in the smoothness, but these generalizations would not add anything to the presentation.

The basis functions are based an infinite sequence of triangulations of the surface, $T_0, T_1, \ldots$. Here, triangulation means a topological triangulation. The skeleton of the triangulation T_i, denoted $Sk(T_i)$, is the union of a finite number of nodes and edges. The nodes are points in S, and the edges are sufficiently smooth curves lying on S whose endpoints are nodes. The only points that two edges could have in common are nodes. The nodes and edges make up a triangulation provided the following property holds: each connected component of $S - Sk(T_i)$ is homeomorphic to a triangle and is incident upon exactly three nodes and three edges. We will refer to the closures of these components as "triangles." We regard T_i itself as the

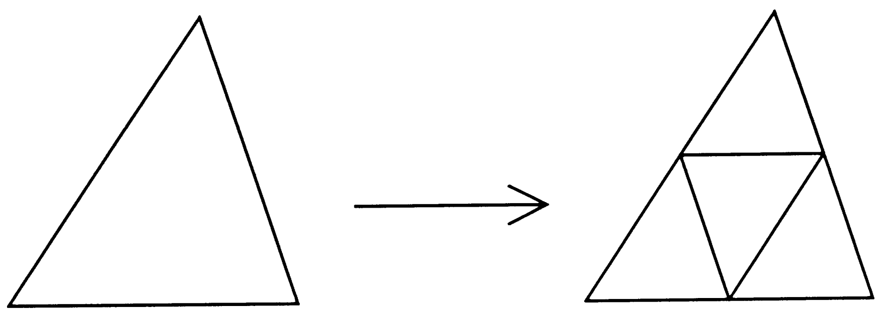

Figure 2: Refining T_i to get T_{i+1}

collection of the triangles. Thus, the union of the triangles in T_i is S, and the intersection of two triangles in T_i must lie in $Sk(T_i)$.

We think of T_0 as a very coarse triangulation that is able to capture the gross topological features of S plus any corners or edges of S in the nonsmooth case.

Triangulation T_i is a refinement of T_{i-1}. In particular, each triangle of T_{i-1} is the union of exactly four triangles of T_i. This is accomplished by positioning an additional node at the midpoint of every edge of $Sk(T_{i-1})$, which subdivides the edge into two edges. Then, three additional edges are inserted into the interior of each triangle of T_{i-1}. Figure 2 gives an example of this subdivision process. We require that the four triangles in T_i arising from one triangle of T_{i-1} all have equal surface area.

Our basis consists of functions p_τ for all triangles $\tau \in T_0$ and q_τ, r_τ, s_τ for all triangles $\tau \in T_0 \cup T_1 \cup T_2 \cup \cdots$ The definitions are as follows. For a particular $\tau \in T_0$, define

$$p_\tau(x) = \begin{cases} 1 & \text{for } x \in \tau \\ 0 & \text{for } x \in S - \tau. \end{cases}$$

For the other basis functions, let τ be a triangle in T_i for some i. Let $\tau_1, \tau_2, \tau_3, \tau_4$ be the four triangles in T_{i+1} whose union is τ. Let τ_1 be the "interior" subtriangle, i.e., the subtriangle that has no common edge with

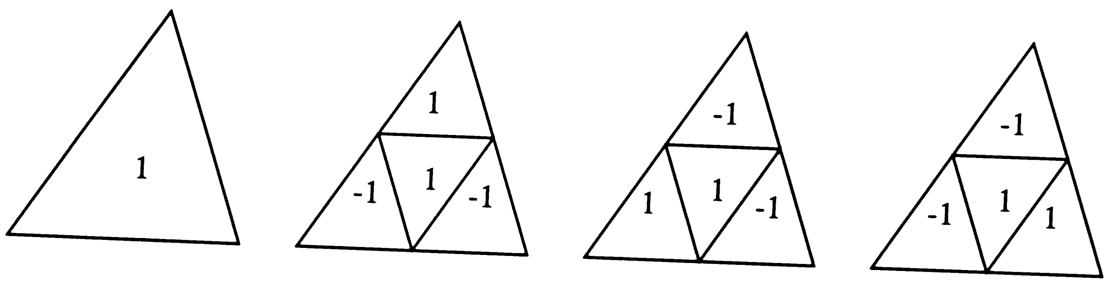

Figure 3: The basis functions $p_\tau, q_\tau, r_\tau, s_\tau$.

τ. Then we define

$$q_\tau(x) = \begin{cases} 1 & \text{for } x \in \tau_1 \cup \tau_2 \\ -1 & \text{for } x \in \tau - (\tau_1 \cup \tau_2). \\ 0 & \text{for } x \in S - \tau \end{cases}$$

$$r_\tau(x) = \begin{cases} 1 & \text{for } x \in \tau_1 \cup \tau_3 \\ -1 & \text{for } x \in \tau - (\tau_1 \cup \tau_3). \\ 0 & \text{for } x \in S - \tau \end{cases}$$

$$s_\tau(x) = \begin{cases} 1 & \text{for } x \in \tau_1 \cup \tau_4 \\ -1 & \text{for } x \in \tau - (\tau_1 \cup \tau_4). \\ 0 & \text{for } x \in S - \tau \end{cases}$$

These basis functions are illustrated in Figure 3. We now establish that they have the wavelet properties. Let $\mathcal{F}$ denote the family of functions defined above, i.e.,

$$\mathcal{F} = \{p_\tau : \tau \in T_0\} \cup \{q_\tau, r_\tau, s_\tau : \tau \in T_0 \cup T_1 \cup \cdots\}.$$

Theorem 1 *Let f, g be two distinct functions in $\mathcal{F}$. Then f, g are orthogonal in the L^2 inner product.*

Proof. There are several cases to consider, all of them quite simple. The first case is that the supports of f, g are disjoint or overlap only on the skeleton, in which case the result follows because the skeleton has measure zero.

The next case is that f, g have the same support; say the support is a triangle τ. There are several cases to consider. For example, one case is the inner product of q_τ with r_τ. We see that this inner product is equal to

$$\text{area}(\tau_1) + \text{area}(\tau_4) - \text{area}(\tau_2) - \text{area}(\tau_3).$$

Since $\tau_1, \tau_2, \tau_3, \tau_4$ have equal area, this sum cancels. The other cases yield the same result.

The last case is that the support of f strictly contains the support of g. Then its follows from the definitions that f is in fact constant on the support of g. Accordingly, we need to check only that the integral of g is 0, which is the case for q_τ, r_τ, s_τ. ∎

Theorem 2 *For any $\epsilon > 0$ and for any $f \in L^2(S)$, there is a function $\hat{f}$ that is a linear combination of functions in $\mathcal{F}$ such that $\|f - \hat{f}\| < \epsilon$.*

The norm in this theorem is the L^2 norm. Another way of stating this theorem is that $\mathcal{F}$ is *complete* for $L^2(S)$.

Proof. Let $\mathcal{L}_i$ be the following class of functions from S to $\mathbb{R}$: $\hat{f}$ is in $\mathcal{L}_i$ iff it is constant on every component of $S - Sk(T_i)$. It may be arbitrary on $Sk(T_i)$. Note that if $\tau \in T_i$ then $q_\tau, r_\tau, s_\tau \in \mathcal{L}_{i+1}$.

It follows from standard analysis that for some i large enough there is a function $\hat{f}$ in $\mathcal{L}_i$ such that $\|f - \hat{f}\| < \epsilon$.

Accordingly, it suffices to show that every function in $\mathcal{L}_i$ can be expressed as a linear combination of functions in $\mathcal{F}$ (modulo disagreements on the skeleton of T_i). In particular, we claim any function $\hat{f} \in \mathcal{L}_i$ can be expressed as a linear combination of p_τ for $\tau \in T_0$ and q_τ, r_τ, s_τ for $\tau \in T_0 \cup \cdots \cup T_{i-1}$. This is proved by induction on i. The base case of $i = 0$ is clear, because $\{p_\tau : \tau \in T_0\}$ is the obvious basis for $\mathcal{L}_0$.

For the induction step, let $\hat{f}$ lie in $\mathcal{L}_i$. Focus on a triangle $\tau \in T_{i-1}$. Let $\hat{f}_\tau$ be the function that agrees with $\hat{f}$ on τ and is zero elsewhere. The idea is to express $\hat{f}_\tau$ as a linear combination of a constant function on τ and q_τ, r_τ, s_τ.

Let v_τ be the function that is 1 on τ and 0 elsewhere on S. Let the subdivision of τ in T_i be $\tau_1, \tau_2, \tau_3, \tau_4$. Let the constant values of $\hat{f}$ on these four subtriangles be denoted a_1, a_2, a_3, a_4. Then it is easy to see that

$$\hat{f}_\tau = b_1 q_\tau + b_2 r_\tau + b_3 s_\tau + b_4 v_\tau + u$$

where u is a function that is nonzero only on the skeleton of T_i and b_1, b_2, b_3, b_4 satisfy the equations:

$$\begin{aligned} b_1 + b_2 + b_3 + b_4 &= a_1 \\ b_1 - b_2 - b_3 + b_4 &= a_2 \\ -b_1 + b_2 - b_3 + b_4 &= a_3 \\ -b_1 - b_2 + b_3 + b_4 &= a_4 \end{aligned}$$

The first equation states that $\hat{f}_\tau$ must agree with $b_1 q_\tau + b_2 r_\tau + b_3 s_\tau + b_4 v_\tau + u$ on τ_1, and so on.

Now it is easy to see that b_1, b_2, b_3, b_4 exist to satisfy this particular system of equations; in fact, the columns of this system are orthogonal so the system is certainly nonsingular.

This shows that $\hat{f}_\tau$ can be expressed in terms of $q_\tau, r_\tau, s_\tau, v_\tau$. Therefore, $\hat{f}$ can be expressed as a linear combination of these functions as τ ranges over all triangles. The functions q_τ, r_τ, s_τ are in $\mathcal{F}$, and the functions v_τ are in $\mathcal{L}_{i-1}$ and therefore the induction hypothesis can be applied. ∎

The two theorems of this section show that the basis constructed has the desirable properties of the wavelet basis—in particular, the local support and the orthogonality. The main difficulty with this basis is that the analog of (1) holds only for $m = 0$ (in fact, it fails for the basis elements p_τ even for $m = 0$, but this case does not matter in the integral equation applications).

3 Smoother wavelets on surfaces

As mentioned in the introduction, it would be very interesting to have a wavelet basis for surfaces with a greater degree of smoothness. It might be possible to construct such a basis recursively in a manner similar to Daubechies' construction.

In particular, we can think about first defining a "scaling" function following Strang's terminology. The scaling function v_τ in the last section was the constant function 1 on a triangle τ. It satisfies the formula $v_\tau =$

$v_{\tau_1} + v_{\tau_2} + v_{\tau_3} + v_{\tau_4}$. The idea would be to come up with a more complicated recursive formula for v_τ, possibly involving subtriangles of neighbors of τ, and not only the four subtriangles of τ. From this scaling function the wavelet could be constructed. For this approach to succeed, the right coefficients of the recursion must be discovered.

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