Guaranteed-Quality Triangular Meshes

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Abstract

There are a number of applications for which it is desirable to divide a given region in the plane into nicely shaped triangles. One important such application is the finite element method, a method widely used to obtain approximate solutions to a wide variety of engineering problems. For this kind of application, not just any triangulation will do; error bounds are best if all the triangles are as close as possible to equilateral triangles. Presently, either these triangulations are produced by hand or they are produced automatically using one of a number of heuristic techniques. For these heuristic techniques, certain cases can require human intervention to eliminate flat triangles. In this paper, we present an efficient new technique (based on *Delaunay triangulations*) for automatically producing desirable triangulations. Unlike most previous techniques, this one comes with a guarantee: the angles in the resulting triangles are all between 30° and 120°, and the edge lengths are all between h and 2h where h is a parameter chosen by the user. Additional useful properties include (1) the worst-case time to produce a triangulation is linear in the final number of triangles, and (2) the user can control the element density, producing smaller triangles in areas where more accuracy is desired.

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1. Introduction.

We study the following problem: given a polygonal region in the plane, divide the region into triangles in such a way that the triangles are as close as possible to equilateral triangles. We refer to the process of dividing a region into triangles as *triangulation* and the resulting set of triangles is called a *triangular mesh* for the region. A program for creating a mesh is called an *automatic mesh generator*. The reader should note that unlike many triangulation problems (e.g., triangulating a set of points in the plane) in which only edges are introduced, an automatic mesh generator can introduce both new vertices and new edges.

This problem is motivated by the requirements of the finite element method, a widely used technique for obtaining approximate numerical solutions to a wide variety of engineering problems. Given a problem, the first step of the finite element method is to divide the problem region into finitely many simply-shaped regions called elements, creating a finite element mesh. In two dimensions, this usually means dividing a given region into either triangles or quadrilaterals. Triangular elements are sometimes preferred because of the ease with which they can be made to fit complex boundaries. Not just any triangulation will do; error bounds are best if the triangles are as close as possible to equilateral triangles. (For more information on the finite element method, see, for instance, [HT82, OR76, SF73, or Zi77].)

Triangulating a planar region by hand (i.e., using a mouse or other input device and a graphics terminal) can be a tedious process; thus, a number of automatic mesh generators have been developed (for instance, [BW&87, Ca74, CFF85, Fr87, Jo86, JS86, Lo85, YS83], and many others; [Sh88, Si79, and Th80] are survey papers on this subject). Several of these mesh generators use an iterative smoothing process to improve the resulting meshes. Even with smoothing, these algorithms do not provide a guarantee about the quality of the result, and human intervention may sometimes be required to improve the triangulation.

In this paper, we present a new algorithm for triangulating planar regions. Unlike most previous methods, this new algorithm comes with a guarantee: for a triangulation produced by the algorithm, all angles are between 30° and 120° and all edge lengths are between h and 2h where h is a parameter chosen by the user. In addition, the algorithm can be implemented to run in worst-case time O(n) where n is the number of triangles in a triangulation produced by an imaginary perfect triangulation algorithm. Further, the user can control the element density, producing smaller triangles in areas where more accuracy is desired.

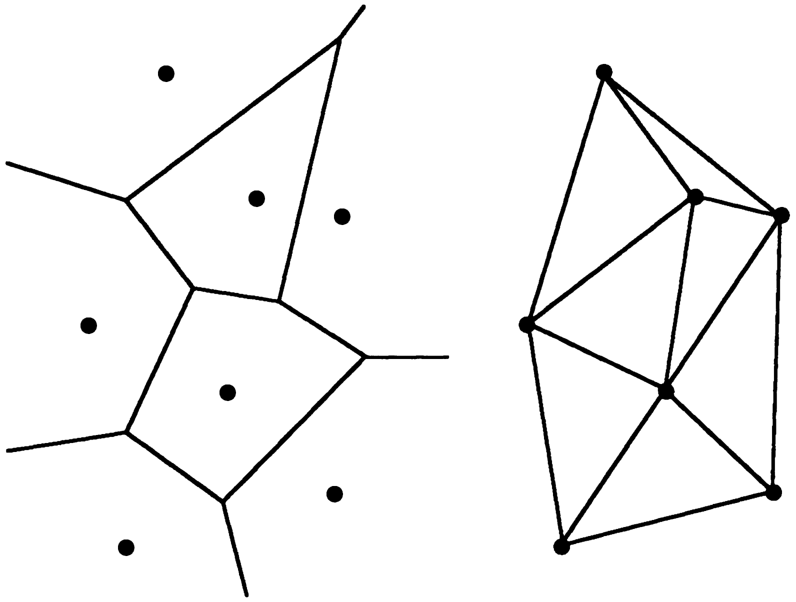
The guarantee associated with this algorithm is particularly important. Without such a guarantee, an automatic mesh generator may sometimes require human inspection of the mesh. One problem with this requirement is that it is inappropriate for the naive user of the finite element method. More significantly, this requirement is inappropriate for even the expert user when solving problems where many meshes need to be generated, such as problems involving adaptive remeshing, or problems where the geometry changes over time. A guaranteed-quality mesh generator can make human inspection of the mesh unnecessary.

Baker, Grosse, and Rafferty [BGR88] have also presented a 2D triangulation algorithm that includes a guarantee. They use a completely different technique for which the resulting angles are guaranteed to be between 13° and 90°. They place a grid over the region to be triangulated and observe that each interior cell can be triangulated with a single diagonal. They then develop methods for triangulating the boundary cells. This technique leads to small triangles near the boundary of the given region.

In the next section we explain some of the background necessary for understanding the guaranteed-quality triangulation algorithm. In Section 3 we explain the basic algorithm. The linear time version of the algorithm is explained in Section 4, and section 5 contains a comparison of the algorithm with an imaginary perfect triangulation algorithm. Section 6 explains how the sizes of the elements may be varied across a region. Finally, Section 7 presents some conclusions and further research.

2. Background - Delaunay Triangulations.

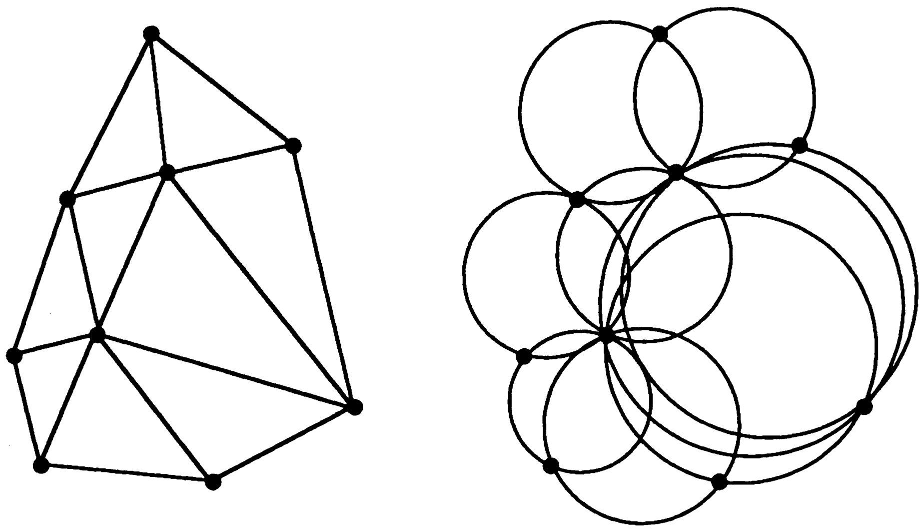
Like some earlier techniques ([CFF85, Fr87, and Jo86], for instance) the guaranteed-quality triangulation technique is based on properties of *Delaunay triangulations*. In this section we present some of the properties of the Delaunay triangulations and we explain a special type of Delaunay triangulation, called a *constrained Delaunay triangulation*, which has characteristics particularly useful for mesh generators.



A Voronoi diagram and the corresponding Delaunay triangulation.

The *Delaunay triangulation* of a set S of points in the plane is most easily introduced by reference to the *Voronoi diagram* of S (also called the *Dirichlet tessalation* of S). The Voronoi diagram of S divides the plane into regions, one region for each point in S, such that for each region R and corresponding point p, every point within R is closer to p than to any other point of S. The boundaries of these regions form a planar graph. The Delaunay triangulation of S is the straight-line dual of the Voronoi diagram of S; that is, we connect a pair of points in S iff they share a Voronoi boundary. The Voronoi diagram and its dual, the Delaunay triangulation, have been found to be among the most useful data structures in computational geometry. See [Ed87 or PS85] for a number of Voronoi diagram and Delaunay triangulation applications.

Each triangle of the Delaunay triangulation of S has the *empty circle property*: a circle circumscribed about a Delaunay triangle contains no points of S in its interior. Indeed, this property can be used as the definition of Delaunay triangulation.



A Delaunay triangulation and the corresponding empty circles.

Definition. Let S be a set of points in the plane. A triangulation **T** is a *Delaunay triangulation* of S if for each triangular face Δ of T there exists a circle C with the following properties:

1. C circumscribes Δ, and
2. no vertex of S is in the interior of C.

A circle circumscribed about about a Delaunay triangle is called a *Delaunay* *circle.* If S contains 4 points that are cocircular then the Delaunay triangulation is not necessarily unique. For our purposes, if there is not a unique Delaunay triangulation then any of them will do.

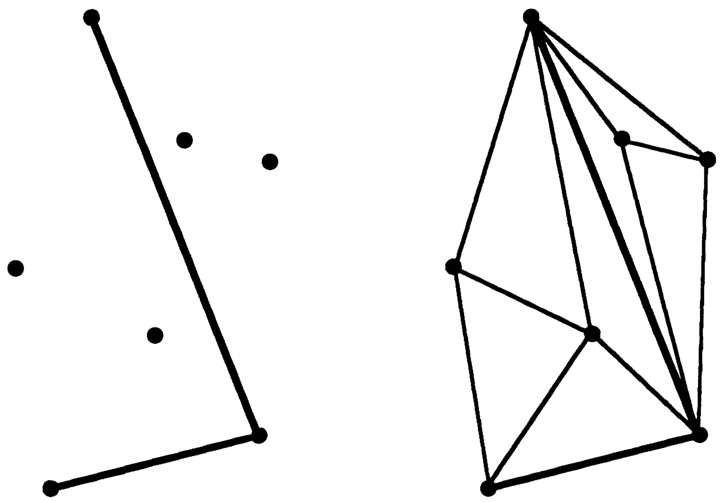
The Delaunay triangulation has some properties indicating that it might lead to good finite element meshes. In particular, the Delaunay triangulation maximizes the minimum angle for a set of points. In other words, among all triangulations of a given set of points, the Delaunay triangulation has the largest minimum angle (see, for instance, [Ed87]).

Unfortunately, the Delaunay triangulation is not quite what is needed to make triangulations for the finite element method. The problem is, for a finite element mesh, certain edges must be used as part of the final triangulation (e.g., those that describe the boundaries or, perhaps, those that describe a crack in the object to be analyzed). Such edges do not always correspond to legal Delaunay edges. Schroeder and Shephard [SS88] discuss some of the difficulties of using a Delaunay triangulation to create a mesh for an object.

Many of these difficulties can be resolved by using a *constrained Delaunay triangulation (CDT).* Intuitively, a CDT is as close as possible to a Delaunay triangulation given that certain prespecified edges must be included in the triangulation. Compare the definition of the CDT with the definition of the (unconstrained) Delaunay triangulation given above.

Definition. Let G be a straight-line planar graph. A triangulation T of the vertices of G is a *constrained Delaunay triangulation* (CDT) of G if each edge of G is an edge of T and for each triangular face Δ of T there exists a circle C with the following properties:

1. C circumscribes Δ, and
2. if any vertex v of G is in the interior of C then it cannot be "seen" from at least one of the vertices of Δ (i.e., if you draw the line segments from v to each vertex of Δ then at least one of the line segments crosses an edge of G).



A graph G and the corresponding constrained Delaunay triangulation.

.The same term, *Delaunay circle*, is used for a circle circumscribed about either a standard Delaunay triangle or a CDT triangle. The CDT, also called a *generalized Delaunay triangulation* [Le78, LL86], was first introduced by Lee. $O(n^2)$ time algorithms for constructing the CDT of a straight-line planar graph G (with n vertices) are given in [DFP85 and LL86]. An asymptotically-optimal, O(n log n) worst-case time algorithm appears in [Ch89a]. Surprisingly, the triangulation technique introduced in Section 4 can be implemented to triangulate a region in worst-case linear time (linear in the number of triangles produced) even though the resulting triangulation is a CDT. Applications of CDTs appear in [Ch89a, Ch89b, DFP85, Le78, and LL86].

3. How to Triangulate.

In this section we present one version of our automatic mesh generator. We show how properties of the CDT can be used to prove that the triangles in the resulting mesh are of guaranteed quality. The algorithm presented in this section is relatively slow, requiring $O(n^2)$ time in the worst case (n is the number of triangles in the final triangulation). Section 4 contains a modification of this algorithm that creates a guaranteed-quality triangulation in worst-case linear time.

There are some undemanding preconditions that the initial problem must satisfy. The input to the algorithm is a set of data points and data edges. Basically, these points and edges are those necessary to describe the boundary of the region, but, in fact, we allow the user to specify additional points and edges. All points and edges specified at this stage will appear in the final triangular mesh. The data points and data edges must satisfy the following two conditions (recall that h is a parameter chosen by the user; intuitively, it represents the desired side-length of triangles in the triangulation):

1. No two data points are closer than h.
2. All data edges have lengths between h and $(\sqrt{3})h$.

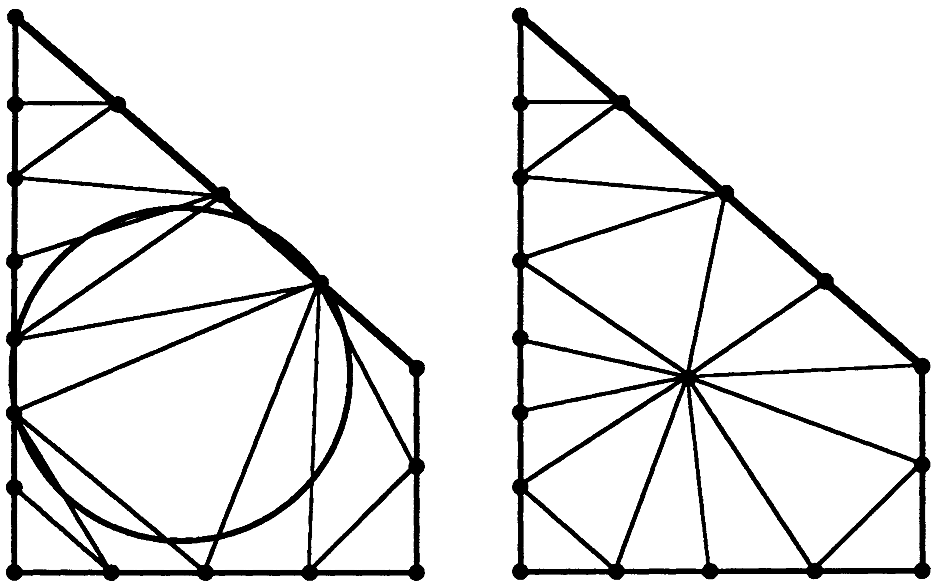
In practice, it is trivial to subdivide long edges to comply with condition two, except for edges with lengths between $(\sqrt{3})h$ and 2h. Later in this section, we show how these edges can, in effect, be hidden, although when this is done, care must be taken to ensure that condition 1 is not compromised.

Preconditions:

The input is a set of data points and data edges. This set describes the region to be triangulated, but may include additional data points and data edges. No two data points are closer than h. All data edges have lengths between h and $(\sqrt{3})h$.

Algorithm:

begin
Compute the CDT of the data points and data edges;
Let T be the portion of the CDT that is within the region to be triangulated;
while there is a Delaunay circle within T that has radius > h **do**
  Add the center of the Delaunay circle as a new data point;
  Recompute T;
  end while;
Report T as the desired triangulation;
end.



Add the center point of a large-radius Delaunay circle and recompute the CDT.

Theorem 1. T, the final triangulation produced by the preceding algorithm, satisfies the following two properties.

1. No two vertices of T are closer than h.
2. Every triangle of T fits within a circle of radius $\leq$ h.

Proof. First, note that property 1 holds when the algorithm begins and continues to hold throughout the execution of the algorithm. To see this, note that all points added are at the centers of Delaunay circles with radius greater than h.

Second, no point is closer than distance h/2 to a data edge. To see this, consider a point p and let e be the closest edge to p. Since e has length $\leq (\sqrt{3})h$, if point p is closer than h/2 to such an edge then it is also closer than h to an endpoint of the edge, a contradiction.

Finally, it is obvious that property 2 holds if the algorithm halts. To see that it halts, note that for each new point added during the execution of the algorithm, the circle of radius h/2 about the point is empty, containing no other points and no portions of data edges. Further, from property 1, it is clear that these circles do not overlap. Only finitely many such circles can fit within a bounded region; thus, since a new circle is added each time through the loop, the algorithm must halt. ◊

Corollary. T, the final triangulation produced by the preceding algorithm, satisfies the following additional properties.

3. All edges of T have lengths between h and 2h.
4. All angles of T are between 30° and 120°.

Proof. Property 3 obviously follows from properties 1 and 2. To see that property 4 also follows, consider any triangle of the triangulation and let α be the smallest angle of that triangle. From property 2, the circle circumscribed about the triangle has radius $\le$ h, and from property 1, the edge opposite angle α has length $\ge$ h. It immediately follows that the central angle corresponding to α is $\ge 60°$ and α itself is $\ge 30°$. Once we know the smallest angle is $\ge 30°$, it is clear that the greatest angle is $\le 120°$. ◊

The second precondition (all data edges have length between h and $(\sqrt{3})h$) can be relaxed. Of course, for an edge longer than 2h, new data points can be introduced to divide the edge into pieces with lengths in the required range. However, if an edge has length between $(\sqrt{3})h$ and 2h, a problem occurs: such an edge cannot be divided into appropriately sized pieces. This difficulty can be avoided by, in effect, hiding such edges. To do this, place a new data point in the interior of the region to make a 45° isosceles triangle (with the problem edge as its hypotenuse); connect the endpoints of the problem edge to this new data point. These new edges have lengths in the required range. Now, when the algorithm is executed, no points can be placed within the triangle (such a new point would be too close to one of the vertices of the triangle) and the triangle itself satisfies the properties of the theorem (it fits within a circle of radius h and no two of its points are closer than h). Of course, care must be taken to ensure that any new data point introduced to hide an edge is no closer than h to any other data point.

The algorithm as outlined in this section can be implemented to run in worst-case time $O(n^2)$, where n is the number of triangles in the final triangulation. The initial CDT can be built in time $O(n \log n)$ [Ch89a] or, using a simpler algorithm, in time $O(n^2)$. Each point that is inserted can require up to $O(n)$ processing to update the CDT, giving a total time of $O(n^2)$ for the entire algorithm.

4. How to Triangulate in Optimal Time.

This section contains an outline of the optimal time algorithm. For the optimal time algorithm, we look at the triangulation problem a little differently; instead of starting with data points and data edges and building a triangulation, we start with an independent triangulation and modify it as we add the data points and data edges. We assume the data points and data edges satisfy the preconditions stated above. In addition, we assume that the data points and edges form a connected graph. (The algorithm can be modified to handle multiple components.) Data points will be added to the triangulation in depth-first-search order.

We start with a regular grid of equilateral triangles with edge length $(\sqrt{3})h$ and large enough to cover the region to be triangulated. We assume that this regular grid is precomputed; the time needed to construct this grid is not counted as part of the running time of the algorithm. Note that the grid itself is the Delaunay triangulation (also the CDT) of its own vertices. We call these vertices *grid points* to distinguish them from the *data points* describing the region to be triangulated. Initially, T consists entirely of this simple CDT, a triangulation using only grid points and no data points. The data points are added one at a time to T in depth-first-search order; when a data point is added we also add any data edges between it and data points already in T. Depth-first-search order is used so that we never have to search T to find where to put a data point; a data point is always placed near a previous data point.

T is modified after each addition to ensure that T is always a CDT and to ensure that T satisfies the properties of Theorem 1. To do this, we execute the following steps:

(1) Eliminate any points closer than h to the new data point;

(2) Make T into a CDT again;

(3) **while** there exists a large (radius $>$ h) Delaunay circle **do**

Add the circle's center point and make T into a CDT again.

The important observation is that all these changes to T are local changes (within radius 4h of the newly added data point). This observation can be used to show that the algorithm runs quickly. Since the only changes in the diagram are within radius 4h of the new point, and since the points of T cannot be too close to each other, the number of points affected by a newly added data point can be bounded by a small constant. In other words, it takes just constant time to update T for each new data point; thus, the total time is O(n) where n is the number of data points.

There are many possible variations on this algorithm depending on the triangular mesh desired. For instance, the edge length in the initial grid can be different. It also works with a regular grid of squares (with diagonals added to make triangles) or rectangles. It is even possible to start with an irregular grid provided the grid is a Delaunay triangulation in which each Delaunay circle has radius less than h.

5. Comparison with an Optimal Triangulation.

In this section we show that the guaranteed-quality triangulation algorithm produces an optimal number of triangles. In other words, any mesh generator that produces a mesh in which the edge length is about h, will produce about the same number of triangles. Thus, the linear time version described in the previous section is optimal in the sense that any mesh generating algorithm will have to take at least linear time just to report the mesh.

An imaginary perfect triangulation algorithm would allow the user to specify the parameter h and would then completely divide a given region into equilateral triangles where all triangles have edges of length h. Of course, not every region can be subdivided in this way, but such a triangulation represents an ideal goal. Each equilateral triangle in such a triangulation has area $(\sqrt{3})h/4$; thus, if A is the area of the region to be triangulated then the number of triangles is $4A/(h\sqrt{3})$.

Our technique matches this ideal goal. The area of the smallest possible triangle that can be produced by this technique is $(\sqrt{3})h/4$; thus, the maximum number of triangles that can be produced for a region of area A is $(4A)/(h\sqrt{3})$. In other words, the guaranteed-quality triangulation algorithm presented here creates about the same number of triangles as an imaginary ideal algorithm.

6. Varying the Density.

It is sometimes desirable to divide a region into triangles of different sizes. For example, for the finite element method, small elements generally give greater accuracy than large elements, but it takes longer to process small elements. Because of this, it can be desirable to have small elements where something interesting is occurring (e.g., near the boundary of the region) to ensure accuracy, with large elements elsewhere (e.g., in the interior of the region) to save processing time. In this section we describe one way in which the element density can be varied without compromising the quality of the resulting mesh. To do this, we create artificial boundaries.

To see how this can be done, recall the precondition needed by the guaranteed-quality triangulation algorithm: all boundary edges must have lengths between h and $(\sqrt{3})h$. This was the necessary precondition for using circles of radius h. If we use circles with radius $(\sqrt{3})h$ then the precondition for the edge lengths must be altered in proportion; in this case, all boundary edges would be required to have lengths between $(\sqrt{3})h$ and 3h. Note that if a boundary consists entirely of edges of length $(\sqrt{3})h$, then on one side of the boundary we can use circles of radius h and on the other side we can use circles of radius $(\sqrt{3})h$.

This idea can be used to vary the element density by inserting artificial boundaries between regions of different density. Wherever we want a change in the size of triangles, we insert an artificial boundary, allowing the element size to change across the boundary by a factor of $\sqrt{3}$. If a larger factor is desired then more artificial boundaries can be used. In practice, it may not be desirable to allow $\sqrt{3}$ size jumps because of the difficulty of getting all the edges of an artificial boundary to have exactly the same length, but it should easy enough to allow jumps of size, say, 3/2.

7. Conclusions and Further Research.

Provided the input satisfies some undemanding preconditions, the triangulation technique introduced in this paper has the following properties:

(1) the resulting triangulation contains only nicely shaped triangles;

(2) the user can vary the triangle sizes over the region;

(3) the number of triangles produced is near optimal; and

(4) the technique can be implemented to run in optimal, linear time.

In addition, the technique may lead to a number of interesting applications and extensions.

The guaranteed-quality technique should be applicable to the triangulation of shells (curved surfaces). Basically, instead of looking at Delaunay circles, we look at the corresponding spheres where a sphere is constrained to have its center on the surface.

In addition to the finite element method, other application areas for guaranteed-quality triangulations include virtually any problem where it is desirable to choose a set of points that evenly fills a given region in the plane or that evenly covers a given surface. For instance, in [DJ87] a set of sample points describing a surface is used to detect errors in numerically controlled machining. The triangulation technique presented here, altered to work for surfaces, gives a good way to select the sample points. There are other possible uses in graphics and geology.

Of particular interest, is the possibility of extending the algorithm presented here to develop a guaranteed-quality mesh generator for problems in three dimensions. For the finite element method in three dimensions, the goal is to divide an object into nicely shaped tetrahedra (or hexahedra or triangular prisms). We refer to the process of dividing an object into tetrahedra as *3D triangulation*.

The basis of the 2-dimensional technique, the Delaunay triangulation, extends nicely to 3 dimensions, producing tetrahedra with nicely shaped faces. Unfortunately, a guarantee about the shapes of the tetrahedral faces does not imply a guarantee about the quality of the tetrahedra; in the terminology of [CFF85], *slivers* can occur. *Slivers* are tetrahedra with well-proportioned faces, but with arbitrarily small volume. An example is the degenerate tetrahedron defined by 4 points equally spaced around the equator of a sphere.

Several heuristic techniques for 3D triangulation have been developed [CFF85, Fi86, PP&88, and YS84]; [Sh88] surveys several 3D triangulation techniques. These techniques are heuristic in the sense that there is no guarantee about the quality of the resulting 3D mesh. A 3D triangulation technique with guaranteed performance, a technique for which all tetrahedra are guaranteed to be close to regular tetrahedra, would be of significant interest in Computer Aided Design.

References

[BGR88] B. S. Baker, E. Grosse, C. S. Rafferty, Nonobtuse Triangulation of Polygons, *Discrete and Computational Geometry*, **3** (1988), 147-168.

[BW&87] P. L. Baehmann, S. L. Wittchen, M. S. Shephard, K. R. Grice, and M. A. Yerry, Robust, geometrically based, automatic two-dimensional mesh generation, *International Journal for Numerical Methods in Engineering*, **24** (1987), 1043-1078.

[Ca74] J. C. Cavendish, Automatic triangulation of arbitrary planar domains for the finite element method, *International Journal for Numerical Methods in Engineering*, **8** (1974), 679-696.

[CFF85] J. C. Cavendish, D. A. Field, and W. H. Frey, An approach to automatic three-dimensional finite element mesh generation, *International Journal for Numerical Methods in Engineering*, **21** (1985), 329-347.

[Ch89a] L. P. Chew, Constrained Delaunay triangulations, *Algorithmica*, **4** (1989), 97-108.

[Ch89b] L. P. Chew, There are planar graphs almost as good as the complete graph, *JCSS*, to appear.

[DFP85] L. De Floriani, B. Falcidieno, and C. Pienovi, Delaunay-based representation of surfaces defined over arbitrarily shaped domains, *Computer Vision, Graphics, and Image Processing*, **32** (1985), 127-140.

[DJ87] R. L. Drysdale and R. B. Jerard, Discrete simulation of NC machining, *Proceedings of the Third Annual Symposium on Computational Geometry*, (1987), 126-135.

[Ed87] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer-Verlag (1987).

[Fi86] D. A. Field, Implementing Watson's algorithm in three dimensions, *Proceedings of the Second Annual Symposium on Computational Geometry*, (1986), 246-259.

[Fr87] W. H. Frey, Selective refinement: a new strategy for automatic node placement in graded triangular meshes, *International Journal for Numerical Methods in Engineering*, **24** (1987), 2183-2200.

[HT82] K. H. Huebner and E. A. Thornton, *The Finite Element Method for Engineers*, John Wiley & Sons (1982).

[Jo86] B. Joe, Delaunay triangular meshes in convex polygons, *SIAM J. Sci. Stat. Comput.*, **7**:2 (1986), 514-539.

[JS86] B. Joe and R. B. Simpson, Triangular meshes for regions of complicated shape, *International Journal for Numerical Methods in Engineering*, **23** (1986), 751-778.

[Le78] D. T. Lee, Proximity and reachability in the plane, Technical Report R-831, Coordinated Science Laboratory, University of Illinois (1978).

[LL86] D. T. Lee and A. K. Lin, Generalized Delaunay triangulation for planar graphs, *Discrete and Computational Geometry*, **1** (1986), 201-217.

[Lo85] S. H. Lo, A new mesh generation scheme for arbitrary planar domains, *International Journal for Numerical Methods in Engineering*, **21**, 1403-1426 (1985).

[OR76] J. T. Oden and J. N. Reddy, *An Introduction to the Mathematical Theory of Finite Elements*, John Wiley & Sons (1976).

[PP&88] J. Peraire, J. Peiro, L. Formaggio, K. Morgan, and O. C. Zienkiewicz, Finite element Euler computations in three dimensions, *International Journal for Numerical Methods in Engineering*, **26** (1988), 2135-2159.

[PS85] F. P. Preparata and M. I. Shamos, *Computational Geometry*, Springer-Verlag (1985).

[SF73] G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall (1973).

[Sh88] M. S. Shephard, Approaches to the automatic generation and control of finite element meshes, *Applied Mechanics Reviews,* **41**:4 (April 1988), 169-185.

[Si79] R. B. Simpson, A survey of two dimensional finite element mesh generation, *Proc. of the Ninth Manitoba Conference on Numerical Math. and Computing*, (1979), 49-124.

[SS88] W. J. Schroeder and M. S. Shephard, Geometry-based fully automatic mesh generation and the Delaunay triangulation, *International Journal for Numerical Methods in Engineering*, **26** (1988), 2503-2515.

[Th80] W. C. Thacker, A brief review of techniques for generating irregular computational grids, *International Journal for Numerical Methods in Engineering*, **15** (1980), 1335-1341.

[YS83] M. A. Yerry and M. S. Shephard, A modified quadtree approach to finite element mesh generation, *IEEE Computer Graphics & Appl.*, **3**:1 (1983), 39-46.

[YS84] M. A. Yerry and M. S. Shephard, Automatic three-dimensional mesh generation by the modified-octree technique, *International Journal for Numerical Methods in Engineering,* **20** (1984), 1965-1990.

[Zi77] O. C. Zienkiewicz, *The Finite Element Method*, McGraw-Hill (1977).