Second Order Kinematic Constraint Between Two Bodies Rolling, Twisting and Slipping Against Each Other While Maintaining Point Contact

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Abstract

The second order kinematic constraint (acceleration constraint) between two rigid bodies that are rolling, twisting, and slipping against each other while maintaining point contact, is derived by differentiation of the first order constraint and by including the geometry of the surfaces at their contact. This constraint is derived with a view to facilitate the simulation of such motion with general purpose dynamics simulators, and more specifically for the Newton dynamic simulator developed at Cornell University. The constraint is first derived for planar motion and then generalized for motion in three dimensions. Some simple, but representative, examples are presented.

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1 Introduction and Motivation

An important kinematic constraint that occurs between interacting rigid bodies is that they roll, twist, and slide against each other, while maintaining point contact. More specifically, this sort of constraint has to be modeled for modeling dextrous hand manipulation of objects through robotic end-effectors, simulation of compliant motion during assembly and manufacturing, design of automobiles and other mechanisms, etc.; all of which form an important part of computer modeling and simulation activities of current interest [Hopcroft 1988].

When simulating the general three dimensional motion of two rigid bodies that move while maintaining point contact (figure 1), using the Newtonian formulation of their equations of motion [Cremer and Stewart 1989], one encounters the following set of 12 scalar equations[1] (eqns. 1), in 15 scalar variables $(\ddot{\mathbf{r}}_1, \dot{\omega}_1, \ddot{\mathbf{r}}_2, \dot{\omega}_2, \mathbf{F_c})$

$$
\begin{aligned}
m_1\ddot{\mathbf{r}}_1 &= \sum \mathbf{F} + \mathbf{F_c} \\
[\mathbf{J_1}]\dot{\omega}_1 + \omega_1 \times [\mathbf{J_1}]\omega_1 &= \sum(\mathbf{r} \times \mathbf{F}) + \mathbf{c_1} \times \mathbf{F_c} \\
m_2\ddot{\mathbf{r}}_2 &= \sum \mathbf{F} - \mathbf{F_c} \\
[\mathbf{J_2}]\dot{\omega}_2 + \omega_2 \times [\mathbf{J_2}]\omega_2 &= \sum(\mathbf{r} \times \mathbf{F}) - \mathbf{c_2} \times \mathbf{F_c} \qquad (1)
\end{aligned}
$$

involving body masses (m_1 and m_2), linear accelerations of the centers of masses ($\ddot{\mathbf{r}}_1$ and $\ddot{\mathbf{r}}_2$), angular accelerations ($\dot{\omega}_1$ and $\dot{\omega}_2$), mass moments of inertia about mass centers ($[\mathbf{J_1}]$ and $[\mathbf{J_2}]$), external forces and moments ($\mathbf{F}$ and $\mathbf{M}$), and the contact force $\mathbf{F_c}$ between the two bodies (assuming no moment) at their point of contact C.

The three extra equations for getting unique answers to the above system are provided by the kinematic constraints imposed on the system due to the nature of their motion, i.e. the two bodies maintain point contact. We have the following two cases:

1. **The bodies roll and twist without slip at the contact:** Here we assume that the normal and the tangential (friction force) components of the contact force, and the other external forces and moments, conspire to produce pure rolling and twist of the two bodies at their point of contact. This condition leads to a second order kinematic constraint of the form,

$$\mathbf{f_r}(\ddot{\mathbf{r}}_1, \ddot{\mathbf{r}}_2, \dot{\omega}_1, \dot{\omega}_2) = \mathbf{0} \qquad (2)$$

which gives us the extra three equations to make our system determinate. If the solution to the system with the above assumption of no slip between the surfaces yields a contact force $\mathbf{F_c}$ that does not obey the friction law, then obviously this assumption is incorrect. This leads us to the second case.

[1] bold letters denote vectorial entities, e.g. $\mathbf{r_1}$, and ordinary letters their scalar magnitudes, e.g. r_1

2. **The bodies roll, twist and slip against each other while maintaining contact:** This condition leads us to a second order kinematic constraint of the form,

$$\mathbf{f_r}(\ddot{\mathbf{r}}_1, \ddot{\mathbf{r}}_2, \dot{\omega}_1, \dot{\omega}_2, \mathbf{a_s}) = \mathbf{0} \tag{3}$$

where $\mathbf{a_s}$ is the relative slip acceleration between the two bodies at their point of contact C. So along with these three constraint equations, we have also introduced three extra scalar variables ($\mathbf{a_s}$). The friction law (like Coulomb friction) comes to our rescue now giving us three more equations involving the contact force $\mathbf{F_c}$ in the form:

$$\mathbf{f_f}(\mathbf{F_c}) = \mathbf{0} \tag{4}$$

For example Coulomb friction gives us one equation relating the magnitude of the normal component $\mathbf{F_{cn}}$, and the tangential component $\mathbf{F_{ct}}$ (in the tangent plane at the point of contact) of the contact force, and two equations giving the direction of the tangential component of the contact force in terms of the relative slip velocity direction $\mathbf{v_s}$. These equations are of the form:

$$\mathbf{F_{ct}} = \mu F_{cn} \frac{\mathbf{v_s}}{|\mathbf{v_s}|}$$

where μ is the coefficient of friction. Hence equations 1, 3 and 4 give us 18 scalar equations in 18 variables, making the system determinate.

In either of the above two cases, we need the second order kinematic constraint (equations 2 and 3) obtained by differentiation of the first order velocity constraint for rolling. This constraint involves the relative motion parameters between the two surfaces and some geometrical information (principal radii of curvature) on the surfaces at their contact.

Facilitation of the correct and accurate numerical simulation of rolling and slipping motion between general rigid surfaces on the Newton dynamics simulator [Cremer and Stewart 1989] through the implementation of appropriate generalized kinematic constraints, provides the basic motivation for deriving these constraints in the rest of this report. Other things specified, the Newton dynamics simulator solves for linear and angular accelerations $\ddot{r}_1, \ddot{r}_2, \dot{\omega}_1, \dot{\omega}_2, a_s$ from the above set of equations.

More information on the subject of rolling and slipping of surfaces can also be obtained from Neǐmark and Fufaev [1972], Montana [1986], Zexiang, Canny and Sastry [1989].

In the following sections of this paper, rolling and slipping in a plane (2D case) is first analyzed and the kinematic constraint thus obtained is appropriately generalized for three dimensional rolling, twisting and slipping of surfaces.

2 Rolling and Slipping in a Plane

Consider two rigid planar bodies bounded by surfaces S_1 and S_2 that are free to move in the $X-Y$ plane while maintaining point contact, by rolling and slipping against each other as shown in figure 2. Their center of masses are at $\mathbf{r_1}$ and $\mathbf{r_2}$ and their angular velocities are $\omega_\mathbf{1}$ and $\omega_\mathbf{2}$ respectively. Hence the angular velocity of S_1 relative to S_2 is $\omega_\mathbf{r} = \omega_\mathbf{1} - \omega_\mathbf{2}$. The vectors joining the center of masses of the two bodies to the instantaneous point of contact C at $\mathbf{r_c}$, are $\mathbf{c_1}$ and $\mathbf{c_2}$ respectively. S_1 slips against S_2 with a relative slip velocity $\mathbf{v_s}$ at their point of contact C. If $\mathbf{t}$ is a unit vector in the common tangent plane to the two surfaces at C, then $\mathbf{v_s} = v_s\mathbf{t}$, due to the no penetration condition between the two rigid bodies. The rolling and slipping condition between the two surfaces (while maintaining contact) can now be stated as eqn. 5, which says that instantaneously the relative velocity between material points at S_1 and S_2 at C is $\mathbf{v_s}$.

$$\dot{\mathbf{r}}_\mathbf{1} + \omega_\mathbf{1} \times \mathbf{c_1} = \mathbf{v_s} + \dot{\mathbf{r}}_\mathbf{2} + \omega_\mathbf{2} \times \mathbf{c_2} \tag{5}$$

Differentiating the above equation with respect to time gives us eqn. 6,

$$\ddot{\mathbf{r}}_\mathbf{1} + \dot{\omega}_\mathbf{1} \times \mathbf{c_1} + \omega_\mathbf{1} \times \dot{\mathbf{c}}_\mathbf{1} = \mathbf{a_s} + \ddot{\mathbf{r}}_\mathbf{2} + \dot{\omega}_\mathbf{2} \times \mathbf{c_2} + \omega_\mathbf{2} \times \dot{\mathbf{c}}_\mathbf{2} \tag{6}$$

where $\mathbf{a_s} = \dot{\mathbf{v}}_\mathbf{s}$ is the relative slip acceleration between material points at S_1 and S_2 at C. To make the above second order constraint on the accelerations of the two bodies meaningful and implementable, we have to express $\dot{\mathbf{c}}_\mathbf{1}$ and $\dot{\mathbf{c}}_\mathbf{2}$ in terms of the geometry of S_1 and S_2 at the point of contact. It can be seen that:

$$\dot{\mathbf{c}}_\mathbf{1} = \dot{\mathbf{r}}_\mathbf{c} - \dot{\mathbf{r}}_\mathbf{1} \;\; \mathit{and} \;\; \dot{\mathbf{c}}_\mathbf{2} = \dot{\mathbf{r}}_\mathbf{c} - \dot{\mathbf{r}}_\mathbf{2} \tag{7}$$

To facilitate the calculation of $\dot{\mathbf{r}}_\mathbf{c}$, we attach a local reference frame $x-y$ to S_2 at its center of mass. If the velocity of the contact point C in the reference frame $x-y$ is denoted by $\mathbf{v_c}$, then:

$$\dot{\mathbf{r}}_\mathbf{c} = \mathbf{v_c} + \dot{\mathbf{r}}_\mathbf{2} + \omega_\mathbf{2} \times \mathbf{c_2} \tag{8}$$

Substitution of eqn. 8 and eqn. 5 into eqn. 7 gives:

$$\dot{\mathbf{c}}_\mathbf{1} = \mathbf{v_c} + \omega_\mathbf{1} \times \mathbf{c_1} - \mathbf{v_s} \tag{9}$$

$$\dot{\mathbf{c}}_\mathbf{2} = \mathbf{v_c} + \omega_\mathbf{2} \times \mathbf{c_2} \tag{10}$$

Returning to the calculation of $\mathbf{v_c}$, from the no-penetration condition between the two surfaces we have $\mathbf{v_c} = v_c\mathbf{t}$. Assume that in an infinitesimal time interval dt, the point of contact moves to point C' by traversing a distance ds_1 and ds_2 on S_1 and S_2 respectively, as shown in figure 2. Then:

$$v_c = \frac{ds_2}{dt} \tag{11}$$

The angles between the tangents at C and C' to the surfaces S_1 and S_2 are given by $d\phi_1$ and $d\phi_2$ respectively. Then the total change $d\phi$ in the angular orientation between the two bodies in time dt is:

$$d\phi = d\phi_1 + d\phi_2$$

From the geometry of the two surfaces we have that:

$$\omega_r = \frac{d\phi}{dt} = \frac{d\phi_1}{dt} + \frac{d\phi_2}{dt} = \kappa_1 \frac{ds_1}{dt} + \kappa_2 \frac{ds_2}{dt} \tag{12}$$

where κ_1 and κ_2 are the curvatures along the direction $\mathbf{t}$ at C of S_1 and S_2 respectively. The fact that S_1 is slipping against S_2 with $\mathbf{v_s}$ at C, gives us:

$$\frac{ds_2}{dt}\mathbf{t} = \frac{ds_1}{dt}\mathbf{t} + \mathbf{v_s} \tag{13}$$

Substituting eqn. 13 and eqn. 11 into eqn. 12 and some algebraic manipulation gives:

$$v_c = (\kappa_1 + \kappa_2)^{-1}(\omega_r + \kappa_1 v_s) \tag{14}$$

Substitution of eqns. 9 and 10 into eqn. 6 gives us the required second order kinematic constraint, expressed by eqn. 15, as follows:

$$\ddot{\mathbf{r}}_1 + \dot{\omega}_1 \times \mathbf{c_1} + \omega_1 \times (\mathbf{v_c} + \omega_1 \times \mathbf{c_1} - \mathbf{v_s}) = \mathbf{a_s} + \ddot{\mathbf{r}}_2 + \dot{\omega}_2 \times \mathbf{c_2} + \omega_2 \times (\mathbf{v_c} + \omega_2 \times \mathbf{c_2}) \tag{15}$$

with v_c given by eqn. 14.

In the following subsections we consider some simple examples of planar rolling and slipping, and derive the appropriate second order kinematic constraint.

2.1 Cylinder Rolling on a Block Without Slipping

Let us consider the example of a cylinder of radius R rolling on a moving block without slipping, as shown in figure 3. Here

$$\mathbf{v_s} = 0, \quad \mathbf{a_s} = 0, \quad \kappa_2 = 0$$

$$v_c = R(\omega_1 - \omega_2) \quad or \quad \mathbf{v_c} = -(\omega_1 - \omega_2) \times \mathbf{c_1} \tag{16}$$

Substitution of eqn. 16 into eqn. 15 gives us the following second order constraint[2].

$$\ddot{\mathbf{r}}_1 + \dot{\omega}_1 \times \mathbf{c_1} + \omega_2 \times [(\omega_1 - \omega_2) \times \mathbf{c_1}] + \omega_1 \times (\omega_2 \times \mathbf{c_1}) = \ddot{\mathbf{r}}_2 + \dot{\omega}_2 \times \mathbf{c_2} + \omega_2 \times (\omega_2 \times \mathbf{c_2}) \tag{17}$$

[2]This constraint equation matches exactly with the constraint equation derived independently by Jim Cremer and Daniela Rus to simulate this motion, using different methods.

2.2 Small Cylinder Rolling on a Bigger Cylinder Without Slipping

Consider the small cylinder of radius r that is rolling, without slipping, on a fixed bigger cylinder of radius R as shown in figure 4. Here

$$\mathbf{v_s} = \mathbf{a_s} = \dot{\mathbf{r}}_2 = \ddot{\mathbf{r}}_2 = \omega_2 = \dot{\omega}_2 = 0$$

Also

$$\kappa_1 = \frac{1}{r}, \qquad \kappa_2 = \frac{1}{R}$$

Hence:

$$v_c = \frac{rR\omega_1}{r+R} \quad or \quad \mathbf{v_c} = \frac{rR}{r+R}(\omega_1 \times \mathbf{n}) \tag{18}$$

and the second order constraint is:

$$\ddot{\mathbf{r}}_1 + \dot{\omega}_1 \times \mathbf{c}_1 + \omega_1 \times (\mathbf{v_c} + \omega_1 \times \mathbf{c}_1) = 0 \tag{19}$$

To verify that the above constraint is indeed correct, we can perform the following check. If $\mathbf{n}$ is the unit outward normal to the bigger cylinder at the point of contact, then we know that:

$$\dot{\mathbf{r}}_1 = r(\omega_1 \times \mathbf{n}), \qquad \ddot{\mathbf{r}}_1 = -\frac{r^2\omega_1^2}{r+R}\mathbf{n} + r(\dot{\omega}_1 \times \mathbf{n}), \qquad \mathbf{c}_1 = -r\mathbf{n}$$

Substituting these in eqn. 19 gives:

$$-\frac{r^2\omega_1^2}{r+R}\mathbf{n} + r(\dot{\omega}_1 \times \mathbf{n}) - r(\dot{\omega}_1 \times \mathbf{n}) + \omega_1 \times (\frac{rR}{r+R}\omega_1 \times \mathbf{n} - r\omega_1 \times \mathbf{n}) = 0$$

which reduces to

$$0 = 0$$

i.e. the constraint equation is correct.

2.3 Small Cylinder Slipping on a Bigger Cylinder With a Fixed Point on it Contacting the Bigger Cylinder

Consider the small cylinder of figure 4 slipping on the bigger cylinder with a relative slip velocity $\mathbf{v_s}$ and a relative slip acceleration $\mathbf{a_s}$ at a fixed point C on the small cylinder that makes contact with the bigger cylinder. Here the second order constraint becomes:

$$\ddot{\mathbf{r}}_1 + \dot{\omega}_1 \times \mathbf{c}_1 + \omega_1 \times (\mathbf{v_c} + \omega_1 \times \mathbf{c}_1 - \mathbf{v_s}) - \mathbf{a_s} = 0 \tag{20}$$

If $\mathbf{t}$ is a unit vector (in the tangent plane) in the direction of $\mathbf{v_s}$, then:

$$\mathbf{v_s} = v_s\mathbf{t}, \qquad \mathbf{a_s} = \dot{v}_s\mathbf{t} - \frac{v_s^2}{R}\mathbf{n}$$

$$\omega_1 = \frac{v_s}{R}, \qquad \dot{\omega}_1 = \frac{\dot{v}_s}{R}$$

$$\Rightarrow \mathbf{v_c} = \frac{rR}{r+R}(\frac{\mathbf{v_s}}{R} + \frac{\mathbf{v_s}}{r}) = \mathbf{v_s}$$

We also know that:

$$\dot{\mathbf{r}}_1 = \frac{R+r}{R}\mathbf{v_s}, \qquad \ddot{\mathbf{r}}_1 = \frac{(R+r)\dot{v}_s}{R}\mathbf{t} - \frac{(R+r)v_s^2}{R^2}\mathbf{n}, \qquad \mathbf{c}_1 = -r\mathbf{n}$$

Substituting all these in the constraint eqn. 20, we can again verify that it reduces to

$$0 = 0$$

i.e. the constraint equation is correct.

3 Three Dimensional Rolling and Slipping of Surfaces

We can now generalize our results from rolling and slipping in a plane to three dimensional rolling, twisting and slipping of surfaces. As shown in figure 5, we attach a local reference system $x - y$ (coinciding with the principal directions of at least one of the surfaces at C) to the tangent plane at the point of contact C. $\mathbf{n}$ is the unit vector normal to this plane. Now the curvatures of S_1 and S_2 at C are matrices given by $[\kappa_1]$ and $[\kappa_2]$, both expressed with respect to the local coordinate system $x - y$. The relative angular velocity $\omega_1 - \omega_2$ between the two surfaces can be thought of as having two components. A *twisting (or pivoting) component* $\omega_\mathbf{p}$ along the normal to the tangent plane at the contact, which does not cause the point of contact on the two surfaces to move, and a *rolling component* $\omega_\mathbf{r}$ that causes the point of contact to move on both the surfaces. These components are given as:

$$\omega_\mathbf{p} = [(\omega_1 - \omega_2)\cdot\mathbf{n}]\mathbf{n} \qquad \omega_\mathbf{r} = (\omega_1 - \omega_2) - \omega_\mathbf{p} \tag{21}$$

If $\mathbf{v_s}$ is the velocity (in the tangent plane) with which S_1 is slipping over S_2 at C, and both $\omega_\mathbf{r}$ and $\mathbf{v_s}$ are expressed in the local coordinate system $x - y$ as:

$$\omega_\mathbf{r} = \begin{bmatrix} \omega_{rx} \\ \omega_{ry} \end{bmatrix} \qquad \mathbf{v_s} = \begin{bmatrix} v_{sx} \\ v_{sy} \end{bmatrix}$$

then analogous to equation 14, the velocity v_c of the contact in a reference frame (parallel to $x - y$) fixed to S_2 at its center of mass, is given by:

$$\mathbf{v_c} = \begin{bmatrix} v_{cx} \\ v_{cy} \end{bmatrix} = ([\kappa_1] + [\kappa_2])^{-1} \left(\begin{bmatrix} -\omega_{ry} \\ \omega_{rx} \end{bmatrix} + [\kappa_1]\mathbf{v_s} \right) \tag{22}$$

Substituting the value of $\mathbf{v_c}$ from eqn. 22 into eqn. 15 gives us the required second order constraint for the three dimensional slipping and sliding motion. We consider some examples [Lamb 1943] below as illustrations.

3.1 Thin Disk Rolling on a Plane

We consider the classical problem of nonholonomic constraints dealt with in most mechanics texts, that of the rolling (without slipping) of a thin disc (of radius R) on a fixed horizontal plane with angular velocity and acceleration ω and $\dot{\omega}$, as shown in figure 6. The thinness of the disc implies that it is making contact at a *point* with the plane and for all practical purposes it can be considered as a circle rolling on the plane without slipping. The principal directions of the disc in the tangent plane to the point of contact are $\mathbf{t}$, the unit tangent vector to the path traced out by the contact point on the plane, and $\mathbf{p}$, the unit vector perpendicular to $\mathbf{t}$. From figure 7 and geometry, we get that if the inclination of the disc with $\mathbf{n}$ is θ, then:

$$[\kappa_2] = [0], \qquad [\kappa_1] = \begin{bmatrix} \kappa_{1t} & 0 \\ 0 & \kappa_{1p} \end{bmatrix} = \begin{bmatrix} \frac{\cos\theta}{R} & 0 \\ 0 & 0 \end{bmatrix}$$

If the component of the angular velocity of the disc in the $\mathbf{p}$ direction is given as $\omega_{\mathbf{p}}$, then:

$$\mathbf{v_c} = \omega_{\mathbf{p}} \times \frac{R}{\cos\theta}\mathbf{n}$$

Substituting this in eqn. 15 gives us the necessary constraint for the rolling of a thin disc without slipping on a plane:

$$\ddot{\mathbf{r}} + \dot{\omega} \times \mathbf{c} + \omega \times [\omega \times \mathbf{c} + \omega_{\mathbf{p}} \times \frac{R}{\cos\theta}\mathbf{n}] = 0 \tag{23}$$

3.2 Torus Rolling on a Plane

The case of the rolling torus, figure 8, is similar to that of the rolling disc. If all the terminology were to remain the same as for the disc, then:

$$[\kappa_2] = [0], \qquad [\kappa_1] = \begin{bmatrix} \kappa_{1t} & 0 \\ 0 & \kappa_{1p} \end{bmatrix} = \begin{bmatrix} \frac{\cos\theta}{R + r\cos\theta} & 0 \\ 0 & \frac{1}{r} \end{bmatrix}$$

Also,

$$\mathbf{v_c} = \omega_\mathbf{p} \times (\frac{R}{\cos\theta} + r)\mathbf{n} + \omega_\mathbf{t} \times r\mathbf{n} = \omega_\mathbf{p} \times \frac{R}{\cos\theta}\mathbf{n} + (\omega - \omega_\mathbf{n}) \times r\mathbf{n}$$

Substituting this in eqn. 15 gives us the necessary constraint for the rolling of a torus without slipping on a plane:

$$\ddot{\mathbf{r}} + \dot{\omega} \times \mathbf{c} + \omega \times [\omega \times \mathbf{c} + \omega_\mathbf{p} \times \frac{R}{\cos\theta}\mathbf{n} + (\omega - \omega_\mathbf{n}) \times r\mathbf{n}] = 0 \tag{24}$$

3.3 Sphere Rolling Inside Fixed Hollow Cylinder

To get an interesting simulation, let us consider the case of a sphere (or radius r) rolling (without slipping) inside a hollow vertical cylinder of radius R, as shown in figure 10. The sphere is rolling with angular velocity and acceleration ω and $\dot{\omega}$ respectively. If $\mathbf{n}$ is a unit vector pointing radially inward into the cylinder axis, $\mathbf{d}$ is a unit vector pointing vertically downwards and $\mathbf{t} = \mathbf{d} \times \mathbf{n}$, then:

$$[\kappa_\mathbf{1}] = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{bmatrix}, \qquad [\kappa_\mathbf{2}] = \begin{bmatrix} \kappa_{2t} & 0 \\ 0 & \kappa_{2l} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{bmatrix}$$

If $\omega_\mathbf{d}$ and $\omega_\mathbf{t}$ are the components of ω along the $\mathbf{d}$ and $\mathbf{t}$ directions respectively, then:

$$\mathbf{v_c} = \omega_\mathbf{d} \times \frac{rR}{R-r}\mathbf{n} + \omega_\mathbf{t} \times r\mathbf{n}$$

Substituting this in eqn. 15 gives us the necessary constraint for the rolling of the sphere inside the hollow cylinder:

$$\ddot{\mathbf{r}} + \dot{\omega} \times \mathbf{c} + \omega \times [\omega \times \mathbf{c} + \omega_\mathbf{d} \times \frac{rR}{R-r}\mathbf{n} + \omega_\mathbf{t} \times r\mathbf{n}] = 0 \tag{25}$$

4 Conclusions

The generalized second order kinematic constraint between rigid surfaces that are moving against each other while maintaining point contact was obtained so that it incorporates, amongst other things, the geometry of the surfaces in contact. It is expressed in a form so that it is easily programmable. Its complete implementation on the Newton dynamics simulator [Cremer and Stewart 1989] would give the simulator the capability to accurately simulate the above described motion.

Implementation of this constraint (by Jim Cremer) on the Newton dynamics simulator for the rolling of a thin disk on a plane have yielded correct simulations. Motions of the disk that had closed form solutions were tried out for analytical verification. This implementation has also allowed the simulation of an automobile (by Xue Dong Yang) and a bicycle (by Suresh Goyal), by making the wheels of these vehicles as thin disks rolling on planes without slipping.

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References

[1] Cremer, J.F. and Stewart, J.A., *The Architecture of Newton, A General Purpose Dynamics Simulator*, proceedings of IEEE International Conference on Robotics and Automation, May 1989.

[2] Hopcroft, J.E., *Electronic Prototyping*, IEEE Transactions on Aerospace and Electronic Systems, Vol. 24, No. 5, September 1988.

[3] Lamb, H., *Higher Mechanics*, Cambridge At The University Press, 1943.

[4] Montana, D.J., *Tactile Sensing and Kinematics of Contact*, Ph.D. Thesis, Division of Applied Sciences, Harvard University, 1986.

[5] Neǐmark, Ju.I. and Fufaev, N.A., *Dynamics of Nonholonomic Systems*, American Mathematical Society, Providence, Rhode Island, 1972.

[6] Zexiang Li, Canny, J.F. and Sastry, S.S., *On Motion Planning For Dextrous Manipulation, Part I: The Problem Formulation*, proceedings of IEEE International Conference on Robotics and Automation, May 1989.

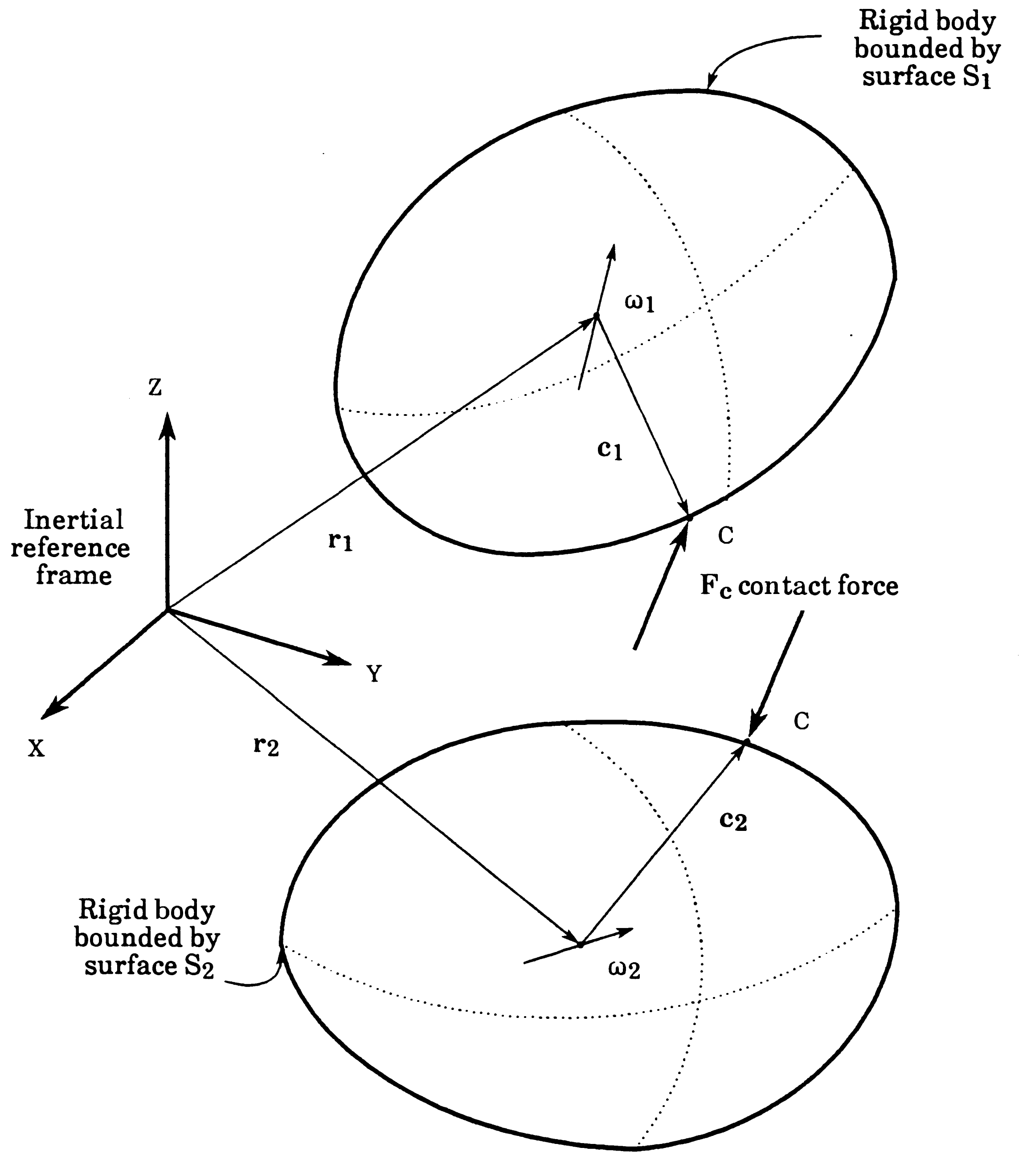

Figure 1 Newtonian mechanics of two rigid bodies that move while maintaining point contact, F_c is the contact force (assuming no moment) at the contact point C.

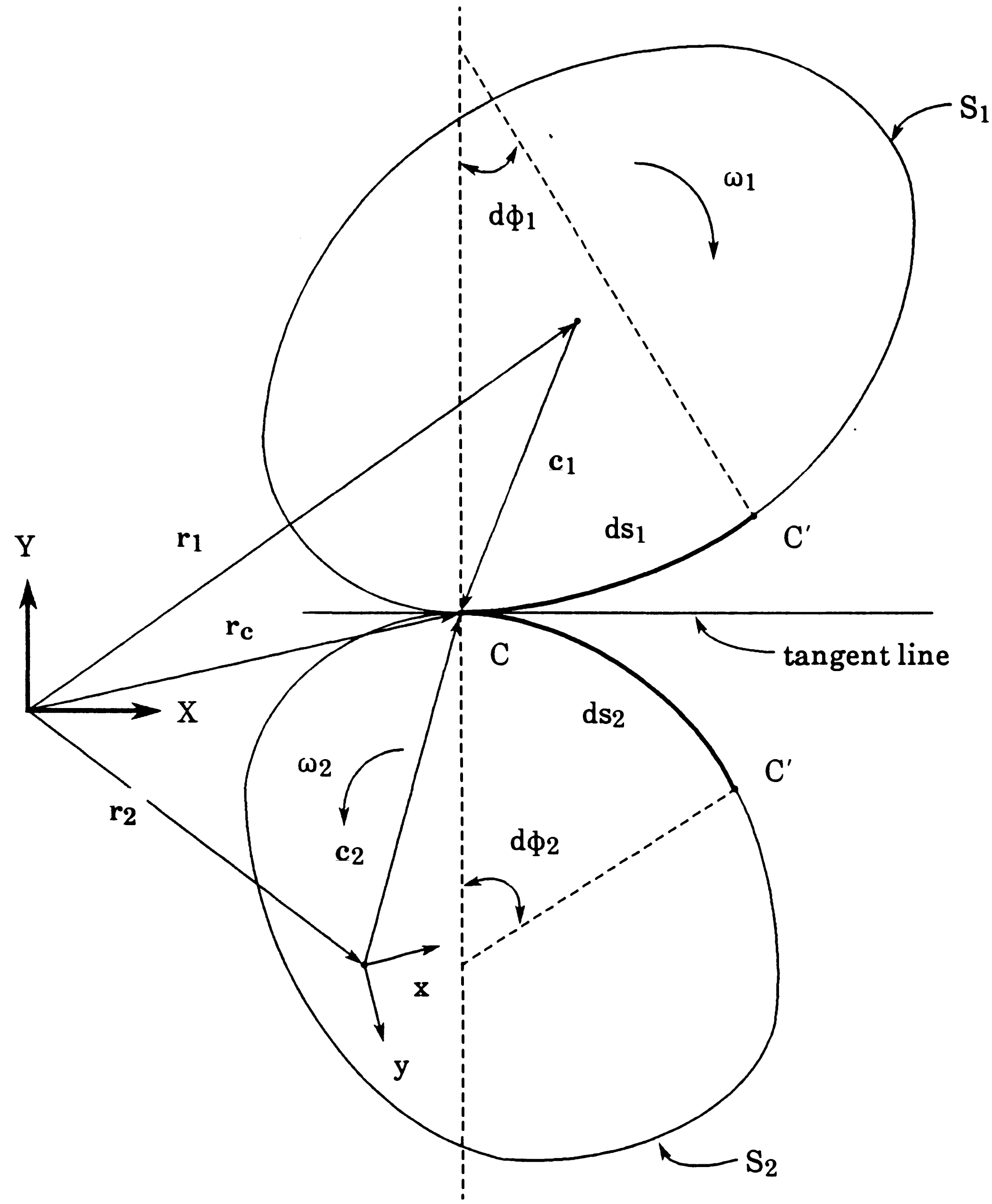

Figure 2 Two rigid surfaces S_1 and S_2 rolling and slipping against each other in the plane while maintaining point contact. In time dt, S_1 rotates by angle $d\phi_1$ and the contact point moves to C′ by traversing distance ds_1; S_2 rotates by angle $d\phi_2$ and the contact point moves to C′ by traversing distance ds_2. X-Y is the inertial frame, x-y is fixed to S_2.

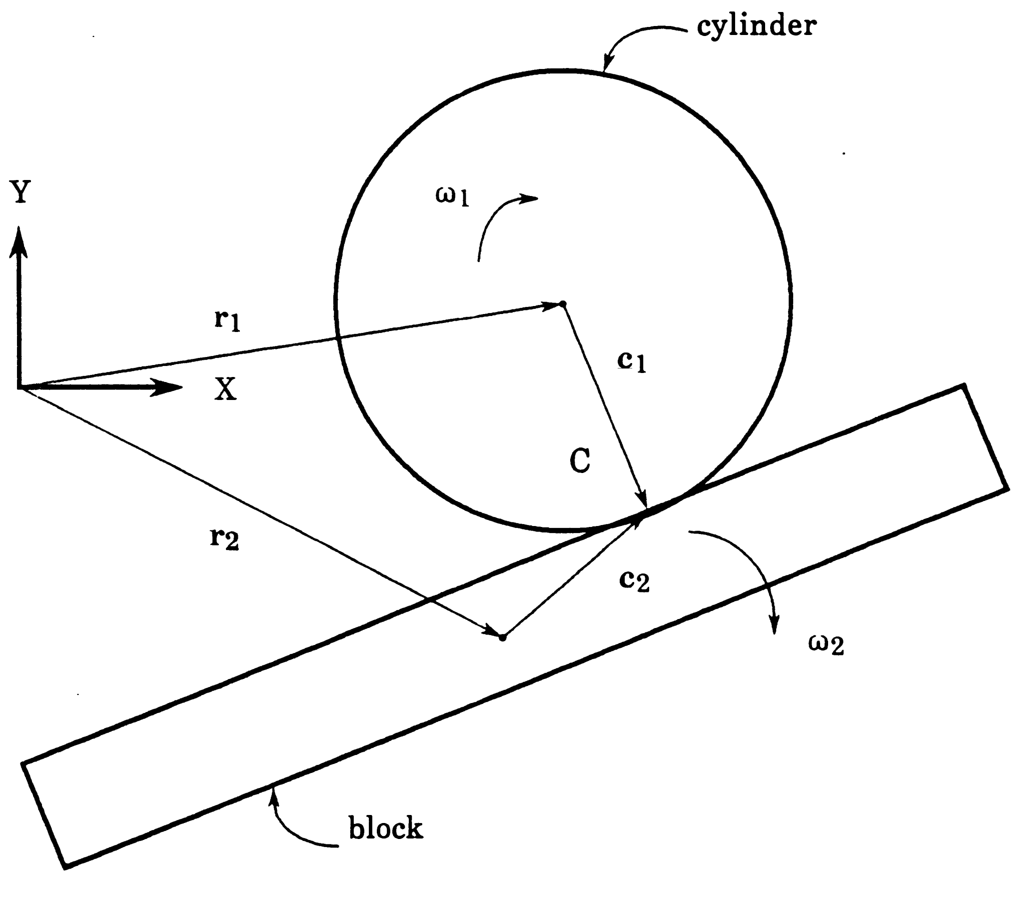

Figure 3 Cylinder rolling on a moving block (in 2D) without slipping.

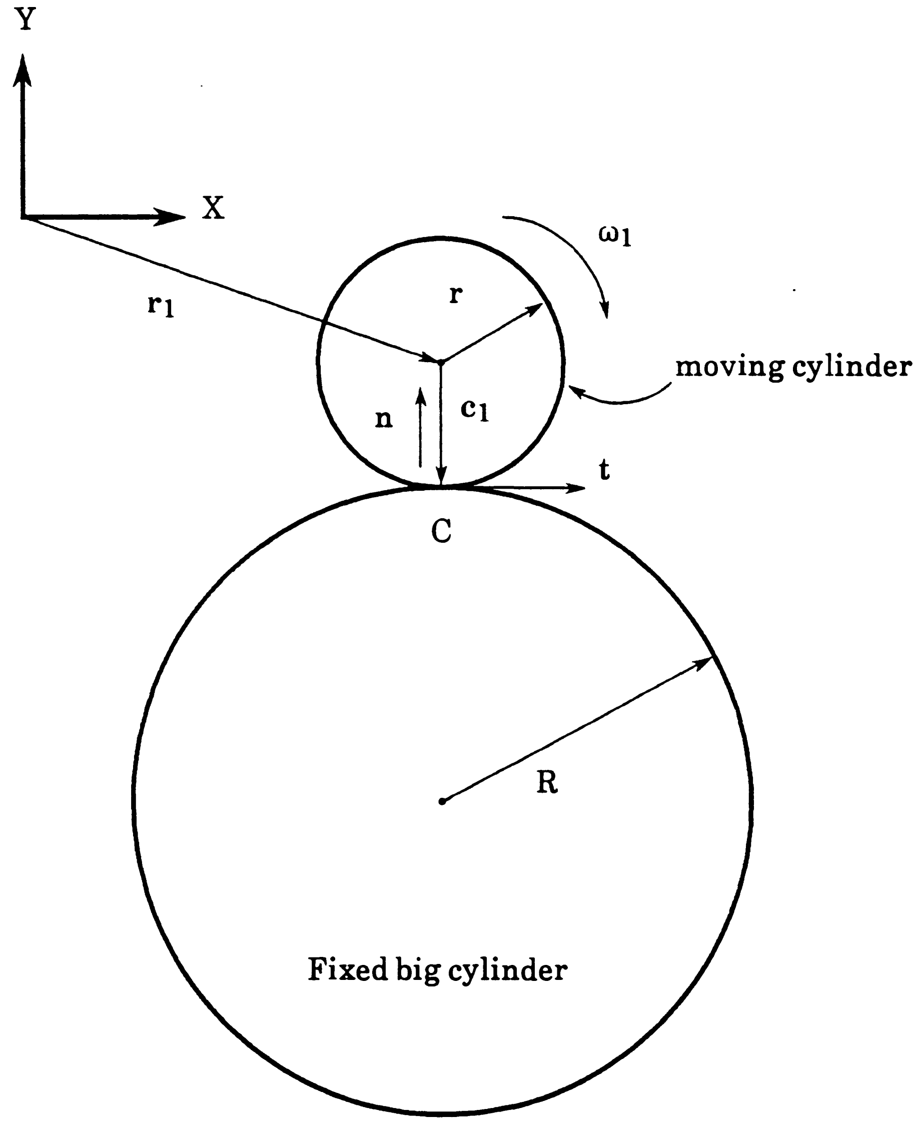

Figure 4 Small cylinder rolling or slipping on a fixed bigger cylinder in 2D.

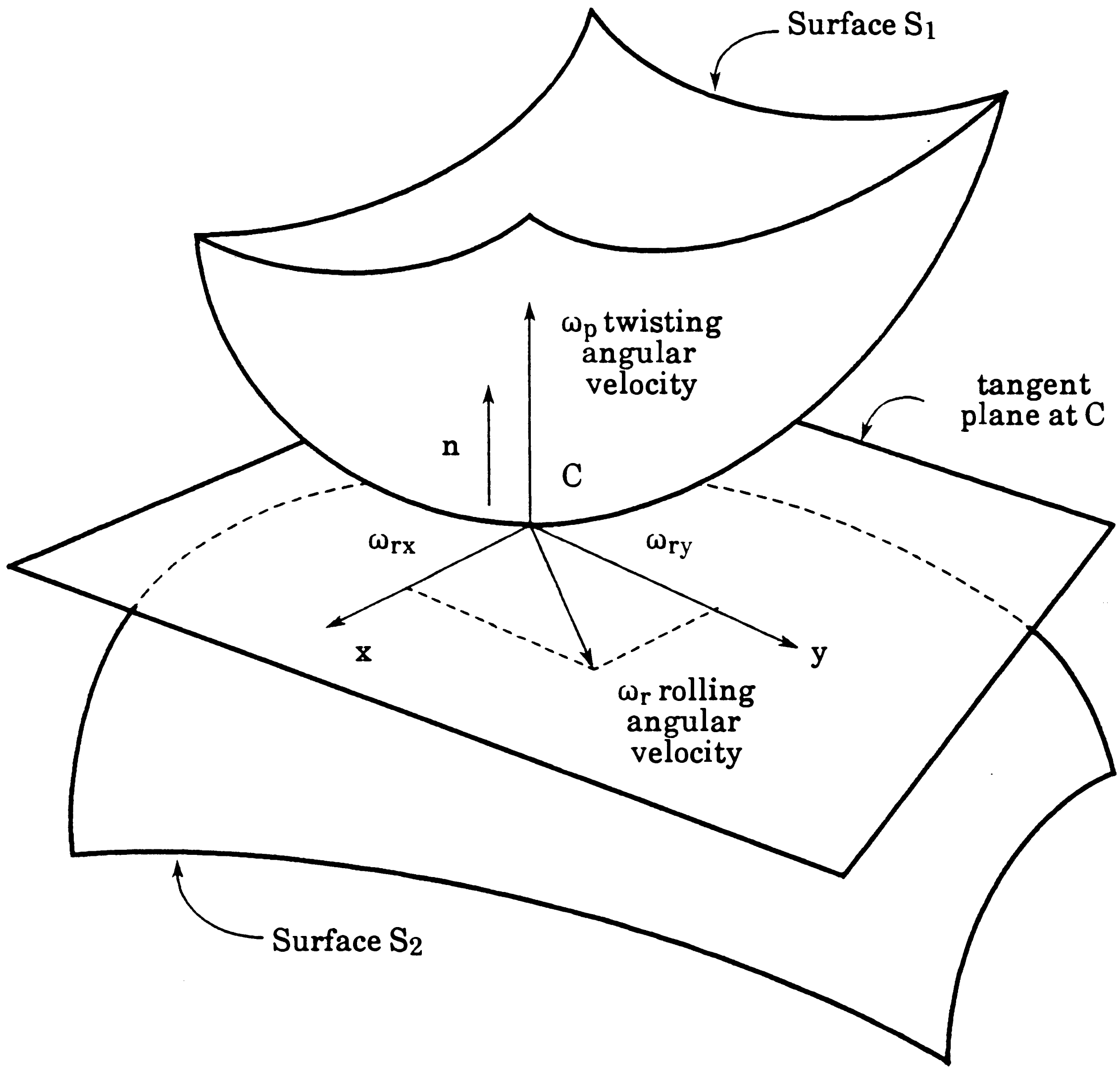

Figure 5 Two surfaces rolling, twisting and slipping in 3D, ω_r is the relative angular velocity between them in the tangent plane to their contact, ω_n is their relative angular velocity normal to this plane .

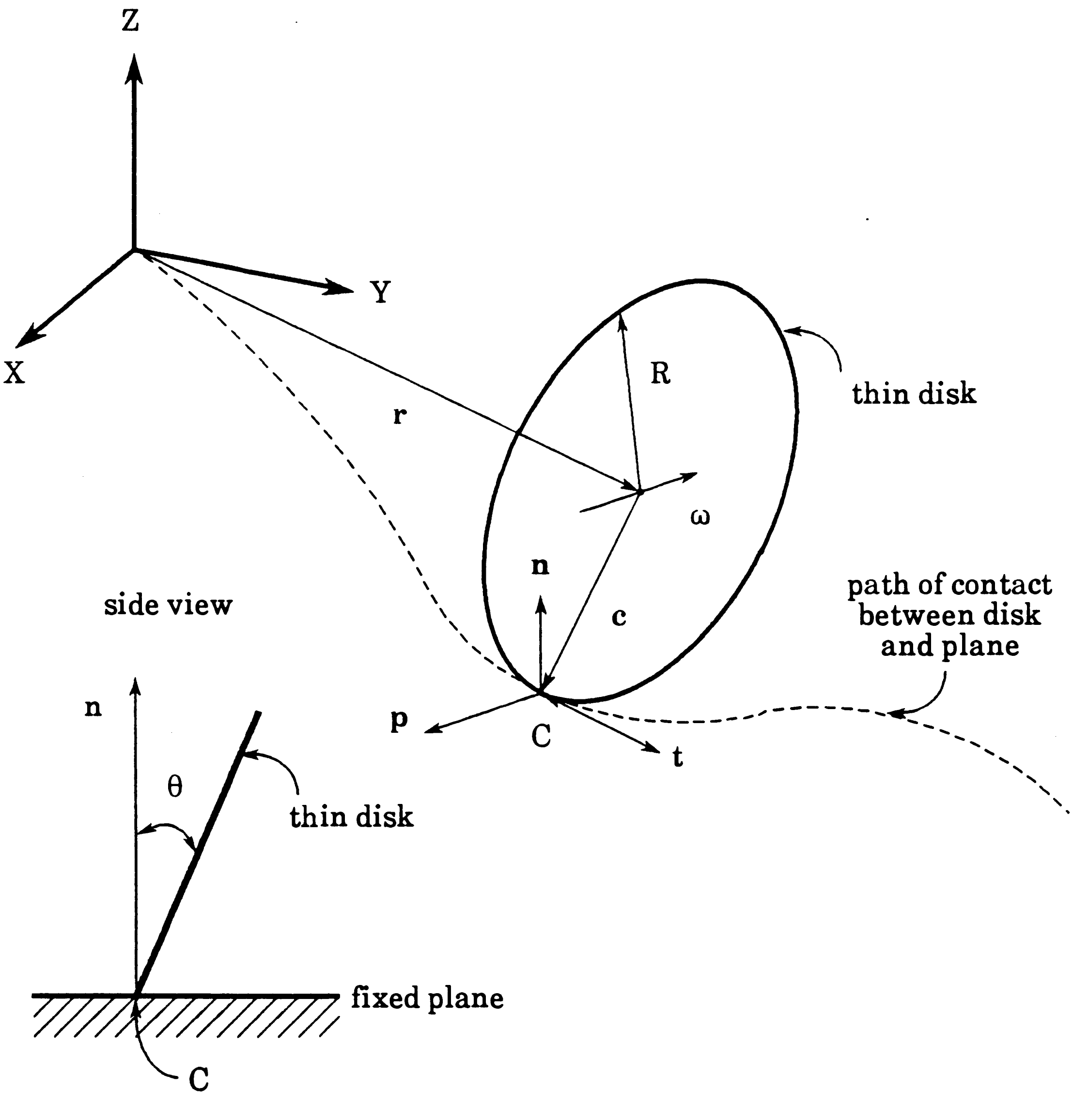

Figure 6 Thin disk rolling on a fixed plane, t is the unit vector tangent to the path of contact on the plane, **p** is perpendicular to t in the plane, **n** is normal to the plane, θ is the angle that the plane of the disk makes with **n**.

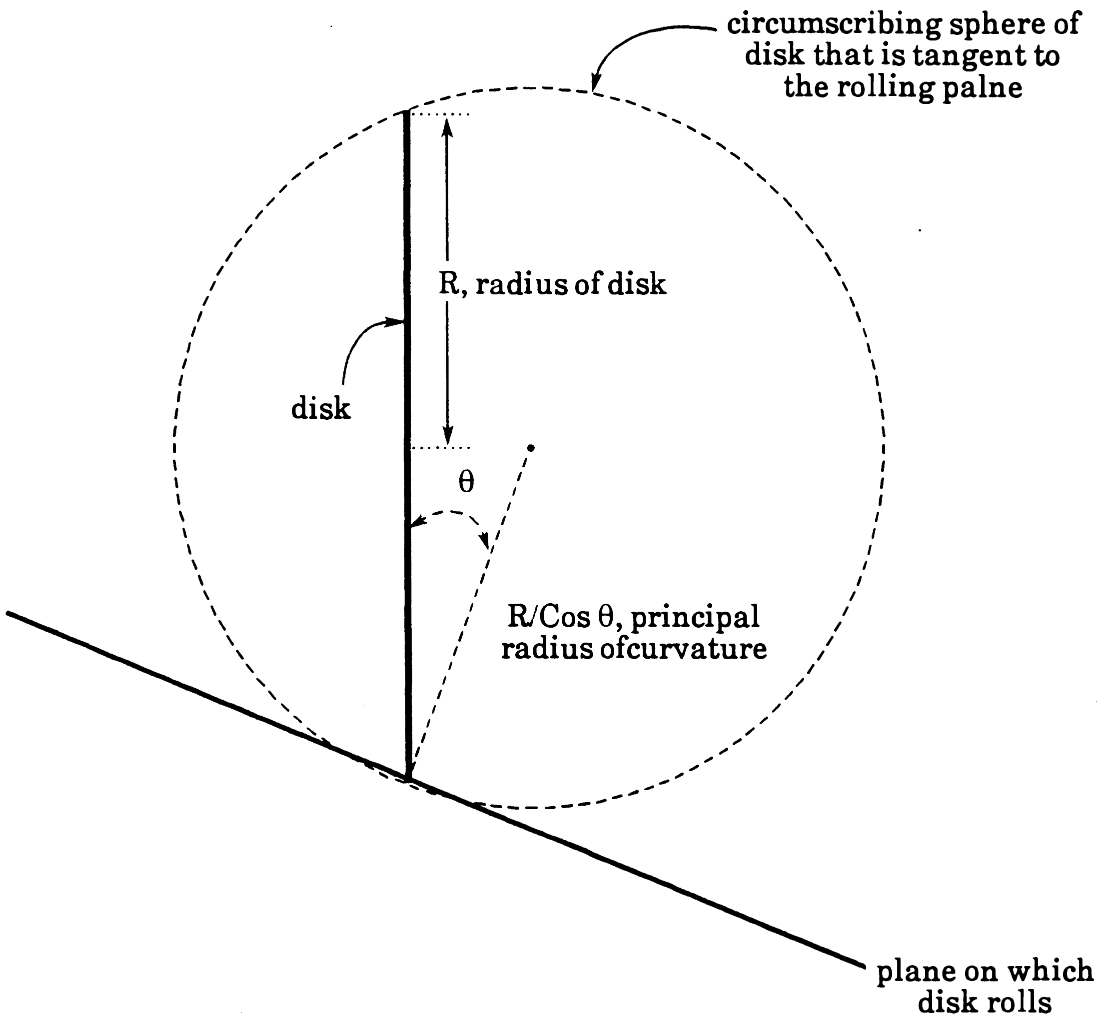

Figure 7 Construction showing principal radius of curvature of inclined thin disk along the direction tangent to its path of contact on the plane, θ is the angle that the plane of the disk makes with the vertical to the rolling plane.

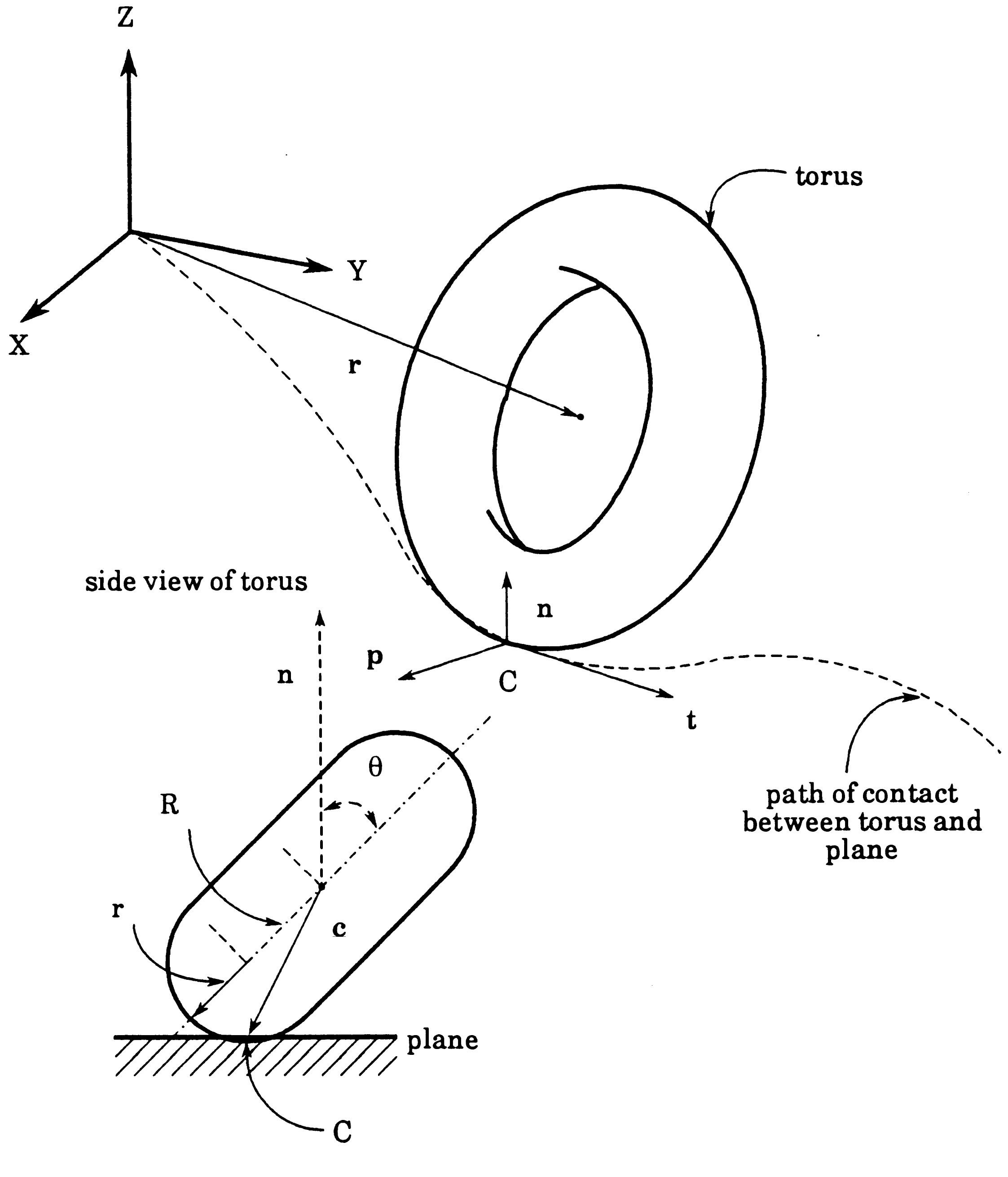

Figure 8 Torus rolling on a fixed plane, same parameters as the thin disk rolling on the plane.

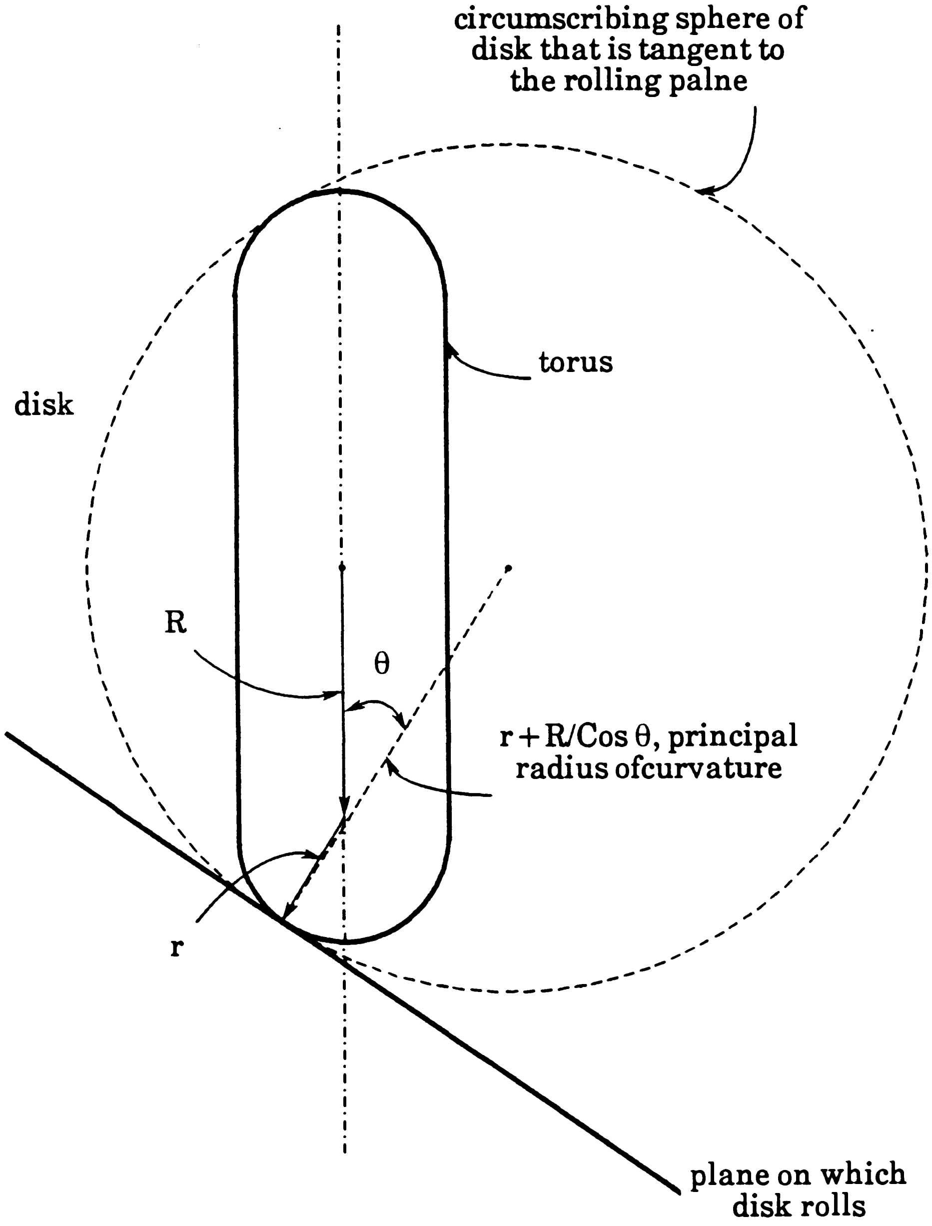

Figure 9 Construction showing principal radius of curvature of inclined torus along the direction tangent to its path of contact on the plane, θ is the angle that the central plane of the torus makes with the vertical to the rolling plane.

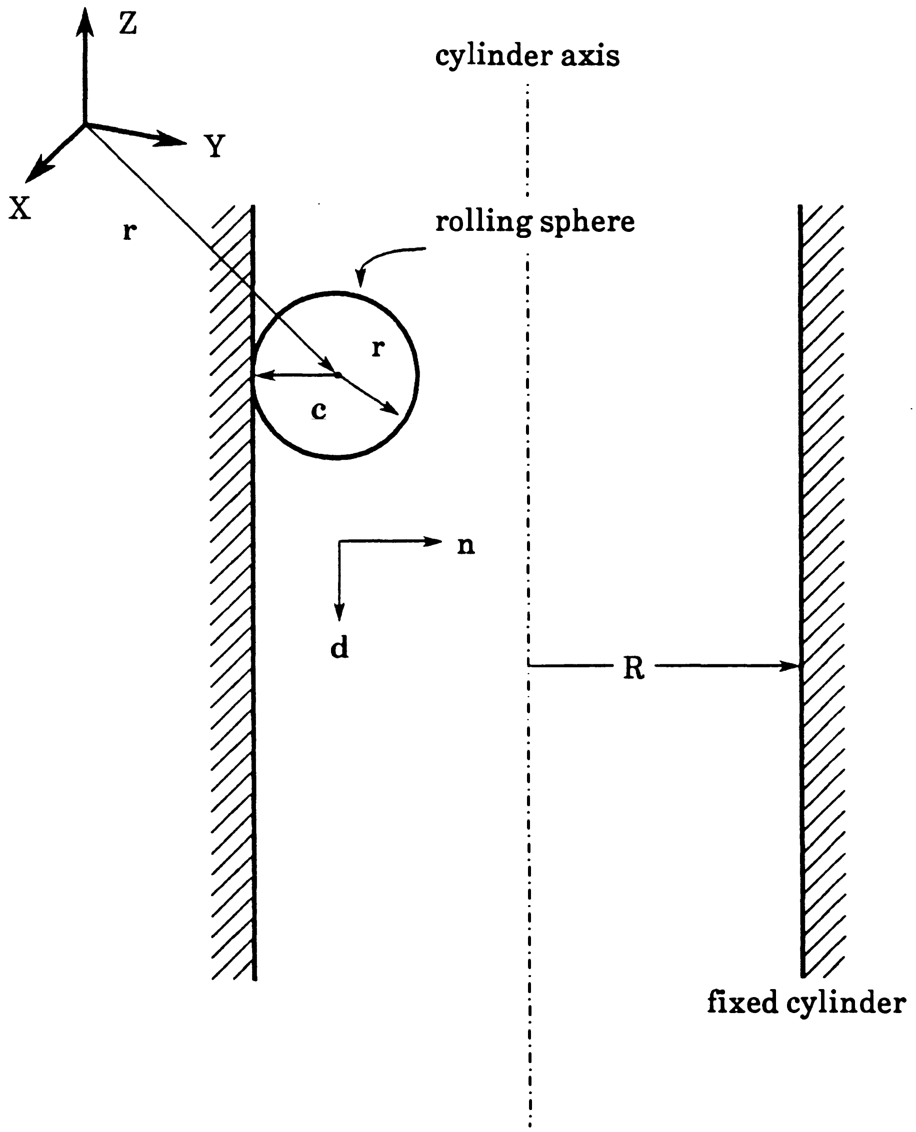

Figure 10 Sphere rolling inside vertical hollow cylinder without slipping, unit vector **n** points radially inwards, unit vector **d** points vertically downwards.