

**L-domains and Lossless
Powerdomains**

Radhakrishnan Jagadeesan*

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Department of Computer Science
Cornell University
Ithaca, NY 14853-7501

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Computer Science Department, Cornell University

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Abstract

The category of L-domains was discovered by Achim Jung [5] while solving the problem of finding maximal cartesian closed categories of algebraic CPO's and continuous functions. In this note we analyse the properties of the lossless powerdomain construction, that is closed on the algebraic-L-Domains. The powerdomain is shown to be isomorphic to a collection of subsets of the domain on which the construction was done. The proof motivates a certain finiteness condition on the inconsistency relations of elements. It is shown that all algebraic CPO's D whose basis $B(D)$ has property M satisfy the condition. In particular, the coherent L- domains [3] satisfy the condition.

1 Introduction

Recent work by A. Jung and C. Gunter shows that the L-domains discovered independently by A. Jung[5] and T. Coquand[1] forms an interesting ccc. P. Buneman proposed the lossless powerdomain construction that is closed on L-domains. Buneman's construction was based on intuitions from databases. We were studying the construction as a possible candidate for modelling oracleisable indeterminacy. The present paper investigates some purely mathematical questions that arose from the study. Another paper, under preparation, will report on the semantic investigations of indeterminacy, using the new powerdomain.

A representation theorem is proved for the lossless powerdomain in Section 3. The proof suggests the imposition of some natural conditions on L-domains. The proof uses the notion of finite separability, i.e the ability to separate elements of the sets that constitute the powerdomain by disjoint basic Scott open sets. (This is made precise in Section 3). This suggests that a natural condition to consider is that the inconsistency relation have finite witnesses. Section 4 discusses this notion of finite inconsistency. Discussions with C. Gunter and E. Gunter revealed the relationship of this property to the coherent L-domains[3].

2 Preliminaries

This section outlines the basic definitions and facts that are used in the note.

A subset X of a partially ordered set is directed iff it is non-empty and if every pair of elements in X has an upper bound in X . A partial order D is said to be (directed) complete if every directed subset X of D has a least upper bound in D . We shall only consider CPO's with a least element, which will be denoted \perp . An element d of D is said to be compact if for every directed set D such that $d \sqsubseteq \bigsqcup D$, there is an element $x \in D$ such that $d \sqsubseteq x$. The set of all elements of D greater than a compact element d is denoted by $d \uparrow$. The set of all elements of D less than an element x is denoted by $x \downarrow$. A CPO D is said to be algebraic if every element is the lub of a directed set of compact elements. The set of compact elements of an algebraic CPO D is denoted by $B(D)$. Step functions of the form $d \searrow e$

where d and e are compact elements of D and E respectively are compact elements of the function space $D \rightarrow E$. The set of minimal upper bounds of a subset A of D below elements of a subset B of D is denoted by $mub_A(B)$.

An algebraic CPO D is said to be an algebraic-L-domain when any of the following equivalent conditions hold [5].

1. For each $x \in D$, the set $x \downarrow \cap B(D)$ is an \vee -semilattice with smallest element.
2. For each upper bound x of a finite subset A of $B(D)$, there is a unique minimal upper bound of A below x .
3. For any finite subset A of $B(D)$, $mub_A(D) = mub_{(mub_A(D))}(D)$

3 The Lossless Powerdomain

The usual power-domain constructions are not closed on L-Domains. Peter Buneman discovered the Lossless Powerdomain construction [2] that is closed on L-domains.

Definition 1 Let D be an L-domain. Define the preorder $P_L(D)$ as follows:

$$|P_L(D)| = \{(e_1 \dots e_n) \mid (\forall 1 \leq i \leq n) [e_i \in B(D)] \wedge (\forall 1 \leq i, j \leq n) [i \neq j \implies mub_{(e_i, e_j)}(D) = \phi]\}$$

and the elements of $|P_L(D)|$ are ordered by the Egli-Milner ordering, \sqsubseteq_{EM} .

Actually $P_L(D)$ is a partial order. The Lossless Powerdomain $\overline{P_L(D)}$ is constructed by ideal-completion of $P_L(D)$.

Lemma 1 $\overline{P_L(D)}$ is an L-domain if D is.

Proof: It suffices to prove that $P_L(D)$ is an upper semi-lattice under the partial order \sqsubseteq_{EM} . Let

$$a = (e_1 \dots e_n), b = (d_1 \dots d_m), c = (f_1 \dots f_p), a, b \sqsubseteq_{EM} c$$

Then, $\sqcup\{a, b\}$ under c is given by,

$$\sqcup\{a, b\} = \cup\{mub_{(e_i, d_j)}(c) \mid 1 \leq i \leq n, 1 \leq j \leq m, e_i \uparrow \cap d_j \uparrow \cap c \neq \phi\}$$

The verification that the above definition is correct is quite easy. ■

It is shown that the ideals in the lossless powerdomain are representatives of their fringe sets. Fringe sets are the sets generated by following the partial order arrows among the elements of the ideal. The following definition captures the idea of “following arrows”.

Definition 2 A GENERATOR d over an ideal $I \in \overline{P_L(D)}$ is a function $d: I \rightarrow D$ such that :

- $i \in I \implies d(i) \in i$
- $(i \in I \wedge j \in I \wedge i \sqsubseteq_{EM} j) \implies d(i) \sqsubseteq d(j)$

The following lemma is an easy consequence of the definition.

Lemma 2 If d is a GENERATOR over I , $d(I)$ is a directed set in D .

Definition 3 Let $I \in \overline{P_L(D)}$. The set generated by I is

$$S_I = \{ \sqcup d(I) \mid d \text{ is a Generator on } I \}$$

Since the lossless powerdomain embodies finite branching only, one expects the sets generated to be Scott-compact. The proof requires the following lemma, that is proved using the axiom of choice. (Proof given in appendix)

Lemma 3 Let $I \in \overline{P_L(D)}$. Let \tilde{I} be cofinal in I . Let P be a predicate defined on D such that

1. $(\forall i \in \tilde{I}) (\exists e \in i) [P(e)]$
2. $(d \in i \in \tilde{I} \wedge e \in j \in \tilde{I} \wedge i \sqsubseteq_{EM} j \wedge d \sqsubseteq e \wedge P(e)) \implies P(d)$

Then there is a generator d over I such that $(\forall i \in \tilde{I}) [P(d(i))]$.

Lemma 4 If $I \in \overline{P_L(D)}$, S_I is non-empty and Scott-compact.

Proof:

1. Define a predicate P on D by

$$P(e) \iff (\exists i \in I) [e \in i]$$

From lemma 3 , we get a generator on D . Hence S_I is non-empty.

2. Let $\{x_\alpha\}$ be a net in S_I . Define a predicate P on D as follows:

$$P(e) \iff (\exists i \in I) [e \in i \wedge e \uparrow \cap \{x_\alpha\} \text{ is co-final }]$$

From lemma 3 , we get a generator d . Consider $\sqcup d(I)$. Any neighbourhood $e \uparrow$ of $\sqcup d(I)$ has non-empty intersection with $d(I)$, and consequently is cofinal in $\{x_\alpha\}$. Hence $\sqcup d(I)$ is an accumulation point of $\{x_\alpha\}$ ■

However, not all non-empty Scott-compact sets are generated by some ideal in the lossless powerdomain. Consider the L-domain in Fig 1. The set $\{x, y\}$ cannot be generated by any ideal in $\overline{P_L(D)}$, even though the set $\{x, y\}$ is finite and hence compact in the Scott Topology. This observation motivates the following definition.

Definition 4 $S \subseteq D$ is FINITELY SEPARABLE if

$$(\forall S_{fin} = \{x_1 \dots x_n\}, S_{fin} \subseteq S) (\exists (e_1 \dots e_m) \in P_L(D)) [n \leq m \wedge (\forall 1 \leq i \leq n)(e_i \sqsubseteq x_i) \wedge (e_1 \dots e_m) \sqsubseteq_{EM} S]$$

The following lemma is an easy consequence of the above definition.

Lemma 5 If $I \in \overline{P_L(D)}$, S_I is FINITELY SEPARABLE

$x \bullet \quad \bullet y$

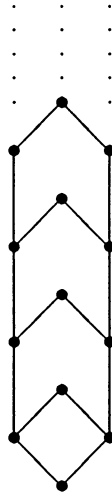


Fig. 1

It turns out that one can generate all *FINITELY SEPARABLE* and Scott-compact sets.

Definition 5 Let L be an L -domain. Then Ψ_D is the set of all Scott-compact finitely-separable subsets of D ordered by \sqsubseteq_{EM}

Note that Ψ_D is a partial order. Now, we define maps between $\overline{P_L(D)}$ and Ψ_D in a natural manner, and show that the definitions do indeed constitute an order isomorphism between the partial orders, giving us the required representation theorem.

Lemma 6 Define $\Phi : \overline{P_L(D)} \rightarrow \Psi_D$ by

$$\Phi(I) = S_I$$

Then, Φ is monotone.

Proof: Let $I_1 \in \overline{P_L(D)}$, $I_2 \in \overline{P_L(D)}$, $I_1 \subseteq I_2$. Let $\Phi(I_1) = S_{I_1}$, $\Phi(I_2) = S_{I_2}$.

- Let $x \in S_{I_1}$
 $\implies (\exists d) [\sqcup d(I_1) = x]$, where d is generator on I_1 . Define

$$\tilde{I}_2 = \{i \mid i \in I_2 \wedge (\exists j \in I_1) [j \sqsubseteq_{EM} i]\}$$

Note that \tilde{I}_2 is cofinal in I_2 . Define predicate P by

$$P(e) \iff (\exists i, j) [i \in \tilde{I}_2 \wedge e \in i \wedge j \in I_1 \wedge d(j) \sqsubseteq e \wedge \tilde{e} \in i \wedge i \in \tilde{I}_2]$$

The generator \tilde{d} on I_2 given by lemma 4 satisfies $x \sqsubseteq \sqcup \tilde{d}$
 $\implies (\forall x \in S_{I_1}) (\exists y \in S_{I_2}) [x \sqsubseteq y]$

- Let $x \in S_{I_2}$
 $\implies (\exists \tilde{d}) [\sqcup \tilde{d}(I_2) = x]$, where \tilde{d} is a generator on I_2 . Restriction of \tilde{d} to I_1 gives a generator d on I_1 such that $d(I_1) \subseteq \tilde{d}(I_2)$. Hence, we have $\sqcup d(I_1) \sqsubseteq \sqcup \tilde{d}(I_2)$. Hence, we have

$$(\forall y \in S_{I_2}) (\exists x \in S_{I_1}) [x \sqsubseteq y] \quad \blacksquare$$

Lemma 7 Let $S \in \Psi_D$, $(e_1 \dots e_n) \sqsubseteq_{EM} S$, $(d_1 \dots d_m) \sqsubseteq_{EM} S$. Then

$$(\forall 1 \leq i \leq n, 1 \leq j \leq m) [mub_{(e_i, d_j)}(S) \text{ is finite}]$$

Proof: Let $1 \leq i \leq n$, $1 \leq j \leq m$. Consider the basic open cover of S consisting of

- $\tilde{e} \uparrow$, $\tilde{e} \in mub_{(e_i, d_j)}(S)$
- $d_{\tilde{j}} \uparrow$, $\tilde{j} \neq j$, $1 \leq \tilde{j} \leq m$
- $e_{\tilde{i}} \uparrow$, $\tilde{i} \neq i$, $1 \leq \tilde{i} \leq n$

This open cover has finite subcover. Since all members of (e_1, \dots, e_n) and (d_1, \dots, d_m) are pairwise inconsistent, we deduce that $mub_{(e_i, d_j)}(S)$ is finite. \blacksquare

Lemma 8 Define $\tilde{\Phi} : \Psi_D \rightarrow \overline{P_L(D)}$ by

$$\tilde{\Phi}(S) = \{ (e_1 \dots e_m) \mid (e_1 \dots e_m) \sqsubseteq_{EM} S \wedge (e_1 \dots e_m) \in P_L(D) \}$$

Then, $\tilde{\Phi}$ is monotone.

Proof:

- We have to first show that $\tilde{\Phi}$ is well defined. Let $S \in \Psi_D$. Then we have

- $\{\perp\} \in \tilde{\Phi}(S)$. So, $\tilde{\Phi}(S)$ is non-empty.
- $(e_1 \dots e_n) \sqsubseteq_{EM} S, (d_1 \dots d_m) \sqsubseteq_{EM} S$
 $\implies [\bigcup \{ \{ \text{mub}_{(e_i, d_j)}(S) \} \mid 1 \leq i \leq n, 1 \leq j \leq m, e_i \uparrow \cap d_j \uparrow \cap S \neq \phi \}] \sqsubseteq_{EM} S$. Hence, $\tilde{\Phi}(S)$ is directed.
- $\tilde{\Phi}(S)$ is obviously downward closed.

Hence, $\tilde{\Phi}(S)$ is an element of $\overline{P_L(D)}$.

- $S_1 \sqsubseteq_{EM} S_2 \implies \tilde{\Phi}(S_1) \subseteq \tilde{\Phi}(S_2)$. Hence, $\tilde{\Phi}$ is monotone. ■

Lemma 9 $\tilde{\Phi} \circ \Phi = Id$

Proof: Note that it suffices to prove that

$$(\forall i \in P_L(D)) [i \in I \iff i \sqsubseteq_{EM} S_I]$$

The 'if' part is obvious. For the reverse direction, consider

$$\begin{aligned} & (e_1 \dots e_m) \in P_L(D) \wedge (e_1 \dots e_m) \sqsubseteq_{EM} S_I \\ \implies & (\exists x_1 \dots x_m \in S_I) (\forall 1 \leq k \leq m) [e_k \sqsubseteq x_k] \end{aligned}$$

Let $d_1 \dots d_m$ be the generators on I corresponding to $x_1 \dots x_m$. Since $d_k(I)$ is a directed set for all $1 \leq k \leq m$, we have

$$(\forall 1 \leq k \leq m) (\exists \tilde{e}_k, i_k) [\tilde{e}_k \in i_k \in I \wedge d_k(i_k) = \tilde{e}_k \wedge e_k \sqsubseteq \tilde{e}_k]$$

Since I is directed, $(\exists i \in I) [i_1 \dots i_m \sqsubseteq_{EM} i]$. Note that $(\forall 1 \leq i \leq m) (\exists \tilde{e}_i \in i) [e_i \sqsubseteq \tilde{e}_i]$. Define

$$\tilde{I} = \{ j \mid j \in I \wedge i \sqsubseteq_{EM} j \}$$

Note that \tilde{I} is cofinal in I .

- We have

$$(\forall j \in \tilde{I}) (\forall 1 \leq k \leq m) (\exists \tilde{e}_k \in j) [e_k \sqsubseteq \tilde{e}_k]$$

- We need to prove that $(\exists j \in \tilde{I}) (\forall \tilde{e} \in j) (\exists 1 \leq k \leq m) [e_k \sqsubseteq \tilde{e}]$ Suppose not. Define predicate P by:

$$P(e) \iff (\exists j \in \tilde{I}) [e \in j \wedge e_1 \not\sqsubseteq e \dots e_m \not\sqsubseteq e]$$

P satisfies the conditions of lemma 3. The generator yielded by lemma 3 gives an element in S_I that is not greater than any of $e_1 \dots e_m$. This is a contradiction since $(e_1 \dots e_m) \sqsubseteq_{EM} S_I$

Hence, we have an element of I above $(e_1 \dots e_m)$. Result follows by downward closure of I . ■

Lemma 10 $\Phi \circ \tilde{\Phi} = Id$

Proof: Let $I = \tilde{\Phi}(S)$, where $S \in \Psi_D$.

1. Let d be a generator over I . We shall prove by contradiction that $(\exists x \in S) [\sqcup d(I) \sqsubseteq x]$. Suppose not. We have the cover

$$\{e \uparrow \mid (\exists i \in I) [e \in i \wedge e \neq d(i)]\}$$

Since S is Scott-compact, there is a finite sub-cover

$$\{e_k \uparrow \mid (\exists i_k \in I) [e_k \in i_k \wedge e \neq d(i_k) \wedge 1 \leq k \leq n]\}$$

Since I is directed,

$$\begin{aligned} & (\exists i \in I) [i_1 \dots i_k \sqsubseteq_{EM} i] \\ \implies & \{e \uparrow \mid e \in (i - \{d(i)\})\} \text{ is a cover of } S \end{aligned}$$

This is a contradiction, since $i \sqsubseteq_{EM} S$ means that there is an element in S greater than $d(i)$

2. Let $x \in S$. From definition of $\tilde{\phi}(S)$,

$$(\forall i \in \tilde{\phi}(S)) (\exists e_i \in i) [e_i \sqsubseteq x]$$

Also, the e_i 's are unique and form a directed set. Hence, a generator d_x can be defined in the obvious way such that $\sqcup d_x(I) \sqsubseteq x$.

3. Let $x, y \in S \wedge x \neq y$. Then, from finite separability of S $d_x \neq d_y$, where d_x, d_y are the generators on I defined as above.

4. Now, we shall show that $\sqcup d_x(I) = x$. Let $e \sqsubseteq x$. Consider the cover

$$\{b \uparrow \mid b \in i \in I, b \neq d_x(i)\} \cup \{e \uparrow\}$$

Note that the above is a cover because of finite separability. Since S is Scott-compact, we have a finite sub-cover

$$\{b_k \uparrow \mid b_k \in i_k \in I, b_k \neq d_x(i_k), 1 \leq k \leq n\} \cup \{e \uparrow\}$$

Since I is directed, $(\exists i \in I) [i_1 \dots i_n \sqsubseteq_{EM} i]$. Hence, we note that

$$\{b \uparrow \mid b \in i, b \neq d_x(i)\} \cup \{e \uparrow\}$$

is a cover. Hence, we have

$$(\forall y \in S) [d(i) \sqsubseteq y \implies e \sqsubseteq y]$$

Hence, we deduce that we have the open cover consisting of

- \tilde{e} , where $\tilde{e} \in i - \{d(i)\}$
- \tilde{m} , where $\tilde{m} \in \text{mub}_{(d(i), e)}(S)$

The above open cover has a finite sub-cover,. Hence we deduce that $\text{mub}_{(d(i), e)}(S)$ is finite. Hence \tilde{i} defined as

$$\tilde{i} = \text{mub}_{(d(i), e)}(S) \cup (i - \{d(i)\})$$

satisfies $\tilde{i} \sqsubseteq_{EM} S \wedge \tilde{i} \in P_L(D)$. Hence, $e \sqsubseteq \bigsqcup d_x (I)$

The above shows that the generators on I generate precisely the elements of S . Hence,

$$\Phi \circ \tilde{\Phi} = Id. \quad \blacksquare$$

Theorem 1 $\overline{P_L(D)}$ is isomorphic to Ψ_D

4 Finitely Detectable Inconsistency

Consider the elements x, y in Figure 1. Every pair of finite elements e_x, e_y below x, y respectively have upper-bounds. However x, y do not have an upper bound. One might demand that the inconsistency relation have finite witnesses, to make it continuous. The above discussion motivates the following definition.

Definition 6 An algebraic CPO D is said to have property FI (for finitely detectable inconsistency) if

$$(\forall x, y \in D) [mub_{(x,y)}(D) = \phi \implies (\exists e_x, e_y \text{ compact}) [e_x \sqsubseteq x \wedge e_y \sqsubseteq y \wedge mub_{(e_x, e_y)}(D) = \phi];$$

It is easy to check that all Scott-Domains have the above property. C. Gunter observed that all coherent L-domains [3] have property FI. The following lemma due to Achim Jung [4] enables us to prove a stronger result.

Lemma 11 An algebraic CPO D is Lawson-compact if and only if $B(D)$ has property M.

Lemma 12 Let D be an algebraic CPO such that every finite subset of $B(D)$ has a complete finite set of minimal upper bounds. (i.e $B(D)$ has Property M). Then, D has property FI.

Proof: Let $x, y \in D$. Let $\{d_j \mid j \in I_x\}$ and $\{e_i \mid i \in I_y\}$ be the compact elements approximating x, y respectively, where I_x and I_y are index sets. Furthermore, let us assume that

$$(\forall d_j, e_i) [j \in I_x \wedge i \in I_y \implies \text{mub}_{(d_j, e_i)}(D) \neq \phi]$$

Let $C = \{\{e_i \uparrow\} \mid i \in I_y\} \cup \{\{d_j \uparrow\} \mid j \in I_x\}$. Then C is a collection of closed sets in the Lawson Topology on D satisfying the finite intersection condition. Result follows from the compactness of the Lawson Topology on D . ■

In particular all SFP objects D have property FI.

Future Work

The lossless powerdomain construction promises to provide the mathematical foundations for a fully-abstract semantics for languages with finite non-determinism in which infinite objects are observable in the operational semantics.(e.g) the language of streams and finite-nondeterminism as in[6]. As part of the mathematical justification, we are working on developing a universal characterisation of the construction. A related question about L-domains that we are examining, is the existence of a first-order, information system like representation for algebraic L-domains.

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Appendix

Lemma 13 *Let β be a limit ordinal. Let S_α be a sequence of finite sets over $\alpha \in \beta$, such that $\alpha_1 \leq \alpha_2 \implies S_{\alpha_1} \subseteq S_{\alpha_2}$. Then*

$$[(\forall \alpha \in \beta) S_\alpha \neq \phi] \implies \alpha \bigcap_{\alpha < \beta} S_\alpha \neq \phi$$

Proof: Induction on the size of S_{α_0} . ■

Lemma 14 *Let $I \in \overline{P_L(D)}$. Let S_i^α be a collection of non-empty finite sets indexed by $i \in I$, such that*

1. $S_i^\alpha \subseteq i$
2. $e_i \in S_i^\alpha, i \sqsubseteq_{EM} j \implies (\exists e_j \in S_j^\alpha) (e_i \sqsubseteq e_j)$
3. $(e_i \in S_i^\alpha, j \sqsubseteq_{EM} i) \implies (\exists e_j \in S_j^\alpha) (e_j \sqsubseteq e_i)$

Then, either

- $(\forall i \in I) |S_i^\alpha| = 1$, or
- There is a collection of non-empty sets $S_i^{\alpha+1}$ such that
 - $(\forall i \in I) [S_i^{\alpha+1} \subseteq S_i^\alpha]$
 - The new collection of sets $S_i^{\alpha+1}$ has properties 1...3.
 - $(\exists i \in I) [S_i^{\alpha+1} \neq S_i^\alpha]$

Proof: Choose $i \in I$ such that $|S_i^\alpha| \neq 1$. Let $e \in S_i^\alpha$. Define

- $S_i^{\alpha+1} = S_i^\alpha - \{e\}$
- For $j \in I$ such that $i \sqsubseteq_{EM} j$, $S_j^{\alpha+1} = S_j^\alpha - \{\tilde{e} \mid e \sqsubseteq \tilde{e}\}$
- For other $k \in I$,

$$S_j^{\alpha+1} = S_j^\alpha - \{\tilde{e} \mid (\forall j)[i \sqsubseteq_{EM} j \implies S_j^{\alpha+1} \cap \tilde{e} \uparrow = \phi]\}.$$

The details that the constructed sets satisfy properties is extensive case analysis and is omitted. ■

Lemma 15 *Let I be an ideal in $\overline{P_L(D)}$. Let \tilde{I} be cofinal in I . Let P be a predicate defined on D such that*

1. $(\forall i \in \tilde{I}) (\exists e \in i) [P(e)]$
2. $(d \in i \wedge i \in \tilde{I} \wedge e \in j \wedge j \in \tilde{I} \wedge i \sqsubseteq_{EM} j \wedge d \sqsubseteq e \wedge P(e)) \implies P(d)$

Then, $(\forall i \in \tilde{I}) (\exists e_i \in i) [P(e_i) \wedge (\forall j \in \tilde{I}) (i \sqsubseteq_{EM} j \implies (\exists e_j \in j) (e_i \sqsubseteq e_j \wedge P(e_j)))]$.

Proof: (By contradiction)

Let $i = (e_1 \dots e_k, e_{k+1} \dots e_m) \wedge i \in I$. Let

- $(\forall s) (1 \leq s \leq k) P(e_s)$
- $(\forall s) ((k+1) \leq s \leq m) \neg P(e_s)$

From assumption

$(\forall s = 1 \dots k) (\exists j_s) (i \sqsubseteq_{EM} j_s \wedge (\forall \tilde{e} \in j_s) [(e_s \sqsubseteq \tilde{e}) \implies \neg P(\tilde{e})])$

Since I is directed and \tilde{I} is cofinal in I , $(\exists j \in \tilde{I}) (j_1 \dots j_k \sqsubseteq_{EM} j)$. From assumption on P $(\exists e \in j) P(e)$.

Let $e_{j_1} \in j_1 \sqsubseteq e, \dots, e_{j_k} \in j_k \sqsubseteq e$

$\implies e_1 \not\sqsubseteq e_{j_1}, \dots, e_k \not\sqsubseteq e_{j_k}$

$\implies e_1 \dots e_k \not\sqsubseteq e$

$\implies (\exists e_s \in i) (s > k \wedge e_s \sqsubseteq e)$

$\implies P(e_s)$ (Contradiction) ■

Lemma 16 *Let $I \in \overline{P_L(D)}$. Let \tilde{I} be co-final in I . Let d be a function $d: \tilde{I} \rightarrow D$ such that :*

- $i \in \tilde{I} \implies d(i) \in i$
- $(i \in \tilde{I} \wedge j \in \tilde{I} \wedge i \sqsubseteq_{EM} j) \implies d(i) \sqsubseteq d(j)$

Then, d can be uniquely extended to a generator on I .

Lemma 17 *Let $I \in \overline{P_L(D)}$. Let \tilde{I} be cofinal in I . Let P be a predicate as in lemma 15. Then there is a generator d over I such that $(\forall i \in \tilde{I}) [P(d(i))]$*

Proof: Define $S_i^0 = \{e \mid e \in i, P(e)\}$, where $i \in \tilde{I}$. Note that the collection S_i^0 satisfies the conditions of the lemma 15. Let Π be the set of all collections S_i^γ such that

- The collection $\{ S_i^\gamma \}$ satisfies the conditions of the lemma 14.
- $S_i^\gamma \subseteq S_i^0$ for all $i \in \tilde{I}$

Let $C_1, C_2 \in \Pi$, where $C_1 = \{ S_i^\gamma \}$ and $C_2 = \{ S_i^\delta \}$.

Define $C_1 \preceq C_2$ if $(\forall i \in I) S_i^\gamma \subseteq S_i^\delta$

Note that every chain in Π has an upper bound by lemma 13. Using Zorn's lemma, we deduce that Π has a maximal element. However, from lemma 14, the maximal element $C = \{ S_i^\alpha \}$ satisfies the condition that $(\forall i \in I) [| S_i^\alpha | = 1]$. From the conditions on the elements of Π and from lemma 16, we observe that d defined by

$$d(i) = e, \text{ where } e \text{ is the unique element in } S_i^\alpha$$

can be uniquely extended to a generator on I ■