HEAT KERNEL ESTIMATE OF THE SCHRÖDINGER OPERATOR IN UNIFORM DOMAINS

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In this thesis we study the properties of the Schrödinger operator $L = -\Delta + q$ on a Harnack-type Dirichlet space for q belonging to Kato class K or Kato-infinity class K^∞. To be specific, it consists of three parts as follows:

The first part is a generalization of [27]. For any Harnack-type Dirichlet space we give conditions under which there exists a positive Dirichlet solution (the profile) in an unbounded uniform domain for the operator L. In this setting, we further give the two-sided heat kernel estimate using the famous h-transform technique.

The idea of second part comes from [64]. In the exterior of a compact set in a non-parabolic Harnack-type space, we can prove some equivalent statements connecting subcrilicality, positiveness of the Green function, gaugeability and the boundedness of the Dirichlet-type solution provided the potential $q \in K^\infty$. Particularly, we can apply the boundedness result of the profile to the first part and conclude a more precise heat kernel estimate.

In the third part we provide some typical examples and explore some properties when the potential decays faster than the quadratic one. Some other examples are given in the domain outside an unbounded domain and we propose some hypothesis as an supplement to the second part.

BIOGRAPHICAL SKETCH

Jingbo Liu was born on May 19th, 1990 in a small village of Dengfeng, a beautiful city in central China. He showed his love and talent to math at an early age. Having solved a couple of difficult puzzles during primary school, he dreamed that one day he could be a mathematician in the future. That faith became even stronger and drove him to learn and explore more math all the way along middle school.

He chose math as his major at University of Science and Technology of China from 2010 and obtained a Bachelor of Science degree in mathematics in 2014. Then he came to Cornell University to pursue a PhD degree in mathematics. In 2019, he completed his thesis under the supervision of his advisor, Laurent Saloff-Coste.

This work is dedicated to my wife, Yujing Wang and my son, Shifan Liu.

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CHAPTER 1
INTRODUCTION

The main focus of this manuscript is to prove Dirichlet-type two-sided global heat kernel estimates for the Schrödinger operator $L = -\Delta + q$ for a class of ″good″ potentials q. Here Dirichlet-type means the local weak solution (which we usually call the profile) of L vanishes along the boundary in some sense. Basically we have two goals: one is to find a suitable potential class such that good properties holds for L, the other one is to find a good domain where a precise two-sided global heat kernel estimate holds.

The Kato class K and Kato-infinity class K^∞ introduced by Kato in [32] have proven to be an ideal potential class for us. In the Euclidean spaces $\mathbb{R}^n (n \geq 3)$, q belongs to K if and only if

$$\lim_{r \to 0^+} \left[\sup_{x \in \mathbb{R}^n} \int_{|y-x| \leq r} \frac{|q(y)|}{|y-x|^{n-2}} dy \right] = 0.$$

We say $q \in K_{\text{loc}}$ if $1_U q \in K$ for any compact set U.
In addition, q belongs to K^∞ if and only if $q \in K_{\text{loc}}$ and

$$\lim_{A \to \infty} \left[\sup_{x \in \mathbb{R}^n} \int_{|y| \geq A} \frac{|q(y)|}{|y-x|^{n-2}} dy \right] = 0.$$

Previous related works mainly focuse on two aspects: the properties of L (like subcriticality and gaugeability) in the whose space $\mathbb{R}^n$, see [40, 41, 59, 60, 63, 64], and the case of bounded Lipschitz domains in $\mathbb{R}^n$ with Dirichlet boundary conditions, see [11, 13, 67]. There seem to be no studies on unbounded domains with Dirichlet boundary condition for the Kato-type potential. Maybe the most important and simplest setting of such type is on the domain outside the unit ball $U = \mathbb{R}^n \setminus \overline{B(0,1)}$ with the potential $q(x) = \frac{\beta}{|x|^\alpha} (\alpha > 2)$. We shall prove that there exists a $\beta_0 < 0$ such that whenever $\beta > \beta_0$ and x, y are away from the boundary, the Dirichlet-type heat kernel $p_q(t, x, y)$ induced by L satisfies the following estimate:

$$\frac{c_1}{t^{n/2}} \exp\Big(- \frac{\rho(x,y)^2}{c_2 t} \Big) \leq p^D_{q,U}(t,x,y) \leq \frac{c_3}{t^{n/2}} \exp\Big(- \frac{\rho(x,y)^2}{c_4 t} \Big). \tag{1.1}$$

This is exactly the same estimate as induced by $-\Delta$ (See [23]). It tells us the effect of such potential can be ignored when x and y are away from the boundary.

The main technique we use here is Doob's h−transform. See, e.g., Chapter 5 in [11] for an introduction. The most exciting thing about the h−transform is that even if it is very simple and even if it has been used by so many authors in so many different contexts, this technique still yields new interesting results. By using the h-transform technique, we can actually conclude similar results in the whole space (although this is not our key point). These results are new even for problems posed in Euclidean space, as treated in [59, 60].

For the second goal, we intend to extend the Euclidean space setting to some abstract metric spaces where good properties still hold. Thanks to the work of A. Grigor'yan, L. Saloff-Coste and S. T. Sturm, we know that any local regular Dirichlet space whose intrinsic distance yields a complete metric structure, and where the volume doubling condition and Poincaré inequality are satisfied is such a good space. Some important examples are Riemannian manifolds with non-negative Ricci curvature and nilpotent Lie groups equipped with a sub-

Riemannian invariant structure. In this good setting, we have the following global heat kernel estimate for $p(t,x,y)$ (which is induced by Laplacian) in the whole space:

$$\frac{c_1}{V(x,\sqrt{t})}\exp\Big(-\frac{\rho(x,y)^2}{c_2 t}\Big) \le p(t,x,y) \le \frac{c_3}{V(x,\sqrt{t})}\exp\Big(-\frac{\rho(x,y)^2}{c_4 t}\Big). \tag{1.2}$$

The only difference between this formula and (1.1) is that we replace $t^{n/2}$ with the volume in the abstract space. The problem remains the same as in Euclidean space: for which potential q does the heat kernel $p_q(t,x,y)$ admit the same form of estimate as in (1.2)? There are some insightful results in the whole-space sense: A. Grigor'yan proved in [19] that for any positive Green-bounded potential, such an estimate holds. As a reminder, a potential q is Green-bounded if either $q \equiv 0$ or the Green function $G(x,y)$ is finite and

$$\sup_{x\in X}\int_X G(x,y)|q(y)|dy < \infty.$$

Almost at the same time, M. Takeda discussed the relationship between gaugeability and the estimate (1.2) for some broader potentials in [55].

Our focus here is the Dirichlet-type heat kernel estimate for $p^D_{q,U}(t,x,y)$ in some good domain U. The classical study of the Schrödinger operator (or even the study of the Laplacian before the work in [27]) with Kato-type potentials and Dirichlet boundary condition has often been restricted to bounded Lipschitz domains in $\mathbb{R}^n$. We can not generalize Lipschitz domain in $\mathbb{R}^n$ to such abstract spaces. Fortunately, the case of a uniform domain proves to be a good generalization: It includes a much broader class of domains than that of Lipschitz domains. This is inspired by the work of P. Gyrya and L. Saloff-Coste in [27], where they studied the Neumann and Dirichlet heat kernel estimates for $-\Delta$ in uniform domains.

To make everything work in such a metric space, we need to redefine the Kato class using heat kernel or Green function and some other related notions as in [34, 55]. Then we can prove that for any local regular Harnack-type Dirichlet space, the same conclusion holds as in (1.2) in some unbounded uniform domain (outside a compact set) if similar criteria are satisfied. This will be the main content of Chapter 4 and Chapter 5.

CHAPTER 2

DIRICHLET SPACES

This chapter provides some background including Dirichlet forms, Harnack-type spaces and weak solutions with different boundary conditions. Dirichlet forms provide our main setting, which is a generalization of a variety of settings such as Riemannian manifolds. More details about Dirichlet forms can be found in [17, 8]. Among those Dirichlet spaces, the Harnack-type ones are of most interest to us because they satisfy many good properties. The Schrödinger operator will also be introduced in this chapter. It is the main object to be studied in this manuscript.

2.1 Dirichlet forms

Let H be a real Hilbert space with inner product $(\ ,\ )$. A non-negative definite symmetric bilinear form densely defined on H is called a *symmetric form* on H. Namely, $\mathscr{E}$ is called a *symmetric form* on H if the following conditions hold :

(1). $\mathscr{E}$ is defined on $\mathscr{D}(\mathscr{E})\times\mathscr{D}(\mathscr{E})$ with values in $\mathbb{R}^1$, where $\mathscr{D}(\mathscr{E})$ is a dense linear subspace of H,

(2). $\mathscr{E}(u,v) = \mathscr{E}(v,u)$, $\mathscr{E}(u+v,w) = \mathscr{E}(u,w) + \mathscr{E}(v,w)$,

$a\mathscr{E}(u,v) = \mathscr{E}(au,v)$, $\mathscr{E}(u,u) \geq 0$, $u,v,w \in \mathscr{D}(\mathscr{E})$, $a \in \mathbb{R}^1$.

Throughout the rest of this manuscript we assume $H = L^2(X,\mu)$ where X is a locally compact separable metric space and μ is a positive Radon measure on X such that $\mathrm{Supp}[\mu] = X$.

A *Markovian* symmetric form has the property that if $u \in \mathscr{D}(\mathscr{E})$ and v is a contraction of u, namely, for all $x, y \in X$

$$|v(x) - v(y)| \leq |u(x) - u(y)| \text{ and } |v(x)| \leq |u(x)|,$$

then $v \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(v, v) \leq \mathscr{E}(u, u)$.

A symmetric form $\mathscr{E}$ is said to be closed if, for $u_n, u_m \in \mathscr{D}(\mathscr{E})$,

$$\mathscr{E}(u_n - u_m, u_n - u_m) + (u_n - u_m, u_n - u_m) \to 0 \text{ as } n, m \to \infty \quad \Rightarrow$$

$$\exists u \in \mathscr{D}(\mathscr{E}), \text{ s.t. } \mathscr{E}(u_n - u, u_n - u) + (u_n - u, u_n - u) \to 0 \text{ as } n \to \infty$$

Definition 2.1.1. *A Dirichlet form is a symmetric bilinear form which is Markovian and closed.*

Let $\mathscr{C}_c(X)$ denote the space of continuous functions with compact support. A *core* for $(X, \mu, \mathscr{E}, \mathscr{E}(\mathscr{D}))$ is a subset $\mathscr{C}$ of both $\mathscr{C}_c(X)$ and $\mathscr{D}(\mathscr{E})$ that is dense in $\mathscr{C}_c(X)$ in uniform norm and dense in $\mathscr{D}(\mathscr{E})$ in the norm

$$(\|f\|_2^2 + \mathscr{E}(f, f))^{1/2}.$$

A Dirichlet form is called *regular* if it admits a core.

$\mathscr{E}$ is said to be *local* if $u, v \in \mathscr{D}(\mathscr{E})$, Supp$[u]$ and Supp$[v]$ being disjoint compact sets implies $\mathscr{E}(u, v) = 0$.

$\mathscr{E}$ is said to be *strictly local* if $u, v \in \mathscr{D}(\mathscr{E})$, Supp$[u]$ and Supp$[v]$ being compact and v being constant on a neighbourhood of Supp$[u]$ implies $\mathscr{E}(u, v) = 0$.

Any strictly local regular Dirichlet form can be written in terms of an energy measure $\Gamma(u,v)$ so that

$$\mathscr{E}(u,v) = \int d\Gamma(u,v),$$

where for $u, v \in \mathscr{D}(\mathscr{E})$, $\Gamma(u,v)$ is a signed Radon measure on X. We refer the readers to [27] for more details.

Definition 2.1.2. *The strictly local regular Dirichlet form $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ is said to admits a carré du champ if, for any $u, v \in \mathscr{D}(\mathscr{E})$, the measure $\Gamma(u,v)$ is absolutely continuous with respect to μ. When that is true, the carré du champ is defined as*

$$\Upsilon(u,v) = \frac{d\Gamma(u,v)}{d\mu},$$

Now we will exhibit a couple of examples about Dirichlet forms.

Example 2.1.1. *Let $X = \mathbb{R}^n$, $\mathscr{D}(\mathscr{E})$ be the Sobolev space $W^1(\mathbb{R}^n)$, and $\mathscr{E}$ be defined as*

$$\mathscr{E}(f,g) = \int \nabla f \cdot \nabla g dx, \forall f, g \in W^1(\mathbb{R}^n).$$

This form is strictly local and regular.

Example 2.1.2. *Let $X = \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be an unbounded domain with smooth boundary ∂U. We define $\mathscr{E}_U^D$ as*

$$\mathscr{E}_U^D(f,g) = \int_U \nabla f \cdot \nabla g dx, \forall f, g \in W_0^1(U).$$

with $\mathscr{D}(\mathscr{E}_U^D) = W_0^1(U)$.
This form is also strictly local and regular.

After the discussion of Dirichlet forms, we are now ready to define the distance function.

Definition 2.1.3. *Given a strictly local regular Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ on $L^2(X,\mu)$, for any $x, y \in X$, set*

$$\rho(x, y) = \rho_{\mathscr{E}}(x, y) = \sup\{u(x) - u(y) : u \in \mathscr{D}(\mathscr{E}) \cap \mathscr{C}_0(X), d\Gamma(u, u) \leq d\mu\}.$$

Here $\mathscr{C}_0(X)$ is the closure of $\mathscr{C}_c(X)$ for the sup norm.

The function ρ depends on $(\mathscr{E}, \mathscr{D}(\mathscr{E})), X$ and the topology of X. Generally it is only a pseudo-distance(See [27, 4]). So people usually make the following two important assumptions:
(A1). The pseudo-distance ρ is finite everywhere, continuous, and defines the original topology of X.
(A2). The metric space (X, ρ) is complete.
After making these two assumptions, we have all the good properties regarding to the distance function ρ. For details see Theorem 2.11 in [27].
Throughout this work we always assume conditions $(A1)$ and $(A2)$ are satisfied.

2.2 Harnack-type Dirichlet spaces

The notion Harnack-type Dirichlet space is the main focus of this chapter and plays a crucial role in this manuscript.

First we will introduce two other geometric properties which have great connection with Harnack inequality.

Definition 2.2.1. *A measure metric space (X, ρ, μ) is said to have the volume doubling property if there exists a constant D_0 such that the volume function $V(x, r) = \mu(B(x, r))$ satisfies*

$$\forall x \in X, r > 0, \quad V(x, 2r) \leq D_0 V(x, r) \tag{2.1}$$

The voluming doubling property implies a variety of volume-control inequalities and gives us some upper bounds or lower bounds. See [44] for more details. Here we give two such inequalities which will be used later.

Theorem 2.2.1. *Suppose the volume doubling is satisfied on (X, ρ, μ), then we have*

i. For any $x, y \in X$ and $0 < r < s < \infty$,

$$\frac{V(x, s)}{V(y, r)} \leq D^2 \left(\frac{\rho(x, y) + s}{r} \right)^A,$$

with $A = log_2 D$.

ii. Assume additionally (X, ρ, μ) is complete non-compact. Then there exist constants C and α such that for any $x \in X$ and $0 < s < r < \infty$,

$$\frac{V(x, s)}{V(x, r)} \leq C \left(\frac{s}{r} \right)^\alpha.$$

Next we introduce the Poincaré Inequality.

Definition 2.2.2. *Let $(X, \rho, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$ and the distance function ρ satisfies the hypothesis (A1) and (A2). We say that* *the Poincaré inequality holds if there exists a constant* P_0 *such that for any* $x \in X, r > 0$ *and* $f \in \mathscr{F}_{\text{loc}}(B(x,r))$*,*

$$\min_{\xi \in \mathbb{R}} \int_{B(x,r)} |f - \xi|^2 d\mu \leq P_0 r^2 \int_{B(x,r)} d\Gamma(f, f). \tag{2.2}$$

For the connection between volume doubling property and a variety of other forms of Poincaré inequalities, see [28, 44].

Before we dive into the Harnack-type Dirichlet space, let us now talk about the heat semigroup and the Green function.

Any Dirichlet form $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ yields a strongly continuous self-adjoint semigroup P_t on $L^2(X, \mu)$ for any $t > 0$ which also preserves positivity, i.e., any $f \in L^2(X, \mu)$ and $0 \leq f \leq 1$ implies $0 \leq P_t f \leq 1$.

The infinitesimal generator A of the semigroup P_t is defined as:

$$Au = \lim_{t \to 0^+} \frac{P_t u - u}{t}$$

and its domain $\mathscr{D}(A)$ is the subset of $\mathscr{D}(\mathscr{E})$ of those functions u for which Au exists as a strong limit. On this domain, A is defined by $\langle -Au, v \rangle = \mathscr{E}(u, v)$ for all $u, v \in \mathscr{D}(\mathscr{E})$. It is a self-adjoint operator and $\mathscr{D}(\mathscr{E}) = \mathscr{D}(\sqrt{-A})$, $\mathscr{E}(u, v) = \langle \sqrt{-A}u, \sqrt{-A}v \rangle$.

Each heat semigroup P_t can also induce a probability measure $p(t, x, dy)$ by the following way:

$$P_t f(x) = \int_X p(t, x, dy) f(y).$$

for any $f(x) \in L^2(X, \mu)$.

If we assume the space (X, μ) is of Harnack-type, then we can rewrite as

$$P_t f(x) = \int_X p(t, x, y) f(y) dy.$$

We usually call $p(t, x, y)$ the heat kernel of P_t.

The Green function $G(x, y)$ on X is defined as

$$G(x, y) = \int_0^\infty p(t, x, y) dt.$$

The space $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ is called parabolic if $G(x, y) = \infty$ and non-parabolic if $G(x, y) < \infty$.

For example, let $X = \mathbb{R}^n$, $d\mu$ is the Lebesgue measure and $A = \Delta$, by simple computation, we have

$$G(x, y) = \begin{cases} +\infty & n = 1, 2 \\ c_n |x - y|^{2-n} & n \geq 3 \end{cases}$$

Now we give the following definition of Harnack-type Dirichlet space:

Definition 2.2.3. *A regular strictly local Dirichlet space $(X, \rho, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ is of Harnack type if the distance ρ satisfies the conditions (A1) and (A2), and the following uniform parabolic Harnack inequality holds:*

$$\sup_{(t,x) \in Q_-} u(t, x) \leq H_0 \inf_{(t,x) \in Q_+} u(t, x), \tag{2.3}$$

for any $z \in X, r > 0$, where u is any non-negative weak solution(see next section for more details) of the heat equation $\partial_t u - Au = 0$ in $(0, r^2) \times B(z, r)$ and $Q_- = (r^2/4, r^2/2) \times B(z, r/2)$, $Q_+ = (3r^2/4, r^2) \times B(z, r/2)$ and both sup and inf are essential.

We are ready now to present the following theorem due to K. T. Sturm([52], [53], [54]), A. Grigor'yan([22]) and L. Saloff-Coste([44]).

Theorem 2.2.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a strictly local regular Dirichlet space. Assume that the distance ρ satisfies the assumptions (A1) and (A2). Then the following properties are equivalent:*

— The form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is of Harnack-type, i.e., the uniform parabolic Harnack inequality (2.3) is satisfied.

— The volume doubling condition (2.1) and the Poincaré inequality (2.2) are satisfied.

— The heat kernel $p(t, x, y)$, for all $t > 0, x, y \in X$, satisfies

$$\frac{c_1}{V(x, \sqrt{t})}\exp\Big(-\frac{\rho(x, y)^2}{c_2 t}\Big) \le p(t, x, y) \le \frac{c_3}{V(x, \sqrt{t})}\exp\Big(-\frac{\rho(x, y)^2}{c_4 t}\Big).$$

for some constant c_1, c_2, c_3, c_4.

2.3 Neumann & Dirichlet boundary conditions

To start with, we first give the following notations. Assume V is any open set in X, then we set

— $\mathscr{F}_{\text{loc}}(V) = \{u \in L^2_{\text{loc}}(V) : \forall$ compact sets $K \subset V, \exists u^\# \in \mathscr{D}(\mathscr{E}), u = u^\#|_K$ a.e.$\}$.

— $\mathscr{F}(V) = \{u \in \mathscr{F}_{\text{loc}}(V) : \int_V |v|^2 d\mu + \int_V d\Gamma_V(u, u) < \infty\}$,

where we define $\Gamma_V(u, v)$ for $u, v \in \mathscr{F}_{\text{loc}}(V)$ on all of V by setting

$$\Gamma_V(u, v)|_\Omega = \Gamma_{V,\Omega}(u, v) = \Gamma(u^\#, v^\#)|_\Omega$$

for all relatively compact sets $\Omega \subset V$ and $u^{\#}, v^{\#}$ being any elements of $\mathscr{D}(\mathscr{E})$ such that $u = u^{\#}|_{\Omega}, v = v^{\#}|_{\Omega}$.

— $\mathscr{F}_c(V) = \{u \in \mathscr{F}(V) : \text{the essential support of } u \text{ is compact in } V\}$.

2.3.1 Local weak solutions without boundary conditions

If we identify $L^2(X, \mu)$ with its dual using the scalar product. Let V be a nonempty open subset of X. Consider the subspace $\mathscr{F}_c(V) \subset \mathscr{D}(\mathscr{E}) \subset L^2(X, \mu)$ and their duals $L^2(X, \mu) \subset \mathscr{D}(\mathscr{E})' \subset \mathscr{F}_c(V)'$. We will use the brackets $\langle \cdot, \cdot \rangle$ to denote duality pairing between these spaces.

Definition 2.3.1. *Let V be a nonempty open subset of X. Let $f \in \mathscr{F}_c(V)'$. A function $u : V \mapsto \mathbb{R}$ is a weak (local) solution of $Au = f$ in V if*

1. *$u \in \mathscr{F}_{\text{loc}}(V)$;*
2. *For any function $\phi \in \mathscr{F}_c(V), \mathscr{E}(\phi, u) = \langle \phi, f \rangle$.*

Next we introduce local weak solutions of the heat equation $\partial_t u = Au$ in a time-space cylinder $I \times V$, where I is a time interval and V is an nonempty open subset of X.

Given a Hilbert space H, let $L^2(I \to H)$ be the Hilbert space of those functions $v : I \to H$ such that

$$\|v\|_{L^2(I \to H)} = \left(\int_I \|v(t)\|_H^2 \, dt \right)^{1/2} < \infty.$$

Let $W^1(I \to H) \subset L^2(I \to H)$ be the Hilbert space of those functions $v : I \to H$ in

$L^2(I \to H)$ whose distributional time derivative v' can be represented by functions in $L^2(I \to H)$, equipped with the norm

$$\|v\|_{W^1(I\to H)} = \left(\int_I (\|v(t)\|_H^2 + \|v'(t)\|_H^2)dt\right)^{1/2} < \infty.$$

Given an open time interval I, set

$$\mathscr{F}(I \times X) = L^2(I \to \mathscr{D}(\mathscr{E})) \cap W^1(I \to \mathscr{D}(\mathscr{E})').$$

Given an open time interval I and an open set $V \subset X$(both nonempty), let

$$\mathscr{F}_{\text{loc}}(I \times V)$$

be the set of all functions $v : I \times V \to \mathbb{R}$ such that, for any open interval $I' \subset I$ relatively compact in I and any open subset V' relatively compact in V there exists a function $u^\# \in \mathscr{F}(I \times X)$ satisfying $u = u^\#$ a.e. in $I' \times V'$. Finally, let

$$\mathscr{F}_c(I \times V) = \{v \in \mathscr{F}(I \times X) : v(t, \cdot) \text{ has compact support in V for a.e. t} \in \text{I}\}.$$

Now we are ready to given the following definition for the weak solution of heat equation:

Definition 2.3.2. *Let I be an open time interval. Let V be an open subset in X and set $Q = I \times V$. A function $u : Q \to \mathbb{R}$ is a weak (local) solution of the heat equation $(\partial_t - A)u = 0$ in Q if*

1. *$u \in \mathscr{F}_{\text{loc}}(Q)$;*
2. *For any open interval J relatively compact in I and any $\phi \in \mathscr{F}_c(Q)$,*

$$\int_J \int_V \phi \partial_t u d\mu dt + \int_J \mathscr{E}(\phi(t, \cdot), u(t, \cdot))dt = 0.$$

2.3.2 Dirichlet-type boundary condition

The Dirichlet-type Dirichlet form is defined as follows:

Definition 2.3.3. *Let $(\mathscr{E}_U^D, \mathscr{D}(\mathscr{E}_U^D))$ be the closure of the form $(\mathscr{E}, \mathscr{F}_c(U))$ in $L^2(U, \mu)$. Set*

$$\mathscr{F}^0(U) = \mathscr{D}(\mathscr{E}_U^D)$$

and $$\mathscr{F}^0_{\text{loc}}(U) = \{f \in L^2_{\text{loc}}(U, \mu) : \forall V \subset U, \textit{ open, relatively compact in } \overline{U}, \\ \exists f^\# \in \mathscr{F}^0(U), f^\# = f \textit{ a.e. in } V\}.$$

As with the global case, we can also define $P_{U,t}^D$ and A_U^D to be the semigroup and the associated infinitesimal generator of the form $(\mathscr{E}_U^D, \mathscr{D}(\mathscr{E}_U^D))$ respectively.
For any open subset U, the semigroup $P_{U,t}^D$ also admits a locally bounded kernel $p_U^D(t, x, y)$, $(t, x, y) \in (0, \infty) \times U \times U$. In addition, we have the following simple comparison:

$$p_U^D(t, x, y) \leq p(t, x, y). \tag{2.4}$$

There are some upper and lower bound of $p_U^D(t, x, y)$ when U is the intrinsic balls. See Chapter 2 in [27] for details.
Let us turn to the boundary condition part. In the first place, we give the following definition related to the Dirichlet boundary condition:

Definition 2.3.4. *Let V be an open subset of U. Set*

$$\mathscr{F}^0_{\text{loc}}(U, V) = \{f \in L^2_{\text{loc}}(V, \mu) : \forall \textit{ open } \Omega \subset V \textit{ relatively compact in } \overline{U} \textit{ with} \\ \rho(\Omega, U \setminus V) > 0, \exists f^\# \in \mathscr{F}^0(U) : f^\# = f \ \mu\text{–}a.e. \textit{ on } \Omega\}$$

Now we are ready to define the notion of a local weak solution of the elliptic equation $Au = f$ with Dirichlet boundary conditions along ∂U.

Definition 2.3.5. *Let V be an open set in U. Let $f \in \mathscr{F}'_c(V)$. we say that a function $u : V \to \mathbb{R}$ is a local weak solution of the equation $Au = f$ in V with weak Dirichlet boundary conditions along ∂U if*

1. *The function u belongs to $\mathscr{F}^0_{\mathrm{loc}}(U, V)$;*
2. *For any function $\phi \in \mathscr{F}_c(V)$, $\int_V d\Gamma(\phi, u) = \int_V \phi f d\mu$.*

In other words, a function u is a local weak solution of the equation $Au = f$ in V with weak Dirichlet boundary conditions along ∂U if it is a local weak solution of the elliptic equation $Au = f$ in V which also belongs to $\mathscr{F}^0_{\mathrm{loc}}(U, V)$.

We can define the notion of a weak solution of the heat equation in $Q = I \times V$ with Dirichlet boundary condition along ∂U similarly as in last subsection.

2.3.3 Neumann-type boundary condition

We first give the definition of Neumann-type Dirichlet form.

Definition 2.3.6. *Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a strictly local regular Dirichlet form on $L^2(X, \mu)$ with energy form Γ. Let U be an open set in X. Then*

$$\mathscr{E}^N_U(f, g) = \int_U d\Gamma(f, g), \quad f, g \in \mathscr{F}(U).$$

and

$$\mathscr{D}(\mathscr{E}^N_U) = \{f \in L^2(X, \mu) : \int_U d\Gamma_U(f, f) < \infty\}.$$

The local weak Neumann-type solution is a little different from that of Dirichlet-type, where an additional boundary condition is imposed. In reality, the Neumann boundary condition may be contained in the definition of local weak solution without any boundary solution(when X has a natural boundary). Here we give an example to clarify it.

Example 2.3.1. *Suppose X is the closed upper-half space $\mathbb{R}^2_+$ equipped with its natural Dirichlet form*

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^2_+} \left(\left| \frac{\partial f}{\partial x} \right|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \right) d\mu, \quad f \in W^1(\mathbb{R}^2_+)$$

Let $V = \{z = (x,y) : x^2 + y^2 < 1; y \geq 0\} \subset \mathbb{R}^2_+$. Note that V is open in $\mathbb{R}^2_+$. Let u be a local weak solution of $\Delta u = 0$ in V. Then u is smooth in V and must have vanishing normal derivative along the segment (-1, 1) of the x-axis.

We can define the notion of a weak solution of the heat equation in $Q = I \times V$ with Neumann boundary condition along ∂U similarly as in Subsection 1.3.1.

Remark 2.3.1. *In this subsection we have used the notion "with Neumann boundary condition along the boundary of U". Nevertheless, the "boundary" here is $\widetilde{U} \setminus U$ where $\widetilde{U}$ is the completion of U which can be very different than the boundary of U in X. See Chapter 2 in [27] for more details.*

2.4 The Schrödinger operator counterpart

Now we turn to the Schrödinger operator

$$L = -\Delta + q$$

where q belongs to the Kato class K defined as any measurable function satisfying

$$\lim_{t\to 0^+} \sup_{x\in X} \int_X \Big(\int_0^t p(s,x,y)ds \Big) |q(y)| dy = 0. \tag{2.5}$$

It is the main object in this manuscript. The class K appears explicitly first in Kato's paper [32]. However it can be traced back to Schechter's book [46] where one of his theorems has as its hypothesis $q \in K$ by some equivalent definitions.

In this section we will discuss the Dirichlet forms and weak solutions in all three different cases: the global one, the Dirichlet one and the Neumann one.

2.4.1 The global case

Similar as Section 1.1, we can define the Dirichlet form $(\mathscr{E}_q, \mathscr{D}(\mathscr{E}_q))$ for potential $q \in K$ as follows.

$$\mathscr{E}_q(f,g) = \int_X d\Gamma(f,g) + \int_X qfg d\mu, \quad f,g \in \mathscr{D}(\mathscr{E}_q)$$

$$\mathscr{D}(\mathscr{E}_q) = \{f \in L^2(X,\mu) : \int_X d\Gamma(f,f) + \int_X |q| f^2 d\mu < \infty\}$$

Any such Dirichlet form $(\mathscr{E}_q, \mathscr{D}(\mathscr{E}_q))$ yields a strongly continuous self-adjoint semigroup $P_{q,t}$ and the corresponding heat kernel $p_q(t,x,y)$ as

$$P_{q,t} f(x) = \int_X p_q(t,x,y) f(y) dy$$

for any $f(x) \in L^2(X, d\mu)$. In addition, this semigroup has the infinitesimal generator $L = -A + q$.

We can also define the space $\mathscr{F}_{q,\text{loc}}, \mathscr{F}_q, \mathscr{F}_{q,c}$ as we did in previous sections. Next we define the local weak solution of the Schrödinger operator L without any boundary conditions:

Definition 2.4.1. *Let V be a nonempty open subset of X. Let $f \in \mathscr{F}_{q,c}(V)'$. A function $u : V \to \mathbb{R}$ is a weak local solution of $Lu = f$ in V if*

1. *$u \in \mathscr{F}_{q,loc}(V)$;*
2. *For any function $\phi \in \mathscr{F}_{q,c}(V), \mathscr{E}_q(\phi, u) = \langle \phi, f \rangle$.*

And the weak solution for the heat equation is defined as follows:

Definition 2.4.2. *Let I be an open time interval. Let V be an open subset in X and set $Q = I \times V$. A function $u : Q \to \mathbb{R}$ is a weak (local) solution of the heat equation $(\partial_t - L)u = 0$ in Q if*

1. *$u \in \mathscr{F}_{q,\text{loc}}(Q)$;*
2. *For any open interval J relatively compact in I and any $\phi \in \mathscr{F}_{q,c}(Q)$,*

$$\int_J \int_V \phi \partial_t u d\mu dt + \int_J \mathscr{E}_q(\phi(t, \cdot), u(t, \cdot)) dt = 0.$$

2.4.2 The Dirichlet-type boundary condition

For any open set U in X, let $(\mathscr{E}^D_{q,U}, \mathscr{D}(\mathscr{E}^D_{q,U}))$ be the closure of the form $(\mathscr{E}_q, \mathscr{F}_{q,c}(U))$ in $L^2(U, \mu)$, and set $\mathscr{F}^0_q(U) = \mathscr{D}(\mathscr{E}^D_{q,U})$. We also define, for any open subset V of U,

$$\mathscr{F}^0_{q,\mathrm{loc}}(U,V) = \{f \in L^2_{\mathrm{loc}}(V,\mu) : \forall \text{ open } \Omega \subset V \text{ relatively compact in } \overline{U} \text{ with}$$
$$\rho(\Omega, U \setminus V) > 0, \exists f^\# \in \mathscr{F}^0_q(U) : f^\# = f \ \mu\text{–a.e. on } \Omega\}$$

Now we define the notion of a local weak solution of the elliptic equation $Lu = f$ with Dirichlet boundary conditions along ∂U.

Definition 2.4.3. *Let V be an open set in U. Let $f \in \mathscr{F}'_{q,c}(V)$. we say that a function $u : V \to \mathbb{R}$ is a local weak solution of the equation $Lu = f$ in V with weak Dirichlet boundary conditions along ∂U if*

1. *The function u belongs to $\mathscr{F}^0_{q,\mathrm{loc}}(U,V)$;*
2. *For any function $\phi \in \mathscr{F}_{q,c}(V)$, $\int_V d\Gamma(\phi, u) + \int_V q\phi u d\mu = \int_V \phi f d\mu$.*

2.4.3 The Neumann-type boundary condition

As we mentioned in Section 1.3.3, the Neumann boundary condition is different from the Dirichlet case, as the Neumann-type local weak solution may be contained in the definition of the global weak solution. As a result, in the Dirichlet form $(\mathscr{E}^N_q(U), \mathscr{D}(\mathscr{E}^N_q(U)))$, we define $\mathscr{D}(\mathscr{E}^N_q) = \mathscr{F}_q(U)$. In addition, we define the Neumann-type weak solution in a similar way as in Section 1.3.3.

Remark 2.4.1. *In the following chapters, we will put our emphasis on the Dirichlet-type Schrödigner operator since it is the most interesting case. At the end of each chapter, we then give, if any, some remarks for the other two types.*

2.5 Goals

The main goal of this manuscript is to present rigorous two-sided heat kernel estimate $p_q(t,x,y)$ for $(-\Delta + q)u = 0$ where q belongs to the Kato class. Through the h-transform technique, we build the connection between $p_q(t,x,y)$ and $p_h(t,x,y)$, where the latter denotes the Doob-type heat kernel for $\Delta u = 0$. The connection is mainly through the profile $h(\cdot)$, roughly speaking, the local weak solution for $(\Delta - q)u = 0$. To achieve that goal we have two main parts to cover:

On one hand, given an open set U in a locally compact separable metric space X and a potential term q belonging to the Kato class, we need clarify under what condition we can expect to have a positive local weak solution $h(\cdot)$. Then we can give the estimate of $p_q(t,x,y)$ through the h-transform.

On the other hand, under a more restricted assumption, can we characterize how h behaves? Under what conditions can a more precise estimate of $p_q(t,x,y)$ be achieved?

Previously there was a lot of work focusing on the heat kernel estimate for a variety of potentials. Q. S. Zhang studied in [59] the heat kernel estimate when the potential behaves like $q(x) \sim \frac{C}{1+|x|^b}$ for $C > 0$ and $0 < b < \infty$. The results there can be summarized as follows: The heat kernel $p_q(t,x,y)$ is bounded from above and below by the multiples of standard Gaussians with a weight function. If $b > 2$, the weight is bounded between two positive constants; if $b = 2$, the weight is bounded between two positive functions of $t, |x|$ and $|y|$, which have polynomial decay; if $b < 2$, the weight is bounded between two positive functions of $t, |x|$ and

$|y|$, which have exponential decay. Q. S. Zhang also studied the negative potential in [60] when $L = -\Delta + q$ is not nonnegative and negative eigenvalues exists. In this case the growth rate of $p_q(t, x, y)$ is comparable with the heat kernel of $-\Delta - c$ for a positive constant c.

A. Grigor′yan used the h-transform technique in [19] to study the heat kernel estimate $p_q(t, x, y)$ when $q(x)$ is positive smooth and Green-bounded. The conclusion is: Let (M, μ) be a complete non-compact non-parabolic manifold and $q(x)$ is non-negative smooth and Green-bounded (for example, compactly supported non-negative smooth function on a non-parabolic manifold is Green-bounded) on M. Then the heat kernel $p_q(t, x, y)$ satisfied the two-sided estimate

$$\frac{c_1}{V(x, \sqrt{t})}\exp\Big(-\frac{\rho(x, y)^2}{c_2 t}\Big) \le p_q(t, x, y) \le \frac{c_3}{V(x, \sqrt{t})}\exp\Big(-\frac{\rho(x, y)^2}{c_4 t}\Big).$$

In this manuscript we take the ideas in [19] and consider a broader class of potentials (K^∞) and obtain the same conclusion about the heat kernel estimate.

CHAPTER 3

THE PROFILE FOR THE SCHRÖDINGER OPERATOR

In Section 2.4 of Chapter 2, we discussed three different types of solutions of the Schrödinger operator. In this chapter we dig into the existence of Dirichlet-type positive weak solutions. We also call such a positive weak solution *the profile*. To be specific, we will talk about the profile of the Schrödinger operator $-\Delta + q$ for the potential class K which was introduced in last chapter. The Kato class is proved to be a general class yet hold a lot of good properties, one of which is the weak solution (profile) satisfies the boundary Harnack principle. We then use boundary Harnack principle and the limiting argument to prove the existence of the profile.

3.1 The profile in an unbounded uniform domain

Let's first introduce the notion of uniform domains.

Recall that, in any metric space X, the length of a continuous curve $\gamma : I = [a, b] \to X$ is given by

$$L(\gamma) = \sup\Big\{ \sum_{i=1}^{n} \rho(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N}, a \le t_0 < \cdots < t_n \le b \Big\}.$$

We have $L(\gamma) \ge \rho(\gamma(a), \gamma(b))$ in general. A metric space is a length space if $\rho(x, y)$ is equal to the infimum of the lengths of continuous curves joining x to y.

A length space is a geodesic length space if, for any pair x, y there exists a continuous curve $\gamma : I = [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$ and

$$\forall x, y \in I, \rho(\gamma(s), \gamma(t)) = |s-t|\rho(x,y).$$

Such a curve is called a minimal geodesic.

Definition 3.1.1. *Let U be an open connected subset of a length metric space (X, ρ). We say that U is uniform if there are constants $c_0, C_0 \in (0, \infty)$ such that, for any $x, y \in U$, there exists a continuous curve $\gamma_{x,y} : [0,1] \to U$ with $\gamma(0) = x, \gamma(1) = y$ and satisfying the following two properties:*

1. *The length $L(\gamma_{x,y})$ of $\gamma_{x,y}$ is at most $C_0\rho(x,y)$.*
2. *For any $z \in \gamma_{x,y}([0,1])$,*

$$\rho(z, \partial U) \geq c_0 \frac{\rho(z,x)\rho(z,y)}{\rho(x,y)}. \tag{3.1}$$

Note that $\max\{\rho(z,x), \rho(z,y)\} \geq \rho(x,y)/2$. Hence (3.1) is equivalent to

$$\rho(z, \partial U) \geq c_0' \min\{\rho(z,x), \rho(z,y)\}.$$

Condition (3.1) is called the banana-type (or cigar) condition.

Definition 3.1.2. *The profile of an unbounded uniform domain U in a local regular Dirichlet space X is any function h satisfying:*

(1). h is a local weak solution of the Schrödinger equation in U;

(2). $h \in \mathscr{F}^0_{q,\mathrm{loc}}(U)$;

(3). $h > 0$ in U.

For example, when $X = \mathbb{R}^+_n := \{x = (x_1, x_2, \ldots, x_n) : x_n > 0\}$ and $q \equiv 0$, then $h(x) = x_n$.

Condition (2) is essential to us and equivalent to say the profile h vanishes along the boundary of U in some sense. In the rest of this manuscript, when we say a weak solution u satisfies the Dirichlet boundary condition, it means $u \in \mathscr{F}^0_{q,\mathrm{loc}}(U)$.

The existence and properties of the profile h take a central central position in our line of reasoning in proving the two-sided heat kernel estimate with Dirichlet boundary condition. In the following three sections we will introduce a variety of properties (mostly based on [11]) of Kato class which include the key ingredients for the existence proof in Section 3.5.

3.2 Gauge theorem in bounded uniform domains

In last chapter we gave the definition of Kato class as follows:

$q \in K$ iff

$$\lim_{t\to 0^+} \sup_{x\in X} \int_X \left(\int_0^t p(s,x,y)ds \right) |q(y)| dy = 0. \tag{3.2}$$

Another equivalent definition (See [2]) is that

$q \in K$ iff

$$\lim_{t\to 0^+} \sup_{x\in X} E^x \left[\int_0^t |q(B_s)| ds \right] = 0, \tag{3.3}$$

where (X, P^x) is the diffusion with generator Δ.

When (X, ρ) is Harnack-type Dirichlet space, (3.2) is satisfied if $q \equiv 1$ by simple calculation, thus $L^\infty(X) \subset K$. The product of a function in K by a function in $L^\infty(X)$ is also in K.

We say that $q \in K_{\text{loc}}$ iff for any bounded domain U, $1_U q \in K$. Next we give some basic properties of Kato class:

Proposition 3.2.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space associated with the distance ρ. Assume q is the potential defined in X. If $q \in K$, then*

$$\sup_{x \in X} \int_{\rho(x,y) \leq 1} |q(y)| dy < \infty. \tag{3.4}$$

Consequently, if $q \in K_{\text{loc}}$, then $q \in L^1_{\text{loc}}(X)$.

To start with the gauge theory, we first claim that the term $\int_0^t q(X(s))ds$ is well-defined. For any $q \in K$ and bounded U, $G_U|q|$ is finite(See [5]). Here $G_U f$ is defined, for any measurable function f, as

$$G_U f(x) = \int_0^\infty P^D_{U,t} f(x) dt = E^x\Big[\int_0^{\tau_U} f(X_t) dt\Big], \tag{3.5}$$

and therefore so are $G_U q^+$ and $G_U q^-$. These potentials, in turn, have naturally associations with the well-defined additive functionals $\int q^+(X(s))ds$ and $\int q^-(X(s))ds$. We set

$$e_q(t) = \exp\Big\{ - \int_0^t q(X(s))ds \Big\}. \tag{3.6}$$

The gauge of (Δ, U, q) is then defined as

$$u_0(x) = E^x[e_q(\tau_U)], \quad x \in U. \tag{3.7}$$

The boundedness of $u_0(x)$, once it is not identically, ∞ is called the gauge theorem, which first appeared in [12]. We have the following theorem:

Theorem 3.2.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space associated with the distance ρ. Let U be a domain in (X, μ) with $\mu(U) < \infty$ and $q \in K$. If $e_q(\tau_U) \not\equiv \infty$, then $e_q(\tau_U)$ is bounded in U.*

We omit the proof here and refer the reader to Chapter 5 of [11]. Though the proof there is in the setting of Euclidean space, nothing changes if we replace the explicit definition of Kato class by the abstract one we give here. In Chapter 5 we will prove a similar theorem in unbounded domain U but for a slightly different potential class.

When $e_q(\tau_U)$ is finite, we call (U, q) is gaugeable. See Chapter 4 of [11] for more details about gaugeability.

3.3 q-Green functions

We define the q-Green function in a domain U as:

$$G_{q,U}(x,y) = \int_0^\infty p_{q,U}(t,x,y)dt, \quad (x,y) \in U \times U, \tag{3.8}$$

where $p_{q,U}(t,\cdot,\cdot)$ is the density for the stopped Feynman-Kac semigroup T_t which is defined as:

$$T_t f(x) = E^x[t < \tau_U; e_q(t) f(X_t)]. \tag{3.9}$$

Similar to (3.5) we can define the q-Green operator as follows, for measurable function f in U:

$$G_{q,U} f(x) = \int_0^\infty T_t f(x) dt = E^x\Big[\int_0^{\tau_U} e_q(t) f(X_t) dt\Big]$$

Here we introduce two inequalities for $p_{q,U}(\cdot;\cdot,\cdot)$, which generalize the $\mathbb{R}^n$ case in [48] to a more abstract space:

Lemma 3.3.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space associated with the distance ρ. Let U is a bounded uniform domain in (X, ρ) and $q \in K$. Assume (U, q) is gaugeable. Then there exist strictly positive constants t_0, α, β and C depending only on U and q such that*

$$p_{q,U}(t; x, y) \leq C\frac{1}{V(x, \sqrt{t})}\exp(-\rho(x, y)^2/4t) \quad \text{if} \quad 0 < t \leq t_0, \tag{3.10}$$

$$p_{q,U}(t; x, y) \leq C\exp(-\alpha t) \quad \text{if} \quad t > t_0. \tag{3.11}$$

Proof. Since $q \in K$ and (U, q) is gaugeable, by definition, there exists $t_0 > 0$ such that for any $0 < t \leq t_0$,

$$\sup_{x \in U} E^x[e_{4q}(t)] \leq 2.$$

Now let $f \in L^2(U)$ and set

$$S_t f(x) = E^x[t < \tau_U; e_{2q}(t) f(X_t)].$$

Then by Hölder's inequality, for $x \in U$ and $0 < t \leq t_0$,

$$\begin{aligned} |S_t f(x)|^2 &\leq E^x[t < \tau_U; e_{4q(t)}]E^x[t < \tau_U; f(X_t)^2] \\ &\leq 2V(x, \sqrt{t})^{-1} \|f\|_2^2 . \end{aligned}$$

The second inequality comes from taking the maximum of the heat kernel $p(t, x, y)$ in the Harnack-type Dirichlet space.

Then we have

$$\|S_t\|_{2,\infty} \leq V(x, \sqrt{t})^{-1/2},$$

and by a similar argument as in Theorem 3.10 in [11],

$$\|S_t\|_{1,\infty} \leq \left\|S_{t/2}\right\|_{1,2} \left\|S_{t/2}\right\|_{2,\infty} \leq \left\|S_{t/2}\right\|_{2,\infty}^2 \leq CV(x, \sqrt{(t)})^{-1}.$$

Let B be an open set in U. We have

$$\begin{aligned}(T_t 1_B(x))^2 &\leq E^x[t < \tau_U; e_{2q}(t) 1_B(X_t)] E^x[t < \tau_U; 1_B(X_t)] \\ &= [S_t 1_B(x)] E^x[t < \tau_U; 1_B(X_t)] \\ &\leq \|S_t\|_{1,\infty} \mu(B) V(x, \sqrt{t})^{-1} e^{-\rho^2/2t} \mu(B) \\ &\leq CV(x, \sqrt{t})^{-2} e^{-\rho^2/2t} \mu(B)^2,\end{aligned}$$

where $\rho = \rho(x, B)$.

Hence by setting $f = 1_B$ in (3.9), we find

$$\int_B p_q(t, x, y) dy \leq CV(x, \sqrt{t})^{-1} e^{-\rho^2/4t} \mu(B).$$

For $x, y \in U$, taking $B = B(y, \delta)$ and letting $\delta \to 0^+$, we obtain (3.10).

Next, since (U, q) is gaugeable, by Theorem 3.17 in [11], there exists C and $\alpha > 0$ such that

$$\|T_t\|_1 \leq Ce^{-\alpha t}, t > 0.$$

Hence for all $t > t_0$ it follows from the semigroup properties of T_t that

$$\begin{aligned}p_q(t, x, y) &\leq \|T_t\|_{1,\infty} \\ &\leq \left\|T_{t-t_0}\right\|_1 \left\|T_{t_0}\right\|_{1,\infty} \\ &\leq \left\|T_{t-t_0}\right\| Ce^{-\alpha(t-t_0)}.\end{aligned}$$

This reduces to (3.11) with a different constant C. □

Based on this lemma, we prove the following important theorem:

Theorem 3.3.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space associated with the distance ρ. Let U is a bounded uniform domain in (X, ρ) and $q \in K$. Assume (U, q) is gaugeable. Then the q-Green function $G_{q,U}(\cdot, \cdot)$ has the following properties.*

(1). $G_{q,U}(\cdot, \cdot)$ is positive, finite, symmetric and continuous in $(x, y) \in U \times U, x \neq y$.

(2). For any fixed $x \in U$ and any relatively compact open set V in U, $G_{q,U}$ is in $\mathscr{F}^0_{q,\mathrm{loc}}(V, V \setminus \{x\})$.

(3). There exists $C > 0$ such that

$$G_{q,U}(x, y) \leq Cg(x, y), \quad (x, y) \in U \times U.$$

where $g(x, y) = G(x, y)$ if $G(x, y)$ is finite and $g(x, y) = \int_0^1 p(t, x, y)dt$ if $G(x, y)$ is infinite.

Proof. (1). The positivity is obvious. The boundedness follows from the definition of $G_{q,U}$ and Lemma 3.3.1. For each t, $p_{q,U}(t; \cdot, \cdot)$ is symmetric and continuous by the properties of heat semigroup T_t (For details, see Theorem 3.17 in [11] for the case in $\mathbb{R}_n$). Thus the symmetry of $G_{q,U}$ is obvious and the continuity comes from Lemma 3.3.1 and the dominated convergence theorem.

(2). This part follows by a similar argument as in Lemma 4.7 of [27].

(3). This part is a direct result of Lemma 3.3.1, by integrating the estimate there. □

The following theorem builds a connection between the q-Green function and the classical Green function induced by Δ.

Theorem 3.3.3. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space associated with the distance ρ. Let U is a bounded uniform domain in (X, ρ) and $q \in K$. Assume (U, q) is gaugeable. Then for all $(x, y) \in U \times U, x \neq y$, we have*

$$G_{q,U}(x, y) = G_U(x, y) - \int_U G_{q,U}(x, u)q(u)G_U(u, y)du, \tag{3.12}$$

and

$$G_{q,U}(x, y) = G_U(x, y) - \int_U G_U(x, u)q(u)G_{q,U}(u, y)du. \tag{3.13}$$

Proof. Since $x \neq y$, we can assume $\rho(x, y) > \delta > 0$, then for any $u \in U$, either $\rho(x, u) > \frac{\delta}{2}$ or $\rho(u, y) > \frac{\delta}{2}$. Hence using Theorem 3.3.2(c) we have

$$G_{q,U}(x, u)|q(u)|G_U(u, y) \leq C|q(u)|(g(x, u) + g(u, y)). \tag{3.14}$$

Since U is bounded and $q \in K$, by the definition of Kato class we have that the set of functions,

$$G_{q,U}(x, \cdot)q(\cdot)G_U(\cdot, y) : (x, y) \in U \times U, \rho(x, y) > \delta,$$

is uniformly integrable over U. On the other hand, for each $u \in U$, the function

$$(x, y) \to G_{q,U}(x, u)q(u)G_U(u, y)$$

is continuous except possibly at $x = u$ or $y = u$. Therefore the integral on the right-hand side of (3.12) is continuous in $(x, y) \in U \times U, \rho(x, y) > \delta$. Taking δ arbitrarily small, we know that both members of (3.12) are continuous in $(x, y) \in U \times U, x \neq y$. Now for any non-negative measurable function f in U, by the Markov property and Fubini's theorem, we have:

$$
\begin{aligned}
&G_{q,U}(qG_U f)(x)\\
=&E^x\Big\{\int_0^{\tau_U} e_q(t)q(X_t)E^{X_t}\Big[\int_0^{\tau_U} f(X_s)ds\Big]dt\Big\}\\
=&E^x\Big\{\int_0^{\tau_U} f(X_s)\Big[\int_0^{s} e_q(t)q(X_t)dt\Big]ds\Big\}\\
=&E^x\Big\{\int_0^{\tau_U} f(X_s)(1-e_q(s))ds\Big\}\\
=&G_U f(x) - G_{q,U} f(x),
\end{aligned}
$$

Here we used the Green operator and q-Green operator to prove the equation. Taking $f \equiv 1$ we get the equation (3.12).

By the same argument, we obtain the other equation. □

Finally we have

Theorem 3.3.4. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space associated with the distance ρ. Let U is a bounded uniform domain in (X, ρ) and $q \in K$. Assume (U, q) is gaugeable. Then for any fixed $x \in U$, the function $G_{q,U}(x, \cdot)$ is a non-negative weak solution of the Schrödinger equation*

$$Lu = 0$$

on $U \setminus \{x\}$.

Proof. Let $U_x = U \setminus \{x\}$. For any $\phi \in C_0^\infty(U_x)$, let S_ϕ be the support of ϕ and $\delta = \rho(x, S_\phi) > 0$. It follows from Theorem 3.3.2 (3) and (3.12) that $G_{q,U}(x, \cdot)$ and

$\int_U G_{q,U}(x,u)q(u)G_U(u,\cdot)du$ are bounded on S_ϕ. Hence we have by Theorem 3.3.3,

$$\begin{aligned}
&\int_{U_x} G_{q,U}(x,y)\Delta\phi(y)(\mathrm{d})\mathrm{y} \\
&= \int_U G_U(x,y)\Delta\phi(y)(\mathrm{d})\mathrm{y} - \int_U \Big[\int_U G_{q,U}(x,u)q(u)G_U(u,y)du\Big]\Delta\phi(y)(\mathrm{d})\mathrm{y} \\
&= \phi(x) - \int_U G_{q,U}(x,u)q(u)\Big[\int_U G_U(u,y)\Delta\phi(y)dy\Big]du \\
&= \int_{U_x} G_{q,U}(x,u)q(u)\phi(u)du,
\end{aligned}$$

since $\phi(x) = 0$. By definition this means that

$$LG_{q,U}(x,\cdot) = 0 \quad \text{on} \quad \mathrm{U_x}.$$

□

3.4 Boundary Harnack principle

To prove the existence of profile, another powerful technique we take advantage of is the boundary Harnack principle.

Theorem 3.4.1. *Let $(X,\mu,\mathscr{E},\mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space associated with the distance ρ. Let U be a uniform domain in (X,ρ). Let $L = -\Delta + q$ and $q \in K$. If (U,q) is gaugeable, then there exists a positive constant c and r_0, such that for any $0 < r < r_0$, $z \in \partial U$ and any positive local weak solution u, v of $Lu = 0$ in $U \cap B(z,2r)$ with weak Dirichlet boundary condition along ∂U, we have*

$$\frac{u(x)}{v(x)} \leq c\frac{u(y)}{v(y)}, \quad x,y \in B(z,r) \cap U.$$

Proof. Assume $U_r = B(z, 3r/2) \cap U$. Write $w_r^x, w_{q,r}^x$ for $\Delta-$harmonic measure on U_r and q-harmonic measure on U_r, respectively. Then $w_r^x \simeq w_{q,r}^x$ (See Theorem 7.22 in [11] for details in Lipschitz domains in $\mathbb{R}^n$. H. Aikawa resolved the issue of possible irregularity in [1] for uniform domains.), and the constants in this equivalence can be taken as the same constants for the analogous statement on U. If u and v are as in the theorem let $\hat{u}$ and $\hat{v}$ be the Δ-harmonic functions on U_r whose boundary values are u and v respectively. Then $u \simeq \hat{u}$ and $v \simeq \hat{v}$ on U_r. Since H. Aikawa proved in [1] the following boundary Harnack principle

$$\frac{\hat{u}(x)}{\hat{v}(x)} \leq c\frac{\hat{u}(y)}{\hat{v}(y)} \quad \forall x, y \in B(z, r) \cap U, \tag{3.15}$$

in uniform domains, it follows that

$$\frac{u(x)}{v(x)} \leq c\frac{u(y)}{v(y)} \quad \forall x, y \in B(z, r) \cap U.$$

□

Remark 3.4.1. *The boundary Harnack principle lies at the heart of many problems when Dirichlet boundary conditions are involved. Ancona([3], 1978) and Wu([58], 1978) proved it in bounded Lipschitz domains in $\mathbb{R}^n$, and then Jerison and Kenig([31], 1982) proved it in bounded NTA domains. Thanks to the work of H. Aikawa in [1], we are able to extend the previous work to any uniform domain in a nice abstract metric space.*

The boundary Harnack principle says the local weak solution of $L = -\Delta + q$ vanishing on a portion of ∂U vanishes at the same rate on a subportion. Considering the Feynman-Kac formula $u(x) = E^x[e_q(\tau_U) f(X(\tau_{U_-}))]$ for solutions $Lu = 0$ and $u = f$ on ∂U and the stochastic representation for solutions to $\Delta v = 0, v = f$ on ∂U,

$v(x) = E^x[f(X(\tau_{U^-}))]$, one might expect that $e_q(\tau_U) \to 1$ quasi-everywhere(q.e.) as $x \to \partial U$ so that u and v should vanish at the same rate at ∂U. This is in fact the case.

3.5 Existence of the profile

In this section we will prove, given the hypothesis $q \in K$, the existence of the profile in an unbounded uniform domain U in a non-compact Harnack-type Dirichlet space, namely, a function $h = h_{U,q}$ satisfying the following three properties:

1. h is a local weak solution of the Schrödinger equation $-\Delta + q = 0$ in U;
2. $h \in \mathscr{F}^0_{q,\mathrm{loc}}(U)$;
3. $h > 0$ in U.

We follow the same line of reasoning in [27]: Using the good property of q-Green functions and the limiting argument to step from bounded domain to unbounded domain.

Generally, it is impossible to find an explicit formula for the profile except for some classic domain U and a very special potential term q. Here we prove through a limiting process involving the ratio of the q-Green functions, which are weak solutions by Theorem 3.3.4. For any unbounded uniform domain U, let x_0 be a fixed point in U away from the boundary and a sequence of balls $B_i = U \cap B(x_0, r_i)$ with r_i increasing to ∞. For each i, let x_i be any point such that $\rho(x_i, x_0) = r_i/2$. This is always possible since U is unbounded. Next we take advantage of the q-Green function $G_{q,B_i}(x, y)$ in B_i and define

$$h_i(x) = \begin{cases} \dfrac{G_{q,B_i}(x_i, x)}{G_{q,B_i}(x_i, x_0)}, & \text{when } x \in B_i \\ 0 \quad , & \text{when } x \in U \setminus B_i \end{cases} \tag{3.16}$$

By this construction, we obtain $h_i \geq 0$ on U, and is a local weak solution of $Lu = 0$ in $B_i \setminus \{x_i\}$. In addition, h_i belongs to $\mathscr{F}^0_{q,\mathrm{loc}}(B_i, B_i \setminus \{x_i\})$. Also notice that $h_i(x_0) = 1$, since we assume the underlying is a Harnack-type Dirichlet space, any family $\{u_i\}$ of local weak solutions in a domain V such that u_i is uniformly bounded at one point x_0 is equicontinuous on compact subsets of V.

Applying this reasoning to $\{h_i\}$ on $U \cap B(x_0, x_i/3)$, we see there exists a subsequence of sequence $\{h_i\}$ defined by (3.16) that converges uniformly in any compact subset of U. For simplicity, we still represent this subsequence by $\{h_i\}$ and set

$$\forall x \in U, \quad h(x) = \lim_{i \to \infty} h_i(x). \tag{3.17}$$

Now let us show that h_i converges to h locally in $\mathscr{F}_q(U)$, and that h is indeed a local weak solution of $Lu = 0$ in U. Firstly it is clear that h_i converges to h locally in $L^2(U)$. By the form of Leibniz rule that holds for the energy measure Γ, for any function $\phi \in \mathscr{C}_c(U) \cap \mathscr{D}(\mathscr{E}_q(U))$ and i, j large enough, we have

$$\mathscr{E}_q(\phi(h_i - h_j), \phi(h_i - h_j)) = \int_U (h_i - h_j)^2 d\Gamma(\phi, \phi) + \mathscr{E}_q(h_i - h_j, \phi^2(h_i - h_j)).$$

The last term on the right-hand side is 0 because $\phi^2(h_i - h_j)$ is in $\mathscr{C}_c(U) \cap \mathscr{D}(\mathscr{E}_q(U))$ and $h_i - h_j$ is a local weak solution of $Lu = 0$ in an open subset of U containing the compact support of ϕ. Since h_i converges to h locally uniformly in U, this proves convergence locally in $\mathscr{F}_q(U)$. Then it follows easily that h is a local weak solution of $Lu = 0$. This is true, since for any $\phi \in \mathscr{C}_c(U) \cap \mathscr{D}(\mathscr{E}_q(U))$,

$$\mathscr{E}_q(h, \phi) = \mathscr{E}_q(\lim_{i\to\infty} h_i, \phi) = \lim_{i\to\infty} \mathscr{E}_q(h_i, \phi) = 0.$$

By the Harnack inequality, it follows that $h > 0$ in U.

Till now, we have proved properties 1 and 3. Next we prove the crucial part, that is $h \in \mathscr{F}^0_{q,\mathrm{loc}}(U)$.

Theorem 3.5.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space. Let U be an unbounded uniform domain in (X, ρ) and $q \in K_{\mathrm{loc}}$. We also assume (D, q) is gaugeable in any bounded domain D. Then any function h obtained as in (3.17) is in $\mathscr{F}^0_{q,\mathrm{loc}}(U)$ and thus is a profile for U.*

Proof. The first step is to show that the sequence $\{h_i\}$ defined by (3.16) is in fact Cauchy in $L^2_{\mathrm{loc}}(U, d\mu)$. Let F be a compact set in X and $V = F \cap U$. It suffices to show that $\{h_i\}$ is Cauchy in $L^2(V, \mu|_V)$. Let Ω be a relatively compact open set in X containing F, $B = B(x_0, R)$ be a large ball in X such that $\Omega \subset B(\xi_0, R/2)$ and ξ_0' be a point in $[B(\xi_0, 2R) \setminus \Omega] \cap U$. Let $g(x) = G_{q,U\cap B}(\xi_0, x)$ be the corresponding q-Green function. Note that g is continuous and positive in $\Omega \cap U$, belongs to $\mathscr{F}^0_{q,\mathrm{loc}}(U, \Omega \cap U)$, and is a local weak solution of $Lu = 0$ in $\Omega \cap U$. Applying the Harnack boundary principle to Ω, F and any of the pairs h_i, g with i large enough yields

$$\sup_{U\cap F}\{h_i/g\} \leq C(\Omega, F).$$

For any $\eta \in (0,1)$ small enough, let $V_\eta = \{x \in V : \rho(x, X \setminus U) \geq \eta\} \subset U$ and note that V_η is a compact subset of U. For i, j large enough, we have

$$\int_V |h_i - h_j|^2 d\mu \leq \int_{V_\eta} |h_i - h_j|^2 d\mu + 2C(\Omega, F) \int_{V \setminus V_\eta} g^2 d\mu. \tag{3.18}$$

As $\int_{V \setminus V_\eta} g^2 d\mu$ tends to 0 with η and (h_i) converges to h in $L^2_{\text{loc}}(U, \mu)$, this indeed shows that (h_i) is Cauchy in $L^2(V, \mu)$.

The second and last step is to show that for any open subset V of U which is relatively compact in X, $\{h_i\}$ is a Cauchy sequence in $\mathscr{F}_q(V)$. Since, for i large enough, $h_i \in \mathscr{F}^0_q(U, V)$, this implies that $h \in \mathscr{F}^0_q(U, V)$ and thus $h \in \mathscr{F}^0_{q,\text{loc}}(U)$ as desired. To this end, let $\phi(x) = (1 - \rho(x, V))_+ = \max\{1 - \rho(x, V), 0\}$. By Theorem 2.11 in [27], this function is in $\mathscr{F}_{q,c}(X)$ with $d\Gamma(\phi, \phi) \leq d\mu$ a.e. and $d\Gamma(\phi, \phi)|_V = 0$. By construction, $h_i \in \mathscr{F}^0_{q,\text{loc}}(B_i, B_i \setminus \{x_i\})$ where $B_i = U \cap B(x_0, r_i)$ and $x_i \in U$ is such that $\rho_U(x_i, x_0) = r_i/2$. Hence, for i large enough, $\phi h_i \in \mathscr{D}(\mathscr{E}^D_{q,B_i}) \subset \mathscr{F}^0_q(U)$. To show that $\{h_i\}$ is Cauchy in $\mathscr{F}_q(V)$, it suffices to show that ϕh_i is Cauchy in $\mathscr{F}^0_q(U)$. We have

$$\mathscr{E}_q(\phi(h_i - h_j), \phi(h_i, h_j)) = \int_U (h_i - h_j)^2 d\Gamma(\phi, \phi) + \mathscr{E}_q(h_i - h_j, \phi^2(h_i - h_j)).$$

We claim that the last term on the right-hand side is 0. Indeed, $\phi^2(h_i - h_j)$ is in $\mathscr{F}^0_q(U)$ and can be approximated by functions $\psi_n \in \mathscr{F}_{q,c}(U)$ with compact supports all contained in an open set $\Omega \subset U$ with $\Omega \subset B_i \setminus \{x_i\}$ for all i large enough. As $h_i - h_j$ is a local weak solution of $Lu = 0$ in Ω,

$$\mathscr{E}_q(h_i - h_j, \phi^2(h_i - h_j)) = \mathscr{E}_q(h_i - h_j, \lim_{n\to\infty} \psi_n) = \lim_{n\to\infty} \mathscr{E}_q(h_i - h_j, \psi_n) = 0.$$

Hence, setting $F = \{x \in U : \rho(x, V) \leq 1\}$, we have

$$\begin{aligned}\int_V d\Gamma(h_i - h_j, h_i - h_j) &\leq \mathscr{E}_q(\phi(h_i - h_j), \phi(h_i - h_j)) \\ &= \int_U (h_i - h_j)^2 d\Gamma(\phi, \phi) \\ &\leq \int_F (h_i - h_j)^2 d\mu.\end{aligned}$$

Since $\{h_i\}$ is Cauchy in $L^2_{\text{loc}}(U)$, this shows that it is also Cauchy in $\mathscr{F}_q(V)$. □

CHAPTER 4

THE HEAT KERNEL ESTIMATE OF THE OPERATOR $-\Delta + q$ IN UNBOUNDED UNIFORM DOMAINS

In this chapter we will present the first main result of this manuscript: to give the sharp two-sided Gaussian type estimates of the Dirichlet (Neumann) heat kernel for Schrödinger operator in an unbounded uniform domain through h-transform technique. This method was first used by A. Grigor'yan in [19] for some non-negative Green-bounded potentials. We utilize the idea there and get the generalized result in this chapter. Besides, we apply the method here to the K^{∞} potentials in next chapter to get more precise results.

4.1 The h-transform technique

To start with, let us clarify some notations here. We will use both the Dirichlet forms $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ and $(\mathscr{E}_q, \mathscr{D}(\mathscr{E}_q))$ in what follows, where the latter represents the Dirichlet form for the operator $L = -\Delta + q$. We will firstly give some related definitions based on $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. The same idea also applies to the Schrödinger's counterpart, but we omit it to make things concise. Secondly, the connection between these two will be illuminated. Along with the discussion, we always assume the potential term $q \in K$ and (U, q) is gaugeable for any bounded domain.

Before we introduce the famous h-transform technique, let us first give the following definition:

Definition 4.1.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a strictly local regular Dirichlet space satisfying $(A1)$ and $(A2)$ and U be an unbounded uniform domain in X. Let v be a positive continuous function defined on U. Define*

$$(\mathscr{E}_U^{D,v}, \mathscr{D}(\mathscr{E}_U^{D,v}))$$

to be the closure of the symmetric closable bilinear form

$$(f, g) \mapsto \int_U v d\Gamma(f, g), \quad f, g \in \mathscr{F}_c(U) \subset L^2(U, v d\mu).$$

The next proposition gathers properties of the form $(\mathscr{E}_U^{D,v}, \mathscr{D}(\mathscr{E}_U^{D,v}))$, which easily follow by inspection.

Proposition 4.1.1. *The form $(\mathscr{E}_U^{D,v}, \mathscr{D}(\mathscr{E}_U^{D,v}))$ defined by Definition 4.1.1 is a strictly local regular Dirichlet form on $L^2(U, v d\mu)$ with energy measure*

$$d\Gamma^v(f, g) = v d\Gamma(f, g), \quad f, g \in \mathscr{F}_{\mathrm{loc}}(U).$$

Now let us describe the basic ingredients of the well-known h-transform originally due to Doob. This technique is a key ingredient for our main results and we describe it in detail. We start with the following simple definition.

Definition 4.1.2. *Let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Dirichlet form on $L^2(X, \mu)$ with associated semigroup $(P_t)_{t>0}$ and infinitesimal generator $(A, \mathscr{D}(A))$. Let h be a measurable positive function on X. Let H denote multiplication by h viewed as a unitary operator*

$$H : L^2(U, h^2 d\mu) \to L^2(U, d\mu), \quad f \mapsto hf.$$

Define $(\mathscr{E}_h, \mathscr{D}(\mathscr{E}_h)), A_h$ *and* $P_{h,t}$ *to be, respectively, the pulled back closed bilinear form, operator and semigroup on* $L^2(X, h^2 d\mu)$ *defined by*

$$\mathscr{E}_h(f, g) = \mathscr{E}(hf, hg), \quad \mathscr{D}(\mathscr{E}_h) = H^{-1}\mathscr{D}(\mathscr{E}); \tag{4.1}$$

$$A_h = H^{-1} \circ A \circ H, \quad \mathscr{D}(A_h) = H^{-1}\mathscr{D}(A); \tag{4.2}$$

$$P_{h,t} = H^{-1} \circ P_t \circ H. \tag{4.3}$$

From this definition it immediately follows that $\mathscr{E}_h$ is a densely defined closed symmetric bilinear form on $L^2(X, h^2 d\mu)$ associated with the self-adjoint semigroup of contractions $P_{h,t}$ on $L^2(X, h^2 d\mu)$, which admits A_h as its (self-adjoint) infinitesimal generator.

Next we present a simple yet useful lemma connecting $p_h(t, x, y)$ with $p(t, x, y)$.

Lemma 4.1.1. *Referring to* Definition 4.1.2, if the semigroup $(P_t)_{t>0}$ admits a kernel $p(t, x, y)$, $(t, x, y) \in (0, \infty) \times X \times X$, with respect to the measure $d\mu$ then the semigroup $(P_{h,t})_{t>0}$ admits a kernel $p_h(t, x, y)$, $(t, x, y) \in (0, \infty) \times X \times X$, with respect to the measure $h^2 d\mu$ and these two kernels are related by

$$p(t, x, y) = p_h(t, x, y)h(x)h(y), \quad (t, x, y) \in (0, \infty) \times X \times X.$$

Proof. By definition, for $f \in L^2(X, h^2 d\mu)$, we have

$$\begin{aligned} P_{h,t}f(x) &= \frac{1}{h(x)} P_t[hf](x) \\ &= \frac{1}{h(x)} \int_X p(t, x, y)h(y)f(y)d\mu(y) \\ &= \int_X \frac{p(t, x, y)}{h(x)h(y)} f(y)h^2(y)d\mu(y) \end{aligned}$$

Hence the semigroup $P_{h,t}$ admits the kernel

$$p_h(t,x,y) = \frac{p(t,x,y)}{h(x)h(y)}$$

with respect to the measure $h^2 d\mu$. □

Till now, we have constructed two very different closed bilinear forms on $L^2(U, h^2 d\mu)$:

- The form $(\mathscr{E}_U^{D,h^2}, \mathscr{D}(\mathscr{E}_U^{D,h^2}))$ obtained by setting $v = h^2$ in Definition 4.1.1.

- The form $(\mathscr{E}_{U,h}^D, \mathscr{D}(\mathscr{E}_{U,h}^D))$ with $\mathscr{D}(\mathscr{E}_{U,h}^D) = H^{-1}\mathscr{D}(\mathscr{E}_U^D)$ obtained by h-transform by setting $(\mathscr{E}, \mathscr{D}(\mathscr{E})) = (\mathscr{E}_U^D, \mathscr{D}(\mathscr{E}_U^D))$ in Definition 4.1.2.

Under some special circumstances, these two forms are actually equal. The following proposition spells out cases where that is true.

Proposition 4.1.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a strictly local regular Dirichlet space satisfying the conditions (A1) and (A2). Let U be a domain in X. Let h be a continuous positive function on U. Referring to the notation introduced above, we have:*

- Assume that $h \in \mathscr{F}_{\mathrm{loc}}(U)$. Then the set $H^{-1}(\mathscr{F}_c(U) \cap L^\infty(U,\mu))$ is dense in the Hilbert space $\mathscr{D}(\mathscr{E}_{U,h}^D) = H^{-1}\mathscr{D}(\mathscr{E}_U^D)$ and, in fact,

$$H^{-1}(\mathscr{F}_c(U) \cap L^\infty(U,\mu)) = \mathscr{F}_c(U) \cap L^\infty(U,\mu) = \mathscr{F}_c(U) \cap L^\infty(U, h^2 d\mu).$$

- Assume that $h \in \mathscr{F}_{q,\mathrm{loc}}(U)$ and is a local weak solution of $(-\Delta + q)u = 0$ in U. Then the forms $(\mathscr{E}_U^{D,h^2}, \mathscr{D}(\mathscr{E}_U^{D,h^2}))$ and $(\mathscr{E}_{q,U,h}^D, \mathscr{D}(\mathscr{E}_{q,U,h}^D))$ coincide.

Proof. For the first part, see the proof in Chapter 5 in [27].

To prove the second statement, we observe that, according to what we just proved above, it suffices to compare the two forms on the common dense subset $\mathscr{F}_{q,c}(U)\cap L^\infty(U,\mu)$ of their respective domains. As $h\in\mathscr{F}_{q,\text{loc}}(U)\cap L^\infty_{\text{loc}}(U)$, for any $g\in\mathscr{F}_{q,c}(U)\cap L^\infty(U,\mu)$, the functions g, g^2, gh, g^2h are all in $\mathscr{F}_{q,c}(U)$. Using the properties of the energy form of a strictly local Dirichlet form, i.e., the product rule and the chain rule, we have

$$\begin{aligned}
\mathscr{E}^D_{q,U,h}(f,g) &= \int_U d\Gamma(hf,hg) + \int_U qfgh^2 d\mu \\
&= \int fg d\Gamma(h,h) + \int gh d\Gamma(f,h) + \int fh d\Gamma(g,h) + \int h^2 d\Gamma(f,g) + \int qfgh^2 d\mu \\
&= \int d\Gamma(h,fgh) + \int qfgh^2 d\mu + \int h^2 d\Gamma(f,g) \\
&= \int h^2 d\Gamma(f,g) \\
&= \mathscr{E}^{D,h^2}_U(f,g).
\end{aligned}$$

To obtain the last line, we have used the fact that $\int d\Gamma(h,fgh)+\int qfgh^2 d\mu = 0$ since h is a local weak solution and $fgh\in\mathscr{F}_{q,c}(U)$. □

4.2 The Neumann-type heat kernel estimate without potential term

Before proceeding to the main result involving with Dirichlet-type heat kernel estimate for the operator $-\Delta+q$, we first need to briefly introduce the Neumann-type heat kernel estimate corresponding $\Delta u = 0$.

Like Dirichlet-type case, we can define the Neumann-type Dirichlet form $(\mathscr{E}_U^N, \mathscr{D}(\mathscr{E}_U^N))$, and the following weighted version:

Definition 4.2.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a regular strictly local Dirichlet space with energy measure Γ and intrinsic distance ρ satisfying the conditions $(A1)$ and $(A2)$. Assume that the form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ admits a carré du champ $\Upsilon : \mathscr{D}(\mathscr{E}) \times \mathscr{D}(\mathscr{E}) \to L^1(X, \mu)$. Let $U \subset X$ be an open set and let $v \in L^\infty_{\mathrm{loc}}(U, \mu)$ be a locally uniformly positive and locally bounded measurable function on U. Set*

$$\mathscr{E}_U^{N,v}(f,g) = \int_U v d\Gamma(f,g) = \int_U \Upsilon(f,g) v d\mu, \quad f, g \in \mathscr{D}(\mathscr{E}_U^{N,v}) \tag{4.4}$$

where

$$\mathscr{D}(\mathscr{E}_U^{N,v}) = \mathscr{F}^v(U) = \left\{ f \in \mathscr{F}_{\mathrm{loc}}(U) \cap L^2(U, v d\mu) : \int_U \Upsilon(f,f) v d\mu < \infty \right\}$$

Note that if we take $v \equiv 1$, then the form defined above is exactly $(\mathscr{E}_U^N, \mathscr{D}(\mathscr{E}_U^N))$. Next we will present some useful theorems and omit the proofs (See Chapter 3 in [27]).

Theorem 4.2.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space with energy measure Γ that admits a carré du champ Υ. Let ρ be its intrinsic distance and U be an unbounded uniform domain. Let v be a measurable function which is locally uniformly bounded and positive in U. Assume that the measure $v d\mu$ on U satisfies the volume doubling condition (2.1). Assume also that there exist positive constants C and N such that the function v satisfies*

$$\sup_B v \leq C \inf_B v, \tag{4.5}$$

on any ball $B = B_U(x,r)$ with $\rho(B, \partial U) > Nr$. Then there exists a constant P_1 such that, for any ball $B = B_U(x,r)$ in (U,ρ), we have

$$\forall f \in \mathscr{F}^v(B), \inf_{\xi \in \mathbb{R}} \int_B |f - \xi|^2 v d\mu \leq P_1 r^2 \int_B v d\Gamma(f,f). \tag{4.6}$$

That is, the Poincaré inequality holds true for the form $(\mathscr{E}_U^{N,v}, \mathscr{D}(\mathscr{E}_U^{N,v}))$ with reference measure $v d\mu$ on U.

Now we present the main theorem about the heat kernel estimates for the Neumann-type semigroup.

Theorem 4.2.2. *Let U be an unbounded uniform domain in a Harnack-type Dirichlet space $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$. Assume that $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ admits a carré du champ. Let v be a measurable locally uniformly positive and locally uniformly bounded function on U. Assume that the measure $v d\mu$ on U satisfies the volume doubling condition (2.1). Assume further that there exist positive constants C and N such that*

$$\sup_B v \leq C \inf_B v$$

on any ball $B = B(x,r)$ with $\rho(B, X \setminus U) > Nr$. Then the Dirichlet space $(U, v d\mu, \mathscr{E}_U^{N,v}, \mathscr{F}^v(U))$ is of Harnack-type. In particular, the associated Neumann-type semigroup admits a continuous kernel $p_U^{N,v}(t,x,y)$ which satisfies

$$\frac{c_1 \exp(-\frac{\rho(x,y)^2}{c_2 t})}{\sqrt{V_v(x, \sqrt{t}) V_v(y, \sqrt{t})}} \leq p_U^{N,v}(t,x,y) \leq \frac{c_3 \exp(-\frac{\rho(x,y)^2}{c_4 t})}{\sqrt{V_v(x, \sqrt{t}) V_v(y, \sqrt{t})}} \tag{4.7}$$

for all $x, y \in U$ and all $t > 0$. Here V_v denote the weighted volume $V_v(x,r) = \int_{B_U(x,r)} v d\mu$.

The reason why we introduce this section is the following theorem which connects two different Dirichlet spaces:

Theorem 4.2.3. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space which admits a carré du champ. Let U be an unbounded uniform domain in (X, ρ), $q \in K$ with (U, q) gaugeable. Let h be the profile for (U, q) provided by Section 2.5 that satisfies the volume doubling condition (2.1). Then the Dirichlet forms*

$$(\mathscr{E}_U^{N,h^2}, \mathscr{D}(\mathscr{E}_U^{N,h^2})) \quad \text{and} \quad (\mathscr{E}_U^{D,h^2}, \mathscr{D}(\mathscr{E}_U^{D,h^2}))$$

coincide and are regular on $\overline{U}$ with core $\mathrm{Lip_c}(\overline{\mathrm{U}})$. Moreover, these forms also coincide with $(\mathscr{E}^D_{q,U,h}, \mathscr{D}(\mathscr{E}^D_{q,U,h}))$, and the Dirichlet space

$$(\overline{U}, h^2 d\mu, \mathscr{E}_U^{D,h^2}, \mathscr{D}(\mathscr{E}_U^{D,h^2})) = (\overline{U}, h^2 d\mu, \mathscr{E}^D_{q,U,h}, \mathscr{D}(\mathscr{E}^D_{q,U,h}))$$

is a Harnack-type Dirichlet space.

For the proof, see Chapter 5 in [27].

4.3 Dirichlet heat kernel estimate for the operator $-\Delta + q$

Finally we have

Theorem 4.3.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space which admits a carré du champ. Let U be an unbounded uniform domain in (X, ρ). Assume that q belongs* *to Kato class K and suppose q is gaugeable for any bounded domain in X. Let h be the profile for (U, q) that satisfies the condition (4.5)*

$$\sup_B v \le C \inf_B v,$$

on any ball $B = B_U(x, r)$ with $\rho(B, \partial U) > Nr$. Assume that the measure $h^2 d\mu$ satisfies the volume doubling condition

$$V_{h^2}(x, 2r) \le C V_{h^2}(x, r). \tag{4.8}$$

Let $P^D_{q,U,h,t}, t > 0$ be the semigroup associated with the Harnack-type Dirichlet form

$$(\mathscr{E}^D_{q,U,h}, \mathscr{D}(\mathscr{E}^D_{q,U,h})) = (\mathscr{E}^{D,h^2}_U, \mathscr{D}(\mathscr{E}^{D,h^2}_U)) = (\mathscr{E}^{N,h^2}_U, \mathscr{D}(\mathscr{E}^{N,h^2}_U))$$

on $(\overline{U}, h^2 d\mu)$. Then the induced Dirichlet-type heat kernel $p^D_{q,U}(t, x, y), (t, x, y) \in (0, \infty) \times \overline{U} \times \overline{U}$ satisfies

$$\frac{c_1 h(x) h(y) \exp(-\frac{\rho(x,y)^2}{c_2 t})}{\sqrt{V_{h^2}(x, \sqrt{t}) V_{h^2}(y, \sqrt{t})}} \le p^D_{q,U}(t, x, y) \le \frac{c_3 h(x) h(y) \exp(-\frac{\rho(x,y)^2}{c_4 t})}{\sqrt{V_{h^2}(x, \sqrt{t}) V_{h^2}(y, \sqrt{t})}} \tag{4.9}$$

Proof. This is a direct result when we combine Theorem 2.2.2, Lemma 4.1.1, Proposition 4.1.2, Theorem 4.2.2 and Theorem 4.2.3. □

Remark 4.3.1. *Theorem 4.3.1 gives, under some assumptions of q, the heat kernel estimate for the Schrödinger operator $L = -\Delta + q$. However, we have no idea of the behavior of h if we only know the existence of the profile. This is different from the case for Δ. That is why we assume additionally the condition (4.5) of h and volume doubling condition for $h^2 d\mu$. Some improvements of this problem will be the content of next chapter.*

To end this chapter, we will give one example to illustrate how to apply Theorem 4.3.1 to heat kernel estimate.

Assume $X = \mathbb{R}^n (n \geq 2)$, μ is the Lebesgue measure and $U = \mathbb{R}^n \setminus \overline{B(0,1)}$. Let $q(x) = \frac{\beta}{|x|^2}$ in U. Apparently, $q \in K$. When $n = 2$, the Dirichlet-type solutions are as follows,

$$u(x) = \begin{cases} |x|^{\sqrt{\beta}} - |x|^{-\sqrt{\beta}} & \beta > 0; \\ \ln |x| & \beta = 0; \\ \sin(\sqrt{-\beta} \ln |x|) & \beta < 0. \end{cases}$$

When $\beta \geq 0$, L is locally gaugeable, and there exists a profile $h = u$. When $\beta < 0$ there does not exist a profile.

Case1 : $\beta > \mathbf{0}$.

Next we need to check the condition (4.5) and the volume doubling property (4.8). When $\beta \geq 0$, $h(x)$ is increasing with at most polynomial rate. So condition (4.5) is satisfied.

For the volume doubling property, we take two steps to check. Firstly, when $1 \leq |x| \leq 2$, $h(x) \simeq 2\sqrt{\beta}(|x| - 1)$. Thus for $z \in \partial U$ and $r < 1$, $V_{h^2}(z, r) \simeq r^4$. Secondly, when $r > 2$, $V_{h^2}(z, r) \simeq r^{2+2\sqrt{\beta}}$. By the argument in [25] in terms of anchored balls and remote balls, we know $h^2 d\mu$ is volume doubling.

As a conclusion, when $\beta > 0$, we can apply Theorem 4.3.1 to get the precise global heat kernel $p^D_{q,U}(t, x, y)$. To be specific, we have for all x, y large enough and for all $t > 0$,

$$p^D_{q,U}(t, x, y) \asymp \frac{|x|^{\sqrt{\beta}} |y|^{\sqrt{\beta}}}{t(|x|^{\sqrt{\beta}} + t^{\sqrt{\beta}/2})(|y|^{\sqrt{\beta}} + t^{\sqrt{\beta}/2})} e^{-\frac{|x-y|^2}{t}}.$$

When both x and y are near the boundary, we have

$$p^D_{q,U}(t,x,y) \asymp \frac{\beta(|x|-1)(|y|-1)}{t[\sqrt{\beta}(|x|-1)+t^{\sqrt{\beta}/2}][\sqrt{\beta}(|y|-1)+t^{\sqrt{\beta}/2}]} e^{-\frac{|x-y|^2}{t}}.$$

Similarly, when x is near the boundary and y are far away from the boundary or vice verse, we can still get the corresponding estimate.

Case2 : $\beta = \mathbf{0}$.

In this case $L = -\Delta$. Condition (4.5) and the volume-doubling condition (4.8) have been established in [27]. We have for all x, y and for all $t > 0$,

$$p^D_{q,U}(t,x,y) \asymp \frac{\log|x|\log|y|}{t(\log(1+\sqrt{t})+\log|x|)(\log(1+\sqrt{t})+\log|y|)} e^{-\frac{|x-y|^2}{t}}.$$

Such an estimate was given in [23].

Next let's see the situation when $n \geq 3$. The Dirichlet-type solution is as follows:

$$u(x) = \begin{cases} |x|^{\frac{-n+2+\sqrt{s_0}}{2}} - |x|^{\frac{-n+2-\sqrt{s_0}}{2}} & \beta > -\frac{(n-2)^2}{4}; \\ |x|^{\frac{-n+2}{2}} \ln|x| & \beta = -\frac{(n-2)^2}{4}; \\ |x|^{\frac{-n+2}{2}} \sin(\frac{\sqrt{-s_0}}{2}\ln|x|) & \beta < -\frac{(n-2)^2}{4}, \end{cases}$$

where $s_0 = (n-2)^2 + 4\beta$.

It is easily seen that when $\beta \geq -\frac{(n-2)^2}{4}$, there exists a profile $h(x) = u(x)$. Furthermore, when x is away from the boundary, $h(x) \asymp |x|^{\frac{-n+2+\sqrt{s_0}}{2}}$ or $|x|^{\frac{-n+2}{2}} \ln|x|$. Condition (4.5) is satisfied, since $u(x)$ is polynomially decreasing. Also the power $\frac{-n+2+\sqrt{s_0}}{2}$ or $\frac{-n+2}{2}$ is greater than $\frac{-n}{2}$. By the method in [25] we know the measure $h^2 d\mu$ is also volume doubling. Therefore we can also apply the Theorem 4.3.1.

Case1 : $\beta > -\frac{\mathbf{(n-2)^2}}{\mathbf{4}}$.

When x and y are away from the boundary, we have for all $t > 0$,

$$p^D_{q,U} \asymp \frac{|x|^{(-n+2+\sqrt{s_0})/2}|y|^{(-n+2+\sqrt{s_0})/2}}{t^{n/2}(|x|^{(-n+2+\sqrt{s_0})/2} + t^{(-n+2+\sqrt{s_0})/4})(|y|^{(-n+2+\sqrt{s_0})/2} + t^{(-n+2+\sqrt{s_0})/4})} e^{-\frac{|x-y|^2}{t}}.$$

When x and y are near the boundary, we have for all $t > 0$,

$$p^D_{q,U} \asymp \frac{s_0(|x|-1)(|y|-1)}{t^{n/2}[\sqrt{s_0}(|x|-1) + t^{(-n+2+\sqrt{s_0})/4}][\sqrt{s_0}(|y|-1) + t^{(-n+2+\sqrt{s_0})/4}]} e^{-\frac{|x-y|^2}{t}}.$$

Case2 : $\beta = -\frac{\mathbf{(n-2)^2}}{\mathbf{4}}$.

In this case, we have for all $x, y \in U$ and for all $t > 0$,

$$p^D_{q,U} \asymp \frac{(|x|^{(-n+2)/2}\ln|x|)(|y|^{(-n+2)/2}\ln|y|)}{t^{n/2}(|x|^{(-n+2)/2}\ln|x| + t^{(-n+2)/4}\ln(1+\sqrt{t}))(|y|^{(-n+2)/2}\ln|y| + t^{(-n+2)/4}\ln(1+\sqrt{t}))} e^{-\frac{|x-y|^2}{t}}.$$

In Chapter 6 we will give some counterexamples which violate these conditions, and the profile there will have exponential growth. However, these conditions need not be satisfied even for a polynomial-growth profile. Assume we still have $U = \mathbb{R}^n \setminus \overline{B(0,1)}(n \geq 2)$ and $h(x) = |x|^\alpha(\alpha < -n/2)$ when $|x| > 2$. Then it satisfies the condition (4.5) but violates the volume doubling for the weighted measure $h^2 d\mu$. See more details in [25].

CHAPTER 5

SUBCRITICALITY AND GAUGEABILITY OF $-\Delta + q$ IN THE EXTERIOR OF A COMPACT SET

We gave the estimate for the heat kernel $p_q(t, x, y)$ in last chapter. However, the assumption that q is in Kato class is still not good enough since we know little about the profile h even if we assume the gaugeability. In this chapter, we will investigate a more restricted class, K^∞. Under this stronger assumption, we prove a couple of equivalent statements, one of which specifies the boundedness of h. As a result, we can draw a more precise conclusion about $p_q(t, x, y)$.

Besides the stronger assumptions of the potential q, we also assume, through this chapter, the space (X, μ) is non-parabolic. In our context of Harnack space, it means $\int^\infty V(x, \sqrt{t})^{-1} dt < \infty$(See Chapter 5 in [44]). In addition, we let U be an unbounded uniform domain which is the exterior of a compact set. This condition is also crucial in our proof.

We first present all related results for the Dirichlet-type solution. Then some remarks about the Neumann one will be included in the end.

5.1 K^∞ class

When the Harnack-type space (X,μ) is non-parabolic, by [44], the Green function $G(x,y)$ is positive and satisfies the following estimate

$$c\int_{\rho(x,y)^2}^{\infty}\frac{dt}{V(x,\sqrt{t})}\le G(x,y)\le C\int_{\rho(x,y)^2}^{\infty}\frac{dt}{V(x,\sqrt{t})},\tag{5.1}$$

for some positive constant c, C.

Under this assumption, we have the following equivalent definition of Kato class(See [34]): $q\in K$ iff

$$\lim_{r\to 0}\sup_{x\in X}\int_{\rho(x,y)<r}G(x,y)|q(y)|dy=0.\tag{5.2}$$

In a similar manner, we define the K^∞ class as follows:

Definition 5.1.1. $K^\infty=\left\{q\in K_{\mathrm{loc}}:\lim\limits_{r\to\infty}\sup_{x\in X}\int_{|y|>r}G(x,y)|q(y)|dy=0\right\}$ where $|y|=\rho(0,y)$ for a fixed point x_0 *as the origin 0.*

We first give two propositions about the potential class K^∞.

Proposition 5.1.1. *Let $(X,\mu,\mathscr{E},\mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Assume there exists $C_0>0$ and $D_0>2$ such that for any $x\in X, 0<r<R<\infty$,*

$$\frac{V(x,R)}{V(x,r)}\ge C_0\Big(\frac{R}{r}\Big)^{D_0}.\tag{5.3}$$

Then for $\alpha>2$, $\{q\in K:q(x)=O(|x|^{-\alpha})\text{ as }|x|\to\infty\}\subseteq K^\infty$.

Proof. Since $q\in K$, for any $\epsilon>0$, we can find $a>0$, such that

$$\sup_{x\in X}\int_{\rho(x,y)\le a}G(x,y)|q(y)|dy\le\epsilon.$$

Suppose for large enough $A > 0$, that $q(x) \leq \frac{c}{|y|^\alpha}$. Then

$$\int_{\rho(x,y)\geq A} G(x,y)|q(y)|dy \leq \epsilon + c\int_{|y|\geq A, \rho(x,y)\geq a} G(x,y)\frac{1}{|y|^\alpha}dy.$$

We split the integration on the right-hand side into three parts as follows:

$$\begin{aligned}\int_{|y|\geq A, \rho(y,x)>a} &= \int_{|y|\geq A, \rho(y,x)\geq |y|/2} + \int_{\substack{|y|\geq A, a<\rho(y,x)<|y|/2 \\ \text{and } \rho(y,x)\geq |x|/2}} + \int_{\substack{|y|\geq A, a<\rho(y,x)<|y|/2 \\ \text{and } \rho(y,x)<|x|/2}} \\ &= (1) + (2) + (3).\end{aligned}$$

Since we assume the volume growth condition (5.3), the Green function $G(x,y)$ has the following estimate(See Chapter 5 in [44]):

$$c\frac{\rho(x,y)^2}{V(x,\rho(x,y))} \leq G(x,y) \leq C\frac{\rho(x,y)^2}{V(x,\rho(x,y))}. \tag{5.4}$$

Part (1):

We know Green function $G(x,y)$ is a decreasing function(up to a constant) of $\rho(x,y)$. Thus

$$\begin{aligned}\int_{|y|\geq A, \rho(x,y)\geq |y|/2} G(x,y)\frac{dy}{|y|^\alpha} &\leq C\int_{|y|\geq A, \rho(x,y)\geq |y|/2} \frac{|y|^2}{V(y,|y|)}\frac{1}{|y|^\alpha}dy \\ &\leq \int_{|y|\geq A, \rho(x,y)\geq |y|/2} \frac{|y|^{2-\alpha}}{V(0,|y|)}dy,\end{aligned}$$

where the second inequality comes from the volume doubling property and 0 is the origin of the space.

We split the domain $\{|y| \geq A\}$ into annulus $\{2^k A \leq |y| < 2^{k+1}A\}$ for $k = 0, 1, 2, \ldots$. Then

$$\begin{aligned}(1) &\leq \sum_{k=0}^{\infty} \frac{(2^{k+1}A)^{2-\alpha}V(0,2^{k+1}A)}{V(0,2^k A)} \\ &\leq C\sum_{k=0}^{\infty}(2^{k+1}A)^{2-\alpha}.\end{aligned}$$

Since $\alpha > 2$, (1) $\to 0$ as $A \to \infty$.

Part (2):

In this region, we have $A \le |y| \le 2|x|$ and $a \le \rho(y,x) \le |y|/2 \le |x|$. Thus, if we set $M := \lfloor \log_2(A) \rfloor$ and $N := \lceil \log_2(2|x|) \rceil$,

$$\begin{aligned}
(2) &\le \sum_{k=M}^{N} \frac{|x|^2 V(0, 2^{k+1})}{2^{k\alpha} V(0,|x|)} \\
&\le \sum_{k=M}^{N} \left(\frac{|x|}{2^k}\right)^2 \cdot \left(\frac{2^k}{|x|}\right)^{\beta} \text{ for some } \beta > 2 \text{ by (5.3)} \\
&\le |x|^{2-\beta} \sum_{k=M}^{N} 2^{-k(2-\beta)} \\
&\le C.
\end{aligned}$$

The last inequality comes from $-k(2-\beta) > 0$ and the upper limit is $\log_2(2|x|)$. Thus (2) $\to 0$ when $A \to \infty$.

Part (3):

In this region, we have $A \le |y| \le \frac{3}{2}|x|$ and $a \le \rho(y,x) \le \frac{1}{2}|y| \le \frac{3}{4}|x|$. Following a similar argument as in Part (2), we have (3) $\to 0$ as $A \to \infty$.

Thus we obtain

$$\overline{\lim_{A\to\infty}} \sup_{x\in X} \int_{|y|\ge A} G(x,y)|q(y)|dy \le \epsilon.$$

So we conclude $q(x) \in K^{\infty}$. □

Remark 5.1.1. *If we drop the condition (5.3), the conclusion is only valid for large ρ where ρ depends on the volume growth rate.*

Proposition 5.1.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Assume the volume growth condition (5.3) is satisfied. Assume also the volume of the unit ball in*

X has a uniform lower bound

$$\inf_{x\in X} V(x,1) \geq M.$$

Then $K \cap L^1(X,\mu) \subseteq K^\infty \subseteq K$

Proof. To show that $K \cap L^1(X,\mu) \subseteq K^\infty$, suppose that $q(x) \in K \cap L^1(X,\mu)$. Then for any $\epsilon > 0$, there exists an $a > 0$ such that

$$\sup_{x\in X} \int_{\rho(x,y)\leq a} G(x,y)|q(y)|dy < \frac{\epsilon}{2}.$$

From the fact that $q(x) \in L^1(X,\mu)$, we know there exists $A > 0$ such that

$$\int_{|y|\geq A} |q(y)|dy < c_1\frac{\epsilon}{2}.$$

Then for each $x \in X$ we have

$$\begin{aligned}\int_{|y|\geq A} G(x,y)|q(y)|dy &\leq \int_{\rho(x,y)\leq a} G(x,y)|q(y)|dy + \int_{\rho(x,y)>a,|y|\geq A} G(x,y)|q(y)|dy \\ &\leq \frac{\epsilon}{2} + C\int_{a^2}^{\infty} \frac{dt}{V(x,\sqrt{t})} \cdot c_1\frac{\epsilon}{2},\end{aligned}$$

where the constant C comes from (5.1).

From the condition (5.3) we have

$$\begin{aligned}\int_{a^2}^{\infty} \frac{dt}{V(x,\sqrt{t})} &\leq C_1 \int_{a^2}^{\infty} \frac{dt}{V(x,1)t^{D_0/2}} \\ &\leq C_2\frac{1}{V(x,1)} \\ &\leq C_3.\end{aligned}$$

Choose A large enough so that $c_1 < (CC_3)^{-1}$. We then have

$$\int_{|y|\geq A} G(x,y)|q(y)|dy < \epsilon.$$

Thus $q(x) \in K^\infty$.

The second part follows from the fact $\{(x,y) : \rho(x,y) \le a\} \subseteq \{(x,y) : \rho(x,y) \le a \text{ and } |x| \le A+1\} \cup \{(x,y) : |y| \ge A\}$, for $0 < a \le 1$. □

Remark 5.1.2. *We use the additional assumption (5.3) and the uniform lower volume bound only for the first part of the proof. The second part holds for any non-parabolic Harnack-type space.*

For $q(x) \in K^\infty$, put

$$|||q||| = \sup_{x\in X} \int_X G(x,y)|q(y)|dy.$$

Obviously we have $|||q||| < \infty$. In other words, if $q \in K^\infty$, then it is Green-bounded. It is easy to verify the following proposition by definition:

Proposition 5.1.3. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Then for measurable q on X, $q \in K^\infty$ if and only if the family $\{G(x,y)|q(y)|, x \in X\}$ is uniformly integrable in X.*

Next we will present the gauge theorem.

Let τ_U be the exit time. Put

$$u_0(x) = E^x[e_q(\tau_U)] = E^x\Big[\exp\Big(-\int_0^{\tau_U} q(X_s)ds\Big)\Big].$$

Then we have:

Theorem 5.1.1. *Let (X, μ) be a non-parabolic Harnack-type space, and U be an unbounded uniform domain that is the exterior of a compact set. Then for any $x \in U$,*

$u_0(x) \not\equiv \infty$ in U if and only if $u_0(x)$ is bounded. If so, $u_{0,D}(x) = E^x[e_q(\tau_U)\mathbb{1}_{\{\tau_U=\infty\}}]$ is a positive continuous solution of $Lu = 0$ in U. In addition, $u_{0,D}(x)$ satisfies the Dirichlet boundary condition.

To prove this theorem, let us first introduce the crucial Khasminskii's Lemma. See Theorem 1.2 in [2] for the proof.

Lemma 5.1.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Suppose U is a domain in X, q is a measurable function, $\{X_t\}$ is a Markov process in U and τ is either a constant time or an exit time. If*

$$\sup_{x\in U} E^x\Big[\int_0^\tau |q(X_s)|ds\Big] \le \alpha < 1,$$

then

$$\sup_{x\in U} E^x\Big[\exp\Big(\int_0^\tau |q(X_s)|ds\Big)\Big] \le \frac{1}{1-\alpha}.$$

Now let us prove Theorem 5.1.1:

Proof. "$\Leftarrow$": Trivial.
"$\Rightarrow$": Assume $x_0 \in U$ and $u_0(x_0) < \infty$. By the argument of Theorem 1 in [65], we know $u_0(x)$ is bounded in any compact set containing x_0. Pick $A > 0$ such that $B_A := B_A(x_0) \subsetneq U$ and set $M := \sup_{x\in B_A} u_0(x) < \infty$.
Set $B_A^* := U \setminus B_A$. For any $x \in B_A^*$, by the Strong Markov Property, we have

$$\begin{aligned} u_0(x) &= E^x[\tau_{B_A^*} = \tau_U; e_q(\tau_{B_A^*})] + E^x[\tau_{B_A^*} < \tau_U; e_q(\tau_{B_A^*})u_0(X(\tau_{B_A^*}))] \\ &\le (M+1)E^x[e_q(\tau_{B_A^*})]. \end{aligned}$$

By the definition of K^∞, $|||1_{B_A^*} q(x)||| < C_1 < 1$ for some positive constant C_1, provided that A is big enough. Together with Khasminskii's Lemma, we conclude $e_q(\tau_{B_A^*}) < C_2$, which implies $u_0(x)$ is finite everywhere;

Now suppose $C := \sup_{x \in U} u_0(x) < \infty$. Then

$$\int_U G_U(x,y)|q(y)|u_0(y)dy \le \int_X G(x,y)|q(y)|u_0(y)dy \le C|||q||| < \infty$$

By Fubini's theorem and strong Markov property, we have

$$\begin{aligned}
\int_U G_U(x,y)q(y)u_{0,D}(y)dy &= E^x\Big[t < \infty; \int_0^\infty q(X_t)E^{X(t)}[e_q(\infty)]dt\Big] \\
&= E^x\Big\{t < \infty; \int_0^\infty q(X_t)\exp\Big[-\int_t^\infty q(X_s)ds\Big]dt\Big\} \\
&= E^x[t < \infty; 1 - e_q(\infty)] \\
&= E^x[\tau_U = \infty; 1 - e_q(\tau_U)] \\
&= P(\tau_U = \infty) - u_{0,D}(x).
\end{aligned} \tag{5.5}$$

Applying Δ on both side of the previous equation we have $(-\Delta + q(x))u_{0,D}(x) = 0$. In addition, when $x_i \to x \in \partial U$, $u_{0,D}(x_i) \to 0$ for any regular point of the domain U. Thus $u_{0,D}(x)$ is a distributional solution of L and also satisfying the Dirichlet boundary condition in the sense $u_{0,D}(x) \in \mathscr{F}^0_{q,\mathrm{loc}}(U)$.

The continuity of $u_{0,D}(x)$ comes from the uniform integration of $\{G(x,y)|q(y)|u_0(y) : x \in X\}$ by Proposition 5.1.3. □

5.2 Subcriticality, criticality and supercriticality

Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Let U be an unbounded uniform domain which is the exterior of a compact set and q belongs to the potential class K^∞. We first give the following definition:

Definition 5.2.1. *For any domain U in X and $q \in K^\infty$, $L = -\Delta + q \geq 0$ in D if and only if for any $\phi \in \mathscr{F}_q^0(U)$,*

$$\int_U d\Gamma(\phi, \phi) + \int_U q\phi^2 d\mu \geq 0. \tag{5.6}$$

We first give the following definition for subcriticality:

Definition 5.2.2. *Let f be any continuous nonnegative function with compact support in U which is positive on a set of positive measure. Then the Schrödinger operator $L = -\Delta + q$ is called subcritical in U if $L \geq 0$ and there exists $\epsilon > 0$ such that $-\Delta + q - \epsilon f \geq 0$.*

Intuitively, being subcritical means it is possible to perturb L by small perturbations and still keeps its non-negativity.
Correspondingly, we have

Definition 5.2.3. *Let f be any continuous nonnegative function with compact support in U which is positive on a set of positive measure. Then the Schrödinger operator $L = -\Delta + q$ is called critical in U if $L \geq 0$ and $-\Delta + q - \epsilon f$ is not nonnegative for all $\epsilon > 0$.*

and

Definition 5.2.4. *The Schrödinger operator $L = -\Delta + q$ is called supercritical in U if $L \geq 0$ is not satisfied.*

If $L \geq 0$, then it is either critical or subcritical. Thus in the following sections, we will only focus on the subcriticality property.
For more about subcriticality, we recommend [43, 47] to the readers.

Our first theorem is a characterization of the subcritical/critical potential. See Theorem 2.5 in [41] for the proof.

Theorem 5.2.1. *Let q_0 be a bounded nonnegative function in U which is positive on a set of positive measure. Then:*

(1) If q is critical, then $q + q_0$ is subcritical, and $q - q_0$ is subcritical.

(2) If q is subcritical, then $q + q_0$ is subcritical.

At the end of this section we present a theorem which tells the relation between subcriticality(criticality) and gaugeability.

Theorem 5.2.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Let U be an unbounded uniform domain in X which is the exterior of a compact set and $q \in K^\infty$. Fix a point $x_0 \in U$ and $B_r := \{x \in U; \rho(x_0, x) \leq r\}$. Then $L = -\Delta + q \geq 0$ if and only if for any $r > 0$,*

$$E^x[e_q(\tau_{B_r} \wedge \tau_U)] \not\equiv \infty \text{ in } B_r. \tag{5.7}$$

Proof. If $L \geq 0$ then by a similar argument as in Theorem 5 of [67], there exists a positive and continuous solution $u > 0$ of $Lu = 0$ in U. For any $r > 0$, we have

$\inf_{x \in B_r} u(x) \geq \inf_{x \in \overline{B_r}} u(x) > 0$. Then by Theorem 4 in [67], (5.7) holds.

Conversely, if (5.7) is true, then all eigenvalues of L are larger than 0 by Theorem 4 in [67]. For any $\phi \in \mathscr{F}_q^0(U)$ we can find r big enough such that $\text{supp}(\phi) \subset B_r$. Then we have

$$\int_U d\Gamma(\phi, \phi) + \int_U q\phi^2 d\mu = \int_{B_r} d\Gamma(\phi, \phi) + \int_{B_r} q\phi^2 d\mu \geq 0$$

This shows $L = -\Delta + q \geq 0$ by the Definition 5.2.1. □

5.3 Shuttle operator S_λ

Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Let U be an unbounded uniform domain in X which is the exterior of a compact set and q belongs to K^∞. Now fix a point x_0 and a constant $A > 0$ such that $\overline{B_A(x_0)} \subset U$. In the following we always denote $B_r := B_r(x_0)$.

For each $r \in (2A, \infty]$, put $J_{A,r} = \{y \in U : A < \rho(x_0, y) < r\}$. We define the following stopping time:

$$\sigma_r = \tau_{B_{2A}} + \tau_{J_{A,r}} \circ \theta_{\tau_{B_{2A}}}. \tag{5.8}$$

and the operator in the Banach space $C(\partial B_A)$(See the Figure 5.1):

$$S_{q,r} f(z) = E^z[\sigma_r < \tau_U \text{ and } X(\sigma_r) \in \partial B_A; e_q(\sigma_r) f(X(\sigma_r))], z \in \partial B_A, f \in C(\partial B_A). \tag{5.9}$$

We first give the following lemma specifying the kernel property of $S_{q,r}$. See Lemma 6 in [64] for the details of proof.

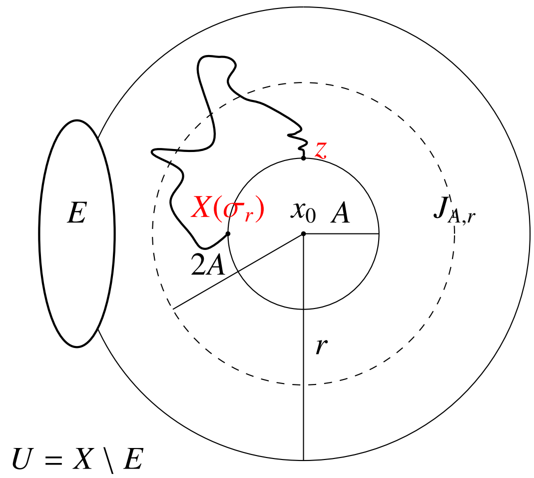

Figure 5.1: Shuttle Operator

Lemma 5.3.1. *For each $r \in (2A, \infty]$, the operator $S_{q,r}$ is an integral operator in $C(\partial B_A)$ with a positive continuous kernel $\Phi_r(\cdot,\cdot)$:*

$$S_{q,r}f(z) = \int_{\partial B_A} \Phi_r(z,w)f(w)\sigma(dw).$$

Now we define the spectrum λ_r for the operator $S_{q,r}$. For the compact set ∂B_A, $r \in (2A, \infty]$, set

$$\lambda_r(q) = \lim_{n\to\infty} \sqrt[n]{\left\|(S_{q,r})^n\right\|}. \tag{5.10}$$

The following lemma brings to light the connection between gaugeability and the spectrum of the operator $S_{q,r}$.

Lemma 5.3.2. *Let $e_q(t)$ and $\lambda_r(q)$ be defined as before for $q \in K^\infty$ and $r \in (2A, \infty)$, then $E^x[e_q(\tau_{B_r} \wedge \tau_U)] \not\equiv \infty$ if and only if $\lambda_r(q) < 1$.*

Proof. "$\Leftarrow$":For each $r < \infty$, define

$$f_r(z) = \begin{cases} E^z[X(\sigma_r) \in \partial B_r; e_q(\sigma_r)], & \text{when } \sigma_r < \tau_U \\ \\ E^z[e_q(\tau_U)], & \text{when } \sigma_r \geq \tau_U. \end{cases} \tag{5.11}$$

We also define

$$f_\infty(z) = E^z[\sigma_r = \tau_U; e_q(\sigma_r)].$$

Then for each $z \in \partial B_A$, by the strong Markov property, we have for each $r \in (2A, \infty)$,

$$E^z[e_q(\tau_{B_r} \wedge \tau_U)] = \sum_{n=0}^{\infty} (S_{q,r})^n f_r(z). \tag{5.12}$$

Here we set $S_{q,r} f(x) = 0$ if $f \notin C(\partial B_A)$. If $\lambda_r(q) < 1$, then by (5.10), the series (5.12) converges uniformly on ∂B_A, which is the desired result.

"$\Rightarrow$": If $E^x[e_q(\tau_{B_r} \wedge \tau_U)] \not\equiv \infty$, then by Theorem 5.1.1(this is for the case $r = \infty$; the case $r < \infty$ works in the same manner) $g(x) = E^x[e_q(\tau_{B_r} \wedge \tau_U)]$ is a bounded continuous function in ∂B_A. By Dini's theorem, the convergence in (5.12) is uniform on ∂B_A.

Since $f_r(z)$ is a strictly bounded continuous function on ∂B_A, set $m := \min_{z \in \partial B_A} f_r(z) > 0$, then by the uniform convergence we can find an integer N such that for all $n > N$, $\left\|(S_{q,r})^n f_r\right\| < m$. As a result, $\left\|(S_{q,r})^n\right\| = \left\|(S_{q,r})^n 1\right\| \leq c_{-1} \left\|(S_{q,r})^n f_r\right\| < 1$, thus $\sqrt[n]{\left\|(S_{q,r})^n\right\|} < 1$. Hence we get the conclusion by the definition of $\lambda_r(q)$. □

The following lemma shows the relation between subcriticality(criticality) and the spectrum radius.

Lemma 5.3.3. *Let $q \in K^\infty$, then $L \equiv -\Delta + q \geq 0$ if and only if $\lambda_\infty(q) \leq 1$.*

Proof. By Theorem 5.2.2 and Theorem 5.3.2 we have that $L \geq 0$ if and only if $\lambda_r(q) < 1$ for $2A < r < \infty$. By continuity and monotonicity of the kernel function Φ_r, the latter condition is equivalent to $\lambda_\infty(q) \leq 1$. □

From now on we define the shuttle operator to be $S_q := S_{q,\infty}$ and its spectrum radius $\lambda(q) := \lambda_\infty(q)$.

We end this section with two lemmas specifying the continuity property of the spectrum $\lambda(q)$. See Lemma 14 and 15 in [64] for the proof in $\mathbb{R}^n$. A similar argument follows for the abstract case.

Lemma 5.3.4. *Let $\{q_n\} \subseteq K^\infty$ be a sequence of potential and $\lim_{n\to\infty} |||q_n - q||| = 0$, then we have*

$$\lim_{n\to\infty} \lambda(q_n) = \lambda(q).$$

Lemma 5.3.5. *Let $q_1, q_2 \in K^\infty$ and $q_1 \leq q_2$ and $q_1 \not\equiv q_2$ on an open set with positive measure in X, then*

$$\lambda(q_1) > \lambda(q_2).$$

5.4 Main results

Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type space. Let $q \in K^\infty$ and U is an unbounded uniform domain which is the exterior of a compact set. Let $L = -\Delta + q$. For any $x, y \in U$, put

$$u_0(x) = E^x[e_q(\tau_U)] = E^x\Big[\exp\Big(-\int_0^{\tau_U} q(X_s)ds\Big)\Big].$$

and y-conditioned gauge

$$u_0(x,y) = E^x_y[e_q(\tau_U)] = E^x_y\Big[\exp\Big(-\int_0^{\tau_U} q(X_s)ds\Big)\Big].$$

where E^x_y is defined as

$$E^x_y[t < \tau_U; \Phi] = G_U(x,y)^{-1}E^x[t < \tau_U; \Phi \cdot G_U(X_t, y)], x \in U.$$

See Chapter 5 in [11] for more details about E^x_y.

We first introduce the famous $3G$-Theorem in the abstract space sense, which will be of great use in proving the main theorem:

Theorem 5.4.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type Dirichlet space. Assume the volume growth satisfies the condition (5.3). Let $U = X \setminus E$ is an unbounded uniform domain and E is a compact set. Let $G_U(x,y)$ and $G(x,y)$ be the Green function in the domain U and in the whole space X respectively, then for any $x, y, z \in U$, we have:*

$$\frac{G_U(x,z)G_U(z,y)}{G_U(x,y)} \le C_1 \frac{G(x,z)G(z,y)}{G(x,y)}. \tag{5.13}$$

and

$$\frac{G_U(x,z)G_U(z,y)}{G_U(x,y)} \le C_2(G(x,z) + G(z,y)). \tag{5.14}$$

Proof. We first prove (5.13) implies (5.14):

It suffices to prove

$$\frac{G(x,z)G(z,y)}{G(x,y)} \le C_3(G(x,z) + G(z,y)),$$

which is equivalent to

$$\frac{1}{G(x,y)} \leq C_3\Big(\frac{1}{G(x,z)} + \frac{1}{G(z,y)}\Big).$$

Under the condition (5.3), we have the Green estimate (5.4). Thus it suffices to prove

$$\frac{V(x,\rho(x,y))}{\rho^2(x,y)} \leq C_3\Big(\frac{V(x,\rho(x,z))}{\rho^2(x,z)} + \frac{V(z,\rho(z,y))}{\rho^2(z,y)}\Big). \tag{5.15}$$

- Case 1: $\rho(x,y) \leq \rho(x,z)$

By the inequality (5.3), we have

$$\frac{V(x,\rho(x,y))}{V(x,\rho(x,z))} \leq C_0\Big(\frac{\rho(x,y)}{\rho(x,z)}\Big)^{D_0} \leq C_0\frac{\rho(x,y)^2}{\rho(x,z)^2},$$

which proves

$$\frac{V(x,\rho(x,y))}{\rho^2(x,y)} \leq C_3\Big(\frac{V(x,\rho(x,z))}{\rho^2(x,z)}\Big).$$

Therefore (5.15) follows easily.

- Case 2: $\rho(x,y) > \rho(x,z)$ and $\rho(x,y) \leq \rho(z,y)$

We prove this case using the exact argument as in case 1 since x and y are holding the symmetric role.

- Case 3: $\rho(x,y) > \rho(x,z)$ and $\rho(x,y) > \rho(z,y)$

By the symmetric role of x and y, we can assume $\rho(x,z) > \rho(z,y)$ without loss of generality. Then

$$
\begin{aligned}
\frac{V(x,\rho(x,y))}{\rho^2(x,y)} &\le \frac{V(x,\rho(x,z)+\rho(z,y))}{\rho^2(x,z)} \\
&\le \frac{V(x,2\rho(x,z))}{\rho^2(x,z)} \\
&\le C_5\frac{V(x,\rho(x,z))}{\rho^2(x,z)}
\end{aligned}
$$

The last inequality comes from the volume doubling property. Therefore (5.15) follows.

Now we are left to prove (5.13).

By Theorem 5.13 in [27] we have the following estimate for $G_U(x,y)$:

$$
c\int_{\rho(x,y)^2}^{\infty}\frac{h(x)h(y)}{V_{h^2}(x,\sqrt{s})}ds \le G_U(x,y) \le C\int_{\rho(x,y)^2}^{\infty}\frac{h(x)h(y)}{V_{h^2}(x,\sqrt{s})}ds.
$$

where $h(x)$ is the profile for $\Delta u = 0$ in U.

In addition, we have

$$
V_{h^2}(x,r) \simeq h(x_r)^2 V(x,r).
$$

where x_r is any point in U such that $\rho(x,x_r) \le r/4$ and $\rho(x_r, X\setminus U) \ge c_0 r/8$. See Theorem 4.17 in [27] for details.

As the global Green function also has the similar form(See the estimate (5.1)), thus when x or y or z is far away from the boundary, the conclusion follows easily.

Now we only consider the case where x,y and z are all near the boundary. Namely, $x,y,z\in E_\delta = \{x\in U : \rho(x,\partial U)\le\delta\}$. To make it consistent, we still use the notation U for bounded uniform domain.

Since U is bounded and uniform, it satisfies the following two condition (See [31]).

(1) The interior condition for a NTA domain: There exists constants $C_0 \geq 1$ and $\alpha > 0$ such that for any $z \in \partial$ and $0 < r \leq \alpha$, we can find a point $A = A_r(z)$ in U satisfying the following condition:

$$|A - z| \leq C_0 r, \text{ and } \delta(A) \geq r,$$

where $\delta(A) = \rho(A, \partial U)$.

(2) The Harnack chain condition: For any given $C_1 > 0$ and any $x, y \in U$ such that $|x - y| \leq C_1[\delta(x) \wedge \delta(y)]$, we can find finite number of balls $B(a_i, r_i)(0 \leq i \leq n)$ in U such that $a_0 = x, a_n = y$ and $B(a_i, \frac{r_i}{2}) \cap B(a_{i+1}, \frac{r_{i+1}}{2}) \neq \emptyset$ for $0 \leq i \leq n-1$. Here the number n does not depend on the choice of (x, y).

We now prove the following lemma.

Lemma 5.4.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type Dirichlet space. Assume the volume growth satisfies the condition (5.3). Let U be any bounded uniform domain. Set $\delta(x) := \rho(x, \partial U)$ and suppose for some constant $C_1 > 0$ we have*

$$\rho(x, y) \leq C_1(\delta(x) \wedge \delta(y)).$$

Then there exists a constant $C_2 > 0$ such that

$$G_U(x, y) \geq C_2 G(x, y).$$

Proof. By symmetry of the Green function, we may assume $\delta(x) < \delta(y)$.

- Case 1: $C_1 < 1$

Since $\rho(x, y) < \delta(x)$, then $B := B(x, \delta(x)) \subset U$, thus $G_U(x, y) \geq G_B(x, y)$. By Lemma 4.8

in [27], we have

$$\begin{aligned}
G_U(x,y) &\geq G_B(x,y) \\
&\geq C_3 \int_{\rho(x,y)^2/2}^{2\delta(x)^2} \frac{ds}{V(x,\sqrt{s})} \\
&\geq C_3 \int_{\rho(x,y)^2/2}^{\frac{2}{C_1^2}\rho(x,y)^2} \frac{ds}{V(x,\sqrt{s})} \\
&\geq C_3 \int_{\rho(x,y)^2/2}^{2\rho(x,y)^2} \frac{ds}{V(x,\sqrt(s))} \\
&\geq C_4 \frac{\rho(x,y)^2}{V(x,\rho(x,y))}.
\end{aligned}$$

The last inequality comes form the volume doubling property.

Since we also assumed the condition (5.3), which implies

$$G(x,y) \leq C_5 \frac{\rho(x,y)^2}{V(x,\rho(x,y))}$$

Thus we have the desired inequality.

- Case 2: $C_1 \geq 1$

In this case we use the Harnack chain condition, and the same argument as Case 1 follows. □

The following reasoning of lines are mainly from Section 6.2 of [11].

Let $C_0 \geq 1, \alpha > 0$ and $A_r(z), 0 < r \leq \alpha$ be as given in (5.4). Let x^* be the point on ∂U in which $\rho(x,x^*) = \delta(x)$. Now for $x \in U$, set $x_r = A_r(x^*)$ if $\delta(x) < r$, while $x_r = x$ if $\delta(x) \geq r$. Then we have

$$\rho(x_r, x) \leq \rho(x_r, x^*) + \rho(x^*, x) \leq (C_0 + 1)r.$$

Next we prove a lemma to show, in certain cases, we can replace the point x by x_r.

Lemma 5.4.3. *Let $0 < r \le \alpha$ be as in (5.4), $\rho(x,y) \ge (2C_0+1)r$ and $\rho(x,z) \ge (2C_0+1)r$. If (5.13) holds for (x_r, y, z), then it also holds for (x, y, z). A similar conclusion is true if we interchange x and y by their symmetric structure.*

Proof. If $\delta(x) \ge r$, it holds trivially. So we assume $\delta(x) < r$ in the following. Since

$$\rho(y, x^*) \ge \rho(y, x) - \delta(x) \ge (2C_0+1)r - r = 2C_0 r$$

and

$$\rho(z, x^*) \ge \rho(z, x) - \delta(x) \ge (2C_0+1)r - r = 2C_0 r,$$

we have both $G_U(\cdot, y)$ and $G_U(\cdot, z)$ are positive harmonic functions in $U \cap B(x^*, C_0 r)$. In addition, we have

$$\rho(x_r, x^*) \ge C_0 r$$

and

$$\rho(x, x^*) = \delta(x) < r \le C_0 r.$$

By boundary Harnack principle, we have

$$\frac{G_U(x,z)}{G_U(x,y)} \ge C \frac{G_U(x_r,z)}{G_U(x_r,y)}.$$

Thus

$$\frac{G_U(x,z)G_U(z,y)}{G_U(x,y)} \le C \frac{G_U(x_r,z)G_U(z,y)}{G_U(x_r,y)}.$$

By assumption, we have

$$\begin{aligned}\rho(x_r, y) &\le \rho(x,y) + (C_0+1)r \le \rho(x,y) + \frac{C_0+1}{2C_0+1}\rho(x,y) \\ &= \frac{3C_0+2}{2C_0+1}\rho(x,y)\end{aligned}$$

and

$$\rho(x_r, z) \geq \rho(x, z) - (C_0 + 1)r \geq \rho(x, z) - \frac{C_0 + 1}{2C_0 + 1}\rho(x, z)$$
$$= \frac{C_0}{2C_0 + 1}\rho(x, z).$$

By hypothesis we have

$$\frac{G_U(x_r, z)G_U(z, y)}{G_U(x_r, y)} \leq C\frac{G(x_r, z)G(z, y)}{G(x_r, y)}.$$

Therefore by the decreasing property of Green function, we have proved (5.13).

□

In the following part of the proof, we always assume $\delta(x) \leq \delta(y)$. Let $d(U)$ represent the diameter of U and set

$$C_1 = C_0 \vee \frac{d(U)}{3\alpha},$$

where C_0 and α come from the interior condition of uniform domain U.

We first prove 3G Theorem in more restrictive conditions in the following two lemmas.

Lemma 5.4.4. *If $\rho(x, y) \leq (10C_1 + 8)\delta(x)$, then (5.13) holds.*

Proof. By Lemma 5.4.2 we have

$$G_U(x, y) \geq CG(x, y)$$

Also we have

$$G_U(x, z) \leq G(x, z)$$

and

$$G_U(z, y) \leq G(z, y).$$

Thus (5.4.1) follows easily. □

Lemma 5.4.5. *If $\rho(x, z) \leq (7C_1 + 4)\delta(x)$, then (5.13) holds.*

Proof. By virtue of Lemma 5.4.4, we may assume $\rho(x, y) > (10C_1 + 8)\delta(x)$. Since

$$\rho(z, x^*) \leq \rho(z, x) + \delta(x) \leq (7C_1 + 5)\delta(x)$$

and

$$\rho(y, x^*) \geq \rho(y, x) - \delta(x) > (10C_1 + 7)\delta(x),$$

we have $G_U(\cdot, y)$ is a positive harmonic function in $U \cap B(x^*, (7C_1 + 5)\delta(x))$. Using Carleson's lemma (see [1]), we have

$$G_U(z, y) \leq CG_U(x, y)$$

Thus we have

$$\frac{G_U(x, z)G_U(z, y)}{G_U(x, y)} \leq CG_U(x, z) \leq CG(x, z).$$

On the other hand, we have

$$\rho(z, y) \leq \rho(x, z) + \rho(x, y) \leq 2\rho(x, y).$$

Thus $G(z, y) \geq C'G(x, y)$. The desired result then follows. □

We continue to prove the 3G Theorem. Firstly we assume

$$\rho(z, y) \geq 2\rho(x, y). \tag{5.16}$$

Now let $r = \frac{\rho(x,y)}{3C_1+2}$. Then by assumption we have $r < \frac{d(U)}{3C_1} \leq \alpha$. Additionally we have

$$\rho(y, x) = (3C_1 + 2)r > (2C_0 + 1)r$$

and

$$\rho(y, z) \geq 2\rho(x, y) > (2C_0 + 1)r.$$

Hence by Lemma 5.4.3, it suffices to prove (5.13) for (x, y_r, z).

By (5.16),

$$\rho(x, z) \geq \rho(y, z) - \rho(x, y) \geq \rho(x, y) > (2C_0 + 1)r.$$

We also have

$$\begin{aligned}
\rho(x, y_r) &\geq \rho(x, y) - \rho(y, y_r) \\
&\geq (3C_1 + 2)r - (C_0 + 1)r \\
&\geq (2C_0 + 1)r.
\end{aligned}$$

Again by Lemma 5.4.3, it suffices to prove (5.13) for (x_r, y_r, z).
Since

$$\begin{aligned}
\rho(x_r, y_r) &\leq \rho(x, y) + \rho(x_r, x) + \rho(y_r, y) \\
&\leq \rho(x, y) + 2(C_0 + 1)r \\
&\leq (3C_1 + 2)r + 2(C_1 + 1)r \\
&\leq (5C_1 + 4)\delta(x_r),
\end{aligned}$$

(5.13) for (x_r, y_r, z) follows from Lemma 5.4.4.

Now we consider the other case

$$\rho(z,y) < 2\rho(x,y). \tag{5.17}$$

This time we set $r = \frac{\rho(x,z)}{6C_1+3}$, then we have $r \leq \alpha$ as before. We additionally have

$$\rho(x,z) = (6C_1+3)r > (2C_0+1)r$$

and

$$\rho(x,y) \geq \rho(x,z) - \rho(z,y) \geq \rho(x,z) - 2\rho(x,y).$$

Thus

$$\rho(x,y) \geq \frac{1}{3}\rho(x,z) > (2C_0+1)r.$$

By Lemma 5.4.3 it suffices to prove (5.13) for (x_r, y, z).

We have by the definition of x_r,

$$\begin{aligned}\rho(x_r,z) &\leq \rho(x,z) + \rho(x,x_r) \leq (6C_1+3)r + (C_0+1)r\\ &\leq (7C_1+4)r \leq (7C_1+4)\delta(x_r).\end{aligned}$$

If $\delta(x_r) \leq \delta(y)$, then (5.13) is satisfied for (x_r, y, z) by Lemma 5.4.5.

If $\delta(y) < \delta(x_r)$, then we consider (5.13) for (y, x_r, z). Same conclusion follows if we discuss two different cases $\rho(z,x_r) \geq 2\rho(y,x_r)$ (as in (5.16)) and $\rho(z,x_r) < 2\rho(y,x_r)$ (as in (5.17)). □

Now we will prove the main theorem as follows:

Theorem 5.4.6 (Main Theorem). *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a non-parabolic Harnack-type Dirichlet space. Assume the volume growth satisfies the condition (5.3). Let $U = X \setminus E$ is an unbounded uniform domain and E is a compact set. Assume $L = -\Delta + q$ and q belongs to the potential class K^∞. Then the following statements are equivalent:*

(1). $u_0(x) \not\equiv \infty$.

(2). $u_0(x, y)$ is bounded in $U \times U \backslash \{(x, x) : x \in U\}$

(3). $\lambda(q) < 1$.

(4). There exists a positive q-Green function in U.

(5). L is subcritical.

(6). There exists a continuous positive solution u of $Lu = 0$ which satisfies $0 < inf_x u(x) \leq sup_x u(x) < \infty$.

Proof. (3) ⇒ (5): Since $\lambda(q) < 1$, by Lemma 5.3.4 for any continuous nonnegative function f with compact support in U, there exists $\epsilon > 0$ such that if $|||\epsilon f|||$ is small enough, then $\lambda(q - \epsilon f) < 1$. By lemma 5.3.3, we have $L - \epsilon f \geq 0$. Thus L is subcritical.

(5) ⇒ (3): If (5) is true, then there exists $q_0(x) > 0$ with $|||q_0||| < \epsilon$ and $q_0(x) \in K^\infty$. Consequently, $L \geq L - q_0 \geq 0$. By lemma 5.3.3 we have $\lambda(q - q_0) \leq 1$. Then by Lemma 5.3.5, $\lambda(q) < \lambda(q - q_0) \leq 1$, i.e., (3) is true.

(1) ⇒ (6): By a similar argument as in Theorem 5.1.1, we can show $u_0(x)$ is a continuous positive solution satisfying

$$\int_U G_U(x, y) q(y) u_0(y) dy = 1 - u_0(x). \tag{5.18}$$

Also by Theorem 5.1.1, $u_0(x)$ is bounded if $u_0(x) \not\equiv \infty$.

From the equation (5.18), we have

$$\lim_{\rho(x,\partial U)\to\infty} u_0(x) = 1 - \lim_{\rho(x,\partial U)\to\infty} \int_X G_U(x,y)q(y)u_0(y)dy$$
$$= 1.$$

The second equation comes from the boundedness of $u_0(x)$ and the definition of K^∞.

On the other hand, as $x \to \partial U$, $\tau_U = 0$ q.e.. Thus $u(x) = 1$ along the boundary q.e.. Therefore we draw the desired conclusion;

(6) $\Rightarrow$ (1): Suppose $u(x)$ is a continuous positive solution satisfying $u(x) \geq c > 0$. Fix a point $x_0 \in U$. For any $R > 0$, set $B_R := U \cap B_R(x_0)$, we have (See Figure 5.2) for large R,

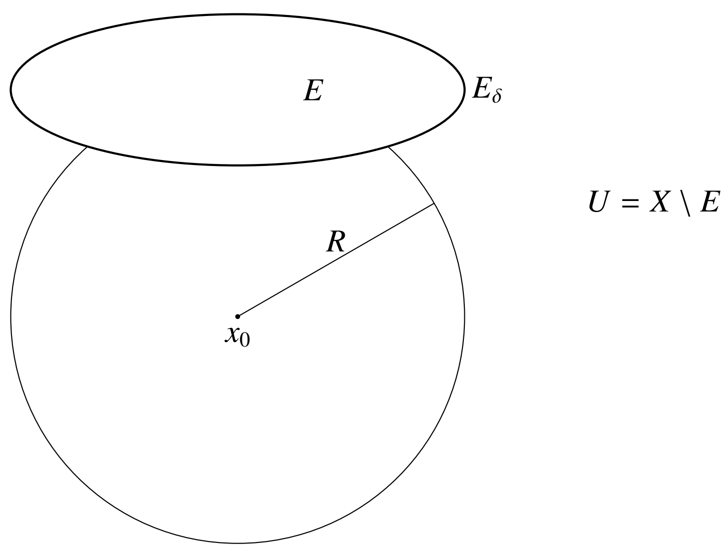

Figure 5.2: Brownian Motion in B_R

$$
\begin{aligned}
u(x_0) &= E^{x_0}[\tau_{B_R} \le \tau_U; e_q(\tau_{B_R})u(X(\tau_{B_R}))] \\
&= E^{x_0}[\tau_{B_R} = \tau_U; e_q(\tau_{B_R})u(X(\tau_{B_R}))] + E^{x_0}[\tau_{B_R} < \tau_U; e_q(\tau_{B_R})u(X(\tau_{B_R}))] \\
&\ge c_1 E^{x_0}[\tau_{B_R} = \tau_U; e_q(\tau_{B_R})] + c_1 E^{x_0}[\tau_{B_R} < \tau_U; e_q(\tau_{B_R})].
\end{aligned}
$$

By Fatou's lemma, we conclude

$$
\begin{aligned}
E^{x_0}[e_q(\tau_U)] &\le \lim_{R\to\infty} E^{x_0}[\tau_{B_R} = \tau_U; e_q(\tau_{B_R})] + \lim_{R\to\infty} E^{x_0}[\tau_{B_R} < \tau_U; e_q(\tau_{B_R})] \\
&\le c_1^{-1} u(x_0) < \infty.
\end{aligned}
$$

Thus we have $u_0(x_0) < \infty$. Together with Theorem 5.1.1, we obtain $u_0(x) \not\equiv \infty$.

(1) $\iff$ (3) comes from Lemma 5.3.2 by setting $r = \infty$.

Till now we have proved the equivalence of (1), (3), (5) and (6).

(2) $\Rightarrow$ (4): We first have

$$
u_0(x, y) = E_y^x\Big[\exp\Big(\int_0^\zeta q(X_s)ds\Big)\Big] = 1 - E_y^x\Big[t < \zeta; \int_0^\zeta q(X_t)\exp\Big(-\int_t^\zeta q(X_s)ds\Big)dt\Big].
$$

by a formal integration.

It follows from Fubini's theorem that

$$
\begin{aligned}
E_y^x\Big[t < \zeta; \int_0^\zeta q(X_t)\exp\Big(-\int_t^\zeta q(X_s)ds\Big)dt\Big] &= E_y^x\Big[t < \zeta; \int_0^\zeta q(X_t)E_t^{X_t}[e_q(\zeta)]dt\Big] \\
&= E_y^x\Big[t < \zeta; \int_0^\zeta q(X_t)u_0(X_t, y)dt\Big] \qquad (5.19) \\
&= \int_U \frac{G_U(x,z)q(z)G_U(z,y)}{G_U(x,y)} u_0(z,y)dz.
\end{aligned}
$$

The last equality comes from the definition of y-conditioned process.

In order to justify the equalities (5.19), we need also prove the finiteness of $\int_U \frac{G_U(x,z)q(z)G_U(z,y)}{G_U(x,y)} u_0(z,y)dz$. This comes from the definition of K^∞, the finiteness of

$u_0(x, y)$ and the 3G-Theorem (5.14).

Now put $F(x, y) = u_0(x, y)G_U(x, y) > 0$ for $x, y \in U$ and $x \neq y$. Then by (5.19) we have

$$F(x, y) = G_U(x, y) - \int_X G_U(x, z)q(z)F(z, y)dz. \tag{5.20}$$

For any $\phi \in C_0^\infty(U)$, put

$$F\phi(\cdot) = \int_U F(\cdot, y)\phi(y)dy.$$

and

$$G\phi(\cdot) = \int_U G(\cdot, y)\phi(y)dy.$$

Then we have $F\phi = G\phi - G(q(F\phi)) = G(\phi - q(F\phi))$. Applying Δ on both sides we obtain $(-\Delta + q)(F\phi) = \phi$, which means $F(\cdot, \cdot)$ is exactly the positive Green function $G_{q,U}$ for the operator L.

(4) $\Rightarrow$ (2): For any $x, y \in U$, we can find $x_0 \in U$ and $R > 0$ large enough such that $x, y \in U \cap B_{x_0}(R)$. Take $B := U \cap B_{x_0}(R)$, we know

$$G_{q,B}(x, y) \leq G_{q,U}(x, y) < \infty, \quad x, y \in B.$$

By Theorem 7 in [66] and a limiting method we have

$$E_y^x[\zeta < \tau_B; e_q(\zeta)] \leq \frac{G_{q,B(x,y)}}{G_U(x, y)}.$$

Since $G_{q,B}(x, y) \leq G_{q,U}(x, y)$, by Fatou's lemma, we have $E_y^x[e_q(\zeta)] \leq \frac{G_{q,U}(x,y)}{G_U(x,y)} < \infty$. By a similar argument as in 5.1.1, we conclude $u_0(x, y)$ is bounded for any $x, y \in U$ and $x \neq y$.

For (4) $\iff$ (5), we refer readers to Chapter 2 in [41]. The proof there applies for the case in $\mathbb{R}^n$, yet the proof doesn't use the particular geometric property of

$\mathbb{R}^n$, therefore the method there applies to the general Dirichlet space without difficulty.

$\square$

Remark 5.4.1. *We assume U is the exterior of a compact domain in Theorem 5.4.6 since it is crucial in proving* (1) $\Rightarrow$ (6) *and* (2) $\Rightarrow$ (4). *For the other implications, just assuming U to be an unbounded uniform domain should be OK.*

Together with Theorem 5.1.1, we have the following corollary.

Corollary 5.4.1. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a non-parabolic Harnack-type Dirichlet space. Assume the volume growth satisfies the condition (5.3). Let U be an unbounded uniform domain which is the exterior of a compact set in (X, ρ). Let q belongs to Kato class K^∞. If any statement in Theorem 5.4.6 is true, then there exists a positive Dirichlet-type solution u(x) satisfying* $\lim_{\rho(x,\partial U)\to\infty} u(x) > 0$.

Together with Theorem 4.3.1, we have the following corollary.

Corollary 5.4.2. *Let $(X, \mu, \mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a Harnack-type Dirichlet space. Assume the volume growth satisfies the condition (5.3). Let U be an unbounded uniform domain which is the exterior of a compact set in (X, ρ). Let q belongs to Kato class K^∞. Let $p^D_{q,U}(t, x, y), t > 0$ be the heat kernel associated with the Harnack-type Dirichlet form. If any statement in Theorem 5.4.6 is true, then we have, for all $(t, x, y) \in (0, \infty) \times U \times U$,*

$$\frac{c_1 h(x)h(y)\exp(-\frac{\rho(x,y)^2}{c_2 t})}{\sqrt{V_{h^2}(x, \sqrt{t})V_{h^2}(y, \sqrt{t})}} \leq p^D_{q,U}(t, x, y) \leq \frac{c_3 h(x)h(y)\exp(-\frac{\rho(x,y)^2}{c_4 t})}{\sqrt{V_{h^2}(x, \sqrt{t})V_{h^2}(y, \sqrt{t})}}.$$

Here x_r is any point in U satisfying $\rho(x_r, x) \leq r$ and $\rho(x_r, X \setminus U) \geq c_0 r/8$ where c_0 comes from the constant in defining uniform domains.

In particular, for x and y are far away from the boundary and for all $t > 0$, we have

$$\frac{c_1 \exp(-\frac{\rho(x,y)^2}{c_2 t})}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}} \leq p^D_{q,U}(t,x,y) \leq \frac{c_3 \exp(-\frac{\rho(x,y)^2}{c_4 t})}{\sqrt{V(x, \sqrt{t})V(y, \sqrt{t})}}. \tag{5.21}$$

It tells that for x, y far away from the boundary and for the potential q satisfying either condition in Theorem 5.4.6, the effect of q is so small that it doesn't affect the heat kernel in essential way.

Remark 5.4.2. *Similar results can be derived as Theorem 5.4.6 if we change the set-up to the global case or the Neumann case. As a result, the same form of heat kernel estimate as in Corollary 5.4.2 is obtained if we replace the profile h to the global weak solution or Neumann-type solution. Partial results about the relation between Dirichlet one and Neumann one will be included in next chapter.*

As we mentioned in the introduction part, there has been plenty of work trying to get the same heat kernel estimate as in (5.21). For the global case, A. Grigor'yan proved in [19] that any nonnegative Green-bounded potential satisfies this property. M. Takeda proved in [55] that if $q = q^+ - q^-$ and q^+ is Green-bounded and $q^- \in K^\infty$, then same conclusion holds if q is gaugeable. For the Dirichlet case, A. Grigor'yan and L. Saloff-Coste proved in [23] that when x and y are away from the boundary, then the heat kernel $p^D_U(t,x,y)$ induced by the Laplacian is comparable to the global one $p(t,x,y)$. Our work in this chapter generalizes the results in these two directions. One point should be mentioned here: As we showed at the start of this chapter, the potential class discussed in [55] actually includes all potential in K^∞. Thus it might be possible to prove all conclusions in this chapter for $q = q^+ - q^-$ where q^+ is Green-bounded and $q^- \in K^\infty$. This will be one of the future goals.

CHAPTER 6

APPLICATIONS

In this chapter we first give some explicit examples in $\mathbb{R}^n$ to compare different potentials. Then a more general result on certain model manifolds will be given, where we also compare the Dirichlet solution and Neumann solution for certain type of potentials. Finally we give an example in upper-half space to reveal why we need to restrict the domain to be exterior of a compact set in Theorem 5.4.6.

6.1 Some examples in $\mathbb{R}^n$

Firstly we will present some classical examples where explicit solutions might exist and thus get some heuristics to our more generalized results in next section.

6.1.1 Examples outside the unit ball

Throughout this subsection, we assume $U = \mathbb{R}^n \setminus \overline{B(0,1)}$ and $q = \frac{\beta}{|x|^2}$ is a radially symmetric function defined in U. Thus the solution is also radial. If we apply polar coordinates, then

$$\Delta u(x) = u'' + \frac{n-1}{r}u' \tag{6.1}$$

where $u'' = \frac{\partial^2 u(r)}{\partial r^2}$ and $u' = \frac{\partial u(r)}{\partial r}$.

Since $q(x) = \frac{\beta}{|x|^2}$, by equation (6.1), we have the following equation for $u(r)$:

$$(-\Delta + q)u = -u'' - \frac{n-1}{r}u' + \frac{\beta}{r^2}u = 0 \tag{6.2}$$

This is an Euler equation, and if we set $u(r) = r^s$, we can obtain the characteristic equation

$$s^2 + (n-2)s - \beta = 0, \tag{6.3}$$

which is quadratic.

Case1 : $\mathbf{n = 2}$

When $n = 2$, (6.3) becomes $s^2 - \beta = 0$.

Subcase 1.1 : $\beta > 0$.

In this case we get two linearly independent solutions $u_1(r) = r^{\sqrt{\beta}}$ and $u_2(r) = r^{-\sqrt{\beta}}$.

• If we impose the Dirichlet boundary condition, then we have

$$u(r) = r^{\sqrt{\beta}} - r^{-\sqrt{\beta}},$$

which is a positive solution and tends to infinity when $r \to \infty$.

• If we impose the Neumann boundary condition, then we have

$$u(r) = \frac{1}{2}r^{\sqrt{\beta}} + \frac{1}{2}r^{-\sqrt{\beta}},$$

which is also a positive solution and tends to infinity when $r \to \infty$.

Subcase 1.2 : $\beta = 0$.

In this special case, equation (6.2) becomes

$$u'' + \frac{u'}{r} = 0,$$

and we have the two solutions $u_1(r) = \ln r$ and $u_2(r) = 1$.

• If we impose the Dirichlet boundary condition, then we have

$$u(r) = \ln r,$$

which is also a positive solution and tends to infinity when $r \to \infty$.

• If we impose the Neumann boundary condition, then we have

$$u(r) \equiv 1,$$

which is also a positive solution yet bounded when $r \to \infty$.

Subcase 1.3 : $\beta < 0$.

In this case, $s^2 - \beta = 0$ has complex roots.In the usual way we find two real solutions: $u_1(r) = \cos(\sqrt{-\beta}\ln r)$ and $u_2(r) = \sin(\sqrt{-\beta}\ln r)$.

• If we impose the Dirichlet boundary condition, then we have

$$u(r) = u_2(r) = \sin(\sqrt{-\beta}\ln r),$$

which is not a positive solution.

• If we impose the Neumann boundary condition, then we have

$$u(r) = u_1(r) = \cos(\sqrt{-\beta}\ln r),$$

which is also not a positive solution either.

Case2 : $\mathbf{n \geq 3}$

The reason why we separate the case $n = 2$ and the case $n \geq 3$ lies in the different property of the positive solution(bounded or not), which is related to the property subcriticality(criticality) (See [41] for more details). Here we set $s_0 = (n-2)^2 + 4\beta$.

Subcase 2.1 : $\beta > -\frac{(n-2)^2}{4}$.

The two basic solutions are $u_1(r) = r^{\frac{-n+2+\sqrt{s_0}}{2}}$ and $u_2(r) = r^{\frac{-n+2-\sqrt{s_0}}{2}}$.

• If we impose the Dirichlet boundary condition, then we have

$$u(r) = r^{\frac{-n+2+\sqrt{s_0}}{2}} - r^{\frac{-n+2-\sqrt{s_0}}{2}},$$

which is a positive solution. In this case the sign of β plays some role: positive β gives an unbounded solution and non-positive β gives a bounded solution.

• If we impose the Neumann boundary condition, then we have

$$u(r) = (n-2+\sqrt{s_0})r^{\frac{-n+2+\sqrt{s_0}}{2}} + (-n+2+\sqrt{s_0})r^{\frac{-n+2-\sqrt{s_0}}{2}},$$

which is also a positive solution, and positive β gives an unbounded solution while non-positive β gives a bounded solution.

Subcase 2.2 : $\beta = -\frac{(n-2)^2}{4}$.

The two basic solutions are $u_1(r) = r^{\frac{-n+2}{2}}$ and $u_2(r) = r^{\frac{-n+2}{2}}\ln r$

• If we impose the Dirichlet boundary condition, then we have

$$u(r) = u_2(r) = r^{\frac{-n+2}{2}}\ln r,$$

which is a bounded positive solution.

• If we impose the Neumann boundary condition, then we have

$$u(r) = r^{\frac{-n+2}{2}}(1 + \frac{n-2}{2}\ln r),$$

which is also a bounded positive solution.

Subcase 2.3 : $\beta < -\frac{(n-2)^2}{4}$.

The two basic solutions are $u_1(r) = r^{\frac{-n+2}{2}}\cos\Big(\frac{\sqrt{-s_0}}{2}\ln r\Big)$ and $u_2(r) = r^{\frac{-n+2}{2}}\sin\Big(\frac{\sqrt{-s_0}}{2}\ln r\Big)$.

• If we impose the Dirichlet boundary condition, then we have

$$u(r) = u_2(r) = r^{\frac{-n+2}{2}}\sin\Big(\frac{\sqrt{-s_0}}{2}\ln r\Big).$$

Similar to the case $n = 2$, this solution is not positive.

• If we impose the Neumann boundary condition, then we have

$$u(r) = u_1(r) + \frac{n-2}{\sqrt{-s_0}}u_2(r),$$

which is also not positive.

Remark 6.1.1. *As mentioned in Chapter 4, when* $q(x) = \frac{\beta}{|x|^2}$, *if there exists a profile* h, *then condition (4.5) is satisfied, and the measure* $h^2 d\mu$ *is volume doubling.*

6.1.2 Examples on $\mathbb{R}^n_+$

Throughout this subsection, we assume $U = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_n > 0\}$ and q is a bounded function of x_n. Here the solution $u(x)$ satisfies:

$$u'' - q(x_n)u = 0$$

where $u'' = \frac{\partial^2 u}{\partial x_n^2}$.

Case1 : $\mathbf{q(x_n) \equiv const}$

Subcase 1.1: $q(x_n) \equiv 1$ In this case, we have the following differential equation

$$u'' - u = 0.$$

There are two basic solutions: $u_1(x) = e^{x_n}$ and $u_2(x) = e^{-x_n}$.

• The Dirichlet-type solution is

$$u(x) = e^{x_n} - e^{-x_n}.$$

and

• the Neumann-type solution is

$$u(x) = e^{x_n} + e^{-x_n}.$$

Both are positive solutions with exponential increasing rate as $x_n \to \infty$.

Remark 6.1.2. *Here the profile is* $h(x) = e^{x_n} - e^{-x_n}$*. The condition (4.5) will be violated and the measure* $h^2 d\mu$ *is not doubling either.*

Subcase 1.2: $q(x) \equiv -1$

In this case the two basic solutions are $u_1(x) = \cos(x_n)$ and $u_2(x) = \sin(x_n)$. $u(x) = u_2(x)$ satisfies the Dirichlet boundary condition while $u(x) = u_1(x)$ satisfies the Neumann boundary solution. However, these are not positive solutions.

Case2 : $\mathbf{q(x) = \frac{\beta}{(1+x_n)^4}}$

Subcase 2.1: $\beta > 0$

There are two basic solutions: $u_1(x) = (x_n+1)e^{\frac{\sqrt{\beta}x_n}{x_n+1}}$ and $u_2(x) = (x_n+1)e^{-\frac{\sqrt{\beta}x_n}{x_n+1}}$.

• The Dirichlet-type solution is

$$u(x) = (x_n+1)\Big(e^{\frac{\sqrt{\beta}x_n}{x_n+1}} - e^{-\frac{\sqrt{\beta}x_n}{x_n+1}}\Big).$$

which is positive and furthermore, $u(x)$ is of linear increasing rate when $x_n \to \infty$. Thus the condition (4.5) is satisfied and the measure $h^2 d\mu$ satisfies the volume doubling property.

• The Neumann-type solution is

$$u(x) = (\sqrt{\beta}-1)(x_n+1)e^{\frac{\sqrt{\beta}x_n}{x_n+1}} + (\sqrt{\beta}+1)(x_n+1)e^{-\frac{\sqrt{\beta}x_n}{x_n+1}}$$

which is a positive solution. Like the Dirichlet case, this solution is also increasing with linear rate!

Subcase 2.2: $\beta < 0$

In this case the two basic solutions are: $u_1(x) = (x_n+1)\cos\frac{\sqrt{-\beta}}{x_n+1}$ and $u_2(x) = (x_n+1)\sin\frac{\sqrt{-\beta}}{x_n+1}$.

• The Dirichlet-type solution is

$$u(x) = (x_n+1)\sin\frac{\sqrt{-\beta}x_n}{x_n+1}.$$

The positivity of $u(x)$ depends on the choice of β, and we make the following observation:

– When $\beta \in (-\pi^2, 0)$, $\frac{\sqrt{-\beta}x_n}{x_n+1} \in (0,\pi)$, so the *sin* part will always be positive. Thus we get a positive solution. In addition, $u(x)$ will be of linear increasing rate when $x_n \to \infty$.

– When $\beta = -\pi^2$, we again get a positive solution, but $u(x)$ will be bounded when $x_n \to \infty$.

– When $\beta \in (-\infty, -\pi^2)$, the solution will not be positive.

As a result, whenever $\beta \geq -\pi^2$, the condition (4.5) is satisfied and the measure $h^2 d\mu$ satisfies the volume doubling property.

• The Neumann-type solution is

$$u(x) = (x_n + 1)\sqrt{1 - \frac{1}{\beta}}\cos(\frac{\sqrt{-\beta}x_n}{x_n + 1} + \gamma)$$

where $\gamma = \tan^{-1}(\frac{1}{\sqrt{-\beta}})$.

The positivity of $u(x)$ also depends on the choice of β. By calculation we know there exists $\beta_0 < 0$ such that

– When $\beta_0 < \beta < 0$, $u(x)$ is positive and of linear increasing rate when $x_n \to \infty$.

– When $\beta = \beta_0$, we again get a positive solution, but $u(x)$ will be bounded when $x_n \to \infty$.

– When $\beta < \beta_0$, the solution will not be positive.

However, unlike all previous examples, $\beta_0 \neq -\pi^2$ in this case. Careful calculation tells us β_0 is around $-(\frac{\pi}{4})^2$.

We end this section by some remarks on these examples.

Remark 6.1.3. *(1). We introduced three types of potentials in this section: constant one(1 & -1), quadratic decaying one($\frac{\beta}{|x|^2}$) and faster-than-quadratic decaying one($\frac{\beta}{(1+x)^4}$), all of which belong to the Kato class K and the fast decaying one belongs to K^∞ by Proposition 5.1.1.*

(2). Among all these three examples, the positive solution only exists for some range: as

the negative part of the potential gets larger and larger, the positive solution will no longer exist beyond certain threshold.
(3). The constant potential gives, if any, exponentially increasing positive solution. If we view it as a profile as we did in Chapter 4, the weighted volume V_{h^2} will violate the volume doubling property. Thus we can not apply the result in Theorem 4.3.1 .
(4). Even though fast decaying potential belongs to K^∞, the example here does not coincide to the conclusion in Chapter 5 since the positive solution is not bounded. This is because $\mathbb{R}^n_+$ is not the exterior of a compact set. We will raise some open questions related to the positive solution behavior of such potentials in next section.

6.2 Examples on certain model manifolds

We introduced some examples in $\mathbb{R}^n$ in last section. Actually we can generalize these examples to a more general setting: model manifolds.
Throughout this section we let the metric space (X, μ) be an n-dimensional Riemannian manifold $(M, \mathbf{g})$. A *Riemannian Model* is such a Riemannian manifold that satisfies:

(1) There is a chart on M that covers all M, and the image of this chart in $\mathbb{R}^n$ is a ball $B_{r_0} := \{x \in \mathbb{R}^n : |x| < r_0\}$ for $r_0 \in (0, +\infty]$;

(2) The metric $\mathbf{g}$ in the polar coordinates (r, θ) in the above chart has the form

$$\mathbf{g} = dr^2 + \psi^2(r)\mathbf{g}_{\mathbb{S}^{n-1}}, \tag{6.4}$$

where $\psi(r)$ is a smooth positive function on $(0, r_0)$.

The following lemma gives the Riemannian measure and the Laplace operator representation in polar coordinates for a Riemannian model. We omit the proof and refer the reader to Section 3.10 in [20] for more details.

Lemma 6.2.1. *On a manifold (M, g) with metric (6.4), the Riemannian measure ν is given in the polar coordinates by*

$$d\nu = \psi(r)^{n-1} dr d\theta, \tag{6.5}$$

where $d\theta$ stands for the Riemannian measure on $\mathbb{S}^{n-1}$, and the Laplace operator on (M, g) has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \left(\frac{d}{dr}\log\psi^{n-1}\right)\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_{\mathbb{S}^{n-1}}. \tag{6.6}$$

Throughout this section we suppose $\psi(r)$ is a smooth positive function and $\psi(r) = r^s$ when $r \geq 1$. In addition, the following conditions need to be satisfied in order for $\mathbf{g}$ to be extended to a metric (See [33]):

$$\psi(0) = 0, \quad \psi'(0) = 1 \quad \text{and} \quad \psi''(0) = 0.$$

By Proposition 4.10 in [25], we know $(M, \mathbf{g})$ is of Harnack-type if and only if $-\frac{1}{n-1} < s \leq 1$.

In order for $(M, \mathbf{g})$ to be non-parabolic, we have the following criteria(See [23]):

Lemma 6.2.2. *If (M, g) be a model manifold which is of Harnack-type, then M is non-parabolic if and only if*

$$\int^{\infty} \frac{ds}{V(x, \sqrt{s})} < +\infty.$$

Together with the Riemannian measure ν given in (6.5), we obtain M is non-parabolic if and only if $s > \frac{1}{n-1}$ for $n \geq 2$. Thus a non-parabolic Harnack-type manifold M with measure $\psi(r) = r^s$ for $r \geq 1$ satisfies $\frac{1}{n-1} < s \leq 1$ for $n \geq 3$. (We rule out the case $n = 2$ since no suitable s exists in this case.)

In the rest of this section we always assume $\frac{1}{n-1} < s \leq 1$ and $n \geq 3$. Notice that if we assume $\frac{1}{n-1} < s \leq 1$, condition (5.3) is also satisfied.

Theorem 6.2.3. *Let (M, g) be a non-parabolic Harnack-type model manifold with $\psi(r) = r^s$ when $r \geq 1$ and $U = M \setminus \overline{B(0,1)}$. Let $q(x) = \frac{\beta}{|x|^\alpha}(\alpha > 2)$, and consider the Schrödinger operator $L = -\Delta + q$. Then there exists $\beta_{0,D} < 0$ such that*

– When $\beta > \beta_{0,D}$, there exists a positive solution $u(x)$ satisfying $\lim_{x \to \partial U} u(x) = 0$ and $\lim_{x\to\infty} u(x) = c > 0$ for some constant c;

– When $\beta = \beta_{0,D}$, there exists a positive solution $u(x)$ satisfying $\lim_{x \to \partial U} u(x) = 0$ and $\lim_{x\to\infty} u(x) = 0$;

– When $\beta < \beta_{0,D}$, there exists no positive solution.

Remark 6.2.1. *By Theorem 5.4.6 we know L is subcritical if and only if $\beta > \beta_{0,D}$; L is critical if and only if $\beta = \beta_{0,D}$ and L is supercritical if and only if $\beta < \beta_{0,D}$.*

Remark 6.2.2. *If we replace $B(0,1)$ in the theorem by any compact set containing the origin by drawing one inner ball contained in this domain and one outer ball containing this domain, then the same argument and same conclusion follow.*

Proof. We prove the theorem in two steps: Firstly we prove that there exists a $\beta_1 < 0$ such that for all $\beta > \beta_1$, the desired positive solution exists; secondly we prove that there exists a $\beta_2 < 0$ such that for all $\beta < \beta_2$, there exists no positive solution. Then by the Lemma 5.3.3, 5.3.4, 5.3.5 and Theorem 5.4.6 we obtain the desired conclusion.

By (6.6) we know in polar coordinates

$$Lu = -u'' - \frac{s(n-1)}{r}u' + \frac{\beta}{r^\alpha}u$$

In particular, when $\alpha = 2$ we have a similar differential equation and a similar quadratic equation as in Section 6.1:

$$t^2 + (sn - s - 1)t - \beta = 0$$

Letting the discriminant be 0, we note that

$$\beta_1 = -\frac{(sn - s - 1)^2}{4} < 0,$$

since $\frac{1}{n-1} < s \leq 1$ implies $0 < sn - s - 1 \leq n - 2$.

By the definition of subcriticality, for any $\beta > \beta_1$, the potential $\frac{\beta}{|x|^2}$ is subcritical, and thus by Theorem 5.2.1, $\frac{\beta}{|x|^\alpha}(\alpha > 2)$ is also subcritical. Then by the Main Theorem we have proven the first part.

On the other hand, choose $\beta_2' < \beta_1$ and $q = \frac{\beta_2'}{|x|^2}$ in $1 \leq |x| \leq 10$ and 0 otherwise. A simple calculation shows that no positive solution exists in this case. Then we can choose suitable $c > 0$ such that $q_2 = \frac{c\beta_2'}{|x|^\alpha}(\alpha > 2)$ is less than $\frac{\beta_2'}{|x|^2}$. By a similar argument we know q_2 is supercritical. Hence $\beta_2 = c\beta_2'$ is what we want, as mentioned at the start of the proof. □

Theorem 6.2.4. *Let (M, g) be a non-parabolic Harnack-type model manifold with $\psi(r) = r^s$ when $r \geq 1$ and $U = M \backslash \overline{B(0,1)}$. Let $q(x) = \frac{\beta}{|x|^\alpha}$ and the Schrödinger operator $L = -\Delta + q$.*

Then there exists $\beta_{0,N} < 0$ such that a similar result can be obtained as Theorem 6.2.3. Moreover, $\beta_{0,D} < \beta_{0,N}$.

Proof. We run the same argument as in Theorem 6.2.3 to prove the first part.
The second part comes from the shuttle operator in Chapter 5. We constructed the Dirichlet-type shuttle operator there, a similar construction for the Neumann-type can also be obtained, and the only difference comes from the kernel of the shuttle operator. As the kernel in Lemma 5.3.1 can be expressed in terms of Possion kernel (See [64]), and the Possion kernel comes from the heat kernel of the stochastic process, thus the Neumann-type kernel is strictly greater than that for the Dirichlet-type for negative potentials. As a result, the spectrum satisfies $\lambda_{q,D} < \lambda_{q,N}$. The conclusion follows if we apply the Theorem 5.4.6 for the Dirichlet version and the Neumann version respectively. □

Remark 6.2.3. *The potentials in Theorem 6.2.3 and Theorem 6.2.4 don't change the sign for a fixed β. In fact, we can obtain the same conclusions even if $q(x)$ changes the sign. To be specific, let $q(x) = q(|x|) \in K^\infty$ be any potential which is positive (flip the sign of $q(x)$ if it is negative) in an open set $V \subset U$. Then there exists a negative β_0 such that whenever $\beta < \beta_0$, $L = -\Delta + \beta q(x)$ doesn't admit a positive solution in U. To prove it, it suffices to compare $\beta q(x)$ with $q_0(x) = -C (C > 0)$ in V: we can always let C be big enough such that $L_0 = -\Delta + q_0$ doesn't admit a positive solution. Then we can choose a suitable β_0 such that $\beta_0 q(x) < -C$ in V. By the definition of subcriticality we know $L = -\Delta + \beta q(x)$ doesn't admit a positive solution in V for any $\beta < \beta_0$. Therefore, there is no positive solution for L in U.*

Till now we have a thorough investigation for the potential class K^{∞} in an abstract sense. However, these conclusions are all in the non-parabolic space. The parabolic case seems more subtle. M. Murata gives two theorems in [41] to discuss the positive solution asymptotic behavior in $\mathbb{R}^1$ and $\mathbb{R}^2$ as follows:

Theorem 6.2.5. *Suppose that $L = -\Delta + q$ in $\mathbb{R}^2$ with q a fast-decaying potential as defined in Lemma 5.1.1. Then we have:*

(1). L is subcritical if and only if $Lu = 0$ has a positive solution u such that

$$u(x) = \log|x|/2 + O(1)$$

as $|x| \to \infty$.

(2). L is critical if and only if $Lu = 0$ has a positive solution such that

$$u(x) = 1 + O(|x|^{-1})$$

as $|x| \to \infty$.

Theorem 6.2.6. *Suppose that $L = -\Delta + q$ in $\mathbb{R}^1$ with q a fast-decaying potential as defined in Lemma 5.1.1. Then we have:*

(1). L is subcritical if and only if $Lu = 0$ has a positive solution u such that

$$u(x) = |x| + O(1)$$

as $|x| \to \infty$.

(2). L is critical if and only if $Lu = 0$ has a positive solution such that

$$u(x) = 1 + O(|x|^{-1})$$

as $|x| \to \infty$.

In that paper M. Murata also gives the case in $\mathbb{R}^n (n \geq 3)$ which we've already covered in Theorem 5.4.6. A natural open question would be: can these conclusions in $\mathbb{R}^1$ and $\mathbb{R}^2$ also be generalized to the Dirichlet positive solution estimate for any abstract parabolic Harnack-type space?

Another related question would be the Dirichlet heat kernel estimate for the operator $L = -\Delta + q$ in the domain above the graph of a nice function in $\mathbb{R}^n$. Although it could be in any dimension, its behavior is much like the one-dimensional case. R. Song proved in [51] that for any domain U in $\mathbb{R}^n$ which is above the graph of a bounded $C^{1,1}$ function, the Dirichlet heat kernel $p^D(t, x, y)$ induced by the Laplacian satisfies the following global estimate:

$$C_1\left(\frac{\rho(x)\rho(y)}{t} \wedge 1\right)t^{-n/2}e^{-\frac{C_2|x-y|^2}{t}} \leq p_U^D(t, x, y) \leq C_3\left(\frac{\rho(x)\rho(y)}{t} \wedge 1\right)t^{-n/2}e^{-\frac{C_4|x-y|^2}{t}},$$

where $\rho(x)$ stands for the distance between x and ∂U. Here we can roughly view $\rho(x)$ as the profile of $-\Delta$ in some sense. It's very similar to the positive solution specified in Theorem 6.2.6. Then a natural open question is as follows:

Q: Let U be the domain above a bounded $C^{1,1}$ function in $\mathbb{R}^n$. Let q be the potential belonging to K^∞. Assume $-\Delta + q$ is subcritical, then the heat kernel $p_{q,U}^D(t, x, y)$ induced by the operator $-\Delta + q$ satisfies the following global estimate:

$$C_1\left(\frac{\rho(x)}{\sqrt{t}} \wedge 1\right)\left(\frac{\rho(y)}{\sqrt{t}} \wedge 1\right)t^{-n/2}e^{-\frac{C_2|x-y|^2}{t}} \leq p_{q,U}^D(t, x, y) \leq C_3\left(\frac{\rho(x)}{\sqrt{t}} \wedge 1\right)\left(\frac{\rho(y)}{\sqrt{t}} \wedge 1\right)t^{-n/2}e^{-\frac{C_4|x-y|^2}{t}}.$$

6.3 The story when K is not compact

Let $U = X \setminus K$. In previous sections we have a classification of the potential behaving like $\frac{\beta}{|x|^\alpha}$, which tells us in the exterior of a compact set K, both the decaying rate α and the coefficient β are important. Is the story the same if K is non-compact? As we discussed in Section 6.1 for the potential $q(x) = \frac{\beta}{(1+x_n)^4}$ in $\mathbb{R}^n_+$, the answer is NO! That helps us understand why we require K to be compact in Chapter 5. We present two more examples in this section: one is again in the upper-half space and the other is the domain outside a cylinder.

In the first example we assume the potential term q is bounded and symmetric rotationally. We set $U = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_n > 0\}(n \geq 3)$.
It is easily to see that there exists a unique solution g_1(compare it with formula (6.2)!) of the initial value problem

$$-g_1''(r) - \frac{n-1}{r}g_1'(r) + \left[\frac{n-1}{r^2} + q(r)\right]g_1(r) = 0 \quad \forall r \in (0, \infty) \tag{6.7}$$

$$g_1(r) = r + o(r^2) \tag{6.8}$$

We have the following theorem from [41]:

Theorem 6.3.1. *(i). The operator $L \geq 0$ in $\mathbb{R}^n_+$ if and only if $g_1(r) > 0$ for any $r > 0$.*

(ii). q is subcritical in $\mathbb{R}^n_+$ if and only if $g_1(r) > 0$ for $r > 0$ and

$$\int_1^\infty r^{1-n} g_1(r)^{-2} dr < \infty. \tag{6.9}$$

(iii). q is critical in $\mathbb{R}^n_+$ if and only if $g_1(r) > 0$ for $r > 0$ and the left hand side of (6.9)

diverges. In this case, any positive solution is a constant multiple of

$$u(x) = g_1(|x|)x_n/|x|. \tag{6.10}$$

Till now all the potentials we have discussed are in terms of the distance to the boundary. Here we will give a different one in $\mathbb{R}^n_+$:

Example 6.3.1. *Assume*

$$q(x) = \begin{cases} 0, & |x| \leq 1; \\ \dfrac{\beta}{|x|^2}, & |x| > 1; \end{cases}$$

By simple calculation we have, when $r < 1$, $g_1(r) = r$;

When $r \geq 1$ we set $g_1(r) = r^s$ and plug it into (6.7), obtaining

$$s^2 + (n-2)s - (\beta + n - 1) = 0.$$

Again we have three cases.

(1). $\beta = -\frac{n^2}{4}$.

In this case we have

$$g_1(r) = \begin{cases} r, & r \leq 1; \\ r^{-\frac{n-2}{2}} + \dfrac{n}{2} r^{-\frac{n-2}{2}} \ln r, & r > 1; \end{cases}$$

This g_1 doesn't satisfy condition (6.9), and so it is critical.

The solution here is

$$u(x) = \begin{cases} x_n, & |x| \leq 1; \\ (|x|^{-\frac{n}{2}} + \dfrac{n}{2}|x|^{-\frac{n}{2}} \ln |x|)x_n, & |x| > 1, \end{cases}$$

and $u(x)$ tends to 0 when $x_n \to \infty$.

(2). $\beta > -\frac{n^2}{4}$.

We have

$$g_1(r) = \begin{cases} r, & r \le 1; \\ ar^{\frac{-n+2+\sqrt{n^2+4\beta}}{2}} + br^{\frac{-n+2-\sqrt{n^2+4\beta}}{2}}, & r > 1, \end{cases} \tag{6.11}$$

where $a = \frac{\sqrt{n^2+4\beta}+n}{2\sqrt{n^2+4\beta}}$ and $b = \frac{\sqrt{n^2+4\beta}-n}{2\sqrt{n^2+4\beta}}$.

The Dirichlet-type solution is

$$u(x) = \begin{cases} x_n, & |x| \le 1; \\ (a|x|^{\frac{-n+\sqrt{n^2+4\beta}}{2}} + b|x|^{\frac{-n-\sqrt{n^2+4\beta}}{2}})x_n, & |x| > 1. \end{cases} \tag{6.12}$$

This positive solution tends to ∞ with polynomial growing rate or tends to 0 when $x_n \to \infty$ depending on the positivity of β.

(3). $\beta < -\frac{n^2}{4}$.

Similar to the case outside the unit ball, there is no positive solution.

The more interesting case is when q decays faster than quadratical one. Namely, we let

$$q(x) = \begin{cases} 0, & |x| \le 1; \\ \dfrac{\beta}{|x|^\alpha}, & |x| > 1; \end{cases} \tag{6.13}$$

where $\alpha > 2$. To address this complicated case, we can view the term $\frac{n-1}{r^2} + q(r)$ as the new potential term as we did in the case outside the unit ball. Since the $q(r)$ term can be controlled by the quadratic term in some sense, it allows more flexibility compared to the case $U = \mathbb{R}^n \setminus \overline{B(0,1)}$. See [29, 30] for more details about the behavior of such type of potentials.

The special case in this setting is the zero potential $q(x) \equiv 0$. Then the gauge $u_0(x) = E^x[e_q(\tau_U)] \equiv 1$ is finite. However, the positive Dirichlet-type solution is $u(x) = x_n$, which is not bounded. This example also tells us why we need to assume U to be the exterior of a compact set.

Now we turn to the second example. Here we set $U = \mathbb{R}^n \setminus K (n \geq 3)$ where $K = \{x = (x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq 1\}$.
When $q = 0$ we know the Dirichlet-type solution is

$$u(x) = \begin{cases} \log|x'| & n = 3 \\ 1 - |x'|^{3-n} & n \geq 4, \end{cases}$$

where $x' = (x_1, x_2, \dots, x_{n-1})$.
Here we get some interesting results: the Dirichlet-type solution in dimension n looks exactly the same as that for $\mathbb{R}^{n-1} \setminus \overline{B(0,1)}$. Namely, the positive solution is bounded if and only if $n \geq 4$. Generally, we can let the potential q be any function of the distance to the cylinder, and study the properties of $L = -\Delta + q$ in U just as we did in $\mathbb{R}^{n-1} \setminus \overline{B(0,1)}$.
Generally, it is not difficult to prove the following corollary if we run the same argument as in Chapter 5 and notice the fact $\lim_{\rho(x,\partial U) \to \infty} P^x(\tau_U = \infty) = 1$:

Corollary 6.3.1. *Let $U = \mathbb{R}^n \setminus \overline{E} (n \geq 3)$ and $E = B(0,1)^m \times \mathbb{R}^{n-m} (1 \leq m \leq n)$. Let $L = -\Delta + q$ and $q \in K^\infty$. Then Theorem 5.4.6 holds if $m \geq 3$.*

BIBLIOGRAPHY

[1] H. Aikawa, *Boundary Harnack principle and Martin boundary for a uniform domain*, J. Math. Soc. Japan 53 (2001), 119-145.

[2] M. Aizenman, B. Simon, *Brownian motion and Harnack inequality for Schrdinger operators*, Comm. Pure. Appl. Math 35 (2) (1982), 209-273.

[3] A. Ancona, *Principle de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien*, Ann. Inst. Fouries (Grenoble) 28 (1978), 169-213.

[4] A. Bendikov and L. Saloff-Coste, *On- and off-diagonal heat kernel behaviors on certain infinite dimensional local Dirichlet spaces*, Amer. J. Math 122 (2000), 1205-1263.

[5] R. Blumenthal and R. Getoor, *Markov processes and potential theory*, Academic Press, New York, 1968.

[6] M. Bramanti, *Potential theory for stationary Schrödinger operators: A survey of results obtained with non-probabilistic methods*, Le Matematiche. Vol XL VII (1992), 25-61.

[7] Z. Q. Chen, *Gaugeability and conditional gaugeability*, Trans. Amer. Math. Soc, 354 (2002), 4639-4679.

[8] Z. Q. Chen and M. Fukushima. *Symmetric Markov Processes, Time Changes, and Boundary Theory*, Pinceton Univesrity Press (2012).

[9] Z.-Q. Chen and R. Song. *Estimates on Green functions and Poisson kernels of symmetric stable processes*, Math. Ann., 312 (1998), 465-501.

[10] P. Chen, R. Song, L. Xie and Y. Xie. *Heat kernel estimates for Dirichlet fractional Laplacian with gradient perturbation*, J. Korean Math. Soc. Vol. 56 (2019), 91-111.

[11] K. L. Chung and Z. Zhao, *From Brownian motion to Schrödinger equation*, Springer, New York, 1994.

[12] K. L. Chung and M. Rao, *Feynman-Kac functional and Schrödinger equation*, Sem. Stoch. Proc., Birkh*ä*user, Boston, Mass., 1981.

[13] M. Cranston, E. Fabes and Z. Zhao, *Conditional gauge and potential theory for the Schrödinger operator*, Trans. Amer. Math. Soc. 307 (1988), 171-194.

[14] E. B. Davies, *Heat kernel and spectral theory*, Cambridge Univ. Press (1989).

[15] E. B. Davies, *Intrinsic ultracontractivity and the Dirichlet Laplacian*, J. Funct. Anal. 100 (1991), 163-180.

[16] E. B. Davies and B. Simons, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. 59 (1984), 335-395.

[17] M. Fukushima, Y. $\overline{\text{O}}$shima and M. Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter Studies in Mathematics, Vol 19 (1994).

[18] F. Gesztesy and Z. Zhao, *On positive solutions of critical Schrödinger operators in two dimensions*, J. Funct. Anal. 127 (1995), 235-256.

[19] A. Grigor'yan, *Heat kernel on weighted manifolds and applications*, AMS Special Session-The Ubiquitous Heat Kernel, October 2-4 (2003), 93-192.

[20] A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, American Mathematical Society, Providence, RI; International Press, Boston, MA, (2009).

[21] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. (1999).

[22] A. Grigor'yan, *The heat equation on noncompact Riemannian manifolds*, Mat. Sb. 182 (1991), 55-87.

[23] A. Grigor'yan and L. Saloff-Coste, *Dirichlet heat kernel in the exterior of a compact set*, Comm. Pure Appl. Math. 55 (2002), 93-133.

[24] A. Grigor'yan and L. Saloff-Coste, *Hitting probabilities for Brownian motion on Riemannian manifolds*, J. Math. Pure Appl. 81 (2002), 115-142

[25] A. Grigor'yan and L. Saloff-Coste, *Stability results for Harnack inequalities*, Ann. Inst. Fourier, Grenoble, 55 (2005) No.3, 825-890.

[26] A. Grigor'yan and L. Saloff-Coste, *Heat kernel on manifolds with ends*, Ann. Inst. Fourier, Grenoble, 59 (2009), No.5, 1917-1997.

[27] P. Gyrya and L. Saloff-Coste, *Neumann and Dirichlet heat kernels in inner uniform domains*, Astérisque 336, Soc. Math. de France, 2011.

[28] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. 145 (2000).

[29] N. Ioku, K. Ishige and E. Yanagida, *Sharp decay estimates of L^p-norms for non-negative Schrödinger heat semigroups*, J. Funct. Anal. 264 (2013), 2764?2783.

[30] K. Ishige and A. Mukai, *Large time behavior of solutions of the heat equation with inverse square potential*, preprint (arXiv:1709.00809).

[31] D. S. Jerison and C. E. Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Adv. Math. 46 (1982) 80-147.

[32] T. Kato, *Schrödinger operators with singular potentials*, Israel J. Math. 13 (1973), 135-148.

[33] J. L. Kazdan and F. W. Warner, *Prescribing curvatures*, Proceedings of Symposia in Pure Mathematics, 27 (1975), 309-319.

[34] K. Kuwae and M. Takahashi, *Kato class measures of symmetric Markov processes under heat kernel estimates*, Journal of Functional Analysis 250 (2007), 86-113.

[35] P. Li and S. T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta. Math. Springer (1986).

[36] J. Lierl and L. Saloff-Coste, *Scale-invariant boundary Harnack principle in inner uniform domains*, Osaka J. Math. 51 (2014) No. 3, 619?656.

[37] J. Lierl, *Scale-invariant boundary Harnack principle on inner uniform domains in fractal-type spaces*, Potential Analysis. 43 (2015) No. 4, 717?747

[38] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. 14 (1961), 577-591.

[39] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. 17 (1964), 101-134.

[40] M. Murata, *Positive solutions and large time behaviors of Schrödinger semigroups, Simon's problem*, J. Funct. Anal. 56 (1984), 300-310.

[41] M. Murata, *Structure of positive solutions to* $(\Delta + V)u = 0$ *in* $\mathbb{R}^n$, Duke Math. J. Vol 53 (1986) No. 4.

[42] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. 80 (1958), 931-954.

[43] Y. Pinchover, *Criticality and ground states for second order elliptic equations*, J. Differential Equations 80 (1989), 237-250.

[44] L. Saloff-Coste, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, Vol 289, Cambridge Univ. Press (2002).

[45] L. Saloff-Coste, *Parabolic Harnack inequality for divergence-form second-order differential operators*, Potential Anal. 4 (1995), 429-467.

[46] M. Schechter, *Spectra of partial differential operators*, North-Holland, Amsterdam, (1971).

[47] B. Simon, *Large time behavior of the L^p norm of Schrödinger semigroups*, J. Funct. Anal. 40 (1981), 66-83.

[48] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. 7 (1982), 447-526.

[49] B. Simon, *Essential self-adjointness of Schrdinger operators with positive potentials*, Math. Annalen. Vol 201, Issue 3 (1973), 211-220.

[50] B. Simon, *Schrödinger operators in the twentieth century*, J. Math. Phy. Vol 41, No. 6, (2000).

[51] R. Song, *Estimates on the Dirichlet heat kernel of domains above the graphs of bounded $C^{1,1}$ functions*, Glas. Mat. Ser. III 39(59) (2004), 273-286.

[52] K. T. Sturm, *Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties*, J. reine angew. Math. 456 (1994), 173-196.

[53] K. T. Sturm, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math. 32 (1995), 275-312.

[54] K. T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. 75 (1996), 273-297.

[55] M. Takeda, *Gaussian bounds of heat kernels for Schrödinger operators on Riemannian manifolds*, Bull. London Math. Soc., 39 (2007), 85-94.

[56] M. Takeda, *Some Topics connected with Gaugeability for Feynman-Kac Functionals*, Proceedings of RIMS Workshop on Stochastic Analysis and Applications, (2008).

[57] J. Väisälä, *Uniform domains*, Tohoku Mathematical Journal, Second Series, 40 (1988), 101-118.

[58] J.-M. G. Wu, *Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains*, Ann. Inst. Fourier, 28 (1978), 147-167.

[59] Q. S. Zhang, *Large time behavior of Schrödinger heat kernels and applications*, Commun. Math. Phys. 210 (2000), 371-398.

[60] Q. S. Zhang, *Global bounds of Schrödinger heat kernels with negative potentials*, J. Funct. Anal. 182 (2001), 344-370.

[61] Q. S. Zhang, *The boundary behavior of heat kernels of Dirichlet Laplacian*, J. Differential Equations. 182 (2002), 416-430.

[62] Q. S. Zhang, *The global behavior of heat kernels in exterior domains*, J. Funct. Anal. 200 (2003), 160-176.

[63] Z. Zhao, *Subcriticality, positivity and gaugeability of the Schrödinger operators*, Bull. Amer. Math. Soc., Vol 23 (1990), 513-517.

[64] Z. Zhao, *Subcriticality, and gaugeability of the Schrödinger operator*, Trans. Amer. Math. Soc. 334 (1992), 75-96.

[65] Z. Zhao, *Conditional gauge and unbounded potential*, Z. Wahrsch. Verw. Gebiete. 65 (1983), 13-18.

[66] Z. Zhao, *Green function for Schrödinger operator and conditional Feynman-Kac gauge*, J. Math. Anal. Appl. 116 (1986), 309-334.

[67] Z. Zhao, *An equivalence theorem for Schrödinger operators and its applications*, Diffusion Processes and Related Problems in Analysis, Birkhäuser, Boston, Mass., (1990), 245-260.