SOME SPECTRAL IDEAS APPLIED TO FINANCE AND TO SELF-SIMILAR AND LONG-RANGE DEPENDENT PROCESSES

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This dissertation consists of four parts. The aim of the first part is to present original transformations on tractable Markov processes (or equivalently, on their semigroup) in order to make the discounted transformed process a martingale, while keeping its tractability. We refer to such procedures as risk-neutral pricing techniques. To achieve our goal, we resort to the concept of intertwining relationships between Markov semigroups that enables us, on the one hand to characterize a risk-neutral measure and on the other hand to preserve the tractability and flexibility of the models, two attractive features of models in mathematical finance. To illustrate the usefulness of our approach, we proceed by applying this risk-neutral pricing techniques to some classes of Markov processes that have been advocated in the literature as substantial models.

In the second part, we introduce spectral projections correlation functions of a stochastic process which are expressed in terms of the non-orthogonal projections into eigenspaces of the expectation operator of the process and its adjoint. We obtain closed-form expressions of these functions involving eigenvalues, the condition number and/or the angle between the projections, along with their large time asymptotic behavior for three important classes of processes: general Markov processes, Markov processes subordinated in the sense of Bochner and non-Markovian processes which are obtained by time-changing a Markov process with an inverse of a subordinator. This enables us to provide a unified and original framework for designing statistical tests that investigate

critical properties of a stochastic process, such as the path properties of the process (presence of jumps), distance from symmetry (self-adjoint or non-self-adjoint) and short-to-long-range dependence. To illustrate the usefulness of our results, we apply them to generalized Laguerre semigroups, which is a class of non-self-adjoint and non-local Markov semigroups, and also to their time-change by subordinators and their inverses.

In the third part, we introduce and study non-local Jacobi operators, which generalize the classical (local) Jacobi operator on $[0, 1]$. We show that these operators extend to the generator of an ergodic Markov semigroup with an invariant probability measure β and study its spectral and convergence properties. In particular, we give a series expansion of the semigroup in terms of explicitly defined polynomials, which are counterparts of the classical Jacobi orthogonal polynomials, and give a complete characterization of the spectrum of the non-self-adjoint generator and semigroup in $L^2(\beta)$. We show that the variance decay of the semigroup is hypocoercive with explicit constants which provides a natural generalization of the spectral gap estimate. After a random warm-up time the semigroup also decays exponentially in entropy and is both hypercontractive and ultracontractive. All of our proofs hinge on developing commutation identities, known as intertwining relations, between local and non-local Jacobi operators/semigroups, with the local Jacobi operator/semigroup serving as a reference object for transferring properties to the non-local ones.

In the last part, by observing that the fractional Caputo derivative of order $\alpha \in (0, 1)$ can be expressed in terms of a multiplicative convolution operator, we introduce and study a class of such operators which also have the same self-similarity property as the Caputo derivative. We proceed by identifying a subclass which is in bijection with the set of Bernstein functions and we provide several representations of their eigenfunctions, expressed in terms of the corresponding Bernstein function, that generalize the Mittag-Leffler function. Each eigenfunction turns out to be the Laplace transform of the right-

inverse of a non-decreasing self-similar Markov process associated via the so-called Lamperti mapping to this Bernstein function. Resorting to spectral theoretical arguments, we investigate the generalized Cauchy problems, defined with these self-similar multiplicative convolution operators. In particular, we provide both a stochastic representation, expressed in terms of these inverse processes, and an explicit representation, given in terms of the generalized Mittag-Leffler functions, of the solution of these self-similar Cauchy problems.

BIOGRAPHICAL SKETCH

Anna Srapionyan was born and raised in Yerevan, the stunning capital city of the Republic of Armenia which, being 2800 years old, showcases the beauty and historical values of Armenia. Anna received her Bachelor's degree in Mathematics from Yerevan State University, and her Master of Science in Financial Mathematics degree from The University of Chicago, after which she continued to pursue a doctoral degree at the Center for Applied Mathematics at Cornell University.

During her five years at Cornell, Anna worked with Professor Pierre Patie on risk-neutral pricing techniques, spectral projections correlations structure for short-to-long range dependent processes, on non-local ergodic Jacobi semigroups, and self-similar Cauchy problems. Besides her academic work, she enjoys baking for her friends, hiking in the gorgeous gorges of Ithaca, reading and traveling.

After her graduation, Anna will join Bank of America Merrill Lynch in New York City as a quantitative strategies associate.

This document is dedicated to all my teachers and professors.

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CONTENTS

CHAPTER 1
INTRODUCTION

The spectral theory is an outgrowth of fundamental work of David Hilbert between 1900 and 1910 on the analysis of integral operators on infinite-dimensional spaces - now called Hilbert spaces. However, like almost every important new development in mathematics, it was preceded by much related work, for example Poincaré's analysis of the Dirichlet problem and associated eigenvalues (1890 – 1896). One could maintain that the subject started with the seminal work of Fourier on the solution of the heat equation using series expansions in sines and cosines, which was published by the Académie Française in 1822. In conjuction with the rapid development of quantum mechanics, research into this area had grown. Many of the results brought here are due to mathematicians such as Hilbert himself (who also started working in physics after 1912), John von Neumann and Hermann Weyl, as well as physicists, such as Erwin Schrödinger and Werner Heisenberg, for whom Hilbert spaces played a central role. Spectral theory is an extremely rich field which has been studied by many qualitative and quantitative techniques - for example Sturm-Liouville theory, separation of variables, Fourier and Laplace transforms, perturbation theory, eigenfunction expansions, variational methods, microlocal analysis, stochastic analysis and numerical methods including finite elements. The goal of spectral theory, at its broadest, might be described as an attempt to "classify" all linear operators. Usually, one naturally restricts their attention to Hilbert spaces since it is much easier than the general case of operators on Banach spaces, and secondly, many of the most important applications belong to this simpler setting of operators on Hilbert spaces.

Self-adjoint operators on Hilbert spaces have an extremely detailed theory, and are of great importance for many applications. On the other hand, the theory of non-self-adjoint operators is a young branch of functional analysis and it is very much less unified than the self-adjoint theory. The latter is much easier to analyze because of the existence of the spectral theorem and the fact that one can often use variational methods to obtain tight bounds on eigenvalues, both numerically and theoretically. Although the class of non-self-adjoint and non-local operators is central and generic in the study of linear operators, its spectral analysis is fragmentarily understood due to the fundamental technical difficulties arising when the properties of symmetry and locality are simultaneously relaxed. Nevertheless, there are increasing numbers of problems in physics that require the analysis of non-self-adjoint operators, and thus it has attracted the ever increasing attention of mathematicians, physicists and engineers.

This dissertation consists of five chapters. Besides this introductory chapter, each of the remaining chapters are based on papers submitted to peer-reviewed journals. All these chapters, even though answering to different interesting questions, take a spectral theoretical approach and/or rely on so-called intertwining relations between (not necessarily self-adjoint) linear operators. In particular, Chapter 2 presents an application of the above-mentioned concepts in mathematical finance.

Tractability and flexibility are among the two most attractive features of models in mathematical finance. For the pricing of derivative products, to avoid arbitrage opportunities, the fundamental theorem of asset pricing requires the existence of an equivalent martingale measure under which the discounted price process is a (local) martingale. This risk-neutral probability measure generally differs from its statistical (real-world or physical) counterpart. The latter describes the likelihood of these risky outcomes and is typically estimated from historical time series data on past realizations. The risk-neutral probability, on the other hand, is the market price of Arrow-Debreu securities associated with risky events. The question then arises as to how to construct the risk-neutral measure used to price derivatives. In the semimartingale setting, the traditional approach of risk-neutral valuation uses a change of measure invoking Girsanov's theorem. This approach often destroys the tractability of the process which is undesirable for quantitative finance applications.

Chapter 2 is based on the paper "Risk-neutral pricing techniques and examples", which is a joint work with R. A. Jarrow, P. Patie and Y. Zhao, and which overcomes these limitations. We suggest several transformations on a tractable Markov process (or equivalently, its respective semigroup) in order to make the discounted transformed process a martingale, while keeping its tractability. We refer to such procedures as *risk-neutral pricing transformations*. In particular, we employ a transformation based on the concept of intertwining relationships that allows us to convert Markovian semigroups into a pricing semigroup while keeping its tractability. Under specific circumstances, this method boils down to some special cases such as Doob's h-transform, Dynkin's criterion and the Esscher transform which is well-known in Lévy market models. Furthermore, we provide equivalent conditions for these transformations in terms of extended generators, and we emphasize that one advantage of working with this version of a generator is the fact that it might be well defined for unbounded functions as well, an important feature in mathematical finance since payoff functions, such as the one of European call options, may not be necessarily bounded. In order to illustrate the usefulness of this approach, we apply it to exponential Lévy, positive self-similar, and generalized CIR processes that all fall within the class of polynomial processes introduced by Cuchiero et al. [44]. Moreover, we carry on by providing an explicit eigenvalues expansion for the non-self-adjoint pricing semigroup of a risk-neutral generalized CIR model with jumps. This allows us to obtain analytical formulas for the pricing of derivative products written on this asset model. We also give a detailed analysis of the approximation errors and show the outcomes of numerical experiments.

As in finance, in other fields as well, stochastic processes play an important role in the investigation of random phenomena depending on time. When using a stochastic process for modeling or for statistical testing purposes, one should take into account its special features which indicate how well the process reflects the reality. Notions of covariance and correlation functions have been intensively studied in the statistical literature. For example, the distance correlation coefficient is especially useful for complicated dependence structures in multivariate data, and the maximal correlation coefficient is a convenient numerical measure of dependence between two random variables particularly for its tensorization property, i.e. it is unchanged when computed for i.i.d. copies. However, these statistical measures of dependence do not provide information about some of the most essential features of the process which include (but are not limited to) observing whether the process is Markovian or not, whether its trajectories are continuous or incorporate jumps, what type of range dependence it exhibits, and how far it is from symmetry (self-adjointness).

Chapter 3 is based on the paper "Spectral projections correlation structure for short-to-long range dependent processes", which is a joint work with P.P atie and addresses this question. More formally, let $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ be a stochastic process issued from $x \in \mathbb{R}$ that admits a marginal stationary measure ν, i.e. $\nu\mathbf{P}_t f = \nu f$ for all $t \geq 0$, where $\mathbf{P}_t f(x) = \mathbb{E}_x[f(\mathbf{X}_t)]$. In this chapter we introduce the (resp. biorthogonal) spectral projections correlation functions which are expressed in terms of projections into the eigenspaces of $\mathbf{P}_t$ (resp. and of its adjoint in the weighted Hilbert space $L^2(\nu)$). We obtain closed-form expressions involving eigenvalues, the condition number and/or the angle between the projections in the following different situations: when $\mathbf{X} = X$ with $X = (X_t)_{t\geq 0}$ being a Markov process, $\mathbf{X}$ is the subordination of X in the sense of Bochner, and $\mathbf{X}$ is a non-Markovian process which is obtained by time-changing X with an inverse of a subordinator. It turns out that these spectral projections correlation functions have different expressions with respect to these classes of processes which enables to identify substantial and deep properties about their dynamics. This interesting fact can be used to design original statistical tests to make inferences, for example, about the path properties of the process (presence of jumps), distance from symmetry (self-adjoint or non-self-adjoint) and short-to-long-range dependence. To reveal the usefulness of our results, we apply them to a class of non-self-adjoint Markov semigroups studied in [129], and then time-change by subordinators and their inverses.

The study of time-dependent homogeneous diffusion processes with transition density and invariant distributions from the Pearson family dates from the 1930's. These diffusion processes are popular throughout the applied sciences including physics, biology, biophysics and financial mathematics. The class of Jacobi diffusions, one of the six subfamilies of the Pearson processes, is the most general polynomial diffusion on the unit interval. It is a stochastic diffusion characterized by a linear drift and a special form of multiplicative noise which keeps the process confined between two boundaries. The Jacobi process provides a convenient representation for the evolution of discrete stochastic probability distributions in continuous time. Therefore, it can be applied to continuous time switching regime processes with unobserved regimes, called the smooth transition processes, in which the probabilities of regimes have their own dynamics. In population biology the Jacobi process is well known as Wright-Fisher diffusion with migration studied by Karlin and Taylor [88]. In the finance context, the Jacobi process have been used by Delbaen and Shirakawa [51] to model interest rates, by De Jong et al. [49], and by Larsen and Sørensen [97] to model the exchange rates in a target zone. The Jacobi processes have also been studied by Gourieroux and Jasiak [72], they introduced a multidimensional version and pointed out several applications. The analysis of the spectrum of the infinitesimal generators of the Jacobi processes is simple and purely discrete with classical orthogonal polynomials as corresponding Jacobi polynomials as eigenfunctions. The spectral and convergence properties of this classical Jacobi operator have been well studied over the past few decades. The aim of the next chapter of the dissertation is to generalize the associated classical Jacobi operators on the unit interval, and study in depth the most substantial properties of this non-local operator and its associated non-self-adjoint semigroup.

Chapter 4 is based on the joint work with P. Cheridito, P. Patie and A. Vaidyanathan called "On non-local ergodic Jacobi semigroups: Spectral theory, convergence-to-equilibrium and contractivity". In this chapter we introduce and study non-local Jacobi operators, which generalize the classical (local) Jacobi operator on $[0, 1]$. We show that these operators extend to the generator of an ergodic Markov semigroup with an invariant probability measure β and study its spectral and convergence properties. In particular, we give a series expansion of the semigroup in terms of explicitly defined polynomials, which are counterparts of the classical Jacobi orthogonal polynomials, and give a complete characterization of the spectrum of the non-self-adjoint generator and semigroup in $L^2(\beta)$. We show that the variance decay of the semigroup is hypocoercive with explicit constants which provides a natural generalization of the spectral gap estimate. After a random warm-up time the semigroup also decays exponentially in entropy and is both hypercontractive and ultracontractive. All of our proofs hinge on developing commutation identities, known as intertwining relations, between local and non-local Jacobi operators/semigroups, with the local Jacobi operator/semigroup serving as a reference object for transferring properties to the non-local ones.

The fractional calculus is a name of theory of integrations and derivatives of arbitrary order, which unify and generalize the notion of integer order differentiation and n-fold integration. The fractional calculus is a 300 years old mathematical discipline. In fact and some time after the publication of the studies on Differential Calculus, where he introduced the notation $\frac{d^n}{dx^n}y(x)$, Leibnitz received a letter from Bernoulli putting him a question about the meaning of a non-integer derivative order. Only in the early XIX century, interesting developments started being published. Laplace proposed an integral formulation (1812), but it was Lacroix who used for the first time the designation derivative of arbitrary order (1819). Since the beginning of the nineties of XXth century, the Fractional Calculus attracted the attention of an increasing number of mathematicians, physicians, and engineers that have been supporting its development and originating several new formulations and mainly using it to explain some natural and engineering phenomena and also using it to develop new engineering applications. Namely, applied scientists and engineers realized that differential equations with fractional derivative provided a natural framework for the discussion of various kinds of real problems modeled by the aid of fractional derivative, such as viscoelastic systems, signal processing, diffusion processes, control processing, fractional stochastic systems, allometry in biology and ecology. There are many possible generalizations of the notion of a derivative of a function that would lead to the answer of the question: what is $\frac{d^n}{dx^n}y(x)$ when n is any real number? However, most of the theoretical setup of Fractional Calculus was done by mathematicians that directed their attention preferably to the so-called Riemann-Liouville and Caputo derivatives. These are multistep derivatives that use several integer order derivatives and a fractional integration. Caputo (1967) formulated a definition, more restrictive than the Riemann-Liouville but more appropriate to discuss problems involving a fractional differential equation with initial conditions. The Caputo derivative is of use to modeling phenomena which takes account of interactions within the past and also problems with non-local properties. In this sense, one can think of the equation as having ”memory”.

Furthermore, the fractional Caputo derivative enjoys a self-similarity property which is appealing from a modelling viewpoint as it has been observed in many physical and economics phenomena. Next, it is natural to consider the associated Fractional Cauchy problems which replace the usual first-order time derivative by a fractional derivative. It turns out, that in some specific settings, the solution to this problem has a stochastic representation which is given in terms of a non-Markovian process defined as the Markov process time-changed by the inverse of a subordinator, see e.g. [6], [36], [152]. Then, two questions naturally arise: 1) Can one define a class of linear operators enjoying the same self-similarity property as the fractional Caputo derivative? 2) If yes, can one find a stochastic representation for the solution of the corresponding self-similar Cauchy problem?

Chapter 5 is based on the paper "Self-similar Cauchy problems and generalized Mittag-Leffler functions", a joint work with P. Patie, and it provides a positive and detailed answer to each of these questions. In this chapter, inspired by the self-similarity property of the fractional derivative, we start by identifying a class of self-similar multiplicative convolution operator which is in bijection with the set of Bernstein functions and which encompasses the fractional Caputo derivative as a specific instance. We provide some analytical properties of these operators and in particular we characterize their eigenfunctions expressed as analytical power series that generalize the Mittag-Leffler function. We proceed, using spectral theoretical arguments, by providing a spectral representation of the strong solution of, what we name, the self-similar Cauchy problem which is defined by replacing in the classical Cauchy problem, the first order time derivative with our generalized multiplicative convolution operators. We also show that this solution admits a stochastic representation in terms of the expectation operator associated to a strong Markov process time-changed with the right-inverse of an increasing self-similar Markov process, which is associated by the Lamperti one-to-one mapping to the Bernstein function that identifies the convolution operator. We end up this chapter by providing various examples illustrating our main results including Markov processes associated to normal operators (such as squared Bessel, classical Laguerre and Jacobi semigroups) as well as to non-self-adjoint and non-local operators (such as generalized Laguerre and generalized Jacobi semigroups).

CHAPTER 2

RISK-NEUTRAL PRICING TECHNIQUES AND EXAMPLES

2.1 Introduction

When using a stochastic process to price derivatives, one necessarily refers to the two Fundamental Theorems of Asset Pricing (FTAPs). The first FTAP was suggested by Harrison and Kreps in 1979, and generalized by Harrison and Pliska (1981) as well as Delbaen and Schachermayer (1994). The first FTAP relates to the notion of an arbitrage opportunity. An *arbitrage opportunity* is the possibility to make profits in a financial market without risk and without any net investment of capital. The principle of *no arbitrage* states that a well-functioning financial market should not allow for such arbitrage opportunities. The first FTAP essentially establishes the equivalence between no arbitrage and the existence of an equivalent martingale (risk-neutral) probability measure, see Harrison and Pliska [79]. More comprehensive versions of the first FTAP are obtained by generalizing the concept of an arbitrage opportunity to a free lunch, a free lunch with bounded risk or a free lunch with vanishing risk, see e.g. Delbaen and Schachermayer [50]. We also refer to [50] for a more detailed history of this theorem. The second FTAP essentially states that in a market with no arbitrage, the market is complete if and only if the equivalent martingale probability measure is unique, see Harrison and Pliska [80]. Completeness enables the use of risk-neutral valuation to price derivatives, see Jarrow and Protter [86]. Risk-neutral valuation means that the price of any derivative equals the discounted expected value of its future payoffs under the risk-neutral measure. This risk-neutral probability measure generally differs from its statistical (real-world or physical) counterpart. The latter describes the likelihood of these risky outcomes and is typically estimated from historical time series data on past realizations. The risk-neutral probability, on the other hand, is the market price of Arrow-Debreu securities associated with risky events. The question then arises as to how to construct the risk-neutral measure used to price derivatives.

The traditional approach to pricing derivatives using risk-neutral valuation is to do a change of measure using Girsanov's theorem. Girsanov's theorem describes how the dynamics of stochastic processes change when the original measure is changed to an equivalent probability measure. In mathematical finance, this theorem tells how to convert from the physical measure to the risk-neutral measure. However, even when one starts with a stochastic process which has various properties that can capture the behavior of a financial asset, and is easily tractable, in many cases after changing measure it loses its tractability.

On the other hand, there are some additional issues in risk-neutral pricing. For example, we may need to model some assets that are not directly traded in the market, which need not to follow the local-martingale requirement due to their non-tradability. Then, the question naturally arises if it is possible to represent some other traded asset as a function or transformation of these non-traded assets, such that the discounted transformed process is a local martingale under the same measure? For example, we may want to represent a stock index in terms of macro-economic data, or represent a firm's stock price in terms of its (non-traded) asset value, etc. This is particularly useful when we consider Merton's structural model of credit risk, see [109], which has long been criticized for being unrealistic because a firm's value is not tradable.

To overcome these limitations, we suggest several transformations on a tractable Markov process (or equivalently, its respective semigroup) in order to make the discounted transformed process a martingale, while keeping its tractability. We refer to such procedures as *risk-neutral pricing transformations*. We introduce an intertwining relation between semigroups to achieve this goal. Moreover, we provide examples that illustrate several classes of processes (Lévy, positive self-similar and generalized CIR) to reveal the usefulness of our result, and furthermore, show that under certain circumstances, the derivative pricing formula can be represented by spectral expansions and evaluated numerically.

The paper is organized as follows. In Section 2, we describe the risk-neutral pricing methods based on the concept of intertwining relationships between Markov semigroups. Section 3 is devoted to the application of the above-mentioned methods on some important classes of Markov processes, namely the exponential Lévy processes, positive self-similar processes and generalized CIR processes. In Section 4, we provide an eigenvalues expansion for the non-self-adjoint pricing semigroup of a risk-neutral generalized CIR model with jumps and study its numerical implementation for the pricing of derivatives. Proofs of the main results of Section 2 are presented in the Appendix.

2.1.1 Preliminaries

Let $X = (X_t)_{t\geq 0}$ be an E-valued, with $E \subseteq \mathbb{R}_+$, homogeneous Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We denote its semigroup by $P = (P_t)_{t\geq 0}$, i.e. for all $x \in E,\ t \geq 0$ and $f \in \mathcal{B}_b(E)$, the set of all bounded measurable functions on E,

$$P_t f(x) = \mathbb{E}_x[f(X_t)],$$

Here $\mathbb{E}_x$ denotes the expectation associated to $\mathbb{P}_x(X_0 = x) = 1$. Similarly, we denote by $\mathcal{B}(E)$ the set of all measurable functions on E. The Markov property then indicates that

(a) $P_t : \mathcal{B}_b(E) \mapsto \mathcal{B}_b(E)$ is a linear operator for every $t \geq 0$, .

(b) $P_{t+s} = P_t \circ P_s$ for $s, t \geq 0$ (semigroup property).

(c) $P_0 = \mathbb{I}$, the identity operator (initial condition).

(d) $P_t \mathbb{1} \leq \mathbb{1}$ for all $t \geq 0$, $\mathbb{1}$ being the constant function equal to 1.

(e) $P_t f \geq 0$ for all $f \geq 0$ and $t \geq 0$, (positivity preserving).

(f) For each $f \in \mathcal{B}_b(E)$ the function $t \mapsto P_t f$ is continuous.

In addition, if $P_t \mathbb{1} = \mathbb{1}$ for all $t \geq 0$, then P is conservative, see e.g. [10]. A linear operator is Markov if it satisfies (d) and (e) (for $t = 1$).

Next, for a semigroup P, we define its domain in $\mathcal{B}(E)$ as

$$\mathcal{B}_P(E) \quad = \quad \{f \in \mathcal{B}(E);\ P_t|f| < \infty,\ \forall t \geq 0\}.$$

We shall also need some basic concepts from potential theory, and we follow [52, Chapter XII] for the following definition.

Definition 2.1.1.1. Let $r \geq 0$ be fixed.

1. The set of r-excessive functions for the semigroup P is defined as

   $$\mathcal{E}_r(P) = \{h_r : E \to \mathbb{R}_+; e^{-rt} P_t h_r(x) \leq h_r(x), \forall t \geq 0, \forall x \in E, \text{ and } \lim_{t \searrow 0} e^{-rt} P_t h_r(x) = h_r(x), \forall x \in E\}.$$

   Moreover, h_r is called r-purely excessive if $h_r \in \mathcal{E}_r(P)$ and $\lim_{t\to\infty} e^{-rt} P_t h_r(x) = 0$ for any $x \in E$.

2. The set of r-invariant functions for P is defined as

   $$\mathcal{I}_r(P) = \{h_r \in \mathcal{E}_r;\ e^{-rt} P_t h_r(x) = h_r(x), \forall t \geq 0 \text{ and } \forall x \in E\}.$$

When $r = 0$, we simply say excessive (resp. invariant) functions for 0-excessive (resp. 0-invariant) functions.

Remark 2.1.1. We mention that $h_r \in \mathcal{E}_r(P), r \geq 0$, if and only if $h_r \in \mathcal{E}_0(P^{(r)})$ that is it is an excessive function for the semigroup $P^{(r)} = (e^{-rt}P_t)_{t\geq 0}$ of the Markov process X killed at an independent exponential time $\mathbf{e}_r$ of parameter $r > 0$. Indeed, let $h_r \in \mathcal{E}_r(P)$, then, for $x \in E$, we have

$$P_t^{(r)}h_r(x) = \mathbb{E}_x[h_r(X_t)\mathbb{1}_{\{t<\mathbf{e}_r\}}] = e^{-rt}\mathbb{E}_x[h_r(X_t)] = e^{-rt}P_th_r(x) \leq h_r(x),$$

which means that $h_r \in \mathcal{E}_0(P^{(r)})$.

The next result recalls the connection between the concept of excessive functions and positive super-martingales. It will enable us to reinterpret the usual conditions based on stochastic calculus for risk-neutral pricing from a potential theoretical viewpoint. For sake of completeness, we also provide its proof.

Lemma 2.1.1. Let $r \geq 0$.

1. $h_r \in \mathcal{I}_r(P)$ if and only if $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a positive martingale under $\mathbb{P}$.
2. $h_r \in \mathcal{E}_r(P)$ if and only if $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a positive super-martingale under $\mathbb{P}$.
3. If $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a local martingale bounded from below under $\mathbb{P}$, then $h_r \in \mathcal{E}_r(P) = \mathcal{E}_0(P^{(r)})$.

Proof. First, let $h_r \in \mathcal{I}_r(P)$, then clearly $P_t|h_r| = P_th_r = e^{rt}h_r < \infty$ for any $t \geq 0$. Moreover, for any $s < t$, the Markov property entails that

$$\mathbb{E}\left[e^{-rt}h_r(X_t)|\mathcal{F}_s\right] = e^{-rt}P_{t-s}h_r(X_s) = e^{-rs}h_r(X_s),$$

where for the last identity we use the fact that $h_r \in \mathcal{I}_r(P)$. Hence, $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a positive martingale under $\mathbb{P}$. The reverse statement is obvious. Item (2) is proved similarly. Finally, to show item (3), we note that every positive local martingale is a super-martingale, which is a direct application of Fatou's lemma. Therefore, $(e^{-rt}h_r(X_t))_{t\geq 0}$ is a super-martingale under $\mathbb{P}$, and by item (2), $h_r \in \mathcal{E}_r(P)$. ■

We emphasize that the results of this paper could be extended to the case $E \subseteq \mathbb{R}_+^d$, $d \in \mathbb{N}$. Indeed, in the multidimensional case, martingales and excessive functions are defined componentwise and the concept of intertwining, which is central in this work, is robust by tensorization. However, for sake of clarity and simplicity, we assume that $E \subseteq \mathbb{R}_+$.

When using risk-neutral valuation, assuming the interest rate is constant, the requirement of discounted prices being martingales under an equivalent probability measure can be reinterpreted from the viewpoint of semigroups. More precisely, let the stock price have dynamics $X = (X_t)_{t\geq 0}$ under a risk neutral provability measure $\mathbb{Q}$, and denote its semigroup by $Q = (Q_t)_{t\geq 0}$. Recall that we call a probability measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ a *risk-neutral* measure, if $(e^{-rt}X_t)_{t\geq 0}$ is a martingale under $\mathbb{Q}$, where $r \geq 0$ is the interest rate. The collection of operators $Q^{(r)} = (e^{-rt}Q_t)_{t\geq 0}$ defined by

$$Q_t^{(r)} f(x) = \mathbb{E}_x[e^{-rt} f(X_t)]$$

is then referred as the *pricing semigroup* of X. This concept of the *pricing semigroups* in financial economics goes back to Garman [65]. Now we present the following potential-theoretical characterization of pricing semigroups.

Proposition 2.1.1. For any $r \geq 0$, $Q^{(r)}$ is a pricing semigroup if $p_1 \in \mathcal{I}_r(Q) = \mathcal{I}_0(Q^{(r)})$, where $p_1(x) = x$.

Proof. Let $p_1 \in \mathcal{I}_r(Q)$. Then, by part (1) of Lemma 2.1.1, $(e^{-rt}X_t)$ is a martingale under the measure $\mathbb{Q}$. Therefore, $Q^{(r)} = (e^{-rt}Q_t)_{t\geq 0}$ defines a pricing semigroup. ∎

2.2 Risk-neutral pricing transformations

As mentioned in the introduction, when pricing derivatives, one has to identify a risk-neutral measure under which the discounted price process is a martingale. To do this, we introduce transformations on a tractable and flexible process (or equivalently, on its respective semigroup), based on the concept of intertwining relationships, which make the discounted transformed process a martingale, while still maintaining its tractability. We refer to such procedures as *risk-neutral pricing transformations*.

We introduce the concept of intertwining relations between Markov semigroups as a comprehensive tool for various risk-neutral pricing techniques. We emphasize that the literature on intertwining is important with a broad range of applications in stochastic and functional analysis, see e. g. Dynkin [58], Rogers and Pitman [137], Diaconis and Fill [54], Carmona et al. [31], Jansen and Kurt [84], Pal and Shkolnikov [120], Patie and Savov [129] and references therein.

We now present the main theorem of the paper which establishes some risk-neutral pricing techniques based on the concept of the intertwining relationship.

Theorem 2.2.1. Let $P = (P_t)_{t\geq 0}$ and $Q = (Q_t)_{t\geq 0}$ be two Markov semigroups acting respectively on $\mathcal{B}_P(E)$ and $\mathcal{B}_Q(F)$, where $E, F \subseteq \mathbb{R}_+$. Assume that there exists a Markov kernel $\Lambda : \mathcal{D}(\Lambda) \to \mathcal{B}(E)$ such that $p_1 \in \mathcal{D}(\Lambda)$, the domain of Λ in $\mathcal{B}(E)$, and for any $t \geq 0$

$$P_t^{(r)}\Lambda p_1 = \Lambda Q_t^{(r)} p_1. \tag{2.1}$$

If $\Lambda p_1 \in \mathcal{I}_r(P)$ and $(Q_t p_1 - e^{rt} p_1) \notin Ker(\Lambda) \setminus \{\mathbf{0}\}$, where $Ker(\Lambda) = \{f \in \mathcal{D}(\Lambda); \Lambda f = 0\}$, then

$Q^{(r)}$ is a pricing semigroup.

Remark 2.2.1. In the literature usually the intertwining relation (2.1) between two operators $P^{(r)}$ and $Q^{(r)}$ is given for all functions f in the appropriate domain, say $C_0(\mathbb{R}_+)$ in the case of Feller semigroups. However, we only require the identity (2.1) to hold for the function p_1, to make our claim as comprehensive as possible.

In what follows, we provide some examples of intertwining kernels. We note that such kernels characterize the family of linear operators $Q^{(r)} = (e^{-rt}Q_t)_{t\geq 0}$, defined via an intertwining relation (2.2) below, as Markov semigroups.

Proposition 2.2.1. Let $r \geq 0$ and let $P = (P_t)_{t\geq 0}$ be a Markov semigroup and $Q = (Q_t)_{t\geq 0}$ be a family of linear operators acting respectively on $\mathcal{B}_P(E)$ and $\mathcal{B}_Q(F)$, where $E, F \subseteq \mathbb{R}_+$. Assume that there exists a Markov kernel $\Lambda : \mathcal{D}(\Lambda) \to \mathcal{B}(E)$ such that for any $f \in \mathcal{D}(\Lambda)$ with $\Lambda f \in \mathcal{B}_P(E)$, we have for $t \geq 0$,

$$P_t^{(r)}\Lambda f = \Lambda Q_t^{(r)} f. \tag{2.2}$$

Then, for the following intertwining operators Λ, the familiy of linear operators $Q^{(r)}$ is a pricing semigroup.

(a) $\Lambda f = f \circ h_r$, where $h_r \in \mathcal{I}_r(P)$ is a homeomorphism.

(b) $\Lambda f = f g_\lambda$ for a strictly positive function $g_\lambda \in \mathcal{E}_{|\lambda|}(P)$ for some $\lambda \leq r$ and such that the mapping $h_r = p_1 g_\lambda \in \mathcal{I}_r(P)$.

(c) $\Lambda f = (f \circ H_{r,\lambda}) g_\lambda$ for a strictly positive function $g_\lambda \in \mathcal{E}_{|\lambda|}(P)$ for some $\lambda \leq r$ and such that $H_{r,\lambda} = \frac{h_r}{g_\lambda}$ is, for some $h_r \in \mathcal{I}_r(P)$, a homeomorphism.

We postpone the proof of Theorem 2.2.1 and Proposition 2.2.1 to the Appendix 2.5.1. We proceed by explaining that two particular instances of this risk-neutral pricing technique in Proposition 2.2.1 are related to some classical transformations.

Remark 2.2.2. We point out that Proposition 2.2.1(a) hinges on a result due to Dynkin [58]. Indeed, if $h_r \in \mathcal{I}_r(P)$ is a homeomorphism, then the family of operators $Q^{(r)} = (Q_t^{(r)})_{t\geq 0}$ defined for $f \in \mathcal{B}_b(E)$ and $x \in E$, by

$$Q_t^{(r)} f(y) = P_t^{(r)}(f \circ h_r)(x), \quad y = h_r(x), \tag{2.3}$$

is a pricing semigroup.

Remark 2.2.3. We also mention that Proposition 2.2.1(b) is related to Doob's h-transform. To see this, we can write the intertwining relation (2.1) as

$$Q_t^{(r)} f(x) = \frac{1}{g_\lambda(x)} P_t^{(r)}(g_\lambda f)(x), \quad x \in E,$$

which is the well-known Doob's h-transform, see e.g. Chung and Walsh [40]. Note that in the Remark 2.3.1 below, we illustrate this idea by showing that Theorem 2.2.1(b) can be seen as a generalization of the well-known Esscher transform that is used as a time-space Doob's h-transform in the context of Lévy processes based models.

In the following, we present a *dual* version of the results stated Theorem 2.2.1 and Proposition 2.2.1, in the sense that the roles of the semigroups P and Q are interchanged. As their justifications follows the same pattern as for the proofs of Theorem 2.2.1 and Proposition 2.2.1, we state them without proofs.

Corollary 2.2.1. Let $r \geq 0$ and let $P = (P_t)_{t\geq 0}$ and $Q = (Q_t)_{t\geq 0}$ be two Markov semigroups acting respectively on $\mathcal{B}_P(E)$ and $\mathcal{B}_Q(F)$, where $E, F \subseteq \mathbb{R}_+$. Assume that there exists a Markov kernel $\Lambda : \mathcal{D}(\Lambda) \to \mathcal{B}(E)$ and a function $h_r \in \mathcal{I}_r(Q) \cap \mathcal{D}(\Lambda)$ such that for any $t \geq 0$

$$P_t^{(r)} \Lambda h_r = \Lambda Q_t^{(r)} h_r.$$

If $\Lambda h_r = p_1$, then $P^{(r)}$ is a pricing semigroup.

Corollary 2.2.2. Let $r \geq 0$ and let $P = (P_t)_{t\geq 0}$ be a Markov semigroup and $Q = (Q_t)_{t\geq 0}$ be a family of linear operators acting respectively on $\mathcal{B}_P(E)$ and $\mathcal{B}_Q(F)$, where $E, F \subseteq \mathbb{R}_+$.

Assume that there exists a Markov kernel $\Lambda : \mathcal{D}(\Lambda) \to \mathcal{B}(E)$ such that for any $f \in \mathcal{D}(\Lambda)$ with $\Lambda f \in \mathcal{B}_P(E)$, we have for $t \geq 0$,

$$P_t^{(r)} \Lambda f = \Lambda Q_t^{(r)} f.$$

Then, for the following intertwining operators Λ, the familiy of linear operators $P^{(r)}$ is a pricing semigroup.

a) $\Lambda f = f \circ h_r^{-1}$, where $h_r \in \mathcal{I}_r(Q)$ is a homeomorphism.

b) $\Lambda f = \frac{f}{g_\lambda}$ for a function $g_\lambda \in \mathcal{E}_{|\lambda|}(P)$ for some $\lambda \leq r$ and such that $h_r = p_1 g_\lambda \in \mathcal{I}_r(P)$.

c) $\Lambda f = \frac{f}{g_\lambda} \circ H_{r,\lambda}^{-1}$ for a function $g_\lambda \in \mathcal{E}_{|\lambda|}(P)$ for some $\lambda \leq r$ and such that $H_{r,\lambda} = \frac{h_r}{g_\lambda}$ is, for some $h_r \in \mathcal{I}_r(P)$, a homeomorphism.

We proceed by providing, in the proposition below, an equivalent condition of (2.1) in terms of *extended generators*. We mention that one advantage of working with this version of a generator is the fact that it might be well defined for unbounded functions as well, an important feature in mathematical finance since payoff functions, such as the one of European call options, may not be necessarily bounded. Next, we recall that the notion of extended generators was first introduced by Kunita [90], and later was used by many authors, with possible minor modifications. We find the following version convenient for our purposes.

Definition 2.2.0.1. A function $f \in \mathcal{B}(E)$ is said to belong to *the domain* $\mathcal{D}(\mathcal{A})$ *of the extended generator* if there exists a measurable function g such that

1. $\int_0^t P_s|g|(x)ds < \infty, \quad \forall t \geq 0,\ \forall x \in E$, and
2. the process

$$\left(M_t^f := f(X_t) - f(X_0) - \int_0^t g(X_s)ds\right)_{t\geq 0} \tag{2.4}$$

is a martingale.

Then, we write $g = \mathcal{A}f$ and $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ represents the *extended generator* of the process $(X_t)_{t\geq 0}$.

Note that g is not uniquely defined but it is defined up to a set of null potential. We identify all functions g such that (2.4) is a martingale and write $\mathcal{A}f$ instead of g. It is easy to note that the domain of the infinitesimal generator is contained in the domain of the extended generator. One then can equivalently define the extended generator as follows.

Proposition 2.2.2. Let $f, g \in \mathcal{B}(E)$. Then $f \in \mathcal{D}(\mathcal{A})$ and $g = \mathcal{A}f$ if and only if the following two conditions hold

1. $P_t|f|(x) < \infty$ and $\int_0^t P_s|g|(x)ds < \infty$ for $t \geq 0,\ x \in E$,
2. $P_t f(x) = f(x) + \int_0^t P_s g(x)ds$ for $t \geq 0,\ x \in E$.

Next, we provide the equivalent statement of Theorem 2.2.1 expressed in terms of the extended generators.

Proposition 2.2.3. Let $P = (P_t)_{t\geq 0}$ and $Q = (Q_t)_{t\geq 0}$ be two Markov semigroups acting respectively on $\mathcal{B}_P(E)$ and $\mathcal{B}_Q(F)$, where $E, F \subseteq \mathbb{R}_+$. Assume that $p_1 \in \mathcal{D}(\mathcal{A}_Q)$, and there exists a Markov kernel $\Lambda : \mathcal{D}(\Lambda) \to \mathcal{B}(E)$ such that $p_1 \in \mathcal{D}(\Lambda)$ with $\Lambda p_1 \in \mathcal{D}(\mathcal{A}_P)$, and for any $t \geq 0$,

$$\mathcal{A}_P \Lambda p_1 = \Lambda \mathcal{A}_Q p_1. \tag{2.5}$$

If $\Lambda p_1 \in \mathcal{I}_r(P)$ and $(\mathcal{A}_Q p_1 - r p_1) \notin Ker(\Lambda) \setminus \{\mathbf{0}\}$, then $Q^{(r)}$ is a pricing semigroup.

The proof of this proposition is given in Appendix 2.5.3.

Another important feature of intertwining relationships between Markov semigroups is that they are not relating probability measures, through the transition kernels, that are equivalent. To highlight this aspect, we present the following examples.

- An intertwining relation between two Markov semigroups via Dynkin's criterion as presented in Remark 2.2.2 insures equivalence between the associated probability measures if h_r thereout is an endomorphism of E, that is the Markov processes have the same state space. Otherwise, the laws have a singular component.
- The intertwining relation through h-transforms with the excessive function g between two Markov semigroups preserves the equivalence between the probability measures if $\{x \in E; g(x) \neq 0\} = E$. This particular intertwining relation has many important applications, e.g. it is related to the well-known Esscher transform which we discuss in Section 2.3.1, and the Ross recovery theorem, see e.g. [138], [32].
- In [129], the authors present examples of intertwining relations between the semigroup of a diffusion and a family of Markov processes with jumps which can be even pure jump processes. Miclo and Patie [117] show intertwining relations between the semigroups of the CIR process on $\mathbb{R}_+$ and a linear birth-and-death process on $\mathbb{N}$. Obviously, in all these cases the probability measures associated with the two semigroups can not be equivalent.

2.3 Examples

In this section, we illustrate the risk-neutral pricing techniques through the concept of intertwining by detailing some examples of semigroups that have been advocated in the literature as flexible models. Namely, we focus on exponential Lévy processes, positive self-similar Markov processes, and CIR (Laguerre) models with jumps, which all belong to the class of polynomial processes, see e.g. Cuchiero et al. [44], Filipović and Larrson [63]. Moreover, in some instances, we also provide analytical formulas for the pricing of some derivative products.

2.3.1 Exponential of Lévy processes

Lévy processes form a subclass of Markov processes which include many familiar processes such as the Brownian motion, the Poisson and stable processes. They also form a flexible class of models which have been applied in quantitative finance when heavy tailed phenomena are observed, see e.g. [11], [59], [146], [147]. Let now $\xi = (\xi_t)_{t\geq 0}$ be a real-valued Lévy process, that is a real-valued random process with almost surely (a.s.) càdlàg paths, and stationary and independent increments. It is well known that the law of ξ is determined by the law of the variable ξ_1 which is infinitely divisible and is itself characterized by the triplet (σ, m, Π), where $\sigma \geq 0,\ m \in \mathbb{R}$, and Π is a Lévy measure concentrated on $\mathbb{R} \setminus \{0\}$ that satisfies the integrability condition $\int_{\mathbb{R}}(1 \wedge y^2)\Pi(dy) < \infty$. More specifically, we have that for $z \in i\mathbb{R}$ and $t \geq 0$,

$$\mathbb{E}\left[e^{iz\xi_t}\right] = e^{t\Psi(z)} \tag{2.6}$$

where Ψ admits the following Lévy-Khintchine representation

$$\Psi(z) = \sigma^2 z^2 + mz + \int_{-\infty}^{\infty} (e^{zy} - 1 - zy1_{\{|y|<1\}})\Pi(dy). \tag{2.7}$$

Then, let $X = \left(X_t = e^{\xi_t}\right)_{t\geq 0}$ and denote its semigroup by $P = (P_t)_{t\geq 0}$, i.e. for any $t \geq 0$, $x > 0$ and $f \in \mathcal{B}_P(\mathbb{R}_+)$,

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_{\ln x}\left[f\left(e^{\xi_t}\right)\right] = \mathbb{E}_0\left[f\left(xe^{\xi_t}\right)\right] = \mathbb{E}_1[f(xX_t)].$$

Then, we deduce from (2.6), that for any $t \geq 0$, $x > 0$ and $z \in i\mathbb{R}$,

$$P_t p_z(x) = e^{t\Psi(z)} p_z(x),$$

where $p_z(x) = x^z = e^{z \ln x}$.

Next, assume that ξ_1 admits some exponential moments, i.e. there exists a non-empty set $\mathbf{C}$ such that for any $u \in \mathbf{C}$,

$$\mathbb{E}\left[e^{u\xi_1}\right] < +\infty.$$

It is well know that this condition is equivalent to the existence of the same exponential moments for the Lévy measure Π away from 0, i.e.

$$\mathbf{C} = \left\{u \in \mathbb{R}; \int_{|y|>1} e^{uy}\Pi(dy) < \infty\right\}, \tag{2.8}$$

see e.g. Sato [144, Lemma 26.4] and where it is also shown that $\mathbf{C}$ is an interval. Then, Ψ admits an analytical extension on the strip $\{z \in \mathbb{C};\ \Re(z) \in \mathbf{C}\}$, which we still denote by Ψ. Moreover, using a dominated convergence argument, we get that for all $u \in \mathbf{C}$, $\Psi''(u) = \sigma^2 + \int_{-\infty}^{\infty} x^2 e^{uy}\Pi(dy) \geq 0$ and therefore $\Psi : \mathbf{C} \to \mathbb{R}$ is a convex function, see e.g. Sato [144, Lemma 26.4]. Let us finally assume that there exists $\theta \in \mathbf{C}$ such that $\Psi(\theta) = 0$. Let M denote the supremum of $\mathbf{C}$ which can be infinity, and note that Ψ is increasing on $[\theta, M)$, hence there exists a continuous increasing function $\Phi : [0, \Psi(M_-)) \to [\theta, M)$ which is the inverse of Ψ, i.e. $\Psi(\Phi(r)) = r$, and where $\Psi(M_-) = \lim_{u\uparrow M} \Psi(u)$ can be either finite or ∞. We are ready to state the following.

Proposition 2.3.1. For any $0 < r \leq \Psi(M_-)$, define the family of operators $Q = (Q_t)_{t\geq 0}$ by the following intertwining relation, for any $f \in \mathcal{B}_b(\mathbb{R}_+)$,

$$P_t\Lambda f = \Lambda Q_t f$$

where $\Lambda f(x) = f \circ p_{\Phi(r)}(x)$, where $p_{\Phi(r)}(x) = x^{\Phi(r)}$. Then, $Q^{(r)}$ is a pricing semigroup.

Proof. Since $\mathbb{E}[e^{u\xi_t}] = e^{t\Psi(u)} < \infty$, for all $u \in [0, M)$, one can define, for any $0 < r \leq \Psi(M_-)$, $x \geq 0$, $h_r(x) = x^{\Phi(r)}$. Then, we have

$$\begin{aligned} P_t h_r(x) &= \mathbb{E}_x[h_r(X_t)] = \mathbb{E}_x[X_t^{\Phi(r)}] = \mathbb{E}_{\ln x}[e^{\Phi(r)\xi_t}] = \mathbb{E}[e^{\Phi(r)(\xi_t + \ln x)}] = x^{\Phi(r)} e^{\Psi(\Phi(r))t} \\ &= h_r(x) e^{rt}. \end{aligned} \tag{2.9}$$

Therefore, $h_r \in \mathcal{I}_r(P)$, and since, for any $0 < r \leq \Psi(M_-)$, h_r is a an increasing function, it is a homeomorphism of $\mathbb{R}_+$. Hence, noting that $h_r = p_{\Phi(r)}$, that is $\Lambda f(x) = f \circ h_r(x)$, we conclude that $Q^{(r)}$ is a pricing semigroup by direct application of Proposition 2.2.1 (a). ■

Remark 2.3.1. We show that Theorem 2.2.1(b) can be seen as a generalization of the well-known Esscher transform that we now recall in the context of Lévy market models. To this end, let us assume that the stock price dynamics is given by $(S_t = e^{\bar{\xi}_t})_{t \geq 0}$ where $\bar{\xi} = (\bar{\xi}_t)_{t \geq 0}$ is, under the physical probability measure $\mathbb{P}^*$, a Lévy process with characteristic exponent $\overline{\Psi}$. Assume that there exists $u > 0$ such that $[0, u+1] \subset \overline{\mathbf{C}}$, the latter set being defined as in (2.8), then $\overline{\Psi}$ admits an analytical extension on the closed strip $\{z \in \mathbb{C};\ \Re(z) \in [u, u+1]\}$ and we set $\overline{\Psi}(u+1) - \overline{\Psi}(u) = r > 0$. Then, one can define a new probability measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}^*$ and such that for all $t > 0$,

$$\frac{d\mathbb{Q}}{d\mathbb{P}^*}|_{\mathcal{F}_t} = p_u(S_t) e^{-\overline{\Psi}(u)t} = e^{u\bar{\xi}_t} e^{-\overline{\Psi}(u)t} = \frac{e^{u\bar{\xi}_t}}{\mathbb{E}^{\mathbb{P}^*}[e^{u\bar{\xi}_t}]}$$

where we recall that $p_u(x) = x^u$. $\mathbb{Q}^*$ is in fact a risk-neutral measure in the sense that $(e^{-rt} S_t)_{t \geq 0}$ is a martingale under the measure Q^*, see e.g. [92] for details and Gerber and Shiu [69] for further discussion on the use of Esscher transform in insurance and financial mathematics. Note that the Esscher transform is a time-space Doob's h-transform with the $\Psi(u)$-invariant function $p_u(x)$. On the other hand assume that $X = (X_t = e^{\xi_t})_{t \geq 0}$, where $(\xi_t)_{t \geq 0}$ is a real-valued Lévy process with characteristic exponent Ψ. Then, for $x > 0$ and $f \in \mathcal{B}_b(\mathbb{R}_+)$,

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \mathbb{E}_{\ln x}[f(e^{\xi_t})].$$

Next, assume further that, for some $\lambda > 0$, $[0, \lambda + 1] \subset \mathbf{C}$, it follows from Proposition 2.3.1 that $p_\lambda \in \mathcal{I}_{\Psi(\lambda)}(P)$ and $p_1 p_\lambda = p_{\lambda+1} \in \mathcal{I}_{\Psi(\lambda+1)}(P)$. Choosing $g_\lambda = p_\lambda$ and $r = \Psi(\lambda + 1)$ in Theorem 2.2.1(b) yields that the family of linear operators $Q^{(r)} = (e^{-rt}Q_t)_{t\geq 0}$ defined for any $f \in \mathcal{B}_b(\mathbb{R}_+)$ by

$$P_t^{(r)} \Lambda f(x) = \Lambda Q_t^{(r)} f(x) \tag{2.10}$$

where $\Lambda f(x) = (f p_\lambda)(x)$, is a pricing semigroup. Since $(p_\lambda(X_t))_{t\geq 0}$ is not a martingale under the original measure, we note that the measures $\mathbb{Q}^{(r)}$ and $\mathbb{P}^{(r)}$ are not absolutely continuous with respect to each other.

2.3.2 Positive self-similar Markov processes

In this section, we consider the family of positive self-similar Markov processes. We recall that a *positive α-self-similar, $\alpha > 0$, Markov process $X = (X_t)_{t\geq 0}$* is a positive valued strong Markov process which fulfills the following scaling property. For any $c, x > 0$ and $t \geq 0$,

$$\text{the law of } (cX_{c^{-\alpha}t})_{t\geq 0} \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}$$

where by $\mathbb{P}_x$ we understand the law of the process starting at x. Writing $P = (P_t)_{t\geq 0}$ for the semigroup associated with X, this property reads, for any $c, x > 0$ and $t \geq 0$, as

$$P_{c^{-\alpha}t} d_c f(x) = d_c P_t f(x) \tag{2.11}$$

where d_c is the dilation operator, i.e. $d_c f(x) = f(cx)$. Some well known instances of positive self-similar Markov processes are squared Bessel processes ($\alpha = 1$), α-stable subordinators, reflected or killed α-stable process, stable Lévy processes conditioned to stay positive, see e.g. Bertoin and Yor [17, 18], Caballero and Chaumont [30], Kuznetsov and

Kwaśnicki [91], Patie and Zhao [133].

An interesting relationship between positive self-similar Markov processes and real-valued Lévy processes was obtained by Lamperti [95]. More specifically, Lamperti showed that any positive self-similar Markov process up to its first hitting time of 0 (which may be a polar point) can be expressed as the exponential of a Lévy process, time-changed by the inverse of its exponential functional. More formally, let X be a positive α-self-similar Markov process, starting from $x > 0$. Then,

$$X_t = x \exp\left(\xi_{\tau_{tx^{-\alpha}}}\right), \quad 0 \leq t < T_0 = \inf\{t > 0;\ X_t = 0\}, \tag{2.12}$$

where

$$\tau_t = \inf\left\{s > 0;\ \int_0^s e^{\alpha \xi_u} du \geq t\right\}$$

and $\xi = (\xi_t)_{t \geq 0}$ is a real-valued Lévy process starting from 0. The relation (2.12) yields a one-to-one correspondence between the class of positive α-self-similar Markov processes and the one of Lévy processes. Next, we have that $\mathbb{P}_x(T_0 = \infty) = 1$ if ξ has an infinite lifetime and it does not drift to $-\infty$, i.e. $\limsup_{t \to \infty} \xi_t = +\infty$ a.s.

We recall from Section 2.3.1 that the law of a Lévy process ξ is determined by its characteristic exponent Ψ, which admits Lévy-Khintchine representation (2.7). Next, we assume that $\mathbf{C} = \mathbb{R}_+$, see (2.8) for definition, that is, for all $t, u \geq 0$,

$$\mathbb{E}\left[e^{u\xi_t}\right] = e^{t\Psi(u)} < \infty, \tag{2.13}$$

where we keep the same notation Ψ for the analytical extension of the Lévy-Khintchine exponent to the right half-plane. This condition of finite exponential moments holds, for instance, when the jumps of ξ are bounded above by some fixed number, and, in particular, include the spectrally negative case, see Sato [144, Theorem 25.17]. Note that the condition $\lim_{u \to \infty} \Psi(u) = +\infty$ is equivalent to either $\sigma > 0$, $m > 0$ or $\Pi(\mathbb{R}) = \infty$, which means that this condition excludes the case when ξ is a compound Poisson process. In this setting, the condition that ξ does not drift to $-\infty$ is equivalent to

$$m = \mathbb{E}[\xi_1] = \Psi'(0+) \in [0, \infty). \tag{2.14}$$

Moreover, we recall that the Wiener-Hopf factorization of Lévy processes gives that Ψ can be decomposed as

$$\Psi(u) = -\phi_+(-u)\phi_-(u), \quad u \geq 0, \tag{2.15}$$

where ϕ_+ and ϕ_- are Bernstein functions which take the form

$$\phi_\pm(u) = \kappa_\pm + \gamma_\pm u + \int_0^\infty (1 - e^{-uy})\mu_\pm(dy), \tag{2.16}$$

with $\kappa_\pm \geq 0$ such that $\kappa_+\kappa_- = 0$, $\gamma_\pm \geq 0$ and $\mu_\pm$ being Lévy measures which satisfy the integrability condition $\int_0^\infty (y \wedge 1)\mu_\pm(dy) < \infty$, see e.g. Kyprianou [92, Section 6]. Note that for $u > 0$, the condition $\int_1^\infty e^{uy}\Pi(dy) < +\infty$ implies that $\int_1^\infty e^{uy}\mu_+(dy) < +\infty$. Hence, $\lim_{u\to\infty} \Psi(u) < \infty$ only when ξ is a decreasing compound Poisson process. Based on the above consideration, we define the following class of functions

$$\mathcal{N} = \{\Psi \text{ of the form (2.7) with } \lim_{u\to\infty} \Psi(u) = +\infty \text{ and such that (2.13) and (2.14) hold}\}. \tag{2.17}$$

Now, for $\Psi \in \mathcal{N}$, we define

$$\mathcal{I}_\Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{W_\Psi(n+1)}, \quad z \in \mathbb{C}, \tag{2.18}$$

where $W_\Psi(1) = 1$ and $W_\Psi(n+1) = \prod_{k=1}^n \Psi(k)$ for $n \in \mathbb{N}$.

Proposition 2.3.2. Let $\Psi \in \mathcal{N}$ and $P = (P_t)_{t\geq0}$ be the semigroup of a 1-self-similar Markov process $X = (X_t)_{t\geq0}$ associated via the Lamperti mapping to the Lévy process with characteristic exponent Ψ. Then, the following statements hold.

(1) $\mathcal{I}_\Psi$, defined by (2.18), is an entire function. Moreover, it is positive and increasing on $\mathbb{R}_+$.

(2) For any $r \geq 0$, $d_r \mathcal{I}_\Psi \in \mathcal{I}_r(P)$, i.e. for any $t \geq 0$, $e^{-rt} P_t d_r \mathcal{I}_\Psi(x) = d_r \mathcal{I}_\Psi(x) = \mathcal{I}_\Psi(rx)$.

(3) For any $r \geq 0$, the family of linear operators $Q^{(r)} = (e^{-rt} Q_t)_{t \geq 0}$ defined, for any $x \in E$ and $t \geq 0$, by

$$Q_t^{(r)} f(y) = P_t^{(r)}(f \circ d_r \mathcal{I}_\Psi)(x), \quad y = d_r \mathcal{I}_\Psi(x), \tag{2.19}$$

is a pricing semigroup.

Remark 2.3.2. The result of Proposition 2.3.2 can be generalized readily to α-self-similar processes for any $\alpha > 0$. That is, if $P^{(\alpha)} = (P_t^{(\alpha)})_{t \geq 0}$ is the semigroup of the positive α-self-similar Markov process associated to the Lévy process with characteristic exponent $\Psi \in \mathcal{N}$, then for any $r \geq 0$,

$$d_r \mathcal{I}_\Psi^{(\alpha)}(x) = d_r \mathcal{I}_\Psi(x^{\frac{1}{\alpha}}) \in \mathcal{I}_r(P^{(\alpha)}).$$

Indeed, one can easily check that there exists a similarity transform between 1- and α-self-similar semigroups. More specifically, one has for any $t, x > 0$,

$$P_t^{(\alpha)} f(x) = \Lambda_{\frac{1}{\alpha}} P_t \Lambda_\alpha f(x),$$

where $\Lambda_\alpha f(x) = f \circ p_\alpha(x)$ with $p_\alpha(x) = x^\alpha$ a homeomorphism on $\mathbb{R}_+$. Then, observing that $\Lambda_\alpha \mathcal{I}_\Psi^{(\alpha)}(x) = \mathcal{I}_\Psi(x)$, this relation yields that, for any $r, x > 0$,

$$\begin{aligned} P_t^{(\alpha)} d_r \mathcal{I}_\Psi^{(\alpha)}(x) &= \Lambda_{\frac{1}{\alpha}} P_t \Lambda_\alpha d_r \mathcal{I}_\Psi^{(\alpha)}(x) = \Lambda_{\frac{1}{\alpha}} P_t d_r \Lambda_\alpha \mathcal{I}_\Psi^{(\alpha)}(x) = \Lambda_{\frac{1}{\alpha}} P_t d_r \mathcal{I}_\Psi(x) \\ &= e^{rt} \Lambda_{\frac{1}{\alpha}} \mathcal{I}_\Psi(x) = e^{rt} \mathcal{I}_\Psi^{(\alpha)}(x), \end{aligned}$$

which gives that indeed $d_r \mathcal{I}_\Psi^{(\alpha)} \in \mathcal{I}_r(P^{(\alpha)})$.

Remark 2.3.3. When $\Pi(0, \infty) = 0$, that is, X is spectrally negative meaning that it does not have positive jumps, the function $\mathcal{I}_\Psi$ appears in [123] as an invariant function that characterizes the Laplace transform of the first passage time of X, see also [93, Chapter 13]. More generally, the function $\mathcal{I}_\Psi$ appear in the spectral representation of α-self-similar Markov semigroups, see [126].

Proof. To show (1), first observe that for all $n \in \mathbb{N}$, we have $\frac{|W_\Psi(n+1)|}{|W_\Psi(n+2)|} = \frac{1}{\Psi(n+1)}$. Hence, the analyticity of $\mathcal{I}_\Psi$ follows from the fact that $\lim_{u\to\infty} \Psi(u) = +\infty$. Moreover, recalling from Section 2.3.1, that for all $u \geq 0$, $\Psi''(u) \geq 0$, the assumption (2.14), that is $\Psi'(0+) \geq 0$, entails that $\Psi'(u) \geq 0$ for all $u \geq 0$. Therefore, Ψ is a non-decreasing function on $\mathbb{R}_+$. Moreover, since plainly $\Psi(1) > 0$, we have that $\frac{1}{W_\Psi(n+1)} > 0$ for all $n \geq 0$. Therefore, $\mathcal{I}_\Psi$ is an entire function with all positive coefficients, and, in particular, is positive and monotone increasing on $\mathbb{R}_+$.

To show (2), we first note if $f_1 \in \mathcal{I}_1(P)$ then $f_r = d_r f_1 \in \mathcal{I}_r(P)$. Indeed from the self-similarity property (2.11), we get that

$$P_t f_r = P_t d_r f_1 = d_r P_{rt} f_1 = e^{rt} d_r f_1 = e^{rt} f_r,$$

thus we assume without loss of generality that $r = 1$. Then, observe that if (2.13) and (2.14) hold, then by [18, Proposition 1], we have for every $t \geq 0$ and $n \geq 0$, recalling that $p_n(x) = x^n$,

$$P_t p_n(x) = \sum_{k=0}^{n} \frac{\Psi(n)\cdots\Psi(n-k+1)}{k!} t^k p_{n-k}(x) = W_\Psi(n+1) \sum_{k=0}^{n} \frac{t^k p_{n-k}(x)}{W_\Psi(n-k+1)k!}. \tag{2.20}$$

Now, using the definition of $\mathcal{I}_\Psi$ in (2.18), (2.20) and applying Tonelli's theorem, we obtain that

$$P_t \mathcal{I}_\Psi(x) = \sum_{n=0}^{\infty} \frac{P_t p_n(x)}{W_\Psi(n+1)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{t^k p_{n-k}(x)}{W_\Psi(n-k+1)k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{n=0}^{\infty} \frac{p_n(x)}{W_\Psi(n+1)} = e^t \mathcal{I}_\Psi(x),$$

where we used that, for any sequence $(a_{n,k})_{n,k\geq 0}$, the following convolution identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n,k} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_{n+k,k} \tag{2.21}$$

holds. Therefore, $e^{-rt} P_t d_r \mathcal{I}_\Psi(x) = d_r \mathcal{I}_\Psi(x)$, hence $d_r \mathcal{I}_\Psi \in \mathcal{I}_r(P)$. Item (3) directly follows from Proposition 2.2.1 (see also Remark 2.2.2). ∎

When $\Psi(u) = \sigma^2 u^2 + mu$, $m \geq 0$ and $\alpha = 1$, then $X = X^{(0)} = (X_t^{(0)})_{t\geq 0}$ is the (scaled by $2\sigma^2$) squared Bessel process of index m. In this case, since $W_\Psi(n+1) = \prod_{k=1}^n \left(\sigma^2 k^2 + mk\right) = \sigma^{2n} n! \frac{\Gamma(n+1+m_\sigma)}{\Gamma(1+m_\sigma)}$, where $m_\sigma = \frac{m}{\sigma^2}$, we have

$$\mathcal{I}_\Psi(x) = \Gamma(1+m_\sigma) \sum_{n=0}^{\infty} \frac{x^n}{\sigma^{2n} n! \Gamma(n+1+m_\sigma)} = \frac{\sigma^{2m_\sigma}\Gamma(1+m_\sigma)}{x^{m_\sigma}} I_{m_\sigma}\left(\frac{2}{\sigma}\sqrt{x}\right)$$

where I_{m_σ} denotes the modified Bessel function of the first kind of order m_σ. We mention that squared Bessel processes have found many applications in financial modeling, see e.g. Yor [155], Geman and Yor [68]. We also note that more generally the function $\mathcal{I}_\Psi$ can be seen as a generalization of the class of hypergeometric functions. We refer to [123] for the representation of the function $\mathcal{I}_\Psi$ as known special functions such as the Mittag-Leffler function and Wright-hypergeometric ones.

2.3.3 Generalized Cox-Ingersoll-Ross models

Let $X^{(0)} = (X_t^{(0)})_{t\geq 0}$ be the squared Bessel process defined above. By denoting its semigroup by $P^{(0)} = (P_t^{(0)})_{t\geq 0}$ we introduce the family of linear operators $\overline{P}^{(0)} = (\overline{P}_t^{(0)})_{t\geq 0}$ defined, for any $t \geq 0$, by

$$\overline{P}_t^{(0)} f(x) = P_{e^t-1}^{(0)} f \circ d_{e^{-t}}(x) = \mathbb{E}_x[f(e^{-t} X_{e^t-1}^{(0)})], \quad x > 0, \tag{2.22}$$

where we recall that $d_c f(x) = f(cx)$ is the dilation operator. Then, it is well known that $\overline{P}^{(0)} = (\overline{P}_t^{(0)})_{t\geq 0}$ is the semigroup of the classical Cox-Ingersoll-Ross (CIR) process which can be seen to be associated, through the underlying 1-self-similar Markov process to $\Psi_0(u) = \sigma^2 u^2 + mu, u, m \geq 0$, see e.g. Göing-Jaeschke and Yor [76]. Let $\mathbf{A}_0$ denote the infinitesimal generator of this CIR process, i.e. for $f \in C_0^2(\mathbb{R}_+)$ we have

$$\mathbf{A}_0 f(x) = \sigma^2 x f''(x) + (m + \sigma^2 - x) f'(x). \tag{2.23}$$

CIR processes are widely used in financial modeling, such as for modeling stochastic volatility or interest rates, see e.g. Heston [81] and Cox et al. [42].

Now, let $X = (X_t)_{t\geq 0}$ be a 1-self-similar positive Markov process with semigroup $P = (P_t)_{t\geq 0}$ and associated, via the Lamperti mapping to the Lévy process $\xi = (\xi_t)_{t\geq 0}$ with characteristic exponent $\Psi \in \mathcal{N}$. Define the family of linear operators $\overline{P} = (\overline{P}_t)_{t\geq 0}$ by

$$\overline{P}_t f(x) = P_{e^t-1} f \circ d_{e^{-t}}(x) = \mathbb{E}_x[f(e^{-t}X_{e^t-1})], \quad x > 0, \tag{2.24}$$

The self-similarity and the Markov property of P entails that $\overline{P} = (\overline{P}_t)_{t\geq 0}$ is also a Markov semigroup, see [129, Theorem 1.6] for a detailed proof. We call it a *generalized CIR (or generalized Laguerre)* semigroup and its associated process a generalized CIR (or generalized Laguerre) process. Hence, it follows from the Lamperti mapping, that there is also a bijection between the class of generalized CIR processes and the set $\mathcal{N}$. In the aforementioned paper, the authors show that the infinitesimal generator associated to this semigroup is then given, for a smooth function f, by

$$\mathbf{A}f(x) = \sigma^2 x f''(x) + (m + \sigma^2 - x)f'(x) + \int_{\mathbb{R}} (f(e^{-y}x) - f(x) + yxf'(x))\,\Pi(x, dy), x > 0, \tag{2.25}$$

where $\Pi(x, dy) = \frac{\Pi(dy)}{x}$, and $\sigma, m \geq 0$, and the Lévy measure Π form the characteristic triplet of Ψ. Note that when $\Pi(\mathbb{R}) = 0$, then $\overline{P}$ boils down to the semigroup of the classical CIR process and, in [129], it is shown that this semigroup intertwines with the one of a generalized CIR semigroup for any $\Psi \in \mathcal{N}$ with $\Pi(0, \infty) = 0$. We observe that it follows from the scaling property (2.11) of $\overline{P}$ that

$$\lim_{t\to\infty} \overline{P}_t f(x) = \lim_{t\to\infty} P_{e^t-1} d_{e^{-t}} f(x) = \lim_{t\to\infty} P_{1-e^{-t}} f(xe^{-t}) = P_1 f(0).$$

Recalling from [30] that the condition on the so-called ascending ladder height exponent $\phi'_+(0^+) < +\infty$ in (2.15) is a necessary and sufficient one for $P_1 f(0) = \lim_{x\downarrow 0} \mathbb{E}_x[f(X_1)]$ to define a non-degenerate probability distribution. It means that, under this condition, $\overline{P}$

admits an invariant non-degenerate probability measure.

It is also worth noting that generalized CIR processes fall in the class of polynomial processes in the sense of Cuchiero et al.[44]. Indeed, one can easily check that for any $\Psi \in \mathcal{N}$, since $\Psi(n) < \infty$ for $n \in \mathbb{N}$, it follows from (3.26), that with $p_n(x) = x^n$, $x > 0$,

$$\mathbf{A}p_n(x) = \Psi(n)p_{n-1}(x) - np_n(x).$$

Finally, for any $r > 0$, we define the function $\mathcal{I}_{\Psi,r}$ by

$$\mathcal{I}_{\Psi,r}(z) = \sum_{n=0}^{\infty} \frac{(r)_n}{W_\Psi(n+1)} z^n \tag{2.26}$$

where $(r)_n = \frac{\Gamma(r+n)}{\Gamma(r)}$ stands for the Pochhammer symbol.

Proposition 2.3.3. Let $\Psi \in \mathcal{N}$ with $\lim_{u\to\infty} \frac{\Psi(u)}{u} = \infty$. Then, for any $r > 0$, the following statements hold.

(1) $\mathcal{I}_{\Psi,r}$ defined by (2.26) is an entire function. Moreover, it is positive and non-decreasing on $\mathbb{R}_+$.

(2) $\mathcal{I}_{\Psi,r} \in \mathcal{I}_r(\overline{P})$, i.e. for any $t \geq 0$, $e^{-rt}\overline{P}_t\mathcal{I}_{\Psi,r}(x) = \mathcal{I}_{\Psi,r}(x)$.

(3) The family of operators $Q^{(r)} = (e^{-rt}Q_t)_{t\geq 0}$ defined, for any $x \in E$ and $t \geq 0$, by

$$Q_t^{(r)} f(y) = \overline{P}_t^{(r)}(f \circ \mathcal{I}_{\Psi,r})(x), \quad y = \mathcal{I}_{\Psi,r}(x), \tag{2.27}$$

is a pricing semigroup.

Remark 2.3.4. Note that when $\Pi(0,\infty) = 0$, these r-invariant functions were defined in [122].

Proof. First, for $r > 0$ and $n \in \mathbb{N}$, note that

$$\frac{|W_\Psi(n+1)(r)_{n+1}|}{|W_\Psi(n+2)(r)_n|} = \frac{r+n}{\Psi(n+1)}.$$

Since we have that $\lim_{u\to\infty}\frac{\Psi(u)}{u}=\infty$, it follows that $\mathcal{I}_{\Psi,r}$ is an entire function. On the other hand, since $\mathcal{I}_{\Psi,r}(0)=1$, the rest of the claim (1) is obvious. Next, we apply Tonelli's theorem to change the order of integration and get

$$\overline{P}_t\mathcal{I}_{\Psi,r}(x)=\sum_{n=0}^{\infty}\frac{(r)_n}{W_\Psi(n+1)}\overline{P}_t p_n(x)=\sum_{n=0}^{\infty}\frac{(r)_n}{W_\Psi(n+1)}P_{e^t-1}d_{e^{-t}}p_n(x) \tag{2.28}$$

where we have successively used the linearity of $\overline{P}_t$ and the identity (2.24). Next, using the self-similarity property (2.11) of P, we have that

$$P_{e^t-1}d_{e^{-t}}p_n(x)=d_{e^{-t}}P_{1-e^{-t}}p_n(x).$$

Since $\Psi\in\mathcal{N}$, we can substitute (2.20) in (2.28) to get

$$\begin{aligned}
\overline{P}_t\mathcal{I}_{\Psi,r}(x) &= \sum_{n=0}^{\infty}\frac{(r)_n}{W_\Psi(n+1)}d_{e^{-t}}P_{1-e^{-t}}p_n(x)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\Gamma(r+n)}{\Gamma(r)}\frac{p_{n-k}(x)}{W_\Psi(n-k+1)}e^{-nt}\frac{(e^t-1)^k}{k!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{p_{n-k}(x)}{W_\Psi(n-k+1)}\frac{\Gamma(r+n-k)}{\Gamma(r)}e^{-(n-k)t}e^{-kt}\binom{r+n-1}{k}(e^t-1)^k\\
&= \sum_{n=0}^{\infty}\frac{p_n(x)}{W_\Psi(n+1)}\frac{\Gamma(r+n)}{\Gamma(r)}e^{-nt}\sum_{k=0}^{\infty}\binom{r+n+k-1}{k}(e^{-t}(e^t-1))^k\\
&= \sum_{n=0}^{\infty}\frac{(r)_n}{W_\Psi(n+1)}x^n e^{-nt}e^{(r+n)t}=e^{rt}\mathcal{I}_{\Psi,r}(x)
\end{aligned}$$

where for the second last identity, we used the binomial formula $(1-z)^{-c-1}=\sum_{n=0}^{\infty}\binom{n+c}{n}z^n$, $|z|<1$, with $\binom{c}{k}=\frac{\Gamma(c+1)}{\Gamma(c+1-k)k!}$ for $k\in\mathbb{N}$ and arbitrary $c\in\mathbb{C}$ are the generalized binomial coefficients. Therefore $\mathcal{I}_{\Psi,r}\in\mathcal{I}_r(\overline{P})$, which concludes the proof of part (2). Finally, item (3) is a direct application of Proposition 2.2.1(a) (see also Remark 2.2.2). ■

2.4 Option pricing in the jump CIR models

In this section we implement the risk-neutral pricing technique via an intertwining relation for some derivative securities. In particular, we assume that our market consists of one riskless asset (the bond), and one risky asset (the stock). We then show that the price of an option has a nice analytical formula expressed as an eigenvalue expansion which is convenient for numerical purposes. It is worth mentioning that there is a rich literature in mathematical finance on the application of spectral theory for pricing derivative securities, see e.g. Jarrow and Rudd [87], Hansen et al. [78], Davydov and Linetsky [47] for diffusions and Chazal et al. [35] for general affine processes.

Recall that, to prevent arbitrage, the price of a derivative equals the expectation of its payoff f under the risk-neutral probability measure, discounted at the risk-free rate. That is, given the risk-neutral probability measure, the time-zero price of an option with underlying $S = (S_t)_{t\geq 0}$ and maturity $T > 0$ is given by

$$\mathrm{P}_T(S_0) = \mathbb{E}_{S_0}[e^{-rT} f(S_T)]$$

where $\mathbb{E}_{S_0}$ denotes the expectation under the risk-neutral measure $\mathbb{Q}$ for S starting at $S_0 \geq 0$, and r is the risk-free rate.

In [129], Patie and Savov developed a comprehensive spectral decomposition of the generalized CIR (Laguerre) semigroups, the family of non-self-adjoint semigroups which are obtained from the self-similar semigroups by (2.24). We now describe the spectral decomposition of a specific instance of these processes which we call a *jump CIR process*. Note that in [130], the eigenvalues expansions is provided for the jump CIR processes having 0 as a reflecting boundary. Let $\mathfrak{m} \geq 1$ and consider, for any $u > 0$,

$$\phi_{\mathfrak{m}}(u) = \frac{(u + \mathfrak{m} + 1)(u + \mathfrak{m} - 1)}{u + \mathfrak{m}} = u + \frac{\mathfrak{m}^2 - 1}{\mathfrak{m}} + \int_0^\infty (1 - e^{-uy}) e^{-\mathfrak{m}y} dy.$$

Then, we have that

$$\Psi_{\mathfrak{m}}(u) = u\phi_{\mathfrak{m}}(u) = u\frac{(u+\mathfrak{m}+1)(u+\mathfrak{m}-1)}{u+\mathfrak{m}} \in \mathcal{N}, \tag{2.29}$$

where we mention that the first identity provides the Wiener-Hopf factorization (2.15) of $\Psi_{\mathfrak{m}}$. Next, its associated infinitesimal generator is given, for f smooth, by

$$\mathbf{A}_{\mathfrak{m}} f(x) = xf''(x) + \left(\frac{\mathfrak{m}^2-1}{\mathfrak{m}} + 1 - x\right) f'(x) + \int_0^\infty (f(e^{-y}x) - f(x) + yxf'(x))\frac{\mathfrak{m}e^{-\mathfrak{m}y}}{x}dy. \tag{2.30}$$

Note that the trajectories of the process X are càdlàg and their discontinuities are produced by negative jumps (it jumps from x to xe^{-y} at a frequency given by the Lévy kernel $\frac{\mathfrak{m}e^{-\mathfrak{m}y}}{x}$). Moreover, since plainly $\phi_+(u) = u$, the associated semigroup $\overline{P}$ is ergodic with a unique invariant measure, which in this case is an absolutely continuous probability measure with a density denoted by ν and which takes the form

$$\nu(x) = \frac{(1+x)}{\mathfrak{m}+1}\frac{x^{\mathfrak{m}-1}e^{-x}}{\Gamma(\mathfrak{m})} = \frac{(1+x)}{\mathfrak{m}+1}\varepsilon_{\mathfrak{m}-1}(x), \quad x > 0,$$

where the last identity serves to set a notation. Next, we define the Hilbert space

$$L^2(\nu) = \{f : \mathbb{R}_+ \to \mathbb{R} \text{ measurable}; \int_0^\infty f^2(x)\nu(x)dx < \infty\},$$

endowed with the norm $\|\cdot\|_{L^2(\nu)} = \|\cdot\|_\nu$ and inner product $\langle\cdot,\cdot\rangle_\nu$. In [129], the authors show that for all $f \in L^2(\nu)$ and $t > 0$, $\overline{P}_t f$ admits the following spectral expansion

$$\overline{P}_t f(x) = \sum_{n=0}^{\infty} e^{-nt}\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n(x) \tag{2.31}$$

where $(\mathcal{P}_n, \mathcal{V}_n)_{n\geq 0}$ form a biorthogonal sequence of $Ł^2(\nu)$ and are expressed in terms of the Laguerre polynomials $\left(\mathcal{L}_n^{(\mathfrak{m})}\right)_{n\geq 0}$ as follows, for $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{P}_n(x) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(\mathfrak{m}+2)}{\Gamma(\mathfrak{m}+k+2)}\frac{\mathfrak{m}+k}{\mathfrak{m}}x^k = \mathfrak{c}_n(\mathfrak{m}+1)\mathcal{L}_n^{(\mathfrak{m}+1)}(x) - \frac{c_n(\mathfrak{m}+1)}{\mathfrak{m}}x\mathcal{L}_{n-1}^{(\mathfrak{m}+2)}(x), \\
\mathcal{V}_n(x) &= \frac{1}{x+1}\mathcal{L}_n^{(\mathfrak{m}-1)}(x) + \frac{x}{x+1}\mathcal{L}_n^{(\mathfrak{m})}(x).
\end{aligned}$$

Here, $\mathfrak{c}_n(\mathfrak{m}+1) = \frac{\Gamma(n+1)\Gamma(\mathfrak{m}+2)}{\Gamma(n+\mathfrak{m}+2)}$ and we recall that $\mathcal{L}_n^{(\mathfrak{m})}$ is the Laguerre polynomial of order $\mathfrak{m}$, defined either by means of the Rodrigues operator $\mathcal{R}^{(n)}$ as follows

$$\mathcal{L}_n^{(\mathfrak{m})}(x) = \frac{\mathcal{R}^{(n)}\varepsilon_{\mathfrak{m}}(x)}{\varepsilon_{\mathfrak{m}}(x)} = \frac{1}{n!}\frac{(x^n\varepsilon_{\mathfrak{m}}(x))^{(n)}}{\varepsilon_{\mathfrak{m}}(x)}, \quad x > 0,$$

or through its polynomial representation

$$\mathcal{L}_n^{(\mathfrak{m})}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\mathfrak{m}}{n-k}\frac{x^k}{k!}.$$

Using that $\Psi_{\mathfrak{m}}$ is given by (2.29), from (2.26) we have, for any $x, r > 0$, that

$$\mathcal{I}_{\Psi_{\mathfrak{m}},r}(x) \quad = \quad \sum_{n=0}^{\infty}\frac{(r)_n}{W_{\Psi_{\mathfrak{m}}}(n+1)}x^n = \sum_{n=0}^{\infty}\frac{(\mathfrak{m}+1)_n(r)_n}{(\mathfrak{m}+2)_n(\mathfrak{m})_n}\frac{x^n}{n!} = {}_2F_2(\mathfrak{m}+1, r, \mathfrak{m}+2, \mathfrak{m}; x)$$

where ${}_2F_2$ is the hypergeometric function. Therefore, since $\mathcal{I}_{\Psi_{\mathfrak{m}},r} \in \mathcal{I}_r(\overline{P})$, it follows from Proposition 2.3.3 that the family of linear operator $Q = (Q_t)_{t\geq 0}$ defined, for any $x \in E$ and $t \geq 0$, by

$$Q_t^{(r)}f(y) = \overline{P}_t^{(r)}(f \circ \mathcal{I}_{\Psi_{\mathfrak{m}},r})(x), \quad y = \mathcal{I}_{\Psi_{\mathfrak{m}},r}(x),$$

is a pricing semigroup. Now, we consider an option with an underlying S and payoff f, and we model the stock by $S = (S_t = \mathcal{I}_{\Psi_{\mathfrak{m}},r}(X_t))_{t\geq 0}$ with the current spot being $S_0 = \mathcal{I}_{\Psi_{\mathfrak{m}},r}(x)$, where $X = (X_t)_{t\geq 0}$ is the jump CIR process. Then, the time-zero price of this option is given, for any maturity $T > 0$ and such that $f \circ \mathcal{I}_{\Psi_{\mathfrak{m}},r} \in L^2(\nu)$, by

$$\begin{aligned} \mathrm{P}_T(S_0) = \mathrm{P}(S_0, K, r, T; \mathfrak{m}) \quad &= \quad \mathbb{E}_{\mathcal{I}_{\Psi_{\mathfrak{m}},r}(x)}\left[e^{-rT}f(S_T)\right] = e^{-rT}\mathbb{E}_x\left[f(\mathcal{I}_{\Psi_{\mathfrak{m}},r}(X_T))\right] \\ &= \quad e^{-rT}\sum_{n=0}^{\infty}e^{-nT}\left\langle f \circ \mathcal{I}_{\Psi_{\mathfrak{m}},r}, \mathcal{V}_n\right\rangle_\nu \mathcal{P}_n(x) \end{aligned} \tag{2.32}$$

where the last line follows from the representation (5.4.3).

To numerically evaluate the infinite sum above, we need to truncate it. Thus, for $N = 1, 2, \ldots$, we define the N-th order spectral approximate for the spectral operator (5.4.3) by

$$\mathcal{S}_T^{(N)}f(x) = \sum_{n=0}^{N}e^{-nT}\left\langle f, \mathcal{V}_n\right\rangle_\nu \mathcal{P}_n(x). \tag{2.33}$$

In the following, we provide an exact exponential rate of decay of the approximation error in the Hilbert space topology.

Lemma 2.4.1. Let $T > \frac{1}{2}\ln\left(\frac{\mathfrak{m}+2}{\mathfrak{m}+1-\epsilon}\right)$ for any $0 < \epsilon < \mathfrak{m}$. Then, for any $N \in \mathbb{N}$ and $f \in L^2(\nu)$, we have

$$\left\|\overline{P}_T f - \mathcal{S}_T^{(N)} f\right\|_\nu \leq e^{-(N+1)T}\sqrt{\frac{\mathfrak{m}+1}{\mathfrak{m}+1-\epsilon}}\left\|f - \mathcal{S}_0^{(N)} f\right\|_\nu .$$

Proof. We first recall that a sequence $(P_n)_{n\geq 0}$ in the Hilbert space $L^2(\nu)$ is called a *Bessel sequence* with bound $A > 0$ if the inequality $\sum_{n=0}^{\infty} |\langle f, P_n\rangle_\nu|^2 \leq A\|f\|_\nu^2$ holds for all $f \in L^2(\nu)$, see e.g. [156]. It follows from [129, Lemma 10.4] that, writing $\mathsf{d}_\epsilon = \mathfrak{m} - 1 - \epsilon$, the sequences $\left(\frac{\mathcal{P}_n}{\sqrt{\mathfrak{c}_n(\mathsf{d}_\epsilon)}}\right)_{n\geq 0}$ and $\left(\sqrt{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n\right)_{n\geq 0}$ are Bessel sequences in $L^2(\nu)$ with bound 1. Next, observe that since $\frac{e^{-2(n-1)T}\mathfrak{c}_n(\mathsf{d}_\epsilon)\mathfrak{c}_1(\mathfrak{m})}{\mathfrak{c}_1(\mathsf{d}_\epsilon)\mathfrak{c}_n(\mathfrak{m})} = \prod_{j=0}^{n-2} e^{-2T}\left(\frac{n+\mathfrak{m}-j}{n+\mathsf{d}_\epsilon-j}\right)$, we have

$$\sup_{n\geq N+1}\frac{e^{-2nT}\mathfrak{c}_n(\mathsf{d}_\epsilon)\mathfrak{c}_1(\mathfrak{m})}{\mathfrak{c}_1(\mathsf{d}_\epsilon)\mathfrak{c}_n(\mathfrak{m})} \leq e^{-2(N+1)T} \Leftrightarrow e^{-2T}\left(\frac{\mathfrak{m}+2}{\mathsf{d}_\epsilon+2}\right) \leq 1,$$

which holds if and only if $T > \frac{1}{2}\ln\left(\frac{\mathfrak{m}+2}{\mathfrak{m}+1-\epsilon}\right)$. Now, using (5.4.3), we have, for any $n \in \mathbb{N}$ and $f \in L^2(\nu)$, that

$$\begin{aligned}
\left\|\overline{P}_T f - \mathcal{S}_T^{(N)} f\right\|_\nu^2 &= \left\|\sum_{n=N+1}^{\infty} e^{-nT}\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n\right\|_\nu^2 \leq \sum_{n=N+1}^{\infty} e^{-2nT}\frac{\mathfrak{c}_n(\mathsf{d}_\epsilon)}{\mathfrak{c}_n(\mathfrak{m})}\left|\left\langle f,\ \sqrt{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n\right\rangle_\nu\right|^2 \\
&\leq \frac{\mathfrak{c}_1(\mathsf{d}_\epsilon)}{\mathfrak{c}_1(\mathfrak{m})}\sum_{n=N+1}^{\infty}\frac{e^{-2nT}\mathfrak{c}_n(\mathsf{d}_\epsilon)\mathfrak{c}_1(\mathfrak{m})}{\mathfrak{c}_1(\mathsf{d}_\epsilon)\mathfrak{c}_n(\mathfrak{m})}\left|\left\langle f,\ \sqrt{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n\right\rangle_\nu\right|^2 \\
&\leq e^{-2(N+1)T}\frac{\mathfrak{c}_1(\mathsf{d}_\epsilon)}{\mathfrak{c}_1(\mathfrak{m})}\sum_{n=N+1}^{\infty}\left|\left\langle f,\ \sqrt{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n\right\rangle_\nu\right|^2 \\
&\leq e^{-2(N+1)T}\frac{\mathfrak{m}+1}{\mathsf{d}_\epsilon+1}\|f - \mathcal{S}_0^{(N)} f\|_\nu^2
\end{aligned}$$

where in the first and the last inequalities we used that, respectively, $\left(\frac{\mathcal{P}_n}{\sqrt{\mathfrak{c}_n(\mathsf{d}_\epsilon)}}\right)_{n\geq 0}$ and $\left(\sqrt{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n\right)_{n\geq 0}$ form a Bessel sequence in $L^2(\nu)$ with bound 1, and that when $T > \frac{1}{2}\ln\left(\frac{\mathfrak{m}+2}{\mathsf{d}_\epsilon+2}\right)$, $\frac{\mathfrak{m}+1}{\mathsf{d}_\epsilon+1}e^{-2T} \geq \frac{\mathsf{d}_\epsilon+2}{\mathsf{d}_\epsilon+1}\frac{\mathfrak{m}+1}{\mathfrak{m}+2} \geq 1$ as $\mathfrak{m} > \mathsf{d}_\epsilon$. This completes the proof. ■

Next, we proceed by illustrating this result with some numerical experiments. To this end, in order to measure the accuracy of the spectral approximation, we introduce for $\epsilon > 0$, the quantity

$$\mathbf{N}_\epsilon = \inf\{N \geq 4;\ \max\{|\mathcal{S}_N - \mathcal{S}_{N-1}|, |\mathcal{S}_N - \mathcal{S}_{N-2}|, |\mathcal{S}_N - \mathcal{S}_{N-3}|\} \leq \epsilon|\mathcal{S}_N|\}. \tag{2.34}$$

That is, $\mathbf{N}_\epsilon$ is the smallest number of terms needed in the spectral expansion such that the truncated series has "converged" in the sense that the $(\mathbf{N}_\epsilon - 1)$-th, $(\mathbf{N}_\epsilon - 2)$-th and $(\mathbf{N}_\epsilon - 3)$-th order truncated summation do not differ from the $\mathbf{N}_\epsilon$-th by more than ϵ. As a numerical example, we evaluate an at-the-money put option, i.e. $f(x) = (K - x)^+$, with $S_0 = K = 10, r = 0.03, T = 1$ and $\mathfrak{m} = 4$, with tolerance level $\epsilon = 10^{-4}$. Our algorithm returns the option price $\mathrm{P}(S_0, K, r, T; \mathfrak{m}) \approx \mathcal{S}_{\mathbf{N}_{10^{-4}}} = 5.9571$. Note that since f is bounded we have $f \circ \mathcal{I}_{\Psi_\mathfrak{m}, r} \in L^2(\nu)$. Then, the relative truncation error at order N, with $N \geq 4$, which is denoted by $\mathfrak{e}(N)$, is defined as follows

$$\mathfrak{e}(N) = \frac{\left|\sum_{n=0}^N e^{-nT} \left\langle (K - \mathcal{I}_{\Psi_\mathfrak{m}, r})^+, \mathcal{V}_n \right\rangle_\nu \mathcal{P}_n(x) - \mathcal{S}_{\mathbf{N}_{10^{-4}}}\right|}{\mathcal{S}_{\mathbf{N}_{10^{-4}}}}.$$

Figure 2.1 which shows $\mathfrak{e}(N)$ for different values of N reveals that the convergence is achieved after very few terms. Figure 2.1 indicates that a relative error of about 5×10^{-4} can be achieved only with 3 terms of the sum. Moreover, in Table 2.1 the value of $\mathbf{N}_\epsilon$ for $\epsilon = 10^{-4}$ are computed for various strikes and expiry times. The approximated option values are shown in Table 2.2. In line with Lemma 2.4.1, we see from Table 2.1 that the spectral method performs better as the expiry time increases in the sense that less terms in the sum in (2.32) are needed. As a remark, we want to emphasize that, as (2.32) indicates, the estimation of the option value also involves numerically evaluating the hypergeometric function which we do with number of significant figures being equal to 10. A numerical evaluator for the generalized hypergeometric function at the desired accuracy is publicly available on MathWorks.

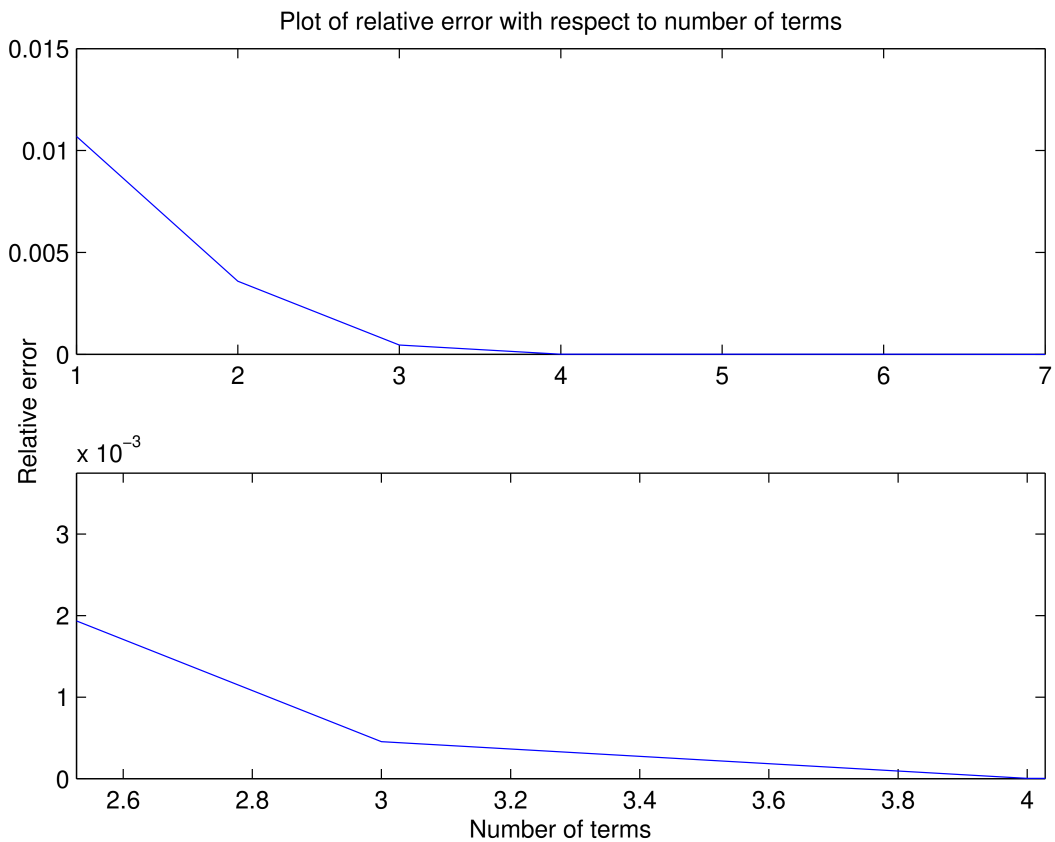

Figure 2.1: Relative series truncation error for $S_0 = K = 10, r = 0.03, T = 1, m = 4$

Maturity T \ Strike K	7	10	13	17
0.33	23	23	23	22
0.5	17	18	18	18
1	11	11	11	11
2	7	7	7	7

Table 2.1: Values of $\mathbf{N}_{10^{-4}}$ for various K and T with parameters $S_0 = 10, r = 0.03, \mathfrak{m} = 4$.

Maturity T \ Strike K	7	10	13	17
0.33	3.656	6.176	8.831	12.443
0.5	4.314	6.984	9.745	13.457
1	5.152	7.969	10.831	14.634
2	5.438	8.241	11.070	14.814

Table 2.2: Values of $\mathcal{S}_{\mathbf{N}_{10^{-4}}}$ for various K and T with parameters $S_0 = 10, r = 0.03, \mathfrak{m} = 4$.

2.5 Proofs of the main results

2.5.1 Proof of Theorem 2.2.1 and Proposition 2.2.1

First, let (2.1) holds. Then, since $\Lambda p_1 \in \mathcal{I}_r(P)$, one has, for any $t \geq 0$,

$$\Lambda Q_t p_1 = P_t \Lambda p_1 = e^{rt} \Lambda p_1.$$

Therefore, given that $(Q_t p_1 - e^{rt} p_1) \notin Ker(\Lambda) \setminus \{\mathbf{0}\}$, it follows that $Q_t p_1 = e^{rt} p_1$, i.e. $p_1 \in \mathcal{I}_r(Q) = \mathcal{I}_0(Q^{(r)})$, which means that $Q^{(r)}$ is a pricing semigroup by Proposition 2.1.1. This concludes the proof of Theorem 2.2.1.

Now, we proceed by proving Proposition 2.2.1. First, assume that (a) holds. Then, as mentioned in Remark 2.2.2, Dynkin's criterion insures that $Q = (Q_t)_{t\geq 0}$ and $Q^{(r)} = (e^{-rt} Q_t)_{t\geq 0}$ are Markov semigroups. Next, since h_r is a homeomorphism, its inverse, h_r^{-1}, exists, and we easily get that $\Lambda^{-1} f := f \circ h_r^{-1}$ is the left inverse of Λ. Observing that $\Lambda p_1 = h_r \in \mathcal{I}_r(P) =$

$\mathcal{I}_0(P^{(r)})$, by Theorem 2.2.1, we conclude that $Q^{(r)}$ is a pricing semigroups.

Next, let (b) holds. Then, since $g_\lambda \in \mathcal{E}_\lambda(P)$ with $\lambda \leq r$, we have that

$$P_t^{(r)} g_\lambda(x) = e^{-rt} P_t g_\lambda(x) \leq e^{-(r-\lambda)t} g_\lambda(x) \leq g_\lambda(x),$$

which means that $g_\lambda \in \mathcal{E}_0(P^{(r)})$. Therefore, as $\{x \in E; g_\lambda(x) \neq 0\} = E$, according to [40], the intertwining relationship boils down to a Doob's h-transform, ensuring that $Q^{(r)}$ is indeed a Markov semigroup on E. Now, following the same pattern as in the proof of part (a), we have $\Lambda^{-1} f = \frac{f}{g_\lambda}$ is the left inverse of Λ. Therefore, as g_λ is strictly positive, Λ is injective on its domain. On the other hand, since $\Lambda p_1 = p_1 g_\lambda \in \mathcal{I}_r(P) = \mathcal{I}_0(P^{(r)})$, it follows from Theorem 2.2.1 that $Q^{(r)}$ is a pricing semigroup.

To prove part (c), we first note that the proofs of parts (a) and (b) entail that $Q^{(r)}$ is a Markov semigroup. Moreover, recalling that $H_{r,\lambda}$ is a homeomorphism, it is easy to see that $\Lambda^{-1} f = \frac{f}{g_\lambda} \circ H_{r,\lambda}^{-1}$ is the left inverse of Λ. Thus, as above, Λ is injective on its domain. On the other hand, we have

$$\Lambda p_1 = (p_1 \circ H_{r,\lambda}) g_\lambda = H_{r,\lambda} g_\lambda = h_r \in \mathcal{I}_r(P) = \mathcal{I}_0(P^{(r)}).$$

Hence, by an application of Theorem 2.2.1, $Q^{(r)}$ is a pricing semigroup, and this concludes the proof of Proposition 2.2.1. ■

2.5.2 Extended generators and resolvents

In order to prove Proposition 2.2.3, we shall need the following result which shows that one may define the (domain of the) extended generator using the resolvent operator $(U_q)_{q>0}$ associated to the transition semigroup $P = (P_t)_{t\geq 0}$. Recall that, for any $f \in \mathcal{B}(E)$,

$$U_q f = \int_0^\infty e^{-qt} P_t f dt.$$

Proposition 2.5.1. Let $f \in \mathcal{B}(E)$ with $\lim_{t\to\infty} e^{-qt}P_t f(x) = 0$ for some $q > 0$ and any $x \in E$. Then $f \in \mathcal{D}(\mathcal{A})$ if and only if there exists a function $g \in \mathcal{B}(E)$ such that the mapping $t \to g(X_t)$ is integrable $\mathbb{P}_x$-a.s, $\lim_{t\to\infty} e^{-qt}P_t g(x) = 0$, $\forall x \in E$, and

$$U_q(qf - g) = f. \tag{2.35}$$

The function $g = \mathcal{A}f$ is uniquely determined up to a set of potential zero, that is, a set $C \subseteq E$ such that $U_q\mathbf{1}_C = 0$ for any $q > 0$.

Proof. First, let $f \in \mathcal{D}(\mathcal{A})$. Then, by the definition of the extended generator, there exists a function $g \in \mathcal{B}(E)$ such that M^f defined by (2.4) is a martingale. Therefore, for any $t \geq 0$ and $x \in E$, $\mathbb{E}_x[M_t^f] = 0$, or, equivalently,

$$\mathbb{E}_x\left[f(X_t) - f(X_0) - \int_0^t g(X_s)ds\right] = P_t f(x) - f(x) - \int_0^t P_s g(x)ds = 0. \tag{2.36}$$

Then, we have

$$\begin{aligned}
U_q(qf - g)(x) &= q\int_0^\infty e^{-qt}P_t f(x)dt - \int_0^\infty e^{-qt}P_t g(x)dt \\
&= q\int_0^\infty e^{-qt}\left[f(x) + \int_0^t P_s g(x)ds\right]dt - \int_0^\infty e^{-qt}P_t g(x)dt \\
&= f(x) + q\int_0^\infty e^{-qt}\int_0^t P_s g(x)dsdt - \int_0^\infty e^{-qt}P_t g(x)dt \\
&= f(x) - e^{-qt}\int_0^t P_s g(x)ds\Big|_0^\infty = f(x) + \lim_{t\to\infty}\left[e^{-qt}(P_t f(x) - f(x))\right] = f(x)
\end{aligned}$$

where we used that by Lebesgue's theorem $\frac{d}{dt}\int_0^t P_s g(x)dx = P_t g(x)$ and $\lim_{t\to\infty} e^{-qt}P_t f(x) = 0$ for any $x \in E$. Hence (2.35) holds.

Conversely, assume there exists a function g such that (2.35) is true. Then, by L'Hôpital's rule, for any $x \in E$, we have

$$\lim_{t\to\infty} e^{-qt}\int_0^t P_s g(x)ds = \frac{1}{q}\lim_{t\to\infty} e^{-qt}P_t g(x) = 0.$$

Therefore, performing an integration by parts, we obtain

$$U_q g(x) = q \int_0^\infty e^{-qt} \int_0^t P_s g(x) ds dt. \tag{2.37}$$

On the other hand, from (2.35), we get

$$U_q g(x) = qU_q f(x) - f(x) = q \int_0^\infty e^{-qt} \left(P_t f(x) - f(x) \right) dt. \tag{2.38}$$

Combining (2.37) and (2.38), and applying inverse Laplace transform, we see that for any $t \geq 0$ and $x \in E$,

$$P_t f(x) - f(x) - \int_0^t P_s g(x) ds = 0. \tag{2.39}$$

Now, take $u, t \geq 0$, then, assuming without loss of generality that $f(X_0) = 0$, the Markov property yields

$$\begin{aligned}
\mathbb{E}_x \left[f(X_{t+u}) - \int_0^{t+u} g(X_s) ds \middle| \mathcal{F}_u \right] &= \mathbb{E}_x \left[f(X_{t+u}) - \int_0^{t+u} g(X_s) ds \middle| \mathcal{F}_u \right] \\
&= \mathbb{E}_x \left[\left(f(X_{t+u}) - \int_u^{t+u} g(X_s) ds \right) \middle| \mathcal{F}_u \right] - \int_0^u g(X_s) ds \\
&= P_t f(X_u) - \int_0^t P_s g(X_u) ds - \int_0^u g(X_s) ds \\
&= f(X_u) - \int_0^u g(X_s) ds,
\end{aligned}$$

where in the last equality we used (2.39). This implies that M^f defined by (2.4) is a martingale, and hence $f \in \mathcal{D}(\mathcal{A})$ with $g = \mathcal{A} f$. ■

Remark 2.5.1. Note that by (2.35), the condition stated in Proposition 2.1.1 regarding the pricing semigroup is equivalent to $p_1 = qU_q^{(r)} p_1 = qU_{q+r} p_1$ for all $q > 0$, where $(U_q^{(r)})_{q>0}$ is the q-resolvent for $P^{(r)}$.

Lemma 2.5.1. Let $f \in \mathcal{D}(\mathcal{A})$. Then, $f \in \mathcal{I}_r(P)$ if and only if $\mathcal{A} f = r f$.

Proof. First, let $f \in \mathcal{I}_r(P)$. Then, for any $q > r$ and for all $x \in E$, we have that

$$\lim_{t \to \infty} e^{-qt} P_t f(x) = \lim_{t \to \infty} e^{-(q-r)t} f(x) = 0. \tag{2.40}$$

Similarly,

$$\lim_{t\to\infty} e^{-(q+r)t} P_t f(x) = 0. \tag{2.41}$$

Next, since $f \in \mathcal{D}(\mathcal{A})$, it follows from (2.41) and Proposition 2.5.1 that

$$U_{q+r}((q+r)f - \mathcal{A}f) = f. \tag{2.42}$$

Equivalently, we can write

$$U_q^{(r)}(qf - (\mathcal{A}f - rf)) = f.$$

Then, it follows from (2.40) and Proposition 2.5.1 that $f \in \mathcal{D}(\mathcal{A}^{(r)})$ and

$$\mathcal{A}^{(r)} f = \mathcal{A}f - rf. \tag{2.43}$$

Moreover, Definition 2.2.0.1 implies that

$$\left(f(X_t^{(r)}) - \int_0^t \mathcal{A}^{(r)} f(X_s) ds \right)_{t\geq 0}$$

is a martingale. On the other hand, since $f \in \mathcal{I}_r(P)$, it follows from Lemma 2.1.1 that $(e^{-rt} f(X_t))_{t\geq 0}$ is a martingale, too. Therefore, we must have

$$\mathcal{A}^{(r)} f = 0.$$

Hence, it follows from (2.43) that

$$\mathcal{A}f = rf, \tag{2.44}$$

which concludes the proof of the first part of the lemma.

Now, we move to prove the other direction, i.e. let $\mathcal{A}f = rf$, and we need to show that $f \in \mathcal{I}_r(P)$. Since $f \in \mathcal{D}(\mathcal{A})$, first note that it follows from Proposition 2.2.2(2) and (2.44) that for any $x \in E$ and $t \geq 0$,

$$P_t f(x) = f(x) + \int_0^t P_s \mathcal{A} f(x) ds = f(x) + r \int_0^t P_s f(x) ds.$$

Therefore,

$$\lim_{t\to\infty} e^{-qt} P_t f(x) = \lim_{t\to\infty} e^{-qt} f(x) + r \lim_{t\to\infty} e^{-qt} \int_0^t P_s f(x) ds = \frac{r}{q} \lim_{t\to\infty} e^{-qt} P_t f(x),$$

where in the second equality we used L'Hôpital's rule. Since we assumed $q > r$, we get

$$\lim_{t\to\infty} e^{-qt} P_t f(x) = 0.$$

Hence, we can successfully apply Proposition 2.5.1 to get

$$U_q(qf - \mathcal{A}f) = f. \tag{2.45}$$

Combining (2.44) and (2.45), we get

$$(q - r)U_q f = f.$$

Thus, it follows from Remark 2.5.1 that $f \in \mathcal{I}_r(P)$, which concludes the proof of the lemma.

■

2.5.3 Proof of Proposition 2.2.3

First, observe that since $\Lambda p_1 \in \mathcal{D}(\mathcal{A}_P)$ and $\Lambda p_1 \in \mathcal{I}_r(P)$, it follows from Lemma 2.5.1 that

$$\mathcal{A}_P \Lambda p_1 = r\Lambda p_1.$$

Thus, the intertwining relation (2.5) yields that

$$\Lambda \mathcal{A}_Q p_1 = r\Lambda p_1,$$

or, equivalently,

$$\Lambda(\mathcal{A}_Q p_1 - r p_1) = 0.$$

Hence, given that $(\mathcal{A}_Q p_1 - r p_1) \notin Ker(\Lambda) \setminus \{\mathbf{0}\}$, we get

$$\mathcal{A}_Q p_1 = r p_1.$$

Therefore, it follows from Lemma 2.5.1 that $p_1 \in \mathcal{I}_r(Q)$, or equivalently, $p_1 \in \mathcal{I}_0(Q^{(r)})$, that is $Q^{(r)}$ is a pricing semigroup by Theorem 2.2.1. ■

CHAPTER 3

SPECTRAL PROJECTIONS CORRELATION STRUCTURE FOR SHORT-TO-LONG RANGE DEPENDENT PROCESSES

3.1 Introduction

Stochastic processes play an important role in the investigation of random phenomena depending on time. When using a stochastic process for modeling or for statistical testing purposes, one should take into account its special features which indicate how well the process reflects the reality. Some of the most essential features include (but are not limited to) observing whether the process is Markovian or not, whether its trajectories are continuous or incorporate jumps, what type of range dependence it exhibits, and how far it is from symmetry (self-adjointness).

With the objective in mind, we introduce the concept of (biorthogonal) spectral projections correlation functions, see Definition 3.1.2.1 below. We proceed by computing explicitly these functions along with their large time asymptotic behavior for three classes of processes, namely Markov processes, Markov processes subordinated in the sense of Bochner and non-Markovian processes which are obtained by time-changing a Markov process with an inverse of a subordinator. These findings enable us to provide a unified and original framework for designing statistical tests that investigates critical properties of a stochastic process including the one described above. Indeed, in these three scenarios the (biorthogonal) spectral projections correlation functions have different expressions, involving some quantities characterizing the process, such as their eigenvalues with their associated condition number or the angle between the spectral projections.

We indicate that the recent years have witnessed the ubiquity of such non-Markovian dynamics in relation to the fractional Cauchy problem, see e.g. [152, 116, 77], and, also due to their central role in diverse physical applications within the field of anomalous diffusion, see e.g. [107], as well as for neuronal models for which their long range dependence feature is attractive, see e.g. [100]. We also mention that Leonenko et al. [99] and Mijena and Nane [112] investigate the orthogonal spectral projections correlation structure in the framework of Pearson diffusions, i.e. diffusions with polynomial coefficients. More specifically, in [99], the authors discuss the case when a Pearson diffusion is time-changed by an inverse of an α-stable subordinator, $0 < \alpha < 1$. Whereas the authors of [112] consider a Pearson diffusion time-changed by an inverse of a linear combination of independent α- and β-stable subordinators, $0 < \alpha, \beta < 1$. In this work, we start with a general Markov process that admits an invariant measure with its associated semigroup not necessarily being self-adjoint and local, and then we perform a time-change with general subordinators and their inverses.

Finally, we emphasize that the notion of long-range dependence, also known as long memory, of stochastic processes has been and it is still a center of great interests in probability theory and its applications in the last decades. We refer for thorough and historical account of this concept to the recent monograph of Samorodnitsky [142]. The definitions of long-range dependence based on the second-order properties of a stationary stochastic process such as asymptotic behavior of covariances, spectral density, and variances of partial sums are among the most developed ones appearing in literature. These second-order properties are conceptually relatively simple and easy to estimate from the data. By far the most popular point of view on range dependence is through the rate of decay of covariance or correlation functions. Conceptually, short memory corresponds to a sufficiently fast rate of decay of the correlation (covariance) function as geometric decay, and long-range dependence corresponds to a sufficiently slow rate of decay of the correlation (covariance) function as power decay.

3.1.1 Preliminaries

Let $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ be a stochastic process defined on a sample filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and state space $E \subseteq \mathbb{R}$, endowed with a sigma-algebra $\mathcal{E}$. Let its associated family of linear operators $\mathbf{P} = (\mathbf{P}_t)_{t\geq 0}$ defined, for any $t \geq 0$ and $f \in \mathcal{B}_b(E)$, the space of bounded Borelian functions on E, by

$$\mathbf{P}_t f(x) = \mathbb{E}_x[f(\mathbf{X}_t)],$$

where $\mathbb{E}_x$ stands for the expectation operator with respect to $\mathbb{P}_x(\mathbf{X}_0 = x) = 1$. Since $x \mapsto \mathbb{E}_x$ is $\mathcal{E}$-measurable, for any Radon measure ν, we use the notation

$$\nu\mathbf{P}_t f = \mathbb{E}_\nu[f(\mathbf{X}_t)] = \int_E \mathbb{E}_x[f(\mathbf{X}_t)]\nu(dx).$$

We say that a Radon measure ν on E is a *marginal stationary measure*, if for all $t \geq 0$,

$$\nu\mathbf{P}_t f = \nu f. \tag{3.1}$$

Note that if $\mathbf{X}$ is a Markov process and (3.1) holds, we say that ν is an *invariant measure*. Then, since ν is non-negative on E, we define the weighted Hilbert space

$$L^2(\nu) = \{f : E \to \mathbb{R} \text{ measurable}; \int_E f^2(x)\nu(dx) < \infty\},$$

endowed with the inner product $\langle \cdot, \cdot \rangle_\nu$, where $\langle f, g \rangle_\nu = \int_0^\infty f(x)g(x)\nu(dx)$, and norm $\|f\|_\nu = \sqrt{\langle f, f \rangle_\nu}$. Next, the operators $\mathbf{P}_t$, $t \geq 0$ being linear, positive and with total mass $\mathbf{P}_t\mathbb{1} = \mathbb{1}$ with $\mathbb{1}$ being the identity function on the appropriate space, we have, by Jensen's inequality, for any $f \in C_0(E) \subseteq \mathcal{B}_b(E)$ where $C_0(E)$ is the set of continuous functions on E

vanishing at infinity,

$$\|\mathbf{P}_t f\|_\nu^2 = \int_E (\mathbf{P}_t f)^2(x)\nu(dx) \leq \int_E \mathbf{P}_t f^2(x)\nu(dx) = \nu f^2.$$

Thus, the Hahn-Banach theorem yields that we can extend $\mathbf{P}_t$ as a contraction of $L^2(\nu)$. From now on, when there is no confusion, we denote by $\mathbf{P}_t$ its extension to $L^2(\nu)$. Now, let $\mathbf{P}^* = (\mathbf{P}_t^*)_{t\geq 0}$ be the adjoint of $\mathbf{P}$ in $L^2(\nu)$, i.e. for any $t \geq 0$ and $f, g \in L^2(\nu)$,

$$\langle \mathbf{P}_t f, g\rangle_\nu = \langle f, \mathbf{P}_t^* g\rangle_\nu. \tag{3.2}$$

We are now ready to state the following hypothesis.

Assumption 1. Let $\mathtt{N} \subseteq \mathbb{N}$ be a finite or a countable set, and for any $t \geq 0$, $(\mathcal{P}_n)_{n\in\mathtt{N}}$ (resp. $(\mathcal{V}_n)_{n\in\mathtt{N}}$) be a set of eigenfunctions of $\mathbf{P}_t$ (resp. $\mathbf{P}_t^*$) in $L^2(\nu)$ in the sense that there exist distinct $(\lambda_n)_{n\in\mathtt{N}} \in \mathbb{R}_+$ such that for any $n \in \mathtt{N}$ and $t \geq 0$, we have

$$\mathbf{P}_t \mathcal{P}_n = e^{-\lambda_n t}\mathcal{P}_n, \tag{3.3}$$

$$\mathbf{P}_t^* \mathcal{V}_n = e^{-\lambda_n t}\mathcal{V}_n. \tag{3.4}$$

We may also find convenient to characterize the $\mathcal{V}_n$'s by duality using (3.2), i.e. $\langle \mathbf{P}_t f, \mathcal{V}_n\rangle_\nu = e^{-\lambda_n t}\langle f, \mathcal{V}_n\rangle_\nu$, for all $f \in L^2(\nu)$. Note that the assumption on $(\lambda_n)_{n\in\mathtt{N}}$ being of multiplicity 1 is in fact for sake of simplicity since we mean to consider only one of the eigenfunctions in the eigenspace associated to each eigenvalue.
Next, without loss of generality, we assume that for any $n \in \mathtt{N}$,

$$\langle \mathcal{P}_n, \mathcal{V}_n\rangle_\nu = 1.$$

Indeed, if $\langle \mathcal{P}_n, \mathcal{V}_n\rangle_\nu = a_n \neq 0$ for $n \in \mathtt{N}$, then we could consider the sequences $\bar{\mathcal{P}}_n = \frac{\mathcal{P}_n}{\sqrt{|a_n|}}$ and $\bar{\mathcal{V}}_n = \frac{\mathcal{V}_n}{\sqrt{|a_n|}}$, for which, obviously, we have $\langle \bar{\mathcal{P}}_n, \bar{\mathcal{V}}_n\rangle_\nu = 1$ for $n \in \mathtt{N}$. We also note that the condition $\langle \mathcal{P}_n, \mathcal{V}_n\rangle_\nu = 1$ does not constrain the norms of the sequences $(\mathcal{P}_n)_{n\in\mathtt{N}}$ and $(\mathcal{V}_n)_{n\in\mathtt{N}}$

to be 1, but it only follows from Cauchy-Schwartz inequality that, for any $n \in \mathbb{N}$,

$$1 = |\langle \mathcal{P}_n, \mathcal{V}_n \rangle_\nu| \le \|\mathcal{P}_n\|_\nu \|\mathcal{V}_n\|_\nu.$$

In Lemma 3.4.1 below, we shall show that $(\mathcal{P}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$ form a biorthogonal sequence in $L^2(\nu)$, i.e. $\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu = \delta_{mn}$, where δ_{mn} is the Kronecker symbol defined in (3.12). In particular, if $\mathbf{P}$ is self-adjoint in $L^2(\nu)$, i.e. for all $t \ge 0$, $\mathbf{P}_t = \mathbf{P}_t^*$, we have $\mathcal{P}_n = \mathcal{V}_n$ for $n \in \mathbb{N}$, and $(\mathcal{P}_n)_{n\in\mathbb{N}}$ form an orthonormal sequence in $L^2(\nu)$. Below we consider $\mathbf{X}$ to belong to one of the following three families of stochastic processes.

Markov process

First, let $\mathbf{X} = X$ with $X = (X_t)_{t\ge0}$ a Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$. We endow the state space E with a sigma-algebra $\mathcal{E}$. Let its associated semigroup be the family of linear operators $\mathbf{P} = P = (P_t)_{t\ge0}$ defined, for any $t \ge 0$ and $f \in \mathcal{B}_b(E)$, by

$$P_t f(x) = \mathbb{E}_x[f(X_t)].$$

Next, we assume that for $t \ge 0$ and $f \in \mathcal{B}_b(E)$ the mapping $t \mapsto P_t f$ is continuous (this is equivalent to the stochastic continuity property of the process X), and the semigroup P admits an invariant probability measure ν, i.e. $\nu P_t f = \nu f$. In such framework, a classical result states that the semigroup P can be extended to a strongly continuous contraction semigroup in $L^2(\nu)$, see e.g. Da Prato [45], and by an abuse of notation, we still denote its extension to $L^2(\nu)$ by P. Note that the adjoint of P in $L^2(\nu)$, P^* is the semigroup of a stochastic process which may not be necessarily a strong Markov one, but instead has the moderate Markov property, see e.g. Chung and Walsh [41, Chapter 13] for more details.

Bochner subordination

In Section 3.2 below, we also study the spectral projections correlation structure of subordinated Markov processes. *Bochner subordination* is a transformation of a Markov process to a new one through random time change by an independent subordinator, i.e. a real-valued Lévy process with non-decreasing sample paths, see e.g. [22], [23], [145]. From the operator semigroup perspective, Bochner subordination is a classical method for generating a new semigroup of linear operators on a Banach space from an existing one. More formally, using the notation of Section 3.1.1 above, for $P = (P_t)_{t\geq 0}$, a strongly continuous contraction semigroup in $L^2(\nu)$, and $(\mu_t)_{t\geq 0}$, a vaguely continuous convolution semigroup of probability measures on $[0, \infty)$, the subordination of P in the sense of Bochner is defined by

$$P_t^{\varphi} f(x) = \int_0^{\infty} P_s f(x) \mu_t(ds), \quad t \geq 0, \ f \in \mathcal{B}_b(E). \tag{3.5}$$

The superscript φ alludes to the Laplace exponent of $(\mu_t)_{t\geq 0}$, which is a Bernstein function with the following representation, for $\lambda \geq 0$,

$$\varphi(\lambda) = \varrho\lambda + \int_0^{\infty} (1 - e^{-\lambda y}) \vartheta(dy), \tag{3.6}$$

where $\varrho \geq 0$, and ϑ is a Lévy measure concentrated on $\mathbb{R}_+$ satisfying $\int_0^{\infty} (1 \wedge y) \vartheta(dy) < \infty$. Note that $(\mu_t)_{t\geq 0}$ gives rise to a Lévy subordinator $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$, which is assumed to be independent of X, and the law of $\mathcal{T}$ is uniquely characterized by its Laplace exponent φ, that is, for $t, \lambda \geq 0$,

$$\mathbb{E}\left[e^{-\lambda \mathcal{T}_t}\right] = e^{-t\varphi(\lambda)}. \tag{3.7}$$

We write $\mathbf{X} = X_{\mathcal{T}} = (X_{\mathcal{T}_t})_{t\geq 0}$ for the Markov process associated with the semigroup $\mathbf{P}_t f(x) = P_t^{\varphi} f(x) = \mathbb{E}_x[f(X_{\mathcal{T}_t})]$. Moreover, one has that ν is also an invariant measure for the semigroup P^{φ}. Indeed, let $f \in \mathcal{B}_b(E)$ and assume, without loss of generality, f is non-negative, then, for $t \geq 0$, we have

$$\nu P_t^{\varphi} f = \langle P_t^{\varphi} f, \mathbb{1} \rangle_\nu = \int_0^\infty \langle P_s f, \mathbb{1} \rangle_\nu \mu_t(ds) = \int_0^\infty \nu f \mu_t(ds) = \nu f,$$

where we used Tonelli's theorem, the fact that ν is an invariant probability measure for P, and $\varphi(0) = 0$ in (3.6). Therefore, as above, P^φ can be extended to a contraction semigroup in $L^2(\nu)$. It is easy to note that the semigroup P^φ shares the same eigenspaces and co-eigenspaces (eigenspaces for the adjoint) as P, and, in particular, we have the following.

Proposition 3.1.1. Let $(\mathcal{P}_n)_{n\in\mathtt{N}}$ and $(\mathcal{V}_n)_{n\in\mathtt{N}}$ be as defined in Assumption 1 with $\mathbf{P} = P$ of Section 3.1.1. Then, $(\mathcal{P}_n)_{n\in\mathtt{N}}$ and $(\mathcal{V}_n)_{n\in\mathtt{N}}$ are the eigenfunctions of the semigroup P^φ and its adjoint in $L^2(\nu)$, respectively, associated to the eigenvalues $(\varphi(\lambda_n))_{n\in\mathtt{N}}$.

Proof. First, note that for $n \in \mathtt{N}$, $\mathcal{P}_n \in L^2(\nu)$, and for any $t \geq 0$, we have

$$P_t^{\varphi}\mathcal{P}_n = \int_0^\infty P_s \mathcal{P}_n \mu_t(ds) = \mathcal{P}_n \int_0^\infty e^{-\lambda_n s}\mu_t(ds) = e^{-t\varphi(\lambda_n)}\mathcal{P}_n,$$

where in the second equality we used (3.3), and the last step follows from (3.7). Next, for $f \in L^2(\nu)$, $n \in \mathtt{N}$ and $t \geq 0$, note that

$$\begin{aligned}
\langle P_t^{\varphi} f, \mathcal{V}_n \rangle_\nu &= \int_E P_t^{\varphi} f(x) \mathcal{V}_n(x) \nu(dx) = \int_E \int_0^\infty P_s f(x) \mu_t(ds) \mathcal{V}_n(x) \nu(dx) \\
&= \int_0^\infty \int_E P_s f(x) \mathcal{V}_n(x) \nu(dx) \mu_t(ds) = \int_0^\infty \langle P_s f, \mathcal{V}_n \rangle_\nu \mu_t(ds) \\
&= \int_0^\infty \langle f, P_s^* \mathcal{V}_n \rangle_\nu \mu_t(ds) = \int_0^\infty e^{-\lambda_n s} \langle f, \mathcal{V}_n \rangle_\nu \mu_t(ds) = \langle f, \mathcal{V}_n \rangle_\nu e^{-t\varphi(\lambda_n)},
\end{aligned}$$

where in the last two steps we used (3.4) and (3.7), and we were allowed to change the order of integration using Fubini's theorem, since by Cauchy-Schwartz inequality, we have

$$\int_0^\infty |\langle P_s f, \mathcal{V}_n \rangle_\nu| \, \mu_t(ds) \leq \int_0^\infty \|P_s f\|_\nu \|\mathcal{V}_n\|_\nu \mu_t(ds) \leq \|f\|_\nu \|\mathcal{V}_n\|_\nu < \infty.$$

∎

Non-Markovian processes obtained by a time-change with an inverse of a subordinator

Let $\mathcal{T}$ denote the subordinator defined in (3.7), and define its right inverse, for $t > 0$, by

$$L_t = \inf\{s > 0; \mathcal{T}_s > t\}.$$

We point out that $t \mapsto \mathcal{T}_t$ is right-continuous and non-decreasing, and hence $t \mapsto L_t$ is also right-continuous and non-decreasing. In particular, when $t \mapsto \mathcal{T}_t$ is a.s. increasing, which is equivalent to $\varphi(\infty) = \infty$ in (3.6), then $t \mapsto L_t$ is continuous and $L_{\mathcal{T}_t} = t$ a.s., whereas $\mathcal{T}_{L_t} > t$ a.s. Next, let l_t denote the distribution of L_t, i.e. for any B Borelian set of $\mathbb{R}_+$, $l_t(B) = \mathbb{P}(L_t \in B)$. Then, for any $\lambda \geq 0$ and $t \geq 0$, its Laplace transform is denoted by

$$\eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds). \tag{3.8}$$

For sake of simplicity, we assume that $\mathbb{P}(L_t < \infty) = \eta_t(0) = \int_0^\infty l_t(ds) = 1$ for all $t \geq 0$. However, all of the results presented below could be easily adapted to the case when $\int_0^\infty l_t(ds) < 1$ for some $t \geq 0$ (and hence, all $t \geq 0$). Let $\mathbf{P} = P^\eta = (P_t^\eta)_{t\geq 0}$ be the family of linear operators defined, for $f \in \mathcal{B}_b(E)$ and $t \geq 0$, by

$$P_t^\eta f(x) = \int_0^\infty P_s f(x) l_t(ds).$$

The corresponding time-changed process will be denoted by $\mathbf{X} = X_L = (X_{L_t})_{t\geq 0}$. As mentioned in the introduction above, this time-change with an inverse of a subordinator in specific situations was discussed in [99] and [112]. In the following we provide some basic properties of P^η.

Proposition 3.1.2. For any $f \in \mathcal{B}_b(E)$ and $t \geq 0$, $\nu P_t^\eta f = \nu f$, i.e. ν is a marginal stationary measure, and it is also a limiting distribution for P^η, i.e. $\lim_{t\to\infty} \nu P_t^\eta f = \nu f$. Moreover, for all $t \geq 0$, P_t^η can be extended to a contraction in $L^2(\nu)$.

Proof. Let $f \in \mathcal{B}_b(E)$ and non-negative, then, for any $t \geq 0$, we have, as above,

$$\nu P_t^\eta f = \langle P_t^\eta f, \mathbb{1} \rangle_\nu = \int_0^\infty \langle P_s f, \mathbb{1} \rangle_\nu \, l_t(ds) = \nu f,$$

where we used Tonelli's theorem, the fact that ν is an invariant measure for P and $\int_0^\infty l_t(ds) = 1$. Next, for a fixed $t \geq 0$ and any $f \in L^2(\nu)$, we note that

$$\begin{aligned} \|P_t^\eta f\|_\nu^2 = \int_0^\infty \left(P_t^\eta f(x)\right)^2 \nu(dx) &= \int_0^\infty \left(\int_0^\infty P_s f(x) l_t(ds)\right)^2 \nu(dx) \\ &\leq \int_0^\infty \int_0^\infty (P_s f(x))^2 \nu(dx) l_t(ds) \\ &= \int_0^\infty \|P_s f\|_\nu^2 l_t(ds) \leq \|f\|_\nu^2, \end{aligned}$$

where we used Jensen's inequality and Tonelli's theorem, and in the last step we used the fact that P is a contraction semigroup in $L^2(\nu)$, and that the total mass of l_t is 1. ■

3.1.2 Covariance and correlation functions

Notions of covariance and correlation functions have been intensively studied in the statistical literature. For example, the introduction of *distance covariance* and *distance correlation*, which are analogous to product-moment covariance and correlation but generalize and extend these classical bivariate measures of dependence, is well detailed in Székely et al. [150]. More formally, let X and Y be two random vectors with finite first moments in $\mathbb{R}^p$ and $\mathbb{R}^q$, $p, q \in \mathbb{N}$, respectively. For any $d \in \mathbb{N}$, $|\cdot|_d$ denotes the Euclidean norm of the vector in $\mathbb{R}^d$, and

$$c_d = \frac{\pi^{(1+d)/2}}{\Gamma((1+d)/2)}.$$

Then, the *distance covariance* between random vectors X and Y is the non-negative number $V(X, Y)$ defined by

$$V^2(X, Y) = \|f_{X,Y} - f_X f_Y\|^2 = \frac{1}{c_p c_q} \int_{\mathbb{R}^{p+q}} \frac{|f_{X,Y}(t, s) - f_X(t) f_Y(s)|^2}{|t|_p^{1+p} |s|_q^{1+q}} dt ds,$$

where f_X and f_Y are the characteristic functions of the random vectors X and Y, respectively and $f_{X,Y}$ denotes their joint characteristic function. Similarly, the *distance correlation coefficient* between random vectors X and Y with finite first moments is the nonnegative number $R(X, Y)$ defined by

$$R(X,Y) = \begin{cases} \frac{V^2(X,Y)}{\sqrt{V^2(X,X)V^2(Y,Y)}}, & \text{if } V^2(X,X)V^2(Y,Y) > 0, \\ 0, & \text{if } V^2(X,X)V^2(Y,Y) = 0. \end{cases}$$

Furthermore, note that $R \in [0,1]$, and $R(X,Y) \leq |\rho(X,Y)|$, where ρ denotes the Pearson correlation coefficient, and equality holds when $\rho = \pm 1$. We remark that distance correlation measures the strength of relation between X and Y, and it generalizes the idea of correlation in two fundamental ways:

(i) $R(X,Y)$ is defined for X and Y in arbitrary dimensions;

(ii) $R(X,Y) = 0$ characterizes independence of X and Y.

The distance correlation coefficient is especially useful for complicated dependence structures in multivariate data. Székely et al. [150] discuss some asymptotic properties and present implementation of the independence test and Monte Carlo results. It is worth to mention that Székely and Rizzo [149] introduce the notion of covariance with respect to a stochastic process and show that population distance covariance coincides with the covariance with respect to Brownian motion. Furthermore, Bhattacharjee [19] elaborates the application of a Bayesian approach in distance correlation which can be useful to test the linear relation between variables.

Another interesting measure of dependence between two random variables X and Y is the *maximal correlation coefficient* introduced by Gebelein [67] and later studied by

Rényi [136], Papadatos and Xifara [121], Beigi and Gohari [13], among other authors. It is defined as

$$\rho_{\max}(X,Y) = \sup_{f,g}\{\rho(f(X), g(Y));\ 0 < \mathbb{E}|f(X)|^2 < \infty, 0 < \mathbb{E}|g(Y)|^2 < \infty\}, \tag{3.9}$$

where the supremum is taken over all Borel measurable functions $f, g : \mathbb{R} \to \mathbb{R}$, and $\rho(X,Y)$ is the classical (Pearson) correlation coefficient between the random variables X and Y. Definition (3.9) is equivalent to

$$\rho_{\max}(X,Y) = \sup_{f,g}\{\mathbb{E}[f(X)g(Y)]; \mathbb{E}[f(X)] = \mathbb{E}[g(Y)] = 0, \mathbb{E}|f(X)|^2 = \mathbb{E}|g(Y)|^2 = 1\},$$

where the supremum is again taken over Borel measurable functions $f, g : \mathbb{R} \to \mathbb{R}$. The main role of $\rho_{\max}(X,Y)$ is that of a convenient numerical measure of dependence between X and Y. In particular, it has the tensorization property, i.e. it is unchanged when computed for i.i.d. copies. Furthermore, $\rho_{\max}(X,Y) = 0$ if and only if X and Y are independent. Even though the maximal correlation coefficient plays a fundamental role in various areas of statistics, despite its usefulness, it is often difficult to calculate it in an explicit form, except in some rare cases. Some well-known exceptions are provided by the results of [67], [96], [53], [29], [157].

Now, let $\mathbf{X}$ be a stochastic process, and ν be a Radon measure on the state space of $\mathbf{X}$. We define the *covariance and correlation functions under* ν in the following way. Let $s, t \geq 0$, then for any functions $f, g \in L^2(\nu)$,

$$C_\nu(f(\mathbf{X}_t), g(\mathbf{X}_s)) = \mathbb{E}_\nu[f(\mathbf{X}_t)g(\mathbf{X}_s)] - \mathbb{E}_\nu[f(\mathbf{X}_t)]\mathbb{E}_\nu[g(\mathbf{X}_s)], \tag{3.10}$$

$$\rho_\nu(f(\mathbf{X}_t), g(\mathbf{X}_s)) = \begin{cases} \frac{C_\nu(f(\mathbf{X}_t), g(\mathbf{X}_s))}{std_\nu(f(\mathbf{X}_t))std_\nu(g(\mathbf{X}_s))}, & \text{if } std_\nu(f(\mathbf{X}_t))std_\nu(g(\mathbf{X}_s)) > 0, \\ 0, & \text{if } std_\nu(f(\mathbf{X}_t))std_\nu(g(\mathbf{X}_s)) = 0, \end{cases} \tag{3.11}$$

where std_ν stands for the standard deviation defined by

$$std_\nu(f(\mathbf{X}_t)) = \sqrt{C_\nu(f(\mathbf{X}_t), f(\mathbf{X}_t))}.$$

Definition 3.1.2.1. When ν is a marginal stationary measure for $\mathbf{X}$ and Assumption 1 holds, for $m, n \in \mathtt{N}$ and $t, s > 0$, we call $\rho_\nu(\mathcal{P}_m(\mathbf{X}_t), \mathcal{P}_n(\mathbf{X}_s))$ (resp. $\rho_\nu(\mathcal{P}_m(\mathbf{X}_t), \mathcal{V}_n(\mathbf{X}_s))$) *(resp. biorthogonal) spectral projections correlation functions*.

The rest of the paper is organized as follows. In Section 3.2, we present the main results which include explicit expressions for the spectral projections correlation structure of non-reversible Markov processes, of their subordinated counterparts, as well as of non-Markovian processes, obtained by time-changing a Markov process with an inverse of a subordinator. In Section 3.3, we illustrate our results for the class of generalized Laguerre processes, which are associated with non-self-adjoint and non-local semigroups. The proofs of the main results are presented in Section 3.4.

3.2 Main results

Let us start with $\mathbf{X} = X$ a Markov process admitting an invariant probability measure ν, i.e. $\nu P_t f = \nu f$ for all $t \geq 0$ and $f \in L^2(\nu)$ where P is the $L^2(\nu)$-semigroup. Recall from Assumption 1 that $\mathtt{N} \subset \mathbb{N}$ is a finite or a countable set, and for any $t \geq 0$, $(\mathcal{P}_n)_{n\in\mathtt{N}}$ and $(\mathcal{V}_n)_{n\in\mathtt{N}}$ denote the sets of eigenfunctions of P_t and P_t^*, respectively. Next, for $m, n \in \mathbb{N}$, let δ_{mn} be the Kronecker symbol, i.e.

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases} \tag{3.12}$$

Then, we have the following characterization of the (biorthogonal) spectral projections correlation functions.

Theorem 3.2.1. Let $m, n \in \mathtt{N}$. Then, for any $t \geq s > 0$, we have

$$\rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_s)) = e^{-\lambda_m(t-s)} \kappa_\nu^{-1}(m) \delta_{mn},$$

and

$$\rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_s)) = e^{-\lambda_m(t-s)} c_\nu(n, m),$$

where $\kappa_\nu(m) = \|\mathcal{P}_m\|_\nu \|\mathcal{V}_m\|_\nu$ and $-1 \leq c_\nu(n, m) = \frac{\langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu} \leq 1$. Consequently, $c_\nu(n, n) = 1$ for any $n \in \mathtt{N}$.

Remark 3.2.1. We shall show in Lemma 3.4.1 below that $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathtt{N}}$ form a biorthogonal sequence in $L^2(\nu)$ in the sense that $\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu = \delta_{mn}$ for any $m, n \in \mathtt{N}$. Then, each (non-orthogonal) spectral projection is given by

$$\boldsymbol{\mathcal{P}}_m f = \langle f, \mathcal{P}_m \rangle_\nu \mathcal{V}_m, \quad \text{for } f \in L^2(\nu).$$

Moreover, in this context, the number

$$\kappa_\nu(m) = \|\mathcal{P}_m\|_\nu \|\mathcal{V}_m\|_\nu$$

is called the *condition number* of the eigenvalue λ_m and corresponds to the norm of the operator $\boldsymbol{\mathcal{P}}_m$, see e.g. Davies [48]. The condition number measures how unstable the eigenvalues are under small perturbations of the operator P_t. We note that when $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathtt{N}}$ form an orthonormal sequence, then $\kappa_\nu(m) = 1$.

Remark 3.2.2. Recall that a biorthogonal system $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathtt{N}}$ is called *tame* in $L^2(\nu)$ if $\mathtt{N} = \infty$, it is complete (i.e. $\overline{\text{Span}}(\mathcal{P}_n)_{n \in \mathtt{N}} = L^2(\nu)$) and

$$\kappa_\nu(m) = O(m^\beta),$$

for all $m \in \mathtt{N}$ and some β, i.e. there exists $b \in \mathbb{R}_+$ and $m_0 \in \mathbb{N}$ such that $|\kappa_\nu(m)| = \kappa_\nu(m) \leq b m^\beta$ for all $m \geq m_0$, see Davies [48]. Otherwise, we say that the system is *wild*. It is easy to note that if $(\mathcal{P}_n)_{n \in \mathtt{N}}$ is a basis in $L^2(\nu)$, then $\kappa_\nu(m)$ is uniformly bounded, so the system is tame with $\beta = 0$.

Remark 3.2.3. When $P = (P_t)_{t \geq 0}$ is a self-adjoint compact semigroup, then $\mathtt{N} = \mathbb{N}$ and $(\mathcal{P}_n)_{n \in \mathbb{N}} = (\mathcal{V}_n)_{n \in \mathbb{N}}$ form an orthonormal basis of $L^2(\nu)$. However, when P is non-self-adjoint, then $(\mathcal{P}_n, \mathcal{V}_n)_{n \in \mathtt{N}}$ do not form, in general, a basis of $L^2(\nu)$. A necessary condition for $(\mathcal{P}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$ to form a basis is that the condition number $\kappa_\nu(m)$ is uniformly bounded. In this sense, the rate of growth of $\kappa_\nu(m)$ also can be seen as a measure of departure of these sequences from the basis property.

Remark 3.2.4. From the definition of the inner product, we note that $c_\nu(n,m) = \cos\measuredangle(\mathcal{P}_n, \mathcal{P}_m)$ and $\arccos c_\nu(n,m)$ measures the angle between the polynomials $\mathcal{P}_n$ and $\mathcal{P}_m$ denoted by $\measuredangle(\mathcal{P}_n, \mathcal{P}_m)$. In particular, the sequence $(\mathcal{P}_n)_{n\geq 0}$ is orthogonal if and only if $c_\nu(n,m) = 0$ for $n \neq m$.

The proof of Theorem 3.2.1 is presented in Section 3.4.1.

Lemma 3.2.1. For any $f, g \in L^2(\nu)$ and $t \geq 0$,

$$\rho_\nu(f(X_t), g(X_t)) = \frac{\langle f, g\rangle_\nu - \nu f \cdot \nu g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}}.$$

In particular, for any $m, n \in \mathbb{N}$ and $t \geq 0$,

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_t)) &= c_\nu(n,m), \\ \rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_t)) &= \kappa_\nu^{-1}(m)\delta_{mn}. \end{aligned}$$

Remark 3.2.5. Note that if $f, g \in L^2(\nu)$ are such that $\nu f = \nu g = 0$, then for any $t \geq 0$

$$\rho_\nu(f(X_t), g(X_t)) = \frac{\langle f, g\rangle_\nu}{||f||_\nu ||g||_\nu}.$$

The proof of Lemma 3.2.1 is presented in Section 3.4.2.

We now proceed by studying the effect of the stochastic time-change in the analysis of the spectral projections correlation function. First, we start with Bochner subordination. To this end, recall that $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ is a subordinator with Laplace exponent φ and transition kernel $\mu_t(ds)$, i.e.

$$\mathbb{E}\left[e^{-\lambda \mathcal{T}_t}\right] = \int_0^\infty e^{-\lambda s}\mu_t(ds) = e^{-t\varphi(\lambda)}, \quad \lambda > 0, \ t \geq 0,$$

where $\varphi(\lambda) = \varrho\lambda + \int_0^\infty (1 - e^{-\lambda y})\vartheta(dy)$ with $\varrho \geq 0$, and ϑ being a Lévy measure satisfying $\int_0^\infty (1 \wedge y)\vartheta(dy) < \infty$. Denote the semigroup of the subordinated process by $P^\varphi = (P_t^\varphi)_{t\geq 0}$, i.e. for $f \in \mathcal{B}_b(E)$ and $t \geq 0$,

$$P_t^\varphi f(x) = \mathbb{E}_x[f(X_{\mathcal{T}_t})].$$

We recall from Section 3.1.1 that P^φ defines an $L^2(\nu)$-Markov semigroup with ν as an invariant measure. By combining Proposition 3.1.1 and Theorem 3.2.1, we obtain the following characterization of the spectral projections correlation structure of the subordinated process.

Corollary 3.2.1. Moreover, for $m, n \in \mathbb{N}$ and $t \geq s > 0$, we have

$$\rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{V}_n(X_{\mathcal{T}_s})) = e^{-\varphi(\lambda_m)(t-s)}\kappa_\nu^{-1}(m)\delta_{mn}, \tag{3.13}$$

and

$$\rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{P}_n(X_{\mathcal{T}_s})) = e^{-\varphi(\lambda_m)(t-s)}c_\nu(n,m). \tag{3.14}$$

Remark 3.2.6. Since $(\mathcal{P}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$ form a biorthogonal sequence in $L^2(\nu)$, and are, respectively, the eigenfunctions of P^φ and its adjoint in $L^2(\nu)$, the correlation function $\rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{P}_n(X_{\mathcal{T}_s}))$ (resp. $\rho_\nu(\mathcal{P}_m(X_{\mathcal{T}_t}), \mathcal{V}_n(X_{\mathcal{T}_s}))$) is the (resp. biorthogonal) spectral projections correlation function of the process $(X_{\mathcal{T}_t})_{t\geq 0}$.

We continue with another stochastic time-change given by an inverse of a subordinator, which, as explained in Section 3.1.1, gives rise to a non-Markovian process. Recall that the inverse of the subordinator $\mathcal{T}$ is defined for $t > 0$ by $L_t = \inf\{s > 0; \mathcal{T}_s > t\}$, its distribution is denoted by l_t, and its Laplace transform by η_t, that is for any $\lambda > 0$,

$$\eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds).$$

Also recall that we assume $\eta_t(0) = \int_0^\infty l_t(ds) = 1$ for all $t \geq 0$. Then, $P^\eta = (P_t^\eta)_{t\geq 0}$, defined, for $t \geq 0$ and $f \in L^2(\nu)$, by

$$P_t^\eta f(x) = \int_0^\infty P_s f(x) l_t(ds),$$

is a linear operator, and the corresponding time-changed process will be denoted by $X_L = (X_{L_t})_{t\geq 0}$. Note that Leonenko et al. [99] and Mijena and Nane [112] characterize the correlation structure of so-called Pearson diffusions when they are time-changed by an inverse of a linear combination of independent stable subordinators. We extend their methodology by first considering a general Markov process with biorthogonal spectral projections, and then time-changing it with an inverse of any independent subordinator. We also point out that by following a line of reasoning similar to the proof of Proposition 3.1.1, it can be shown that the biorthogonal sequence $(\mathcal{P}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$ represent a set of eigenfunctions of the linear operator P^η_t, $t \geq 0$ and its adjoint in $L^2(\nu)$, respectively. Thus, $\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s}))$ (resp. $\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s}))$) is the (resp. biorthogonal) spectral projections correlation function of the process X_L. Finally, we set the following notation.

(a) We write $f \overset{a}{\sim} g$ for $a \in [0,\infty]$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$. We may write $f(x) \overset{x\to a}{\sim} g(x)$ to emphasize dependency on the variable x.

  (a1) f is called a long-tailed function if $\tau_y f(x) \overset{x\to\infty}{\sim} f(x)$ for any fixed $y > 0$, where $\tau_y f(x) = f(x+y)$ is the shift operator.

  (a2) f is called slowly varying at 0 if $d_a f(x) \overset{x\to 0}{\sim} f(x)$ for any fixed $a > 0$, where $d_a f(x) = f(ax)$ is the dilation operator.

  (a3) We say that f is strongly regularly varying at a with index $0 < \alpha < 1$ if $f \overset{a}{\sim} p_\alpha$, where $p_\alpha(x) = Cx^\alpha$ for some constant $C > 0$.

(b) We write $f \overset{a}{\asymp} g$ if there exists a constant $C > 0$ such that $\frac{1}{C} g(x) \leq f(x) \leq C g(x)$ for $x \geq a$.

(c) We write $f = O(g)$ if $\overline{\lim}_{x\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty$.

We are now ready to state our last main result.

Theorem 3.2.2. Let $m, n \in \mathtt{N}$. Then, for $t \geq s > 0$,

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) = c_\nu(n,m)\Big(\lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m)\Big), \tag{3.15}$$

and

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) = \kappa_\nu^{-1}(m)\delta_{mn}\Big(\lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m)\Big), \tag{3.16}$$

where $U(dr) = \int_0^\infty \mathbb{P}(\mathcal{T}_t \in dr)dt$ is the renewal measure of the subordinator $\mathcal{T}$. Moreover, for any fixed $s > 0$,

$$\begin{aligned} c_\nu(n,m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1) \leq \quad & \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \quad \leq c_\nu(n,m)\eta_{t-s}(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1), \\ \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1) \leq \quad & \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \quad \leq \kappa_\nu^{-1}(m)\delta_{mn}\eta_{t-s}(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1). \end{aligned}$$

Furthermore, if for a fixed $s > 0$, $\overline{\lim}_{t\to\infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} = C$ for some constant $C = C(s, \lambda_m)(\lambda_m\mathbb{E}[L_s] + 1)$, then there exists $t_0 > 0$ such that

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \quad & \overset{t_0}{\asymp} \quad c_\nu(n,m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1), \\ \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \quad & \overset{t_0}{\asymp} \quad \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1). \end{aligned}$$

In particular, if $t \mapsto \eta_t(\lambda_m)$ is a long-tailed function, we have

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \quad \overset{t\to\infty}{\sim} \quad c_\nu(n,m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1), \tag{3.17}$$

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \quad \overset{t\to\infty}{\sim} \quad \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1). \tag{3.18}$$

The proof of this theorem is presented in Section 3.4.4. We complete this part with the following result which provides a sufficient condition for η_t to be a long-tailed function.

Proposition 3.2.1. For any $\lambda > 0$, there exists a positive random variable X_λ such that $\eta_t(\lambda)$ is the tail of its distribution, i.e. $\eta_t(\lambda) = \mathbb{P}(X_\lambda > t)$, $t > 0$. Moreover, $\eta_t(\lambda)$ is a long-tailed distribution if φ is strongly regularly varying at 0.

The proof of this proposition is presented in Section 3.4.3.

3.2.1 Interpretation of the (biorthogonal) spectral projections correlation functions for statistical properties

The results presented above regarding the (biorthogonal) spectral projections correlation functions and their asymptotic behavior provide an interesting approach for designing statistical tests in order to identify substantial properties of a stochastic process. More formally, we start by assuming that the sample $\widehat{\mathbf{X}} = (\widehat{\mathbf{X}}_1, \cdots, \widehat{\mathbf{X}}_T)$, with $T \in \mathbb{N}$ large, is coming from a stochastic process $\mathbf{X}$ which belongs to some family with a marginal stationary measure $(\nu_i)_{i \in I}$ and associated biorthogonal sequence $\left((\mathcal{P}_n^{(i)}, \mathcal{V}_n^{(i)})_{n \in \mathtt{N}}\right)_{i \in I}$ as defined in Assumption 1, where I is the index set of the family. For example, in the case when $\mathbf{X}$ belongs to the family of generalized Laguerre processes presented in Section 3.3.1 below, we can consider one element from each of the following sub-families: a pure diffusion, a diffusion component and jumps with finite activity, a diffusion component and jumps with infinite activity, and a pure jump process. Now, based on the (biorthogonal) spectral projections correlation structure, one can identify

(a) how far from symmetry (self-adjointness) the process is,

(b) what type of range dependence (short-to-long) it displays, and

(c) the path properties of the process (cádlág or continuous paths).

For designing statistical tests, one can rely on the estimates of $\kappa_{\nu_i}(m)$ and/or $c_{\nu_i}(n, m)$, $i \in I$, $n, m \in \mathtt{N}$. Since $(\kappa_{\nu_i}(m))_{m \in \mathtt{N}}$ contain information about both of the sequences $(\mathcal{P}_n^{(i)})_{n \in \mathtt{N}}$ and $(\mathcal{V}_n^{(i)})_{n \in \mathtt{N}}$, below we describe some statistical tests involving the condition number. However, the estimates of $c_{\nu_i}(n, m)$ can be useful to further refine the search of the process. More precisely, based on the main results presented in Section 3.2, one can make the following implications.

(a) To study the possible *departure from symmetry* of $\widehat{\mathbf{X}}$, see Remark 3.2.3, following the results provided by Lemma 3.2.1, we first take $t = s = k$ for some $k \in \{1, \cdots, T\}$ and $m = n \in \mathtt{N}$. Then, since the marginal stationary measure guarantees that the statistical properties of the process do not change over time, for each $i \in I$, we compute the empirical estimates of the condition number $\kappa_{\nu_i}(m)$ for some $m \in \mathtt{N}$, by

$$\begin{aligned}\widehat{\kappa}_{\nu_i}^{-1}(m) &= \widehat{\rho}_{\nu_i}(\mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_k), \mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_k)) \\ &= \frac{\sum_{j=1}^T \left(\mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{P}}_m^{(i)}(\widehat{\mathbf{X}})\right)\left(\mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{V}}_m^{(i)}(\widehat{\mathbf{X}})\right)}{\sqrt{\sum_{j=1}^T \left(\mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{P}}_m^{(i)}(\widehat{\mathbf{X}})\right)^2}\sqrt{\sum_{j=1}^T \left(\mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_j) - \overline{\mathcal{V}}_m^{(i)}(\widehat{\mathbf{X}})\right)^2}},\end{aligned} \tag{3.19}$$

where $\overline{\mathcal{P}}_m^{(i)}(\widehat{\mathbf{X}}) = \frac{1}{T}\sum_{j=1}^T \mathcal{P}_m^{(i)}(\widehat{\mathbf{X}}_j)$ and $\overline{\mathcal{V}}_m^{(i)}(\widehat{\mathbf{X}}) = \frac{1}{T}\sum_{j=1}^T \mathcal{V}_m^{(i)}(\widehat{\mathbf{X}}_j)$ are the sample means. Next, we compute the theoretical condition number by

$$\kappa_{\nu_i}(m) = \|\mathcal{P}_m^{(i)}\|_{\nu_i}\|\mathcal{V}_m^{(i)}\|_{\nu_i}.$$

Finally, to identify the couple $(\mathcal{P}_n^{(i)}, \mathcal{V}_n^{(i)})_{n\in\mathtt{N}}$, we choose $\epsilon_S > 0$, and check if

$$|\kappa_{\nu_i}(m) - \widehat{\kappa}_{\nu_i}(m)| < \epsilon_S\,. \tag{3.20}$$

For the next step, for sake of simplicity, we suppose that there is only one $\bar{i} \in I$ such that the condition (3.20) is satisfied.

(b) To asses the *range dependence* of the sample, we study the asymptotic behavior of the empirical correlation $\widehat{\rho}_{\nu_{\bar{i}}}(\mathcal{P}_m^{(\bar{i})}(\widehat{\mathbf{X}}_k), \mathcal{V}_m^{(\bar{i})}(\widehat{\mathbf{X}}_j))$, $k, j \in \{1, \cdots, T\}$, $k > j$, $m \in \mathtt{N}$. More formally, we first compute $\widehat{\kappa}_{\nu_{\bar{i}}}(m)$ by (3.19), fix some $j \in \{1, \cdots, T\}$ (one can simply set $j = 1$ or $j = 2$), and we proceed by studying

$$g_{\lambda_m}(k) = \widehat{\kappa}_{\nu_{\bar{i}}}(m) \cdot \widehat{\rho}_{\nu_{\bar{i}}}(\mathcal{P}_m^{(\bar{i})}(\widehat{\mathbf{X}}_k), \mathcal{V}_m^{(\bar{i})}(\widehat{\mathbf{X}}_j)). \tag{3.21}$$

Now, if $k \mapsto g_{\lambda_m}(k)$, $j < k \in \{1, \cdots, T\}$ exhibits exponential decay with respect to λ_m, then we have short-range dependence. In contrast, if it exhibits a polynomial decay, then the process has long-range dependence, and, in particular, it is not a (subordinated) Markov process. We remark that although these two cases are the most popular ones discussed in the literature, depending on the rate of decay of the correlation function, the process can exhibit short-to-long-range dependence. The concept of long-range dependence has been repeatedly used to describe properties of financial time series such as stock prices, foreign exchange rates, market indices and commodity prices. In this context, based on the behavior of (biortogonal) spectral projections correlation functions, in their working paper [83], the authors provide a more detailed empirical study to detect the (short-to-long-) range dependence in volatility in financial markets.

(c) Finally, to study the *path properties* of the process, i.e. the presence of jumps and their activity, we study the behavior of $\widehat{\kappa}_{\nu_{\bar{i}}}(m)$ for large m. To illustrate this with a specific example, let us consider the class of generalized Laguerre processes introduced in Section 3.3.1. Note that this class encompasses a range of symmetries and jumps. Then, one can identify the following cases.

   (i) If $\widehat{\kappa}_{\nu_{\bar{i}}}(m) = 1$, $m \in \mathbb{N}$, then the process is a pure diffusion, see Section 3.3.2.

   (ii) If $\widehat{\kappa}_{\nu_{\bar{i}}}(m) = O(m^{\beta})$ for some β, then the process has both a diffusion component and a jump component with finite activity, see Section 3.3.3.

   (iii) If $\widehat{\kappa}_{\nu_{\bar{i}}}(m) = O(e^{\epsilon m})$ for any $\epsilon > 0$, then, similarly, the process has both diffusion and jump components while in this case jumps have infinite activity, see Section 3.3.4.

   (iv) If $\widehat{\kappa}_{\nu_{\bar{i}}}(m) = O\left(e^{m^{\beta}}\right)$ for some β, then the process is a pure jump process.

The problem of deciding whether the continuous-time process which models an economic or financial time series has continuous paths or exhibits jumps is an important issue. For example, Aït-Sahalia and Jacod [2] design a test to identify the presence of jumps in a discretely observed semimartingale, based on power variations sampled at different frequencies. Furthermore, in this setting, the authors of [3] propose statistical tests to discriminate between the finite and infinite activity of jumps in a semimartingale. We emphasize that our approach allows one to design a statistical test in order to identify both the presence and the types of jumps.

3.3 Examples

In this section, we illustrate the results of Section 3.2 for the class of generalized Laguerre semigroups which have been studied in depth by Patie and Savov in [129]. To investigate the behavior of (biorthogonal) spectral projections correlation structure in various scenarios, we first discuss two important examples of subordinators and their inverses.

Example 3.3.1. Let $\mathcal{T}$ be an α-stable subordinator, i.e. in (3.7), $\varphi(\lambda) = \lambda^{\alpha}$, $0 < \alpha < 1$. We recall from [99] that for any $\lambda > 0$ and $t \geq 0$, the Laplace transform of its inverse is given by

$$\eta_t(\lambda) = E_{\alpha}(-\lambda t^{\alpha}),$$

where E_{α} is the Mittag-Lefler function defined by $E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ for $z \in \mathbb{C}$. On the other hand, since $U(ds) = \frac{r^{\alpha-1}}{\Gamma(\alpha)}ds$, $s > 0$, we have that $\mathbb{E}[L_s] = U(0,s) = \frac{s^{\alpha}}{\Gamma(1+\alpha)}$. Now, Corollary 3.2.1 yields that for any $m, n \in \mathtt{N}$ and $t \geq s > 0$,

$$\begin{aligned}
\rho_{\nu}\left(\mathcal{P}_m\left(X_{\mathcal{T}_t}\right), \mathcal{V}_n\left(X_{\mathcal{T}_s}\right)\right) &= e^{-\lambda_m^{\alpha}(t-s)} \kappa_{\nu}^{-1}(m)\delta_{mn}, \\
\rho_{\nu}\left(\mathcal{P}_m\left(X_{\mathcal{T}_t}\right), \mathcal{P}_n\left(X_{\mathcal{T}_s}\right)\right) &= e^{-\lambda_m^{\alpha}(t-s)} c_{\nu}(n,m).
\end{aligned}$$

Next, it follows from Theorem 3.2.2 that

$$\begin{aligned}\rho_\nu\left(\mathcal{P}_m\left(X_{L_t}\right), \mathcal{P}_n\left(X_{L_s}\right)\right) &= c_\nu(n,m)\left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m)\right) \\ &= c_\nu(n,m)\left(\lambda_m \int_0^s E_\alpha(-\lambda_m(t-r)^\alpha)\frac{r^{\alpha-1}}{\Gamma(\alpha)}dr + E_\alpha(-\lambda_m t^\alpha)\right) \\ &= \frac{c_\nu(n,m)\ \lambda_m t^\alpha}{\Gamma(\alpha)} \int_0^{s/t} \frac{E_\alpha(-\lambda_m t^\alpha(1-z)^\alpha)}{z^{1-\alpha}}dz + c_\nu(n,m)E_\alpha(-\lambda_m t^\alpha).\end{aligned}$$

Note that since φ is strongly regularly varying at 0, we have, by Proposition 3.2.1, that $t \mapsto \eta_t$ is a long-tailed function. Furthermore, it is well known, see e.g. [99], that when $t \to \infty$,

$$\eta_t(\lambda) = E_\alpha(-\lambda t^\alpha) \sim \frac{1}{\Gamma(1-\alpha)\lambda t^\alpha}.$$

Hence, from Theorem 3.2.2, we deduce that for a fixed $s > 0$, when $t \to \infty$, we have

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) \sim \frac{c_\nu(n,m)}{\Gamma(1-\alpha)t^\alpha}\left(\frac{1}{\lambda_m} + \frac{s^\alpha}{\Gamma(1+\alpha)}\right), \tag{3.22}$$

and, similarly,

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) \sim \frac{\kappa_\nu^{-1}(m)\delta_{mn}}{\Gamma(1-\alpha)t^\alpha}\left(\frac{1}{\lambda_m} + \frac{s^\alpha}{\Gamma(1+\alpha)}\right).$$

When X is a Pearson diffusion, we note that (3.22) boils down to the case discussed in [99]. Finally, since the correlation functions decay in a polynomial rate of $\alpha \in (0,1)$, here the process X_L exhibits long-range dependence.

Example 3.3.2. Let $\mathcal{T}$ be a Poisson subordinator with mean $\frac{1}{\theta}$, i.e.in (3.7), $\varphi(\lambda) = \theta(1 - e^{-\lambda})$. Then, for the inverse Poisson subordinator, we have that L_t follows Gam $([t+1], 1/\theta)$, see Leonenko et al. [98]. Using the moment generating function of a Gamma random variable, we get

$$\eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds) = \left(1 + \frac{\lambda}{\theta}\right)^{-[t+1]}, \tag{3.23}$$

and thus $U(0,s) = \mathbb{E}[L_s] = \frac{[s+1]}{\theta}$. Then, it follows from Corollary 3.2.1 that for any $m, n \in \mathtt{N}$

and $t \geq s > 0$,

$$\begin{aligned}
\rho_\nu\left(\mathcal{P}_m\left(X_{\mathcal{T}_t}\right), \mathcal{V}_n\left(X_{\mathcal{T}_s}\right)\right) &= e^{-\theta(1-e^{-\lambda_m})(t-s)}\kappa_\nu^{-1}(m)\delta_{mn}, \\
\rho_\nu\left(\mathcal{P}_m\left(X_{\mathcal{T}_t}\right), \mathcal{P}_n\left(X_{\mathcal{T}_s}\right)\right) &= e^{-\theta(1-e^{-\lambda_m})(t-s)}c_\nu(n,m).
\end{aligned}$$

Now, when $t \to \infty$, note that $\eta_t(\lambda) \sim \left(1+\frac{\lambda}{\theta}\right)^{-t}$. Consequently $\lim_{t\to\infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} \neq 1$, and therefore (3.17) and (3.18) do not hold. However, we are able to compute the exact formulas for the (biorthogonal) spectral projections correlation functions of X_L as follows. First, noting that for any $k \in \mathbb{N}$ such that $k < t$, $-[t-k+1] = k - [t+1]$, we have

$$\begin{aligned}
\lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m) &= \frac{\lambda_m}{\theta}\sum_{k=0}^{[s]}\left(1+\frac{\lambda_m}{\theta}\right)^{-[t-k+1]} + \left(1+\frac{\lambda_m}{\theta}\right)^{-[t+1]} \\
&= \left(1+\frac{\lambda_m}{\theta}\right)^{-[t+1]}\left(2-\left(1+\frac{\lambda_m}{\theta}\right)^{[s+1]}\right). \qquad (3.24)
\end{aligned}$$

Thus, it follows from Theorem 3.2.2 that

$$\begin{aligned}
\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= c_\nu(n,m)\left(1+\frac{\lambda_m}{\theta}\right)^{-[t+1]}\left(2-\left(1+\frac{\lambda_m}{\theta}\right)^{[s+1]}\right), \\
\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) &= \kappa_\nu^{-1}(m)\delta_{mn}\left(1+\frac{\lambda_m}{\theta}\right)^{-[t+1]}\left(2-\left(1+\frac{\lambda_m}{\theta}\right)^{[s+1]}\right).
\end{aligned}$$

Since the spectral projections correlation functions decay in an exponential rate, the time-changed process X_L exhibits short-range dependence although it is non-Markovian.

3.3.1 A short review of the generalized Laguerre semigroups

In this section, we provide a short description of the so-called generalized Laguerre processes introduced and studied by Patie and Savov [129], see also Patie et al. [130]. We point out that these processes have been recently used to model asset price dynamics in Jarrow et al. [85]. To this end, let $\tilde{\mathbf{A}}$ be the infinitesimal generator of classical Laguerre process which in financial literature is known as a Cox-Ingersoll-Ross (CIR) process, i.e. for at least $f \in C_0^2(\mathbb{R}_+)$, we have

$$\tilde{\mathbf{A}}f(x) = \sigma^2 x f''(x) + (\beta + \sigma^2 - x)f'(x), \tag{3.25}$$

where $\beta, \sigma \geq 0$. We say that a semigroup $P = (P_t)_{t\geq 0}$ is a *generalized Laguerre (gL) semigroup* if its infinitesimal generator is given, for a smooth function f on $x > 0$, by

$$\mathbf{A}f(x) = \tilde{\mathbf{A}}f(x) + \int_0^\infty (f(e^{-y}x) - f(x) + yxf'(x))\,\Pi(x, dy), \tag{3.26}$$

where $\Pi(x, dy) = \frac{\Pi(dy)}{x}$, with Π being a Lévy measure concentrated on $(0, \infty)$ and satisfying the integrability condition $\int_0^\infty (y^2 \wedge y)\Pi(dy) < \infty$. We call the corresponding process $X = (X_t)_{t\geq 0}$ a *generalized Laguerre process*. Note that when $\Pi(0, \infty) = 0$, then P boils down to the semigroup of a classical Laguerre process. Moreover, from [129, Theorem 1.6] we have that the semigroup P admits a unique invariant measure, which in this case is absolutely continuous with a density that we denote by ν, and write the Hilbert space $L^2(\nu)$ as in Section 3.1. Recall that P can be extended to a contraction semigroup in $L^2(\nu)$, and by an abuse of notation, we still denote it by P. Now, [129, Theorem 1.11] yields that if $\overline{\Pi}(y) = \int_y^\infty \Pi(dr)$ is strongly regularly varying at 0 with some index $\alpha \in (0, 1)$, then, for any $f \in L^2(\nu)$ and $t > T_\Pi$ for some explicit T_Π (with $T_\Pi = 0$ when $\sigma^2 > 0$), we have the following spectral expansion,

$$P_t f = \sum_{n=0}^\infty e^{-nt}\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \quad \text{in } L^2(\nu),$$

where $(\mathcal{P}_n, \mathcal{V}_n)_{n\geq 0}$ form a biorthogonal sequence of $L^2(\nu)$, and are expressed as follows:

$$\mathcal{P}_n(x) = \sum_{k=0}^n (-1)^k \frac{\binom{n}{k}}{W_\phi(k+1)} x^k \in L^2(\nu),$$

and,

$$\mathcal{V}_n(x) = \frac{\mathcal{R}^{(n)}\nu(x)}{\nu(x)} = \frac{(x^n \nu(x))^{(n)}}{n!\,\nu(x)} \in L^2(\nu),$$

with the last equation serving as a definition of the Rodrigues operator. Here, $W_\phi(1) = 1$ and, for $n \in \mathbb{N}$, $W_\phi(n+1) = \prod_{k=1}^n \phi(k)$, where ϕ is the Bernstein function, see (3.6), which takes the form

$$\phi(\lambda) = \beta + \sigma^2\lambda + \int_0^\infty \left(1 - e^{-\lambda y}\right)\overline{\overline{\Pi}}(y)dy,$$

with Π, β, σ^2 as in (3.26). Furthermore, by [129, Theorem 7.3 and Proposition 8.4] we have, that for any $n \geq 0$ and $t \geq 0$, $\mathcal{P}_n$ (resp. $\mathcal{V}_n$) is an eigenfunction for P_t (resp. P_t^*) associated to the eigenvalue e^{-nt}, i.e. $\mathcal{P}_n, \mathcal{V}_n \in L^2(\nu)$ and

$$P_t\mathcal{P}_n(x) = e^{-nt}\mathcal{P}_n(x) \quad \text{and} \quad P_t^*\mathcal{V}_n(x) = e^{-nt}\mathcal{V}_n(x),$$

with $(P_t^*)_{t\geq 0}$ being the adjoint of $(P_t)_{t\geq 0}$ in $L^2(\nu)$. Therefore, in this case, we have $\lambda_n = n$, $n \in \mathbb{N}$.

Next, we describe the eigenvalue expansions of specific instances of the generalized Laguerre semigroups which illustrate the different situations that are ranging from the self-adjoint case to perturbation of a self-adjoint differential operator through non-local operators without diffusion component. We study their spectral projections correlation structure, and discuss some of their important properties as are range dependence and symmetry (self-adjointness), among others.

3.3.2 The self-adjoint diffusion case

For any $\beta > 0$, the infinitesimal generator of the classical Laguerre process takes the form

$$\mathbf{A}_\beta f(x) = xf''(x) + (\beta + 1 - x)f'(x).$$

Note that this is the infinitesimal generator of a one-dimensional diffusion often referred in the literature as the CIR process. The eigenfunctions are given by

$$\overline{\mathcal{L}}_n^{(\beta)}(x) = \sqrt{\mathfrak{c}_n(\beta)}\mathcal{L}_n^{(\beta)}(x),$$

where $\mathfrak{c}_n(\beta) = \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+1)}$ and $\mathcal{L}_n^{(\beta)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\beta}{n-k}\frac{x^k}{k!}$ is the associated Laguerre polynomial of order β. Denote by

$$\gamma_\beta(dx) = \frac{x^\beta e^{-x}}{\Gamma(\beta+1)}dx, \quad x > 0, \tag{3.27}$$

the law of a Gamma random variable with parameter $(\beta + 1)$. Then, the semigroup is self-adjoint in $L^2(\gamma_\beta)$ and the sequence $(\overline{\mathcal{L}}_n^{(\beta)})_{n\geq 0}$ forms an orthonormal basis of $L^2(\gamma_\beta)$. In particular, this means that, for $n, m \in \mathbb{N}$, we have that

$$\kappa_{\gamma_\beta}(m) = 1 \quad \text{and} \quad c_{\gamma_\beta}(n,m) = \delta_{nm}.$$

Hence, it follows from Theorem 3.2.1, that for $m, n \in \mathtt{N}$ and $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_t), \overline{\mathcal{L}}_n^{(\beta)}(X_s)) = e^{-m(t-s)}\delta_{nm}.$$

Now, from Corollary 3.2.1, Theorem 3.2.2 and examples 3.3.1 and 3.3.2 we have the following additional results.

- Let $\mathcal{T}$ be an **α-stable subordinator**, i.e. $\varphi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, see Example 3.3.1. Then, for any $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{\mathcal{T}_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{\mathcal{T}_s})) = e^{-m^\alpha(t-s)}\delta_{nm}.$$

- Let $\mathcal{T}$ be a **Poisson subordinator** with parameter θ, i.e. $\varphi(\lambda) = \theta(1 - e^{-\lambda})$, see Example 3.3.2. Then, for any $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{\mathcal{T}_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{\mathcal{T}_s})) = e^{-\theta(1-e^{-m})(t-s)}\delta_{nm}.$$

- Let L be the **inverse of an α-stable subordinator**, see Example 3.3.1. Then,

$$\rho_{\gamma_\beta}\left(\overline{\mathcal{L}}_m^{(\beta)}(X_{L_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{L_s})\right) = \frac{\delta_{nm}\, mt^\alpha}{\Gamma(\alpha)}\int_0^{s/t}\frac{E_\alpha(-mt^\alpha(1-z)^\alpha)}{z^{1-\alpha}}dz + \delta_{nm}E_\alpha(-mt^\alpha).$$

  Furthermore, for a fixed $s > 0$, when $t \to \infty$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{L_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{L_s})) \sim \frac{\delta_{nm}}{\Gamma(1-\alpha)t^\alpha}\left(\frac{1}{m} + \frac{s^\alpha}{\Gamma(1+\alpha)}\right).$$

- Let L be the **inverse of a Poisson subordinator** with parameter θ, see Example (3.3.2). Then, for any $t \geq s > 0$,

$$\rho_{\gamma_\beta}(\overline{\mathcal{L}}_m^{(\beta)}(X_{L_t}), \overline{\mathcal{L}}_n^{(\beta)}(X_{L_s})) = \delta_{nm}\left(1+\frac{m}{\theta}\right)^{-[t+1]}\left(2-\left(1+\frac{m}{\theta}\right)^{[s+1]}\right).$$

3.3.3 Small perturbation of the Laguerre semigroup.

Let $b \geq 1$, and take $\sigma^2 = 1$, $\beta = \frac{b^2-1}{b}$ and $\overline{\Pi}(y) = e^{-by}$, $y \geq 0$ in (3.26), i.e. we consider, for f smooth,

$$\mathbf{A}^{(b)}f(x) = xf''(x) + \left(\frac{b^2-1}{b}+1-x\right)f'(x) + \frac{b}{x}\int_0^\infty (f(e^{-y}x)-f(x)+yxf'(x))e^{-by}dy.$$

The associated semigroup is ergodic with a unique invariant measure ν_b,

$$\nu_b(dx) = \frac{(1+x)}{b+1}\gamma_{b-1}(dx), \quad x > 0,$$

with $\gamma_{b-1}(dx)$ as in (3.27). Then, the eigenfunctions and co-eigenfunctions $(\mathcal{P}_n^{(b)}, \mathcal{V}_n^{(b)})_{n\geq 0}$ are expressed in terms of Laguerre polynomials $\left(\mathcal{L}_n^{(b)}\right)_{n\geq 0}$ as follows, $n \geq 0$,

$$\begin{aligned}
\mathcal{P}_n^{(b)}(x) &= \mathfrak{c}_n(b+1)\mathcal{L}_n^{(b+1)}(x) - \frac{\mathfrak{c}_n(b+1)}{b}x\mathcal{L}_{n-1}^{(b+2)}(x),\\
\mathcal{V}_n^{(b)}(x) &= \frac{1}{x+1}\mathcal{L}_n^{(b-1)}(x) + \frac{x}{x+1}\mathcal{L}_n^{(b)}(x),
\end{aligned}$$

where $\mathfrak{c}_n(\cdot)$ and $\left(\mathcal{L}_n^{(b)}\right)_{n\geq 0}$ are defined as in Section 3.3.2, see [129, Example 3.2]. Next, it follows from [129, Theorem 2.2 and (3.9)] that

$$\|\mathcal{P}_n^{(b)}\|_{\nu_b} = O(1), \quad \|\mathcal{V}_n^{(b)}\|_{\nu_b} = O(n^{(b+1)/2}).$$

Then, for any $m \in \mathbb{N}$, we have

$$\kappa_{\nu_b}(m) = O(m^{(b+1)/2}),$$

and therefore, the biorthogonal sequence $(\mathcal{P}_n^{(b)}, \mathcal{V}_n^{(b)})_{n\in\mathbb{N}}$ is tame, see Remark 3.2.2 for definition. Thus, it follows from Theorem 3.2.1 that

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_t), \mathcal{V}_n^{(b)}(X_s)) = e^{-m(t-s)}\kappa_{\nu_b}^{-1}(m)\delta_{nm}.$$

Now, from Corollary 3.2.1, Theorem 3.2.2 and examples 3.3.1 and 3.3.2 we have the following results.

- Let $\mathcal{T}$ be an **α-stable subordinator**, i.e. $\varphi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, see Example 3.3.1. Then, for any $t \geq s > 0$,

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(b)}(X_{\mathcal{T}_s})) = e^{-m^\alpha(t-s)}\kappa_{\nu_b}^{-1}(m)\delta_{nm}.$$

- Let $\mathcal{T}$ be a **Poisson subordinator** with parameter θ, i.e. $\varphi(\lambda) = \theta(1 - e^{-\lambda})$, see Example 3.3.2. Then, for any $t \geq s > 0$

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(b)}(X_{\mathcal{T}_s})) = e^{-\theta(1-e^{-m})(t-s)}\kappa_{\nu_b}^{-1}(m)\delta_{nm}.$$

- Let L be the **inverse of an α-stable subordinator**, see Example 3.3.1. Then, for a fixed $s > 0$, when $t \to \infty$,

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{L_t}), \mathcal{V}_n^{(b)}(X_{L_s})) \sim \frac{\kappa_{\nu_b}^{-1}(m)\delta_{mn}}{\Gamma(1-\alpha)t^\alpha}\left(\frac{1}{m} + \frac{s^\alpha}{\Gamma(1+\alpha)}\right).$$

- Let L be the **inverse of a Poisson subordinator** with parameter θ, see Example 3.3.2. Then, for any $t \geq s > 0$,

$$\rho_{\nu_b}(\mathcal{P}_m^{(b)}(X_{L_t}), \mathcal{V}_n^{(b)}(X_{L_s})) = \kappa_{\nu_b}^{-1}(m)\delta_{mn}\left(1 + \frac{m}{\theta}\right)^{-[t+1]}\left(2 - \left(1 + \frac{m}{\theta}\right)^{[s+1]}\right).$$

3.3.4 The Gauss-Laguerre semigroup.

We next consider the Gauss-Laguerre semigroup $P^{\alpha,b} = (P_t^{\alpha,b})_{t\geq 0}$ which has been introduced and extensively studied in [127], and which is an instance of the generalized Laguerre semigroups, see [129, Example 3.3]. In particular, its infinitesimal generator, for any $\alpha \in (0,1)$ and $b \in [1-\frac{1}{\alpha}, \infty]$, and for any given smooth function f, takes the form

$$\mathbf{A}^{(\alpha,b)} f(x) = (b_\alpha - x) f'(x) + \frac{\sin(\alpha\pi)}{\pi} x \int_0^1 f''(xy) g_{\alpha,b}(y) dy, \ \ x > 0,$$

where $b_\alpha = \frac{\Gamma(\alpha b+\alpha+1)}{\Gamma(\alpha b+1)}$ and

$$g_{\alpha,b}(y) = \frac{\Gamma(\alpha)}{b+\frac{1}{\alpha}+1} y^{b+\frac{1}{\alpha}+1} {}_2F_1(\alpha(b+1)+1, \alpha+1; \alpha(b+1)+2; y^{\frac{1}{\alpha}}),$$

with ${}_2F_1$ the Gauss hypergeometric function. The associated semigroup $P^{\alpha,b} = (P_t^{\alpha,b})_{t\geq 0}$ is a non-self-adjoint contraction in $L^2(\mathbf{e}_{\alpha,b})$, where

$$\mathbf{e}_{\alpha,b}(dx) = \frac{x^{b+\frac{1}{\alpha}-1} e^{-x^{\frac{1}{\alpha}}}}{\Gamma(\alpha b+1)} dx, \ \ x > 0,$$

is its unique invariant measure. For any $x \geq 0$, we set $\mathcal{P}_0^{(\alpha,b)}(x) = 1$ and for any $n \geq 1$,

$$\begin{aligned} \mathcal{P}_n^{(\alpha,b)}(x) &= \Gamma(\alpha b+1) \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}}{\Gamma(\alpha k + \alpha b + 1)} x^k, \\ \mathcal{V}_n^{(\alpha,b)}(x) &= \frac{(-1)^n}{n!\, \mathbf{e}_{\alpha,b}(x)} (x^n \mathbf{e}_{\alpha,b}(x))^{(n)}, \end{aligned}$$

which are the eigenfunctions and co-eigenfunctions of $P^{\alpha,b}$. It is worth mentioning that in [127, Proposition 3.3] the authors show that the $\mathcal{V}_n^{(\alpha,b)}$'s can be expressed in terms of sequences of polynomials as well. Then, it follows from [127, Proposition 2.3] and [129, Theorem 2.2] that

$$\|\mathcal{P}_n^{(\alpha,b)}\|_{\mathbf{e}_{\alpha,b}} = O(1), \quad \|\mathcal{V}_n^{(\alpha,b)}\|_{\mathbf{e}_{\alpha,b}} = O\left(e^{T_\alpha n}\right),$$

where $T_\alpha = -\ln(2^\alpha - 1)$. Then, we have that for any $m \in \mathbb{N}$,

$$\kappa_{\mathbf{e}_{\alpha,b}}(m) = O\left(e^{T_\alpha m}\right),$$

hence, in this case. the biorthogonal sequence $(\mathcal{P}_n^{(\alpha,b)}, \mathcal{V}_n^{(\alpha,b)})_{n\in\mathbb{N}}$ is a wild system in $L^2(\mathbf{e}_{\alpha,b})$. Now, it follows from Theorem 3.2.1 that

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{V}_n^{(\alpha,b)}(X_s), \mathcal{P}_m^{(\alpha,b)}(X_t)) = e^{-m(t-s)} \kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m) \delta_{nm}.$$

Next, from Corollary 3.2.1, Theorem 3.2.2 and examples 3.3.1 and 3.3.2 we have the following additional results.

- Let $\mathcal{T}$ be an α**-stable subordinator**, i.e. $\varphi(\lambda) = \lambda^\alpha$, $0 < \alpha < 1$, see Example 3.3.1. Then, for any $t \geq s > 0$,

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(\alpha,b)}(X_{\mathcal{T}_s})) = e^{-m^\alpha(t-s)}\kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m)\delta_{nm}.$$

- Let $\mathcal{T}$ be a **Poisson subordinator** with parameter θ, i.e. $\varphi(\lambda) = \theta(1 - e^{-\lambda})$, see Example 3.3.2. Then, for any $t \geq s > 0$

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{\mathcal{T}_t}), \mathcal{V}_n^{(\alpha,b)}(X_{\mathcal{T}_s})) = e^{-\theta(1-e^{-m})(t-s)}\kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m)\delta_{nm}.$$

- Let L be the **inverse of an** α**-stable subordinator**, see Example 3.3.1. Then, for a fixed $s > 0$, when $t \to \infty$,

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{L_t}), \mathcal{V}_n^{(\alpha,b)}(X_{L_s})) \sim \frac{\kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m)\delta_{mn}}{\Gamma(1-\alpha)t^\alpha}\left(\frac{1}{m} + \frac{s^\alpha}{\Gamma(1+\alpha)}\right).$$

- Let L be the **inverse of a Poisson subordinator** with parameter θ, see Example 3.3.2. Then, for any $t \geq s > 0$,

$$\rho_{\mathbf{e}_{\alpha,b}}(\mathcal{P}_m^{(\alpha,b)}(X_{L_t}), \mathcal{V}_n^{(\alpha,b)}(X_{L_s})) = \kappa_{\mathbf{e}_{\alpha,b}}^{-1}(m)\delta_{mn}\left(1 + \frac{m}{\theta}\right)^{-[t+1]}\left(2 - \left(1 + \frac{m}{\theta}\right)^{[s+1]}\right).$$

3.4 Proofs of the main results

3.4.1 Proof of Theorem 3.2.1

We split the proof of Theorem 3.2.1 into several intermediary lemmas.

Lemma 3.4.1. The sequence $(\mathcal{P}_n, \mathcal{V}_n)_{n\in\mathbb{N}}$, defined in Assumption 1, form a biorthogonal sequence in $L^2(\nu)$, i.e. for any $n, m \in \mathbb{N}$,

$$\langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = \delta_{nm}. \tag{3.28}$$

Proof. First, recall that in Section 3.1.1 we assumed, without loss of generality, that for any $n \in \mathbb{N}$, $\langle \mathcal{P}_n, \mathcal{V}_n \rangle_\nu = 1$. Therefore, we need to show that $\langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = 0$ when $n \neq m$. Then, note that for all $t \geq 0$ and $m, n \in \mathbb{N}$,

$$\langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = e^{\lambda_n t} \langle P_t \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = e^{\lambda_n t} \langle \mathcal{P}_n, P_t^* \mathcal{V}_m \rangle_\nu = e^{(\lambda_n - \lambda_m)t} \langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu$$

where in the first and last equality we used (3.3) and (3.4) respectively. Therefore,

$$\left(1 - e^{(\lambda_n - \lambda_m)t}\right) \langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = 0.$$

Hence, since we assumed that the eigenvalues are of multiplicity 1, $\lambda_n \neq \lambda_m$ if $n \neq m$. Thus, $\langle \mathcal{P}_n, \mathcal{V}_m \rangle_\nu = 0$, which concludes the proof. ∎

Lemma 3.4.2. Let $f \in L^2(\nu)$. Then, for any $t \geq 0$,

$$std_\nu(f(X_t)) = \sqrt{\nu f^2 - (\nu f)^2} = \sqrt{\nu f^2}. \tag{3.29}$$

In particular, if f is such that $\nu f = 0$, then

$$std_\nu(f(X_t)) = \|f\|_\nu. \tag{3.30}$$

Proof. The first claim immediately follows, for any $t \geq 0$, from the sequence of equalities

$$std_\nu(f(X_t)) = \sqrt{\nu P_t f^2 - (\nu P_t f)^2} = \sqrt{\nu f^2 - (\nu f)^2}.$$

Finally, if $\nu f = 0$, then we have

$$std_\nu(f(X_t)) = \sqrt{\nu f^2} = \|f\|_\nu. \tag{3.31}$$

∎

Lemma 3.4.3. Let $f \in L^2(\nu)$. Then, for any $m \in \mathtt{N}$ and $t \geq s > 0$,

$$C_\nu(\mathcal{P}_m(X_t), f(X_s)) = e^{-\lambda_m(t-s)}\langle \mathcal{P}_m, f\rangle_\nu.$$

Proof. First, Lemma 3.4.1 yields that $\langle \mathcal{P}_m, \mathcal{V}_n\rangle_\nu = \delta_{mn}$, $m, n \in \mathtt{N}$. In particular, since ν is invariant, for any $m \in \mathtt{N}$, $\nu P_t \mathcal{P}_m = \nu\mathcal{P}_m = \langle \mathcal{P}_m, 1\rangle_\nu = \delta_{0m} = 0$, where we used the fact that the constant function $\mathbb{1}$ is an eigenfunction for P_t since $P_t\mathbb{1} = \mathbb{1}$ for all $t \geq 0$. Similarly, since $P_0 = P_0^* = \mathbb{1}$, then $\nu\mathcal{P}_m = \nu\mathcal{V}_m = \delta_{0m} = 0$. Then, from the definition of the covariance function given in (3.10), we obtain, for any $t \geq s > 0$,

$$\begin{aligned} C_\nu(\mathcal{P}_m(X_t), f(X_s)) &= \mathbb{E}_\nu[\mathcal{P}_m(X_t)f(X_s)] - \mathbb{E}_\nu[\mathcal{P}_m(X_t)]\mathbb{E}_\nu[f(X_s)] \\ &= \mathbb{E}_\nu[\mathcal{P}_m(X_t)f(X_s)] - \nu P_t\mathcal{P}_m \ \nu P_s f \\ &= \mathbb{E}_\nu[\mathcal{P}_m(X_t)f(X_s)]. \end{aligned}$$

Next, using the Markov property and (3.3), we get

$$\begin{aligned} \mathbb{E}_\nu[\mathcal{P}_m(X_t)f(X_s)] &= \mathbb{E}_\nu[\mathbb{E}_{X_s}[\mathcal{P}_m(X_{t-s})]f(X_s)] \\ &= \mathbb{E}_\nu[P_{t-s}\mathcal{P}_m(X_s)f(X_s)] \\ &= e^{-\lambda_m(t-s)}\mathbb{E}_\nu[\mathcal{P}_m(X_s)f(X_s)] \\ &= e^{-\lambda_m(t-s)}\nu P_s\mathcal{P}_m f = e^{-\lambda_m(t-s)}\nu\mathcal{P}_m f \\ &= e^{-\lambda_m(t-s)}\langle \mathcal{P}_m, f\rangle_\nu, \end{aligned}$$

where in the second last equality we used the fact that ν is an invariant measure for P. ■

We are now ready to prove Theorem 3.2.1. First, recall from the proof of Lemma 3.4.3 that for any $m \in \mathtt{N}$, $\nu\mathcal{P}_m = \nu\mathcal{V}_m = 0$. Next, it follows from Lemma 3.4.2 that for any $t \geq 0$ and $m \in \mathtt{N}$,

$$std_\nu(\mathcal{P}_m(X_t)) = \|\mathcal{P}_m\|_\nu \quad \text{and} \quad std_\nu(\mathcal{V}_m(X_t)) = \|\mathcal{V}_m\|_\nu.$$

Then, using Lemma 3.4.3 with $f = \mathcal{V}_n$ and $f = \mathcal{P}_m$, respectively, we get, for any $t \geq s > 0$ and $n, m \in \mathtt{N}$, that

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_s)) &= \frac{e^{-\lambda_m(t-s)}\langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu} = e^{-\lambda_m(t-s)} c_\nu(n,m), \\ \rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_s)) &= \frac{e^{-\lambda_m(t-s)}\delta_{mn}}{\|\mathcal{P}_m\|_\nu \, \|\mathcal{V}_m\|_\nu} = e^{-\lambda_m(t-s)} \kappa_\nu^{-1}(m)\delta_{mn}, \end{aligned}$$

where we recall that for $m, n \in \mathtt{N}$, $\kappa_\nu(m) = \|\mathcal{P}_m\|_\nu \|\mathcal{V}_m\|_\nu$ and $c_\nu(n,m) = \frac{\langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu}$. Then, by symmetry, it is easy to note that for any $t, s > 0$, we have

$$\rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_s)) = e^{-\lambda_m(t-s)^+ - \lambda_n(s-t)^+} c_\nu(n,m).$$

Finally, the Cauchy-Schwartz inequality entails that $|\langle \mathcal{P}_n, \mathcal{P}_m \rangle_\nu| \leq \|\mathcal{P}_n\|_\nu \|\mathcal{P}_m\|_\nu$ and hence $-1 \leq c_\nu(n,m) \leq 1$. Moreover, when $n = m$, we have that for any $n \in \mathtt{N}$,

$$c_\nu(n,n) = \frac{\langle \mathcal{P}_n, \mathcal{P}_n \rangle_\nu}{\|\mathcal{P}_n\|_\nu \|\mathcal{P}_n\|_\nu} = \frac{\|\mathcal{P}_n\|_\nu^2}{\|\mathcal{P}_n\|_\nu^2} = 1,$$

and this concludes the proof of Theorem 3.2.1. ■

3.4.2 Proof of Lemma 3.2.1

The definitions of the covariance and correlation functions in (3.10) and (3.11) give that for any $t \geq 0$,

$$\begin{aligned} \rho_\nu(f(X_t), g(X_t)) &= \frac{\mathbb{E}_\nu[f(X_t)g(X_t)] - \mathbb{E}_\nu[f(X_t)]\mathbb{E}_\nu[g(X_t)]}{std_\nu(f(X_t))std_\nu(g(X_t))} \\ &= \frac{\nu P_t fg - \nu P_t f \cdot \nu P_t g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}} \\ &= \frac{\nu fg - \nu f \cdot \nu g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}} \\ &= \frac{\langle f, g \rangle_\nu - \nu f \cdot \nu g}{\sqrt{\nu f^2 - (\nu f)^2} \cdot \sqrt{\nu g^2 - (\nu g)^2}} \end{aligned} \tag{3.32}$$

where in the third equality we used the fact that ν is an invariant measure for P_t. Next, for $n, m \in \mathbb{N}$, taking $f = \mathcal{P}_m$ with $g = \mathcal{P}_n$ and $g = \mathcal{V}_n$ in (3.32), and using Lemma 3.4.1 and Lemma 3.4.2, we get, for any $t \geq 0$,

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_t), \mathcal{P}_n(X_t)) &= \frac{\langle \mathcal{P}_m, \mathcal{P}_n \rangle_\nu}{\|\mathcal{P}_m\|_\nu \cdot \|\mathcal{P}_n\|_\nu} = c_\nu(n, m), \\ \rho_\nu(\mathcal{P}_m(X_t), \mathcal{V}_n(X_t)) &= \frac{\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu}{\|\mathcal{P}_m\|_\nu \cdot \|\mathcal{V}_n\|_\nu} = \kappa_\nu^{-1}(m)\delta_{mn} \end{aligned}$$

where we recall that for any $n \in \mathbb{N}$, $\nu\mathcal{P}_n = \nu\mathcal{V}_n = 0$. ■

3.4.3 Proof of Proposition 3.2.1

For a function f, we write $\mathcal{L}_f(q) = \int_0^\infty e^{-qz} f(z)dz$ and we use the same notation for the Laplace transform of a measure. Then, for any $\lambda > 0$, denoting

$$U_\lambda(dw) = \int_0^\infty \mathbb{P}(\mathcal{T}_z \in dw)e^{-\lambda z}dz, \quad w \geq 0, \tag{3.33}$$

the λ-potential measure of $\mathcal{T}$, we have, for any $q > 0$,

$$\begin{aligned} \mathcal{L}_{U_\lambda}(q) &= \int_0^\infty e^{-qw} \int_0^\infty \mathbb{P}(\mathcal{T}_z \in dw)e^{-\lambda z}dz \\ &= \int_0^\infty e^{-\lambda z} \int_0^\infty e^{-qw}\mathbb{P}(\mathcal{T}_z \in dw) \\ &= \int_0^\infty e^{-\lambda z}e^{-z\varphi(q)}dz = \frac{1}{\lambda + \varphi(q)}. \end{aligned} \tag{3.34}$$

Next, as $\varphi(0) = 0$, see (3.6), $\int_0^\infty U_\lambda(dw) = \frac{1}{\lambda}$. Thus, writing, for any $t \geq 0$, $\overline{U}_\lambda(t) = \lambda \int_t^\infty U_\lambda(dw)$ and changing the order of integration justified by an application of Tonelli's theorem, we get that for any $q > 0$,

$$\mathcal{L}_{\overline{U}_\lambda}(q) = \frac{1}{q} - \lambda\frac{1}{q(\lambda + \varphi(q))} = \frac{\varphi(q)}{q(\lambda + \varphi(q))}. \tag{3.35}$$

On the other hand, it is well known that the Laplace transform of $t \mapsto \eta_t(\lambda)$, where we recall that $\eta_t(\lambda) = \int_0^\infty e^{-\lambda s} l_t(ds)$, takes the form, for any $q > 0$,

$$\mathcal{L}_{\eta_\cdot(\lambda)}(q) = \frac{\varphi(q)}{q(\lambda + \varphi(q))}, \tag{3.36}$$

see e.g. Mijena and Nane [112]. Therefore, the injectivity of the Laplace transform implies that for any $t \geq 0$,

$$\eta_t(\lambda) = \lambda \int_t^\infty U_\lambda(dw). \tag{3.37}$$

Here, writing $\widetilde{U}_\lambda(dw) = \lambda_m U_\lambda(dw)$, $w > 0$, we have

$$\eta_t(\lambda) = \int_t^\infty \widetilde{U}_\lambda(dr). \tag{3.38}$$

Since $\eta_t(\lambda)$ is decreasing in t, $\eta_0(\lambda) = 1$ and $\lim_{t\to\infty} \eta_t(\lambda) = 0$, we deduce that $\eta_t(\lambda)$ is a tail of a probability measure, i.e. there exists a random variable X_λ such that $\eta_t(\lambda) = \mathbb{P}(X_\lambda > t) = \int_t^\infty \widetilde{U}_\lambda(dr)$, $t > 0$. Next, assuming that φ is strongly regularly varying at 0, i.e. $\varphi(q) \overset{0}{\sim} Cq^\alpha$, $0 < \alpha < 1$ for some constant $C > 0$, using (3.34), we obtain

$$1 - \mathcal{L}_{\widetilde{U}_\lambda}(q) = 1 - \int_0^\infty e^{-qr} \widetilde{U}_\lambda(dr) = \frac{\varphi(q)}{\lambda + \varphi(q)} \overset{0}{\sim} q^\alpha.$$

Then, it follows from a Tauberian theorem, see e.g. [20, Corollary 8.1.7], that equivalently we have

$$\eta_t(\lambda) \overset{t\to\infty}{\sim} \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

Thus, for any $a > 0$, we get that

$$\lim_{t\to\infty} \frac{\eta_{\log(at)}(\lambda)}{\eta_{\log t}(\lambda)} = \lim_{t\to\infty} \frac{(\log a + \log t)^{-\alpha}}{(\log t)^{-\alpha}} = 1.$$

Therefore, $t \mapsto \eta_{\log(t)}(\lambda)$ is slowly varying at infinity, and thus $t \mapsto \eta_t(\lambda)$ is long-tailed, see e.g. [64, Lemma 2.15], which completes the proof of the proposition. ■

3.4.4 Proof of Theorem 3.2.2

Writing for $t, s \geq 0$, $H_{t,s}(u,v) = \mathbb{P}(L_t \leq u, L_s \leq v)$, we have that the independence of X and L entails that for any $m, n \in \mathtt{N}$,

$$\begin{aligned}\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= \int_0^\infty \int_0^\infty \rho_\nu(\mathcal{P}_m(X_u), \mathcal{P}_n(X_v)) H_{t,s}(du, dv),\\ \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) &= \int_0^\infty \int_0^\infty \rho_\nu(\mathcal{P}_m(X_u), \mathcal{V}_n(X_v)) H_{t,s}(du, dv).\end{aligned}$$

Next, recalling that c_ν is symmetric, i.e. $c_\nu(n,m) = c_\nu(m,n)$ for any $m, n \in \mathtt{N}$, Theorem 3.2.1 gives that for any $u, v \geq 0$,

$$\rho_\nu(\mathcal{P}_m(X_u), \mathcal{P}_n(X_v)) = c_\nu(n,m)\left(e^{-\lambda_m(u-v)}1_{\{u>v\}} + e^{-\lambda_n(v-u)}1_{\{u\leq v\}}\right)$$

and hence

$$\begin{aligned}\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= \int_0^\infty \int_0^\infty c_\nu(n,m)\left(e^{-\lambda_m(u-v)}1_{\{u>v\}} + e^{-\lambda_n(v-u)}1_{\{u\leq v\}}\right) H_{t,s}(du, dv)\\ &= c_\nu(n,m)\int_0^\infty \int_0^\infty \left(e^{-\lambda_m(u-v)}1_{\{u>v\}} + e^{-\lambda_n(v-u)}1_{\{u\leq v\}}\right) H_{t,s}(du, dv).\end{aligned} \tag{3.39}$$

Now, let $(u,v) \mapsto F(u,v)$ be a function of bounded variation such that $u \mapsto F(u,v)$ and $v \mapsto F(u,v)$ are also of bounded variation. Then, writing $F(du,v)$, $F(u,dv)$ and $F(du,dv)$, we mean the one dimensional measures generated by the sections $u \mapsto F(u,v)$, $v \mapsto F(u,v)$ and the two dimensional measure generated by $(u,v) \mapsto F(u,v)$ respectively. For such a function F, recall the bivariate integration by parts formula

$$\begin{aligned}\int_0^\infty \int_0^\infty F(u,v) H_{t,s}(du, dv) &= \int_0^\infty \int_0^\infty H_{t,s}([u,\infty] \times [v,\infty]) F(du, dv)\\ &+ \int_0^\infty H_{t,s}([u,\infty] \times (0,\infty]) F(du, 0)\\ &+ \int_0^\infty H_{t,s}((0,\infty] \times [v,\infty]) F(0, dv)\\ &+ F(0,0) H_{t,s}((0,\infty] \times (0,\infty]),\end{aligned} \tag{3.40}$$

see e.g. Gill et al. [70, Lemma 2.2]. Let us apply this formula to

$$F(u,v) = e^{-\lambda_m(u-v)}1_{\{u>v\}} + e^{-\lambda_n(v-u)}1_{\{u\leq v\}}, \quad (u,v) \in \mathbb{R}_+^2,$$

which is clearly of bounded variation. Then, writing $\overline{H}_{t,s}(u,v) = \mathbb{P}(L_t \geq u, L_s \geq v)$ and $\overline{H}_t(u) = \mathbb{P}(L_t \geq u)$,

$$\begin{aligned}\int_0^\infty \int_0^\infty F(u,v)H_{t,s}(du,dv) &= \int_0^\infty \int_0^\infty \overline{H}_{t,s}(u,v)F(du,dv) + \int_0^\infty \overline{H}_t(u)F(du,0) \\ &+ \int_0^\infty \overline{H}_s(v)F(0,dv) + 1, \end{aligned} \tag{3.41}$$

where we used that as $\mathbb{P}(L_t = 0) = 0$, $\mathbb{P}(L_t > 0) = 1$ for all $t > 0$, $F(0,0) = 1$ and H is a distribution function. Note that $F(du,v) = \left(-\lambda_m e^{-\lambda_m(u-v)}1_{\{u>v\}} + \lambda_n e^{-\lambda_n(v-u)}1_{\{u\leq v\}}\right)du$ for all $v \geq 0$. Thus, an integration by parts yields that

$$\begin{aligned}\int_0^\infty \overline{H}_t(u)F(du,0) &= \int_0^\infty (1-\mathbb{P}(L_t < u))(-\lambda_m e^{-\lambda_m u})du \\ &= e^{-\lambda_m u}\overline{H}_t(u)\Big|_0^\infty + \int_0^\infty e^{-\lambda_m u} l_t(u)du = \eta_t(\lambda_m) - 1,\end{aligned}$$

and similarly,

$$\int_0^\infty \overline{H}_s(v)F(0,dv) = \eta_s(\lambda_n) - 1. \tag{3.42}$$

Hence, (3.39) reduces to

$$c_\nu(n,m)\int_0^\infty \int_0^\infty F(u,v)H_{t,s}(du,dv) = c_\nu(n,m)(I(t,s) + \eta_t(\lambda_m) + \eta_s(\lambda_n) - 1),$$

where we have set

$$I(t,s) = \int_0^\infty \int_0^\infty \overline{H}_{t,s}(u,v)F(du,dv). \tag{3.43}$$

Then, observing that $I(s,t) = I(t,s)$, we assume, without loss of generality, that $s \leq t$ and we write $I(t,s) = I_1(t,s) + I_2(t,s) + I_3(t,s)$, where

$$\begin{aligned}I_1(t,s) &= \int_0^\infty \int_0^v \overline{H}_{t,s}(u,v)F(du,dv),\ I_2(t,s) = \int_0^\infty \int_{u=v} \overline{H}_{t,s}(u,v)F(du,dv), \\ I_3(t,s) &= \int_0^\infty \int_v^\infty \overline{H}_{t,s}(u,v)F(du,dv).\end{aligned}$$

Then, as the inverse of the subordinator $\mathcal{T}$ is non-decreasing, $\overline{H}_{t,s}(u,v) = \mathbb{P}(L_t \geq u, L_s \geq v) = \mathbb{P}(L_s \geq v) = \overline{H}_s(v)$ for $u \leq v$ and $F(du,dv) = -\lambda_n^2\, e^{-\lambda_n(v-u)}dudv$ for $u < v$. Thus,

$$\begin{aligned} I_1(t,s) &= \int_0^\infty \int_0^v \overline{H}_s(v)F(du,dv) \\ &= -\lambda_n^2 \int_0^\infty \int_0^v \overline{H}_s(v)e^{\lambda_n(u-v)}dudv \\ &= -\lambda_n \int_0^\infty \overline{H}_s(v)\left(1-e^{-\lambda_n v}\right)dv \\ &= -\lambda_n \int_0^\infty \overline{H}_s(v)dv + \lambda_n \int_0^\infty e^{-\lambda_n v}\overline{H}_s(v)dv \\ &= -\lambda_n \mathbb{E}[L_s] + \lambda_n \int_0^\infty e^{-\lambda_n v}\mathbb{P}(L_s \geq v)dv \\ &= -\lambda_n \mathbb{E}[L_s] - \eta_s(\lambda_n) + 1, \end{aligned}$$

where in the last identity we have performed an integration by parts. We also note that

$$\mathbb{E}[L_s] = \int_0^\infty \overline{H}_s(v)dv = \int_0^\infty \mathbb{P}(L_s \geq v)dv = U(0,s). \tag{3.44}$$

Next, writing simply $f_v(u)du = F(du,v) = \left(-\lambda_m\, e^{-\lambda_m(u-v)}1_{\{u>v\}} + \lambda_n\, e^{-\lambda_n(v-u)}1_{\{u\leq v\}}\right)du$, we remark that the mapping $u \mapsto f_v(u)$ has a jump of size $(\lambda_m+\lambda_n)$ at the point $u=v$. Then,

$$I_2(t,s) = \int_0^\infty \int_{u=v} \overline{H}_s(v)F(du,dv) = (\lambda_m+\lambda_n)\int_0^\infty \overline{H}_s(v)dv = (\lambda_m+\lambda_n)\mathbb{E}[L_s].$$

Finally, as $F(du,dv) = -\lambda_m^2 e^{-\lambda_m(u-v)}dudv$ for $u > v$, we deduce that

$$I_3(t,s) = -\lambda_m^2 \int_0^\infty \overline{H}_{t,s}(u,v)\int_v^\infty e^{-\lambda_m(u-v)}dudv,$$

and we proceed by computing the joint tail distribution of the pair (L_t, L_s), that is $\overline{H}_{t,s}(u,v) = \mathbb{P}(L_t \geq u, L_s \geq v)$. Note that since L is the inverse of $\mathcal{T}$, then $\{L_t \geq u\} = \{\mathcal{T}_u \leq t\}$, and thus $\mathbb{P}(L_t \geq u, L_s \geq v) = \mathbb{P}(\mathcal{T}_u \leq t, \mathcal{T}_v \leq s)$. Now, since as a Lévy process $\mathcal{T}$ has stationary and independent increments, it follows, recalling that $s \leq t$,

$$\begin{aligned} \overline{H}_{t,s}(u,v) = \mathbb{P}(\mathcal{T}_u \leq t, \mathcal{T}_v \leq s) &= \mathbb{P}((\mathcal{T}_u - \mathcal{T}_v) + \mathcal{T}_v \leq t, \mathcal{T}_v \leq s) \\ &= \int_0^s \mathbb{P}(\mathcal{T}_v \in dr)\int_0^{t-r} \mathbb{P}(\mathcal{T}_{u-v} \in dw). \end{aligned}$$

Using Fubini's theorem and performing the change of variable $z = u - v$, we get

$$\begin{aligned} I_3(t,s) &= -\lambda_m^2 \int_0^\infty \int_0^s \mathbb{P}(\mathcal{T}_v \in dr) \int_0^{t-r} \mathbb{P}(\mathcal{T}_{u-v} \in dw) \int_v^\infty e^{-\lambda_m (u-v)} du dv \\ &= -\lambda_m^2 \int_0^s \int_0^{t-r} \int_0^\infty e^{-\lambda_m z} \mathbb{P}(\mathcal{T}_z \in dw) dz \int_0^\infty \mathbb{P}(\mathcal{T}_v \in dr) dv \\ &= -\lambda_m^2 \int_0^s \int_0^{t-r} \int_0^\infty e^{-\lambda_m z} \mathbb{P}(\mathcal{T}_z \in dw) dz U(dr) \end{aligned} \tag{3.45}$$

where in the last step, from the definition of the renewal measure, we have used that $\int_0^\infty \mathbb{P}(\mathcal{T}_v \in dr) dv = U(dr)$. Now, taking $\lambda = \lambda_m$ in (3.33) and (3.37) in the proof of Proposition (3.2.1), we have $U_{\lambda_m}(dw) = \int_0^\infty \mathbb{P}(\mathcal{T}_z \in dw) e^{-\lambda_m z} dz$, $w > 0$ and $\eta_t(\lambda_m) = \lambda_m \int_t^\infty U_{\lambda_m}(dw)$. Hence, using (3.44), the expression of $I_3(t,s)$ in (3.45) reduces to

$$\begin{aligned} I_3(t,s) &= -\lambda_m^2 \int_0^s \int_0^{t-r} U_{\lambda_m}(dw) U(dr) \\ &= \lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) - \lambda_m \int_0^s U(dr) \\ &= \lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) - \lambda_m \mathbb{E}[L_s]. \end{aligned}$$

Finally, putting all pieces together, we obtain that for $t \geq s > 0$,

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{P}_n(X_{L_s})) &= c_\nu(n,m)(I_1(t,s) + I_2(t,s) + I_3(t,s) + \eta_t(\lambda_m) + \eta_s(\lambda_n) - 1) \\ &= c_\nu(n,m)\Big(-\lambda_n \mathbb{E}[L_s] - \eta_s(\lambda_n) + 1 + (\lambda_m + \lambda_n)\mathbb{E}[L_s] \\ &+ \lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) - \lambda_m \mathbb{E}[L_s] + \eta_t(\lambda_m) + \eta_s(\lambda_n) - 1\Big) \\ &= c_\nu(n,m)\Big(\lambda_m \int_0^s \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m)\Big) \end{aligned} \tag{3.46}$$

which provides the claim (3.15). We proceed by studying the spectral projections correlation structure of $\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s}))$ for $t \geq s > 0$ and $m, n \in \mathtt{N}$. Note that, for any $u, v \geq 0$,

$$\rho_\nu(\mathcal{P}_m(X_u), \mathcal{V}_n(X_v)) = F_1(u,v) + F_2(u,v), \tag{3.47}$$

where we have written, see Theorem 3.2.1,

$$\begin{aligned} F_1(u,v) &= \rho_\nu(\mathcal{P}_m(X_u),\mathcal{V}_n(X_v))1_{\{u\geq v\}} = \kappa_\nu^{-1}(m)\delta_{mn}e^{-\lambda_m(u-v)}1_{\{u\geq v\}}, \\ F_2(u,v) &= \rho_\nu(\mathcal{P}_m(X_u),\mathcal{V}_n(X_v))1_{\{u<v\}}. \end{aligned}$$

Then, for any $t \geq s > 0$ and $m,n \in \mathbb{N}$, we have

$$\rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{V}_n(X_{L_s})) = \int_0^\infty\int_0^\infty F_1(u,v)H_{t,s}(du,dv) + \int_0^\infty\int_0^\infty F_2(u,v)H_{t,s}(du,dv). \tag{3.48}$$

Now, recalling the bivariate integration by parts formula (3.40), one has

$$\begin{aligned} \int_0^\infty\int_0^\infty F_1(u,v)H_{t,s}(du,dv) &= \int_0^\infty\int_0^\infty \overline{H}_{t,s}(u,v)F_1(du,dv) + \int_0^\infty \overline{H}_t(u)F_1(du,0) \\ &+ \kappa_\nu^{-1}(m)\delta_{mn}, \end{aligned}$$

where we used that $\int_0^\infty \overline{H}_s(v)F_1(0,dv) = 0$ and $F_1(0,0) = \kappa_\nu^{-1}(m)\delta_{mn}$. Now, following the same pattern as in the proof of the first part of Theorem 3.2.1 above, and since on $\{u < v\}$, $F_1(du,dv) = 0$, one gets

$$\begin{aligned} \int_0^\infty\int_0^\infty F_1(u,v)H_{t,s}(du,dv) &= \int_0^\infty\int_{u=v} \overline{H}_{t,s}(u,v)F_1(du,dv) + \int_0^\infty\int_v^\infty \overline{H}_{t,s}(u,v)F_1(du,dv) \\ &+ \int_0^\infty \overline{H}_t(u)F_1(du,0) + \kappa_\nu^{-1}(m)\delta_{mn} \\ &= \kappa_\nu^{-1}(m)\delta_{mn}\lambda_m\mathbb{E}[L_s] \\ &+ \kappa_\nu^{-1}(m)\delta_{mn}\left(\lambda_m\int_0^s \eta_{t-r}(\lambda_m)U(dr) - \lambda_m\mathbb{E}[L_s]\right) \\ &+ \kappa_\nu^{-1}(m)\delta_{mn}\left(\eta_t(\lambda_m) - 1\right) \\ &= \kappa_\nu^{-1}(m)\delta_{mn}\left(\lambda_m\int_0^s \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m)\right). \end{aligned}$$

Next, we turn to the computation of the second integral on the right-hand side of (3.48). As the functions $(u,v) \mapsto F_2(u,v)$, $u \mapsto F_2(u,v)$ and $v \mapsto F_2(u,v)$ are of bounded variation since by (3.47), $F_2(u,v)$ is a difference of two functions of bounded variation, then, by means of the bivariate integration by parts formula (3.40), we get

$$\int_0^\infty\int_0^\infty F_2(u,v)H_{t,s}(du,dv) = \int_0^\infty\int_0^\infty \overline{H}_{t,s}(u,v)F_2(du,dv) + \int_0^\infty \overline{H}_s(v)F_2(0,dv), \tag{3.49}$$

where we used that $\int_0^\infty \overline{H}_t(u)F_2(du,0) = 0$ and $F_2(0,0) = 0$. Now, since on $\{u > v\}$, $F_2(du,dv) = 0$, then, for $t \geq s$, (3.49) reduces to

$$\begin{aligned}\int_0^\infty \int_0^\infty F_2(u,v)H_{t,s}(du,dv) &= \int_0^\infty \int_0^v \overline{H}_{t,s}(u,v)F_2(du,dv) \\ &+ \int_0^\infty \int_{u=v} \overline{H}_{t,s}(u,v)F_2(du,dv) + \int_0^\infty \overline{H}_s(v)F_2(0,dv) \\ &= \int_0^\infty \int_0^v \overline{H}_s(v)F_2(du,dv) + \int_0^\infty \int_{u=v} \overline{H}_s(v)F_2(du,dv) \\ &+ \int_0^\infty \overline{H}_s(v)F_2(0,dv).\end{aligned}$$

Thus, $\int_0^\infty \int_0^\infty F_2(u,v)H_{t,s}(du,dv)$ does not depend on t. On the other hand, taking $t = s$ in (3.48), we get, for any $s \geq 0$,

$$\kappa_\nu^{-1}(m)\delta_{mn} = \kappa_\nu^{-1}(m)\delta_{mn} + \int_0^\infty \int_0^\infty F_2(u,v)H_{s,s}(du,dv),$$

wher we used that by taking $t = s$ in (3.46), Lemma 3.2.1 yields that for any $t \geq 0$,

$$\lambda_m \int_0^t \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m) = 1.$$

This can also be independently proven as in Remark 3.4.1. Hence, for any $s \geq 0$,

$$\int_0^\infty \int_0^\infty F_2(u,v)H_{s,s}(du,dv) = 0,$$

and we deduce that for any $t \geq s > 0$,

$$\int_0^\infty \int_0^\infty F_2(u,v)H_{t,s}(du,dv) = \int_0^\infty \int_0^\infty F_2(u,v)H_{s,s}(du,dv) = 0.$$

Therefore, putting pieces together, (3.48) reduces to

$$\rho_\nu(\mathcal{P}_m(X_{L_t}), \mathcal{V}_n(X_{L_s})) = \kappa_\nu^{-1}(m)\delta_{mn}\left(\lambda_m \int_0^s \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m)\right).$$

Now we are ready to study the right-hand side of (3.15) and (3.16) for large t when $s > 0$ is fixed under the assumption that $\lim_{t\to\infty} \frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} = 1$. Since $t \mapsto \eta_t(\lambda_m)$ is decreasing on $\mathbb{R}^+$, we have

$$\int_0^s \eta_{t-r}(\lambda_m)U(dr) \geq \eta_t(\lambda_m)U(0,s) = \eta_t(\lambda_m)\mathbb{E}[L_s]$$

and

$$\int_0^s \eta_{t-r}(\lambda_m)U(dr) \le \eta_{t-s}(\lambda_m)U(0,s) \le \eta_{t-s}(\lambda_m)\mathbb{E}[L_s].$$

Consequently,

$$\begin{aligned} c_\nu(n,m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1) \le \quad & \rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{P}_n(X_{L_s})) \quad \le c_\nu(n,m)\eta_{t-s}(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1), \\ \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1) \le \quad & \rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{V}_n(X_{L_s})) \quad \le \kappa_\nu^{-1}(m)\delta_{mn}\eta_{t-s}(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1). \end{aligned}$$

Now, if for a fixed $s>0$, there exists a constant $C = C(s,\lambda_m) > 0$ such that $\overline{\lim}_{t\to\infty}\frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} = C$, then there exists $t_0 > 0$ such that for $t \ge t_0$, $\frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} \le C$, and thus

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{P}_n(X_{L_s})) &\overset{t_0}{\asymp} c_\nu(n,m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1), \\ \rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{V}_n(X_{L_s})) &\overset{t_0}{\asymp} \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1). \end{aligned}$$

In particular, if for a fixed $s>0$, $\lim_{t\to\infty}\frac{\eta_{t-s}(\lambda_m)}{\eta_t(\lambda_m)} \equiv 1$, we have

$$\begin{aligned} \rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{P}_n(X_{L_s})) &\overset{t\to\infty}{\sim} c_\nu(n,m)\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1), \\ \rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{V}_n(X_{L_s})) &\overset{t\to\infty}{\sim} \kappa_\nu^{-1}(m)\delta_{mn}\eta_t(\lambda_m)(\lambda_m\mathbb{E}[L_s]+1), \end{aligned}$$

and this concludes the proof of Theorem 3.2.2. ■

Remark 3.4.1. One can easily check that for any $n,m \in \mathtt{N}$, when $t = s$ in (3.15), we have for any $t \ge 0$,

$$\rho_\nu(\mathcal{P}_m(X_{L_t}),\mathcal{P}_n(X_{L_t})) = c_\nu(n,m),$$

i.e.

$$\lambda_m \int_0^t \eta_{t-r}(\lambda_m)U(dr) + \eta_t(\lambda_m) = 1.$$

Indeed, let us plug in $t = s$ in (3.15). Then, noting that the convolution, we get that, for any $q > 0$,

$$\begin{aligned}\mathcal{L}_{\int_0^{\cdot} \eta_{\cdot - r}(\lambda_m) U(dr)}(q) &= \mathcal{L}_{\eta_\cdot(\lambda_m)}(q)\mathcal{L}_U(q) \\ &= \frac{\varphi(q)}{q(\lambda_m + \varphi(q))}\frac{1}{\varphi(q)} \\ &= \frac{1}{q(\lambda_m + \varphi(q))}.\end{aligned}$$

Next, using (3.36), one has

$$\mathcal{L}_{\lambda_m \int_0^{\cdot} \eta_{\cdot - r}(\lambda_m) U(dr) + \eta_\cdot(\lambda_m)}(q) = \frac{\lambda_m}{q(\lambda_m + \varphi(q))} + \frac{\varphi(q)}{q(\lambda_m + \varphi(q))} = \frac{1}{q}.$$

Thus, by the injectivity of the Laplace transform we conclude that

$$\lambda_m \int_0^t \eta_{t-r}(\lambda_m) U(dr) + \eta_t(\lambda_m) = 1.$$

CHAPTER 4

ON NON-LOCAL ERGODIC JACOBI SEMIGROUPS: SPECTRAL THEORY, CONVERGENCE-TO-EQUILIBRIUM, AND CONTRACTIVITY

4.1 Introduction

In this paper we study the non-local Jacobi operators on $E = [0, 1]$ given for suitable f by

$$\mathbb{J}f(x) = \mathbf{J}_\mu f(x) - f' \diamond h(x), \tag{4.1}$$

where $\mathbf{J}_\mu$ is the classical Jacobi operator

$$\mathbf{J}_\mu f(x) = x(1-x)f''(x) - (\lambda_1 x - \mu)\, f'(x),$$

and $\diamond$ denotes the product convolution operator

$$f \diamond h(x) = \int_0^x f(r)h(xr^{-1})r^{-1}dr,$$

with λ_1, μ, and the function h satisfying Assumption 4.2.1.1 below. The classical Jacobi operator is a central object in the study of Markovian diffusions. For instance, it is a model candidate for testing functional inequalities such as the Sobolev and logarithmic Sobolev inequalities, see e.g. [8, 143]. When $\mu = \frac{\lambda_1}{2} = n$, an integer, there exists a homeomorphism between this symmetric Jacobi operator and the radial part of the Laplace-Beltrami operator on the n-sphere, revealing connections to diffusions on higher-dimensional manifolds that, in particular, lead to a curvature-dimension inequality, as described in [10, Chapter 2.7]. From the spectral theory viewpoint, the Markov semigroup $\mathbf{Q}^{(\mu)} = (e^{t\mathbf{J}_\mu})_{t\geqslant 0}$ is diagonalizable with respect to an orthonormal, polynomial basis for $\mathrm{L}^2(\beta_\mu)$, where β_μ denotes its unique invariant probability measure. As a consequence of these facts the semigroup $\mathbf{Q}^{(\mu)}$ converges to equilibrium in various senses, such as in variance and in entropy, and is both hypercontractive and ultracontractive. See Section 4.5 where we review essential facts about the classical Jacobi operator, semigroup, and process. We also mention, without aiming to be exhaustive, that the Jacobi process has been a popular model in diverse scientific areas including population genetics, where it is known as the Wright-Fisher diffusion, see e.g. [62, Chapter 10] and [73, 74, 82, 119], and in finance, to model bounded interest rates [51] and asset return volatility [72].

Due to the non-local part of $\mathbb{J}$ and its non-self-adjointness as a densely defined and closed operator in $L^2(\beta)$ with β denoting its invariant measure, a fact that is proved below, the traditional techniques that are used to study $\mathbf{J}_\mu$ seem out of reach. Nevertheless, our investigation of $\mathbb{J}$ yields generalizations of the classical and substantial results mentioned above. A central tool in our developments is the notion of an intertwining relation, which is a type of commutation relationship for linear operators. Fixing $\boldsymbol{\lambda}_1$ and for some parameters $\widetilde{\mu}, \overline{\mu}$ to be specified below, we develop identities of the form

$$\mathbb{J}\Lambda = \Lambda \mathbf{J}_{\widetilde{\mu}}, \quad \text{and} \quad V\mathbb{J} = \mathbf{J}_{\overline{\mu}} V,$$

on the space of polynomials, the first of which allows us to prove that $\mathbb{J}$ generates an ergodic Markov semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t\geqslant 0}$ with unique invariant measure β. We also establish, for $t \geqslant 0$,

$$\mathbb{Q}_t\Lambda = \Lambda \mathbf{Q}_t^{(\widetilde{\mu})} \quad \text{and} \quad V\mathbb{Q}_t = \mathbf{Q}_t^{(\overline{\mu})} V,$$

on $L^2(\beta_{\widetilde{\mu}})$ and $L^2(\beta)$, respectively, where $\Lambda : L^2(\beta_{\widetilde{\mu}}) \to L^2(\beta)$ and $V : L^2(\beta) \to L^2(\beta_{\overline{\mu}})$ are bounded linear operators. These latter identities are crucial towards obtaining the spectral theory, convergence-to-equilibrium, hypercontractivity, and ultracontractivity estimates for $\mathbb{Q}$.

The paper is organized as follows. We state our main results in Section 4.2. All proofs are given in Section 4.3 and a specific family of non-local Jacobi semigroups is considered in Section 4.4. Finally we collect known results on the classical Jacobi operator, semigroup, and process in Section 4.5.

4.2 Main results on non-local Jacobi operators and semigroups

4.2.1 Preliminaries and existence of Markov semigroup

In this section we state our main results concerning the non-local operator $\mathbb{J}$ defined in (4.1). We write $\mathbb{R}_+ = (0, \infty)$ and $\mathbf{1}$ for the indicator function, and throughout we shall operate under the following assumption.

Assumption 4.2.1.1. The function $h : (1, \infty) \to [0, \infty)$ is such that $-(e^r h(e^r))'$ is a finite Radon measure on $\mathbb{R}_+$, and $\hbar = \int_1^\infty h(r)dr < \infty$. Furthermore, if $h \not\equiv 0$,

$$\boldsymbol{\lambda}_1 > \mathbf{1}_{\{\mu<1+\hbar\}} + \mu \quad \text{and} \quad \mu > \hbar,$$

while otherwise $\boldsymbol{\lambda}_1 > \mu > 0$.

Note that, for $h \not\equiv 0$, we have $\hbar > 0$ and thus $\boldsymbol{\lambda}_1 > 1$. Next, we consider the convex, twice differentiable and eventually increasing function $\Psi : [0, \infty) \to \mathbb{R}$ given by

$$\Psi(u) = u^2 + (\mu - \hbar - 1)u + u \int_1^\infty (1 - r^{-u})h(r)dr, \tag{4.2}$$

which is easily seen to always have 0 as a root, and has a root $\mathsf{r} > 0$ if and only if $\mu < 1 + \hbar$. Set

$$\mathsf{r}_0 = \mathsf{r}\mathbf{1}_{\{\mu<1+\hbar\}} \quad \text{and} \quad \mathsf{r}_1 = 1 - \mathsf{r}_0, \tag{4.3}$$

and define $\phi : [0, \infty) \to [0, \infty)$ to be the function given by

$$\phi(u) = \frac{\Psi(u)}{u - \mathsf{r}_0}. \tag{4.4}$$

For instance, when $\mathsf{r}_0 = 0$, then

$$\phi(u) = u + (\mu - \hbar - 1) + \int_1^\infty (1 - r^{-u})h(r)dr,$$

and we note that both ϕ and $\mathbb{J}$ are uniquely determined by λ_1, μ, and h so that there is a one-to-one correspondence between ϕ and $\mathbb{J}$, given fixed λ_1. As we show in Lemma 4.3.2 ϕ is a Bernstein function, i.e. $\phi : [0, \infty) \to [0, \infty)$ is infinitely differentiable on $\mathbb{R}_+$ and $(-1)^{n+1}\frac{d^n}{du^n}\phi(u) \geqslant 0$, for all $n = 1, 2, \ldots$ and $u > 0$. Any Bernstein function ϕ admits an analytic extension to the right half-plane $\Re(z) > 0$, see e.g. [129, Chapter 4], and we write W_ϕ for the unique solution, in the space of positive definite functions, to the functional equation

$$W_\phi(z + 1) = \phi(z)W_\phi(z), \quad \Re(z) > 0,$$

with $W_\phi(1) = 1$, and we refer to Patie and Savov [128] for a thorough account on this set of functions that generalize the gamma function, which appears as a special case when $\phi(z) = z$. In particular, for any $n \in \mathbb{N}$,

$$W_\phi(n + 1) = \prod_{k=1}^{n} \phi(k), \tag{4.5}$$

with the convention $\prod_{k=1}^{0} \phi(k) = 1$ and where throughout we write $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let $C(E)$ denote the Banach space of continuous functions on E equipped with the sup-norm $\|\cdot\|_\infty$, and let, for $k \in \mathbb{N}$, $C^k(E)$ denote the space of functions on E admitting k continuous derivatives with $C^\infty(E) = \cap_{k=0}^\infty C^k(E)$, $C^0(E) = C(E)$. What we call a Markov semigroup in $C(E)$, $\mathbb{Q} = (\mathbb{Q}_t)_{t \geqslant 0}$, is a one parameter semigroup of positivity-preserving, conservative operators acting on the space of bounded, measurable functions such that, for any $t \geqslant 0$ and $f \in C(E)$, $\|\mathbb{Q}_t f\|_\infty \leqslant \|f\|_\infty$ and $\lim_{t\to 0} \|\mathbb{Q}_t f - f\|_\infty = 0$, see [27, Chapter 1.1] for a definition of these properties. A positive, σ-finite measure β on E is invariant for a Markov semigroup $\mathbb{Q}$ if, for all $f \in C(E)$ and $t \geqslant 0$,

$$\beta\mathbb{Q}_t f = \beta f = \int_E f(y)\beta(dy),$$

where the last equality serves as a definition for the notation βf. It is then classical, see e.g. [10] or [45], that given a Markov semigroup with invariant measure β one may extend it to a Markov semigroup in $\mathrm{L}^2(\beta)$, the weighted Hilbert space defined as

$$\mathrm{L}^2(\beta) = \left\{ f : E \to \mathbb{R} \text{ measurable with } \beta f^2 < \infty \right\}.$$

Such a semigroup is said to be ergodic if, for every $f \in \mathrm{L}^2(\beta)$, $\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{Q}_t f dt = \beta f$ in the $\mathrm{L}^2(\beta)$-norm.

Next, for any $x \in [0, \infty)$ and $a \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ we write $(a)_x$ to denote the Pochhammer symbol

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}.$$

Writing $\mathcal{P}$ for algebra of polynomials and letting $p_n(x) = x^n$ we define formally the following sequence, for any $n \in \mathbb{N}$,

$$\beta p_n = \frac{(\mathrm{r}_1)_n}{(\lambda_1)_n} \frac{W_\phi(n+1)}{n!}, \tag{4.6}$$

and note that in Lemma 4.3.2 we show that $\mathrm{r}_1 \in (0, 1]$. Recall that a sequence is said to be Stieltjes moment determinate if it is the moment sequence of a unique probability measure on $[0, \infty)$, and let $E^{\mathrm{o}} = (0, 1)$. Our first main result provides the existence of an ergodic Markov semigroup generated by the non-local Jacobi operator $\mathbb{J}$.

Theorem 4.2.1.

1. The sequence $(\beta p_n)_{n\geqslant 0}$ is a determinate Stieltjes moment sequence of a probability measure β supported on E, and is absolutely continuous with a positive density on E^{o}.

2. The extension of $\mathbb{J}$ to an operator in $\mathrm{L}^2(\beta)$, still denoted by $\mathbb{J}$, is the infinitesmal generator, having $\mathcal{P}$ as a core, of an ergodic Markov semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t\geqslant 0}$ in $\mathrm{L}^2(\beta)$ whose unique invariant measure is β.

It is worth mentioning that the proof of Item 2 makes use of an intertwining relation stated in Proposition 4.3.1, which is an original approach to showing that an operator is a Markovian generator, and in particular that the assumptions of the Hille-Yosida-Ray Theorem are fulfilled, see Lemma 4.3.7 and its proof for more details. Note that the idea of constructing a Markov semigroup in $C(E)$ via intertwining relations is not new and was used, for instance, by Borodin and Olshanski [25] to construct a Markov process on the Thoma cone.

We also point out that the invariant measure β is a natural extension of the beta distribution, which is recovered in the case when $\phi(u) = u$ as in this case in (4.6), $W_\phi(n+1) = n!$. The condition in Assumption 4.2.1.1 that $-(e^r h(e^r))'$ is a finite Radon measure is necessary for the existence of an invariant probability measure for $\mathbb{Q}$. Indeed, as we illustrate in our proof of Theorem 4.2.1, any candidate for such a measure must have moments given by (4.6). When $-(e^r h(e^r))'$ is not a finite measure then estimates for W_ϕ along imaginary lines, see [128, Theorem 3.3], imply that the analytical extension of (4.6) to $\Re(z) > \mathsf{r}_1$ does not decay along imaginary lines, a necessary condition to be a probability measure.

4.2.2 Spectral theory of the Markov semigroup and generator

We proceed by developing the $\mathrm{L}^2(\beta)$-spectral theory for both the semigroup $\mathbb{Q}$ and the operator $\mathbb{J}$. Recalling that, for fixed $\boldsymbol{\lambda}_1$, there is a one-to-one correspondence between $\mathbb{J}$ and the Bernstein function ϕ in (4.4), we define, for $n \in \mathbb{N}$, the polynomial $\mathcal{P}_n^\phi : E \to \mathbb{R}$ as

$$\mathcal{P}_n^\phi(x) = \sqrt{\mathrm{C}_n(\mathsf{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{(\mathsf{r}_1)_n}{(\mathsf{r}_1)_k} \frac{x^k}{W_\phi(k+1)}, \tag{4.7}$$

where $\mathrm{C}_n(\mathsf{r}_1)$ is given by

$$\mathrm{C}_n(\mathsf{r}_1) = (2n + \boldsymbol{\lambda}_1 - 1) \frac{n!(\boldsymbol{\lambda}_1)_{n-1}}{(\mathsf{r}_1)_n (\boldsymbol{\lambda}_1 - \mathsf{r}_1)_n}.$$

Note that when $h \equiv 0$ then in (4.2) we get $\Psi(u) = u(u-(1-\mu))$ and $\mathcal{P}_n^{\phi}$ boils down to $\mathcal{P}_n^{(\mu)}$, the classical Jacobi orthogonal polynomial reviewed in Section 4.5. Next, we write $\mathbf{R}_n$ for the following scaled Rodrigues operator,

$$\mathbf{R}_n f(x) = \frac{2^n}{n!}\frac{d^n}{dx^n}(x^n f(x)) \tag{4.8}$$

and set

$$\Delta = \lambda_1 - \mathrm{r}_1 - (\mu-1)\mathbf{1}_{\{\mu \geqslant 1+\hbar\}} - \hbar \mathbf{1}_{\{\mu<1+\hbar\}}.$$

We write $\beta(dx) = \beta(x)dx$ for the density given in Theorem 4.2.11, and define, for any $n \geqslant 1$, the function $\beta_{\lambda_1+n,\lambda_1} : E \to [0,\infty)$ as

$$\beta_{\lambda_1+n,\lambda_1}(x) = \frac{(\lambda_1)_n}{n!}x^{\lambda_1-1}(1-x)^{n-1}.$$

Proposition 4.2.1. Let $\mathcal{V}_0^{\phi} \equiv 1$ and, for $n = 1,2,\ldots$, define $\mathcal{V}_n^{\phi} : E^{\mathrm{o}} \to \mathbb{R}$ as

$$\mathcal{V}_n^{\phi}(x) = \frac{1}{\beta(x)}\frac{(\lambda_1-\mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\,\mathbf{R}_n(\beta_{\lambda_1+n,\lambda_1}\diamond\beta)(x) = \frac{1}{\beta(x)}w_n(x). \tag{4.9}$$

Then $w_n \in C^{\infty}(E^{\mathrm{o}})$ and, if $\Delta > \frac{1}{2}$, then additionally $w_n \in \mathrm{L}^2(E)$. If $\lfloor\Delta\rfloor \geqslant 2$ then $\mathcal{V}_n^{\phi} \in C^{\lfloor\Delta\rfloor-1}(E^{\mathrm{o}})$.

Remark 4.2.1. The definition in (4.9) makes sense regardless of the differentiability of β, since $\mathbf{R}_n(\beta_{\lambda_1+n,\lambda_1}\diamond\beta) = \mathbf{R}_n\beta_{\lambda_1+n,\lambda_1}\diamond\beta$ and $\beta_{\lambda_1+n,\lambda_1} \in C^{\infty}(E^{\mathrm{o}})$. However, the differentiability of $\mathcal{V}_n^{\phi}$ is limited by the smoothness of β, which is quantified by the index $\lfloor\Delta\rfloor - 1$. Note that, when $h \equiv 0$ then $\beta = \beta_{\mu}$ and, by moment identification and determinacy, it is easily checked that (4.9) boils down to the Rodrigues representation of $\mathcal{P}_n^{(\mu)}$ given in (4.70). In this sense $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ and $(\mathcal{V}_n^{\phi})_{n\geqslant 0}$ both generalize $(\mathcal{P}_n^{(\mu)})_{n\geqslant 0}$ in different ways, coming from the different representations of these orthogonal polynomials.

We say that two sequences $(f_n)_{n\geqslant 0}, (g_m)_{m\geqslant 0} \in \mathrm{L}^2(\beta)$ are biorthogonal if $\beta f_n g_m = 1$, when $n = m$, and $\beta f_n g_m = 0$ otherwise, and then write $f_n \otimes g_n$ for the projection operator given by

$f \mapsto (\beta g_n f)\, f_n$. Moreover, a sequence that admits a biorthogonal sequence will be called minimal and a sequence that is both minimal and complete, in the sense that its linear span is dense in $L^2(\beta)$, will be called *exact*. It is easy to show that a sequence $(f_n)_{n\geqslant 0}$ is minimal if and only if none of its elements can be approximated by linear combinations of the others. If this is the case, then a biorthogonal sequence will be uniquely determined if and only if $(f_n)_{n\geqslant 0}$ is complete. Next, a sequence $(f_n)_{n\geqslant 0} \in L^2(\beta)$ is said to be a Bessel sequence if there exists $B > 0$ such that, for all $f \in L^2(\beta)$,

$$\sum_{n=0}^{\infty} (\beta f_n f)^2 \leqslant B\, \beta f^2,$$

and the quantity B is called the Bessel bound of $(f_n)_{n\geqslant 0}$, see [39] for further information on these objects that play a central role in non-harmonic analysis.

We write $\sigma(\mathbb{Q}_t)$ for the spectrum of the operator $\mathbb{Q}_t$ in $L^2(\beta)$ and $\sigma_p(\mathbb{Q}_t)$ for its point spectrum, and similarly define $\sigma(\mathbb{J})$ and $\sigma_p(\mathbb{J})$. For an isolated eigenvalue $\varrho \in \sigma_p(\mathbb{Q}_t)$ we write $M_a(\varrho, \mathbb{Q}_t)$ and $M_g(\varrho, \mathbb{Q}_t)$ for the algebraic and geometric multiplicity of ϱ, respectively. We also define, for $n \in \mathbb{N}$,

$$\lambda_n = n(n-1) + \boldsymbol{\lambda}_1 n = n^2 + (\boldsymbol{\lambda}_1 - 1)n, \tag{4.10}$$

noting that $\lambda_1 = \boldsymbol{\lambda}_1$, which explains our choice of notation. As reviewed in Section 4.5, for the self-adjoint Jacobi operator we have that $\sigma(\mathbf{J}_\mu) = \sigma_p(\mathbf{J}_\mu) = \{-\lambda_n;\ n \in \mathbb{N}\}$. We write, for any $t \geqslant 0$, $\mathbb{Q}_t^*$ for the $L^2(\beta)$-adjoint of $\mathbb{Q}_t$. We have the following spectral theorem for $\mathbb{Q}$.

Theorem 4.2.2. Let $t > 0$.

1. Then, with equality holding in operator norm, we have

$$\mathbb{Q}_t = \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{P}_n^{\phi} \otimes \mathcal{V}_n^{\phi},$$

where $(\mathcal{P}_n^\phi)_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ is an exact Bessel sequence with Bessel bound 1, and $(\mathcal{V}_n^\phi)_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ is its unique biorthogonal sequence, which is also exact. Moreover, for any $n \in \mathbb{N}, \mathcal{P}_n^\phi$ (resp. $\mathcal{V}_n^\phi$) is an eigenfunction for $\mathbb{Q}_t$ (resp. $\mathbb{Q}_t^*$) associated to the eigenvalue $e^{-\lambda_n t}$.

2. The semigroup $\mathbb{Q}$ is immediately compact, i.e. $\mathbb{Q}_t$ is compact for any $t > 0$.
3. The following spectral mapping theorem holds

$$\sigma(\mathbb{Q}_t) \setminus \{0\} = \sigma_p(\mathbb{Q}_t) \setminus \{0\} = e^{t\sigma_p(\mathbb{J})} = e^{t\sigma(\mathbb{J})} = \left\{e^{-\lambda_n t}; n \in \mathbb{N}\right\}.$$

Furthermore, $\sigma(\mathbb{Q}_t) = \sigma(\mathbb{Q}_t^*)$ and, for any $n \in \mathbb{N}$,

$$\mathrm{M}_a(e^{-\lambda_n t}, \mathbb{Q}_t) = \mathrm{M}_g(e^{-\lambda_n t}, \mathbb{Q}_t) = \mathrm{M}_a(e^{-\lambda_n t}, \mathbb{Q}_t^*) = \mathrm{M}_g(e^{-\lambda_n t}, \mathbb{Q}_t^*) = 1.$$

4. The operator $\mathbb{Q}_t$ is self-adjoint in $\mathrm{L}^2(\beta)$ if and only if $h \equiv 0$.

The expansion in Theorem 4.2.21 is not valid for $t = 0$ as $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ is a Bessel sequence but not a Riesz sequence, as it is not the image of an orthogonal sequence by a bounded linear operator having a bounded inverse, see Proposition 4.3.5 below. The sequence of non-self-adjoint projection operators $\mathcal{P}_n^\phi \otimes \mathcal{V}_n^\phi$ is not uniformly bounded in n, see Remark 4.3.3, and, in contrast to the self-adjoint case, the eigenfunctions of $\mathbb{Q}_t$ and $\mathbb{Q}_t^*$ do not form a basis of $\mathrm{L}^2(\beta)$. Finally, we note that from Theorem 4.2.24 $\mathcal{P}_n^\phi \neq \mathcal{V}_n^\phi$ for all $n = 1, 2, \ldots$.

4.2.3 Convergence-to-equilibrium and contractivity properties

For an open interval $I \subseteq \mathbb{R}$, we say that a function $\Phi : I \to \mathbb{R}$ is admissible if

$$\Phi \in C^4(I) \text{ with both } \Phi \text{ and } -1/\Phi'' \text{ convex.} \tag{4.11}$$

Given an admissible function we write

$$\mathrm{Ent}_{\beta}^{\Phi}(f) = \beta\Phi(f) - \Phi(\beta f) \tag{4.12}$$

for the so-called Φ-entropy of a function $f \in \mathrm{L}^1(\beta)$ with $\Phi(f) \in \mathrm{L}^1(\beta)$. An important case is when $\Phi(r) = r^2$, $I = \mathbb{R}$, so that (4.12) gives the variance $\mathrm{Var}_\beta(f)$ of a function $f \in \mathrm{L}^2(\beta)$. Recall that in the classical case, i.e. $h \equiv 0$, we have the following equivalence between the Poincaré inequality for $\mathbf{J}_\mu$ and the spectral gap inequality for $\mathbf{Q}^{(\mu)}$, for $f \in \mathrm{L}^2(\beta_\mu)$ and $t \geqslant 0$,

$$\lambda_1 = \inf_f \frac{-\beta_\mu(f\mathbf{J}_\mu f)}{\mathrm{Var}_{\beta_\mu}(f)} \iff \mathrm{Var}_{\beta_\mu}(\mathbf{Q}_t^{(\mu)} f) \leqslant e^{-2\lambda_1 t}\,\mathrm{Var}_{\beta_\mu}(f),$$

where the infimum is over all functions in the L^2-domain of $\mathbf{J}_\mu$, see for instance [10, Chapter 4.2]. The above variance decay is optimal in the sense that the decay rate does not hold for any constant strictly smaller than $2\lambda_1$. Another important instance of (4.12) is when $\Phi(r) = r\log r$, $I = \mathbb{R}_+$, which recovers the classical notion of entropy for a non-negative function, written simply as $\mathrm{Ent}_\beta(f)$. Here the classical equivalence is between the logarithmic Sobolev inequality and entropy decay, namely for $f \in \mathrm{L}^1(\beta)$ with $\mathrm{Ent}_\beta(f) < \infty$ and $t \geqslant 0$,

$$\lambda_{\mathrm{logS}}^{(\mu)} = \inf_f \frac{-4\beta_\mu(f\mathbf{J}_\mu f)}{\mathrm{Ent}_{\beta_\mu}(f^2)} > 0 \iff \mathrm{Ent}_{\beta_\mu}(\mathbf{Q}_t^{(\mu)} f) \leqslant e^{-\lambda_{\mathrm{logS}}^{(\mu)} t}\,\mathrm{Ent}_{\beta_\mu}(f).$$

Note that the optimal entropy decay rate is obtained only when $\mu = \frac{\lambda_1}{2} > 1$, in which case $\lambda_{\mathrm{logS}}^{(\mu)} = 2\lambda_1$, while otherwise $\lambda_{\mathrm{logS}}^{(\mu)} < 2\lambda_1$, see [143, Theorem 9.1]. We refer to the excellent article by Chafaï [33], the book by Ané et al. [4], the relevant sections of Bakry et al. [10], and also to Section 4.5 where we review these notions for the classical Jacobi semigroup. However, due to the non-self-adjointness and non-local properties of $\mathbb{J}$, it seems challenging to develop an approach based on the Poincaré or log-Sobolev inequalities. For this reason, we take an alternative route to tackling convergence to equilibrium by using the recently introduced concept of completely monotone intertwining relations, see [110, Section 3.5] and [111].

Next, recalling that when $h \not\equiv 0$ we have $\boldsymbol{\lambda}_1 > 1$, we let $\rho : [0,\infty) \to [0,\infty)$ be defined as

$$\rho(u) = \sqrt{u + \frac{(\boldsymbol{\lambda}_1 - 1)^2}{4}} - \frac{\boldsymbol{\lambda}_1 - 1}{2}$$

and note that it is a Bernstein function, as it is obtained by translating and centering the well-known Bernstein function $u \mapsto \sqrt{u}$. In the literature ρ is known as the Laplace exponent of the so-called relativistic 1/2-stable subordinator, see [7, 24]. For any Bernstein function ϕ, we denote by

$$d_\phi = \inf\{u \geqslant 0;\ \phi(-u) = 0 \text{ or } \phi(-u) = \infty\} \in [0,\infty], \tag{4.13}$$

and we let, for any $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$,

$$\boldsymbol{d} = \mathrm{r}_1 \mathbf{1}_{\{\mu < 1+\hbar\}} + (d_\phi + 1 - \epsilon_0)\mathbf{1}_{\{\mu \geqslant 1+\hbar\}} \tag{4.14}$$

noting that when $d_\phi = 0$ then $\epsilon_0 = 0$. We write, for any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ and $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$, τ for the random variable, independent of the Markov process which is the realization of $\mathbb{Q}$, whose Laplace transform is

$$\mathbb{E}\left[e^{-u\tau}\right] = \frac{(\boldsymbol{d})_{\rho(u)}}{(\mathfrak{m})_{\rho(u)}} \frac{(\boldsymbol{\lambda}_1 - \mathfrak{m})_{\rho(u)}}{(\boldsymbol{\lambda}_1 - \boldsymbol{d})_{\rho(u)}}, \quad u \geqslant 0. \tag{4.15}$$

Theorem 4.2.3. Let $t \geqslant 0$. For any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ and $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$, we have the following.

1. For any $f \in \mathrm{L}^2(\beta)$

$$\mathrm{Var}_\beta(\mathbb{Q}_t f) \leqslant \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d})}{\boldsymbol{d}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t}\, \mathrm{Var}_\beta(f),$$

   with $\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d}) > \boldsymbol{d}(\boldsymbol{\lambda}_1 - \mathfrak{m})$.

2. The function $u \mapsto -\log \mathbb{E}[e^{-u\tau}]$ is a Bernstein function, which gives that τ is infinitely divisible and hence there exists a subordinator $\boldsymbol{\tau} = (\tau_t)_{t\geqslant 0}$ with $\tau_1 \overset{(d)}{=} \tau$. For any $f \in \mathrm{L}^1(\beta)$ with $\mathrm{Ent}_\beta(f) < \infty$

$$\mathrm{Ent}_\beta(\mathbb{Q}_{t+\tau} f) \leqslant e^{-\lambda^{(\mathfrak{m})}_{\log S} t}\, \mathrm{Ent}_\beta(f).$$

Furthermore, if $\boldsymbol{\lambda}_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$ then, with $\mathfrak{m} = \boldsymbol{\lambda}_1/2$,

$$\mathrm{Ent}_\beta(\mathbb{Q}_{t+\tau} f) \leqslant e^{-2\boldsymbol{\lambda}_1 t}\,\mathrm{Ent}_\beta(f).$$

Suppose, in addition, that $\mathbf{1}_{\{\mu<1+\hbar\}} + \mu < \boldsymbol{\lambda}_1/2 \in \mathbb{N}$, and let $\Phi : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, be an admissible function, as in (4.11). Then, for any $f : E \to I$ such that $f, \Phi(f) \in \mathrm{L}^1(\beta)$ and with $\mathrm{Ent}_\beta^\Phi(f) < \infty$,

$$\mathrm{Ent}_\beta^\Phi(\mathbb{Q}_{t+\tau} f) \leqslant e^{-(\boldsymbol{\lambda}_1-1)t}\,\mathrm{Ent}_\beta^\Phi(f).$$

Remark 4.2.2. Since $\frac{\mathfrak{m}(\boldsymbol{\lambda}_1-\boldsymbol{d})}{\boldsymbol{d}(\boldsymbol{\lambda}_1-\mathfrak{m})} > 1$ the estimate in Theorem 4.2.31 gives the hypocoercivity, in the sense of Villani [154], for non-local Jacobi semigroups. This notion continues to attract research interests, especially in the area of kinetic Fokker-Planck equations, see e.g. [1, 12, 55, 113]. We are able to identify the hypocoercive constants, namely the exponential decay rate as twice the spectral gap, and the coefficient in front of the exponential, which is a measure of the deviation of the spectral projections from forming an orthogonal basis and is 1 in the case an orthogonal basis. Note that in general the hypocoercive constants may be difficult to identify, and may have little to do with the spectrum. Similar results have been obtained by Patie and Savov in [129] and Achleitner et al. [1]. Our hypocoercive estimate is obtained via intertwining, which suggests that hypocoercivity may be studied purely from this viewpoint, an idea that is further investigated in the recent work [132].

Remark 4.2.3. The second part of Theorem 4.2.3 gives the exponential decay in entropy of $\mathbb{Q}$ but after an independent random warm-up time. Note that, for $\boldsymbol{\lambda}_1 \leqslant 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$ the entropy decay rate is the same as for $\mathbf{Q}^{(\mathfrak{m})}$ while under the mild assumption that $\boldsymbol{\lambda}_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$ we get the optimal rate for more than simply a fixed value of μ. The proof relies on developing so-called *completely monotone intertwining relations*, a concept which has been introduced and studied in the recent work by Miclo and Patie [111], where the classical Jacobi semigroup $\mathbf{Q}^{(\mathfrak{m})}$ serves as a reference object, see Proposition 4.3.6 below.

Remark 4.2.4. The additional condition $\lambda_1/2 \in \mathbb{N}$ required for the Φ-entropic convergence in Theorem 4.2.32 is to ensure that we can invoke the known result in (4.77) for the classical Jacobi semigroup $\mathbf{Q}^{(\lambda_1/2)}$. However, our approach allows us to immediately transfer any improvement in (4.77) to the non-local Jacobi semigroup $\mathbb{Q}$.

Next, we recall the famous equivalence between entropy decay and hypercontractivity due to Gross [75], i.e. for any $t \geqslant 0$ and $f \in \mathrm{L}^1(\beta_\mu)$ such that $\mathrm{Ent}_{\beta_\mu}(f) < \infty$,

$$\mathrm{Ent}_{\beta_\mu}(\mathbf{Q}_t^{(\mathfrak{m})} f) \leqslant e^{-\lambda_{\log S}^{(\mathfrak{m})} t}\, \mathrm{Ent}_{\beta_\mathfrak{m}}(f) \iff \|\mathbf{Q}_t^{(\mathfrak{m})}\|_{2\to q} \leqslant 1 \text{ where } 2 \leqslant q \leqslant 1 + e^{\lambda_{\log S}^{(\mathfrak{m})} t},$$

where we use the shorthand $\|\cdot\|_{p\to q} = \|\cdot\|_{\mathrm{L}^p(\beta_\mathfrak{m})\to \mathrm{L}^q(\beta_\mathfrak{m})}$ for $1 \leqslant p, q \leqslant \infty$. To state our next result we write, when $\lambda_1 - \mathfrak{m} > 1$, $c_\mathfrak{m} > 0$ for the Sobolev constant of $\mathbf{J}_\mathfrak{m}$ of order $\frac{2(\lambda_1-\mathfrak{m})}{(\lambda_1-\mathfrak{m}-1)}$, and recall that as a result of the Sobolev inequality for $\mathbf{J}_\mathfrak{m}$ it follows that $\mathbf{Q}^{(\mathfrak{m})}$ is also ultracontractive, i.e. $\|\mathbf{Q}_t^{(\mathfrak{m})}\|_{1\to\infty} < \infty$ for all $t > 0$, see Section 4.5 for a review of these concepts. We have the following concerning the contractivity of $\mathbb{Q}$.

Theorem 4.2.4. For any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \lambda_1)$ and $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$, the following holds:

1. For $t \geqslant 0$, we have the hypercontractivity estimate

$$\|\mathbb{Q}_{t+\tau}\|_{2\to q} \leqslant 1, \quad \text{where} \quad 2 \leqslant q \leqslant 1 + e^{\lambda_{\log S}^{(\mathfrak{m})} t},$$

and furthermore, if $\lambda_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$, then, with $\mathfrak{m} = \frac{\lambda_1}{2}$,

$$\|\mathbb{Q}_{t+\tau}\|_{2\to q} \leqslant 1, \quad \text{where} \quad 2 \leqslant q \leqslant 1 + e^{2\lambda_1 t}.$$

2. If in addition $\lambda_1 - \mathfrak{m} > 1$ then, for $0 < t \leqslant 1$, we have the ultracontractivity estimate

$$\|\mathbb{Q}_{t+\tau}\|_{1\to\infty} \leqslant c_\mathfrak{m} t^{-\frac{\lambda_1-\mathfrak{m}}{\lambda_1-\mathfrak{m}-1}}.$$

We write $\mathbb{Q}^\tau = (\mathbb{Q}_t^\tau)_{t\geqslant 0}$ for the semigroup subordinated, in the sense of Bochner, with respect to $\boldsymbol{\tau} = (\tau_t)_{t\geqslant 0}$, i.e.

$$\mathbb{Q}_t^\tau = \int_0^\infty \mathbb{Q}_s \mathbb{P}(\tau_t \in ds),$$

so that $\mathbb{Q}^{\tau}_1 = \mathbb{Q}_{\tau}$, see [144, Chapter 6]. Note that $\mathbb{Q}^{\tau}$ is also an ergodic Markov semigroup in $L^2(\beta)$ with β as an invariant measure, and we have the following concerning the subordinated semigroup.

Corollary 4.2.1. For any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ and $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$ the statement of Theorem 4.2.2 holds for $\mathbb{Q}^{\tau}$ upon replacing $(\lambda_n)_{n\geqslant 0}$ by $(\log \frac{(\mathfrak{m})_n(\boldsymbol{\lambda}_1-\boldsymbol{d})_n}{(\boldsymbol{d})_n(\boldsymbol{\lambda}_1-\mathfrak{m})_n})_{n\geqslant 0}$ for $t \geqslant 1$, and the statements of Theorem 4.2.32 and Theorem 4.2.41 hold for $\mathbb{Q}^{\tau}$ upon replacing $\boldsymbol{\lambda}_1$ by $\log \frac{\mathfrak{m}(\boldsymbol{\lambda}_1-\boldsymbol{d})}{\boldsymbol{d}(\boldsymbol{\lambda}_1-\mathfrak{m})}$ and τ by 1.

We point out that the Markov process which is the realization of $\mathbb{Q}$ (resp. $\mathbb{Q}^{\tau}$) has non-symmetric and spectrally negative (resp. two-sided) jumps and can easily be shown to be a polynomial process on E in the sense of Cuchiero et al. [43]. We emphasize that what also belongs to this class are the realizations of Markov semigroups obtained by subordinating $\mathbb{Q}$ with respect to any conservative subordinator $\boldsymbol{\tau}$ with Laplace exponent ϕ^{τ} (growing fast enough at infinity, e.g. logarithmically) and we obtain, from Theorem 4.2.2, the spectral expansion for the subordinated semigroup by replacing $(\lambda_n)_{n\geqslant 0}$ with $(\phi^{\tau}(\lambda_n))_{n\geqslant 0}$. We mention that in the aforementioned paper the authors investigate the martingale problem for general polynomial operators on the unit simplex, of which $\mathbb{J}$ (and thus the generator $\phi^{\tau}(\mathbb{J})$ of the subordinated semigroup $\mathbb{Q}^{\tau}$) is a specific instance. In particular, $\mathbb{J}$ is a Lévy type operator with affine jumps of Type 2, in the sense of [43], and for such operators they prove the existence and uniqueness for the martingale problem under the weaker condition $\boldsymbol{\lambda}_1 \geqslant \mu$. However, the conditions in Assumption 4.2.1.1 allow us to obtain the existence and uniqueness of an invariant measure.

4.3 Proofs

4.3.1 Preliminaries

We state and prove some preliminary results that will be useful throughout the paper. We start by giving an alternative form of the operator $\mathbb{J}$, which will make some later proofs more transparent.

Lemma 4.3.1. Let $\Pi(dr) = -(e^r h(e^r))', r > 0$. Then, Π is a finite Radon measure on $(0, \infty)$ with $\int_0^\infty r\Pi(dr) = \hbar < \infty$, and the operator $\mathbb{J}$ defined in (4.1) may be written, for suitable f, as

$$\mathbb{J}f(x) = x(1-x)f''(x)-(\lambda_1 x-\mu+\hbar)f'(x)+\int_0^\infty (f(e^{-r}x) - f(x) + xrf'(x))\frac{\Pi(dr)}{x}, \quad x \in E. \quad (4.16)$$

Proof. Since

$$\hbar = \int_1^\infty h(r)dr = \int_0^\infty e^r h(e^r)dr < \infty$$

it follows that $\lim_{r\to\infty} e^r h(e^r) = 0$. Consequently, for any $y > 0$,

$$\overline{\Pi}(y) = \int_y^\infty \Pi(dr) = -\int_y^\infty (e^r h(e^r))'dr = e^y h(e^y) - \lim_{r\to\infty} e^r h(e^r) = e^y h(e^y).$$

Thus, by a change of variables and integration by parts, one gets

$$\int_0^\infty r\Pi(dr) = \int_0^\infty \overline{\Pi}(r)dr = \int_1^\infty h(r)dr = \hbar < \infty.$$

Next, we again use $\hbar < \infty$ to get that

$$\int_0^\infty (f(e^{-r}x) - f(x) + xrf'(x))\frac{\Pi(dr)}{x} = \hbar f'(x) + \int_0^\infty \frac{f(e^{-r}x) - f(x)}{x}\Pi(dr).$$

Integrating the right-hand side by parts, and noting that the boundary terms evaluate to zero, yields

$$\int_0^\infty \frac{f(e^{-r}x) - f(x)}{x}\Pi(dr) = -\int_0^\infty e^{-r}f'(e^{-r}x)\overline{\Pi}(r)dr = -\int_0^\infty f'(e^{-r}x)h(e^r)dr = -f' \diamond h(x)$$

where the last equality follows from a straightforward change of variables, and uses the definition of product convolution. ■

In the sequel we keep the notation $\Pi(dr) = -(e^r h(e^r))'$, $r > 0$ and $\overline{\Pi}(y) = e^y h(e^y)$, $y > 0$. Let $\phi_{\mathrm{r}_1}^{\checkmark} : [0,\infty) \to [0,\infty)$ be the function given by

$$\phi_{\mathrm{r}_1}^{\checkmark}(u) = \frac{u + \mathrm{r}_1}{u+1}\phi(u+1). \tag{4.17}$$

The following result collects some useful properties of the functions ϕ and $\phi_{\mathrm{r}_1}^{\checkmark}$.

Lemma 4.3.2. Let ϕ be given by (4.4).

1. ϕ is a Bernstein function and satisfies $\lim_{u\to\infty} \frac{\phi(u)}{u} = 1$.
2. We have $\mathrm{r}_1 \in (0,1]$, with $\mathrm{r}_1 = 1$ if and only if $\mu \geqslant 1 + \hbar$ where we recall that r_1 is defined in (4.3). Additionally, if $\mu \geqslant 1+\hbar$ then $\phi(0) = \mu - \hbar - 1$ while if $\mu < 1+\hbar$ then $\phi(0) = 0$.
3. Suppose $\mu < 1 + \hbar$. Then $\phi_{\mathrm{r}_1}^{\checkmark}$ defined in (4.17) is a Bernstein function that is in correspondence with the non-local Jacobi operator $\mathbb{J}_{\phi_{\mathrm{r}_1}^{\checkmark}}$ with parameters λ_1, $\mu_{\phi_{\mathrm{r}_1}^{\checkmark}} = 1+\mu$, and the non-negative function $h_{\phi_{\mathrm{r}_1}^{\checkmark}}(r) = r^{-1}\overline{\Pi}_{\phi_{\mathrm{r}_1}^{\checkmark}}(\log r), r > 1$, where $\Pi_{\phi_{\mathrm{r}_1}^{\checkmark}}$ is the finite Radon measure given by

   $$\Pi_{\phi_{\mathrm{r}_1}^{\checkmark}}(dr) = e^{-r}\left(\Pi(dr) + \overline{\Pi}(r)dr\right),\ r > 0.$$

   Furthermore, writing $\hbar_{\phi_{\mathrm{r}_1}^{\checkmark}} = \int_1^\infty h_{\phi_{\mathrm{r}_1}^{\checkmark}}(r)dr$, we have $\hbar_{\phi_{\mathrm{r}_1}^{\checkmark}} < \infty$ with $\mu_{\phi_{\mathrm{r}_1}^{\checkmark}} \geqslant 1 + \hbar_{\phi_{\mathrm{r}_1}^{\checkmark}}$ and $\lambda_1 > \mu_{\phi_{\mathrm{r}_1}^{\checkmark}}$.

Proof. First we rewrite (4.2) using a straightforward integration by parts to get, for any $u \geqslant 0$,

$$\Psi(u) = u^2 + (\mu-\hbar-1)u + u\int_1^\infty (1-r^{-u})h(r)dr = u^2+(\mu-\hbar-1)u + \int_0^\infty (e^{-ur}+1-ur)\Pi(dr). \tag{4.18}$$

Since, by Lemma 4.3.1 we have $\int_0^\infty r\Pi(dr) < \infty$, we recognize Ψ as the Laplace exponent of a spectrally negative Lévy process with a finite mean given by $\Psi'(0^+) = \mu - \hbar - 1$. In particular, on $[0,\infty)$, Ψ is a convex, eventually increasing, twice differentiable function which is always zero at 0 and hence it has a strictly positive root r_0 if and only if $\mu < 1 + \hbar$. By the Wiener-Hopf factorization of Lévy processes, see e.g. [93, Chapter 6.4], we get, when $\Psi'(0^+) \geqslant 0$ (resp. $\Psi'(0^+) < 0$) that $\Psi(u) = u\phi(u)$ (resp. $\Psi(u) = (u - \mathrm{r}_0)\phi(u)$) for a Bernstein function ϕ. The limit then follows from the well-known result that $\lim_{u\to\infty} u^{-2}\Psi(u) = 1$, which can be obtained by dominated convergence since Π is a finite measure, and this completes the proof of the first item. Next, we will show that $\Psi(1) > 0$, which, by the convexity of Ψ is equivalent to $\mathrm{r}_0 \in [0,1)$. Indeed, from (4.18) and an application of Fubini's theorem we get

$$\Psi(1) = \mu - \hbar + \int_0^\infty (1 - e^{-r})\overline{\Pi}(r)dr > 0,$$

where we used the assumption that $\mu > \hbar$ and the positivity of $\overline{\Pi}$. Next, if $\mu \geqslant 1 + \hbar$ then, as $\mathrm{r}_0 = 0$ in this case, we get, from (4.18), that

$$\phi(u) = u + (\mu - \hbar - 1) + \int_0^\infty (e^{-ur} + 1 - ur)\Pi(dr),$$

and the expression for $\phi(0)$ readily follows. On the other hand if $\mathrm{r}_0 > 0$, then the fact that $\Psi(0) = -\mathrm{r}_0\phi(0) = 0$ forces $\phi(0) = 0$, which completes the proof of the second item. Next, write $\Psi_1(u) = \frac{u}{u+1}\Psi(u+1)$ so that, according to [34, Proposition 2.2], we get that Ψ_1 is also the Laplace exponent of a spectrally negative Lévy process whose Gaussian component is 1, mean is $\mu_{\phi^{\checkmark}_{\mathrm{r}_1}}$, and Lévy measure is $\Pi_{\phi^{\checkmark}_{\mathrm{r}_1}}$. Observe that $\Psi_1'(0^+) = \Psi(1) > 0$ and

$$\Psi_1(u) = \frac{u}{u+1}(u + 1 - \mathrm{r}_0)\phi(u+1) = u\frac{u + \mathrm{r}_1}{u+1}\phi(u+1) = u\phi^{\checkmark}_{\mathrm{r}_1}(u),$$

so, by the Wiener-Hopf factorization of Ψ_1, it follows that $\phi^{\checkmark}_{\mathrm{r}_1}$ is a Bernstein function. Moreover, integration by parts of $\Pi_{\phi^{\checkmark}_{\mathrm{r}_1}}$ gives

$$\hbar_{\phi^{\checkmark}_{\mathrm{r}_1}} = \int_0^\infty \overline{\Pi}_{\phi^{\checkmark}_{\mathrm{r}_1}}(r)dr = \int_0^\infty e^{-r}\overline{\Pi}(r)dr \leqslant \hbar < \infty,$$

where the boundary terms are easily seen to evaluate to 0. Finally, using the assumption that $\mu > \hbar$ we get that $\mu_{\phi^{\checkmark}_{\mathfrak{r}_1}} = 1+\mu-\hbar_{\phi^{\checkmark}_{\mathfrak{r}_1}}+\hbar_{\phi^{\checkmark}_{\mathfrak{r}_1}} \geqslant 1+\mu-\hbar+\hbar_{\phi^{\checkmark}_{\mathfrak{r}_1}} > 1+\hbar_{\phi^{\checkmark}_{\mathfrak{r}_1}}$, while the condition $\boldsymbol{\lambda}_1 > \mu_{\phi^{\checkmark}_{\mathfrak{r}_1}}$ follows from the assumption that $\boldsymbol{\lambda}_1 > \mathbf{1}_{\{\mu<1+\hbar\}} + \mu = 1+\mu = \mu_{\phi^{\checkmark}_{\mathfrak{r}_1}}$. ■

4.3.2 Proof of Theorem 4.2.11

Before we begin we provide an analytical result, which will allow us to show that β is supported on $[0,1]$ and will also be used in subsequent proofs. We say that a linear operator Λ is a Markov multiplicative kernel, or a Markov kernel for short, if $\Lambda f(x) = \mathbb{E}[f(xI)]$ for some random variable I. With the definition of d_ϕ in (4.13), we let, for any $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$,

$$\boldsymbol{d}_0 = \mathbf{1}_{\{\mu<1+\hbar\}} + (d_\phi + 1 - \epsilon_0)\mathbf{1}_{\{\mu\geqslant 1+\hbar\}}, \tag{4.19}$$

recalling that when $d_\phi = 0$ then $\epsilon_0 = 0$, so that at least $\boldsymbol{d}_0 \geqslant 1$. By [129, Lemma 10.3], the mapping

$$u \mapsto \phi_{\boldsymbol{d}_0}(u) = \frac{u}{u+\boldsymbol{d}_0-1}\phi(u) \tag{4.20}$$

is a Bernstein function, writing simply $\phi_1 = \phi$, and by Proposition 4.4(1) of the same paper we also have that, for any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$, the mapping

$$u \mapsto \phi^*_{\mathfrak{m}}(u) = \frac{\phi(u)}{u+\mathfrak{m}-1} \tag{4.21}$$

is a Bernstein function. We define the following linear operators acting on the space of polynomials $\mathcal{P}$, recalling that for $n \in \mathbb{N}$, $p_n(x) = x^n$,

$$\Lambda_{\phi_{\boldsymbol{d}_0}} p_n(x) = \frac{(\boldsymbol{d}_0)_n}{W_\phi(n+1)} p_n(x),\ \mathrm{V}_{\phi^*_{\mathfrak{m}}} p_n(x) = \frac{W_\phi(n+1)}{(\mathfrak{m})_n} p_n(x),\ \text{and}\ \mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}} p_n(x) = \frac{\phi^{\checkmark}_{\mathfrak{r}_1}(0)}{\phi^{\checkmark}_{\mathfrak{r}_1}(n)} p_n(x) \tag{4.22}$$

where $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ is defined for any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$, and $\phi^{\checkmark}_{\mathfrak{r}_1}$ was defined in (4.17). We write $\mathcal{B}(C(E))$ for the unital Banach algebra of bounded linear operators on $C(E)$ and say that a linear operator between two Banach spaces is a quasi-affinity if it has trivial kernel and dense range.

Lemma 4.3.3. The operators $\Lambda_{\phi_{d_0}}$, $\mathrm{V}_{\phi^*_{\mathrm{m}}}$ and $\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ defined in (4.22) are Markov kernels associated to random variables $X_{\phi_{d_0}}$, $X_{\phi^*_{\mathrm{m}}}$ and $X_{\phi^{\checkmark}_{\mathrm{r}_1}}$, respectively, that are all supported on E, and hence moment determinate. Furthermore, all operators belong to $\mathcal{B}(C(E))$, and $\Lambda_{\phi_{d_0}}$ is a quasi-affinity on $C(E)$ while $\mathrm{V}_{\phi^*_{\mathrm{m}}}$ and $\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ have dense range in $C(E)$.

Proof. The claims regarding the operators $\Lambda_{\phi_{d_0}}$ and $\mathrm{V}_{\phi^*_{\mathrm{m}}}$, and their respective random variables, have been proved in [129], see e.g. Proposition 6.7(1), Theorem 5.2, and Section 7.1 therein. Let $W : [0,\infty) \to [0,\infty)$ be the function characterized by its Laplace transform via

$$\int_0^\infty e^{-ux} W(x)dx = \frac{1}{\Psi(u)}, \quad u > 0,$$

and note that W is increasing and, since Ψ has a Gaussian component, it is at least continuously differentiable, see e.g. [93, Section 8.2]. Then $X_{\phi^{\checkmark}_{\mathrm{r}_1}}$ is the random variable whose law is given by

$$\mathbb{P}(X_{\phi^{\checkmark}_{\mathrm{r}_1}} \in dx) = \phi^{\checkmark}_{\mathrm{r}_1}(0) W'(-\log x)dx, \quad x \in E,$$

which is clearly supported on E, and the claims concerning $\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ were shown in [130, Lemma 4.2], where we note that $W(0) = 0$ since Ψ has a Gaussian component. ■

Now, suppose $\mu \geqslant 1 + \hbar$ so that, by Lemma 4.3.2, $\mathrm{r}_1 = 1$. Then, for all $n \in \mathbb{N}$, (4.6) reduces to

$$\beta p_n = \frac{W_\phi(n+1)}{(\lambda_1)_n}.$$

Since $\lambda_1 > \mu \geqslant 1$, we get that $\phi^*_{\lambda_1}$ as in (4.21) is a Bernstein function. Indeed, in the case when $\mu = 1$ we clearly must have $\hbar = 0$, and the function $u \mapsto \frac{u}{u+\lambda_1-1}$ is Bernstein since $\lambda_1 > 1$, see e.g. [145, Chapter 16], while on the other hand the same Proposition 4.4(1) guarantees that $\phi^*_{\lambda_1}$ is a Bernstein function. Thus, one straightforwardly checks that, for all $n \in \mathbb{N}$,

$$\beta p_n = W_{\phi^*_{\lambda_1}}(n+1)$$

which implies from [15] that, in this case, $(\beta p_n)_{n\geqslant 0}$ is indeed a determinate Stieltjes moment sequence of a probability measure β, and its absolute continuity follows from [124, Proposition 2.4]. Now suppose $\mu < 1 + \hbar$ so that $\boldsymbol{\lambda}_1 > 1 + \mu > 1$ and observe that (4.6) factorizes as

$$\beta p_n = \frac{W_\phi(n+1)}{(\boldsymbol{\lambda}_1)_n} \frac{(\mathrm{r}_1)_n}{n!},$$

where the first term in the product is a Stieltjes moment sequence by the above arguments, and the second term is the moment sequence of a beta distribution, see e.g. (4.66). Consequently, in this case one also has that $(\beta p_n)_{n\geqslant 0}$ is a Stieltjes moment sequence, and we temporarily postpone the proof of its moment determinacy, and its absolute continuity, to after the proof of Lemma 4.3.4. For our next result we write $(\beta_{\phi^{\checkmark}_{\mathrm{r}_1}} p_n)_{n\geqslant 0}$ for the sequence obtained from (4.6) by replacing ϕ with $\phi^{\checkmark}_{\mathrm{r}_1}$ defined in (4.17), and with the same $\boldsymbol{\lambda}_1$.

Lemma 4.3.4. With $\boldsymbol{d}$ as in (4.14), the following factorization of operators holds on the space $\mathscr{P}$,

$$\beta \Lambda_{\phi_{\boldsymbol{d}_0}} = \beta_{\boldsymbol{d}}, \quad \beta_{\mathrm{m}} \mathrm{V}_{\phi^*_{\mathrm{m}}} = \beta, \quad \text{and} \quad \beta_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \beta, \tag{4.23}$$

where the second identity holds for $\mu \geqslant 1 + \hbar$, while the third holds for $\mu < 1 + \hbar$.

Remark 4.3.1. Once we establish the moment determinacy of β for $\mu < 1 + \hbar$, then the factorizations of operators in Lemma 4.3.4 extends to the space of bounded measurable functions. Indeed, (4.23) implies

$$B_\phi \times X_{\phi_{\boldsymbol{d}_0}} \stackrel{(d)}{=} B_{\boldsymbol{d}},$$

where B_ϕ and $B_{\boldsymbol{d}}$ are random variables with laws β and $\beta_{\boldsymbol{d}}$, respectively, and $\times$ denotes the product of independent random variables.

Proof. Observe, from (4.22), that for any $n \in \mathbb{N}$,

$$\beta\Lambda_{\phi_{d_0}} p_n = \frac{(\boldsymbol{d}_0)_n}{W_\phi(n+1)}\beta p_n = \frac{(\boldsymbol{d}_0)_n}{W_\phi(n+1)}\frac{(\mathsf{r}_1)_n}{(\boldsymbol{\lambda}_1)_n}\frac{W_\phi(n+1)}{n!} = \frac{(\boldsymbol{d}_0)_n}{n!}\frac{(\mathsf{r}_1)_n}{(\boldsymbol{\lambda}_1)_n}.$$

By considering the cases $\mathsf{r}_1 = 1$ and $\mathsf{r}_1 < 1$ separately we obtain the desired right-hand side, noting that $\beta_{\boldsymbol{d}_0}$ is well-defined, i.e. $\boldsymbol{\lambda}_1 > d_\phi + 1$, due to $\boldsymbol{\lambda}_1 > \mu = (\mu - \hbar) + \hbar$ and [129, Proposition 4.4(1)]. For the second claim we get that for any $n \in \mathbb{N}$ and since, by Lemma 4.3.23, $\mu \geqslant 1 + \hbar$ if and only if $\mathsf{r}_1 = 1$,

$$\beta_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} p_n = \frac{W_\phi(n+1)}{(\mathfrak{m})_n}\beta_{\mathfrak{m}} p_n = \frac{W_\phi(n+1)}{(\mathfrak{m})_n}\frac{(\mathfrak{m})_n}{(\boldsymbol{\lambda}_1)_n} = \frac{W_\phi(n+1)}{(\boldsymbol{\lambda}_1)_n} = \beta p_n,$$

which, by linearity, completes the proof. For the last claim we have, by Lemma 4.3.23 and using the notation therein, that $\mu_{\phi^{\checkmark}_{\mathsf{r}_1}} \geqslant 1 + \hbar_{\phi^{\checkmark}_{\mathsf{r}_1}}$ and thus 0 is the only non-negative root of $u \mapsto u\phi^{\checkmark}_{\mathsf{r}_1}(u)$. Consequently

$$\beta_{\phi^{\checkmark}_{\mathsf{r}_1}} p_n = \frac{W_{\phi^{\checkmark}_{\mathsf{r}_1}}(n+1)}{(\boldsymbol{\lambda}_1)_n}.$$

Some straightforward computations give that, for any $n \in \mathbb{N}$,

$$W_{\phi^{\checkmark}_{\mathsf{r}_1}}(n+1) = \frac{(\mathsf{r}_1+1)_n}{(n+1)!}\frac{W_\phi(n+2)}{\phi(1)}, \quad \text{and} \quad \mathrm{U}_{\phi^{\checkmark}_{\mathsf{r}_1}} p_n(x) = \frac{\phi^{\checkmark}_{\mathsf{r}_1}(0)}{\phi^{\checkmark}_{\mathsf{r}_1}(n)} = \frac{\mathsf{r}_1\phi(1)(n+1)}{(n+\mathsf{r}_1)\phi(n+1)} p_n(x).$$

Putting these observations together yields

$$\beta_{\phi^{\checkmark}_{\mathsf{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathsf{r}_1}} p_n = \frac{1}{(\boldsymbol{\lambda}_1)_n}\frac{\mathsf{r}_1(\mathsf{r}_1+1)_n}{(n+\mathsf{r}_1)}\frac{(n+1)}{(n+1)!}\frac{W_\phi(n+2)}{\phi(n+1)} = \frac{1}{(\boldsymbol{\lambda}_1)_n}(\mathsf{r}_1)_n\frac{1}{n!}W_\phi(n+1) = \beta p_n,$$

where we repeatedly use the recurrence relations for both the gamma function and the function W_ϕ, see e.g. (4.5). ■

Now suppose that, when $\mu < 1 + \hbar$, the measure β is moment indeterminate. Then, as the sequence $\left(\frac{(\boldsymbol{d}_0)_n}{W_\phi(n+1)}\right)_{n\geqslant 0}$ is a non-vanishing Stieltjes moment sequence, it follows, by (4.23) and invoking [15, Lemma 2.2], that the beta distribution $\beta_{\boldsymbol{d}}$ is moment indeterminate, which is a contradiction. Therefore we conclude that, in all cases, β is moment determinate and consequently we have the extended factorization of operators as described in Remark 4.3.1. To get the absolute continuity of β in the case $\mu < 1 + \hbar$ we note that the factorization $\beta p_n = \frac{W_\phi(n+1)}{(\lambda_1)_n}\frac{(\mathrm{r}_1)_n}{n!}$ implies, by moment determinacy, that β is the product convolution of two absolutely continuous measures. Next, take $\epsilon_0 = d_\phi$ so that $\boldsymbol{d} = \mathrm{r}_1$, see (4.14). As in the proof Lemma 4.3.3, the distribution of X_ϕ, denoted by ι, satisfies $\mathrm{supp}(\iota) = [0,1]$, where $\mathrm{supp}(\iota)$ denotes the support of the measure ι. Consequently, since $\mathrm{supp}(\beta_{\mathrm{r}_1}) = [0,1]$, it follows from (4.23) that $\mathrm{supp}(\beta) = [a,b]$ for some $0 \leqslant a < b \leqslant 1$, which may be deduced from the corresponding factorization of random variables, see again Remark 4.3.1. To show that, in fact $\mathrm{supp}(\beta) = [0,1]$, we suppose that $b < 1$. Then, by (4.23) we have, for $x \in [0,1]$,

$$\beta_{\mathrm{r}_1}\mathbf{1}_{(x,1]} = \int_0^1 \beta\mathbf{1}_{(x/y,1]}\iota(dy) \leqslant \beta\mathbf{1}_{(x,1]},$$

and taking $x = b$ we get

$$0 < \beta_{\mathrm{r}_1}\mathbf{1}_{(b,1]} \leqslant \beta\mathbf{1}_{(b,1]} = 0,$$

which is a contradiction. If $\mu \geqslant 1 + \hbar$ then, since $\mathrm{supp}(\beta_{\mathfrak{m}}) = [0,1]$ and $\mathrm{supp}(\beta) = [a,1]$, we deduce from (4.23) and similar arguments as above, that the distribution of $X_{\phi^*_{\mathfrak{m}}}$, say $\nu_{\mathfrak{m}}$, satisfies $\mathrm{supp}(\nu_{\mathfrak{m}}) = [c,1]$, for some $c \in [0,1)$. Assume $a > 0$. Then, from (4.23) we get that, for $x \in [0,1]$,

$$\beta\mathbf{1}_{[0,x)} = \int_c^1 \beta_{\mathfrak{m}}\mathbf{1}_{[0,x/y)}\nu_{\mathfrak{m}}(dy) \geqslant \beta_{\mathfrak{m}}\mathbf{1}_{[0,x)}.$$

Thus, when $x = a$, recalling that $0 < a < 1$, we get

$$0 = \beta\mathbf{1}_{[0,a)} \geqslant \beta_{\mathfrak{m}}\mathbf{1}_{[0,a)} > 0,$$

which is a contradiction. Therefore, $a = 0$, and we conclude that $\mathrm{supp}(\beta) = [0,1]$ in this case. The case when $\mu < 1 + \hbar$ follows by similar arguments, with $\beta_{\mathfrak{m}}$ and $X_{\phi^*_{\mathfrak{m}}}$ replaced by $\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}$ and $X_{\phi^{\checkmark}_{\mathrm{r}_1}}$, respectively, where we note that $\mathrm{supp}(\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}) = [0,1]$ since $\mu_{\phi^{\checkmark}_{\mathrm{r}_1}} \geqslant 1 + h_{\phi^{\checkmark}_{\mathrm{r}_1}}$. This completes the proof. ■

4.3.3 Proof of Theorem 4.2.12

We start by stating and proving the following more general intertwining that will be useful in subsequent proofs, recalling the definition of $\Lambda_{\phi_{d_0}}$ in (4.22).

Proposition 4.3.1. With $\boldsymbol{d}$ and $\boldsymbol{d}_0$ as in (4.14) and (4.19), respectively, we have, for any $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$,

$$\mathbb{J}\Lambda_{\phi_{d_0}} = \Lambda_{\phi_{d_0}}\mathbf{J}_{\boldsymbol{d}}, \quad \text{on} \quad \mathscr{P}. \tag{4.24}$$

Remark 4.3.2. Note that $\boldsymbol{\lambda}_1$ is the common parameter of the Jacobi type operators in (4.24) while the constant part of the affine drift, as well as the non-local components are different. The commonality of $\boldsymbol{\lambda}_1$ is what ensures the isospectrality of these operators, as their spectrum depends only on $\boldsymbol{\lambda}_1$, see Theorem 4.2.22 and (5.5.6).

We split the proof of Proposition 4.3.1 into two lemmas and, among other things, our proof hinges on the interesting observation that intertwining relations are stable under perturbation with an operator that commutes with the intertwining operator, see Lemma 4.3.6 below. Let $\mathbf{L}_\mu$ be the operator defined as

$$\mathbf{L}_\mu f(x) = xf''(x) + \mu f'(x) \tag{4.25}$$

and write $\mathbb{I}_h f(x) = -f' \diamond h(x)$ where h is as in Assumption 4.2.1.1, and set $\mathbb{L} = \mathbf{L}_\mu + \mathbb{I}_h$,

Lemma 4.3.5. With the notation of Proposition 4.3.1 the following holds on $\mathscr{P}$,

$$\mathbb{L}\Lambda_{\phi_{d_0}} = \Lambda_{\phi_{d_0}}\mathbf{L}_{\boldsymbol{d}}. \tag{4.26}$$

Proof. Using that $\hbar = \int_1^\infty h(r)dr$ and the symmetry of $\diamond$ we get, by straightforward calculation, that, for any $n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{L}p_n(x) &= n(n-1)p_{n-1}(x) + \mu n p_{n-1}(x) - np_{n-1}(x)\int_1^\infty r^{-(n-1)}h(r)r^{-1}dr \\ &= n^2 p_{n-1}(x) + (\mu - \hbar - 1)np_{n-1}(x) - np_{n-1}(x)\int_1^\infty (1-r^{-n})h(r)dr \\ &= (n-\mathsf{r}_0)\phi(n)p_{n-1}(x).\end{aligned}$$

Thus, combining this with (4.22) one obtains, for any $n \in \mathbb{N}$,

$$\mathbb{L}\Lambda_{\phi_{d_0}} p_n(x) = \frac{(\boldsymbol{d}_0)_n}{W_\phi(n+1)}(n-\mathsf{r}_0)\phi(n)p_{n-1}(x) = \frac{(\boldsymbol{d}_0)_n}{W_\phi(n)}(n-\mathsf{r}_0)p_{n-1}(x),$$

while on the other hand,

$$\Lambda_{\phi_{d_0}}\mathbf{L}_{\boldsymbol{d}}p_n(x) = n(n+\boldsymbol{d}-1)\frac{(\boldsymbol{d}_0)_{n-1}}{W_\phi(n)}p_{n-1}(x) = (n-\mathsf{r}_0)\frac{(\boldsymbol{d}_0)_n}{W_\phi(n)}p_{n-1}(x),$$

where the second equality follows by considering the cases $\mathsf{r}_1 = 1$ and $\mathsf{r}_1 < 1$ separately. The linearity of the involved operators completes the proof. ■

The next lemma allows us to identify a family of operators commuting with the Markov operators defined above, although, more generally, it is a statement on commuting operators and intertwinings. Denote by $\mathbf{D}_n$ the operator acting via $\mathbf{D}_n f(x) = x^n \frac{d^n}{x^n} f(x)$ and write $\mathrm{d}_y f(x) = f(yx), y > 0$ for the dilation operator.

Lemma 4.3.6. Let $\Lambda_\eta f(x) = \int_0^\infty f(xy)\eta(dy)$, where η is any signed measure on $\mathbb{R}_+$ endowed with the Borel sigma-algebra. Suppose for a linear operator $\mathbf{A}$ on $C(\mathbb{R}_+)$ and suitable f we have

$$\eta\mathbf{A}f = \mathbf{A}\eta f \quad \text{and} \quad \mathrm{d}_y\mathbf{A}f = \mathbf{A}\mathrm{d}_y f, \quad \forall y > 0.$$

Then, for such functions,

$$\mathbf{A}\Lambda_\eta f = \Lambda_\eta \mathbf{A} f.$$

In particular, suppose that $\int_0^\infty y^n|\eta|(dy) < \infty$, for all $n \in \mathbb{N}$, where $|\eta|$ stands for the total variation of the measure η. Then, for any $n \in \mathbb{N}$ we have, for suitable f,

$$\mathbf{D}_n \Lambda_\eta f = \Lambda_\eta \mathbf{D}_n f.$$

Proof. Since

$$\Lambda_\eta f(x) = \int_0^\infty f(xy)\eta(dy) = \int_0^\infty \mathrm{d}_x f(y)\eta(dy) = \eta \mathrm{d}_x f$$

it follows that any operator $\mathbf{A}$ commuting with η and with d_x, for any $x > 0$, commutes with η, for suitable functions f. Next, the assumption on the measure η allows us to invoke Fubini's theorem and conclude that $\eta \mathbf{R}_n = \mathbf{R}_n \eta$. Finally, observing that, for any $n \in \mathbb{N}$ and $x, y > 0$,

$$\mathrm{d}_y \mathbf{D}_n f = y^n x^n f^{(n)}(yx) = \mathbf{D}_n \mathrm{d}_y f$$

completes the proof. ∎

Proof of Proposition 4.3.1. It is now an easy exercise to complete the proof of Proposition 4.3.1. Let us write

$$\mathbf{A} = \mathbf{D}_2 + \lambda_1 \mathbf{D}_1.$$

Then, for any $f \in \mathcal{P}$, we get by combining Lemma 4.3.5 and Lemma 4.3.6, that

$$\mathbb{J}\Lambda_{\phi_{d_0}} f = (\mathbb{L} - \mathbf{A})\, \Lambda_{\phi_{d_0}} f = \Lambda_{\phi_{d_0}} (\mathbf{L}_{\boldsymbol{d}} - \mathbf{A})\, f = \Lambda_{\phi_{d_0}} \mathbf{J}_{\boldsymbol{d}} f,$$

where we also use the linearity of the involved operators. ∎

Having established the necessary intertwining relation we are now able to show that $\mathbb{J}$ extends to the generator of a Markov semigroup.

Lemma 4.3.7. The operator $(\mathbb{J}, \mathcal{P})$ is closable in $C(E)$, and its closure is the infinitesimal generator of a Markov semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t \geqslant 0}$ in $C(E)$.

Proof. We aim at invoking the Hille-Yosida-Ray Theorem for Markov generators, see [27, Theorem 1.30], which requires that both $\mathscr{P}$ and, for some (or all) $q > 0$, $(q - \mathbb{J})(\mathscr{P})$ are dense in $C(E)$, and that $\mathbb{J}$ satisfies the positive maximum principle on $\mathscr{P}$. Since the density of $\mathscr{P}$ in $C(E)$ follows from the compactness of E, we focus on showing that $(q - \mathbb{J})(\mathscr{P})$ is dense in $C(E)$. To this end, set $\epsilon_0 = d_\phi$, and note, by Lemma 4.3.3, that Λ_ϕ is injective and bounded on $C(E)$, which gives that its inverse Λ_ϕ^{-1} is a closed, densely defined, linear operator on $\Lambda_\phi(\mathscr{P})$. Furthermore, since Λ_ϕ is a Markov kernel it follows that it preserves the set of polynomials, i.e. $\Lambda_\phi(\mathscr{P}) = \mathscr{P}$, and consequently by injectivity we get $\Lambda_\phi^{-1}(\mathscr{P}) = \mathscr{P}$. Putting these observations together we deduce, from the first intertwining in Proposition 4.3.1, that

$$\mathbb{J} = \Lambda_\phi \mathbf{J}_{\mathsf{r}_1} \Lambda_\phi^{-1} \quad \text{on} \quad \mathscr{P},$$

and hence, for any $q > 0$,

$$(q - \mathbb{J})(\mathscr{P}) = (q - \Lambda_\phi \mathbf{J}_{\mathsf{r}_1} \Lambda_\phi^{-1})(\mathscr{P}) = \Lambda_\phi (q - \mathbf{J}_{\mathsf{r}_1}) \Lambda_\phi^{-1}(\mathscr{P}) = \Lambda_\phi (q - \mathbf{J}_{\mathsf{r}_1})(\mathscr{P}), \tag{4.27}$$

where we use the trivial commutation of Λ_ϕ with q. Next, the assumption on $\boldsymbol{\lambda}_1$ guarantees that $\boldsymbol{\lambda}_1 > \mathsf{r}_1$, since we always have $\boldsymbol{\lambda}_1 > 1$ and $\mathsf{r}_1 = 1 - \mathsf{r}_0 \in (0, 1]$. Thus it follows that $\mathscr{P}$ belongs to the $C(E)$-domain of $\mathbf{J}_{\mathsf{r}_1}$, which is explicitly described in (4.65), and as $\mathscr{P}$ is an invariant subspace for the classical Jacobi semigroup $\mathbf{Q}^{(\mathsf{r}_1)}$ we get that $\mathscr{P}$ is a core for $\mathbf{J}_{\mathsf{r}_1}$, see [27, Lemma 1.34]. Hence, by the converse of the Hille-Yosida-Ray Theorem, we get that $(q - \mathbf{J}_{\mathsf{r}_1})(\mathscr{P})$ is dense in $C(E)$ for any $q > 0$. It is a straightforward exercise to show that the image of a dense subset under a bounded operator with dense range is also dense in the codomain. Thus it follows that $\Lambda_\phi(q - \mathbf{J}_{\mathsf{r}_1})(\mathscr{P})$, and from (4.27) we get that $(q - \mathbb{J})(\mathscr{P})$ is dense in $C(E)$ for any $q > 0$. Next, let $f \in \mathscr{P}$, set $f(x_0) = \sup_{x \in E} f(x)$, and observe that

$$f(ax_0) - f(x_0) \leqslant 0 \text{ for any } a \in E. \tag{4.28}$$

Using Lemma 4.3.1 we can write $\mathbb{J}f(x_0)$ as

$$\mathbb{J}f(x_0) = x_0(1-x_0)f''(x_0) - (\lambda_1 x_0 - \mu) f'(x_0) + \int_0^\infty (f(e^{-r}x_0) - f(x_0)) \frac{\Pi(dr)}{x_0}, \qquad (4.29)$$

where we note that since $\hbar = \int_0^\infty r\Pi(dr)$ these two terms cancel. Then, from (4.28) it follows that, for $x_0 \in E$,

$$\int_0^\infty (f(e^{-r}x_0) - f(x_0)) \frac{\Pi(dr)}{x_0} \leqslant 0.$$

Now suppose that $x_0 \in E \setminus \{0\}$. From the previous equation it suffices, in this case, to only consider the terms involving derivatives in (4.29). When $x \in E^o$ then $f''(x_0) \leqslant 0$ and $f'(x_0) = 0$, and thus plainly $\mathbb{J}f(x_0) \leqslant 0$. On the other hand, if $x_0 = 1$ then we must have $f'(1) \geqslant 0$ and so $\mathbb{J}f(1) \leqslant -(\lambda_1 - \mu) f'(1) \leqslant 0$, where the latter follows trivially from $\lambda_1 > \mu$. Finally assume that $x_0 = 0$, so that then $f'(0) \leqslant 0$. For x small we have $\frac{f(e^{-r}x) - f(x)}{x} = e^{-r}f'(0) + R(x)$, where the function R satisfies $\limsup_{x\to 0} \frac{|R(x)|}{x} < \infty$, from which it follows that $\mathbb{J}f(0) \leqslant (\mu + \int_0^\infty e^{-r}\Pi(dr))f'(0) \leqslant 0$, since both μ and $\int_0^\infty e^{-r}\Pi(dr)$ are clearly positive. Thus $\mathbb{J}$ satisfies the maximum principle (and in particular the positive maximum principle) on $\mathscr{P}$, which gives that $\mathbb{J}$ extends to the generator of a Feller semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t\geqslant 0}$, in the sense of [27, Theorem 1.30]. However, the fact that $\mathbb{Q}$ is conservative, i.e. $\mathbb{Q}_t \mathbf{1}_E = \mathbf{1}_E$, follows from $\mathbb{J}\mathbf{1}_E = 0$, since

$$\mathbb{Q}_t \mathbf{1}_E - \mathbf{1}_E = \int_0^t \mathbb{Q}_s \mathbb{J} \mathbf{1}_E \, ds = 0,$$

see e.g. [27, Lemma 1.26]. ■

Proof of Theorem 4.2.12. To complete the proof it suffices to establish the claims concerning the invariant measure. For $f \in \mathscr{P}$ we have,

$$\beta \mathbb{J} \Lambda_\phi f = \beta \Lambda_\phi \mathbf{J}_{\mathrm{r}_1} f = \beta_{\mathrm{r}_1} \mathbf{J}_{\mathrm{r}_1} f = 0, \qquad (4.30)$$

where successively we have used Proposition 4.3.1 (setting $\epsilon_0 = d_\phi$), Lemma 4.3.4, and the fact that β_{r_1} is the invariant measure of $\mathbf{J}_{\mathrm{r}_1}$. The fact that (4.30) holds on the dense subset $\Lambda_\phi(\mathcal{P}) = \mathcal{P}$ of $C(E)$ implies that β is an invariant measure for $\mathbb{Q}$, see for instance [10, Section 1.4.1]. To show uniqueness, we note that any other invariant measure $\widetilde{\beta}$ for $\mathbb{J}$ must first, have all positive moments finite, and also satisfy

$$\widetilde{\beta}\mathbb{J}\Lambda_\phi f = \widetilde{\beta}\Lambda_\phi \mathbf{J}_{\mathrm{r}_1} f = 0,$$

for any $f \in \mathcal{P}$, where we used that $\Lambda_\phi(\mathcal{P}) = \mathcal{P}$. By uniqueness of the invariant measure for $\mathbf{J}_{\mathrm{r}_1}$ we then get the factorization of operators $\widetilde{\beta}\Lambda_\phi = \beta_{\mathrm{r}_1}$, on $\mathcal{P}$, and the moment determinacy of β then forces $\widetilde{\beta} = \beta$. Finally the extension of $\mathbb{Q}$ to a Markov semigroup on $\mathrm{L}^2(\beta)$ is classical, see for instance the remarks before the theorem, and it is well-known that if $\mathbb{Q}$ has a unique invariant measure then it is an ergodic Markov semigroup, see e.g. [45, Theorem 5.16]. ■

4.3.4 Proof of Proposition 4.2.1

Before giving the proof of Proposition 4.2.1 we state and prove two auxiliary results, the first of which characterizes w_n in a distributional sense. To this end we recall that the Mellin transform of a finite measure β, resp. of an integrable function f, on $\mathbb{R}_+$ is given by

$$\mathcal{M}_\beta(z) = \beta p_{z-1} = \int_0^\infty x^{z-1}\beta(dx), \quad \text{resp. } \mathcal{M}_f(z) = \int_0^\infty x^{z-1} f(x)dx,$$

which is valid for at least $z \in 1 + i\mathbb{R}$. We denote by $\mathrm{E}_{p,q}$ (resp. $\mathrm{E}'_{p,q}$), with $p < q$ reals, the linear space of functions $f \in C^\infty(\mathbb{R}_+)$ such that there exist $c, c' > 0$ for which, for all $k \in \mathbb{N}$,

$$\lim_{x\to 0}\left| x^{k+1-p-c}\frac{d^k}{dx^k}f(x)\right| = 0 \quad \text{and} \quad \lim_{x\to\infty}\left| x^{k+1+c'-q}\frac{d^k}{dx^k}f(x)\right| = 0,$$

(resp. the linear space of continuous linear functionals on $\mathrm{E}_{p,q}$ endowed with a structure of a countably multinormed space as described in [114, p. 231]). Next, we write, for any

$n \in \mathbb{N}$ and $x \in E$,

$$\mathrm{p}_n^{(\mathrm{r}_1)}(x) = \beta_{\mathrm{r}_1}(x)\mathcal{P}_n^{(\mathrm{r}_1)}(x) = \frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\mathbf{R}_n\beta_{\lambda_1+n,\mathrm{r}_1}(x),$$

where $\mathbf{R}_n$ denotes the Rodrigues operator defined in (4.8) and the last identity follows from (4.70). For suitable a we also extend the Pochhammer notation $(a)_z$ to any z with $\Re(z) > 0$ and, for the remainder of the proofs, we shall write $\langle \cdot,\cdot \rangle_\beta$ for the $\mathrm{L}^2(\beta)$-inner product, adopting the same notation for other weighted Hilbert spaces.

Proposition 4.3.2. For any $n \in \mathbb{N}$, the Mellin convolution equation

$$\widehat{\Lambda}_\phi \hat{f}(x) = \mathrm{p}_n^{(\mathrm{r}_1)}(x) \tag{4.31}$$

has a unique solution, in the sense of distributions, given by

$$w_n(x) = \frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\,\mathbf{R}_n(\beta_{\lambda_1+n,\lambda_1} \diamond \beta)(x) \in \mathrm{E} = \cup_{q>\mathrm{r}_0}\mathrm{E}_{\mathrm{r}_0,q}. \tag{4.32}$$

Its Mellin transform is given, for any $\Re(z) > \mathrm{r}_0$, by

$$\mathcal{M}_{w_n}(z) = \frac{(-2)^n}{n!}\frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z)\mathcal{M}_\beta(z). \tag{4.33}$$

Proof. The proof is an adaptation of the proof of [129, Lemma 8.5] to the current setting. We write $\iota^*(y) = \iota(1/y)1/y$ where ι is the density of X_ϕ, which is well-known to exist, and let Λ_ϕ^* be the operator characterized, for any $f \in \mathrm{L}^2(\beta)$, by

$$\Lambda_\phi^* f(x) = \frac{1}{\beta_{\mathrm{r}_1}(x)}\int_0^1 f(xy)\beta(xy)\iota^*(y)dy = \frac{1}{\beta_{\mathrm{r}_1}(x)}\widehat{\Lambda}_\phi(f\beta)(x)$$

where $\widehat{\Lambda}_\phi f(x) = \int_0^1 f(xy)\iota^*(y)dy$ and $\beta(x)$ is the density of the invariant measure β. Then, for any non-negative functions $f \in \mathrm{L}^2(\beta_{\mathrm{r}_1})$ and $g \in \mathrm{L}^2(\beta)$, we get

$$\begin{aligned}\langle \Lambda_\phi f, g\rangle_\beta &= \int_0^\infty \left(\int_0^\infty f(xy)\iota(y)dy\right) g(x)\beta(x)dx \\ &= \int_0^\infty f(r)\beta_{\mathrm{r}_1}{}^{-1}(r)\left(\int_0^\infty \iota(r/x)g(x)\beta(x)/x dx\right)\beta_{\mathrm{r}_1}(r)dr \\ &= \int_0^\infty f(r)\beta_{\mathrm{r}_1}{}^{-1}(r)\left(\int_0^\infty g(rv)\beta(rv)\iota^*(v)dv\right)\beta_{\mathrm{r}_1}(r)dr \\ &= \langle f, \tfrac{1}{\beta_{\mathrm{r}_1}}\widehat{\Lambda}_\phi g\beta\rangle_{\beta_{\mathrm{r}_1}}.\end{aligned}$$

However, $f \in \mathrm{L}^2(\beta)$ implies that $|f| \in \mathrm{L}^2(\beta)$, so we conclude that the above holds for any $f \in \mathrm{L}^2(\beta)$ and $g \in \mathrm{L}^2(\beta_{\mathrm{r}_1})$. Thus Λ_ϕ^* is the $\mathrm{L}^2(\beta)$-adjoint of the Markov kernel Λ_ϕ which justifies the notation and, by Lemma 4.3.10, we have $\Lambda_\phi^* \in \mathscr{B}(\mathrm{L}^2(\beta), \mathrm{L}^2(\beta_{\mathrm{r}_1}))$. Next, since the mapping $z \mapsto \mathcal{M}_\iota(z) = \mathcal{M}_{\Lambda_\phi}(z) = \mathcal{M}_{\iota^*}(1-z)$ is analytic on $\Re(z) > 0$ and $|\mathcal{M}_{\Lambda_\phi}(z)| \leqslant \mathcal{M}_{\Lambda_\phi}(\Re(z)) < \infty$, for any $\Re(z) > 0$, see for instance [129, Proposition 6.8], we deduce from [114, Theorem 11.10.1] that $\iota \in \mathrm{E}'_{0,q}$, for every $q > 0$ and $\iota^* \in \mathrm{E}'_{p,1}$ for every $p < 1$. Consequently, since for any $f \in \mathrm{E}_{0,q}$, $q > 0$,

$$\Lambda_\phi f(x) = \int_0^1 f(xy)\iota(y)dy = \langle \iota, f(x.)\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}},$$

we have, for any $w \in \mathrm{E}'_{0,q}$, with $q > 0$,

$$\langle \widehat{\Lambda}_\phi w, f\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \langle w \surd\, \iota, f\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \langle w, \Lambda_\phi f\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}}, \ \forall f \in \mathrm{E}_{0,q},$$

where we recall that the last relation is a definition given in [114, 11.11.1], and where we used the notation $\widehat{\Lambda}_\phi w := w \surd\, \iota$ with $w \surd\, \iota$ being the Mellin convolution operator in the space of distributions, see [114, Chapter 11.11] for definitions and notation. Here also note that for $w \in \mathrm{L}^1(\iota^*)$, we have the identities $w \surd\, \iota(x) = \int_0^\infty w(x/y)\iota(y)dy/y = \int_0^\infty w(xy)\iota^*(y)dy = \widehat{\Lambda}_\phi w(x)$, which justifies the notation above. Next, recalling that $\widehat{\Lambda}_\phi w = w \surd\, \iota$ and taking $w \in \mathrm{E}'_{0,q}$, $q > 0$, and, with $0 < \Re(z) < q$, $p_z(x) = x^z \in \mathrm{E}_{0,q}$, we have

$$\mathcal{M}_{\widehat{\Lambda}_\phi w}(z) = \langle w \surd \iota, p_{z-1}\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \langle w, \Lambda_\phi p_{z-1}\rangle_{\mathrm{E}'_{0,q},\mathrm{E}_{0,q}} = \mathcal{M}_{\Lambda_\phi}(z)\mathcal{M}_w(z),$$

where we used that $\Lambda_\phi p_{z-1}(x) = p_{z-1}(x)\mathcal{M}_{\Lambda_\phi}(z)$. On the other hand, for any $n \in \mathbb{N}$, we get, from [114, 11.7.7] and a simple computation,

$$\mathcal{M}_{\mathrm{p}_n^{(\mathrm{r}_1)}}(z) = \frac{(-2)^n}{n!}(\lambda_1 - \mathrm{r}_1)_n \sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\frac{(\mathrm{r}_1)_{z-1}}{(\lambda_1)_{z+n-1}}.$$

Putting pieces together, we deduce that the Mellin transform of a solution to (4.31) takes the form

$$\begin{aligned}\mathcal{M}_{\hat{f}}(z) = \frac{\mathcal{M}_{\mathrm{p}_n^{(\mathrm{r}_1)}}(z)}{\mathcal{M}_{\Lambda_\phi}(z)} &= \frac{(-2)^n}{n!}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}(\lambda_1 - \mathrm{r}_1)_n\frac{(\mathrm{r}_1)_{z-1}}{(\lambda_1)_{z+n-1}}\frac{W_\phi(z)}{\Gamma(z)} \\ &= \frac{(-2)^n}{n!}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\frac{(\lambda_1)_{z-1}}{(\lambda_1+n)_{z-1}}\mathcal{M}_\beta(z) \\ &= \frac{(-2)^n}{n!}\frac{(\lambda_1 - \mathrm{r}_1)_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathrm{r}_1)}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z)\mathcal{M}_\beta(z).\end{aligned}$$

Next, we have that for $\Re(z) > \mathrm{r}_0$, $z \mapsto \mathcal{M}_\beta(z)$ is analytical with $|\mathcal{M}_\beta(z)| \leqslant \mathcal{M}_\beta(\Re(z)) < \infty$, so we deduce, from [114, Theorem 11.10.1] that $\beta \in \mathrm{E}'_{\mathrm{r}_0,q}$, for any $q > \mathrm{r}_0$. Hence, by means of [114, 11.7.7], we have that $\hat{f} \in \mathrm{E}'_{\mathrm{r}_0,q}$ with $\hat{f} = w_n$ is a solution to (4.31), and the uniqueness of the solution follows from the uniqueness of Mellin transforms in the distributional sense. ■

Lemma 4.3.8. For $a > \mathrm{r}_0$ fixed and $b \in \mathbb{R}$, we have the estimate

$$\left|\mathcal{M}_\beta(a+ib)\right| \leqslant C|b|^{-\Delta},$$

which holds uniformly on bounded a-intervals and for $|b|$ large enough, where $C > 0$ is a constant depending on ϕ and a.

Proof. By uniqueness of W_ϕ in the space of positive-definite functions, the Mellin transform of β is given by

$$\mathcal{M}_\beta(z) = \frac{(\mathrm{r}_1)_{z-1}}{(\lambda_1)_{z-1}}\frac{W_\phi(z)}{\Gamma(z)},$$

where $z = a + ib$, with $a > \mathrm{r}_0 \geqslant 0$. Invoking [128, Equation (6.20)] we get the following estimate, which holds uniformly on bounded a-intervals and for $|b|$ large enough,

$$\left|\frac{W_\phi(a+ib)}{\Gamma(a+ib)}\right| \leqslant C_\phi |b|^{\phi(0)+\overline{\nu}(0)}, \tag{4.34}$$

with $C_\phi > 0$ a constant depending on ϕ, and where, for any $y > 0$, $\overline{\nu}(y) = \int_y^\infty \nu(ds)$ with ν denoting the Lévy measure of ϕ. Lemma 4.3.23 gives in all cases the expression of $\phi(0)$ and when $\mu \geqslant 1 + \hbar$, $\nu(dy) = \overline{\overline{\Pi}}(y)dy$ follows from (4.4). Thus to utilize the estimate in (4.34) we need to identify $\overline{\nu}(0)$ when $\mu < 1 + \hbar$, which we do as follows. First, let us write $\Psi(u) = (u - \mathrm{r}_0)\phi(u) = (u - \mathrm{r}_0)\phi_{\mathrm{r}_0}(u - \mathrm{r}_0)$, where $\phi_{\mathrm{r}_0}(u) = \phi(u + \mathrm{r}_0)$. From the fact that $\Psi(\mathrm{r}_0) = 0$ we conclude that $\Psi(u + \mathrm{r}_0) = u\phi_{\mathrm{r}_0}(u)$ is itself a function of the form (4.18), which gives $\nu_{\mathrm{r}_0}(dy) = \overline{\overline{\Pi}}_{\mathrm{r}_0}(y)dr, y > 0$, where Π_{r_0} is the Lévy measure of $\Psi(u + \mathrm{r}_0)$ obtained via (4.18) and ν_{r_0} denotes the Lévy measure of ϕ_{r_0}. As ϕ_{r_0} is a Bernstein function it is given, for $u \geqslant -\mathrm{r}_0$, by

$$\phi_{\mathrm{r}_0}(u) = \kappa + u + u\int_0^\infty e^{-uy}\overline{\rho}_{\mathrm{r}_0}(y)dy,$$

for some $\kappa \geqslant \mathrm{r}_0$. Thus, for $u \geqslant 0$,

$$\begin{aligned}
\phi(u) = \phi_{\mathrm{r}_0}(u - \mathrm{r}_0) &= \kappa + (u - \mathrm{r}_0) + (u - \mathrm{r}_0)\int_0^\infty e^{-(u-\mathrm{r}_0)y}\overline{\nu}_{\mathrm{r}_0}(y)dy \\
&= (\kappa - \mathrm{r}_0) + u + u\int_0^\infty e^{-uy}e^{\mathrm{r}_0 y}\overline{\nu}_{\mathrm{r}_0}(y)dy - \mathrm{r}_0\int_0^\infty e^{-uy}e^{\mathrm{r}_0 y}\overline{\nu}_{\mathrm{r}_0}(y)dy \\
&= (\kappa - \mathrm{r}_0) + u + u\int_0^\infty e^{-uy}e^{\mathrm{r}_0 y}\overline{\nu}_{\mathrm{r}_0}(y)dy - \mathrm{r}_0 u\int_0^\infty e^{-uy}\int_0^y e^{\mathrm{r}_0 s}\overline{\nu}_{\mathrm{r}_0}(s)dsdy \\
&= (\kappa - \mathrm{r}_0) + u + u\int_0^\infty e^{-uy}\left(e^{\mathrm{r}_0 r}\overline{\nu}_{\mathrm{r}_0}(y) - \mathrm{r}_0\int_0^y e^{\mathrm{r}_0 s}\overline{\nu}_{\mathrm{r}_0}(s)ds\right)dy.
\end{aligned}$$

The third equality follows from Fubini's theorem, justified as all integrands therein are non-negative, and using $e^{-uy} = \int_y^\infty ue^{-us}ds$. Thus we deduce

$$\overline{\nu}(y) = e^{\mathrm{r}_0 y}\overline{\nu}_{\mathrm{r}_0}(y) - \mathrm{r}_0\int_0^y e^{\mathrm{r}_0 s}\overline{\nu}_{\mathrm{r}_0}(s)ds = \int_y^\infty e^{\mathrm{r}_0 s}\nu_{\mathrm{r}_0}(ds),$$

where the latter follows by some straightforward integration by parts and shows that ν is indeed the Lévy measure of ϕ. Next, an application of [129, Proposition 4.1(9)] together with another integration by parts yields $\int_0^\infty e^{-\mathrm{r}_0 y}\overline{\Pi}(y)dy \leqslant \int_0^\infty \overline{\Pi}(y)dy = \hbar$. Putting pieces together we get $\overline{\nu}(0) = \overline{\nu}_{\mathrm{r}_0}(0) \leqslant \hbar$, so that in all cases $\overline{\nu}(0) \leqslant \hbar$. Therefore from the estimate in (4.34) we deduce

$$\left|\frac{W_\phi(a+ib)}{\Gamma(a+ib)}\right| \leqslant C_\phi |b|^{\phi(0)+\hbar}, \tag{4.35}$$

which, as before, holds uniformly on bounded a-intervals and for $|b|$ large enough. Next, we recall the following classical estimate for the gamma function,

$$\lim_{|b|\to\infty} C_a |b|^{\frac{1}{2}-a} e^{\frac{\pi}{2}|b|} |\Gamma(a+ib)| = 1, \tag{4.36}$$

where $C_a > 0$ is a constant depending on a. Combining this estimate with the one in (4.35) we thus get, uniformly on bounded a-intervals and for $|b|$ large enough,

$$\left|\mathcal{M}_\beta(z)\right| \leqslant C|b|^{-\lambda_1+\mathrm{r}_1+\phi(0)+\hbar},$$

for a constant $C > 0$. Since C is a function of C_ϕ and the constants in the estimate for the Γ-function, it follows that it only depends on ϕ and a. Finally, the fact that $\Delta = \lambda_1 - \mathrm{r}_1 - \phi(0) + \hbar$ follows by Lemma 4.3.23. ■

Proof of Proposition 4.2.1. Note that $\mathbf{R}_n\beta_{\lambda_1+n,\lambda_1} \in C^\infty(E^\circ)$ and, trivially, $\beta \in \mathrm{L}^1(E)$. Then, well-known properties of convolution give $\mathbf{R}_n(\beta_{\lambda_1+n,\lambda_1} \diamond \beta) = \mathbf{R}_n\beta_{\lambda_1+n,\lambda_1} \diamond \beta$, and that w_n is a well-defined $C^\infty(E^\circ)$-function, which completes the proof of this claim. To show that $\Delta > \frac{1}{2}$ implies $w_n \in \mathrm{L}^2(E)$ we note that the classical estimate for the gamma function given in (4.36) yields that, for $z = a + ib$ with $a > n$ fixed,

$$\lim_{|b|\to\infty}\left|\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z)\right| = \lim_{|b|\to\infty}(\lambda_1)_n\left|\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(z+\lambda_1-1)}{\Gamma(z+\lambda_1+n-1)}\right| = C,$$

where C is a positive constant depending only on a, λ_1, and n. Thus, from (4.32) we get that $\mathcal{M}_{w_n}$ has the same rate of decay along imaginary lines as $\mathcal{M}_\beta$, and combining Lemma 4.3.8 together with Parseval's identity for Mellin transforms shows that $w_n \in \mathrm{L}^2(E)$. Finally, since $w_n \in C^\infty(E^o)$, it follows that the differentiability of $\mathcal{V}_n^\phi$ is determined by the differentiability of β. Invoking Lemma 4.3.8 we get, for $a > \mathrm{r}_0$ and $|b|$ large enough that

$$\left|(a+ib)^n \mathcal{M}_\beta(a+ib)\right| \leqslant C|b|^{n-\Delta},$$

uniformly on bounded a-intervals and with $C > 0$ a constant, so that, for any $n \leqslant \lfloor \Delta \rfloor - 1$, the right-hand side is integrable in b. A classical Mellin inversion argument then gives $\beta \in C^{\mathrm{n}}(E^o)$. ■

4.3.5 Proof of Theorem 4.2.2

To prove this result we shall need to develop further intertwinings for $\mathbb{J}$, and then will lift these to the level of semigroups. We write $\mathbb{J}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ for the non-local Jacobi operator with parameters $\boldsymbol{\lambda}_1$, $\mu_{\phi^{\checkmark}_{\mathrm{r}_1}}$ and $h_{\phi^{\checkmark}_{\mathrm{r}_1}}$, as in Lemma 4.3.2, which is in one-to-one correspondence with the Bernstein function $\phi^{\checkmark}_{\mathrm{r}_1}$ defined in (4.17).

Lemma 4.3.9. For any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$, the following identities hold on $\mathscr{P}$,

$$\mathbf{J}_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} = \mathrm{V}_{\phi^*_{\mathfrak{m}}} \mathbb{J}, \quad \text{and} \quad \mathbb{J}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathbb{J}, \tag{4.37}$$

in the cases $\mu \geqslant 1 + \hbar$ and $\mu < 1 + \hbar$, respectively.

Proof. It suffices to prove that $\mathbf{L}_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} = \mathrm{V}_{\phi^*_{\mathfrak{m}}} \mathbb{L}$ and $\mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathbb{L}$ hold on $\mathscr{P}$, where we write $\mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \mathbf{L}_{\mu_{\phi^{\checkmark}_{\mathrm{r}_1}}} + \mathbb{I}_{h_{\phi^{\checkmark}_{\mathrm{r}_1}}}$ and refer to (4.25) and subsequent discussion for the definitions, as then the same arguments for the proof of Proposition 4.3.1 will go through. In the case $\mu \geqslant 1+\hbar$, we have, for any $n \in \mathbb{N}$ and using the recurrence relation of the gamma function,

$$\mathbf{L}_{\mathfrak{m}} \mathrm{V}_{\phi^*_{\mathfrak{m}}} p_n(x) = \frac{W_\phi(n+1)}{(\mathfrak{m})_n} \mathbf{L}_{\mathfrak{m}} p_n(x) = \frac{W_\phi(n+1)}{(\mathfrak{m})_n} n(n+\mathfrak{m}-1) p_{n-1}(x) = \frac{W_\phi(n+1)}{(\mathfrak{m})_{n-1}} n p_{n-1}(x).$$

On the other hand, since $W_\phi(n+1) = \phi(n)W_\phi(n)$ and $\mathrm{r}_1 = 1$,

$$\mathrm{V}_{\phi^*_{\mathfrak{m}}}\mathbb{L}p_n(x) = \frac{W_\phi(n)}{(\mathfrak{m})_{n-1}}n\phi(n)p_{n-1}(x) = \frac{W_\phi(n+1)}{(\mathfrak{m})_{n-1}}np_{n-1}(x),$$

which proves this claim in this case. Finally,

$$\mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}}\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}p_n(x) = \frac{\phi^{\checkmark}_{\mathrm{r}_1}(0)}{\phi^{\checkmark}_{\mathrm{r}_1}(n)}\mathbb{L}_{\phi^{\checkmark}_{\mathrm{r}_1}}p_n(x) = \frac{\phi^{\checkmark}_{\mathrm{r}_1}(0)}{\phi^{\checkmark}_{\mathrm{r}_1}(n)}n\phi^{\checkmark}_{\mathrm{r}_1}(n)p_{n-1}(x) = \phi^{\checkmark}_{\mathrm{r}_1}(0)np_{n-1}(x),$$

while on the other hand, using the definition of $\phi^{\checkmark}_{\mathrm{r}_1}$ in (4.17),

$$\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}\mathbb{L}p_n(x) = (n-\mathrm{r}_0)\phi(n)\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}p_{n-1}(x) = (n-\mathrm{r}_0)\phi(n)\frac{\phi^{\checkmark}_{\mathrm{r}_1}(0)}{\phi^{\checkmark}_{\mathrm{r}_1}(n-1)}p_{n-1}(x) = \phi^{\checkmark}_{\mathrm{r}_1}(0)np_{n-1}(x),$$

which completes the proof, by linearity. ■

The following result lifts the intertwinings in Proposition 4.3.1 and Lemma 4.3.9 to the level of semigroups. We write here $\mathbb{Q} = \mathbb{Q}^\phi = (\mathbb{Q}^\phi_t)_{t\geqslant 0}$ to emphasize the one-to-one correspondence, given fixed $\boldsymbol{\lambda}_1$, between ϕ and $\mathbb{Q}$.

Proposition 4.3.3. Let $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$ and $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$. Then, with $\boldsymbol{d}$ as in (4.14), the following identities hold for all $t \geqslant 0$ on the appropriate L^2-spaces,

$$\mathbb{Q}^\phi_t\Lambda_{\phi_{d_0}} = \Lambda_{\phi_{d_0}}\mathbf{Q}^{(\boldsymbol{d})}_t, \quad \mathbf{Q}^{(\mathfrak{m})}_t\mathrm{V}_{\phi^*_{\mathfrak{m}}} = \mathrm{V}_{\phi^*_{\mathfrak{m}}}\mathbb{Q}^\phi_t, \quad \text{and} \quad \mathbb{Q}^{\phi^{\checkmark}_{\mathrm{r}_1}}_t\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} = \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}\mathbb{Q}^\phi_t, \tag{4.38}$$

with the latter two holding when $\mu \geqslant 1+\hbar$, and $\mu < 1+\hbar$, respectively.

We shall need an auxiliary result concerning the corresponding intertwining operators, which extends their boundedness from $C(E)$ to the corresponding weighted Hilbert spaces. For two Banach spaces B and $\widetilde{B}$ we write $\mathcal{B}(B,\widetilde{B})$ for the space of bounded linear operators from B to $\widetilde{B}$.

Lemma 4.3.10. Under the assumptions above, the operators $\Lambda_{\phi_{d_0}}$, $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$, and $\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}$ belong to $\mathcal{B}(\mathrm{L}^p(\beta_{\boldsymbol{d}}),\mathrm{L}^p(\beta))$, $\mathcal{B}(\mathrm{L}^p(\beta),\mathrm{L}^p(\beta_{\mathfrak{m}}))$, and $\mathcal{B}(\mathrm{L}^p(\beta),\mathrm{L}^p(\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}))$, respectively, for any $p \in \{1,2,\ldots,\infty\}$; in all cases, and for all p, the Markov kernels have operator norm 1.

Proof. Let $f \in \mathcal{P}$ with $p < \infty$. Then, applying Jensen's inequality to the Markov kernel $\Lambda_{\phi_{d_0}}$ together with Lemma 4.3.4 gives

$$\beta\left(\Lambda_{\phi_{d_0}} f\right)^p = \int_E \left(\Lambda_{\phi_{d_0}} f(x)\right)^p \beta(dx) \leqslant \int_E \Lambda_{\phi_{d_0}} f^p(x)\beta(dx) = \beta\Lambda_{\phi_{d_0}} f^p = \beta_{\boldsymbol{d}} f^p,$$

where we used that $f^p \in \mathcal{P}$. Since $\beta_{\boldsymbol{d}}$ is a probability measure on the compact set E it follows that $\mathcal{P}$ is a dense subset of $\mathrm{L}^p(\beta_{\boldsymbol{d}})$, see e.g. [57, Corollary 22.10], so by density we conclude that $\mathcal{B}(\mathrm{L}^p(\beta_{\boldsymbol{d}}), \mathrm{L}^p(\beta))$ with operator norm less than or equal to 1, and equality then follows from $\Lambda_{\phi_{d_0}} \mathbf{1}_E = \mathbf{1}_E$. The case when $p = \infty$ is a straightforward consequence of $\Lambda_{\phi_{d_0}}$ being a Markov kernel and the claims regarding the other operators are proved similarly, by invoking the remaining items of Lemma 4.3.4. ∎

Next, since $\mathbb{J}$ and $\mathbf{J}_{\boldsymbol{d}}$ are generators of $C(E)$-Markov semigroups, it follows that their resolvent operators, given for $q > 0$, by

$$\mathbb{R}_q = (q - \mathbb{J})^{-1}, \quad \text{and} \quad \mathrm{R}_q = (q - \mathbf{J}_{\boldsymbol{d}})^{-1}$$

are bounded, linear operators on $C(E)$. We write $\mathrm{R}_q^{\mathfrak{m}}$ (resp. $\mathbb{R}_q^{\phi_{\mathfrak{r}_1}^{\checkmark}}$) for the resolvent associated to $\mathbf{J}_{\mathfrak{m}}$ (resp. $\mathbb{J}_{\phi_{\mathfrak{r}_1}^{\checkmark}}$).

Lemma 4.3.11. Let $q > 0$. Under the assumptions in Proposition 4.3.3, the following identities hold on $\mathcal{P}$

$$\mathbb{R}_q \Lambda_{\phi_{d_0}} = \Lambda_{\phi_{d_0}} \mathrm{R}_q, \quad \mathrm{V}_{\phi_{\mathfrak{m}}^*} \mathbb{R}_q = \mathrm{R}_q^{\mathfrak{m}} \mathrm{V}_{\phi_{\mathfrak{m}}^*}, \quad \text{and} \quad \mathrm{U}_{\phi_{\mathfrak{r}_1}^{\checkmark}} \mathbb{R}_q = \mathbb{R}_q^{\phi_{\mathfrak{r}_1}^{\checkmark}} \mathrm{U}_{\phi_{\mathfrak{r}_1}^{\checkmark}}. \tag{4.39}$$

Proof. We shall only provide the proof of the first claim, which relies on the intertwining in Proposition 4.3.1, as the other claims follow by invoking Lemma 4.3.9 and involve the same arguments, mutatis mutandis. First, suppose that $\mathbb{R}_q(\mathcal{P}) \subseteq \mathcal{P}$ and $\mathrm{R}_q(\mathcal{P}) \subseteq \mathcal{P}$, and let $f \in \mathcal{P}$ so that there exists $g \in \mathcal{P}$ such that $(q - \mathbf{J}_{\boldsymbol{d}})g = f$. Applying $\Lambda_{\phi_{d_0}}$ to both sides of this equality gives that

$$\Lambda_{\phi_{d_0}} f = \Lambda_{\phi_{d_0}}(q - \mathbf{J}_d)g = (\Lambda_{\phi_{d_0}} q - \Lambda_{\phi_{d_0}} \mathbf{J}_d)g = (q\Lambda_{\phi_{d_0}} - \mathbb{J}\Lambda_{\phi_{d_0}})g = (q - \mathbb{J})\Lambda_{\phi_{d_0}} g,$$

where in the third equality we have invoked Proposition 4.3.1, which is justified as $g \in \mathscr{P}$. This equality may be rewritten as $\mathbb{R}_q \Lambda_{\phi_{d_0}} f = \Lambda_{\phi_{d_0}} g$ and consequently, for any $f \in \mathscr{P}$, we get

$$\mathbb{R}_q \Lambda_{\phi_{d_0}} f = \Lambda_{\phi_{d_0}} g = \Lambda_{\phi_{d_0}} \mathrm{R}_q f.$$

Thus it remains to show the inclusions $\mathbb{R}_q(\mathscr{P}) \subseteq \mathscr{P}$ and $\mathrm{R}_q(\mathscr{P}) \subseteq \mathscr{P}$ for which we recall, from the proof of Proposition 4.3.1, that $\mathbb{J} = \mathbb{L} - \mathbf{A}$ with $\mathbb{L} p_n = (n - \mathrm{r}_0)\phi(n) p_{n-1}$, for any $n \geqslant 1$. A straightforward computation gives that $\mathbf{A} p_n = (\mathbf{D}_2 + \lambda_1 \mathbf{D}_1) p_n = (n(n-1) + \lambda_1 n) p_n$ and hence

$$(q - \mathbb{J}) p_n = (q + n(n-1) + \lambda_1 n) p_n - (n - \mathrm{r}_0)\phi(n) p_{n-1},$$

from which it follows, by the injectivity of $\mathbb{R}_q$ on $\mathscr{P} \subset C(E)$, that

$$\mathbb{R}_q\left((q + n(n-1) + \lambda_1 n) p_n - (n - \mathrm{r}_0)\phi(n) p_{n-1}\right) = p_n.$$

Rearranging the above yields the equation

$$\mathbb{R}_q p_n = \frac{1}{(q + n(n-1) + \lambda_1 n)} p_n + \frac{(n - \mathrm{r}_0)\phi(n)}{(q + n(n-1) + \lambda_1 n)} \mathbb{R}_q p_{n-1}, \tag{4.40}$$

which is justified as, for any $q > 0$, both roots of the quadratic equation $n^2 + (\lambda_1 - 1)n + q = 0$ are always negative. Note that $\mathbb{R}_q p_0 = q^{-1}$ so by iteratively using the equality in (4.40) we conclude that, for any $n \in \mathbb{N}$, $\mathbb{R}_q p_n \in \mathscr{P}$, and by linearity $\mathbb{R}_q(\mathscr{P}) \subseteq \mathscr{P}$ follows. Similar arguments applied to R_q then allow us to also conclude that $\mathrm{R}_q(\mathscr{P}) \subseteq \mathscr{P}$, which completes the proof. ■

Proof of Proposition 4.3.3. We are now able to complete the proof of Proposition 4.3.3. As was shown in the proof of Lemma 4.3.11 above and using the notation therein, $\mathbb{R}_q(\mathscr{P}) \subseteq \mathscr{P}$

and $\mathrm{R}_q(\mathscr{P}) \subseteq \mathscr{P}$, so that on $\mathscr{P} \subset C(E)$ we have

$$\mathbb{R}_q^2 \Lambda_{\phi_{d_0}} = \mathbb{R}_q \mathbb{R}_q \Lambda_{\phi_{d_0}} = \mathbb{R}_q \Lambda_{\phi_{d_0}} \mathrm{R}_q = \Lambda_{\phi_{d_0}} \mathrm{R}_q \mathrm{R}_q = \Lambda_{\phi_{d_0}} \mathrm{R}_q^2,$$

and, by induction, for any $n \in \mathbb{N}$,

$$\mathbb{R}_q^n \Lambda_{\phi_{d_0}} = \Lambda_{\phi_{d_0}} \mathrm{R}_q^n.$$

In particular, for any $f \in \mathscr{P}$ and $t > 0$,

$$(n/t)\mathbb{R}_{n/t}^n \Lambda_{\phi_{d_0}} f = \Lambda_{\phi_{d_0}} (n/t) \mathrm{R}_{n/t}^n f.$$

Now, taking the strong limit in $C(E)$ as $n \to \infty$ of the above yields, by the exponential formula [134, Theorem 8.3] and the continuity of the involved operators guaranteed by Lemma 4.3.3, for any $f \in \mathscr{P}$ and $t \geqslant 0$,

$$\mathbb{Q}_t \Lambda_{\phi_{d_0}} f = \Lambda_{\phi_{d_0}} \mathbf{Q}_t^{(\boldsymbol{d})} f, \tag{4.41}$$

where $\mathbf{Q}^{(\boldsymbol{d})} = (\mathbf{Q}_t^{(\boldsymbol{d})})_{t \geqslant 0}$ is the classical Jacobi semigroup on $C(E)$ with parameters $\boldsymbol{\lambda}_1$ and $\boldsymbol{d}$. By density of $\mathscr{P}$ in $\mathrm{L}^2(\beta_{\mathrm{r}_1})$ and since Lemma 4.3.10 with $p = 2$ gives $\Lambda_{\phi_{d_0}} \in \mathscr{B}(\mathrm{L}^2(\beta_{\boldsymbol{d}}), \mathrm{L}^2(\beta))$ it follows that the identity in (4.41) extends to $\mathrm{L}^2(\beta_{\mathrm{r}_1})$, which completes the proof of the first item. The remaining items follow by similar arguments and so the proof is omitted. ■

For $\boldsymbol{\lambda}_1 > s \geqslant 1$ we define, for $n \in \mathbb{N}$, the quantity $\mathfrak{c}_n(s)$ as

$$\mathfrak{c}_n(s) = \frac{(s)_n}{n!}\sqrt{\frac{\mathrm{C}_n(s)}{\mathrm{C}_n(1)}} = \sqrt{\frac{(s)_n}{n!}\frac{(\boldsymbol{\lambda}_1 - 1)_n}{(\boldsymbol{\lambda}_1 - s)_n}}, \tag{4.42}$$

where the first equality comes from some straightforward algebra given the definition of $\mathrm{C}_n(s)$ in (5.5.5). Note that, with $s = 1$ we get $\mathfrak{c}_n(1) = 1$, for all n. We shall need the following result.

Lemma 4.3.12. For any $\boldsymbol{\lambda}_1 > s > r \geqslant 1$ the mapping $n \mapsto \frac{\mathfrak{c}_n(s)}{\mathfrak{c}_n(r)}$ is strictly increasing on $\mathbb{N}$ with

$$\lim_{n\to\infty} \frac{\mathfrak{c}_n(s)}{n^s} = \sqrt{\frac{\Gamma(\boldsymbol{\lambda}_1 - s)}{\Gamma(s)\Gamma(\boldsymbol{\lambda}_1 - 1)}}. \tag{4.43}$$

Proof. Using the definition in (4.42) we get that

$$\frac{\mathfrak{c}_n^2(s)}{\mathfrak{c}_n^2(r)} = \prod_{j=0}^{n-1} \frac{(s+j)(\boldsymbol{\lambda}_1 - r + j)}{(r+j)(\boldsymbol{\lambda}_1 - s + j)},$$

Since $s > r$ each term in the product is strictly greater than 1 and together with Stirling's formula for the gamma function this completes the proof. ∎

Next we write $\mathrm{V}^*_{\phi^*_{\mathfrak{m}}} : \mathrm{L}^2(\beta_{\mathfrak{m}}) \to \mathrm{L}^2(\beta)$ and $\mathrm{U}^*_{\phi^{\checkmark}_{\mathfrak{r}_1}} : \mathrm{L}^2(\beta_{\phi^{\checkmark}_{\mathfrak{r}_1}}) \to \mathrm{L}^2(\beta)$ for the Hilbertian adjoints of the operators $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ and $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}}$, respectively.

Proposition 4.3.4. Let $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ and $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$. Then, with $\boldsymbol{d}_0$ as in (4.19), the sequence $(\mathfrak{c}_n(\boldsymbol{d}_0)\mathcal{P}_n^\phi)_{n\geqslant 0}$ is a complete, Bessel sequence in $\mathrm{L}^2(\beta)$, with Bessel bound 1. Furthermore, for any $n \in \mathbb{N}$, we have, when $\mu \geqslant 1 + \hbar$, that

$$\mathcal{V}_n^\phi = \mathfrak{c}_n(\mathfrak{m})\mathrm{V}^*_{\phi^*_{\mathfrak{m}}}\mathcal{P}_n^{(\mathfrak{m})}, \tag{4.44}$$

while otherwise

$$\mathcal{V}_n^\phi = \frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\mathfrak{r}_1)}\mathrm{U}^*_{\phi^{\checkmark}_{\mathfrak{r}_1}}\mathrm{V}^*_{\phi^*_{\mathfrak{m}}}\mathcal{P}_n^{(\mathfrak{m})}. \tag{4.45}$$

and $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is the unique biorthogonal sequence to $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ in $\mathrm{L}^2(\beta)$, which is equivalent to $\mathcal{V}_n^\phi$ being the unique $\mathrm{L}^2(\beta)$-solution to $\Lambda^*_\phi g = \mathcal{P}_n^{(\mathfrak{r}_1)}$, for any $n \in \mathbb{N}$. In all cases $\left(\frac{\mathfrak{c}_n(\mathfrak{r}_1)}{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n^\phi\right)_{n\geqslant 0}$ is a complete, Bessel sequence in $\mathrm{L}^2(\beta)$ with Bessel bound 1.

Remark 4.3.3. Note that Proposition 4.3.4 yields norm bounds in $\mathrm{L}^2(\beta)$ for the functions $\mathcal{P}_n^\phi$ and $\mathcal{V}_n^\phi$ for any $n \in \mathbb{N}$. Indeed, writing $\|\cdot\|_\beta$ for the $\mathrm{L}^2(\beta)$-norm we get, from the boundedness claims of Lemma 4.3.10, for any $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$ and any $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$,

$$\|\mathcal{P}_n^\phi\|_\beta \leqslant \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_0)} \leqslant Cn^{-\boldsymbol{d}_0}, \quad \text{and} \quad \|\mathcal{V}_n^\phi\|_\beta \leqslant \frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\mathfrak{r}_1)} \leqslant Cn^{\mathfrak{m}-\mathfrak{r}_1}$$

where $C > 0$ and we used for the two estimates Lemma 4.3.12. We show in the proof below that

$$\frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_\mathfrak{n}(\boldsymbol{d}_0)\mathfrak{c}_n(\mathfrak{r}_1)} = \frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\boldsymbol{d})},$$

and since $\mathfrak{m} > \boldsymbol{d}$, invoking again Lemma 4.3.12, we have that the above ratio grows with n.

Proof. Since, for all $n \in \mathbb{N}$, $\mathcal{P}_n^{(\mathrm{r}_1)} \in \mathrm{L}^2(\beta_{\mathrm{r}_1})$ we get from the intertwining in (4.38) and the linearity of Λ_ϕ that

$$\Lambda_\phi \mathcal{P}_n^{(\mathrm{r}_1)}(x) = \sqrt{\mathrm{C}_n(\mathrm{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{(\mathrm{r}_1)_n}{(\mathrm{r}_1)_k} \frac{k!}{W_\phi(k+1)} \frac{x^k}{k!} = \mathcal{P}_n^\phi(x). \tag{4.46}$$

Recall that the sequence $(\mathcal{P}_n^{(\mathrm{r}_1)})_{n\geqslant 0}$ forms an orthonormal basis of $\mathrm{L}^2(\beta_{\mathrm{r}_1})$ and thus, as the image under a bounded operator of an orthonormal basis, we get that $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ is a Bessel sequence in $\mathrm{L}^2(\beta)$ with Bessel bound given by the operator norm of Λ_ϕ, which by Lemma 4.3.10 is 1. When $\mathrm{r}_1 > 1$ we have $\mathfrak{c}_n(\boldsymbol{d}_0) = \mathfrak{c}_n(1) = 1$, so that the first claim is proved in this case. In the case when $\mathrm{r}_1 = 1$ we suppose, without loss of generality, that $d_\phi > 0$ and $\epsilon_0 \in (0, d_\phi)$. Then $\mathcal{P}_n^\phi$ reduces to

$$\mathcal{P}_n^\phi(x) = \sqrt{\mathrm{C}_n(1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{n!}{k!} \frac{x^k}{W_\phi(k+1)}$$

and from the intertwining (4.38) we get

$$\Lambda_{\phi_{\boldsymbol{d}_0}} \mathcal{P}_n^{(\boldsymbol{d}_0)}(x) = \sqrt{\mathrm{C}_n(\boldsymbol{d}_0)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\boldsymbol{\lambda}_1 - 1)_{n+k}}{(\boldsymbol{\lambda}_1 - 1)_n} \frac{(\boldsymbol{d}_0)_n}{(\boldsymbol{d}_0)_k} \frac{(\boldsymbol{d}_0)_k}{W_\phi(k+1)} x^k = \mathfrak{c}_n(\boldsymbol{d}_0)\mathcal{P}_n^\phi(x).$$

By Lemma 4.3.10 $\Lambda_{\phi_{\boldsymbol{d}_0}} \in \mathcal{B}(\mathrm{L}^2(\beta_{\boldsymbol{d}_0}), \mathrm{L}^2(\beta))$ with operator norm 1 and thus, by similar arguments as above, we deduce that $(\mathfrak{c}_n(\boldsymbol{d}_0)\mathcal{P}_n^\phi)_{n\geqslant 0}$ is also a Bessel sequence in $\mathrm{L}^2(\beta)$ with Bessel bound 1. We continue with the claims regarding $\mathcal{V}_n^\phi$, starting again with the case when $\mathrm{r}_1 = 1$. Following similar arguments as in the proof of Proposition 4.3.2, we get that, for any $f \in \mathrm{L}^2(\beta_{\mathfrak{m}})$

$$\mathrm{V}_{\phi_{\mathfrak{m}}^*}^* f(x) = \frac{1}{\beta(x)} \widehat{\mathrm{V}}_{\phi_{\mathfrak{m}}^*}(\beta_{\mathfrak{m}} f)(x),$$

where $\widehat{\mathrm{V}}_{\phi_{\mathfrak{m}}^*} f(x) = \int_0^1 f(xy) \nu_{\mathfrak{m}}^*(y) dy$ with $\nu_{\mathfrak{m}}^*(y) = \nu_{\mathfrak{m}}(1/y)/y$, and where $\nu_{\mathfrak{m}}$ denotes the density of the random variable $V_{\phi_{\mathfrak{m}}^*}$, whose existence is due to [124, Proposition 2.4]. Thus it suffices to show that, for all $n \in \mathbb{N}$,

$$w_n(x) = \mathfrak{c}_n(\mathfrak{m})\widehat{\mathrm{V}}_{\phi_{\mathfrak{m}}^*}(\beta_{\mathfrak{m}}\mathcal{P}_n^{(\mathfrak{m})})(x) = \mathfrak{c}_n(\mathfrak{m})\widehat{\mathrm{V}}_{\phi_{\mathfrak{m}}^*}\mathrm{p}_n^{(\mathfrak{m})}(x).$$

To this end, taking the Mellin transform of the right-hand side yields, for $\Re(z) > \mathsf{r}_0$,

$$\begin{aligned}
\mathcal{M}_{\widehat{\mathrm{V}}_{\phi_{\mathfrak{m}}^*}\mathrm{p}_n^{(\mathfrak{m})}}(z) &= \mathcal{M}_{V_{\phi_{\mathfrak{m}}^*}}(z)\mathcal{M}_{\mathrm{p}^{(\mathfrak{m})}}(z) \\
&= \frac{(-2)^n}{n!}\frac{(\lambda_1 - \mathfrak{m})_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathfrak{m})}\frac{W_\phi(z)}{(\mathfrak{m})_{z-1}}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\mathfrak{m}}}(z) \\
&= \frac{(-2)^n}{n!}\frac{(\lambda_1 - \mathfrak{m})_n}{(\lambda_1)_n}\sqrt{\mathrm{C}_n(\mathfrak{m})}\frac{\Gamma(z)}{\Gamma(z-n)}\mathcal{M}_{\beta_{\lambda_1+n,\lambda_1}}(z)\mathcal{M}_\beta(z).
\end{aligned}$$

After substituting the definitions of $\mathfrak{c}_n(\mathfrak{m})$, $\mathrm{C}_n(\mathfrak{m})$ in (4.42) and (5.5.5), respectively, we get, by some straightforward algebra,

$$\mathfrak{c}_n(\mathfrak{m})2^n\frac{(\lambda_1 - \mathfrak{m})_n}{(\lambda_1)_n}\frac{\sqrt{\mathrm{C}_n(\mathfrak{m})}}{n!} = 2^n\frac{(\lambda_1 - \mathfrak{m})_n}{(\lambda_1)_n}\frac{(\mathfrak{m})_n}{n!}\frac{\mathrm{C}_n(\mathfrak{m})}{n!\sqrt{\mathrm{C}_n(1)}} = 2^n\frac{\sqrt{\mathrm{C}_n(1)}}{n!}\frac{(\lambda_1 - 1)_n}{(\lambda_1)_n},$$

and the right-hand side is the constant in front of the definition of w_n in (4.32) when $\mathsf{r}_1 = 1$. Invoking the uniqueness claim in Proposition 4.3.2 yields (4.44), as desired. The case when $\mathsf{r}_1 < 1$ follows by similar arguments, albeit with more tedious algebra, and its proof is omitted. Next, using the second intertwining relation (4.38) we get that

$$\mathrm{V}_{\phi_{\mathfrak{m}}^*}\mathcal{P}_n^\phi(x) = \sqrt{\mathrm{C}_n(\lambda_1)}\sum_{k=0}^{n}\frac{(-1)^{n+k}}{(n-k)!}\frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n}\frac{n!}{k!}\frac{W_\phi(k+1)}{(\mathfrak{m})_k}\frac{x^k}{W_\phi(k+1)} = \mathfrak{c}_n^{-1}(\mathfrak{m})\mathcal{P}_n^{(\mathfrak{m})}(x).$$

As $(\mathcal{P}_n^{(\mathfrak{m})})_{n\geqslant 0}$ is an orthonormal sequence in $\mathrm{L}^2(\beta_{\mathfrak{m}})$, we have for any $n, p \in \mathbb{N}$,

$$\delta_{np} = \langle \mathcal{P}_n^{(\mathfrak{m})}, \mathcal{P}_p^{(\mathfrak{m})}\rangle_{\beta_{\mathfrak{m}}} = \mathfrak{c}_n(\mathfrak{m})\langle \mathrm{V}_{\phi_{\mathfrak{m}}^*}\mathcal{P}_n^\phi, \mathcal{P}_p^{(\mathfrak{m})}\rangle_{\beta_{\mathfrak{m}}} = \mathfrak{c}_n(\mathfrak{m})\langle \mathcal{P}_n^\phi, \mathrm{V}_{\phi_{\mathfrak{m}}^*}^*\mathcal{P}_p^{(\mathfrak{m})}\rangle_\beta,$$

and thus we get that $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is a biorthogonal sequence in $\mathrm{L}^2(\beta)$ of $(\mathcal{P}_n^\phi)_{n\geqslant 0}$. As before, the continuity of $\mathrm{V}_{\phi_{\mathfrak{m}}^*}^*$ given by Lemma 4.3.10 combined with the fact that $(\mathcal{P}_n^{(\mathfrak{m})})_{n\geqslant 0}$ forms an orthonormal basis for $\mathrm{L}^2(\beta_{\mathfrak{m}})$ implies that $(\mathfrak{c}_n^{-1}(\mathfrak{m})\mathcal{V}_n^\phi)_{n\geqslant 0}$ is a Bessel sequence in $\mathrm{L}^2(\beta)$ with Bessel bound 1. To show uniqueness, we first observe that any sequence $(g_n)_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ biorthogonal to $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ must satisfy

$$\delta_{np} = \langle \mathcal{P}_n^\phi, g_p\rangle_\beta = \langle \mathcal{P}_n^{(\mathsf{r}_1)}, \Lambda_\phi^* g_p\rangle_{\beta_{\mathsf{r}_1}}$$

that is $(\Lambda_\phi^* g_n)_{n\geqslant 0}$ must be biorthogonal to $(\mathcal{P}_n^{(\mathrm{r}_1)})_{n\geqslant 0}$. However, since $(\mathcal{P}_n^{(\mathrm{r}_1)})_{n\geqslant 0}$ is an orthonormal basis for $\mathrm{L}^2(\beta_\mu)$ the only sequence in $\mathrm{L}^2(\beta_\mu)$ biorthogonal to it is itself. Thus, if there exists another sequence $(g_n)_{n\geqslant 0} \in \mathrm{L}^2(\beta)$ biorthogonal to $(\mathcal{P}_n^\phi)_{n\geqslant 0}$ it follows that, for all $n \in \mathbb{N}$,

$$\Lambda_\phi^* \mathcal{V}_n^\phi = \mathcal{P}_n^{(\mathrm{r}_1)} = \Lambda_\phi^* g_n \implies \Lambda_\phi^* \left(\mathcal{V}_n^\phi - g_n\right) = 0.$$

Since Lemma 4.3.3 gives that $\mathrm{Ran}(\Lambda_\phi)$ is dense in $\mathrm{L}^2(\beta)$ it follows that $\mathrm{Ker}(\Lambda_\phi^*) = \{0\}$ and we conclude that $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is the unique sequence in $\mathrm{L}^2(\beta)$ biorthogonal to $(\mathcal{P}_n^\phi)_{n\geqslant 0}$. Finally, assume now that $\mathrm{r}_1 < 1$. Then, using the definition of $\phi_{\mathrm{r}_1}^{\checkmark}$ in (4.17) we get that

$$\mathcal{P}_n^{\phi_{\mathrm{r}_1}^{\checkmark}}(x) = \sqrt{\mathrm{C}_n(\mathrm{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{n!(k+1)}{(\mathrm{r}_1 + 1)_k} \frac{\phi(1)x^k}{W_\phi(k+2)}.$$

On the other hand, since $\mathrm{U}_{\phi_{\mathrm{r}_1}^{\checkmark}} p_n = \frac{\phi_{\mathrm{r}_1}^{\checkmark}(0)}{\phi_{\mathrm{r}_1}^{\checkmark}(n)} p_n$, see (4.22), simple algebra yields that

$$\mathrm{U}_{\phi_{\mathrm{r}_1}^{\checkmark}} \mathcal{P}_n^\phi(x) = \frac{(\mathrm{r}_1)_n}{n!} \sqrt{\mathrm{C}_n(\mathrm{r}_1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{n!(k+1)}{(\mathrm{r}_1 + 1)_k} \frac{\phi(1)x^k}{W_\phi(k+2)} = \mathfrak{c}_n(\mathrm{r}_1) \mathcal{P}_n^{\phi_{\mathrm{r}_1}^{\checkmark}}(x). \tag{4.47}$$

We know that, since $\lambda_1 > \mathfrak{m} > 1 + \mu = \mu_{\phi_{\mathrm{r}_1}^{\checkmark}}$, $(\mathcal{V}_n^{\phi_{\mathrm{r}_1}^{\checkmark}})_{n\geqslant 0} = (\mathfrak{c}_n(\mathfrak{m}) \mathrm{V}_{\phi_{\mathfrak{m}}^*}^* \mathcal{P}_n^{(\mathfrak{m})})_{n\geqslant 0}$ is the unique sequence biorthogonal to $(\mathcal{P}_n^{\phi_{\mathrm{r}_1}^{\checkmark}})_{n\geqslant 0}$, and combining this with (4.47) gives the biorthogonality of $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ in $\mathrm{L}^2(\beta)$ as well as uniqueness, using similar arguments as above. Finally, the completeness of $(\mathcal{V}_n^\phi)_{n\geqslant 0}$ is a consequence of the fact that $\mathcal{V}_n^\phi$ is, in all cases and by Lemmas 4.3.3 and 4.3.10, the image under a continuous operator with dense range of the sequence $\left(\frac{\mathfrak{c}_n(\mathfrak{m})}{\mathfrak{c}_n(\mathrm{r}_1)} \mathcal{P}_n^{(\mathfrak{m})}\right)_{n\geqslant 0}$, which is itself easily seen to be complete. ■

Proof of Theorem 4.2.2. We are now able to give the proof of all items of Theorem 4.2.2, which we tackle sequentially. Setting $\epsilon_0 = d_\phi$ in (4.14) we get, by the first intertwining in Proposition 4.3.3 and the spectral expansion of the self-adjoint semigroup $\mathbf{Q}^{(\mathrm{r}_1)}$ in (5.5.7), that for any $f \in \mathrm{L}^2(\beta_{\mathrm{r}_1})$ and $t \geqslant 0$,

$$\mathbb{Q}_t \Lambda_\phi f = \Lambda_\phi \mathbf{Q}_t^{(\mathrm{r}_1)} f = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{P}_n^{(\mathrm{r}_1)} \rangle_{\beta_{\mathrm{r}_1}} \mathcal{P}_n^\phi = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle \Lambda_\phi f, \mathcal{V}_n^\phi \rangle_\beta \mathcal{P}_n^\phi,$$

where the second identity is justified by $(\langle f, \mathcal{P}_n^{(\mathfrak{r}_1)}\rangle_{\beta_{\mathfrak{r}_1}})_{n\geqslant 0} \in \ell^2(\mathbb{N})$ and the fact that $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ is a Bessel sequence in $\mathrm{L}^2(\beta)$, see [39, Theorem 3.1.3], and the last identity uses the fact that, by Proposition 4.3.4, $\mathcal{V}_n^{\phi}$ is the unique $\mathrm{L}^2(\beta)$-solution to the equation $\Lambda_\phi^* \mathcal{V}_n^{\phi} = \mathcal{P}_n^{(\mathfrak{r}_1)}$. Next, from the first intertwining in (4.38) and the fact that, for any $n \in \mathbb{N}$, $\mathbf{Q}_t^{(\mathfrak{r}_1)}\mathcal{P}_n^{(\mathfrak{r}_1)} = e^{-\lambda_n t}\mathcal{P}_n^{(\mathfrak{r}_1)}$, see (5.5.7), we get that $\mathcal{P}_n^{\phi}$ is an eigenfunction for $\mathbb{Q}_t$ with eigenvalue $e^{-\lambda_n t}$. Taking the adjoint of the first identity in (4.38) and using the self-adjointness of $\mathbf{Q}_t^{(\mathfrak{r}_1)}$ on $\mathrm{L}^2(\beta_{\mathfrak{r}_1})$ yields $\Lambda_\phi^* \mathbb{Q}_t^* = \mathbf{Q}_t^{(\mathfrak{r}_1)}\Lambda_\phi^*$ and thus, for any $n \in \mathbb{N}$ and $t \geqslant 0$,

$$\Lambda_\phi^* \mathbb{Q}_t{}^* \mathcal{V}_n^{\phi} = \mathbf{Q}_t^{(\mathfrak{r}_1)}\Lambda_\phi^* \mathcal{V}_n^{\phi} = \mathbf{Q}_t^{(\mathfrak{r}_1)}\mathcal{P}_n^{(\mathfrak{r}_1)} = e^{-\lambda_n t}\mathcal{P}_n^{(\mathfrak{r}_1)} = e^{-\lambda_n t}\Lambda_\phi^* \mathcal{V}_n^{\phi},$$

and since $\mathrm{Ker}(\Lambda_\phi^*) = \{0\}$ we deduce $\mathbb{Q}_t{}^* \mathcal{V}_n^{\phi} = e^{-\lambda_n t}\mathcal{V}_n^{\phi}$. Next, let S_t be the linear operator on $\mathrm{L}^2(\beta)$ defined by

$$S_t f = \sum_{n=0}^{\infty} \langle \mathbb{Q}_t f, \mathcal{V}_n^{\phi}\rangle_\beta \mathcal{P}_n^{\phi}$$

so that, by the above observations,

$$S_t f = \sum_{n=0}^{\infty} \langle \mathbb{Q}_t f, \mathcal{V}_n^{\phi}\rangle_\beta \mathcal{P}_n^{\phi} = \sum_{n=0}^{\infty} e^{-\lambda_n t}\langle f, \mathcal{V}_n^{\phi}\rangle_\beta \mathcal{P}_n^{\phi}.$$

For convenience, we set $\boldsymbol{\mathcal{V}}_n^{\phi} = \frac{\mathfrak{c}_n(\mathfrak{r}_1)}{\mathfrak{c}_n(\mathfrak{m})}\mathcal{V}_n^{\phi}$, $n \in \mathbb{N}$. Then, for any $t > 0$ and $f \in \mathrm{L}^2(\beta)$ we have, for $C > 0$ a constant independent of n,

$$\sum_{n=0}^{\infty} e^{-2\lambda_n t}\left|\left\langle f, \mathcal{V}_n^{\phi}\right\rangle_\beta\right|^2 = \sum_{n=0}^{\infty} e^{-2\lambda_n t}\frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\mathfrak{r}_1)}\left|\left\langle f, \boldsymbol{\mathcal{V}}_n^{\phi}\right\rangle_\beta\right|^2 \leqslant C\sum_{n=0}^{\infty}\left|\left\langle f, \boldsymbol{\mathcal{V}}_n^{\phi}\right\rangle_\beta\right|^2 \leqslant C\beta f^2 < \infty,$$

where the first inequality follows from the asymptotic in (4.43) combined with the decay of the sequence $(e^{-2\lambda_n t})_{n\geqslant 0}$, $t > 0$, and the second inequality follows from the Bessel property of $(\boldsymbol{\mathcal{V}}_n^{\phi})_{n\geqslant 0}$ guaranteed by Proposition 4.3.4. Hence we deduce that $\left(e^{-\lambda_n t}\langle f, \mathcal{V}_n^{\phi}\rangle_\beta\right)_{n\geqslant 0} \in \ell^2(\mathbb{N})$ and, as $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ is a Bessel sequence, it follows that S_t defines a bounded linear operator on $\mathrm{L}^2(\beta)$ for any $t > 0$, again by [39, Theorem 3.1.3]. However, $S_t = \mathbb{Q}_t$ on $\mathrm{Ran}(\Lambda_\phi)$, a dense subset of $\mathrm{L}^2(\beta)$. Therefore, by the bounded linear extension theorem, we have

$S_t = \mathbb{Q}_t$ on $\mathrm{L}^2(\beta)$ for any $t > 0$. Note that, by similar Bessel sequence arguments as above, for any $N \geqslant 1$,

$$\left\| \mathbb{Q}_t f - \sum_{n=0}^{N} e^{-\lambda_n t} \left\langle f, \mathcal{V}_n^{\phi} \right\rangle_{\beta} \mathcal{P}_n^{\phi} \right\|_{\beta}^2 \leqslant \left(\beta f^2\right) \sup_{n \geqslant N+1} e^{-2\lambda_n t} \frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\mathrm{r}_1)}.$$

Since the supremum on the right-hand side is decreasing in n, for any $t > 0$, we get that in the operator norm topology

$$\mathbb{Q}_t = \lim_{N \to \infty} \sum_{n=0}^{N} e^{-\lambda_n t} \mathcal{P}_n^{\phi} \otimes \mathcal{V}_n^{\phi},$$

where each $\sum_{n=0}^{N} e^{-\lambda_n t} \mathcal{P}_n^{\phi} \otimes \mathcal{V}_n^{\phi}$ is of finite rank. This completes the proof of Item 1 and also shows that $\mathbb{Q}_t$ is a compact operator for any $t > 0$, which completes the proof of Item 2. Next, the intertwining identity (4.38) and the completeness of $(\mathcal{P}_n^{\phi})_{n \geqslant 0}$ and $(\mathcal{V}_n^{\phi})_{n \geqslant 0}$ enable us to invoke [129, Proposition 11.4] to obtain the equalities for algebraic and geometric multiplicities in Item 3, and also to conclude that

$$\sigma_p(\mathbb{Q}_t) = \sigma_p(\mathbb{Q}_t^*) = \sigma_p(\mathbf{Q}_t^{(\mathrm{r}_1)}) = \{e^{-\lambda_n t};\ n \in \mathbb{N}\}.$$

Since $\mathbb{Q}_t$ is compact we get that $\mathbb{Q}_t^*$ is compact, and thus for both of these operators their spectrum is equal to their point spectrum. To establish the remaining equalities we use the immediate compactness of $\mathbb{Q}$ to invoke [61, Corollary 3.12] and obtain $\sigma(\mathbb{Q}_t) \setminus \{0\} = e^{t\sigma(\mathbb{J})}$, while we also have from [61, Theorem 3.7] that, $\sigma_p(\mathbb{Q}_t) \setminus \{0\} = e^{t\sigma_p(\mathbb{J})}$. Putting all of these together completes the proof of Item 3. Finally it remains to prove the last item concerning the self-adjointness of $\mathbb{Q}$. Clearly if $h \equiv 0$ then $\mathbb{Q}$ is self-adjoint, as in this case β reduces to β_μ and $\mathbb{Q}$ reduces to the classical Jacobi semigroup $\mathbf{Q}^{(\mu)}$, which is self-adjoint on $\mathrm{L}^2(\beta_\mu)$. Now suppose that $\mathbb{Q}$ is self-adjoint on $\mathrm{L}^2(\beta)$, that is $\mathbb{Q}_t = \mathbb{Q}_t^*$ for all $t \geqslant 0$. By differentiating in t the identity, for any $n, m \in \mathbb{N}$,

$$\langle \mathbb{Q}_t p_n, p_m \rangle_{\beta} = \langle p_n, \mathbb{Q}_t p_m \rangle_{\beta}$$

we deduce, by a simple application of Fubini's Theorem using the finiteness of the measure β, that

$$\langle \mathbb{J} p_n, p_m \rangle_\beta = \langle p_n, \mathbb{J} p_m \rangle_\beta \,. \tag{4.48}$$

Note that (4.48) holds trivially if either $n = 0$ or $m = 0$, or if $n = m$, so we may suppose that $n \neq m$; all together we take, without loss of generality, $n > m > 0$. Now, for any $n \geqslant 1$, a straightforward calculation shows that

$$\mathbb{J} p_n(x) = \Psi(n) p_{n-1}(x) - \lambda_n p_n(x), \tag{4.49}$$

where we recall from (4.4) that $\Psi(n) = (n - \mathsf{r}_0)\phi(n)$ and from (4.10) that $\lambda_n = n^2 + (\boldsymbol{\lambda}_1 - 1)n$. Using (4.49) on both sides of (4.48) and rearranging gives

$$(\lambda_n - \lambda_m)\, \beta p_{n+m} = (\Psi(n) - \Psi(m))\, \beta p_{n+m-1}. \tag{4.50}$$

By (4.6) and the recurrence relations for W_ϕ and the gamma function, the ratio $\beta p_{n+m}/\beta p_{n+m-1}$ evaluates to

$$\frac{\beta p_{n+m}}{\beta p_{n+m-1}} = \frac{(n + m + \mathsf{r}_0)}{(n + m + \boldsymbol{\lambda}_1 - 1)} \frac{\phi(n + m)}{(n + m)} = \frac{\Psi(n + m)}{\lambda_{n+m}},$$

so that substituting into (4.50) shows that the following must be satisfied

$$\Psi(n + m)\, (\lambda_n - \lambda_m) = \lambda_{n+m}\, (\Psi(n) - \Psi(m))\,. \tag{4.51}$$

Next, we write Ψ as

$$\Psi(n) = n^2 + (\mu - \hbar - 1)n + n \int_1^\infty (1 - r^{-n}) h(r) dr = n^2 + (\mu - 1)n + n \int_1^\infty r^{-n} h(r) dr,$$

where the first equality is simply the definition of Ψ in (4.2) and the second follows from the assumption that $\hbar = \int_1^\infty h(r)dr < \infty$. Let us write $G(n) = n^2 + (\mu - 1)n$ and $H(n) = n \int_1^\infty r^{-n} h(r) dr$. By direct verification we get

$$\begin{aligned} G(n + m)\, (\lambda_n - \lambda_m) &= (n - m) \left[(n + m)^3 + (\boldsymbol{\lambda}_1 + \mu - 2)(n + m)^2 (\boldsymbol{\lambda}_1 - 1)(\mu - 1)(n + m) \right] \\ &= \lambda_{n+m}\, (G(n) - G(m))\,, \end{aligned}$$

so that (4.51) is equivalent to

$$H(n+m)\left(\lambda_n-\lambda_m\right)=\lambda_{n+m}\left(H(n)-H(m)\right). \tag{4.52}$$

Observe that

$$H(n+m)\left(\lambda_n-\lambda_m\right)=(n-m)(n+m)\left(n+m+\boldsymbol{\lambda}_1-1\right)\int_1^{\infty} r^{-(n+m)}h(r)dr,$$

while

$$\lambda_{n+m}\left(H(n)-H(m)\right)=(n+m)(n+m+\boldsymbol{\lambda}_1-1)\left(n\int_1^{\infty} r^{-n}h(r)dr-m\int_1^{\infty} r^{-m}h(r)dr\right).$$

Hence canceling $(n+m)(n+m+\boldsymbol{\lambda}_1-1)$ on both sides of (4.52), then dividing by nm and rearranging the resulting equation yields

$$\int_1^{\infty} r^{-m}h(r)dr=\int_1^{\infty} r^{-n}h(r)dr+\left(\frac{1}{n}-\frac{1}{m}\right)\int_1^{\infty} r^{-(n+m)}h(r)dr.$$

Applying the dominated convergence theorem when taking the limit as $n\to\infty$ of the right-hand side we find that, for all $m>0$ with $m\neq n$,

$$\int_1^{\infty} r^{-m}h(r)dr=0,$$

which implies that $h\equiv 0$. This completes the proof of Item 4 and thus the proof of the theorem. ■

To conclude this section we give a result concerning the intertwining operators in Proposition 4.3.3 which illustrates that, except in the self-adjoint case of $h\equiv 0$ and $\mu\leqslant 1$, none of these operators admit bounded inverses. This latter fact combined with the relation (4.46) imply that $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ is a not a Riesz sequence in $\mathrm{L}^2(\beta)$, as it is not the image of an orthogonal sequence by an invertible bounded operator, see [39]. Recall that a quasi-affinity is a linear operator between two Banach spaces with trivial kernel and dense range.

Proposition 4.3.5. Let $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ and $\epsilon_0 \in (0, d_\phi] \cup \{d_\phi\}$.

1. The operators $\Lambda_{\phi_{d_0}} : \mathrm{L}^2(\beta_{\boldsymbol{d}_0}) \to \mathrm{L}^2(\beta)$, $\mathrm{V}_{\phi^*_{\mathfrak{m}}} : \mathrm{L}^2(\beta) \to \mathrm{L}^2(\beta_{\mathfrak{m}})$, and $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}} : \mathrm{L}^2(\beta_{\phi^{\checkmark}_{\mathfrak{r}_1}}) \to \mathrm{L}^2(\beta)$ are all quasi-affinities.

2. The operator $\Lambda_{\phi_{d_0}}$ admits a bounded inverse if and only if $h \equiv 0$ and $\mu \leqslant 1$ when $\boldsymbol{d}_0 = 1$, where $\boldsymbol{d}_0$ was defined in (4.19). In all cases $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$ and $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}}$ do not admit bounded inverses.

Proof. Since polynomials belong to the L^2-range of the operators $\Lambda_{\phi_{d_0}}$, $\mathrm{V}_{\phi^*_{\mathfrak{m}}}$, and $\mathrm{U}_{\phi^{\checkmark}_{\mathfrak{r}_1}}$, we get, by moment determinacy, that each of these has dense range in their respective codomains. For the remaining claims we proceed sequentially by considering each operator individually, starting with $\Lambda_{\phi_{d_0}}$. Proposition 4.3.4 gives that, for any $n \in \mathbb{N}$

$$\mathcal{P}^{\phi}_n = \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_0)} \Lambda_{\phi_{d_0}} \mathcal{P}^{(\boldsymbol{d}_0)}_n,$$

and also that $(\mathcal{P}^{\phi}_n)_{n\geqslant 0}$ and $(\mathcal{V}^{\phi}_n)_{n\geqslant 0}$ are biorthogonal. Consequently,

$$\delta_{np} = \left\langle \mathcal{P}^{\phi}_n, \mathcal{V}^{\phi}_p \right\rangle_\beta = \left\langle \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_0)} \Lambda_{\phi_{d_0}} \mathcal{P}^{(\boldsymbol{d}_0)}_n, \mathcal{V}^{\phi}_p \right\rangle_\beta = \frac{1}{\mathfrak{c}_n(\boldsymbol{d}_0)} \left\langle \mathcal{P}^{(\boldsymbol{d}_0)}_n, \Lambda^*_{\phi_{d_0}} \mathcal{V}^{\phi}_p \right\rangle_{\beta_{\boldsymbol{d}_0}}.$$

However, as $(\mathcal{P}^{(\boldsymbol{d}_0)}_n)_{n\geqslant 0}$ forms an orthonormal basis for $\mathrm{L}^2(\beta_{\boldsymbol{d}_0})$ it must be its own unique biorthogonal sequence, which forces

$$\frac{1}{\mathfrak{c}_n(\boldsymbol{d}_0)} \Lambda^*_{\phi_{d_0}} \mathcal{V}^{\phi}_n = \mathcal{P}^{(\boldsymbol{d}_0)}_n,$$

for all $n \in \mathbb{N}$. Thus we conclude that $\mathscr{P} \subset \mathrm{Ran}(\Lambda^*_{\phi_{d_0}})$, so that by moment determinacy of $(\beta_{\boldsymbol{d}_0})$, we get that $\mathrm{Ker}(\Lambda_{\phi_{d_0}}) = \{0\}$. Next, by straightforward computation we have, for any $n \in \mathbb{N}$,

$$\|p_n\|^{-2}_{\beta_{\boldsymbol{d}_0}} \left\|\Lambda_{\phi_{d_0}} p_n\right\|^2_\beta = \frac{W_\phi(2n+1)}{W^2_\phi(n+1)} \frac{(\boldsymbol{d}_0)^2_n}{(\boldsymbol{d}_0)_{2n}} = \frac{W_{\phi_{d_0}}(2n+1)}{W^2_{\phi_{d_0}}(n+1)} \frac{(n!)^2}{(2n)!}, \tag{4.53}$$

where the second equality follows by using the definition of ϕ_{d_0}, see (4.20), together with the recurrence relation for $W_{\phi_{d_0}}$. Now, the same arguments as in the proof of [129, Theorem 7.1(2)] may be applied, see e.g. Section 7.3 therein, to get that the ratio in (4.53) tends to 0 as $n \to \infty$ if and only if $\phi_{d_0}(0) = 0$ and $\Pi \equiv 0 \iff h \equiv 0$. This is because, with the notation of the aforementioned paper, the expression for $\frac{\psi(u)}{u^2}$ is equal to $\frac{\phi_{d_0}(u)}{u}$ in our notation, and we have $\sigma^2 = 1$ from $\lim_{u\to\infty} \frac{\phi_{d_0}(u)}{u} = 1$. From the definition of ϕ_{d_0} in (4.20) we find that, if $d_0 = 1$, then $\phi_{d_0}(0) = \phi(0) = 0$ and from Lemma 4.3.23 we get that $\phi(0) = \mu - 1 - \hbar$ if $\mu \geqslant 1 + \hbar$ while $\phi(0)$ is always zero when $\mu < 1 + \hbar$, which shows that if $d_0 = 1$ then $\phi(0) = 0 \iff \mu \leqslant 1$. On the other hand, from (4.20), it is clear that if $d_0 > 1$ then always $\phi_{d_0}(0) = 0$. This completes the proof of the claims regarding $\Lambda_{\phi_{d_0}}$. Next, by Proposition 4.3.4, $\mathcal{V}_n^{\phi} \in \mathrm{Ran}(\mathrm{V}^*_{\phi^*_{\mathfrak{m}}})$, for each $n \in \mathbb{N}$, and as proved in Proposition 4.3.3, the sequence $(\mathcal{V}_n^{\phi})_{n\geqslant 0}$ is complete. Thus $\mathrm{Ran}(\mathrm{V}^*_{\phi^*_{\mathfrak{m}}})$ is dense in $\mathrm{L}^2(\beta_{\mathfrak{m}})$, or equivalently $\mathrm{Ker}(\mathrm{V}_{\phi^*_{\mathfrak{m}}}) = \{0\}$. By direct calculation we get that,

$$||p_n||_{\beta}^{-2} \left\| \mathrm{V}_{\phi^*_{\mathfrak{m}}} p_n \right\|^2_{\beta_{\mathfrak{m}}} = \frac{W^2_{\phi}(n+1)}{W_{\phi}(2n+1)} \frac{(\mathfrak{m})_{2n}}{(\mathfrak{m})^2_n} = \prod_{k=1}^{n} \frac{\phi^*_{\mathfrak{m}}(k)}{\phi^*_{\mathfrak{m}}(k+n)}, \tag{4.54}$$

where $\phi^*_{\mathfrak{m}}$ was defined in (4.21). Now the fact that $\lim_{u\to\infty} \frac{\phi(u)}{u} = 1$ allow us to deduce $\lim_{u\to\infty} \phi^*_{\mathfrak{m}}(u) = 1$ and, as noted earlier, $\phi^*_{\mathfrak{m}}$ is a Bernstein function and hence non-decreasing. As the case $\phi^*_{\mathfrak{m}} \equiv 1$ is excluded by the assumption on $\mathfrak{m}$, we get that, as $n \to \infty$, the ratio in (4.54) tends to 0. Next, by taking the adjoint of (4.38) we get

$$\mathrm{U}^*_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathbb{Q}_t^{\phi^{\checkmark}_{\mathrm{r}_1}*} = \mathbb{Q}_t^{\phi *} \mathrm{U}^*_{\phi^{\checkmark}_{\mathrm{r}_1}}$$

and using this identity we get that $\mathrm{U}^*_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathcal{V}_n^{\phi}$ is an eigenfunction for $\mathbb{Q}_t^{\phi^{\checkmark}_{\mathrm{r}_1}*}$ associated to the eigenvalue $e^{-\lambda_n t}$. Then, Theorem 4.2.23 forces $\mathrm{U}^*_{\phi^{\checkmark}_{\mathrm{r}_1}} \mathcal{V}_n^{\phi} = \mathcal{V}_n^{\phi^{\checkmark}_{\mathrm{r}_1}}$, and the latter is a complete sequence, whence $\mathrm{Ker}(\mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}}) = \{0\}$. Finally, another straightforward calculation gives that

$$||p_n||_{\beta_{\phi}}^{-2} \left\| \mathrm{U}_{\phi^{\checkmark}_{\mathrm{r}_1}} p_n \right\|^2_{\beta_{\phi^{\checkmark}_{\mathrm{r}_1}}} = \frac{\phi^{\checkmark 2}_{\mathrm{r}_1}(0)}{\phi^{\checkmark 2}_{\mathrm{r}_1}(n)} \frac{\phi(2n+1)}{\mathrm{r}_1 \phi(1)} \frac{2n+\mathrm{r}_1}{2n+1} = \phi^{\checkmark}_{\mathrm{r}_1}(0) \frac{2n+\mathrm{r}_1}{(n+\mathrm{r}_1)^2} \left(\frac{n+!}{\phi(n+1)} \right)^2 \frac{\phi(2n+1)}{2n+1},$$

and using the fact that $\lim_{u\to\infty} \frac{\phi(u)}{u} = 1$ we conclude that the right-hand side tends to 0 as $n \to \infty$. ■

4.3.6 Proof of Theorem 4.2.31

Theorem 4.2.2 gives, for any $f \in \mathrm{L}^2(\beta)$ and $t > 0$,

$$\mathbb{Q}_t f = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{V}_n^{\phi} \rangle_{\beta} \mathcal{P}_n^{\phi}$$

so that, since $\lambda_0 = 0$ and $\mathcal{P}_0^{\phi} \equiv 1 \equiv \mathcal{V}_0^{\phi}$,

$$\mathbb{Q}_t f - \beta f = \sum_{n=1}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{V}_n^{\phi} \rangle_{\beta} \mathcal{P}_n^{\phi}. \tag{4.55}$$

Next, we note that

$$\sup_{n \geqslant 1} e^{-2n\lambda_1 t} \frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\boldsymbol{d})} \leqslant e^{-2\lambda_1 t} \frac{\mathfrak{c}_1^2(\mathfrak{m})}{\mathfrak{c}_1^2(\boldsymbol{d})} \iff 2\boldsymbol{\lambda}_1 t \geqslant \log\left(\frac{(\mathfrak{m}+1)(\boldsymbol{\lambda}_1 - \boldsymbol{d} + 1)}{(\boldsymbol{d}+1)(\boldsymbol{\lambda}_1 - \mathfrak{m} + 1)}\right), \tag{4.56}$$

since

$$e^{-2(n-1)\lambda_1 t} \frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\boldsymbol{d})} \frac{\mathfrak{c}_1^2(\boldsymbol{d})}{\mathfrak{c}_1^2(\mathfrak{m})} = \prod_{j=1}^{n-1} e^{-2\lambda_1 t} \frac{(\mathfrak{m}+j)(\boldsymbol{\lambda}_1 - \boldsymbol{d} + j)}{(\boldsymbol{d}+j)(\boldsymbol{\lambda}_1 - \mathfrak{m} + j)},$$

and $\mathfrak{m} > \boldsymbol{d}$, which is trivial when $\mathsf{r}_1 < 1$, as then $\mathfrak{m} > 1 > \boldsymbol{d} = \mathsf{r}_1$, while if $\mathsf{r}_1 = 1$ we have $\mathfrak{m} - 1 > d_{\phi} > \boldsymbol{d} - 1$ from [129, Proposition 4.4(1)]. Now, we claim that the following computation is valid, writing $\|\cdot\|_{\beta}$ again for the $\mathrm{L}^2(\beta)$-norm and $\boldsymbol{\mathcal{V}}_n^{\phi} = \frac{\mathfrak{c}_n(\mathsf{r}_1)}{\mathfrak{c}_n(\mathfrak{m})} \mathcal{V}_n^{\phi}$,

$$\begin{aligned}
\|\mathbb{Q}_t f - \beta f\|^2 &\leqslant \sum_{n=1}^{\infty} \frac{1}{\mathfrak{c}_n^2(\boldsymbol{d}_0)} \left|\langle \mathbb{Q}_t f, \mathcal{V}_n^{\phi} \rangle_{\beta}\right|^2 = \sum_{n=1}^{\infty} e^{-2\lambda_n t} \frac{\mathfrak{c}_n^2(\mathfrak{m})}{\mathfrak{c}_n^2(\boldsymbol{d})} \left|\left\langle f, \boldsymbol{\mathcal{V}}_n^{\phi} \right\rangle_{\beta}\right|^2 \\
&\leqslant \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d})}{\boldsymbol{d}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t} \sum_{n=1}^{\infty} \left|\left\langle f, \boldsymbol{\mathcal{V}}_n^{\phi} \right\rangle_{\beta}\right|^2 = \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d})}{\boldsymbol{d}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t} \sum_{n=1}^{\infty} \left|\left\langle f - \beta f, \boldsymbol{\mathcal{V}}_n^{\phi} \right\rangle_{\beta}\right|^2 \\
&\leqslant \frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d})}{\boldsymbol{d}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\lambda_1 t} \|f - \beta f\|_{\beta}^2 .
\end{aligned}$$

To justify this we start by observing that the first inequality follows from (4.55) together with $(\mathfrak{c}_n(\boldsymbol{d}_0)\mathcal{P}_n^{\phi})_{n \geqslant 0}$ being a Bessel sequence with Bessel bound 1, which was proved in Proposition 4.3.4. Next we use the fact that $\mathcal{V}_n^{\phi}$ is an eigenfunction for $\mathbb{Q}_t^*$ associated to the eigenvalue $e^{-\lambda_n t}$, and then the identity

$$\mathfrak{c}_n(\mathsf{r}_1)\mathfrak{c}_n(\boldsymbol{d}_0) = \mathfrak{c}_n(\boldsymbol{d}),$$

which follows by considering the cases $\mathsf{r}_1 = 1$ and $\mathsf{r}_1 < 1$ separately. Indeed, when $\mathsf{r}_1 = 1$ then $\boldsymbol{d} = \boldsymbol{d}_0$ and $\mathfrak{c}_n^2(\mathsf{r}_1) = 1$, while otherwise $\boldsymbol{d}_0 = 1$ so that $\boldsymbol{d} = \mathsf{r}_1$ and $\mathfrak{c}_n^2(\boldsymbol{d}_0) = 1$. The second inequality follows from (4.56) and then we use the biorthogonality of $(\mathcal{P}_n^{\phi})_{n\geqslant 0}$ and $(\mathcal{V}_n^{\phi})_{n\geqslant 0}$, given by Proposition 4.3.4, which implies that for any $c \in \mathbb{R}$, $\langle c\mathbf{1}_E, \boldsymbol{\mathcal{V}}_n^{\phi}\rangle_{\beta} = 0$ if $n \neq 0$. The last inequality follows from the fact that $(\boldsymbol{\mathcal{V}}_n^{\phi})_{n\geqslant 0}$ is a Bessel sequence with Bessel bound 1, again due to Proposition 4.3.4. Next, when $0 \leqslant 2\boldsymbol{\lambda}_1 t < \log\left(\frac{(1+\mathfrak{m})(1+\boldsymbol{\lambda}_1-\boldsymbol{d})}{(1+\boldsymbol{d})(1+\boldsymbol{\lambda}_1-\mathfrak{m})}\right)$ and since $\mathfrak{m} > \boldsymbol{d}$, we get

$$\frac{\mathfrak{m}(\boldsymbol{\lambda}_1 - \boldsymbol{d})}{\boldsymbol{d}(\boldsymbol{\lambda}_1 - \mathfrak{m})} e^{-2\boldsymbol{\lambda}_1 t} \geqslant \frac{\mathfrak{m}}{\mathfrak{m}+1} \frac{\boldsymbol{d}+1}{\boldsymbol{d}} \frac{\boldsymbol{\lambda}_1 - \boldsymbol{d}}{\boldsymbol{\lambda}_1 - \boldsymbol{d}+1} \frac{\boldsymbol{\lambda}_1 - \mathfrak{m}+1}{\boldsymbol{\lambda}_1 - \mathfrak{m}} \geqslant 1,$$

so that the contractivity of the semigroup $\mathbb{Q}$ yields, for $f \in \mathrm{L}^2(\beta)$ and any $t > 0$,

$$\|\mathbb{Q}_t f - \beta f\|_{\beta}^2 \leqslant e^{-2\boldsymbol{\lambda}_1 t} \|f - \beta f\|_{\beta}^2 .$$

Finally, since β is an invariant probability measure,

$$\|\mathbb{Q}_t f - \beta f\|_{\beta}^2 = \beta(\mathbb{Q}_t f - \beta f)^2 = \beta(\mathbb{Q}_t f)^2 - 2(\beta f)\beta\mathbb{Q}_t f + (\beta f)^2 = \beta(\mathbb{Q}_t f)^2 - (\beta f)^2 = \mathrm{Var}_{\beta}(\mathbb{Q}_t f),$$

which completes the proof.

4.3.7 Proof of Theorem 4.2.32

We first give a result that strengthens the intertwining relations in Proposition 4.3.3 and falls into the framework of the work by Miclo and Patie [111]. Write $\mathrm{V}_{\boldsymbol{d}}$ for the Markov kernel associated to a random variable with law $\beta_{\boldsymbol{d}}$, which, by the same arguments as in the proof of Lemma 4.3.10, satisfies $\mathrm{V}_{\boldsymbol{d}} \in \mathscr{B}(\mathrm{L}^2(\beta_{\boldsymbol{d}}), \mathrm{L}^2(\beta_{\mathfrak{m}}))$. We write $\overline{\mathrm{V}}_{\phi} = \Lambda_{\phi_{\boldsymbol{d}_0}} \mathrm{V}_{\boldsymbol{d}}^*$ and, for $\mu \geqslant 1 + \hbar$, let $\widetilde{\mathrm{V}}_{\phi} = \mathrm{V}_{\phi_{\mathfrak{m}}^*}$ and otherwise let $\widetilde{\mathrm{V}}_{\phi} = \mathrm{V}_{\phi_{\mathfrak{m}}^*}\mathrm{U}_{\phi_{\mathsf{r}_1}^{\checkmark}}$. Recall that a function $F : \mathbb{R}_+ \to [0, \infty)$ is said to be completely monotone if $F \in C^{\infty}(\mathbb{R}_+)$ and $(-1)^n \frac{d^n}{dx^n} F(u) \geqslant 0$, for $u > 0$ and $n \in \mathbb{N}$. By Bernstein's theorem, any completely monotone function F is the Laplace transform of a positive measure on $[0,\infty)$, and if $\lim_{u\to 0} F(u) < \infty$ (resp. $\lim_{u\to 0} F(u) = 1$) then F is the Laplace transform of finite (resp. probability) measure on $\mathbb{R}_+$, see e.g. [145, Chapter 1].

Proposition 4.3.6. Under the assumptions of the theorem, we have a completely monotone intertwining relationship between $\mathbb{Q}$ and $\mathbf{Q}^{(\mathfrak{m})}$, in the sense of [111], that is for $t \geqslant 0$ and on the respective L^2-spaces

$$\mathbb{Q}_t^{\phi}\overline{\mathrm{V}}_{\phi} = \overline{\mathrm{V}}_{\phi}\mathbf{Q}_t^{(\mathfrak{m})} \quad \text{and} \quad \widetilde{\mathrm{V}}_{\phi}\mathbb{Q}_t^{\phi} = \mathbf{Q}_t^{(\mathfrak{m})}\widetilde{\mathrm{V}}_{\phi} \quad \text{with} \quad \widetilde{\mathrm{V}}_{\phi}\overline{\mathrm{V}}_{\phi} = F_{\phi}(-\mathbf{J}_{\mathfrak{m}}), \tag{4.57}$$

where $-\log F_{\phi}$ is a Bernstein function with $F_{\phi} : [0,\infty) \to [0,\infty)$ being the completely monotone function given by

$$F_{\phi}(u) = \frac{(\boldsymbol{d})_{\rho(u)}}{(\mathfrak{m})_{\rho(u)}}\frac{(\boldsymbol{\lambda}_1 - \mathfrak{m})_{\rho(u)}}{(\boldsymbol{\lambda}_1 - \boldsymbol{d})_{\rho(u)}}, \quad u \geqslant 0.$$

Proof. We give the proof only in the case $\mu \geqslant 1 + \hbar$, so that $\boldsymbol{d} = \boldsymbol{d}_0$, as the other case follows by similar arguments. From Proposition 4.3.3 we get, with $\mathbb{J} = \mathbf{J}_{\boldsymbol{d}_0}$,

$$\mathbf{Q}_t^{(\mathfrak{m})}\mathrm{V}_{\boldsymbol{d}_0} = \mathrm{V}_{\boldsymbol{d}_0}\mathbf{Q}_t^{(\boldsymbol{d}_0)},$$

and taking the adjoint and using that both $\mathbf{Q}^{(\mathfrak{m})}$ and $\mathbf{Q}^{(\boldsymbol{d}_0)}$ are self-adjoint on $\mathrm{L}^2(\beta_{\mathfrak{m}})$ and $\mathrm{L}^2(\beta_{\boldsymbol{d}_0})$, respectively, we get that

$$\mathbf{Q}_t^{(\boldsymbol{d}_0)}\mathrm{V}^*_{\boldsymbol{d}_0} = \mathrm{V}^*_{\boldsymbol{d}_0}\mathbf{Q}_t^{(\mathfrak{m})}.$$

Combining this with the first intertwining relation in Proposition 4.3.3 then yields

$$\mathbb{Q}_t\overline{\mathrm{V}}_{\phi} = \overline{\mathrm{V}}_{\phi}\mathbf{Q}_t^{(\mathfrak{m})},$$

and, together with second intertwining relation in Proposition 4.3.1, we conclude that

$$\mathbf{Q}_t^{(\mathfrak{m})}\widetilde{\mathrm{V}}_{\phi}\overline{\mathrm{V}}_{\phi} = \widetilde{\mathrm{V}}_{\phi}\mathbb{Q}_t\overline{\mathrm{V}}_{\phi} = \widetilde{\mathrm{V}}_{\phi}\overline{\mathrm{V}}_{\phi}\mathbf{Q}_t^{(\mathfrak{m})}. \tag{4.58}$$

As $\mathbf{Q}_t^{(\mathfrak{m})}$ is self-adjoint with simple spectrum the commutation identity (4.58) implies, by the Borel functional calculus, see e.g. [139], that $\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi = F(\mathbf{J}_\mathfrak{m})$ for some bounded Borelian function F, and to identify F it suffices to identify the spectrum of $\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi$. To this end we observe that, for any $g \in \mathrm{L}^2(\beta_{\boldsymbol{d}_0})$,

$$\langle \mathrm{V}_{\boldsymbol{d}_0}^* \mathcal{P}_n^{(\mathfrak{m})}, g\rangle_{\beta_{\boldsymbol{d}_0}} = \langle \mathcal{P}_n^{(\mathfrak{m})}, \mathrm{V}_{\boldsymbol{d}_0} g\rangle_{\beta_\mathfrak{m}} = \sum_{m=0}^{\infty} \langle g, \mathcal{P}_m^{(\boldsymbol{d}_0)}\rangle_{\beta_{\boldsymbol{d}_0}} \langle \mathcal{P}_n^{(\mathfrak{m})}, \mathrm{V}_{\boldsymbol{d}_0} \mathcal{P}_m^{(\boldsymbol{d}_0)}\rangle_{\beta_\mathfrak{m}} = \frac{\mathfrak{c}_n(\boldsymbol{d}_0)}{\mathfrak{c}_n(\mathfrak{m})} \langle \mathcal{P}_n^{(\boldsymbol{d}_0)}, g\rangle_{\beta_{\boldsymbol{d}_0}},$$

where we used that $(\mathcal{P}_n^{(\boldsymbol{d}_0)})_{n\geqslant 0}$ forms an orthonormal basis for $\mathrm{L}^2(\beta_{\boldsymbol{d}_0})$ and the identity $\mathrm{V}_{\boldsymbol{d}_0} \mathcal{P}_m^{(\boldsymbol{d}_0)} = \mathfrak{c}_m(\boldsymbol{d}_0) \mathcal{P}_m^{(\mathfrak{m})} / \mathfrak{c}_m(\mathfrak{m})$ follows by a straightforward, albeit tedious, computation. Consequently, for any $n \in \mathbb{N}$,

$$\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi \mathcal{P}_n^{(\mathfrak{m})} = \frac{\mathfrak{c}_n(\boldsymbol{d}_0)}{\mathfrak{c}_n(\mathfrak{m})} \mathrm{V}_{\phi_\mathfrak{m}^*} \Lambda_{\phi_{\boldsymbol{d}_0}} \mathcal{P}_n^{(\boldsymbol{d}_0)} = \frac{\mathfrak{c}_n^2(\boldsymbol{d}_0)}{\mathfrak{c}_n(\mathfrak{m})} \mathrm{V}_{\phi_\mathfrak{m}^*} \mathcal{P}_n^{\phi} = \frac{\mathfrak{c}_n^2(\boldsymbol{d}_0)}{\mathfrak{c}_n^2(\mathfrak{m})} \mathcal{P}_n^{(\mathfrak{m})},$$

where the second and third equalities follow from calculations that were detailed in the proof of Proposition 4.3.4. Using the definition of $\mathfrak{c}_n$ in (4.42) we thus get that, for $n \in \mathbb{N}$,

$$F(\lambda_n) = \frac{\mathfrak{c}_n^2(\boldsymbol{d}_0)}{\mathfrak{c}_n^2(\mathfrak{m})} = \frac{(\boldsymbol{d}_0)_n}{(\mathfrak{m})_n} \frac{(\boldsymbol{\lambda}_1 - \mathfrak{m})_n}{(\boldsymbol{\lambda}_1 - \boldsymbol{d}_0)_n}$$

recalling from (4.10) that $(\lambda_n)_{n\geqslant 0}$ are the eigenvalues of $-\mathbf{J}_\mathfrak{m}$, which proves that $F_\phi = F$. Next, one readily computes that the non-negative inverse of the mapping $n \mapsto \lambda_n$ is given by the function ρ defined prior to the statement of the theorem, which was remarked to be a Bernstein function. For another short proof of this fact, observe that, for $u \geqslant 0$,

$$\rho'(u) = \left((\boldsymbol{\lambda}_1 - 1)^2 + 4u\right)^{-\frac{1}{2}},$$

which is completely monotone. Since $u \mapsto F_\phi(u^2 + (\boldsymbol{\lambda}_1 - 1)u)$ is the Laplace transform of the product convolution of the beta distributions $\beta_{\boldsymbol{d}_0}$ and $\beta_\mathfrak{m}$ we may invoke [145, Theorem 3.7] to conclude F_ϕ is completely monotone. Finally, to show that $-\log F_\phi$ is a Bernstein function we note that, for any $a, b > 0$, the function $u \mapsto \log(a+b)_u - \log(a)_u$ is a Bernstein function, see e.g. Example 88 in [145, Chapter 16]. Since

$$-\log F_\phi(u) = \log \frac{(\mathfrak{m})_{\rho(u)}}{(\boldsymbol{d}_0)_{\rho(u)}} + \log \frac{(\boldsymbol{\lambda}_1 - \boldsymbol{d}_0)_{\rho(u)}}{(\boldsymbol{\lambda}_1 - \mathfrak{m})_{\rho(u)}},$$

with $\boldsymbol{d}_0 < \mathfrak{m}$, and the composition of Bernstein functions remains Bernstein together with the fact that the set of Bernstein functions is a convex cone, see e.g. [145, Corollary 3.8] for both of these claims, it follows that $-\log F_\phi$ is a Bernstein function. ■

Proof of Theorem 4.2.32. Since $\mathfrak{m} \in (\mathbf{1}_{\{\mu<1+\hbar\}} + \mu, \boldsymbol{\lambda}_1)$ we may apply Proposition 4.3.6 to conclude that $\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi = F_\phi(-\mathbf{J}_\mathfrak{m})$ and a straightforward substitution gives $\mathbb{E}\left[e^{-u\tau}\right] = F_\phi(u)$, $u \geqslant 0$, with $-\log F_\phi$ a Bernstein function. From the Borel functional calculus we get, since $\mathbf{Q}_t^{(\mathfrak{m})}$ is self-adjoint on $\mathrm{L}^2(\beta_\mathfrak{m})$, that

$$\mathbf{Q}_\tau^{(\mathfrak{m})} = \int_0^\infty \mathbf{Q}_t^{(\mathfrak{m})} \mathbb{P}(\tau \in dt) = \int_0^\infty e^{t\mathbf{J}_\mathfrak{m}} \mathbb{P}(\tau \in dt) = F_\phi(-\mathbf{J}_\mathfrak{m}) = \widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi.$$

Combining this identity with (4.57) yields, for non-negative $f \in \mathrm{L}^2(\beta)$,

$$\widetilde{\mathrm{V}}_\phi \overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi f = \int_0^\infty \mathbf{Q}_t^{(\mathfrak{m})} \widetilde{\mathrm{V}}_\phi f \, \mathbb{P}(\tau \in dt) = \int_0^\infty \widetilde{\mathrm{V}}_\phi \mathbb{Q}_t f \, \mathbb{P}(\tau \in dt) = \widetilde{\mathrm{V}}_\phi \int_0^\infty \mathbb{Q}_t f \, \mathbb{P}(\tau \in dt),$$

and the general case follows by linearity and by decomposing f into the difference of non-negative functions. By Proposition 4.3.5 $\widetilde{\mathrm{V}}_\phi$ has trivial kernel on $\mathrm{L}^2(\beta)$ so we deduce

$$\overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi = \int_0^\infty \mathbb{Q}_t \mathbb{P}(\tau \in dt) = \mathbb{Q}_\tau, \tag{4.59}$$

and thus $\mathbb{Q}$ satisfies a completely monotone intertwining relation with $\mathbf{Q}^{(\mathfrak{m})}$, in the sense of [111]. Consequently we may invoke [111, Theorems 7, 24] to transfer the entropy decay and Φ-entropy decay of $\mathbf{Q}^{(\mathfrak{m})}$, reviewed in Section 4.5, to the semigroup $\mathbb{Q}$ but after a time shift of the independent random variable τ. Note that, when $\boldsymbol{\lambda}_1 > 2(\mathbf{1}_{\{\mu<1+\hbar\}} + \mu)$, we may take $\mathfrak{m} = \frac{\lambda_1}{2}$ so that the reference semigroup is $\mathbf{Q}^{(\lambda_1/2)}$, which has optimal entropy decay rate. ■

4.3.8 Proof of Theorem 4.2.4

The proof of Theorem 4.2.41 follows by using Equation (4.59) above to invoke [111, Theorem 8]. Next, by Equation (4.59) and using Proposition 4.3.6 we get

$$\|\mathbb{Q}_{t+\tau}\|_{1\to\infty} = \|\mathbb{Q}_t \overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi\|_{1\to\infty} = \|\overline{\mathrm{V}}_\phi \mathbf{Q}_t^{(\mathfrak{m})} \widetilde{\mathrm{V}}_\phi\|_{1\to\infty} \leqslant \|\mathbf{Q}_t^{(\mathfrak{m})}\|_{1\to\infty},$$

where the last inequality follows by applying Lemma 4.3.10 twice, once in the case $p = \infty$ for $\overline{\mathrm{V}}_\phi$ and once with $p = 1$ for $\widetilde{\mathrm{V}}_\phi$. The remaining claims follow from the ultracontractivity properties of $\mathbf{Q}^{(\mathfrak{m})}$. ■

4.3.9 Proof of Corollary 4.2.1

The following arguments are taken from the proof of [110, Proposition 5]. We denote by $\mathbf{Q}^{(\mathfrak{m},\boldsymbol{\tau})}$ for the classical Jacobi semigroup $\mathbf{Q}^{(\mathfrak{m})}$ subordinated with respect to $\boldsymbol{\tau} = (\tau_t)_{t\geqslant 0}$. By [111, Theorem 3] we obtain, from Proposition 4.3.6, a completely monotone intertwining relationship between the subordinate semigroups, i.e. writing $\overline{\mathrm{V}}_\phi$ and $\widetilde{\mathrm{V}}_\phi$ as above, we have, for any $t \geqslant 0$ and on the appropriate L^2-spaces,

$$\mathbb{Q}_t^{\boldsymbol{\tau}} \overline{\mathrm{V}}_\phi = \overline{\mathrm{V}}_\phi \mathbf{Q}_t^{(\mathfrak{m},\boldsymbol{\tau})} \quad \text{and} \quad \widetilde{\mathrm{V}}_\phi \mathbb{Q}_t^{\boldsymbol{\tau}} = \mathbf{Q}_t^{(\mathfrak{m},\boldsymbol{\tau})} \widetilde{\mathrm{V}}_\phi \quad \text{with} \quad \overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi = \mathbb{Q}_1^{\boldsymbol{\tau}}. \tag{4.60}$$

Using this we get, for any $f \in \mathrm{L}^2(\beta)$ and $t \geqslant 1$,

$$\begin{aligned}
\mathbb{Q}_t^{\boldsymbol{\tau}} f = \mathbb{Q}_{t-1}^{\boldsymbol{\tau}} \overline{\mathrm{V}}_\phi \widetilde{\mathrm{V}}_\phi f = \overline{\mathrm{V}}_\phi \mathbf{Q}_{t-1}^{(\mathfrak{m},\boldsymbol{\tau})} \widetilde{\mathrm{V}}_\phi f &= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{-\lambda_n \tau_{t-1}}\right] \langle \widetilde{\mathrm{V}}_\phi f, \mathcal{P}_n^{(\mathfrak{m})} \rangle_{\beta_{\mathfrak{m}}} \overline{\mathrm{V}}_\phi \mathcal{P}_n^{(\mathfrak{m})} \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{-\lambda_n \tau_{t-1}}\right] \frac{\mathfrak{c}_n^2(\boldsymbol{d})}{\mathfrak{c}_n^2(\mathfrak{m})} \langle f, \mathcal{V}_n^\phi \rangle_\beta \mathcal{P}_n^\phi \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[e^{-\lambda_n \tau_t}\right] \langle f, \mathcal{V}_n^\phi \rangle_{\beta_{\mathfrak{m}}} \mathcal{P}_n^\phi,
\end{aligned}$$

where in the second equality we used the boundedness of $\overline{\mathrm{V}}_\phi$ together the expansion for the subordinated classical Jacobi semigroup which follows from (5.5.7) and standard arguments, then the properties of $\widetilde{\mathrm{V}}_\phi$ and $\overline{\mathrm{V}}_\phi$ detailed in previous sections, and finally the expression for $\mathbb{E}[e^{-\lambda_n \tau}]$ in (4.15). The other claims of the corollary then follow from [111, Theorems 7, 24] applied to (4.60).

4.4 Examples

In this section we consider a parametric family of non-local Jacobi operators for which h is a power function. More specifically, let $\delta \geqslant 1$ and consider the integro-differential operator $\mathbb{J}_\delta$ given by

$$\mathbb{J}_\delta f(x) = x(1-x)f''(x) - (\lambda_1 x - \delta - 1)f'(x) - x^{-(\delta+1)} \int_0^1 f'(r) r^\delta dr$$

Then $\mathbb{J}_\delta$ is a non-local Jacobi operator with $\mu = \delta + 1$ and $h(r) = r^{-\delta-1}$, $r > 1$, or one easily gets that equivalently $\overline{\Pi}(r) = e^{-\delta r}$, $r > 0$. One readily computes that $\hbar = \int_1^\infty h(r)dr = \delta^{-1}$ and thus the condition $\mu \geqslant 1 + \hbar$ is always satisfied, which implies that $\mathrm{r}_1 = 1$. Writing ϕ_δ for the Bernstein function in one-to-one correspondence with $\mathbb{J}_\delta$, we have that for $u \geqslant 0$,

$$\phi_\delta(u) = u + \frac{\delta^2 - 1}{\delta} + \int_1^\infty (1 - r^{-u}) r^{-\delta-1} dr = \frac{(u+\delta+1)(u+\delta-1)}{u+\delta}. \tag{4.61}$$

From the right-hand side of (4.61) we easily see that $d_{\phi_\delta} = \delta - 1$. Now, we assume that $\lambda_1 > \delta + 2 > 3$ and, for sake of simplicity, take $\lambda_1 - \delta \notin \mathbb{N}$. The following result characterizes all the spectral objects for these non-local Jacobi operators.

Proposition 4.4.1.

1. The density of the unique invariant measure of the Markov semigroup associated to $\mathbb{J}_\delta$ is given by

$$\beta(x) = \frac{((\lambda_1 - \delta - 2)x + 1)}{(\delta + 1)(1 - x)}\beta_\delta(x), \ x \in E^{\circ}.$$

2. We have that $\mathcal{P}_0^{\phi_\delta} \equiv 1$ and, for $n \geqslant 1$,

$$\mathcal{P}_n^{\phi_\delta}(x) = \frac{n!}{(\delta+2)_n}\sqrt{\mathrm{C}_n(1)}\left(\frac{\mathcal{P}_n^{(\lambda_1,\delta+2)}(x)}{\sqrt{\mathrm{C}_n(\delta+2)}} + \frac{x}{\delta}\frac{\mathcal{P}_{n-1}^{(\lambda_1+1,\delta+3)}(x)}{\sqrt{\widetilde{\mathrm{C}}_{n-1}(\delta+3)}}\right), \ x \in E.$$

making explicit the dependence on the two parameters for the classical Jacobi polynomials, see (5.5.4), and where $\widetilde{\mathrm{C}}_n(\delta+3) = n!(2n+\lambda_1)(\lambda_1+1)_n/(\delta+3)_n(\lambda_1-\delta-2)_n$.

3. For any $n \in \mathbb{N}$ the function $\mathcal{V}_n^{\phi_\delta}$ is given by

$$\mathcal{V}_n^{\phi_\delta}(x) = \frac{w_n(x)}{\beta(x)}, \ x \in E^{\circ},$$

where w_n has the so-called Barnes integral representation, see e.g. [28], for any $a > 0$,

$$\begin{aligned} w_n(x) &= -C_{\lambda_1,\delta,n}\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty}\frac{\Gamma(\delta+2-z)\Gamma(-z)\Gamma(\delta-z)}{\Gamma(\delta+1-z)\Gamma(-n-z)\Gamma(z+\lambda_1+n)}x^z dz, \\ &= C_{\lambda_1,\delta,n}\frac{\sin(\pi(\delta-\lambda_1))}{\pi}\sum_{k=0}^{\infty}\frac{(\delta+1)_{k+n}}{(\delta+1)_k}\frac{\Gamma(k+\delta-n-\lambda_1+1)}{k!}(k-1)x^{k+\delta}, \quad |x| < 1, \end{aligned}$$

and $C_{\lambda_1,\delta,n} = \delta(\lambda_1-1)\Gamma(\lambda_1+n-1)\sqrt{\mathrm{C}_n(1)}(-2)^n/(n!\Gamma(\delta+2))$.

Proof. First, from (4.61) and (4.5) we get that, for any $n \in \mathbb{N}$,

$$W_{\phi_\delta}(n+1) = \frac{\delta}{n+\delta}(\delta+2)_n \tag{4.62}$$

so that from (4.6) we deduce that

$$\beta p_n = \frac{W_{\phi_\delta}(n+1)}{(\lambda_1)_n} = \frac{\delta}{n+\delta}\frac{(\delta+2)_n}{(\lambda_1)_n}. \tag{4.63}$$

The first term on the right of (4.63) is the n^{th}-moment of the probability density $f_\delta(x) = \delta x^{\delta-1}$ on E while the second term is the n^{th}-moment of a $\beta_{\delta+2}$ density. Thus, by moment identification and determinacy, we conclude that $\beta(x) = f_\delta \diamond \beta_{\delta+2}(x)$ and after some easy algebra we get, for $x \in E^{o}$, that

$$\beta(x) = \frac{\Gamma(\lambda_1)\delta x^{\delta-1}}{\Gamma(\delta+2)\Gamma(\lambda_1-\delta-2)}\int_x^1 y(1-y)^{\lambda_1-\delta-3}dy = \frac{((\lambda_1-\delta-2)x+1)}{(\delta+1)(1-x)}\beta_\delta(x),$$

which completes the proof of the first item. Next, substituting (4.62) in (4.7), gives $\mathcal{P}_0^\delta \equiv 1$, and for $n = 1, 2, \ldots$,

$$\begin{aligned}\mathcal{P}_n^{\phi_\delta}(x) &= \sqrt{\mathrm{C}_n(1)}\left(\sum_{k=0}^{n}\frac{(-1)^{n+k}}{(n-k)!}\frac{(\lambda_1-1)_{n+k}}{(\lambda_1-1)_n}\frac{n!}{k!}\frac{x^k}{(\delta+2)_k} + \sum_{k=0}^{n}\frac{(-1)^{n+k}}{(n-k)!}\frac{(\lambda_1-1)_{n+k}}{(\lambda_1-1)_n}\frac{n!}{k!}\frac{k}{\delta}\frac{x^k}{(\delta+2)_k}\right)\\ &= \frac{n!}{(\delta+2)_n}\sqrt{\mathrm{C}_n(1)}\left(\frac{\mathcal{P}_n^{(\delta+2)}(x)}{\sqrt{\mathrm{C}_n(\delta+2)}} + \frac{x}{\delta}\frac{\mathcal{P}_{n-1}^{(\lambda_1+1,\delta+3)}(x)}{\sqrt{\widetilde{\mathrm{C}}_{n-1}(\delta+3)}}\right),\end{aligned}$$

where, to compute the second equality we made a change of variables and used the recurrence relation of the gamma function, and the definition of the classical Jacobi polynomials, see Section 4.5 and also [151]. This completes the proof of Item 2. To prove Item 3 we recall from (4.9) that, for any $n \in \mathbb{N}$, $\mathcal{V}_n^{\phi_\delta}(x) = \frac{1}{\beta(x)}w_n(x)$, where, by (4.33), the Mellin transform of w_n is given, for any $\Re(z) > 0$, as

$$\mathcal{M}_{w_n}(z) = C_{\lambda_1,\delta,n}(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)},$$

used twice the functional equation for the gamma function and the definition of the constant $C_{\lambda_1,\delta,n}$ in the statement. Next, writing $z = a + ib$ for any $b \in \mathbb{R}$ and $a > 0$, we recall from (4.36) that there exists a constant $C_a > 0$ such that

$$\lim_{|b|\to\infty} C_a|b|^{\lambda_1+n-1}\left|(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)}\right| = 1, \tag{4.64}$$

where we recall that $\lambda_1 > \delta + 2 > 3$ and $n \geqslant 0$. Hence, since $z \mapsto \mathcal{M}_{w_n}(z)$ is analytic on the right half-plane, by Mellin's inversion formula, see e.g. [114, Chapter 11], one gets for any $a > 0$,

$$w_n(x) = C_{\lambda_1,\delta,n}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)}x^{-z}dz,$$

where the integral is absolutely convergent for any $x > 0$. Note that this is a Barnes-integral since we can write, again using the functional equation for the gamma function,

$$w_n(x) = -C_{\lambda_1,\delta,n}\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty}\frac{\Gamma(\delta+2-z)}{\Gamma(\delta+1-z)}\frac{\Gamma(-z)}{\Gamma(-z-n)}\frac{\Gamma(\delta-z)}{\Gamma(z+\lambda_1+n)}x^z dz,$$

see for instance [28]. Next, since $(z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)} = (z+\delta+1)(z-n)\cdots(z-1)$, it follows that the function $z \mapsto (z+\delta+1)\frac{\Gamma(z)}{\Gamma(z-n)}$ does not have any poles, while the function $z \mapsto \frac{\Gamma(z+\delta)}{\Gamma(z+\lambda_1+n)}$ has simple poles at $z = -k-\delta$ for all $k \in \mathbb{N}$. Consequently, by Cauchy's residue theorem we have, for any $|x| < 1$,

$$w_n(x) = C_{\lambda_1,\delta,n}\sum_{k=0}^{\infty}\frac{(1-k)\Gamma(-k-\delta)}{\Gamma(-k-\delta-n)}\frac{(-1)^k}{k!}\frac{x^{k+\delta}}{\Gamma(-k-\delta+\lambda_1+n)},$$

where we used that the integrals along the two horizontal segments of any closed contour vanish, as by (4.64) they go to 0 when $|b| \to \infty$. We justify the radius of convergence of the series as follows. Since $\lambda_1 - \delta \notin \mathbb{N}$, using Euler's reflection formula for the gamma function, i.e. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, $z \notin \mathbb{Z}$, we conclude that

$$w_n(x) = C_{\lambda_1,\delta,n}\frac{\sin(\pi(\delta-\lambda_1))}{\pi}\sum_{k=0}^{\infty}\frac{(\delta+1)_{k+n}}{(\delta+1)_k}\frac{\Gamma(k+\delta-n-\lambda_1+1)}{k!}(k-1)x^{k+\delta},$$

where we used that $\sin(x+k\pi) = (-1)^k\sin(x)$ for $k \in \mathbb{N}$. Using the recurrence relation of the gamma function we deduce that the radius of convergence of this series is 1, which completes the proof. ∎

4.5 Classical Jacobi operator and semigroup

4.5.1 Introduction and boundary classification

Before we begin reviewing the classical Jacobi operator, semigroup, and process we clarify the notational convention that is used for these objects throughout the paper. Namely, instead of writing $\mathbf{J}_{\lambda_1,\mu}$ we suppress the dependency on λ_1 and write simply $\mathbf{J}_\mu$, as we think of λ_1 being fixed and common throughout, and similarly for the beta distribution, Jacobi semigroup, and polynomials. The exception is when these any of these objects depend in a not-straightforward way on λ_1, in which case we highlight the dependency explicitly. Now, let $\lambda_1 > \mu > 0$ and let $\mathbf{Q}^{(\mu)} = (\mathbf{Q}_t^{(\mu)})_{t\geqslant 0}$ be the transition semigroup of a Jacobi process $(Y_t)_{t\geqslant 0}$ on E, i.e. for bounded measurable functions f

$$\mathbf{Q}_t^{(\mu)} f(x) = \mathbb{E}_x\left[f(Y_t)\right], \quad x \in E.$$

Then $\mathbf{Q}^{(\mu)}$ is a Feller semigroup and its infinitesimal generator $\mathbf{J}_\mu$ has, for any $f \in C^2(E)$, the following form

$$\mathbf{J}_\mu f(x) = x(1-x)f''(x) - (\lambda_1 x - \mu)f'(x), \quad x \in E.$$

Note that when the state space of the Jacobi process is taken to be $[-1, 1]$ then the associated infinitesmal generator $\widetilde{\mathbf{J}}_\mu$ is given by

$$\widetilde{\mathbf{J}}_\mu f(x) = (1-x^2)f''(x) + (2\mu - \lambda_1 - \lambda_1 x)f'(x),$$

and setting $g(x) = \frac{x+1}{2}$ yields

$$\widetilde{\mathbf{J}}_\mu (f \circ g)(g^{-1}(x)) = x(1-x)f''(x) - (\lambda_1 x - \mu)f'(x) = \mathbf{J}_\mu f(x).$$

Since the operator $\mathbf{J}_\mu$ is degenerate at the boundaries $\partial E = \{0, 1\}$, it is important to specify how the process behaves at these points. After some straightforward computations, as outlined in [26, Chapter 2] and using the notation therein, we get the boundaries are classified as follows,

$$0 \text{ is } \begin{cases} \text{exit-not-entrance for} & \mu \leqslant 0, \\ \text{regular for} & 0 < \mu < 1, \\ \text{entrance-not-exit for} & \mu \geqslant 1, \end{cases} \quad \text{and,} \quad 1 \text{ is } \begin{cases} \text{exit-not-entrance for} & \lambda_1 \leqslant \mu, \\ \text{regular for} & 0 < \lambda_1 < 1 + \mu, \\ \text{entrance-not-exit for} & \lambda_1 \geqslant 1 + \mu. \end{cases}$$

Thus assumptions on λ_1 and μ guarantee that both 0 and 1 are at least entrance, and may be regular or entrance-not-exit depending on the particular values of λ_1 and μ. Let us write $\mathcal{D}_C(\mathbf{J}_\mu)$ for the $C(E)$-domain of $\mathbf{J}_\mu$ and to specify it we recall that the so-called scale function s of $\mathbf{J}_\mu$ satisfies

$$s'(x) = x^{-\lambda_1}(1-x)^{-(\lambda_1-\mu)}, \quad x \in E.$$

Let f^+ and f^- denote the right and left derivatives of a function f with respect to s, i.e.

$$f^+(x) = \lim_{h\downarrow 0} \frac{f(x+h)-f(x)}{s(x+h)-s(x)}, \quad \text{and} \quad f^-(x) = \lim_{h\downarrow 0} \frac{f(x)-f(x-h)}{s(x)-s(x-h)}.$$

Then,

$$\mathcal{D}_C(\mathbf{J}_\mu) = \left\{ f \in C^2(E);\ \mathbf{J}_\mu f \in C(E),\ f^+(0^+) = f^-(1^-) = 0 \right\}, \tag{4.65}$$

and in particular, $\mathcal{P} \subset \mathcal{D}_{C(E)}$, since for any $f \in \mathcal{P}$ we have

$$f^+(0^+) = \lim_{x\downarrow 0} x^{\lambda_1} f'(x) = 0 \quad \text{and} \quad f^-(1^-) = \lim_{x\uparrow 1} (1-x)^{\lambda_1-\mu} f'(x) = 0.$$

From the boundary conditions in (4.65) we get that if any point in ∂E is regular then it is necessarily a reflecting boundary for the Jacobi process with $\lambda_1 > \mu > 0$.

4.5.2 Invariant measure and L^2-properties

The classical Jacobi semigroup $\mathbf{Q}^{(\mu)} = (\mathbf{Q}_t^{(\mu)})_{t\geqslant 0}$ has a unique invariant measure β_μ, which is the distribution of a beta random variable on E, i.e.

$$\beta_\mu(dx) = \beta_\mu(x)dx = \frac{\Gamma(\lambda_1)}{\Gamma(\mu)\Gamma(\lambda_1-\mu)} x^{\mu-1}(1-x)^{\lambda_1-\mu-1}dx, \quad x \in E^{\mathrm{o}},$$

and we recall that, for any $n \in \mathbb{N}$,

$$\int_0^1 x^n \beta_\mu(dx) = \frac{(\mu)_n}{(\lambda_1)_n}. \tag{4.66}$$

Since β_μ is invariant for $\mathbf{Q}^{(\mu)}$ we get that $\mathbf{Q}^{(\mu)}$ extends to a contraction semigroup on $\mathrm{L}^2(\beta_\mu)$ and, moreover, the stochastic continuity of Y ensures that this extension is strongly continuous in $\mathrm{L}^2(\beta_\mu)$ and thus we obtain a Markov semigroup in $\mathrm{L}^2(\beta_\mu)$, which we still denote by $\mathbf{Q}^{(\mu)} = (\mathbf{Q}_t^{(\mu)})_{t\geqslant 0}$. The eigenfunctions of $\mathbf{J}_\mu$ are the Jacobi polynomials given, for any $n \in \mathbb{N}$ and $x \in E$, by

$$\mathcal{P}_n^{(\mu)}(x) = \sqrt{\mathrm{C}_n(\mu)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{(\mu)_n}{(\mu)_k} \frac{x^k}{k!}, \tag{4.67}$$

where we have set

$$\mathrm{C}_n(\mu) = (2n + \lambda_1 - 1) \frac{n!(\lambda_1)_{n-1}}{(\mu)_n(\lambda_1 - \mu)_n}. \tag{4.68}$$

In particular, when $\mu = 1$ then, we get, for any $n \in \mathbb{N}$,

$$\mathcal{P}_n^{(1)}(x) = \sqrt{\mathrm{C}_n(1)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{n!}{k!} \frac{x^k}{k!}, \tag{4.69}$$

where we note that $\mathrm{C}_n(1) = \frac{\Gamma(\lambda_1 - 1)}{\Gamma(\lambda_1)}(2n + \lambda_1 - 1)$. These polynomials are the orthogonal polynomials with respect to the measure β_μ and, by choice of $\mathrm{C}_n(\mu)$, satisfy the normalization condition

$$\int_0^1 \mathcal{P}_n^{(\mu)}(x)\mathcal{P}_m^{(\mu)}(x)\beta_\mu(dx) = \langle \mathcal{P}_n^{(\mu)}, \mathcal{P}_m^{(\mu)} \rangle_{\beta_\mu} = \delta_{nm},$$

and thus form an orthonormal basis for $\mathrm{L}^2(\beta_\mu)$. Furthermore we have, for $n \in \mathbb{N}$, the following formula

$$\begin{aligned}\mathcal{P}_n^{(\mu)}(x) &= \frac{2^n}{n!} \sqrt{\mathrm{C}_n(\mu)} \frac{1}{\beta_\mu(x)} \frac{d^n}{dx^n} \left(x^n (1-x)^n \beta_\mu(x) \right) \\ &= \frac{1}{\beta_\mu(x)} (\beta_{\lambda_1 - \mu} p_n) \sqrt{\mathrm{C}_n(\mu)} \mathbf{R}_n \beta_{\lambda_1 + n, \mu}(x),\end{aligned} \tag{4.70}$$

where we recall the definition in (4.8) of $\mathbf{R}_n$. All of these relations follow, by the change of variables $x \mapsto 2x - 1$ and simple algebra, from the corresponding relations for the polynomials $P_n^{(\mu-1, \lambda_1 - \mu - 1)}$, defined in [89, Section 0.1], which are orthogonal for the weight $(1-x)^{\mu-1}(1+x)^{\lambda_1 - \mu - 1}$, and are also called Jacobi polynomials in the literature. Indeed, the relationship between $\mathcal{P}_n^{(\mu)}$ and $P_n^{(\mu-1,\lambda_1-\mu-1)}$ is given by

$$\mathcal{P}_n^{(\mu)}(x) = (-1)^n \sqrt{\frac{(2n+\lambda_1-1)n!(\lambda_1)_{n-1}}{(\mu)_n(\lambda_1-\mu)_n}} P_n^{(\mu-1,\lambda_1-\mu-1)}(1-2x).$$

Next, the eigenvalue associated to the eigenfunction $\mathcal{P}_n^{(\mu)}(x)$ is, for $n \in \mathbb{N}$,

$$-\lambda_n = -n^2 - (\lambda_1 - 1)n = -n(n-1) - \lambda_1 n. \tag{4.71}$$

Observe that when $n = 1$ (5.5.6) reduces to $-\lambda_1$ and that $\lambda_0 = 0$, so that $-\lambda_1$ denotes the largest, non-zero eigenvalue of $\mathbf{J}_\mu$, which is also called the spectral gap. The semigroup $\mathbf{Q}^{(\mu)}$ then admits the spectral decomposition given, for any $f \in \mathrm{L}^2(\beta_\mu)$ and $t \geqslant 0$, by

$$\mathbf{Q}_t^{(\mu)} = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle \cdot, \mathcal{P}_n^{(\mu)} \rangle_{\beta_\mu} \mathcal{P}_n^{(\mu)} = \sum_{n=0}^{\infty} e^{-\lambda_n t} \mathcal{P}_n^{(\mu)} \otimes \mathcal{P}_n^{(\mu)} \tag{4.72}$$

where the equality holds in the $\mathrm{L}^2(\beta_\mu)$-sense and in operator norm. The $\mathrm{L}^2(\beta_\mu)$-domain of $\mathbf{J}_\mu$, which we write as $\mathcal{D}_{\mathrm{L}^2}(\mathbf{J}_\mu)$, can then be identified as

$$\mathcal{D}_{\mathrm{L}^2}(\mathbf{J}_\mu) = \left\{ f \in \mathrm{L}^2(\beta_\mu);\ \sum_{n=0}^{\infty} n^4 \left| \langle f, \mathcal{P}_n^{(\mu)} \rangle_{\beta_\mu} \right|^2 < \infty \right\}.$$

4.5.3 Variance and entropy decay; hypercontractivity and ultracontractivity

As mentioned in the introduction, the fact that $\mathbf{Q}^{(\mu)}$ has nice spectral properties and satisfies certain functional inequalities gives quantitative rates of convergence to the equilibrium measure β_μ. For instance, from (5.5.7) one gets the following variance decay estimate, valid for any $f \in \mathrm{L}^2(\beta_\mu)$ and $t \geqslant 0$,

$$\mathrm{Var}_{\beta_\mu}(\mathbf{Q}_t^{(\mu)} f) \leqslant e^{-2\lambda_1 t} \, \mathrm{Var}_{\beta_\mu}(f),$$

which may also be deduced directly from the Poincaré inequality for $\mathbf{J}_\mu$, see [10, Chapter 4.2]. This convergence is optimal in the sense that the decay rate does not hold for any constant strictly smaller than $2\lambda_1$. Next, let us write $\lambda^{(\mu)}_{\mathrm{logS}}$ for the logarithmic Sobolev constant of $\mathbf{J}_\mu$ defined as

$$\lambda^{(\mu)}_{\mathrm{logS}} = \inf_{f\in\mathcal{D}_{L^2}(\mathbf{J})}\left\{\frac{-4\beta_\mu(f\mathbf{J}_\mu f)}{\mathrm{Ent}_{\beta_\mu}(f^2)};\ \mathrm{Ent}_{\beta_\mu}(f^2)\neq 0\right\}. \tag{4.73}$$

Note that always $\lambda^{(\mu)}_{\mathrm{logS}} \leqslant 2\lambda_1$, and in the case of the symmetric Jacobi operator, i.e. $\mu = \frac{\lambda_1}{2} > 1$, we get

$$\lambda^{(\frac{\lambda_1}{2})}_{\mathrm{logS}} = 2\lambda_1, \tag{4.74}$$

while otherwise $\lambda^{(\mu)}_{\mathrm{logS}} < 2\lambda_1$, see e.g. [143, Theorem 9.1], although the equality for the symmetric case goes back to [9, 115]. As a consequence of (4.73) we have on the one hand the convergence in entropy, for any $t \geqslant 0$ and $f \in \mathrm{L}^1(\beta_\mu)$ such that $\mathrm{Ent}_{\beta_\mu}(f) < \infty$,

$$\mathrm{Ent}_{\beta_\mu}(\mathbf{Q}^{(\mu)}_t f) \leqslant e^{-\lambda^{(\mu)}_{\mathrm{logS}}t}\,\mathrm{Ent}_{\beta_\mu}(f), \tag{4.75}$$

and on the other hand from Gross [75] the hypercontractivity estimate, that is for all $t \geqslant 0$,

$$\|\mathbf{Q}^{(\mu)}_t\|_{2\to q} \leqslant 1 \text{ where } 2 \leqslant q \leqslant 1 + e^{\lambda^{(\mu)}_{\mathrm{logS}}t}. \tag{4.76}$$

From (4.74) we thus get that the symmetric Jacobi semigroup attains the optimal entropic decay and hypercontractivity rate. Further, when $\frac{\lambda_1}{2} = n \in \mathbb{N}$ there exists a homeomorphism between $\mathbf{J}_\mu$ and the radial part of the Laplace-Beltrami operator on the n-sphere, which leads to the curvature-dimension condition $CD(\lambda_1 - 1, \lambda_1)$, see [10] for the definition. Thus for any admissible function $\Phi : I \to \mathbb{R}$, we get

$$\mathrm{Ent}^{\Phi}_{\beta_{\lambda_1/2}}(\mathbf{Q}^{(\lambda_1/2)}_t f) \leqslant e^{-(\lambda_1-1)t}\,\mathrm{Ent}^{\Phi}_{\beta_{\lambda_1/2}}(f) \tag{4.77}$$

for any $t \geqslant 0$ and $f : E \to I$ such that $f, \Phi(f) \in \mathrm{L}^1(\beta_{\lambda_1/2})$. Finally, since $\mathbf{J}_\mu$ also satisfies a Sobolev inequality, see e.g. [8], we have, for $t > 0$, the ultracontractivity property

$$\|\mathbf{Q}^{(\mu)}_t\|_{1\to\infty} < \infty, \quad \text{and} \quad \lim_{t\to\infty}\|\mathbf{Q}^{(\mu)}_t\|_{1\to\infty} = 0. \tag{4.78}$$

By a combination of Theorem 6.3.1 and Corollary 7.15 of [10] we get, when $\lambda_1 - \mu > 1$, the estimate

$$\|\mathbf{Q}_t^{(\mu)}\|_{1\to\infty} \leqslant c_\mu t^{-\frac{\lambda_1-\mu}{\lambda_1-\mu-1}},$$

where c_μ is the Sobolev constant for $\mathbf{Q}^{(\mu)}$ of exponent $p = \frac{2(\lambda_1-\mu)}{(\lambda_1-\mu-1)}$, i.e.

$$c_\mu = \inf_{f\in\mathcal{D}(\mathbf{J})} \left\{ \frac{\|f\|_2^2 - \|f\|_p^2}{\beta_\mu(f\mathbf{J}_\mu f)};\ f, \mathbf{J}_\mu f \neq 0 \right\}.$$

The value of c_μ is known exactly in the case $\mu = \frac{\lambda_1}{2}$ and upper and lower bounds are known in the general case, see again [8].

CHAPTER 5

SELF-SIMILAR CAUCHY PROBLEMS AND GENERALIZED MITTAG-LEFFLER FUNCTIONS

5.1 Introduction

The fractional Caputo derivative of order $\alpha \in (0, 1)$ which is usually defined in terms of the additive convolution operator $*$ and the function $h_\alpha(y) = \frac{y^{-\alpha}}{\Gamma(1-\alpha)}$, $y > 0$, as follows

$$\frac{{}^C d^\alpha}{dt^\alpha} f(t) = f' * h_\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(y)}{(t-y)^\alpha} dr, \tag{5.1.1}$$

plays a central and growing role in various contexts, see e.g. [141, 148]. In particular, in analysis, it appears in the fractional Cauchy problem, where one replaces the derivative of order 1 by the fractional one, i.e. $\frac{{}^C d^\alpha}{dt^\alpha} f = \mathbf{L} f$, with $\mathbf{L}$ the infinitesimal generator of a strong Markov process X, see [158] for the introduction of this problem in relation to some Hamiltonian chaotic dynamics of particles given in terms of stable processes.

Bauemer and Meerschaert in [6] showed the intriguing fact that the solution of this problem admits a stochastic representation which is given in terms of a non-Markovian process defined as the Markov process X time-changed by the inverse of an α-stable subordinator. This offers another fascinating connection between stochastic and functional analysis. Observing that the mapping h_α is the tail of the Lévy measure of this stable subordinator, it is then natural to generalize the fractional operator as an additive convolution operator by replacing the function h_α with the tail of the Lévy measure of any subordinator. It turns out that this interesting program has been developed recently by Toaldo [152] and the corresponding generalized fractional Cauchy problem has, when this tail has infinite mass, a similar stochastic representation where the time-changed process is the inverse of the subordinator, see [152, 36].

Another important feature of the fractional Caputo derivative is its self-similarity property

$$\frac{{}^C d^\alpha}{dt^\alpha} d_c f(t) = c^\alpha \frac{{}^C d^\alpha}{dt^\alpha} f(ct), \quad c, t > 0, \tag{5.1.2}$$

where $d_c f(t) = f(ct)$ is the dilation operator. It is not difficult to convince yourself that this property follows from the homogeneity property of the function h_α which itself is inherited from the scaling property of the α-stable subordinator, and thus it does not hold for any $*$-convolution operators associated to any other subordinators. This property is appealing from a modelling viewpoint as it has been observed in many physical and economics phenomena [60] and is also central in (non-trivial) limit theorems for any properly normalized stochastic processes, see [94]. Two questions then arise naturally:

1. Can one define a class of linear operators enjoying the same self-similarity property (5.1.2) as the fractional derivative?
2. If yes, can one find a stochastic representation for the solution of the corresponding self-similar Cauchy problem?

The aim of this paper is to provide a positive and detailed answer to each of these questions. For (1), we observe that the fractional derivative (5.1.1) admits also the representation as a multiplicative convolution operator

$$f' * h_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \int_0^t f'(y) \left(1 - \frac{y}{t}\right)^{-\alpha} dy = t^{-\alpha} f' \diamond g_\alpha(t) \tag{5.1.3}$$

where $g_\alpha(r) = \frac{(1-r)^{-\alpha}}{\Gamma(1-\alpha)}, r \in (0,1)$, and $\diamond$ stands for the multiplicative convolution operator, i.e. for two functions f and g, $f \diamond g(t) = \int_0^t f(r) g\left(\frac{r}{t}\right) dr$. It is then not difficult to show that the self-similarity property (5.1.2) holds for any $\diamond$-convolution operator of the form (5.1.3) by replacing g_α by any measurable function m on $(0, 1)$.

The answer to the question (2) is more subtle. Indeed, we first have to realize that the mapping $y \mapsto g_\alpha(e^{-y})$ is the tail of the Lévy measure on $\mathbb{R}_+$ of a specific subordinator. Then, we identify this subordinator as the Lévy process which is associated, via the Lamperti transform defined in (5.2.3) below, to the stable subordinator seen as an increasing positive self-similar Markov process. This leads us to naturally introduce and study the class of multiplicative convolution operators which is in bijection with the set of subordinators, by considering the measurable functions m such that $y \mapsto m(e^{-y})$ is the tail of a Lévy measure of a subordinator. Under mild conditions on m, we show that the Laplace transform of the process ζ, the right-inverse of the increasing positive self-similar Markov process χ, associated via the Lamperti mapping to the subordinator with Lévy measure given in terms of m, is an eigenfunction for the corresponding multiplicative convolution operator. We also show that the Laplace transform of the ζ's, admits a series representation generalizing the Mittag-Leffler function, which is well-known to be the eigenfunction of the Caputo fractional operator (5.1.3). Moreover, in this comprehensive context, we are able to also provide a Mellin-Barnes integral representation of this Laplace transform, involving the so-called Bernstein-gamma functions recently introduced by Patie and Savov, see [128], from which we infer an exact asymptotic equivalent.

By means a spectral theoretical approach, we proceed by showing that the expectation operator associated to a strong Markov process X time-changed by ζ, is the solution to what we name a self-similar Cauchy problem, where the classical time derivative is replaced by the self-similar multiplicative convolution operator defined in terms of m. We mention that such a time-change has already been used in Loeffen and al. [102] to provide detailed distributional properties of the extinction time of some real-valued non-Markovian self-similar processes.

We now recall that fractional calculus, which defines and studies derivatives and integrals of fractional order, has been applied in various areas of engineering, science, finance, applied mathematics, and bio-engineering, see the monographs [103, 107]. This seems to be attributed to the non-locality of fractional operators which provide a powerful tool for a description of memory and hereditary properties of different substances, see e.g. Liu et al. [101] and Podlubny [135]. Equations of fractional order also appear in a lot of physical phenomena, see e.g. Meerschaert and Sikorskii [107], and in particular for modeling anomalous diffusions, see e.g. Benson et al. [14] and D'Ovidio [56]. On the other hand, the fractional Cauchy problems, for which the integer time derivative is replaced by its fractional counterpart, i.e. $\frac{{}^C d^\alpha}{dt^\alpha} f = \mathbf{L} f$, has a nice stochastic interpretation as it is related to the inverse of a stable subordinator, a connection that was explored by many authors, see e.g. Baeumer and Meerschaert [6], Meerschaert et al. [104], Saichev and Zaslavsky [140], Zaslavsky [159], among others. From this stochastic viewpoint, this gives rise to subdiffusive dynamics which appear in some important limit theorems such as the scaling limit of continuous-time random walks (in which the i.i.d. jumps are separated by i.i.d. waiting times) as in Meerschaert and Scheffler [106] and also the (surprising) intermediate time behaviour of some periodic diffusive flows which gives rise to the fractional kinetic process in Hairer et al. [77].

The rest of this paper is organized as follows. In Section 5.2, we study some of the substantial properties of the inverse of an increasing self-similar Markov process. In Section 5.3, we introduce a self-similar multiplicative convolution generalization of the fractional Caputo derivative. In Section 5.4, we study the corresponding self-similar Cauchy problem and provide the stochastic representation of its solution. Finally, to illustrate some examples, Section 5.5 considers families of some self-adjoint, as well as, some non-local and non-self-adjoint Markov semigroups.

5.2 Inverse of increasing self-similar Markov processes

Let $\chi = (\chi_t)_{t\geq 0}$ be a non-decreasing self-similar Markov process of index $\alpha \in (0,1)$ issued from 0 and denote by $\zeta = (\zeta_t)_{t\geq 0}$ its right-inverse, that is, for any $t \geq 0$,

$$\zeta_t = \inf\{s > 0;\ \chi_s > t\}. \tag{5.2.1}$$

Denoting the law of the process by $\mathbb{P}_x$ when starting from $x > 0$, we say that a stochastic process χ is self-similar of index α (or α-self-similar) if the following identity

$$(c\chi_{c^{-\alpha}t}, \mathbb{P}_x)_{t\geq 0} \overset{d}{=} (\chi_t, \mathbb{P}_{cx})_{t\geq 0} \tag{5.2.2}$$

holds in the sense of finite-dimensional distribution for any $c > 0$. Now, we recall that Lamperti [95] identifies a one-to-one mapping between the class of positive self-similar Markov processes and the one of Lévy processes. In particular, one has, under $\mathbb{P}_x$, $x > 0$, that

$$\chi_t = x\exp\left(\mathcal{T}_{A_{x^{-\alpha}t}}\right), \quad t \geq 0, \tag{5.2.3}$$

where $A_t = \inf\{s > 0;\ \int_0^s \exp(\alpha\mathcal{T}_r)dr > t\}$. Here $\mathcal{T}$ is a subordinator, that is a non-decreasing stochastic process with stationary and independent increments and càdlàg sample paths, and thus its law is characterized by the Bernstein function $\phi(u) = -\log\mathbb{E}[e^{-u\mathcal{T}_1}], u \geq 0$, which in this case, for sake of convenience in the later discussion, is expressed for any $u \geq 0$, as

$$\phi(u) = bu + u\int_0^1 r^{u-1}m(r)dr = bu + u\int_0^\infty e^{-uy}m(e^{-y})dy, \tag{5.2.4}$$

where $b \geq 0$ and $r \mapsto m(r)$ is a non-decreasing function on $(0,1)$ and $\int_0^1 (-\ln r \wedge 1) r m(dr) < +\infty$, where $m(r) = \int_0^y m(ds)$, $r \in (0,1)$. Note that under this condition, the mapping $y \mapsto m(e^{-y})$ defined on $\mathbb{R}_+$, is the tail of a Lévy measure of a subordinator. Furthermore, to ensure that χ can be started from 0, we assume further that

$$\mathbb{E}[\mathcal{T}_1] = \phi'(0^+) = b + \int_0^1 \frac{m(r)}{r} dr < +\infty, \tag{5.2.5}$$

see [16, Theorem 1]. Then, we denote the set of Bernstein functions that satisfy this condition by

$$\mathbf{B} = \{\phi \text{ of the form (5.2.4) such that } \phi'(0^+) < +\infty\}.$$

We shall also need the constant

$$\mathfrak{a}_\phi = \sup\{u \leq 0; |\phi(u)| = \infty\} \in (-\infty, 0]. \tag{5.2.6}$$

Note that by [144, Theorem 25.17] and after performing an integration by parts, we have that $\int_0^A r^{\mathfrak{a}_\phi - 1} m(r) dr < \infty$ for some $A \in (0,1)$. Moreover, the same result also yields that ϕ admits an analytical extension to the half-plane $\{z \in \mathbb{C};\ \Re(z) > \mathfrak{a}_\phi\}$. Next, we recall from [95, Theorem 6.1] that the characteristic operator of χ is given for at least functions f such that $f, tf' \in C_b(\mathbb{R}_+)$, the space of continuous and bounded functions on $\mathbb{R}_+$, by

$$\mathbf{A} f(t) = t^{-\alpha} \left(b t f'(t) + \int_0^\infty (f(te^y) - f(t)) m(de^{-y}) \right), \tag{5.2.7}$$

where $m(de^{-y})$ stands for the image of the measure $m(dy)$ by the mapping $y \mapsto e^{-y}$. Next, since χ has a.s. non-decreasing sample paths, this entails that the paths of ζ, as its right-inverse, are a.s. non-decreasing. Moreover, they are continuous if and only if the ones of χ are a.s. increasing which from the Lamperti mapping in (5.2.3) is equivalent to $\mathcal{T}$ being a.s. increasing. This is well known, see e.g. [93, Section 5], to be the case when the latter is not a compound Poisson process, that is when

$$\phi(\infty) = \infty \Longleftrightarrow b > 0 \text{ or } \int_A^1 dm(r) = \infty, \tag{5.2.8}$$

for some $A \in (0,1)$. We also define a subset of $\mathbf{B}$ which will be useful in the sequel, as follows

$$\mathbf{B}_{\partial^\circ} = \{\phi \in \mathbf{B};\ \mathfrak{a}_\phi \leq -\alpha \text{ and } \lim_{u\downarrow 0} u\phi(u-\alpha) = 0\}.$$

Note that if $\mathfrak{a}_\phi < -\alpha$, then we always have $\lim_{u\downarrow 0} u\phi(u-\alpha) = 0$. We refer to the monograph [93] for a nice account on Lévy processes. Now, for any $\phi \in \mathbf{B}$ we consider the function W_ϕ which is the unique positive-definite function, i.e. the Mellin transform of a positive measure, that solves the functional equation, for $\Re(z) > \mathfrak{a}_\phi$,

$$W_\phi(z+1) = \phi(z)W_\phi(z), \quad W_\phi(1) = 1. \tag{5.2.9}$$

It is easily checked that for any integer n, $W_\phi(n+1) = \prod_{k=1}^n \phi(k)$, see [128] for a thorough study of this functional equation. Throughout, for a random variable X, we use the notation

$$\mathcal{M}_X(z) = \mathbb{E}[X^z]$$

for at least any $z \in i\mathbb{R}$, the imaginary line, meaning that $\mathcal{M}_X(z-1)$ is its Mellin transform. Next, we recall that for any integrable function f on $(0,\infty)$, its Mellin transform is defined by

$$\widehat{f}(z) = \int_0^\infty q^{z-1} f(q)dq,$$

for any complex z such that this integrable is finite. We also recall that χ is the Lamperti process of index $\alpha \in (0,1)$ associated to the Bernstein function $\phi \in \mathbf{B}$, and we denote by $\zeta = (\zeta_t)_{t\geq 0}$ its right-inverse, see (5.2.1). We recall that ζ was used in [102] as a time changed of self-similar Markov processes in the investigation of their extinction time. We are now ready to gather some substantial properties of ζ.

Proposition 5.2.1. Let $\phi \in \mathbf{B}$, $\alpha \in (0,1)$, and write, for any $u \geq 0$, $\phi_\alpha(u) = \phi(\alpha u) \in \mathbf{B}$.

(i) For any $t > 0$ and $z \in \mathbb{C}$,

$$\mathcal{M}_{\zeta_t}(z) = \frac{t^{z\alpha}}{\phi'_\alpha(0^+)} \frac{\Gamma(z)}{W_{\phi_\alpha}(z)}. \tag{5.2.10}$$

In particular, for any $t > 0$, $z \mapsto \mathcal{M}_{\zeta_t}(z)$ in analytical on the half-plane $\Re(z) > \mathfrak{a} = \mathfrak{a}_\phi \vee -1 = \inf\{u > -1; |\phi(u)| = \infty\} \in (-1, 0]$.

(ii) ζ is $\frac{1}{\alpha}$-self-similar and in particular, for all $q, t > 0$ $\mathbb{E}[e^{-q\zeta_t}] = \mathbb{E}[e^{-qt^\alpha \zeta_1}]$. Moreover, for any $|q| < \phi(\infty)$,

$$\mathbb{E}[e^{-q\zeta_1}] = \mathcal{E}_{\phi_\alpha}(e^{i\pi} q) = \frac{1}{\phi'_\alpha(0^+)} \sum_{n=0}^{\infty} (-1)^n \frac{q^n}{n W_{\phi_\alpha}(n)} \tag{5.2.11}$$

where $\mathcal{E}_{\phi_\alpha}$ extends to an analytical function on $\mathbb{D}_{\phi(\infty)} = \{z \in \mathbb{C}; |z| < \phi(\infty)\}$. Consequently, the law of ζ_t is, for all $t > 0$, moment determinate. Moreover, as a Laplace transform of a Radon measure, the mapping $q \mapsto \mathcal{E}_{\phi_\alpha}(e^{i\pi} q)$ is, when $\phi(\infty) = \infty$, completely monotone, i.e. infinitely continuously differentiable with $(-1)^n \frac{d^n}{dq^n} \mathcal{E}_{\phi_\alpha}(e^{i\pi} q) \geq 0$.

(iii) Furthermore, if $\mathfrak{a}_{\phi_\alpha} < 0$, then writing $\mathfrak{a} = \mathfrak{a}_{\phi_\alpha} \vee -1$, $\mathcal{E}_{\phi_\alpha}$ admits the following Mellin-Barnes integral representation, for any $0 < a < |\mathfrak{a}|$,

$$\mathcal{E}_{\phi_\alpha}(z) = -\frac{1}{\phi'_\alpha(0^+)} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\phi_\alpha(\xi)}{\xi} \frac{\Gamma(\xi)\Gamma(1-\xi)}{W_{\phi_\alpha}(\xi+1)} (-z)^\xi d\xi \tag{5.2.12}$$

which is absolutely convergent (at least) on the sector $\{z \in \mathbb{C}; |\arg(-z)| < \frac{\pi}{2}\}$ and $q \mapsto \mathcal{E}_{\phi_\alpha}(e^{i\pi} q) \in C_0^\infty(\mathbb{R}_+)$, the space of infinitely continuously differentiable functions on $\mathbb{R}_+$ vanishing at infinity along with their derivatives.

(iv) Finally, assume that ϕ_α is meromorphic on the half-plane $\Re(z) > -\mathfrak{p} - \epsilon$ for some $\epsilon > 0$ with a unique and simple pole at $-\mathfrak{p}$. Then, if $\mathfrak{p} \in \mathbb{N}$ (resp. $\mathfrak{p} \notin \mathbb{N}$) and $0 < \left|\lim_{z\to 0} \prod_{k=0}^{\mathfrak{p}} \phi_\alpha(z-k)\right| < \infty$ (resp. $0 < |C_\mathfrak{p} = \lim_{z\to\mathfrak{p}} (z-\mathfrak{p})\phi_\alpha(-z)| < \infty$), then

$$\mathcal{E}_{\phi_\alpha}(e^{i\pi} q) \overset{+\infty}{\sim} \frac{C(\mathfrak{p})}{\phi'_\alpha(0^+)} q^{-\mathfrak{p}},$$

where $C(\mathfrak{p}) = \frac{(-1)^\mathfrak{p}}{\mathfrak{p} W_{\phi_\alpha}(-\mathfrak{p})}$ (resp. $C(\mathfrak{p}) = \frac{\Gamma(\mathfrak{p})\Gamma(-\mathfrak{p})}{W_{\phi_\alpha}(1-\mathfrak{p})} C_\mathfrak{p}$), and where for two functions f and g we write $f \overset{a}{\sim} g$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$.

We proceed by showing that the class of functions $\mathcal{E}_{\phi_\alpha}$, which is in bijection with the set of Bernstein functions $\mathbf{B}$, encompasses some famous special functions such as the Mittag-Leffler one and some q-series.

Example 5.2.1 (Mittag-Leffler function). It turns out that the function $\mathcal{E}_{\phi_\alpha}$ is a generalization of the Mittag-Leffler function. Indeed, recall that $\alpha \in (0,1)$ and define

$$\phi_\alpha(z) = \frac{\Gamma(\alpha + \alpha z)}{\Gamma(\alpha z)}, \quad \Re(z) > -1,$$

which is a Bernstein function, see e.g. Loeffen et al. [102]. Furthermore, since $\phi'(0^+) = \Gamma(\alpha) < \infty$, we have $\phi \in \mathbf{B}$. Then, an easy algebra yields that $W_{\phi_\alpha}(z) = \frac{\Gamma(\alpha z)}{\Gamma(\alpha)}$, $\Re(z) > 0$, with $W_{\phi_\alpha}(1) = 1$. Therefore, by means of Proposition 5.2.1 and the recurrence relation of the gamma function, one gets, for $q \in \mathbb{R}$ and $t > 0$,

$$\mathcal{E}_{\phi_\alpha}(q) = \frac{1}{\phi'_\alpha(0^+)} \sum_{n=0}^{\infty} \frac{q^n \Gamma(\alpha)}{n\Gamma(\alpha n)} = \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(\alpha n + 1)} = \mathcal{E}_\alpha(q),$$

where $\mathcal{E}_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$, $z \in \mathbb{C}$ is the Mittag-Leffler function. (5.2.12) yields that $\mathcal{E}_\alpha$ admits the following Mellin-Barnes integral representation, for any $0 < a < 1$,

$$\begin{aligned}\mathcal{E}_\alpha(z) &= -\frac{1}{\phi'_\alpha(0^+)} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\phi_\alpha(\xi)}{\xi} \frac{\Gamma(\xi)\Gamma(1-\xi)}{W_{\phi_\alpha}(1-\xi)} (-z)^\xi d\xi \\ &= -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\xi)\Gamma(1-\xi)}{\Gamma(1-\alpha\xi)} (-z)^\xi d\xi,\end{aligned}$$

where we use the Stirling formula of the gamma function, recalled in (5.2.16) below, to obtain that this integral is absolutely convergent on the sector $\{z \in \mathbb{C}; |\arg(z)| < (2-\alpha)\frac{\pi}{2}\}$. Next, since the gamma function is a meromorphic function with simple poles at the non-positive integers, and $z \mapsto 1/\Gamma(z)$ is an entire function, we have that ϕ_α has a pole at -1 and it is meromorphic on $\Re(z) > -1-\epsilon$ for some $\epsilon > 0$. Furthermore, $0 < |\lim_{z\to 0} \phi_\alpha(z)\phi_\alpha(z-1)| = \left|\lim_{z\to 0} \frac{\Gamma(\alpha z+\alpha)}{\Gamma(\alpha z-\alpha)}\right| = \left|\frac{\Gamma(\alpha)}{\Gamma(-\alpha)}\right| < \infty$. Thus, the conditions of Proposition 5.2.1 are satisfied with $\mathfrak{p} = 1$, and it yields that for any $q, t > 0$,

$$\mathcal{E}_{\phi_\alpha}(e^{i\pi}q) \overset{+\infty}{\sim} \frac{q^{-1}}{\Gamma(1-\alpha)},$$

which is the well-known asymptotic behavior of the Mittag-Leffler function, see e.g. Gorenflo et al. [71, Chapter 3].

Example 5.2.2 ($\mathfrak{q}$-series). Let now ϕ be the Laplace exponent of a Poisson process of parameter $\log \mathfrak{q}$, $0 < \mathfrak{q} < 1$, that is $\phi(u) = 1 - \mathfrak{q}^u$, $u \geq 0$, which admits an extension as an entire function. Next, introducing the following notation from the $\mathfrak{q}$-calculus, $(a; \mathfrak{q})_n = \prod_{k=0}^{n-1}(1 - a\mathfrak{q}^k)$, see [66], and observing that $W_{\phi_\alpha}(n + 1) = (\mathfrak{q}^\alpha; \mathfrak{q}^\alpha)_n$, with n an integer, we get that, for $|z| < \phi(\infty) = 1$,

$$\mathcal{E}_{\phi_\alpha}(z) = \frac{1}{\alpha|\ln \mathfrak{q}|} \sum_{n=0}^{\infty} \frac{1 - \mathfrak{q}^n}{n} \frac{z^n}{(\mathfrak{q}^\alpha; \mathfrak{q}^\alpha)_n}.$$

Proof. For any bounded Borelian function f, we have

$$\begin{aligned} \mathbb{E}[f(\zeta_t)] &= \mathbb{E}[f(t^\alpha \zeta_1)] = \int_0^\infty f(t^\alpha s)\mathbb{P}(\zeta_1 \in ds) \\ &= \alpha \int_0^\infty s^{-\alpha-1} f(t^\alpha s)\mathbb{P}(\chi_1 \in ds^{-\alpha}) \\ &= \int_0^\infty f((t/u)^\alpha)\mathbb{P}(\chi_1 \in du) = \mathbb{E}[f\,(t^\alpha \chi_1^{-\alpha})], \end{aligned} \tag{5.2.13}$$

where we used the identities $\mathbb{P}(\zeta_1 \leq s) = \mathbb{P}(\chi_s \geq 1) = \mathbb{P}(\chi_1 \geq s^{-\alpha})$. Then, according to [128, Theorem 2.24], we deduce that for any $\Re(z) > 0$,

$$\mathcal{M}_{\zeta_t}(z) = \mathbb{E}[\zeta_t^z] = t^{\alpha z}\mathbb{E}[\chi_1^{-z\alpha}] = \frac{t^{\alpha z}}{\phi'_\alpha(0^+)} \frac{\Gamma(z)}{W_{\phi_\alpha}(z)}. \tag{5.2.14}$$

Therefore, in particular, $z \mapsto \mathcal{M}_{\zeta_t}(z)$ is analytical on $\Re(z) > \mathfrak{a}$, since using (5.2.9) and the recurrence property of the gamma function, we have

$$\frac{\Gamma(z)}{\alpha W_{\phi_\alpha}(z)} = \frac{\Gamma(z + 1)}{W_{\phi_\alpha}(z + 1)} \frac{\phi_\alpha(z)}{\alpha z},$$

and $\lim_{u\downarrow 0} \frac{\phi_\alpha(u)}{\alpha u} = \phi'(0^+) < \infty$. Next, by an expansion of the exponential function combined with an application of a standard Fubini argument, the identity (5.2.14) and the recurrence relation for the gamma function, one gets

$$\mathbb{E}\left[e^{q\zeta_1}\right] = \sum_{n=0}^{\infty} \mathbb{E}[\zeta_1^n]\frac{q^n}{n!} = \frac{1}{\phi'_\alpha(0^+)} \sum_{n=0}^{\infty} \frac{1}{n} \frac{q^n}{W_{\phi_\alpha}(n)} = \mathcal{E}_{\phi_\alpha}(q),$$

where, by using the functional equation (5.2.9), the series is easily checked to be absolutely convergent, and hence an analytical function, on $\{z \in \mathbb{C};\ |z| < \phi(\infty)\}$. Then, admitting exponential moments, the law of ζ_t is moment-determinate for all $t > 0$. Next, since χ is an α-self-similar process, by (5.2.1), plainly ζ is $\frac{1}{\alpha}$-self-similar. To derive the Mellin-Barnes integral representation of $\mathcal{E}_{\phi_\alpha}$, we first observe from (5.2.14) that the mapping

$$z \mapsto \mathbb{E}[\zeta_1^z] = \frac{1}{\phi_\alpha'(0^+)} \frac{\Gamma(z)}{W_{\phi_\alpha}(z)}$$

is analytical on $\Re(z) > 0$ since $z \mapsto \Gamma(z)$ and $z \mapsto W_{\phi_\alpha}(z)$ are analytical on $\Re(z) > 0$, and the latter is also zero-free on the same half-plane, see [128, Theorem 4.1]. Next, let us assume that $\mathfrak{a}_{\phi_\alpha} < 0$, and observe, using (5.2.9), that

$$\int_0^\infty \mathbb{E}[e^{-q\zeta_1}] q^{\xi-1} dq = \mathbb{E}[\zeta_1^{-\xi}]\Gamma(\xi) = \frac{\Gamma(\xi)\Gamma(-\xi)}{\phi_\alpha'(0^+) W_{\phi_\alpha}(-\xi)} = \frac{1}{\phi_\alpha'(0^+)} \frac{\phi_\alpha(-\xi)}{-\xi} \frac{\Gamma(\xi)\Gamma(1-\xi)}{W_{\phi_\alpha}(1-\xi)},$$

which is analytical on $0 < \Re(\xi) < |\mathfrak{a}| = |\mathfrak{a}_{\phi_\alpha} \vee -1|$. Indeed, first, since $\xi \mapsto \Gamma(\xi)$ is analytical on the right half-plane $\Re(\xi) > 0$, plainly, $\xi \mapsto \Gamma(\xi)\Gamma(1-\xi)$ is analytical on $0 < \Re(\xi) < 1$. Next, as above, we have that $\xi \mapsto W_{\phi_\alpha}(1-\xi)$ is analytical and is zero-free on $\Re(\xi) < 1$, and we get the sought analyticity from the definition of $\mathfrak{a}_{\phi_\alpha}$. We write

$$\widehat{\mathcal{E}}^*_{\phi_\alpha}(\xi) = \frac{1}{\phi_\alpha'(0^+)} \frac{\phi_\alpha(-\xi)}{-\xi} \frac{\Gamma(\xi)\Gamma(1-\xi)}{W_{\phi_\alpha}(1-\xi)}. \tag{5.2.15}$$

Next, we recall that the Stirling's formula yields that for any $a \in \mathbb{R}$ fixed, when $|b| \to \infty$,

$$|\Gamma(a+ib)| \overset{\infty}{\sim} C_a |b|^{a-\frac{1}{2}} e^{-|b|\frac{\pi}{2}}, \tag{5.2.16}$$

where $C_a > 0$, see e.g. [129, Lemma 9.4]. Furthermore, [129, Proposition 6.12(2)] gives that for any $a > 0$,

$$\overline{\lim}_{|b|\to\infty} \frac{e^{-|b|\frac{\pi}{2}} |b|^{-\frac{1}{2}}}{|W_{\phi_\alpha}(a+bi)|} \leq c_+(a), \tag{5.2.17}$$

for some positive finite constant $c_+(a)$. Therefore, taking $\xi = a + ib$ for any $b \in \mathbb{R}$ and $0 < a < |\mathfrak{a}|$, using (5.2.16) and (5.2.17), there exists $\tilde{C}_a > 0$ such that for a fixed and $|b|$ large

$$|\widehat{\mathcal{E}}^*_{\phi_\alpha}(\xi)| = \left| \frac{\phi_\alpha(-\xi)}{-\xi} \frac{\Gamma(\xi)\Gamma(1-\xi)}{W_{\phi_\alpha}(\xi+1)} (-z)^{-\xi} \right| \leq \tilde{C}_a\, |b|^{2a-\frac{1}{2}} e^{-|b|\frac{\pi}{2}}, \tag{5.2.18}$$

where we used the upper bound of ϕ found in [128, Proposition 3.1]. Thus, by Mellin's inversion formula, see e.g. [118, Chapter 11], one gets that for any $0 < a < |\mathfrak{a}|$,

$$\mathbb{E}[e^{-z\zeta_1}] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \widehat{\mathcal{E}}^*_{\phi_\alpha}(\xi)(-z)^{-\xi}d\xi,$$

and thus by uniqueness of analytical extension, we get that

$$\mathcal{E}_{\phi_\alpha}(z) = \frac{1}{\phi'_\alpha(0^+)}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\phi_\alpha(-\xi)}{-\xi}\frac{\Gamma(\xi)\Gamma(1-\xi)}{W_{\phi_\alpha}(\xi+1)}(-z)^{-\xi}d\xi, \tag{5.2.19}$$

which is a function analytical on the sector $\{z \in \mathbb{C};\ |\arg(-z)| < \frac{\pi}{2}\}$. Indeed, first, by the discussion above, we have that $\xi \mapsto \widehat{\mathcal{E}}^*_{\phi_\alpha}(\xi)$ is analytical on the strip $0 < \Re(\xi) < |\mathfrak{a}|$. Next, taking $\xi = a + ib$, using (5.2.9) and (5.2.18), we have that when $|b|$ is large, there exists a constant $\tilde{C}_a > 0$ such that

$$\left|\widehat{\mathcal{E}}^*_{\phi_\alpha}(\xi)(-z)^{-\xi}\right| \leq \tilde{C}_a\, |z|^{-a}\, |b|^{2a-\frac{1}{2}} e^{-|b|\frac{\pi}{2}+b\arg(-z)}. \tag{5.2.20}$$

Putting pieces together, we indeed get the claimed analytical property of $\mathcal{E}_{\phi_\alpha}$. Now, to study the asymptotic behavior of $\mathcal{E}_{\phi_\alpha}$, we write, for $q > 0$,

$$\mathcal{E}_{\phi_\alpha}(e^{i\pi}q) = \frac{1}{\phi'_\alpha(0^+)}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(\xi)\Gamma(-\xi)}{W_{\phi_\alpha}(-\xi)}q^{-\xi}d\xi, \tag{5.2.21}$$

recall that the gamma function has simple poles at non-positive integers, and investigate the poles of $f_\phi(\xi) = \frac{\Gamma(\xi)}{W_{\phi_\alpha}(\xi)}$. Using (5.2.9), we get that f_ϕ satisfies to the following functional equation,

$$f_\phi(\xi+1) = \frac{\xi}{\phi_\alpha(\xi)}f_\phi(\xi).$$

Next, since $0 < \phi'(0^+) < \infty$, we have that $0 < \lim_{\xi\to 0}\frac{\xi}{\phi_\alpha(\xi)} < \infty$. Moreover, since $\frac{1}{\phi_\alpha}$ is the Laplace transform of a positive measure whose support is contained in $[0,\infty)$, see e.g. [129, Proposition 4.1(4)], it has its singularities on the negative real line. Thus, $\mathfrak{s} < 0$ is a pole for f_ϕ if $\phi_\alpha(\mathfrak{s}) = \infty$. Next, since ϕ_α is meromorphic on $\Re(\xi) > -\mathfrak{p} - \epsilon$ with a unique pole at $-\mathfrak{p}$ with $\mathfrak{p} > 0$, we can extend the domain of analiticity of W_{ϕ_α} on $\Re(\xi) > -\mathfrak{p} - \epsilon$, and by Cauchy's theorem, we have

$$\mathcal{E}_{\phi_\alpha}(e^{i\pi}q) = \frac{1}{\phi'_\alpha(0^+)} Res(\eta, \mathfrak{p}) + \frac{1}{\phi'_\alpha(0^+)} \frac{1}{2\pi i} \int_{\mathfrak{p}+\epsilon-i\infty}^{\mathfrak{p}+\epsilon+i\infty} \frac{\Gamma(\xi)\Gamma(-\xi)}{W_{\phi_\alpha}(-\xi)} q^{-\xi} d\xi. \tag{5.2.22}$$

Next, if $\mathfrak{p} \in \mathbb{N}$, then we have

$$W_{\phi_\alpha}(\xi - \mathfrak{p}) = \frac{W_{\phi_\alpha}(\xi + 1)}{\prod_{k=0}^{\mathfrak{p}} \phi_\alpha(\xi - k)}, \quad \Re(\xi) > 0.$$

Hence, since $W_{\phi_\alpha}(1) = 1$, we deduce that

$$0 < \lim_{\xi \to 0} |W_{\phi_\alpha}(\xi - \mathfrak{p})| = |W_{\phi_\alpha}(-\mathfrak{p})| = \frac{1}{\left|\lim_{\xi \to 0} \prod_{k=0}^{\mathfrak{p}} \phi_\alpha(\xi - k)\right|} < \infty,$$

and since $Res(\Gamma, -n) = \frac{(-1)^n}{n!}$, $n = 1, 2, \cdots$, we have

$$Res(\mathcal{E}_{\phi_\alpha}, \mathfrak{p}) = \frac{\Gamma(\mathfrak{p})}{W_{\phi_\alpha}(-\mathfrak{p})} \frac{(-1)^{\mathfrak{p}}}{\mathfrak{p}!} q^{-\mathfrak{p}} = \frac{(-1)^{\mathfrak{p}}}{\mathfrak{p} W_{\phi_\alpha}(-\mathfrak{p})} q^{-\mathfrak{p}}.$$

Therefore, combining this with (5.2.22), we obtain

$$\mathcal{E}_{\phi_\alpha}(e^{i\pi}q) \overset{+\infty}{\sim} \frac{(-1)^{\mathfrak{p}}}{\phi'_\alpha(0^+) \mathfrak{p} W_{\phi_\alpha}(-\mathfrak{p})} q^{-\mathfrak{p}}.$$

Otherwise, if $\mathfrak{p} \notin \mathbb{N}$, we have

$$\lim_{\xi \to \mathfrak{p}} (\xi - \mathfrak{p}) \frac{\Gamma(-\xi)}{W_{\phi_\alpha}(-\xi)} = \Gamma(-\mathfrak{p}) \lim_{\xi \to \mathfrak{p}} \frac{(\xi - \mathfrak{p})\phi_\alpha(-\xi)}{W_{\phi_\alpha}(1 - \xi)} = -\frac{1}{\mathfrak{p}} \frac{\Gamma(1 - \mathfrak{p})}{W_{\phi_\alpha}(1 - \mathfrak{p})} \lim_{\xi \to \mathfrak{p}} (\xi - \mathfrak{p})\phi_\alpha(-\xi),$$

which is finite since $\frac{\Gamma(1-\mathfrak{p})}{W_{\phi_\alpha}(1-\mathfrak{p})} < \infty$ as $-\mathfrak{p}$ was the first pole of f_ϕ, and by assumption $0 < |C_{\mathfrak{p}} = \lim_{\xi \to \mathfrak{p}} (\xi - \mathfrak{p})\phi_\alpha(-\xi)| < \infty$. Hence,

$$Res(\mathcal{E}_{\phi_\alpha}, \mathfrak{p}) = \frac{\Gamma(\mathfrak{p})\Gamma(-\mathfrak{p})}{W_{\phi_\alpha}(1 - \mathfrak{p})} C_{\mathfrak{p}}\, q^{-\mathfrak{p}},$$

and with (5.2.22), we get

$$\mathcal{E}_{\phi_\alpha}(e^{i\pi}q) \overset{+\infty}{\sim} \frac{\Gamma(\mathfrak{p})\Gamma(-\mathfrak{p})}{\phi'_\alpha(0^+) W_{\phi_\alpha}(1 - \mathfrak{p})} C_{\mathfrak{p}}\, q^{-\mathfrak{p}}.$$

To concludes the proof, we use the estimate (5.2.18) to apply the Riemann-Lebesgue lemma to get

$$\lim_{q \to \infty} \frac{q^{-\mathfrak{p}}}{\phi'_\alpha(0^+)} \int_{\mathfrak{p}+\epsilon-i\infty}^{\mathfrak{p}+\epsilon+i\infty} \frac{\Gamma(\xi)\Gamma(-\xi)}{W_{\phi_\alpha}(-\xi)} q^{-\xi} d\xi = \lim_{q \to \infty} q^{-\epsilon} \int_{-\infty}^{\infty} e^{ib \ln q} \frac{\Gamma(\mathfrak{p} + \epsilon + ib)\Gamma(-\mathfrak{p} - \epsilon - ib)}{\phi'_\alpha(0^+) W_{\phi_\alpha}(-(\mathfrak{p} + \epsilon + ib))} db = 0.$$

■

5.3 Self-similar multiplicative convolution generalization of fractional operators

In this section, we introduce a class of multiplicative convolution operators that generalize the fractional Caputo derivative and provide some interesting properties. In particular, we show that they have the same self-similarity property than the fractional Caputo derivative and we identify conditions under which these operators admit the functions $\mathcal{E}_{\phi_\alpha}$ as eigenfunctions. Inspired by the multiplicative convolution representation of the fractional Caputo derivative presented in (5.1.3), we introduce its generalization as follows. We denote by $AC[0,t]$ the space of absolutely continuous functions on $[0,t]$, $t>0$, and by $L^1(0,t)$ the space of Lebesgue integrable functions on $(0,t)$, $t>0$.

Definition 5.3.0.1. 1) Let m be a non-negative measurable function defined on $(0,1)$, $b \in \mathbb{R}$ and write $\Phi(z) = bz + z\int_0^1 r^{z-1}m(r)dr$ for $z \in \mathbb{C}_\Phi = \{z \in \mathbb{C}; r \mapsto r^{z-1}m(r) \in L^1(0,1)\}$. For $\alpha \in (0,1)$ and $f \in \mathcal{D}(\partial_t^{\diamond\Phi}) = C^1(\mathbb{R}_+) \cap \{f \in AC[0,t]; y \mapsto f'(y)m\left(\frac{y}{t}\right) \in L^1(0,t)\}$, we define

$$\partial_t^{\diamond\Phi} f(t) = t^{1-\alpha} b f'(t) + t^{-\alpha} f' \diamond m(t), \tag{5.3.1}$$

where we recall that $f' \diamond m(t) = \int_0^t f'(r)m\left(\frac{r}{t}\right)dr$.

2) If $\phi \in \mathbf{B}$ is defined by (5.2.4), $b \geq 0$ and $r \mapsto m(r)$ is a non-decreasing function on $(0,1)$ such that $\int_0^1 (-\ln r \wedge 1) r m(dr) < +\infty$, then $\Phi \equiv \phi$ and we write $\partial_t^{\diamond\Phi} = \partial_t^{\diamond\phi}$.

We proceed by providing some substantial properties of these generalized fractional operators.

Proposition 5.3.1. (i) $\partial_t^{\diamond\Phi}$ is a linear operator that satisfies the scaling property

$$\partial_t^{\diamond\Phi} d_c f(t) = c^\alpha \partial_t^{\diamond\Phi} f(ct), \quad c,t > 0.$$

(ii) For any $z \in \mathbb{C}_{\Phi}$ and $t > 0$,

$$\partial_t^{\diamond\Phi} p_z(t) = \Phi(z) p_{z-\alpha}(t). \tag{5.3.2}$$

Consequently, if $\phi \in \mathbf{B}$, then for any $z \in \mathbb{C}_{(\mathfrak{a}_\phi,\infty)}$, we have $\partial_t^{\diamond\phi} p_z(t) = \phi(z) p_{z-\alpha}(t)$. Moreover, let $m_\alpha(r) = r^{-\alpha} m(r)$, $r \in (0,1)$ and for $z \in \mathbb{C}_{\boldsymbol{\phi}} = \{z \in \mathbb{C}; r \mapsto r^{z-1} m_\alpha(r) \in L^1(0,1)\}$, define

$$\boldsymbol{\phi}(z) = \frac{z}{z-\alpha} \Phi(z-\alpha). \tag{5.3.3}$$

Then, for $z \in \mathbb{C}_{\boldsymbol{\phi}}$ and $t > 0$,

$$\partial_t^{\diamond\boldsymbol{\phi}} p_z(t) = \boldsymbol{\phi}(z) p_{z-\alpha}(t). \tag{5.3.4}$$

(iii) Assume that $\phi \in \mathbf{B}_{\partial^\diamond}$. Then, writing $F_q(t) = \mathcal{E}_{\phi_\alpha}(qt^\alpha)$, we have, for any $q \in \mathbb{R}$ and $t > 0$,

$$\partial_t^{\diamond\boldsymbol{\phi}} F_q(t) = q F_q(t). \tag{5.3.5}$$

where, as in (5.3.3), we have set $\boldsymbol{\phi}(z) = \frac{z}{z-\alpha}\phi(z-\alpha)$. Moreover, if in addition $\phi \in \mathbf{B}_{\partial^\diamond}$ is defined by (5.2.4) and $r \mapsto m_\alpha(r) = r^{-\alpha} m(r)$ is a non-decreasing function on $(0,1)$, then the mapping $\boldsymbol{\phi}$ is a Bernstein function, and $\boldsymbol{\phi} \in \mathbf{B}$ if $\mathfrak{a}_\phi < -\alpha$.

(iv) Let $\phi \in \mathbf{B}$. Then, we have the following relation, at least for functions f such that $f, tf' \in C_b(\mathbb{R}_+)$,

$$\partial_t^{\diamond\phi} \Lambda f(t) = -t^{-2\alpha} \Lambda \mathbf{A} f(t), \tag{5.3.6}$$

where $\Lambda f = f \circ \iota$ is an involution defined by $\iota(y) = \frac{1}{y}$, and $\mathbf{A}$ is the characteristic operator, defined in (5.2.7), of the self-similar Markov process associated via the Lamperti mapping with ϕ.

Remark 5.3.1. Note that if $\phi \in \mathbf{B}$ with $\phi(\infty) < \infty$, then (5.3.5) still holds for any $q \in \mathbb{R}$ and $t > 0$, such that $|q|t^\alpha < \phi(\infty)$.

Example 5.3.1. Let χ be an α-stable subordinator, and note that it is also an increasing positive self-similar Markov process. Moreover, the Laplace exponent of the subordinator associated with χ, via the Lamperti mapping, is well known to be $\phi(u) = \frac{\Gamma(u+\alpha)}{\Gamma(u)}$, $u > 0$, see e.g. [102], and note that in this case $\mathfrak{a}_\phi = -\alpha$ with $\lim_{u\downarrow 0} u\phi(u-\alpha) = 0$. Using the integral representation for the ratio of two gamma functions, see e.g. [153, (15)], we can write ϕ as

$$\phi(u) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1-e^{-uy}) \frac{e^{-\alpha y}}{(1-e^{-y})^{\alpha+1}} dy,$$

from where we deduce, since $m(e^{-y})$ is the tail of a Lévy measure, that, for any $y > 0$,

$$m(e^{-y}) = \frac{\alpha}{\Gamma(1-\alpha)} \int_y^\infty \frac{e^{-\alpha r}}{(1-e^{-r})^{\alpha+1}} dr = \frac{\alpha}{\Gamma(1-\alpha)} \int_y^\infty \frac{e^{r}}{(e^{r}-1)^{\alpha+1}} dr = \frac{(e^y-1)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Hence, we have, for any $r \in (0,1)$,

$$m(r) = r^\alpha \frac{(1-r)^{-\alpha}}{\Gamma(1-\alpha)}.$$

Next, noting that $r \mapsto m_\alpha(r) = r^{-\alpha} m(r) = \frac{(1-r)^{-\alpha}}{\Gamma(1-\alpha)}$ is a non-decreasing function on $(0,1)$, item (iii) implies that $\boldsymbol{\phi}$ is a Bernstein function, and we obtain

$$\partial_t^{\circ\boldsymbol{\phi}} f(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \int_0^t f'(y) \left(\frac{y}{t}\right)^{-\alpha} \left(\frac{t}{y}-1\right)^{-\alpha} dr = \frac{{}^C d^\alpha}{dt^\alpha} f(t).$$

Example 5.3.2. Let $\phi(u) = 1 - \mathfrak{q}^u$, $u \geq 0$, be as in Example 5.2.2 with $0 < \mathfrak{q} < 1$. Then, we can write

$$\phi(u) = \int_0^\infty (1-e^{-uy}) \delta_{-\log \mathfrak{q}}(y),$$

where $\delta_{-\log \mathfrak{q}}$ is the Dirac measure supported on $\{-\log \mathfrak{q}\}$. Therefore, as above, we deduce that, for any $y \geq 0$,

$$m(e^{-y}) = \int_y^\infty \delta_{-\log \mathfrak{q}}(r) = \mathbb{1}_{\{y \leq -\log \mathfrak{q}\}}.$$

Thus, a change of variable yields, that for any $r \in (0,1)$,

$$m(r) = \mathbb{1}_{\{\mathfrak{q} \leq r < 1\}}.$$

Therefore, since $r \mapsto r^{-\alpha}m(r)$ is a non-decreasing function, item (iii) implies that $\boldsymbol{\phi}$ is a Bernstein function, and we have

$$\begin{aligned} \partial_t^{\diamond\boldsymbol{\phi}} f(t) &= t^{-\alpha}\int_0^t f'(y)\left(\frac{y}{t}\right)^{-\alpha}\mathbb{1}_{\{y\geq t\mathfrak{q}\}}dy = \int_{t\mathfrak{q}}^t f'(y)y^{-\alpha}dy \\ &= t^{-\alpha}f(t) - (t\mathfrak{q})^{-\alpha}f(t\mathfrak{q}) + \alpha\int_{t\mathfrak{q}}^t f(y)y^{-\alpha-1}dy, \end{aligned}$$

where in the last step we performed an integration by parts.

Proof. First, plainly $\partial_t^{\diamond\Phi}$ is a linear operator and for $c, t > 0$, we note that

$$\begin{aligned} \partial_t^{\diamond\Phi} d_c f(t) &= t^{1-\alpha}bcf'(ct) + t^{-\alpha}\int_0^t cf'(cy)m\left(\frac{y}{t}\right)dy \\ &= c^{\alpha}b(ct)^{1-\alpha}f'(ct) + c^{\alpha}(ct)^{-\alpha}\int_0^{ct} f'(r)m\left(\frac{r}{ct}\right)dr \\ &= c^{\alpha}\partial_t^{\diamond\Phi} f(ct), \end{aligned}$$

and this completes the proof of item (i). To prove item (ii), we perform a change of variable and get

$$\begin{aligned} \partial_t^{\diamond\Phi} p_z(t) &= t^{1-\alpha}\, b\, zt^{z-1} + t^{-\alpha}\int_0^t z\, y^{z-1}m\left(\frac{y}{t}\right)dy \\ &= t^{z-\alpha}\, b\, z + t^{z-\alpha}z\int_0^1 r^{z-1}m(r)dr = \Phi(z)p_{z-\alpha}(t). \end{aligned}$$

To prove item (iii), we first take $\phi \in \mathbf{B}_{\partial^{\diamond}}$ and thus deduce that the mapping

$$z \mapsto \boldsymbol{\phi}(z) = \frac{z}{z-\alpha}\phi(z-\alpha) = bz + z\int_0^1 r^{z-1}m_{\alpha}(r)dr \tag{5.3.7}$$

is analytical on the right-half plane $\Re(z) > \mathfrak{a}_{\phi} + \alpha, \mathfrak{a}_{\phi} + \alpha \leq 0$ since $0 < \phi'(0^+) < \infty$, with $\lim_{u\downarrow 0}\boldsymbol{\phi}(u) = \lim_{u\downarrow 0}\frac{u}{u-\alpha}\phi(u-\alpha) = 0$. Moreover, from item (ii), we get that such a z,

$$\partial_t^{\diamond\boldsymbol{\phi}} p_z(t) = \boldsymbol{\phi}(z)p_{z-\alpha}(t). \tag{5.3.8}$$

Next, let us assume first that $\phi(\infty) = \infty$, and using the series expansion of $\mathcal{E}_{\phi_\alpha}$ in (5.2.11) combined with the previous identity (5.3.8) with $z = \alpha n$, we get, writing $F_q(t) = \mathcal{E}_{\phi_\alpha}(qt^\alpha)$, for any $q \in \mathbb{R}$ and $t > 0$,

$$\begin{aligned}
\partial_t^{\diamond\boldsymbol{\phi}} F_q(t) &= t^{1-\alpha} b F_q'(t) + t^{-\alpha} \int_0^t F_q'(y) m\left(\frac{y}{t}\right) dy \\
&= t^{1-\alpha} b \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \frac{q^n p_{\alpha n}(t)}{n W_{\phi_\alpha}(n)} + t^{-\alpha} \int_0^t \frac{\partial}{\partial y} \sum_{n=0}^{\infty} \frac{q^n p_{\alpha n}(y)}{n W_{\phi_\alpha}(n)}\, m\left(\frac{y}{t}\right) dy \\
&= \frac{1}{\phi_\alpha'(0^+)} \sum_{n=0}^{\infty} \frac{q^n \partial_t^{\diamond\boldsymbol{\phi}} p_{\alpha n}(t)}{n W_{\phi_\alpha}(n)} \\
&= \frac{1}{\phi_\alpha'(0^+)} \sum_{n=1}^{\infty} \frac{q^n \boldsymbol{\phi}(\alpha n) p_{\alpha(n-1)}(t)}{n W_{\phi_\alpha}(n)} \\
&= \frac{1}{\phi_\alpha'(0^+)} \sum_{n=1}^{\infty} \frac{q^n p_{\alpha(n-1)}(t)}{(n-1) W_{\phi_\alpha}(n-1)} = q F_q(t), \qquad (5.3.9)
\end{aligned}$$

where we used that $\lim_{u\downarrow 0} \boldsymbol{\phi}(u) = 0$ as $\phi \in \mathbf{B}_{\partial^\diamond}$, the functional equation of W_{ϕ_α}, the relation (5.3.7), the fact that power series can be term-by-term differentiated inside the interval of its convergence, and changing the order of integration and summation by the dominated convergence argument. More precisely, note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{|q|^n}{n W_{\phi_\alpha}(n)} \int_0^t \alpha n y^{\alpha n-1} \left(\frac{t}{y}\right)^\alpha m\left(\frac{y}{t}\right) dy &= \sum_{n=0}^{\infty} \frac{|q|^n t^{\alpha n}}{n W_{\phi_\alpha}(n)} \alpha n \int_0^1 r^{\alpha n-\alpha-1} m(r) dr \\
&= \lim_{n\downarrow 0} \frac{1}{n W_{\phi_\alpha}(n)} \alpha n \int_0^1 r^{\alpha n-\alpha-1} m(r) dr \\
&\quad + \sum_{n=1}^{\infty} \frac{|q|^n t^{\alpha n}}{n W_{\phi_\alpha}(n)} \alpha n \int_0^1 r^{\alpha n-\alpha-1} m(r) dr < +\infty,
\end{aligned}$$

where the first term of the above expression is 0 since $\lim_{n\downarrow 0} \frac{1}{nW_{\phi_\alpha}(n)} = \lim_{n\downarrow 0} \frac{\phi_\alpha(n)}{nW_{\phi_\alpha}(n+1)} = \phi_\alpha'(0^+) < \infty$, and, by (5.3.7),

$$\lim_{u\downarrow 0} u \int_0^1 r^{u-\alpha-1} m(r) dr = \lim_{u\downarrow 0} (\boldsymbol{\phi}(u) - bu) = 0,$$

and, the second term is also finite since by assumption $a_\phi \leq -\alpha$, that is $\int_0^1 r^{\alpha n-\alpha-1} m(r) dr < \infty$ for any $n \geq 1$, see below (5.2.6). Now, we move to the proof of the second part of item (*iv*), and we recall from Proposition (5.2.1) that we denoted $\mathfrak{a} = \mathfrak{a}_{\phi_\alpha} \vee -1$, and from (5.2.12), one has that, for any $0 < a < |\mathfrak{a}|$, $q \in \mathbb{R}$, $t > 0$,

$$F_q(t) = \mathcal{E}_{\phi_\alpha}(qt^\alpha) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} (-qt^\alpha)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz.$$

Next, observe that for any $z = a + ib$ with $|b|$ large,

$$\left|\frac{\partial}{\partial t}p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)\right| \leq \alpha|q|^{-a}t^{-\alpha a-1}\left|z\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)\right| \leq \tilde{C}_a\, t^{-\alpha a-1}|q|^{-a}\, |b|^{2a+\frac{1}{2}}e^{-|b|\frac{\pi}{2}},$$

where, $\tilde{C}_a > 0$ and we used the bound (5.2.20). This justifies the application of the dominated convergence Theorem, and we get

$$\frac{\partial}{\partial t}F_q(t) = \frac{\partial}{\partial t}\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} p_{-\alpha z}(t)q^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz = -\frac{\alpha}{2\pi i}\int_{a-i\infty}^{a+i\infty} zt^{-\alpha z-1}q^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz.$$

Moreover, for sake of convenience, denoting $\partial_t^{\diamond m_\alpha} f(t) = t^{-\alpha} f' \diamond m_\alpha(t)$, by the same dominated convergence argument, we get

$$\begin{aligned}\partial_t^{\diamond m_\alpha}F_q(t) &= \frac{t^{-\alpha}}{2\pi i}\int_0^t \frac{\partial}{\partial y}\int_{a-i\infty}^{a+i\infty} p_{-\alpha z}(y)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz\, m_\alpha\left(\frac{y}{t}\right)dy\\ &= \frac{t^{-\alpha}}{2\pi i}\int_0^t \int_{a-i\infty}^{a+i\infty} \frac{\partial}{\partial y}p_{-\alpha z}(y)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz\, m_\alpha\left(\frac{y}{t}\right)dy.\end{aligned}$$

Now, by (5.2.20), we have

$$\left|\frac{\partial}{\partial y}p_{-\alpha z}(y)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)m_\alpha\left(\frac{y}{t}\right)\right| \leq \tilde{C}_a\, y^{-\alpha a-1}|q|^{-a}\, |b|^{2a+\frac{1}{2}}e^{-|b|\frac{\pi}{2}}m_\alpha\left(\frac{y}{t}\right).$$

Therefore, since $\mathfrak{a}_\phi \leq -\alpha$, making a change of variable, we get,

$$\int_0^t y^{-\alpha a-1}m_\alpha\left(\frac{y}{t}\right)dy = t^{-\epsilon}\int_0^1 y^{-(\alpha+\alpha a+1)}m(y)dy < \infty. \tag{5.3.10}$$

Thus,

$$\int_0^t\int_{a-i\infty}^{a+i\infty} \tilde{C}_a\, y^{-\alpha a-1}|q|^{-a}\, |b|^{2a+\frac{1}{2}}e^{-|b|\frac{\pi}{2}}m_\alpha\left(\frac{y}{t}\right)dz\, dy < \infty,$$

and, by Fubini's theorem, we get

$$\partial_t^{\diamond m_\alpha}F_q(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \partial_t^{\diamond m}p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz.$$

Finally, putting pieces together, we have

$$\begin{aligned}
\partial_t^{\diamond\boldsymbol{\phi}} F_q(t) &= t^{1-\alpha}\, b\, \frac{\partial}{\partial t} F_q(t) + \partial_t^{\diamond m_\alpha} F_q(t) \\
&= t^{1-\alpha}\, b\, \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\partial}{\partial t} p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz + \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \partial_t^{\diamond m_\alpha} p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz \\
&= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \partial_t^{\diamond\boldsymbol{\phi}} p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz.
\end{aligned}$$

Next, since from (5.3.8) we have that $\partial_t^{\diamond\boldsymbol{\phi}} p_{-\alpha z}(t) = \frac{z}{z+1}\phi_\alpha(-z-1)p_{-\alpha(z+1)}(t)$ and recalling from (5.2.9) that for $\Re(z) < \mathfrak{a}_{\phi_\alpha}$, $W_{\phi_\alpha}(1-z) = \phi_\alpha(-z)W_{\phi_\alpha}(-z)$, and the recurrence relation of the gamma function, $(z-1)\Gamma(z-1) = \Gamma(z)$ and $-z\Gamma(-z) = \Gamma(1-z)$, $z \in \mathbb{C}$, we obtain

$$\begin{aligned}
\partial_t^{\diamond\boldsymbol{\phi}} F_q(t) &= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \partial_t^{\diamond\boldsymbol{\phi}} p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz \qquad (5.3.11)\\
&= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{z}{z+1}\phi_\alpha(-z-1)p_{-\alpha(z+1)}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz \\
&= -\frac{q}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{z}{z+1}\phi_\alpha(-1-z)(-qt^\alpha)^{-(z+1)}\frac{1}{\phi'_\alpha(0^+)}\frac{\Gamma(z)\Gamma(-z)}{W_{\phi_\alpha}(-z)}dz \\
&= \frac{q}{2\pi i}\int_{a-i\infty}^{a+i\infty} (-qt^\alpha)^{-(z+1)}\frac{1}{\phi'_\alpha(0^+)}\frac{\Gamma(z+1)\Gamma(-z-1)}{W_{\phi_\alpha}(-z-1)}dz \\
&= \frac{q}{2\pi i}\int_{a+1-i\infty}^{a+1+i\infty} (-qt^\alpha)^{-z}\frac{1}{\phi'_\alpha(0^+)}\frac{\Gamma(z)\Gamma(-z)}{W_{\phi_\alpha}(-z)}dz \\
&= \frac{q}{2\pi i}\int_{a+1-i\infty}^{a+1+i\infty} (-qt^\alpha)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz = qF_q(t), \qquad (5.3.12)
\end{aligned}$$

where the justification of the last identity is given as follows. First, the mapping $z \mapsto F(z) = p_{-\alpha z}(t)(-q)^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)$ is analytical in the strip $\Re(z) \in (a, a+1)$, and for some $b > 0$, we have

$$\int_{a-bi}^{a+1-bi} F(z)dz + \int_{a+1-bi}^{a+1+bi} F(z)dz + \int_{a+1+bi}^{a+bi} F(z)dz + \int_{a+bi}^{a-bi} F(z)dz = 0. \qquad (5.3.13)$$

Now, to estimate the third integral, a change of variable yields

$$\int_{a+1+bi}^{a+bi} F(z)dz = \int_{a+1+bi}^{a+bi} p_{-\alpha z}(t)q^{-z}\widehat{\mathcal{E}}^*_{\phi_\alpha}(z)dz \;=\; -\int_a^{a+1} t^{-\alpha(y+bi)}q^{-(y+bi)}\widehat{\mathcal{E}}^*_{\phi_\alpha}(y+bi)dy.$$

Thus, using (5.2.18), we obtain

$$\left|\int_{a+1+bi}^{a+bi} F(z)dz\right| \leq t^{-\alpha a} q^{-a} C_a b^{a-\frac{1}{2}} e^{-b\frac{\pi}{2}},$$

and therefore $\int_{a+1+i\infty}^{a+i\infty} F(z)dz = 0$. Similarly, one can show that $\int_{a-i\infty}^{a+1-i\infty} F(z)dz = 0$. Hence, we deduce from (5.3.13) that

$$\int_{a+1-i\infty}^{a+1+i\infty} F(z)dz = \int_{a-i\infty}^{a+i\infty} F(z)dz,$$

which completes the proof of the identity (5.3.5). Finally, the additional condition of the second part of item (iii), that is $r \mapsto m_\alpha(r)$ is a non-decreasing function on $(0, 1)$, yields that the mapping $y \mapsto m_\alpha(e^{-y})$ defined on $\mathbb{R}_+$ is the tail of a Lévy measure of a subordinator. Thus, it follows from [125, Proposition 2.1] that $\boldsymbol{\phi}$ is a Bernstein function. Furthermore, easy algebra yields that $\boldsymbol{\phi}'(0^+) = -\frac{\phi(-\alpha)}{\alpha}$ which is finite if and only if $\mathfrak{a}_\phi < -\alpha$, and this concludes the proof of item (iii). Finally, to prove item (iv), making a change of variables and performing an integration by parts in (5.2.7), we have

$$\begin{aligned}
\mathbf{A}f(t) &= t^{-\alpha}\left(btf'(t) - \int_0^\infty (f(te^y) - f(t))dm(e^{-y})\right) \\
&= t^{-\alpha}\left(btf'(t) + \int_0^\infty te^y f'(te^y)m(e^{-y})dy\right) \\
&= t^{-\alpha}\left(btf'(t) + \int_t^\infty f'(r)m\left(\frac{t}{r}\right)dr\right).
\end{aligned}$$

Then, recalling that $\Lambda f = f \circ \iota$ with $\iota(y) = \frac{1}{y}$, and making another change of variable, we obtain that

$$\begin{aligned}
\mathbf{A}\Lambda f(t) &= t^{-\alpha}\left(bt\frac{-1}{t^2}f'\left(\frac{1}{t}\right) + \int_t^\infty \frac{-1}{r^2}f'\left(\frac{1}{r}\right)m\left(\frac{t}{r}\right)dr\right) \\
&= -t^{-\alpha}\left(b\frac{1}{t}f'\left(\frac{1}{t}\right) + \int_0^{\frac{1}{t}} f'(y)m\left(\frac{y}{1/t}\right)dy\right)
\end{aligned}$$

and thus

$$\Lambda\mathbf{A}\Lambda f(t) = -t^{\alpha}\left(btf'(t) + \int_0^t f'(y)m\left(\frac{y}{t}\right)dy\right) = -t^{2\alpha}\partial_t^{\diamond\phi} f(t),$$

from where we conclude the proof of the intertwining relation by using the fact that Λ is an involution. ■

5.4 Self-similar Cauchy problem and stochastic representation

Let $X = (X_t)_{t\geq 0}$ be a strong Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and taking values in $E \subset \mathbb{R}^d, d \in \mathbb{N}$, endowed with a sigma-algebra $\mathcal{E}$. We denote its associated semigroup by $P = (P_t)_{t\geq 0}$ which is defined, for any $t \geq 0$ and f a bounded Borelian function, by

$$P_t f(x) = \mathbb{E}_x[f(X_t)],$$

where $\mathbb{E}_x$ stands for the expectation operator with respect to $\mathbb{P}_x(X_0 = x) = 1$. Since $x \mapsto \mathbb{E}_x$ is $\mathcal{E}$-measurable, for any Radon measure ν, we use the notation

$$\nu P_t f = \mathbb{E}_\nu[f(X_t)] = \int_E \mathbb{E}_x[f(X_t)]\nu(dx).$$

We say that a Radon measure ν is an *invariant measure* if for all $t \geq 0$, $\nu P_t f = \nu f$. Now, since ν is non-negative on E, we define the weighted Hilbert space

$$L^2(\nu) = \{f : E \to \mathbb{R} \text{ measurable; } \int_E f^2(x)\nu(dx) < \infty\},$$

endowed with the inner product $\langle\cdot,\cdot\rangle_\nu$, where $\langle f, g\rangle_\nu = \int_E f(x)g(x)\nu(dx)$, and norm $\|f\|_\nu = \sqrt{\langle f, f\rangle_\nu}$. We simply write $L^2(\mathbb{R}_+)$ when ν is the Lebesgue measure on $\mathbb{R}_+$. Then, a classical result yields that we can extend P as a strongly continuous contraction Markov semigroup in $L^2(\nu)$, and when there is no confusion, we still denote this extension by P. We denote by $(\mathbf{L}, \mathcal{D}(\mathbf{L}))$ the infinitesimal generator of the semigroup P, i.e.

$$\mathcal{D}(\mathbf{L}) = \{f \in L^2(\nu); \mathbf{L}f = \lim_{t\to 0} \frac{P_t f - f}{t} \in L^2(\nu)\}.$$

In order to provide a stochastic and explicit representation of the solution to the self-similar Cauchy problem, we shall consider two different cases, for which we recall that as bounded family of operators P admits an adjoint semigroup $P^* = (P_t^*)_{t\geq 0}$, which is defined, for all $t \geq 0$, by $\langle P_t f, g\rangle_\nu = \langle f, P_t^* g\rangle_\nu$. We say that P is normal (resp. self-adjoint) if $P_t P_t^* = P_t^* P_t$ (resp. $P_t = P_t^*$), and of course the second property is stronger.

Assumption 2. P is a normal semigroup on $L^2(\nu)$.

Note that under Assumption 2, $\mathbf{L}$ is a non-negative, densely defined and normal operator on $L^2(\nu)$, and there is a unique resolution $\mathbb{I}$ of the identity, supported on $\sigma(\mathbf{L})$, the spectrum of $\mathbf{L}$, where for any $\lambda \in \sigma(\mathbf{L})$, $\Re(\lambda) \geq 0$,

$$\mathbf{L} = \int_{\sigma(\mathbf{L})} -\lambda d\mathbb{I}(\lambda), \tag{5.4.1}$$

with the domain $\mathcal{D}(\mathbf{L}) = \{f \in L^2(\nu); \int_{\sigma(\mathbf{L})} |\lambda|^2 d\mathbb{I}_{f,f}(\lambda) < \infty\}$, see e.g. [139, Chapter IX]. The identity (5.4.1) is a shorthand notation that means

$$\langle \mathbf{L}f, g\rangle_\nu = \int_{\sigma(\mathbf{L})} -\lambda d\mathbb{I}_{f,g}(\lambda), \quad f \in \mathcal{D}(\mathbf{L}),\ g \in L^2(\nu),$$

where $d\mathbb{I}_{f,g}(\lambda)$ is a regular Borel complex measure of bounded variation concentrated on $\sigma(\mathbf{L})$, with $d|\mathbb{I}_{f,g}|(\sigma(\mathbf{L})) \leq \|f\|_\nu \|g\|_\nu$. Then, for ψ a real measurable function defined on $\sigma(\mathbf{L})$, the operator $\psi(\mathbf{L})$ is given by

$$\psi(\mathbf{L}) = \int_{\sigma(\mathbf{L})} \psi(-\lambda) d\mathbb{I}(\lambda) \text{ with the domain } \mathcal{D}(\psi(\mathbf{L})) = \{f \in L^2(\nu); \int_{\sigma(\mathbf{L})} |\psi(-\lambda)|^2 d\mathbb{I}_{f,f}(\lambda) < \infty\}.$$

We point out that spectral theoretical arguments have already been used in the context of the fractional Cauchy problems associated to self-adjoint operators, see e.g. [37], [104], [105], [108].

Next, we say that sequences $(\mathcal{P}_n)_{n\geq 0}$ and $(\mathcal{V}_n)_{n\geq 0}$ are biorthogonal in $L^2(\nu)$ if they both belong to $L^2(\nu)$ and $\langle \mathcal{P}_m, \mathcal{V}_n\rangle_\nu = \mathbf{I}_{\{m=n\}}$. Moreover, a sequence that admits a biorthogonal sequence will be called minimal and a sequence that is both minimal and complete, in the sense that its linear span is dense in $L^2(\nu)$, will be called *exact*. It is easy to show that a sequence $(\mathcal{P}_n)_{n\geq 0}$ is minimal if and only if none of its elements can be approximated by linear combinations of the others. Next, recall that $(\mathcal{P}_n)_{n\geq 0}$ form a Bessel sequence in $L^2(\nu)$ with bound $B > 0$, if for any $f \in L^2(\nu)$,

$$\sum_{n=0}^{\infty} |\langle f, \mathcal{P}_n \rangle_\nu|^2 \leq B \|f\|_\nu^2. \tag{5.4.2}$$

Then, the so-called synthesis operator $S : l^2(\mathbb{N}) \to L^2(\nu)$ defined by

$$S : \underline{c} = (c_n)_{n\geq 0} \mapsto S(\underline{c}) = \sum_{n=0}^{\infty} c_n \mathcal{P}_n$$

is a bounded operator with norm $\|S\|_\nu \leq \sqrt{B}$, i.e. the series is norm-convergent for any sequence $(c_n)_{n\geq 0}$ in $l^2(\mathbb{N})$. Furthermore, when $(\mathcal{P}_n)_{n\geq 0}$ is an orthogonal system, in (5.4.2) we also have a lower bound and the operator S is invertible.

Assumption 3. Assume that P admits the following spectral expansion, for any $f \in \mathcal{D}$ with $\overline{\mathcal{D}} = L^2(\nu)$, and $t > T$ for some $T > 0$,

$$P_t f = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n \quad \text{in } L^2(\nu), \tag{5.4.3}$$

where $(\lambda_n)_{n\geq 0} \in \mathbb{C}$, with $\Re(\lambda_n) \geq 0, n \geq 0$, is the sequence of the ordered (in modulus) eigenvalues associated to the sequence of eigenfunctions $(\mathcal{P}_n)_{n\geq 0}$ which is an exact Bessel sequence in $L^2(\nu)$ with Bessel bound $B > 0$, and $(\mathcal{P}_n, \mathcal{V}_n)_{n\geq 0}$ form a biorthogonal sequence in $L^2(\nu)$.

Note that when P is self-adjoint, then $\mathcal{P}_n = \mathcal{V}_n$, $\forall n \in \mathbb{N}$, and $(\mathcal{P}_n)_{n\geq 0}$ form an orthogonal basis of $L^2(\nu)$ and (5.4.3) is valid for all $t \geq 0$. In general, $(\mathcal{P}_n, \mathcal{V}_n)_{n\geq 0}$ do not need to form a basis. Now, let ζ be the right-inverse of the non-decreasing α-self-similar Markov process associated via the Lamperti's mapping with ϕ defined by (5.2.4). Recall that if $\phi \in \mathbf{B}_{\partial^\circ}$, then Proposition 5.2.1 implies that for $q, t > 0$,

$$\mathbb{E}[e^{-q\zeta_t}] = \int_0^\infty e^{-qs}\mathbb{P}(\zeta_t \in ds) = \mathcal{E}_{\phi_\alpha}(-qt^\alpha), \tag{5.4.4}$$

which either admits the series or the Mellin-Barnes integral representation provided in Proposition 5.2.1. We denote the time-changed process by $X_\zeta = (X_{\zeta_t})_{t\geq 0}$, and for $f \in L^2(\nu)$, define the family of linear operators $P^{\phi_\alpha} = (P_t^{\phi_\alpha})_{t\geq 0}$ by the Bochner integral

$$P_t^{\phi_\alpha} f(x) = \mathbb{E}_x[f(X_{\zeta_t})] = \int_0^\infty P_s f(x)\mathbb{P}(\zeta_t \in ds). \tag{5.4.5}$$

Throughout this section we assume that $\phi \in \mathbf{B}_{\partial^\circ}$, and recall that $\boldsymbol{\phi}(u) = \frac{u}{u-\alpha}\phi(u-\alpha)$, $u > 0$, is well-defined. Then, we define the set of functions,

$$\mathcal{D}_{\mathbf{L}} = \left\{ f \in L^2(\nu);\ (\lambda_n \langle f, \mathcal{V}_n\rangle_\nu)_{n\geq 0} \in l^2(\mathbb{N}) \right\} \subseteq \mathcal{D}(\mathbf{L}),$$

and since clearly $Span(\mathcal{P}_n) \subseteq \mathcal{D}_{\mathbf{L}}$ and by Assumption 3, $Span(\mathcal{P}_n)$ is dense in $L^2(\nu)$, we have $\mathcal{D}_{\mathbf{L}}$ is also dense in $L^2(\nu)$. We are now ready to state the last main result of this paper.

Theorem 5.4.1. Let $\phi \in \mathbf{B}_{\partial^\circ}$. If Assumption 2 (resp. Assumption 3) holds, then for any $f \in \mathcal{D}(\mathbf{L})$ (resp. $f \in \mathcal{D}_{\mathbf{L}}$), the function $u(t,x) = P_t^{\phi_\alpha} f(x)$, is a strong solution in $L^2(\nu)$ to

$$\begin{aligned} \partial_t^{\circ\boldsymbol{\phi}} u(t,x) &= \mathbf{L}u(t,x), \quad t > 0\ (resp.\ t > T), \\ u(0,x) &= f(x), \end{aligned}$$

in the following sense: $t \mapsto u(t,\cdot) \in C_0^1((0,\infty), L^2(\nu))$ (resp. $C_0^1((T,\infty), L^2(\nu))$), and both $t \mapsto u(t,\cdot)$ and $t \mapsto \mathbf{L}u(t,\cdot)$ are analytical on the half plane $\Re(z) > 0$ (resp. $\Re(z) > T$). Moreover, if Assumption 2 holds, then for any $f \in \mathcal{D}(\mathbf{L})$ and $t > 0$, $P_t^{\phi_\alpha} f$ admits the following spectral representation,

$$P_t^{\phi_\alpha} f = \int_{\sigma(\mathbf{L})} \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f \quad \text{in } L^2(\nu). \tag{5.4.6}$$

Otherwise if Assumption 3 holds, then for any $f \in \mathcal{D}_{\mathbf{L}}$ and $t > T$,

$$P_t^{\phi_\alpha} f = \sum_{n=0}^\infty \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \quad \text{in } L^2(\nu). \tag{5.4.7}$$

Proof. First, note that since P_t is for all $t \geq 0$ a contraction, using Bochner's inequality, see [5, Theorem 1.1.4], one can note from (5.4.5) that for any $f \in L^2(\nu)$,

$$\|P_t^{\phi_\alpha} f\|_\nu = \left\| \int_0^\infty P_s f \mathbb{P}(\zeta_t \in ds) \right\|_\nu \leq \int_0^\infty \|P_s f\|_\nu \mathbb{P}(\zeta_t \in ds) \leq \|f\|_\nu.$$

Thus, for any $t \geq 0$, $P_t^{\phi_\alpha}$ is a bounded operator in $L^2(\nu)$. Now, let Assumption 2 holds. Then, by the functional calculus, we have that for all $t > 0$

$$P_t = e^{t\mathbf{L}} = \int_{\sigma(\mathbf{L})} e^{-t\lambda} d\mathbb{I}(\lambda).$$

Therefore, ζ being the right-inverse of the non-decreasing self-similar Markov process associated to $\phi \in \mathbf{B}_{\partial^\circ}$, we have, using the identity (5.4.5), that for any $f \in L^2(\nu)$ and $t > 0$,

$$\begin{aligned} P_t^{\phi_\alpha} f = \int_0^\infty P_s f \mathbb{P}(\zeta_t \in ds) &= \int_0^\infty \int_{\sigma(\mathbf{L})} e^{-s\lambda} d\mathbb{I}(\lambda) f \mathbb{P}(\zeta_t \in ds) \\ &= \int_{\sigma(\mathbf{L})} \int_0^\infty e^{-s\lambda} \mathbb{P}(\zeta_t \in ds) d\mathbb{I}(\lambda) f \\ &= \int_{\sigma(\mathbf{L})} \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f \end{aligned} \tag{5.4.8}$$

where for the transition from second to third equality, we used Fubini's theorem under the inner product $\langle \cdot, \cdot \rangle_\nu = \| \cdot \|_\nu^2$, by a simple polarization argument, which is allowed since the measure $d\mathbb{I}$ is of bounded variation on $\sigma(\mathbf{L})$ and, as a Laplace transform of a probability measure, for all $t, \Re(\lambda) \geq 0$, $|\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha)| \leq 1$, and for the last step we used the identity (5.4.4). Now, as for any $t \geq 0$, $P_t^{\phi_\alpha}$ is bounded in $L^2(\nu)$, we have $P_t^{\phi_\alpha} \mathbf{L} \subseteq \mathbf{L} P_t^{\phi_\alpha}$ and thus $\mathcal{D}(P_t^{\phi_\alpha} \mathbf{L}) = \mathcal{D}(\mathbf{L}) \subseteq \mathcal{D}(\mathbf{L} P_t^{\phi_\alpha}) = \{f \in L^2(\nu); P_t^{\phi_\alpha} f \in \mathcal{D}(\mathbf{L})\}$, see [139, Theorem 13.24, (15) and (10)]. Hence, we conclude that $P_t^{\phi_\alpha}$ maps $\mathcal{D}(\mathbf{L})$ into itself, and since $P_t^{\phi_\alpha} f \in \mathcal{D}(\mathbf{L})$ for all $f \in \mathcal{D}(\mathbf{L})$, by the functional calculus, we obtain

$$\mathbf{L} P_t^{\phi_\alpha} f = \int_{\sigma(\mathbf{L})} -\lambda \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f. \tag{5.4.9}$$

Next, since by Proposition 5.2.1(iii), $t \mapsto \mathcal{E}_{\phi_\alpha}(-t) \in C_0^\infty(\mathbb{R}_+)$, then, for $\Re(\lambda) > 0$, the mapping $t \mapsto \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) \in C_0^\infty(\mathbb{R}_+)$ and

$$\frac{d}{dt} \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) = \frac{d}{dt} \mathbb{E}[e^{-\lambda t^\alpha \zeta_1}] = -\lambda \alpha t^{\alpha-1} \mathbb{E}[\zeta_1 e^{-\lambda t^\alpha \zeta_1}], \tag{5.4.10}$$

which is bounded on $t \in [t_0, \infty)$ for any $t_0 > 0$ and $\Re(\lambda) \geq 0$ since by item (i) of Proposition 5.2.1, $\mathbb{E}[\zeta_1] = \frac{1}{\phi_\alpha'(0^+)} < \infty$. Furthermore, since we have, for any $t, s > 0$,

$$\|P_t^{\phi_\alpha} f - P_s^{\phi_\alpha} f\|_\nu^2 = \int_{\sigma(\mathbf{L})} (\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) - \mathcal{E}_{\phi_\alpha}(-\lambda s^\alpha))^2 d\mathbb{I}_{f,f}(\lambda),$$

and

$$\left\| \frac{P_t^{\phi_\alpha} f - P_s^{\phi_\alpha} f}{t-s} - \int_{\sigma(\mathbf{L})} \frac{d}{dt}\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f \right\|_\nu^2 = \int_{\sigma(\mathbf{L})} \left(\frac{\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) - \mathcal{E}_{\phi_\alpha}(-\lambda s^\alpha)}{t-s} - \frac{d}{dt}\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) \right)^2 d\mathbb{I}_{f,f}(\lambda),$$

we obtain that, in the Hilbert space topology, $t \mapsto P_t^{\phi_\alpha}$ is also continuously differentiable vanishing along with its derivative at ∞, i.e. it is in $C_0^1((0,\infty), L^2(\nu))$. Indeed, the last identity entails that for any $t > 0$,

$$\frac{d}{dt} P_t^{\phi_\alpha} = \int_{\sigma(\mathbf{L})} \frac{d}{dt}\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda), \tag{5.4.11}$$

where we note that for any $t > 0$ and $f \in \mathcal{D}(\mathbf{L})$,

$$\begin{aligned} \left\| \int_{\sigma(\mathbf{L})} \frac{d}{dt}\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f \right\|_\nu^2 &= \int_{\sigma(\mathbf{L})} \left(\frac{d}{dt}\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) \right)^2 d\mathbb{I}_{f,f}(\lambda) \\ &\leq \left(\frac{\alpha t^{\alpha-1}}{\phi_\alpha'(0^+)} \right)^2 \int_{\sigma(\mathbf{L})} |\lambda|^2 d\mathbb{I}_{f,f}(\lambda) < \infty, \end{aligned} \tag{5.4.12}$$

where we used (5.4.10), and once again that $|\mathbb{E}[\zeta_1 e^{-\lambda t^\alpha \zeta_1}]| \leq \mathbb{E}[\zeta_1] = \frac{1}{\phi_\alpha'(0^+)}$ for any $t, \Re(\lambda) \geq 0$. Then, by (5.4.11) and Proposition 5.3.1(iii), we have that

$$\begin{aligned} \partial_t^{\diamond \boldsymbol{\phi}} P_t^{\phi_\alpha} f &= t^{1-\alpha} b \frac{d}{dt} P_t^{\phi_\alpha} f + t^{-\alpha} \int_0^t \frac{d}{dt} P_t^{\phi_\alpha} f \, m\left(\frac{y}{t}\right) dy \\ &= t^{1-\alpha} b \int_{\sigma(\mathbf{L})} \frac{d}{dt}\mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f + t^{-\alpha} \int_0^t \int_{\sigma(\mathbf{L})} \frac{d}{dy}\mathcal{E}_{\phi_\alpha}(-\lambda y^\alpha) d\mathbb{I}(\lambda) f \, m\left(\frac{y}{t}\right) dy \\ &= \int_{\sigma(\mathbf{L})} \partial_t^{\diamond \boldsymbol{\phi}} \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f = \int_{\sigma(\mathbf{L})} -\lambda \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) d\mathbb{I}(\lambda) f, \end{aligned}$$

where in the second step to change the order of integration, we used Fubini's theorem for Bochner integrals, see [5, Theorem 1.1.9], which is justified since by (5.4.12) we have

$$\begin{aligned} \int_0^t \left\| \int_{\sigma(\mathbf{L})} \frac{d}{dy}\mathcal{E}_{\phi_\alpha}(-\lambda y^\alpha) d\mathbb{I}(\lambda) f \right\|_\nu m\left(\frac{y}{t}\right) dy &\leq \frac{\alpha}{\phi_\alpha'(0^+)} \left(\int_{\sigma(\mathbf{L})} |\lambda|^2 d\mathbb{I}_{f,f}(\lambda) \right)^{\frac{1}{2}} \int_0^t y^{\alpha-1} m\left(\frac{y}{t}\right) dy \\ &\leq \frac{\alpha t^\alpha}{\phi_\alpha'(0^+)} \left(\int_{\sigma(\mathbf{L})} |\lambda|^2 d\mathbb{I}_{f,f}(\lambda) \right)^{\frac{1}{2}} \int_0^1 r^{\alpha-1} m(r) dr < \infty. \end{aligned}$$

Thus, $\mathbf{L}P_t^{\phi_\alpha} f = \partial_t^{\diamond\boldsymbol{\phi}} P_t^{\phi_\alpha} f$, and taking $t = 0$ in (5.4.5), we easily check that $u(0, x) = f(x)$, $x \in E$. Now, let us assume that Assumption 3 holds, and define the family of linear operators $S^{\phi_\alpha} = (S_t^{\phi_\alpha})_{t>T}$, for $f \in \mathcal{D}_{\mathbf{L}}$ and $t > T$, by

$$S_t^{\phi_\alpha} f = \sum_{n=0}^{\infty} \int_0^\infty e^{-\lambda_n s}\, \mathbb{P}(\zeta_t \in ds) \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n = \sum_{n=0}^{\infty} \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha) \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n. \tag{5.4.13}$$

Note that $S_t^{\phi_\alpha} f \in L^2(\nu)$ for any $f \in \mathcal{D}_{\mathbf{L}}$. Indeed, recalling that $\Re(\lambda_n) \geq 0$, $n = 0, 1, \cdots$, as a Laplace transform of a probability measure, $|\mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)| \leq 1$ for any $t \geq 0$ and $n = 0, 1, \cdots$, we have

$$\sum_{n=0}^{\infty} |\mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)|^2 |\langle f, \mathcal{V}_n \rangle_\nu|^2 \leq \sum_{n=0}^{\infty} |\langle f, \mathcal{V}_n \rangle_\nu|^2 \leq M + \sum_{n=\mathfrak{m}}^{\infty} |\lambda_n|^2 |\langle f, \mathcal{V}_n \rangle_\nu|^2 < \infty,$$

where $\mathfrak{m} = \min\{k \geq 0;\ |\lambda_k| \geq 1\}$ and in which case there exists $M \geq 0$ such that $\sum_{n=0}^{\mathfrak{m}-1} |\langle f, \mathcal{V}_n \rangle_\nu|^2 \leq M$. Moreover, by the Bessel property of $(\mathcal{P}_n)_{n \geq 0}$, we have that $S_t^{\phi_\alpha}$ is a bounded operator on $\mathcal{D}_{\mathbf{L}}$ with $\|S_t^{\phi_\alpha}\|_\nu \leq \sqrt{B}$. Furthermore, since $\langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu = \mathbf{I}_{\{m=n\}}$, we have, for any $m \in \mathbb{N}$,

$$S_t^{\phi_\alpha} \mathcal{P}_m = \sum_{n=0}^{\infty} \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha) \langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu \mathcal{P}_n = \mathcal{E}_{\phi_\alpha}(-\lambda_m t^\alpha) \mathcal{P}_m.$$

On the other hand, recalling the spectral expansion of P_t given in (5.4.3), we have, for $t > T$,

$$P_t^{\phi_\alpha} \mathcal{P}_m = \int_0^\infty \sum_{n=0}^{\infty} e^{-\lambda_n s} \langle \mathcal{P}_m, \mathcal{V}_n \rangle_\nu \mathcal{P}_n\, \mathbb{P}(\zeta_t \in ds) = \int_0^\infty e^{-\lambda_m s} \mathcal{P}_m\, \mathbb{P}(\zeta_t \in ds) = \mathcal{E}_{\phi_\alpha}(-\lambda_m t^\alpha) \mathcal{P}_m.$$

Thus, $P_t^{\phi_\alpha}$ and $S_t^{\phi_\alpha}$ coincide on $Span(\mathcal{P}_n)$, and since $\overline{Span}(\mathcal{P}_n) = L^2(\nu) \supseteq \mathcal{D}_{\mathbf{L}}$, the bounded linear transformation Theorem implies that $P_t^{\phi_\alpha} = S_t^{\phi_\alpha}$ on $\mathcal{D}_{\mathbf{L}}$ when $t > T$. Next, since for all n, $\mathcal{P}_n$ is an eigenfunction, $\mathcal{P}_n \in L^2(\nu)$, $P_t \mathcal{P}_n = e^{-\lambda_n t} \mathcal{P}_n$ and hence $\mathcal{P}_n \in \mathcal{D}(\mathbf{L})$ with $\mathbf{L}\mathcal{P}_n = -\lambda_n \mathcal{P}_n$. Thus, by linearity, for any $t \geq 0$ and $N = 1, 2, \cdots$, $h_t^N \in \mathcal{D}(\mathbf{L})$, where $h_t^N = \sum_{n=0}^{N} \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha) \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n$, $f \in \mathcal{D}(\mathbf{L})$, and

$$\mathbf{L} h_t^N = \sum_{n=0}^{N} \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha) \langle f, \mathcal{V}_n \rangle_\nu \mathbf{L}\mathcal{P}_n = \sum_{n=0}^{N} -\lambda_n \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha) \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n.$$

Then, letting $N \to \infty$, we obtain

$$\begin{aligned} h_t^N &= \sum_{n=0}^{N} \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \to P_t^{\phi_\alpha} f, \quad \text{and} \\ \mathbf{L}h_t^N &= \sum_{n=0}^{N} -\lambda_n \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \to \sum_{n=0}^{\infty} -\lambda_n \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n. \end{aligned}$$

Observing that, since $|\mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)| \leq 1$, for any $n = 0, 1, 2, \cdots$, $t \geq 0$ and $f \in \mathcal{D}_{\mathbf{L}} \subseteq \mathcal{D}(\mathbf{L})$,

$$\sum_{n=0}^{\infty} | - \lambda_n \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu|^2 \leq \sum_{n=0}^{\infty} \lambda_n^2 |\langle f, \mathcal{V}_n\rangle_\nu|^2 < \infty,$$

and thus the Bessel property of $(\mathcal{P}_n)_{n\geq 0}$ implies that $\sum_{n=0}^{\infty} -\lambda_n \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \in L^2(\nu)$. Therefore, since the operator $\mathbf{L}$ is closed, we obtain that $P_t^{\phi_\alpha} f \in \mathcal{D}(\mathbf{L})$ and

$$\mathbf{L}P_t^{\phi_\alpha} f = \sum_{n=0}^{\infty} -\lambda_n \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n. \tag{5.4.14}$$

Now, similar to the justification under Assumption 2 above, one can show that for any $f \in \mathcal{D}_{\mathbf{L}}$, the mapping $t \mapsto P_t^{\phi_\alpha}$ is a $C_0^1((T, \infty), L^2(\nu))$ function, and for any $t > T$, (5.4.11) holds. Then, for any $f \in \mathcal{D}_{\mathbf{L}}$ and $t > T$, we have

$$\begin{aligned} \partial_t^{\diamond \boldsymbol{\phi}} P_t^{\phi_\alpha} f &= \partial_t^{\diamond \boldsymbol{\phi}} \sum_{n=0}^{\infty} \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \\ &= \sum_{n=0}^{\infty} \partial_t^{\diamond \boldsymbol{\phi}} \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \\ &= \sum_{n=0}^{\infty} -\lambda_n \mathcal{E}_{\phi_\alpha}(-\lambda_n t^\alpha)\langle f, \mathcal{V}_n\rangle_\nu \mathcal{P}_n \in L^2(\nu), \end{aligned}$$

where we noted that we are allowed to change the order of the operator $\partial_t^{\diamond \boldsymbol{\phi}}$ and summation similar to the case of the normal operator above. Indeed, to change the order of summation and integration, using the Bessel property of $(\mathcal{P}_n)_{n\geq 0}$ and recalling the definition of $\mathcal{D}_{\mathbf{L}}$, we apply Fubini's theorem. Thus, we conclude that for $f \in \mathcal{D}_{\mathbf{L}}$ and $t > T$, $\partial_t^{\diamond \boldsymbol{\phi}} P_t^{\phi_\alpha} f = \mathbf{L}P_t^{\phi_\alpha} f$. Moreover, taking $t = 0$ in (5.4.5), one can easily check that $u(0, x) = f(x)$ for $x \in E$. Finally, under Assumption 2 (resp. Assumption 3), given the eigenvalues expansion of $P_t^{\phi_\alpha}$, we have that $t \mapsto u(t, \cdot) = P_t^{\phi_\alpha} f$ and $t \mapsto \mathbf{L}u(t, \cdot) = P_t^{\phi_\alpha} \mathbf{L} f$ are analytical on the half plane $\Re(z) > 0$ (resp. $\Re(z) > T$), and this concludes the proof. ■

5.5 Examples

Let $\zeta = (\zeta_t)_{t\geq 0}$ be the inverse of the non-decreasing α-self-similar Markov process $\chi = (\chi_t)_{t\geq 0}$ defined in Section 5.2, and associated via the Lamperti mapping to the subordinator with a Laplace exponent $\phi \in \mathbf{B}_{\partial^\circ}$, defined by (5.2.4). Furthermore, recall that $\boldsymbol{\phi}$ is defined by (5.3.3). In this section, we consider some examples that illustrate the variety of applications of our main results and they cover the both situations when Assumption 2 or Assumption 3 holds. Namely, section 5.5.1 and 5.5.2 include examples of self-adjoint, and non-self-adjoint and non-local semigroups respectively.

5.5.1 Some self-adjoint examples

Squared Bessel semigroups

We consider first the case where $P = (P_t)_{t\geq 0}$ is the semigroup of the squared Bessel process of order 2, that is its infinitesimal generator is given, for a smooth function f, by

$$\mathbf{L}f(x) = 2xf''(x) + 2f'(x), \quad x > 0. \tag{5.5.1}$$

It is well known that P_t is a strongly continuous contraction semigroup and self-adjoint in $L^2(\mathbb{R}_+)$. Next, we define the function J, for $z \in \mathbb{C}$, by

$$J(z) = \sum_{n=0}^{\infty} \frac{(e^{i\pi}z)^n}{(n!)^2},$$

and observe that $J\left(\frac{z^2}{4}\right) = \mathrm{J}_0(z)$, where J_0 is the Bessel function of the first kind of order 0. We also recall that H the Hankel transform associated to J is an involution of $L^2(\mathbb{R}_+)$, i.e.

HH is the identity, defined by

$$Hf(x) = \int_0^\infty J(\lambda x) f(\lambda) d\lambda.$$

Then, P admits the following spectral expansion, for any $t > 0$ and $f \in L^2(\mathbb{R}_+)$,

$$P_t f = H\mathbf{e}_t H f,$$

where we set $\mathbf{e}_t(x) = e^{-tx}$, see e.g. [131]. Then, since Assumption 2 is satisfied, Theorem 5.4.1 implies that, for any $f \in \mathcal{D}(\mathbf{L})$, $P_t^{\phi_\alpha} f$ solves the self-similar Cauchy problem,

$$\begin{aligned} \partial_t^{\diamond\phi} u(t,x) &= \mathbf{L}u(t,x), \quad t > 0, \\ u(0,x) &= f(x). \end{aligned}$$

Furthermore, the solution has the following spectral representation, for all $t > 0$,

$$P_t^{\phi_\alpha} f(x) = \int_0^\infty \mathcal{E}_{\phi_\alpha}(-\lambda t^\alpha) Hf(\lambda) J(\lambda x) d\lambda \quad \text{in } L^2(\mathbb{R}_+). \tag{5.5.2}$$

The classical Laguerre semigroup

Let $P = (P_t)_{t \geq 0}$ be the classical Laguerre semigroup of order 0, i.e. its infinitesimal generator takes the form, for a smooth function f,

$$\mathbf{L}f(x) = xf''(x) + (1-x)f'(x), \quad x > 0,$$

see e.g. [129, Section 3.1]. Then, the semigroup P is a self-adjoint and strongly continuous contraction semigroup on the weighted Hilbert space $L^2(\nu)$ with $\nu(dx) = e^{-x}dx$, $x > 0$, which is the unique invariant measure. Moreover, it admits the eigenvalues expansions, valid for any $t > 0$,

$$P_t f = \sum_{n=0}^\infty e^{-nt} \langle f, \mathcal{L}_n \rangle_\nu \mathcal{L}_n \quad \text{in } L^2(\nu),$$

where for any $n \geq 0$, $\mathcal{L}_n$ is the Laguerre polynomial of order 0, defined through the polynomial representation

$$\mathcal{L}_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!}.$$

Since P is self-adjoint in $L^2(\nu)$, Assumption 2 is satisfied with $\sigma(\mathbf{L}) = \{\lambda_n = n, n \geq 0\}$, and it follows from Theorem 5.4.1 that for any $f \in \mathcal{D}(\mathbf{L})$, $P_t^{\phi_\alpha} f$ solves the self-similar Cauchy problem,

$$\begin{aligned} \partial_t^{\diamond\phi} u(t,x) &= \mathbf{L}u(t,x), \quad t > 0, \\ u(0,x) &= f(x). \end{aligned}$$

Furthermore, the solution has the following spectral representation, for all $t > 0$,

$$P_t^{\phi_\alpha} f = \sum_{n=0}^{\infty} \mathcal{E}_{\phi_\alpha}(-nt^\alpha) \langle f, \mathcal{L}_n \rangle_\nu \mathcal{L}_n \quad \text{in } L^2(\nu).$$

Classical Jacobi semigroups

Now, assume $\boldsymbol{\lambda}_1 > \mu > 0$ and let us consider the classical Jacobi semigroup $P = (P_t)_{t \geq 0}$ on $E = (0,1)$, which is a Feller semigroup and its infinitesimal generator $\mathbf{L}_\mu$ has, for any $f \in C^2(E)$, the following form

$$\mathbf{L}_\mu f(x) = x(1-x)f''(x) - (\boldsymbol{\lambda}_1 x - \mu) f'(x), \quad x \in (0,1),$$

see e.g. [38, Section 5]. The classical Jacobi semigroup P admits a unique invariant measure β_μ, which is the distribution of a beta random variable of parameters $\mu > 0$ and $\boldsymbol{\lambda}_1 - \mu > 0$, i.e.

$$\beta_\mu(dy) = \beta_\mu(y)dy = \frac{\Gamma(\boldsymbol{\lambda}_1)}{\Gamma(\mu)\Gamma(\boldsymbol{\lambda}_1 - \mu)} y^{\mu-1}(1-y)^{\boldsymbol{\lambda}_1 - \mu - 1} dy, \quad y \in (0,1). \tag{5.5.3}$$

Moreover, P extends to a strongly continuous contraction semigroup on $L^2(\beta_\mu)$ which we still denote by P. The eigenfunctions of P are the Jacobi polynomials which form an orthonormal basis in $L^2(\beta_\mu)$ and are given, for any $n \in \mathbb{N}$ and $x \in E$, by

$$\mathcal{P}_n^{\lambda_1,\mu}(x) = \sqrt{C_n(\mu)} \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(\lambda_1 - 1)_{n+k}}{(\lambda_1 - 1)_n} \frac{(\mu)_n}{(\mu)_k} \frac{x^k}{k!}, \tag{5.5.4}$$

where we have set

$$C_n(\mu) = (2n + \lambda_1 - 1) \frac{n!(\lambda_1)_{n-1}}{(\mu)_n(\lambda_1 - \mu)_n}. \tag{5.5.5}$$

Next, the eigenvalue associated to the eigenfunction $\mathcal{P}_n$ is, for $n \in \mathbb{N}$,

$$\lambda_n = n^2 + (\lambda_1 - 1)n = n(n-1) + \lambda_1 n. \tag{5.5.6}$$

The semigroup P then admits the spectral decomposition given, for any $f \in L^2(\beta_\mu)$ and $t \geq 0$, by

$$P_t f = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{P}_n^{\lambda_1,\mu} \rangle_{\beta_\mu} \mathcal{P}_n^{\lambda_1,\mu}. \tag{5.5.7}$$

Since P is self-adjoint, Assumption 2 is satisfied with $\sigma(\mathbf{L}) = \{\lambda_n = n(n-1) + \lambda_1 n, n \geq 0\}$, and it follows from Theorem 5.4.1 that for any $f \in \mathcal{D}(\mathbf{L}_\mu)$, $P_t^{\phi_\alpha} f$ solves the self-similar Cauchy problem,

$$\begin{aligned} \partial_t^{\diamond \boldsymbol{\phi}} u(t,x) &= \mathbf{L}_\mu u(t,x), \quad t > 0, \\ u(0,x) &= f(x). \end{aligned}$$

Furthermore, the solution has the following spectral representation, for all $t > 0$,

$$P_t^{\phi_\alpha} f = \sum_{n=0}^{\infty} \mathcal{E}_{\phi_\alpha}(-(n(n-1) + \lambda_1 n)t^\alpha) \langle f, \mathcal{P}_n^{\lambda_1,\mu} \rangle_{\beta_\mu} \mathcal{P}_n^{\lambda_1,\mu} \quad \text{in } L^2(\beta_\mu).$$

5.5.2 Some non-self-adjoint and non-local examples

A generalized Laguerre semigroup

We next follow [129, Section 3.2] to present a special instance of the so-called generalized Laguerre semigroups. In particular, let $\mathfrak{m} \geq 1$ and $P = (P_t)_{t\geq 0}$ be the non-self-adjoint semigroup whose infinitesimal generator is given, for a smooth function f, by

$$\mathbf{L}_{\mathfrak{m}} f(x) = xf''(x) + \left(\frac{\mathfrak{m}^2 - 1}{\mathfrak{m}} + 1 - x\right) f'(x) + \int_0^\infty (f(e^{-y}x) - f(x) + yxf'(x)) \frac{\mathfrak{m} e^{-\mathfrak{m} y}}{x} dy, x > 0.$$

The semigroup P is ergodic with a unique invariant measure, which in this case is an absolutely continuous probability measure with a density denoted by ν and which takes the form

$$\nu(y) = \frac{(1+y)}{\mathfrak{m}+1} \frac{y^{\mathfrak{m}-1} e^{-y}}{\Gamma(\mathfrak{m})}, \quad y > 0.$$

Moreover, P_t admits the following spectral representation for any $f \in L^2(\nu)$ and $t > 0$,

$$P_t f = \sum_{n=0}^{\infty} e^{-nt} \langle f, \mathcal{V}_n \rangle_\nu \mathcal{P}_n \quad \text{in } L^2(\nu).$$

Here, $(\mathcal{P}_n, \mathcal{V}_n)_{n\geq 0}$ form an orthogonal sequence in $L^2(\nu)$, and are expressed in terms of the Laguerre polynomials $\left(\mathcal{L}_n^{(\mathfrak{m})}\right)_{n\geq 0}$ as follows, for $n \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{P}_n(x) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(\mathfrak{m}+2)}{\Gamma(\mathfrak{m}+k+2)} \frac{\mathfrak{m}+k}{\mathfrak{m}} x^k = \mathfrak{c}_n(\mathfrak{m}+1) \mathcal{L}_n^{(\mathfrak{m}+1)}(x) - \frac{c_n(\mathfrak{m}+1)}{\mathfrak{m}} x \mathcal{L}_{n-1}^{(\mathfrak{m}+2)}(x), \\
\mathcal{V}_n(x) &= \frac{1}{x+1} \mathcal{L}_n^{(\mathfrak{m}-1)}(x) + \frac{x}{x+1} \mathcal{L}_n^{(\mathfrak{m})}(x).
\end{aligned}$$

Here, $\mathfrak{c}_n(\mathfrak{m}+1) = \frac{\Gamma(n+1)\Gamma(\mathfrak{m}+2)}{\Gamma(n+\mathfrak{m}+2)}$ and we recall that $\mathcal{L}_n^{(\mathfrak{m})}$ is the Laguerre polynomial of order $\mathfrak{m}$,

$$\mathcal{L}_n^{(\mathfrak{m})}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\mathfrak{m}}{n-k} \frac{x^k}{k!}, \quad x > 0.$$

Therefore, since Assumption 3 is satisfied with $\sigma(\mathbf{L}) = \{\lambda_n = n, n \geq 0\}$, Theorem 5.4.1 implies that $f \in \mathcal{D}(\mathbf{L}_{\mathfrak{m}})$, $P_t^{\phi_\alpha} f$ solves the self-similar Cauchy problem,

$$\begin{aligned} \partial_t^{\diamond \boldsymbol{\phi}} u(t,x) &= \mathbf{L}_{\mathfrak{m}} u(t,x), \quad t > 0, \\ u(0,x) &= f(x). \end{aligned}$$

Furthermore, the solution has the following spectral representation, for all $t > 0$,

$$P_t^{\phi_\alpha} f = \sum_{n=0}^{\infty} \mathcal{E}_{\phi_\alpha}(-nt^\alpha) \langle f, \mathcal{P}_n \rangle_\nu \mathcal{V}_n \quad \text{in } L^2(\nu).$$

Generalized Jacobi semigroups

In this section, following Patie et al. [38], we provide a short description of a special instance of generalized Jacobi semigroups. In particular, let $\boldsymbol{\lambda}_1 > \mathfrak{m} > 2$ with $\boldsymbol{\lambda}_1 - \mathfrak{m} \notin \mathbb{N}$, and $P = (P_t)_{t \geq 0}$ be the non-self-adjoint semigroup associated with the infinitesimal generator given for a smooth function f

$$\mathbf{L}_{\mathfrak{m}} f(x) \;=\; x(1-x) f''(x) - (\boldsymbol{\lambda}_1 x - \mathfrak{m} - 1) f'(x) - x^{-(\mathfrak{m}+1)} \int_0^1 f'(r) r^{\mathfrak{m}} dr, \quad x \in E.$$

Then, we have by [38, Proposition 4.1] that the density of the unique invariant measure of the Markov semigroup P is given by

$$\beta(y) = \frac{((\boldsymbol{\lambda}_1 - \mathfrak{m} - 2)y + 1)}{(\mathfrak{m}+1)(1-y)} \beta_{\mathfrak{m}}(y), \quad y \in (0,1),$$

where $\beta_{\mathfrak{m}}$ is the distribution of the beta random variable of parameters $\mathfrak{m} > 0$ and $\boldsymbol{\lambda}_1 - \mathfrak{m} > 0$, see (5.5.3). Furthermore, for any $t > 0$ and $f \in L^2(\beta)$, P_t admits the following spectral representation

$$P_t f = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle f, \mathcal{P}_n \rangle_\beta \mathcal{V}_n \quad \text{in } L^2(\beta),$$

where we recall that $(\lambda_n)_{n\geq 0}$ are defined by (5.5.6), and $(\mathcal{P}_n, \mathcal{V}_n)_{n\geq 0}$ form a biorthogonal sequence in $L^2(\beta)$ and are defined as follows. We have that $\mathcal{P}_0 \equiv 1$ and, for $n \geq 1$,

$$\mathcal{P}_n(x) = \frac{n!}{(\mathfrak{m}+2)_n}\sqrt{C_n(1)}\left(\frac{\mathcal{P}_n^{(\lambda_1,\mathfrak{m}+2)}(x)}{\sqrt{C_n(\mathfrak{m}+2)}} + \frac{x}{\mathfrak{m}}\frac{\mathcal{P}_{n-1}^{(\lambda_1+1,\mathfrak{m}+3)}(x)}{\sqrt{\widetilde{C}_{n-1}(\mathfrak{m}+3)}}\right), \ x \in E.$$

making explicit the dependence on the two parameters for the classical Jacobi polynomials (5.5.4), and where $\widetilde{C}_n(\mathfrak{m}+3) = n!(2n+\lambda_1)(\lambda_1+1)_n/(\mathfrak{m}+3)_n(\lambda_1-\mathfrak{m}-2)_n$ and C_n-s are defined by (5.5.5). For any $n \in \mathbb{N}$ the function $\mathcal{V}_n$ is given by

$$\mathcal{V}_n(x) = \frac{1}{\beta(x)}C_{\lambda_1,\mathfrak{m},n}\frac{\sin(\pi(\mathfrak{m}-\lambda_1))}{\pi}\sum_{k=0}^{\infty}\frac{(\mathfrak{m}+1)_{k+n}}{(\mathfrak{m}+1)_k}\frac{\Gamma(k+\mathfrak{m}-n-\lambda_1+1)}{k!}(k-1)x^{k+\mathfrak{m}}, \ x \in E^o,$$

where $C_{\lambda_1,\mathfrak{m},n} = \mathfrak{m}(\lambda_1-1)\Gamma(\lambda_1+n-1)\sqrt{C_n(1)}(-2)^n/(n!\Gamma(\mathfrak{m}+2))$. Hence, since Assumption 3 is satisfied, Theorem 5.4.1 implies that for $f \in \mathcal{D}(\mathbf{L}_{\mathfrak{m}})$, $P_t^{\phi_\alpha} f$ solves the self-similar Cauchy problem,

$$\begin{aligned} \partial_t^{\diamond\boldsymbol{\phi}} u(t,x) &= \mathbf{L}_{\mathfrak{m}} u(t,x), \quad t>0, \\ u(0,x) &= f(x). \end{aligned}$$

Lastly, the solution has the following spectral representation, for all $t > 0$,

$$P_t^{\phi_\alpha} f = \sum_{n=0}^{\infty}\mathcal{E}_{\phi_\alpha}(-(n(n-1)+\lambda_1 n)t^\alpha)\langle f, \mathcal{V}_n\rangle_\beta \mathcal{P}_n \quad \text{in } L^2(\beta).$$

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