Decomposing Properties into Safety and Liveness using Predicate Logic†

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ABSTRACT

A new proof is given that every property can be expressed as a conjunction of safety and liveness properties. The proof is in terms of first-order predicate logic.

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1. Introduction

Two classes of properties are of particular interest when considering programs: safety properties and liveness properties. Informally, a *safety property* stipulates that "bad things" do not happen during execution of a program and a *liveness property* stipulates that "good things" do happen (eventually) [2]. Distinguishing between safety and liveness properties is useful because knowing whether a property is safety or liveness helps when deciding how to prove that the property holds for a program.

In [1], formal definitions of safety and liveness are given and it is proved that every property can be expressed as the conjunction of a safety property and a liveness property. The formal definitions of safety and liveness are given in terms of first-order predicate logic, but the proof that every property can be decomposed into safety and liveness is not—it uses topology. The purpose of this paper is to give a proof of this theorem using only first-order predicate logic.

2. Specifying Properties

A *program state* is a mapping from variables to values. An execution of a concurrent program can be viewed as an infinite sequence of program states

$$\sigma = s_0 \, s_1 \, \ldots,$$

which we call a *history*. In a history, s_0 is an initial state of the program and each subsequent state results from executing a single atomic action in the preceding state. (For a terminating execution, an infinite sequence is obtained by repeating the final state.) A *property* is a set of such sequences.

One way to specify a property is by using first-order predicate logic. For a state s, define $s.v$ to be the value of variable v in that state. A formula of first-order predicate logic where s is the only free variable defines a set of states. For example,

$$(\forall i: \; 1 \le i < N: \; s.a[i] \le s.a[i+1])$$

specifies the set of states in which the elements of array $a[1:N]$ are sorted. Usually "$s.$" is implicit and therefore left out of such a formula, resulting in the more familiar use of first-order predicate logic as an assertion language.

A set of sequences of states—a property—can also be defined using first-order predicate logic. To facilitate such specifications, for any sequence $\sigma = s_0 \, s_1 \ldots$ define for $0 \le i$:

$\sigma[i] \;\equiv\; s_i$.
$\sigma[..i] \;\equiv\; s_0 \, s_1 \ldots s_{i-1}$. The empty sequence if $i=0$.
$|\sigma| \;\equiv\;$ the length of σ (ω if σ is infinite).

A formula of first-order predicate logic in which σ is the only free variable defines the set of sequences that satisfy the formula and therefore specifies a property. For example,

$$(\forall i: \; 0 \le i: \; \sigma[i].v = 0)$$

specifies the property in which the value of v remains 0 throughout execution.

We write $\alpha \models P$ if $\alpha \in S^{\omega}$ is in the property specified by P. Thus,

$$\alpha \models P \ = \ P^{\sigma}_{\alpha}.$$
$$\alpha \not\models P \ = \ \neg P^{\sigma}_{\alpha}.$$

3. Safety and Liveness

According to [1], a property P is a safety property provided

$$\textbf{Safety: } (\forall \sigma\colon \sigma \in S^{\omega}\colon \sigma \not\models P \Rightarrow (\exists i\colon 0 \le i\colon (\forall \beta\colon \beta \in S^{\omega}\colon \sigma[..i]\beta \not\models P))), \tag{3.1}$$

where S is the set of program states, S^* the set of finite sequences of states, S^{ω} the set of infinite sequences of states, and juxtaposition is used to denote catenation of sequences. A property P is a liveness property provided

$$\textbf{Liveness: } (\forall \alpha\colon \alpha \in S^*\colon (\exists \beta\colon \beta \in S^{\omega}\colon \alpha\beta \models P)). \tag{3.2}$$

Given a property P, we are interested in defining properties $Safe(P)$ and $Live(P)$ such that

- $Safe(P)$ is a safety property,
- $Live(P)$ is a liveness property, and
- $P = Safe(P) \wedge Live(P)$.

Observe that if

$$Safe(P) = P \vee M_P$$
$$Live(P) = P \vee \neg M_P$$

then

$$\begin{aligned} Safe(P) \wedge Live(P) &= (P \vee M_P) \wedge (P \vee \neg M_P) \\ &= (P \wedge P) \vee (P \wedge \neg M_P) \vee (M_P \wedge P) \vee (M_P \wedge \neg M_P) \\ &= P \end{aligned}$$

Hence, we have only to look for an M_P that makes $P \vee M_P$ (i.e. $Safe(P)$) a safety property and $P \vee \neg M_P$ (i.e. $Live(P)$) a liveness property.

It turns out that using

$$M_P\colon (\forall i\colon 0 \le i\colon (\exists \beta\colon \beta \in S^{\omega}\colon \sigma[..i]\beta \models P))$$

suffices. First, we show formally that $Safe(P)$ satisfies definition (3.1) of safety. The proof that follows is a sequence of first-order predicate logic formulas with explanations interspersed (and delimited by « and ») of how each formula is derived from its predecessor.

Choose any $\sigma \in S^{\omega}$:

$\sigma \not\models Safe(P)$

«by definition of $Safe(P)$»

$= \quad \sigma \not\models (P \vee (\forall i\colon\ 0 \le i\colon\ (\exists \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \models P)))$

«by definition of $\not\models$»

$= \quad \neg(P \vee (\forall i\colon\ 0 \le i\colon\ (\exists \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \models P)))^{\sigma}_{\sigma}$

«by substitution»

$= \quad \neg(P \vee (\forall i\colon\ 0 \le i\colon\ (\exists \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \models P)))$

«by De Morgan's Laws»

$= \quad \neg P \wedge (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P))$

«$A \wedge B \Rightarrow B$»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P))$

«because $(\forall x{::}\ A) = (\forall x{::}\ A \wedge (\forall y{::}\ A^{x}_{y}))$»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P \wedge (\forall \gamma\colon\ \gamma \in S^{\omega}\colon\ \sigma[..i]\gamma \not\models P)))$

«because $true \wedge P = P$ and $(\sigma[..i]\beta)[..i] = \sigma[..i]$»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P \wedge (i=i) \wedge (\forall \gamma\colon\ \gamma \in S^{\omega}\colon\ (\sigma[..i]\beta)[..i]\gamma \not\models P)))$

«by substitution»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P \wedge (k=i)^{k}_{i} \wedge (\forall \gamma\colon\ \gamma \in S^{\omega}\colon\ (\sigma[..i]\beta)[..k]\gamma \not\models P)^{k}_{i}))$

«by $\exists$-Generalization»

$\Rightarrow \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P \wedge (\exists k\colon\ k=i\colon\ (\forall \gamma\colon\ \gamma \in S^{\omega}\colon\ (\sigma[..i]\beta)[..k]\gamma \not\models P))))$

«by Range Widening»

$\Rightarrow \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P \wedge (\exists k\colon\ 0 \le k\colon\ (\forall \gamma\colon\ \gamma \in S^{\omega}\colon\ (\sigma[..i]\beta)[..k]\gamma \not\models P))))$

«by De Morgan's Law»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P \wedge \neg(\forall k\colon\ 0 \le k\colon\ (\exists \gamma\colon\ \gamma \in S^{\omega}\colon\ (\sigma[..i]\beta)[..k]\gamma \models P))))$

«by definition of $\not\models$»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P \wedge \sigma[..i]\beta \not\models (\forall k\colon\ 0 \le k\colon\ (\exists \gamma\colon\ \gamma \in S^{\omega}\colon\ \sigma[..k]\gamma \models P))))$

«because $\alpha \not\models A \wedge \alpha \not\models B \ = \ \alpha \not\models (A \vee B)$»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models (P \vee (\forall k\colon\ 0 \le k\colon\ (\exists \gamma\colon\ \gamma \in S^{\omega}\colon\ \sigma[..k]\gamma \models P)))))$

«by definition of $Safe(P)$»

$= \quad (\exists i\colon\ 0 \le i\colon\ (\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models Safe(P)))$

It is not surprising that $Safe(P)$ is a safety property. If $\sigma \not\models Safe(P)$ then, by definition, $\sigma \not\models M_P$. However, this means there exists an i such that

$$(\forall \beta\colon\ \beta \in S^{\omega}\colon\ \sigma[..i]\beta \not\models P).$$

We could consider prefix $\sigma[..i]$ to be a "bad thing". Thus, σ violates a safety property whenever $\sigma \not\models Safe(P)$.

We now show formally that $Live(P)$ satisfies definition (3.2) of liveness.

$\quad (\forall \alpha\colon\ \alpha \in S^{*}\colon\ true)$

«since $true = A \vee \neg A$»

$= \quad (\forall \alpha\colon\ \alpha \in S^{*}\colon\ (\exists \beta\colon\ \beta \in S^{\omega}\colon\ \alpha\beta \models P) \vee \neg(\exists \beta\colon\ \beta \in S^{\omega}\colon\ \alpha\beta \models P))$

«renaming bound variable β to γ»

$= \quad (\forall \alpha\colon\ \alpha \in S^{*}\colon\ (\exists \beta\colon\ \beta \in S^{\omega}\colon\ \alpha\beta \models P) \vee \neg(\exists \gamma\colon\ \gamma \in S^{\omega}\colon\ \alpha\gamma \models P))$

«since β is not free in $(\exists \gamma\colon\ \gamma \in S^{\omega}\colon\ \alpha\gamma \models P)$»

$= \quad (\forall \alpha\colon\ \alpha \in S^{*}\colon\ (\exists \beta\colon\ \beta \in S^{\omega}\colon\ \alpha\beta \models P \vee \neg(\exists \gamma\colon\ \gamma \in S^{\omega}\colon\ \alpha\gamma \models P)))$

«by De Morgan's Law»

$= \quad (\forall \alpha\colon\ \alpha \in S^{*}\colon\ (\exists \beta\colon\ \beta \in S^{\omega}\colon\ \alpha\beta \models P \vee (\forall \gamma\colon\ \gamma \in S^{\omega}\colon\ \alpha\gamma \not\models P)))$

«since *true* $\wedge A = A$»

$=$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models P \vee (|\alpha|=|\alpha| \wedge (\forall\gamma\colon\ \gamma\in S^\omega\colon\ \alpha\gamma\not\models P))))$

«by substitution, since $(\alpha\beta)[..|\alpha|]=\alpha$»

$=$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models P \vee ((i=|\alpha|)^i_{|\alpha|} \wedge (\forall\gamma\colon\ \gamma\in S^\omega\colon\ (\alpha\beta)[..i]\gamma\not\models P)^i_{|\alpha|})))$

«by $\exists$-Generalization»

$\Rightarrow$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models P \vee (\exists i\colon\ i=|\alpha|\colon\ (\forall\gamma\colon\ \gamma\in S^\omega\colon\ (\alpha\beta)[..i]\gamma\not\models P))))$

«by Range Widening»

$\Rightarrow$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models P \vee (\exists i\colon\ 0\le i\colon\ (\forall\gamma\colon\ \gamma\in S^\omega\colon\ (\alpha\beta)[..i]\gamma\not\models P))))$

«by De Morgan's Law»

$=$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models P \vee \neg(\forall i\colon\ 0\le i\colon\ (\exists\gamma\colon\ \gamma\in S^\omega\colon\ (\alpha\beta)[..i]\gamma\models P))))$

«by definition of $\alpha\beta\models A$»

$=$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models P \vee \alpha\beta\models\neg(\forall i\colon\ 0\le i\colon\ (\exists\gamma\colon\ \gamma\in S^\omega\colon\ \sigma[..i]\gamma\models P))))$

«because $\alpha\beta\models A \vee \alpha\beta\models B = \alpha\beta\models(A \vee B)$»

$=$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models (P \vee \neg(\forall i\colon\ 0\le i\colon\ (\exists\gamma\colon\ \gamma\in S^\omega\colon\ \sigma[..i]\gamma\models P))))$

«by definition of *Live*(P)»

$=$ $(\forall\alpha\colon\ \alpha\in S^*\colon\ (\exists\beta\colon\ \beta\in S^\omega\colon\ \alpha\beta\models Live(P)))$

«by Liveness definition (3.2)»

$=$ *Live*(P) is liveness.

An informal justification that *Live*(P) is liveness is the following. If $\sigma\not\models Live(P)$ then, by definition, $\sigma\models M_P$. From, $\sigma\models M_P$, we conclude that it always remains possible for some "good thing" (i.e. β in M_P) to happen. This is the defining characteristic of liveness, so σ violates a liveness property whenever $\sigma\not\models Live(P)$.

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References

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