Decomposing Properties into Safety and Liveness using Predicate Logic†

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Decomposing Properties into Safety and Liveness

using Predicate Logic

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ABSTRACT

A new proof is given that every property can be expressed as a conjunction of safety and liveness properties. The proof is in terms of first-order predicate logic.

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1. Introduction

Two classes of properties are of particular interest when considering programs: safety properties and liveness properties. Informally, a safety property stipulates that "bad things" do not happen during execution of a program and a liveness property stipulates that "good things" do happen (eventually) [2]. Distinguishing between safety and liveness properties is useful because knowing whether a property is safety or liveness helps when deciding how to prove that the property holds for a program.

In [1], formal definitions of safety and liveness are given and it is proved that every property can be expressed as the conjunction of a safety property and a liveness property. The formal definitions of safety and liveness are given in terms of first-order predicate logic, but the proof that every property can be decomposed into safety and liveness is not—it uses topology. The purpose of this paper is to give a proof of this theorem using only first-order predicate logic.

2. Specifying Properties

A program state is a mapping from variables to values. An execution of a concurrent program can be viewed as an infinite sequence of program states

\[ \sigma = s_0 s_1 \ldots, \]

which we call a history. In a history, \( s_0 \) is an initial state of the program and each subsequent state results from executing a single atomic action in the preceding state. (For a terminating execution, an infinite sequence is obtained by repeating the final state.) A property is a set of such sequences.

One way to specify a property is by using first-order predicate logic. For a state \( s \), define \( s.\mathbf{v} \) to be the value of variable \( \mathbf{v} \) in that state. A formula of first-order predicate logic where \( s \) is the only free variable defines a set of states. For example,

\( (\forall i: 1 \leq i < N: s.\mathbf{a}[i] \leq s.\mathbf{a}[i+1]) \)

specifies the set of states in which the elements of array \( \mathbf{a}[1:N] \) are sorted. Usually "s." is implicit and therefore left out of such a formula, resulting in the more familiar use of first-order predicate logic as an assertion language.

A set of sequences of states—a property—can also be defined using first-order predicate logic. To facilitate such specifications, for any sequence \( \sigma = s_0 s_1 \ldots \) define for \( 0 \leq i \):

\[ \sigma[i] = s_i, \]
\[ \sigma[\ldots i] = s_0 s_1 \ldots s_{i-1}. \] The empty sequence if \( i=0 \).
\[ |\sigma| = \text{the length of } \sigma \text{ (} \omega \text{ if } \sigma \text{ is infinite).} \]

A formula of first-order predicate logic in which \( \sigma \) is the only free variable defines the set of sequences that satisfy the formula and therefore specifies a property. For example,

\( (\forall i: 0 \leq i: \sigma[i].\mathbf{v}=0) \)

specifies the property in which the value of \( \mathbf{v} \) remains 0 throughout execution.
We write $\alpha \models P$ if $\alpha \in S^\alpha$ is in the property specified by $P$. Thus,
\[
\alpha \models P = P^\alpha_{\alpha}.
\]
\[
\alpha \not\models P = \neg P^\alpha_{\alpha}.
\]

3. Safety and Liveness

According to [1], a property $P$ is a safety property provided
\[
\text{Safety: } (\forall \sigma: \; \sigma \in S^\alpha: \; \sigma \not\models P \Rightarrow (\exists i: \; 0 \leq i: \; (\forall \beta: \; \beta \in S^\alpha: \; \sigma[\cdot i] \beta \not\models P)))\),
\]
where $S$ is the set of program states, $S^*$ the set of finite sequences of states, $S^\alpha$ the set of infinite sequences of states, and juxtaposition is used to denote catenation of sequences. A property $P$ is a liveness property provided
\[
\text{Liveness: } (\forall \alpha: \; \alpha \in S^*: \; (\exists \beta: \; \beta \in S^\alpha: \; \alpha \beta \models P))\).
\]

Given a property $P$, we are interested in defining properties $\text{Safe}(P)$ and $\text{Live}(P)$ such that

- $\text{Safe}(P)$ is a safety property,
- $\text{Live}(P)$ is a liveness property, and
- $P = \text{Safe}(P) \land \text{Live}(P)$.

Observe that if
\[
\text{Safe}(P) = P \lor M_P
\]
\[
\text{Live}(P) = P \lor \neg M_P
\]
then
\[
\text{Safe}(P) \land \text{Live}(P) = (P \lor M_P) \land (P \lor \neg M_P)
\]
\[
= (P \land P) \lor (P \land \neg M_P) \lor (M_P \land P) \lor (M_P \land \neg M_P)
\]
\[
= P
\]
Hence, we have only to look for an $M_P$ that makes $P \lor M_P$ (i.e. $\text{Safe}(P)$) a safety property and $P \lor \neg M_P$ (i.e. $\text{Live}(P)$) a liveness property.

It turns out that using
\[
M_P: (\forall i: \; 0 \leq i: \; (\exists \beta: \; \beta \in S^\alpha: \; \sigma[\cdot i] \beta \models P))
\]
suffices. First, we show formally that $\text{Safe}(P)$ satisfies definition (3.1) of safety. The proof that follows is a sequence of first-order predicate logic formulas with explanations interspersed (and delimited by « and ») of how each formula is derived from its predecessor.

Choose any $\sigma \in S^\alpha$:
\[
\sigma \not\models \text{Safe}(P)
\]
«by definition of Safe (P)»

= \( \sigma^\#(P \lor (\forall_i: 0 \leq i: (\exists \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P))) \)

«by definition of \( ^\# \)»

= \( \neg((P \lor (\forall_i: 0 \leq i: (\exists \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P))) \)

«by substitution»

= \( \neg(P \lor (\forall_i: 0 \leq i: (\exists \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P))) \)

«by De Morgan's Laws»

= \( \neg P \land (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P)) \)

«A \land B \Rightarrow B»

= \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P)) \)

«because \((\forall x:: A) = (\forall x:: A \land (\forall y:: A)\)»

= \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \land (\forall \gamma:\ \gamma \in S^\omega:\ \sigma[i]\gamma \vdash P))) \)

«because true \land P = P and \((\sigma[i]\beta)[i] = \sigma[i]\)»

= \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \land (i = i) \land (\forall \gamma:\ \gamma \in S^\omega:\ \sigma[i]\beta)[i] \gamma \vdash P))) \)

«by substitution»

= \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \land (k = i)^k \land (\forall \gamma:\ \gamma \in S^\omega:\ \sigma[i]\beta)[k] \gamma \vdash P))) \)

«by \( \exists \) - Generalization»

⇒ \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \land (\exists k: k = i: (\forall \gamma:\ \gamma \in S^\omega:\ \sigma[i]\beta)[k] \gamma \vdash P))) \)

«by Range Widening»

⇒ \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \land (\forall k: 0 \leq k: (\forall \gamma:\ \gamma \in S^\omega:\ \sigma[i]\beta)[k] \gamma \vdash P))) \)

«by De Morgan's Law»

⇒ \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \land (\forall k: 0 \leq k: (\exists \gamma:\ \gamma \in S^\omega:\ \sigma[i]\beta)[k] \gamma \vdash P))) \)

«by definition of \#»

⇒ \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \land (\forall k: 0 \leq k: (\exists \gamma:\ \gamma \in S^\omega:\ \sigma[i]\beta)[k] \gamma \vdash P))) \)

«because \( \alpha \# A \land \alpha \# B = \alpha \# (A \lor B)\)»

⇒ \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \vdash P \lor (\forall k: 0 \leq k: (\exists \gamma:\ \gamma \in S^\omega:\ \sigma[k] \gamma \vdash P))) \)

«by definition of Safe (P)»

⇒ \( (\exists i: 0 \leq i: (\forall \beta:\ \beta \in S^\omega:\ \sigma[i]\beta \# \text{Safe (P)}) \)

It is not surprising that Safe (P) is a safety property. If \( \sigma \# \text{Safe (P)} \) then, by definition, \( \sigma \# M_P \). However, this means there exists an \( i \) such that

\( (\forall \beta:\ \beta \in S^\omega:\ \sigma[i] \beta \# P) \)

We could consider prefix \( \sigma[i] \) to be a "bad thing". Thus, \( \sigma \) violates a safety property whenever \( \sigma \# \text{Safe (P)} \).

We now show formally that Live (P) satisfies definition (3.2) of liveness.

\( (\forall \alpha: \alpha \in S^*:\ true) \)

«since true = A \lor \neg A»

= \( (\forall \alpha: \alpha \in S^*:\ (\exists \beta:\ \beta \in S^\omega:\ \alpha \beta \vdash P) \lor \neg(\exists \beta:\ \beta \in S^\omega:\ \alpha \beta \vdash P)) \)

«renaming bound variable \( \beta \) to \( \gamma \)»

= \( (\forall \alpha: \alpha \in S^*:\ (\exists \beta:\ \beta \in S^\omega:\ \alpha \beta \vdash P) \lor \neg(\exists \gamma:\ \gamma \in S^\omega:\ \alpha \gamma \vdash P)) \)

«since \( \beta \) is not free in \( (\exists \gamma:\ \gamma \in S^\omega:\ \alpha \gamma \vdash P)) \)

= \( (\forall \alpha: \alpha \in S^*:\ (\exists \beta:\ \beta \in S^\omega:\ \alpha \beta \vdash P \lor \neg(\exists \gamma:\ \gamma \in S^\omega:\ \alpha \gamma \vdash P)) \)

«by De Morgan's Law»

= \( (\forall \alpha: \alpha \in S^*:\ (\exists \beta:\ \beta \in S^\omega:\ \alpha \beta \vdash P \lor (\forall \gamma:\ \gamma \in S^\omega:\ \alpha \gamma \vdash P)) \)
\[\begin{align*}
\text{«since true } \land A = A» \\
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash P \lor (i = 1 \alpha_1) \land (\forall \gamma \in S^{\alpha_0}: \alpha \gamma \vdash \neg P)))
\text{«by substitution, since } (\alpha \beta)[\vdash \alpha ] = \alpha\]
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash P \lor (i = 1 \alpha_1) \land (\forall \gamma \in S^{\alpha_0}: \alpha \beta)[\vdash \gamma \vdash \neg P] ))
\text{«by \exists-GENERALIZATION»}
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash P \lor (\forall i: 0 \leq i: (\forall \gamma \in S^{\alpha_0}: \alpha \beta)[\vdash \gamma \vdash \neg P]) )
\text{«by Range Widening»}
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash P \lor (\forall i: 0 \leq i: (\forall \gamma \in S^{\alpha_0}: \alpha \beta)[\vdash \gamma \vdash \neg P]) )
\text{«by De Morgan's Law»}
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash P \lor \neg (\forall i: 0 \leq i: (\exists \gamma \in S^{\alpha_0}: (\alpha \beta)[\vdash \gamma \vdash P])))
\text{«by definition of } \neg \gamma \vdash \neg P\text{»}
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash P \lor \neg (\forall i: 0 \leq i: (\exists \gamma \in S^{\alpha_0}: (\alpha \beta)[\vdash \gamma \vdash P]))
\text{«because } \alpha \beta \vdash A \lor \alpha \beta \vdash B = \alpha \beta \vdash (A \lor B)\text{»}
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash (P \lor \neg (\forall i: 0 \leq i: (\exists \gamma \in S^{\alpha_0}: (\sigma \vdash \gamma \vdash P))))
\text{«by definition of } \neg \gamma \vdash \neg P\text{»}
&= (\forall \alpha \in S^*: (\exists \beta: \beta \in S^{\alpha_0}: \alpha \beta \vdash \neg \gamma \vdash \neg (\forall i: 0 \leq i: (\exists \gamma \in S^{\alpha_0}: (\sigma \vdash \gamma \vdash P))))
\text{«by Liveness definition (3.2)»}
&= \text{Live}(P) \text{ is liveness.}
\end{align*}\]

An informal justification that \text{Live}(P) is liveness is the following. If } \sigma \vdash \text{Live}(P) \text{ then, by definition, } \sigma \vdash M_P. \text{ From, } \sigma \vdash M_P, \text{ we conclude that it always remains possible for some "good thing" (i.e. } \beta \text{ in } M_P \text{) to happen. This is the defining characteristic of liveness, so } \sigma \text{ violates a liveness property whenever } \sigma \vdash \neg \text{Live}(P).\]

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References