# Geometric Ambiguities in Boundary Representations*

Christoph Hoffmann
John Hopcroft
TR 86-725
January 1986

Department of Computer Science
Cornell University
Ithaca, NY 14853

* Supported in part by National Science Foundation Grants MCS 82-17996, ECS 83-12096, and DCR 85-02568.

*Abstract*

Boundary representations are usually separated into two components, a topological component and a geometric component. The conditions necessary to insure that the topological component is unambiguous are well understood. However, in algebraic modeling systems (as opposed to polyhedral modeling systems), an unambiguous topological structure can have noncongruent geometric interpretations with identical vertex coordinates, face equations and edge descriptions using the usual representation of edges as the intersection of faces along with tangent vectors at vertices. This means that the conversion of a CSG tree that unambiguously describes a solid can lead to an ambiguous boundary representations unless additional information is retained in the boundary representation.

This paper examines the source of these ambiguities and presents one method for their elimination.

## 1. *Introduction*

Geometric modeling must be based on a flexible, yet unambiguous scheme for representing objects and sets of objects. Among the schemes used in current modeling systems is the familiar *boundary* representation method, in which the surface of an object is described, e.g. [1, 2]. This method has been used extensively in computer graphics, yet it is by no means a simple task to lay down a complete set of precise criteria permitting formal verification that a given boundary representation scheme is unambiguous and informationally complete in the sense of [8].

A number of papers, e.g. [4, 9, 10, 11], have addressed the intrinsic problems of boundary representations, and the field has succeeded in giving precise conditions under which boundary representations of polygonal objects are correct and unambiguous. Here one of the major issues has been to understand the topological properties of surface descriptions [9]. These topological properties generalize to the curved surface domain. However, geometric problems arise that can lead to ambiguities even when the topology of the surface has been unambiguously specified. These difficulties relate to the following phenomena:

(1) A curved face may have points or curves of singularity. For example, the vertex of a cone is a singular point. and the self intersection curves of certain canal surfaces are curves of singularity.

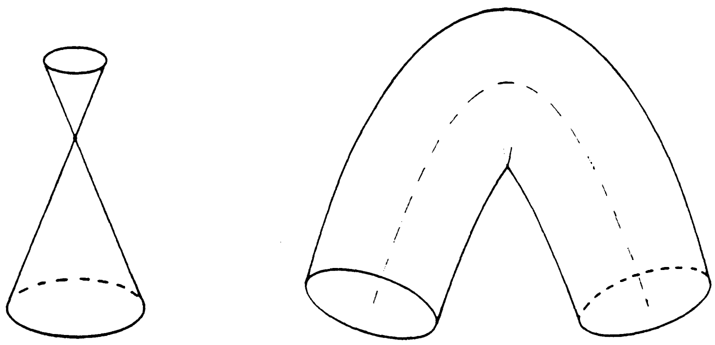

Figure 1.1
Singular Points and Curves

(2) The intersection curve of two faces may be closed, and no intrinsic direction of the edge can be defined naturally.

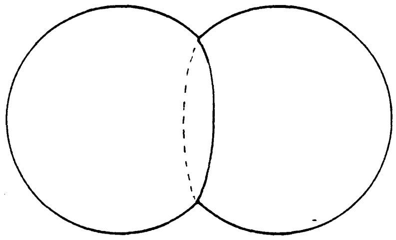

Figure 1.2
Undirected Closed Intersection Curve

(3) The intersection of two curved faces may have multiple components or singular points. For example, the orthogonal, axial intersection of two circular cylinders of equal radius has two intersecting components, and the points of intersection are singularities.

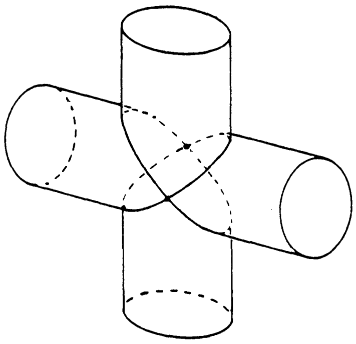

Figure 1.3
Intersection Curve with two Components and two Singular Points

It has been noted previously that singularities on edges can cause problems in boundary representations [7] that require nonlocal information to resolve. However, what has not been recognized is that there are different, noncongruent interpretations of curved surface boundary representations that are consistent with an unambiguously defined surface topology. Thus even global processing of the boundary representation is not sufficient to resolve the ambiguity and additional information must be incorporated into the model.

## 2. *Preliminaries*

In a boundary representation, the surface of an object is specified by giving the following information:

(1) A finite set of vertices, usually specified by Cartesian coordinates in Euclidian space.
(2) A set of edges, where each edge is incident to two vertices as specified.
(3) A set of faces, where each face is bounded by (say) a single cycle of edges.

In addition, edge and face adjacency information is provided.

Typically, an edge is specified by the intersection of two faces, one on the left and one on the right. Here *left* and *right* are defined relative to the edge direction as seen from the exterior of the object. By convention, the edge $(u,v)$ is considered directed from vertex $u$ to vertex $v$. Moreover, the cyclical order of incident edges is specified at each vertex, say in clockwise order, and the surface equation for each face, when specified, has been chosen such that the gradient vector points to the exterior of the object. Additional conventions are understood, e.g., faces do not self intersect, two distinct faces intersect only in edges, etc.

For polygonal objects, it is simple to define edges and faces geometrically: An edge is uniquely specified as the line segment connecting two vertices, and the face plane may be specified by its equation or by three distinct non collinear face vertices.

In contrast, representing curved faces and edges is more difficult. We assume that the surface, of which the face is a patch is specified by an implicit equation. We construct an example of an ambiguous representation from algebraic surfaces, but could have used other surfaces equally well. An edge is the intersection of two adjacent faces. When the adjacent faces belong to the same surface, an auxiliary surface is used whose intersection fixes the edge.

We will show that these rather intuitive conventions are insufficient to describe edges and faces unambiguously. An immediate problem arises from the fact that, in general, the intersection curve of two surfaces does not have an intrinsic direction. Sometimes, an intended direction can be implied by the global adjacency structure. That is, although locally at a vertex no information need be present concerning the direction in which to traverse the space curve so as to reach the opposite vertex, only one direction may actually lead to the correct vertex. An example of this is the description of the lens in Section 3 below when three or more vertices are provided. In general, however, global structure is insufficient and explicit information is required. Although in the example constructed, edge tangents at the vertices would remove the ambiguity, for objects with higher degree surfaces even the tangents are not sufficient to remove the ambiguity.

## 3. *Edge Cycles vs. Left and Right Face Specifications*

Consider modeling the intersection of two spheres, $S_1$ and $S_2$, as shown in Figure 3.1. The resulting lens has the edge $S_1 \cap S_2$, on which we arbitrarily place a vertex $u$ as shown. The edge may then be given as $(u,u)$. Note that closed edges are perfectly legitimate from the point of view of Euler operations [4, 11].

When orienting the edge by the vector drawn at $u$, the correct boundary description of the lens would specify the face on $S_1$ to the left of the (directed) edge, as seen from the outside, and the face on $S_2$ to the right. Since the edge $S_1 \cap S_2$ does not have an intrinsic direction, we could have oriented it in the opposite direction. In this case, the same boundary representation has described the object shown in Figure 3.2, i.e., the union of the two spheres.

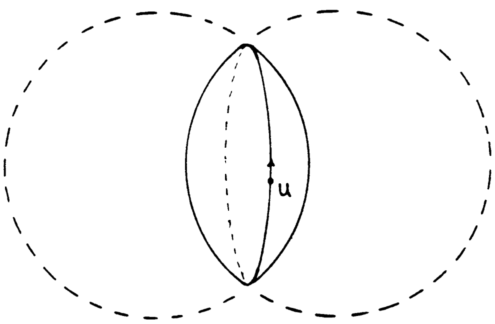


Figure 3.1

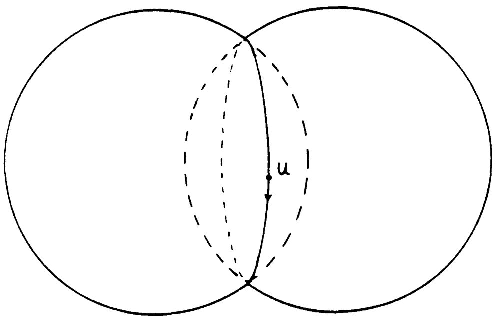


Figure 3.2

Effect of Edge Orientation

A possible remedy is to place more vertices on the edge. A minimum of three vertices is required to encode the edge directionality. Such a representation is shown in Figure 3.3 and no longer is ambiguous.

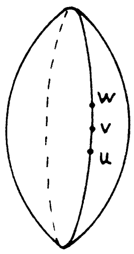


Figure 3.3
Three Vertices to imply Edge Direction

Note, however, that we must traverse an edge, say $(u,v)$, before we can decide whether the initial direction of traversal was the correct one. We show next that this convention does not solve the problem in general.

## 4. *Singularities*

It has been pointed out by Requicha [7] that singular points in the interior of an edge cause uncertainty when traversing the edge. Requicha's example is shown in Figure 4.1.

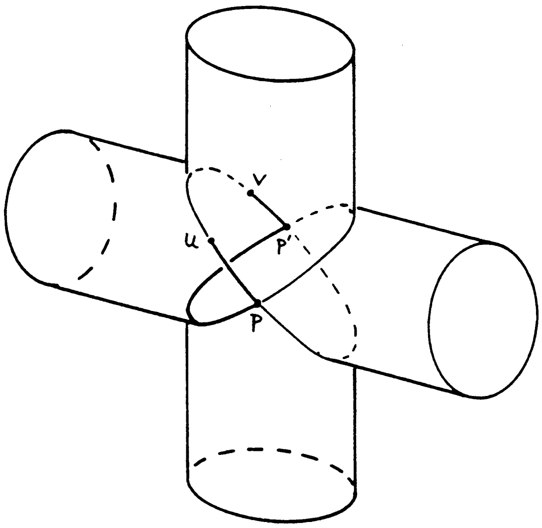


Figure 4.1
Requicha's Edge Ambiguity

When traversing the edge $e$ from $u$ to $v$, at the singular point $p$ we cannot decide locally the direction in which to continue the traversal. Note that this type of singularity arises from very simple intersecting surfaces, but can be handled by requiring that there are no singular points on the interior of faces or edges.

A point $p$ is *singular* on the surface $f=0$ if the gradient $\nabla f(p)$ is zero. On algebraic surfaces, the singular points cannot fill up more than certain individual curves and points [6]. Clearly, faces with singular points are needed in geometric modeling. For example, the vertex of a cone is a singular point that must be modeled in many situations. In order to isolate singularities, we require that all singular points of a face be vertices in the boundary description, and that, moreover, all interior edge points are regular, i.e., possess a unique tangent. Hence, the edge $(u,v)$ of Figure 4.1 is illegitimate because of the singular edge points $p$ and $p'$.

We show that in the presence of singular vertices ambiguous boundary descriptions exist that satisfy the following restrictions:

(1) Every edge cycle consists of at least three edges and every vertex is incident to at least three edges.

(2) All interior points of an edge are regular points.

(3) All interior points of a face are regular points.

Moreover. the topology of the object is unique by the results of [9, 10].

We construct the object first using constructive solid geometry in which each primitive volume, specified by an algebraic equation $f=0$. is the closure of the set of all points $p$ for which $f(p)<0$. Thus we deal with regular sets. We use the standard set [7, 8] of primitives augmented by the cylinder $C$: $y^3+z^3-6yz=0$.

The quartic faces of the object are toroidal. First, we build the object of Figure 4.2 by subtracting tori $T_2$ and $T_3$ from the halved torus $T_1 \cap H$, where $H$ is the half space $y=0$. The corresponding CSG expression is $((T_1 \cap H)-T_2)-T_3$. The tori dimensions are as indicated in the Figure.

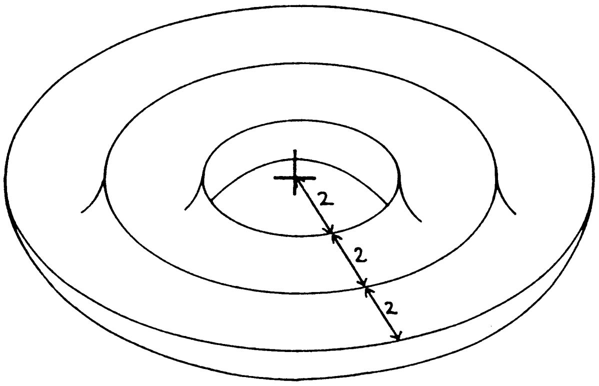


Figure 4.2

Next, we take the cylinder $C$ whose cross cut is the Cartesian Leaf. The cylinder is shown in Figure 4.3. The positive surface side, i.e., the side pointing to the outside of $C$ has been shaded, and the gradients are directed as drawn.

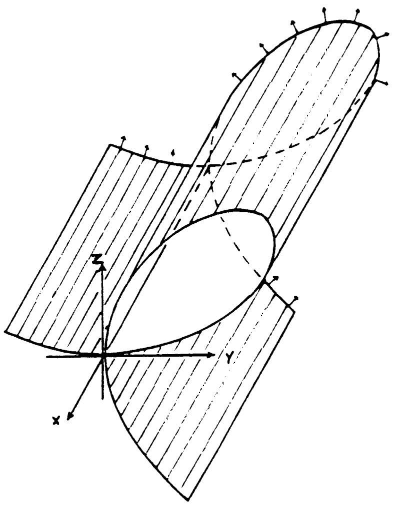


Figure 4.3
The Cylinder $y^3+z^3-6yz=0$

Consequently, the intersection $(((T_1 \cap H)-T_2)-T_3) \cap C$ is the object shown in Figure 4.4 below.

We now give a boundary description of the object. Here. edges are assumed oriented as specified by the vertex pair. e.g., edge $e_1$ is from vertex $v_1$ to vertex $v_3$. The opposite direction is indicated by negating the edge symbol, e.g.. $-e_1$ means the edge $e_1$ in opposite direction.

Vertices:

$v_1$: $(-6,0,0)$
$v_2$: $(-4,0,0)$
$v_3$: $(-2,0,0)$
$v_4$: $(2,0,0)$
$v_5$: $(4,0,0)$
$v_6$: $(6,0,0)$

Face equations:

$$
\begin{aligned}
a&: (x^2+y^2+z^2-4)^2+32(y^2-x^2-z^2-4)+256 = 0,\\
b&: -(x^2+y^2+z^2-1)^2-18(y^2-x^2-z^2-1)-81 = 0,\\
c&: -(x^2+y^2+z^2-1)^2-50(y^2-x^2-z^2-1)-625 = 0,\\
d&: y^3+z^3-6yz = 0,\\
e&: y^3+z^3-6yz = 0,
\end{aligned}
$$

Edges with left and right adjacent face:

$e_1$: $(v_1, v_3)$, $(a,d)$
$e_2$: $(v_3, v_2)$, $(b,d)$
$e_3$: $(v_2, v_1)$, $(c,d)$
$e_4$: $(v_6, v_4)$, $(e,a)$
$e_5$: $(v_4, v_5)$, $(e,b)$
$e_6$: $(v_5, v_6)$, $(e,c)$
$e_7$: $(v_1, v_6)$, $(c,a)$
$e_8$: $(v_2, v_5)$, $(b,c)$
$e_9$: $(v_3, v_4)$, $(a,b)$

Edge cycles clockwise about each face:

$a$: $(-e_1, e_7, e_4, -e_9)$
$b$: $(-e_2, e_9, e_5, -e_8)$
$c$: $(-e_3, e_8, e_6, -e_7)$
$d$: $(e_1, e_2, e_3)$
$e$: $(-e_4, -e_6, -e_5)$

Incident edges clockwise about each vertex:

$v_1$: $(e_1, -e_3, e_7)$
$v_2$: $(e_3, -e_2, e_8)$
$v_3$: $(e_2, -e_1, e_9)$
$v_4$: $(e_5, -e_9, -e_4)$
$v_5$: $(e_6, -e_8, -e_5)$
$v_6$: $(e_4, -e_7, -e_6)$

Note that the representation is topologically unambiguous.

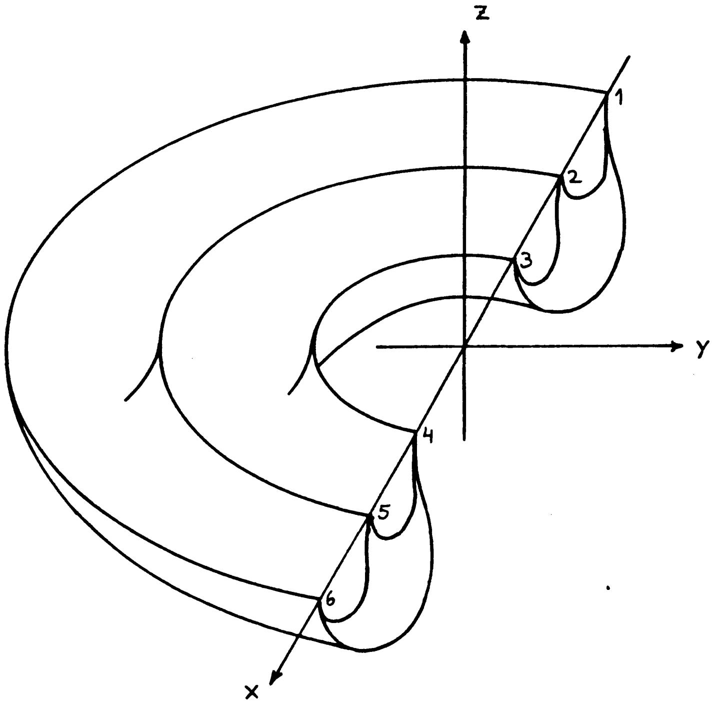


Figure 4.4

$(((T_1 \cap H) - T_2) - T_3)$

Since the vertices are singular, it is possible to interpret the edge positions differently by following a different branch of the intersection curve of $C$ with the torus. The complete intersection curve of the torus $T_1$ with $C$ is shown in Figure 4.5.

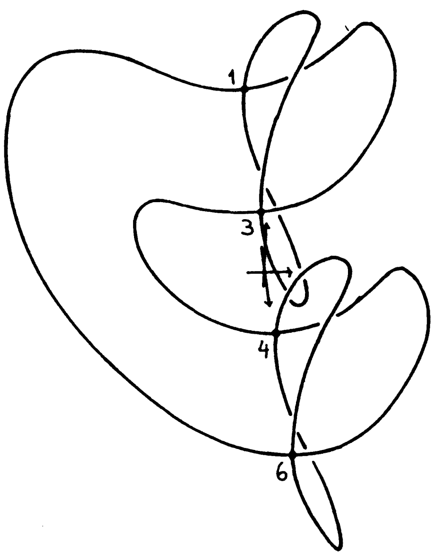


Figure 4.5
Intersection Curve $(C, T_1)$

Four directions may be followed at each vertex, but only two of them are consistent with the convention of outward pointing face normals and the topology of the boundary description. For global topological consistency, the direction choices must agree at all vertices, so there can be only two correct interpretations of the above boundary description. The second interpretation is shown in Figure 4.6.

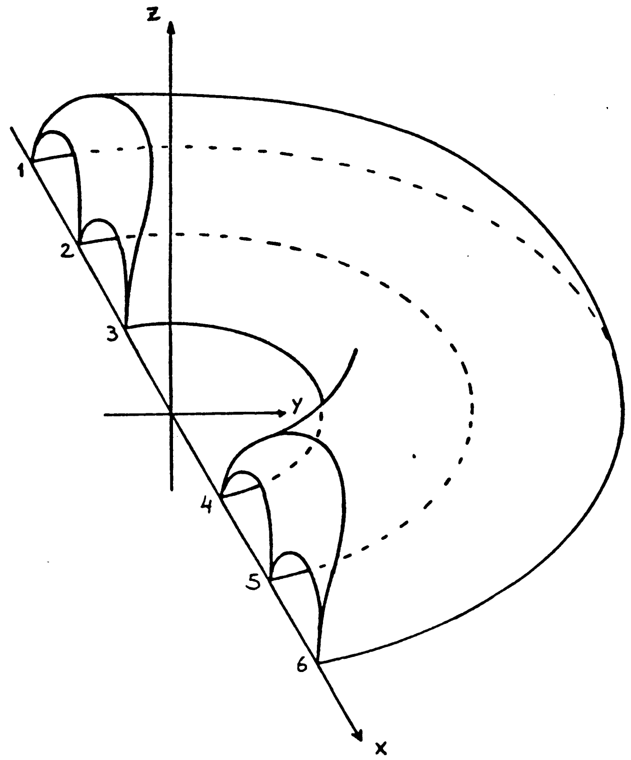


Figure 4.6
Second Interpretation of the Boundary Description

It is easy to verify its consistency with the topology of the boundary description above. However, due to the asymmetry of the two surface branches for the faces $d$ and $e$, the two interpretations have different volumes.

## 5. *Discussion*

In free form surface design, objects are created whose faces are parametric patches with given rectangular or triangular topology, and whose edges are line segments in parameter space. Such objects are geometrically unambiguous, since faces and edges are uniquely specified subsets of 3-space. However, simple operations on solids, e.g. intersection, usually result in new objects whose faces and edges no longer are so simple.

In particular, the intersection curve of two parametric patches is the image of a planar algebraic curve in parameter space and usually does not permit a rational parameterization. So, when performing these operations, the resulting objects must be carefully examined in order to ascertain that the new edges do not have singularities [5]. It appears, therefore, that modeling systems based on parametric boundary representations must also face the geometric problems demonstrated above for implicit surface modeling. More research is needed to study the parallels.

In [7], Requicha proposed the following representation scheme for curved edges:

(1) Segment edges so that each edge interior is homeomorphic to an open line segment.

(2) Represent each edge as the intersection of two surfaces plus one interior point.

Evidently, each edge is unambiguously fixed in 3-space in this representation, and so no geometric ambiguity can arise. However, it is not clear whether this scheme is convenient for, e.g., rendering algorithms, for it requires walking the edge, beginning at the intermediate point, so as to locate the correct branch and direction at each vertex. An alternative is to provide directional information at vertices for each edge. For high degree algebraic curves to be unambiguously specified requires not only tangents but also curvature of orders up to the degree of the curve.

In [3] we have advocated a simpler, but not completely general scheme: For each edge, add the tangent vectors at the vertices. This serves to identify the edge unambiguously as long as the space curve does not have cusps at the vertex and is very convenient, e.g., for rendering. For cusps, information about the higher order derivatives is required. While tangent information suffices to disambiguate the object of Section 4, it is clear that we can construct similar ambiguous objects in which specifying the tangent directions alone no longer suffice to identify the edges.

*References*


1. S. Ansaldi, L. DeFloriani, B. Falciendo (1985)
Geometric Modeling of Solid Objects by Using a Face Adjacency Graph Representation, *Comp. Graphics* 19:3, 131 − 139.

2. B. Baumgart (1972)
Winged Edge Polyhedron Representation, *Stanford AI Report* CS-320.

3. J. Cramer, C. Hoffmann, J. Hopcroft, D. Krafft (1986)
A Display Algorithm for Algebraic Objects, *Tech. Rept.*, Comp. Sci. Dept., Cornell University, in preparation.

4. C. Eastman and K. Weiler (1979)
Geometric Modeling Using the Euler Operators, First Annl. Conf. on Comp. Graphics in CAD/CAM Syst., MIT, 248 − 259.

5. R. T. Farouki (1985)
A Characterization of Parametric Surface Sections, Research Rept., Corporate Res. and Dev., General Electrics, Schenectady NY, 58p.

6. D. Hilbert and S. Cohn-Vossen (1952),
*Geometry and the Imagination*, Chelsea Publishing Co., New York, 357p.

7. A. Requicha (1980)
Representations of Rigid Solid Objects, in *Computer Aided Design*, Springer Lect. Notes in Comp. Sci. 89, J. Encarnacao, ed., 2 − 78.

8. A. Requicha (1983)
Solid Modeling: Current Status and Research Directions, *IEEE Comp. Graphics and Applic.* , 25 − 34.

9. K. Weiler (1983)
Adjacency Relationships in Boundary Graph Based Solid Models, to appear.

10. K. Weiler (1985)
Edge Based Data Structures for Solid Modeling in Curved-Surface Environments, *IEEE Comp. Graphics and Appl.* 5:1, 21 − 40.

11. P. Wilson (1985)
Euler Formulas and Geometric Modeling, *IEEE Comp. Graphics and Applic.* 5, 24 − 36.