THE COMPLEXITY OF FACETS RESOLVED*

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1. INTRODUCTION

The polytopal approach of Edmonds and Fulkerson has been used with great success in the development of polynomial algorithms for a number of combinatorial optimization problems (see, for example, the annotated bibliography in [Gr]). It has been tempting to apply the same methodology to hard problems in the area, such as the traveling salesman problem. This would require a complete understanding of the facets of the corresponding polytope, that is, the non-redundant linear inequalities that describe the problem. Characterizing the facets of the traveling salesman polytope has been one of the most perplexing (and well-studied) problems in combinatorics in the past forty years. Several authors have added, by ingenious constructions, new classes of facets to those already known [C, DFJ, G, GP, PY]. However, no end to this process was in sight.

In [PY], a complexity-theoretic analysis of the phenomenon was attempted. As a result, the class D^P was introduced. This class contains all languages that can be considered as the difference of two languages in NP (or, equivalently, the intersection of a language in NP and one in co-NP). Many important classes of problems are in D^P, and some of them are complete (typical examples are problems like recognizing pairs of formulae, one satisfiable and the other unsatisfiable, called SAT-UNSAT, facets of the clique polytope, and the versions of optimization problems in which one is asked whether a given number is the optimal cost). The class D^P has since been the subject of considerable study, and more results are now known concerning both new complete problems [Baj, Cosm, VV] and structural properties [BG]. This class contains the problem TSP FACETS (given an inequality, is it a facet of the corresponding traveling salesman polytope?) and several other problems related to "critical combinatorics" (examples: minimal non-3-colorable planar graphs, maximal non-Hamiltonian graphs, minimal unsatisfiable Boolean forlmulae, etc.).

Unfortunately (and despite the title of [PY]) it has not yet been showed that TSP FACETS (the problem which motivated all this) is complete for D^P. Thus, the complexity of the facets problem, which motivated the definition of the class, had not been identified with it. In fact, nor is any other "critical" problem known to be complete. In [PY] it is conjectured that such problems are complete, but it is also pointed out that "the constructions [involved in completeness proofs] would have to be extremely delicate.... Most reduction methods (e.g., the ones employing 'gadgets') do not even preserve criticality, let alone create a critical [object] from a non-critical one."

In this paper we show that TSP FACETS is indeed D^P-complete, thus justifying the original motivation of that class. The hardest step is the proof that a starting critical problem, namely "minimal unsatisfiability" (given a Boolean formula in 3-CNF, is it true that it is unsatisfiable, but removing any clause makes it satisfiable?), is D^P-complete. Our approach for circumventing the impediments apparently inherent in criticality is to perform the reduction in two stages: We reduce satisfiability and unsatisfiability to separate instances of minimal unsatisfiability, and then we reduce two instances of minimal unsatisfiability to one. Our construction uses a novel analysis of the covering properties of sets of truth assignments of a formula, in order to induce minimality. We present an outline of the proof in Section 2.

In Section 3 we show that minimal unsatisfiability can be reduced to the problem of determining whether a graph has no Hamilton circuit, but the addition of any edge creates one. This, together with a proof in [PY], reducing the latter problem to TSP FACETS, completes the result. Our reduction is a delicate variant of standard Hamilton circuit reductions. Dealing with the issue of criticality necessitated that the constructions be extremely simple; as a result, from our proof one can extract reductions to the Hamilton circuit problem that are much simpler than those known before.

In section 4 we discuss several other critical problems which have been shown to be D^P-complete by reductions from minimal unsatisfiability. Once again the reductions must be carefully done in order that criticality is transferred to the new problem.

2. MINIMAL UNSATISFIABILITY

MINIMAL UNSATISFIABILITY is the following computational problem: "Given a Boolean formula in conjunctive normal form with at most three literals per clause and at most two occurences of each literal (and therefore at most four occurencess of each variable), is it true that it is unsatisfiable, but removing any clause renders it satisfiable?" It is one of the problems studied in [PY], and in fact conjectured to be complete for D^P. In contrast, the following problem, known as SAT-UNSAT, is known to be D^P-complete [PY]: "Given two Boolean formulae F and G, is it true that F is satisfiable and G is unsatisfiable?". In this Section we show the following:

Theorem 1. MINIMAL UNSATISFIABILITY is D^P-complete.

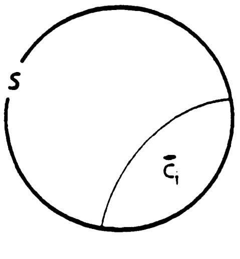
It is easy to argue that it is in D^P. To show completeness, we shall reduce SAT-UNSAT to MINIMAL UNSATISFIABILITY in two stages.

Lemma 1. There is a polynomial many-one reduction from UNSATISFIABILITY to MINIMAL UNSATISFIABILITY.

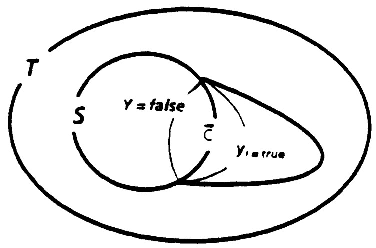
Conveying the idea of the proof requires the use of a Venn-diagram. Let

$$f = \prod_{i=1}^{n} C_i = \prod_{i=1}^{n} (x_{i1} + x_{i2} + x_{i3})$$

be an instance of UNSAT in CNF, where each clause, C_i, is the disjunct (logical or) of three literals. Let the set S, shown below, represent the set of all possible assignments of the variables of f (whether or not the assignments actually satisfy f). Then, each clause $C_{i_0} = x_{i_0 1} + x_{i_0 2} + x_{i_0 3}$ "covers" some of the possible assignments of f: In particular, the assignments in which all three literals of C_{i_0} are identically *false* cannot be satisfying assignments of f. f is satisfiable if not all of S is covered by f's clauses. f is minimally unsatisfiable if all of S is covered, and each clause of f covers some portion of S which is not covered by any other clause.



This brings us to the proof. Given f, we want to generate a new formula, g, such that g is minimally unsatisfiable **iff** f is unsatisfiable. Let the variables of g include those of f plus the new variables $\{y_i\}$, one introduced for each clause C_i of f. Let $S' \subset S$ be the set of assignments in which all the new variables are set to *false*: $Y = \sum_{i=1}^{n} y_i = false$. S' will be closely related to S in the reduction: For each term in f there will be a term in g which will cover an analagous region in S', as well as some region in T outside S', as diagrammed below. The terms may overlap in S', but they will be guaranteed not to overlap outside S', so that each term will cover some assignments which no other term covers. Then, by carefully covering the rest of the region outside S' with other clauses, we will obtain a formula g which is guaranteed to be minimal (removing a clause will leave g satisfiable), but will be satisfiable **iff** f is satisfiable.



For each clause C_i of f, include in g the clause $(C_i + (Y - y_i))$, where $Y - y_i = \sum_{k \neq i} y_k =$ the disjunct of all the new variables **except** y_i. This will cover the same region in S' as the term C_i covers in S, and it also covers a region outside S' where the i^{th} new variable is set to true.

We will cover the rest of the region outside S' by first covering the rest of the assignment where the i^{th} new variable is set to true, and then the region where two new variables are set to true. For each $C_i = (x_{i1} + x_{i2} + x_{i3})$, introduce terms which cover the region that the previous terms do not cover, and where $y_i = true$ and $(Y - y_i) = false$:

$$(x_{i1} + (Y - y_i) + \bar{y}_i) \cdot (\bar{x}_{i2} + (Y - y_i) + \bar{y}_i) \cdot (x_{i3} + (Y - y_i) + y_i)$$

Then to cover where two new variables are set to true, include the term $(\bar{y}_i + \bar{y}_j)$ for every pair of variables y_i and y_j. Notice that each of the terms covers some region of T which no other term covers, thus maintaining that g is minimal. Some case analysis will verify that this reduction does indeed work.

Lemma 2. There is a polynomial many-one reduction from SATISFIABILITY to MINIMAL UNSATISFIABILITY.

The proof is almost the same as for lemma 1. Simply use the same reduction, but introduce an extra term which covers exactly the region S'. Then g is guaranteed to be unsatisfiable, but it will be minimal **iff** f is satisfiable. If f is unsatisfiable, then the extra term could be removed, and g would remain unsatisfiable.

Lemma 3. There is a polynomial many-one reduction from two instances of MINIMAL UNSATISFIABILITY (f_1 and f_2) to one instance (g), such that g is minimally unsatisfiable **iff** f_1 **and** f_2 are.

The reduction is as follows: For each possible pair of clauses, chosen one each from f_1 and f_2, put a clause in g which is the disjunct of the pair of clauses. Then, removing a clause from g is analogous to removing one clause from each of f_1 and f_2. Again, a simple case analysis will prove the lemma.

The last lemma is used to combine the two reductions in lemmas 1 and 2 to obtain a complete reduction from SAT-UNSAT to MINIMAL UNSATISFIABILITY, thus proving MINIMAL UNSATISFIABILITY D^p-complete. The standard reductions for reducing CNF to "3-CNF with no literal repeated more than twice", with a few adjustments, maintain minimalness: thus, an instance of MINIMAL UNSATISFIABILITY in this form is also D^p-complete, which is the form used to prove the main result.

3. FROM THIS TO FACETS

The MNHG (MAXIMUM NON-HAMILTONIAN GRAPH) problem is the following: Given an undirected graph $G=(V,E)$ is it true that it has no Hamilton circuit, but adding any edge to E creates a Hamilton circuit? The following has been shown:

Theorem 2. [PY] MNHG is polynomially reducible to TSP FACETS.

In this Section we show the following result:

Theorem 3. MINIMAL UNSATISFIABILITY is polynomially reducible to MNHG.

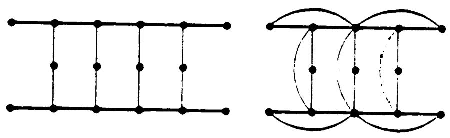
Notice that, together with Theorem 2 and Theorem 1, we arrive at the desired result:

Corollary. TSP FACETS is D^p-complete.

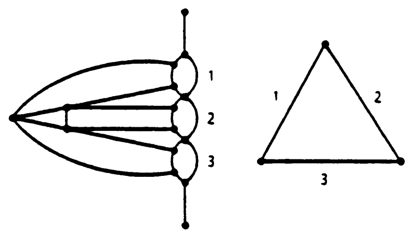
To give a sketch of the proof of Theorem 3, let us recall the standard reductions to Hamilton circuit problems [GJT, PS]. In those reductions, there are subgraphs (with the unfortunate name "gadgets") playing the role of logical gates, Boolean variables, and clauses. There is a variable gadget, which simply consists of two parallel edges, forcing a Hamilton circuit to choose a value ("true" or "false"). We have this in our construction as well. There is a clause gadget, which forces one of the three literals in each clause to be "true", and an "exclusive-or" gadget, propagating the value chosen at the variable to the clause. If the same gadgets are used here, the challenge is to show that adding a new edge results in a Hamilton circuit. Actually, most edges would not. Adding new edges and carrying out the case-by-case analysis for the rest is nearly impossible. We had to discover a number of significant simplifications to the standard constructions:

1. The gadgets need not be arranged in series, with the variables first and then the clauses, appropriately connected. It suffices to connect ALL the endpoints of all gadgets, and the proof still goes through. This takes some proving.

2. The old-fashioned exclusive-or gadget is shown on the left below. The gadget can be simplified by removing one of the vertical paths, then several extra edges must be added in order to induce criticality: the result is the gadget shown on the right.



3. The classical clause gadget is shown on the left below. One of the edges 1, 2, 3 must fail to be traversed. Now, the same holds for the simple device on the right. It is this important simplification that made the argument tractable.



It is easy to show that the resulting graph is non-Hamiltonian iff the original formula was unsatisfiable. It is now a matter of a careful case-by-case analysis on all possible edges added to the graph to show that the graph is maximal iff the formula was minimal.

4. RELATED RESULTS

Several new completeness results have been obtained by reductions from minimal unsatisfiability. Graph MINIMAL 3-COLORABILITY, for instance, is the following problem: Given a graph G, is it true that G is not colorable with 3 colors, but deleting any node from G results in a 3-colorable graph. Cai and Meyer show this problem D^P-complete by designing more sensitive gadgets than those used in the usual colorability reductions [CM].

Another critical problem, α-CRITICAL CLIQUE, is the following: Given a graph G and integer α, is it the case that G has no clique of size α, but that the addition of **any** edge will create a graph with some clique of size α. By reduction from a carefully chosen variant of minimal unsatisfiability, Vazirani shows this problem (and its corresponding critical vertex cover problem) also is complete for D^P [Vaz]. This yields an alternate proof for D^P-completeness of facets of the clique polytope ([PY]'s proof involves a reduction from EXACT CLIQUE.)

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