March 1985 A.E. Res. 85-6

THE INVARIANCE OF ESTIMATES OF A SYSTEM OF LINEAR LOGIT EQUATIONS

Timothy J. Considine

Timothy D. Mount

Department of Agricultural Economics
New York State College of Agriculture and Life Sciences
A Statutory College of the State University
Cornell University, Ithaca, New York 14853

Abstract

For a set of N shares or proportions using a linear logit model, a system of (N -1) equations of share ratios is used for estimation. The estimated structure of this system of equations is shown to be invariant to the choice of base if the system is estimated using an iterative version of Zellner's Seemingly Unrelated Regressions. This is true even if cross-equation constraints on the parameters exist or the equations are dynamic.

This report is a technical appendix to the following article:

Considine, T. J. and T. D. Mount. "The Use of Linear Logit Models for Dynamic Input Demand Systems", The Review of Economics and Statistics, Vol. 66, No. 3, August, 1984, pp. 434-443.

THE INVARIANCE OF ESTIMATES OF A SYSTEM OF LINEAR LOGIT EQUATIONS

by

Timothy J. Considine and Timothy D. Mount

A linear logit model of input demand can be derived by representing a set of N cost shares by a logistics function:

$$w_i = \frac{e^{f_i}}{\sum_{j=1}^{N} e^{f_j}} \qquad \text{for } i = 1, 2, \ldots, N$$

where $w_i = P_i Q_i / C$ is the share of total cost allocated to the $i^{\underline{th}}$ input, P_i and Q_i are the price and quantity of the $i^{\underline{th}}$ input, C is the total cost of all N inputs, and f_i is a function of the N input prices and the level of output, Y. A convenient form for f_i is:

$$f_i = a_i + \sum_{j=1}^{N} c_{ij} \ln P_j + g_i \ln Y + e_i$$

where a_i, c_{ij} and g_i are unknown parameters, and e_i is a stochastic residual. Standard elasticities can be derived using this form of f_i to be linear functions of the parameters and a specified set of shares. It is possible to impose linear restrictions on these elasticities to ensure that they exhibit the properties of neoclassical demand equations (see Considine and Mount, 1984). Furthermore, it is relatively easy to make the equations dynamic without losing these properties.

For estimation purposes, the N share equations can be written in a linearized form using the following system of (N-1) equations:

$$\ln(w_i/w_N) = (a_i - a_N) + \sum_{j=1}^{N} (c_{ij} - c_{Nj})\ln P_j + (g_i - g_N)\ln Y + (e_i - e_N)$$

$$i = 1, 2, \ldots, N-1$$

The restrictions derived from neoclassical theory imply that symmetry constraints are imposed on the c_{ij} coefficients across equations (see Considine and Mount, equation (16), 1984). The presence of these constraints raises the issue of whether or not the estimated structure of demand is invariant to the choice of the base (the $N^{\underline{th}}$) input.

The linearized form of the model used for estimation gives the (N-1) equations shown above. If it assumed that there are $T > N+1$ observations of the N shares and that they are ordered into T groups of (N-1) share ratios, then the statistical model under normality can be written as follows:

$$Y = V\alpha + X\beta + Z\gamma + E$$

$$E[Y] = V\alpha + X\beta + Z\gamma$$

$$\mathrm{Var}[Y] = \Omega$$

Y is $(N-1)T \times 1$ with elements $(\ln(w_{it}) - \ln(w_{Nt}))$ for $i = 1, 2, \ldots, N-1$ and $t = 1, 2, \ldots, T$.

$V = 1_T \otimes I_{N-1}$, where 1_T is a vector of T ones, I_{N-1} is an identity matrix of order N-1, and $\otimes$ is a Kronecker product.

α is $(N-1) \times 1$ with elements $(a_i - a_N)$.

X is $(N-1)T \times (N(N-1)/2)$ containing the weighted price ratios. (In practice, it is unnecessary to specify these regressors individually because the cross-equation constraints on the price coefficients can be incorporated explicitly in statistical packages such as TROLL or SAS.)

β is $(N(N-1)/2) \times 1$ containing the distinct price coefficients after symmetry is imposed, $c^*_{ij} = c_{ij}/w^*_j$ for all $i < j$ (see Considine and Mount, p. 437, 1984).

Z is $(N-1)T \times (N-1)$ and similar to V except that the non-zero elements are $\ln(Y_t)$ for all i, where Y is output.

γ is $(N-1) \times 1$ with elements $(g_i - g_N)$.

E is $(N-1)T \times 1$ with elements $(e_{it} - e_{Nt})$.

$\Omega = I_T \otimes \Sigma$, where Σ is $(N-1) \times (N-1)$ with elements $\mathrm{Cov}(e_{it} - e_{Nt}, e_{jt} - e_{Nt})$ for all $i, j < N$. Σ is assumed to be nonsingular so that the inverse of Ω exists.

The base input can be changed from N to K by premultiplying the model by $A = I_T \otimes C$, where C is an identity matrix of order (N-1) with the Kth column replaced by a column of minus ones. For example, C can be written as follows for the case N = 6 and K = 2:

$$C = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

In the transformed model, it is easy to show that typical elements of AY, $AV\alpha$, $AZ\gamma$ and E are $(\ln(w_{it}) - \ln(w_{Kt}))$, $(a_i - a_K)$, $\ln(Y_t)(g_i - g_K)$, and $(e_{it} - e_{Kt})$ for $i \neq K$, respectively. It can also be shown that $AX\beta$

involves the same cross-equation constraints on the price coefficients based on price ratios $(\ln(P_{it}) - \ln(P_{Kt}))$ for $i \neq K$, rather than $(\ln(P_{it}) - \ln(P_{Nt}))$ for $i \neq N$. In the dynamic version of the model, an additional regressor is included representing the logarithm of the lagged quantity ratio times an unknown parameter λ. The premultiplication of this variable by A is equivalent to changing a typical element from $\lambda[\ln Q_{it-1} - \ln Q_{N-1}]$ to $\lambda[\ln Q_{it-1} - \ln Q_{Kt-1}]$, where Q_i is the quantity of the $i^{\underline{th}}$ input. Hence, the dynamic model creates no new problems, and this discussion of the static model is valid for the dynamic model as well.

With N as the base input, the most convenient normalization of the coefficients is to set $a_N = g_N = d = 0$, but with the transformed model, it would be natural to set $a_K = g_K = d = 0$. This would not affect the fit or elasticities in the same sense that dropping a different dummy variable from a model to obtain a new solution does not affect the results, even though the computed coefficients change.

Having explained how premultiplying the model by A corresponds to changing the base input, the remaining step is to show that the estimates of the coefficients in the original and transformed models are identical. If the matrixes of regressors and coefficients are redefined as $W = [V\ X\ Z]$ and $\theta' = [\alpha'\ \beta'\ \gamma']$, then the generalized least squares (GLS) estimator of θ is

$$\hat{\theta}_G = (W'\Omega^{-1}W)^{-1}W'\Omega^{-1}Y.$$

In the transformed model, $E[AY] = AW\theta$ and $\mathrm{Var}[AY] = A\Omega A' = I_T \otimes C\Sigma C'$. Since A is nonsingular, it is straightforward to show that the GLS estimator of θ from regressing AY on AW is identical to $\hat{\theta}_G$ (except for any renormalization that is introduced). In practice, however, an approximate GLS estimator is used because the unknown parameters in Ω must be estimated, and the two approximate GLS estimators of θ will be the same only if the corresponding estimators of Ω are identical.

The remaining issue for showing that the GLS estimator of θ is unaffected by the choice of base input is to demonstrate that the estimates of Var[Y] and Var[AY] both give the same estimate of Ω. This is equivalent to showing that $A\hat{V}[Y]A' = \hat{V}[AY]$, where $\hat{V}[\cdot]$ is the estimated variance. Since $A = I_T \otimes C$ is known and $\mathrm{Var}[Y] = \Omega = I_T \otimes \Sigma$, the condition $A\hat{V}[Y]A' = \hat{V}[AY]$ corresponds to having the estimates of Σ identical regardless of whether they are estimated directly in the model with base input N, or derived as $C\hat{\psi}C'$, where $\hat{\psi}$ is the estimate of $C\Sigma C'$ in the transformed model (note C is involutory implying $C^2 = I_{N-1}$, and therefore, $C\hat{\psi}C' = CC\hat{\hat{\Sigma}}C'C' = \hat{\hat{\Sigma}}$). This condition can be guaranteed if a maximum likelihood estimator is used because of the invariance property of this estimator.

The model specification is an example of Zellner's seemingly unrelated regressions (SUR) with cross-equation constraints. Oberhofer and Kmenta (1974) have shown that the iterative SUR estimator is identical to a maximum likelihood estimator under normality. Hence, use of an iterative SUR estimator ensures that the estimated elasticities in the linear logit model are invariant to the choice of base input. This property has been confirmed empirically for an example with four inputs. It should be noted that a standard two-step SUR estimator does not exhibit invariance when there are cross-equation constraints on the coefficients.

References

Considine, Timothy J. and Timothy D. Mount. "The Use of Linear Logit Models for Dynamic Input Demand Systems." The Review of Economics and Statistics, Vol. LXVI, No.3, August 1984, pp. 434-443.

Oberhofer, W. and J. Kmenta. "A General Procedure for Obtaining Maximum Likelihood Estimates in Generalized Regression Models." Econometrica Vol. 42, 1974, pp. 579-590.