Voronoi Diagrams and Arrangements

Herbert Edelsbrunner*
Raimund Seidel
TR 85-669
March 1985

Department of Computer Science
Cornell University
Ithaca, New York 14853

* Institute für Informationsverarbeitung, Technical University of Graz,
  Schiesstättgasse 4A, A-8010 Graz, AUSTRIA.
Figure 1.1: A Voronoi Diagram
Voronoï Diagrams and Arrangements

Herbert Edelsbrunner
Institute für Informationsverarbeitung
Technical University of Graz
Schießstättgasse 4A
A-8010 Graz AUSTRIA

Raimund Seidel
Computer Science Department
Cornell University
Ithaca NY. 14853
U.S.A.

Abstract:

We propose a uniform and general framework for defining and dealing with Voronoï Diagrams. In this framework a Voronoï Diagram is a partition of a domain $D$ induced by a finite number of real valued functions on $D$. Valuable insight can be gained when one considers how these real valued functions partition $D \times \mathbb{R}$. With this view it turns out that the standard Euclidean Voronoï Diagram of point sets in $\mathbb{R}^d$ along with its order-$k$ generalizations are intimately related to certain arrangements of hyperplanes. This fact can be used to obtain new Voronoï Diagram algorithms. We also discuss how the formalism of arrangements can be used to solve certain intersection and union problems.

I. Introduction

Figure 1.1 depicts a type of diagram that is known under such names as Dirichlet tessellation, Thiessen polygons, or as we call it, Voronoï Diagram. The formation rule for such a diagram is simple. The location of a finite number of “sites” is known. For each “site” $s$ one wants to form the region of all points for which $s$ is the nearest among the finite set of “sites”. If “nearest” is understood with respect to the Euclidean distance measure, then for each “site” its associated region is polygonal.

In the context of computational geometry Voronoï Diagrams were first introduced in a paper by Hoey and Shamos [Sh-H]. The usefulness of the Voronoï Diagram (from now on VoD) for solving a large number of problems, the fact that it can be constructed efficiently, and maybe also its aesthetically pleasing appearance subsequently kindled the interest of many researchers. They tried to apply the VoD to other problems, and if this was not possible directly, attempted to adapt and generalize the notion of a VoD appropriately. Among the generalizations are higher order VoDs [Sh-H],[L78], VoDs of line segments and/or circular arcs [Dry],[L-D],[K],[S-L],[Yap], VoDs of point sets in $\mathbb{R}^d$, $d \geq 2$, [Bow],[Wa], weighted Voronoï Diagrams [A-E], power diagrams, or VoDs with respect to the Laguerre geometry [A],[IIM], VoDs with respect to the $L^p$ metric, $1 \leq p \leq \infty$, [L80],[L-W], and VoDs with respect to other “funny” metrics [C-D],[A-B].

The types of VoDs just mentioned are quite different and there is little doubt that even more generalizations will be investigated in the future. However, instead of pursuing ever more diversification one can also attempt unification and ask: What do all these different types of VoDs have in common? What constitutes a VoD in its most general form? What are the underlying mathematical concepts? Does one, for instance, need the notion of a metric in order to define VoDs?

In this paper we try to answer some of these questions. We offer a very general definition of VoDs which shows that they can be defined naturally on any domain $D$ via a finite set of real valued functions on $D$. Thus no notion of metric is needed. We show that higher order and higher degree VoDs can be obtained in a similar manner and demonstrate how all these VoDs are related to the partition of the product space $D \times \mathbb{R}$ induced by the finite set of real valued functions on $D$.

This view already proves useful when applied to the standard Euclidean VoD of point sets in $\mathbb{R}^d$. It turns out that this kind of VoD, along with its higher order and higher degree versions, can be defined using a finite set of affine functions from $\mathbb{R}^d$ to $\mathbb{R}$. Consequently these diagrams are intimately related to arrangements of hyperplanes in $\mathbb{R}^{d+1}$, an insight which leads to new algorithms for Euclidean VoDs.

Section 2 of this paper deals with arrangements, i.e. partitions of $D \times \mathbb{R}$ induced by a finite
Figure 2.1: The 2-level of an arrangement

Figure 2.2: The 2-belt of an arrangement
number of real valued functions on $D$. In section 3 we show how VoDs can be defined on any domain $D$ via finite sets of real valued functions. We demonstrate how VoDs are related to arrangements and we discuss a number of examples in some detail. Section 4 deals with higher order and higher degree VoDs and their relationship to arrangements which turns out to be particularly attractive. Section 5 contains a short outline how the formalism of arrangements can be used to solve certain intersection and union problems. In the last section we discuss possible directions for further research.

II. Arrangements

In this section we consider the interactions among a finite number of real valued functions on an arbitrary domain $D$. The structure imposed on $D \times R$ by such a finite set of functions promises to be an interesting object of study, provided the domain $D$ and functions as well as their interactions are restricted appropriately. For instance, when $D = R^d$ and all the functions are affine, the structure of $D \times R$ has been studied to a fair extent as so-called hyperplane arrangement [G71],[G72]. Similarly, in a slight generalization, when $D = R^d$ and all functions are continuous and satisfy certain finite intersection axioms, the structure imposed on $D \times R$ has been studied as so-called arrangement of pseudo-hyperplanes, and also in the context of oriented matroids [M].

In this section we do not intend to study the effect of any other function or domain restriction. We rather want to give some general definitions and results in order to provide a convenient framework to argue about Voronoi Diagrams, the main topic of this paper.

Let $D$ be some domain. For a real valued function $f$ on $D$ we call the subset of $D \times R$

$$f^+ = \{ (x,r) \mid f(x) > r \}$$

the lower hemispace of $f$; the set

$$f^- = \{ (x,r) \mid f(x) < r \}$$

the upper hemispace of $f$, and, with a slight abuse of terminology,

$$f^0 = \{ (x,r) \mid f(x) = r \}$$

the surface of $f$. Note the $f^+$, $f^-$, and $f^0$ properly partition $D \times R$.

Throughout this paper let $E$ denote a finite index set with $n$ elements. For each $e \in E$ let $f_e$ be a real valued function on $D$ and let $f_E$ denote the indexed collection of these $n$ functions. For each point $y \in D \times R$, let

$$\text{minus}(y) = \{ e \in E \mid y \in f_e^- \},$$
$$\text{plus}(y) = \{ e \in E \mid y \in f_e^+ \},$$
$$\text{zero}(y) = \{ e \in E \mid y \in f_e^0 \}.$$ 

Furthermore, let $\pi_f(y)$ denote the partition of $E$ induced by the functions in $f_E$ applied to point $y$, i.e.

$$\pi_f(y) = (E^-, E^0, E^+)$$

where

$$E^- = \text{minus}(y), E^0 = \text{zero}(y), \text{and} E^+ = \text{plus}(y).$$

In a natural way $\pi_f$ defines an equivalence relation on $D \times R$, making two points $y$ and $z$ equivalent iff $\pi_f(y) = \pi_f(z)$. We call the partition of $D \times R$ induced by $\pi_f$ an $f_E$-arrangement. We call each equivalence class of an $f_E$-arrangement a cell. Each cell can be uniquely named by a partition $(E^-, E^0, E^+)$ of $E$:

$$C_{(E^-, E^0, E^+)} = \{ y \in D \times R \mid \pi_f(y) = (E^-, E^0, E^+) \}.$$ 

Equivalently, a cell can be represented as an intersection:

$$C_{(E^-, E^0, E^+)} = \bigcap_{i \in I^-} f_i^- \cap \bigcap_{i \in I^0} f_i^0 \cap \bigcap_{i \in I^+} f_i^+.$$ 

As there are only a finite number of partitions of the index set $E$, the number of cells in an $f_E$-arrangement is finite and hence $f_E$-arrangements can be studied as combinatorial objects.

We call a cell $C_{(E^-, E^0, E^+)}$ a full cell iff $E^0 = \emptyset$. For an integer $k$, $0 \leq k \leq n$, we call a full cell $C_{(E^-, E^0, E^+)}$ a $k$-belt cell iff $|E^-| = k$. We call a cell $C_{(E^-, E^0, E^+)}$ a $k$-level cell iff $|E^-| \leq k$ and $|E^-| + |E^0| \geq k$. Note that a cell $C_{(E^-, E^0, E^+)}$ is in $|E^0|$ different levels. Figure 2.1 shows the 2-level cells of an arrangement of 3 real valued functions on $R$. Figure 2.2 depicts the 2-belt cells of the same arrangement. Note that cells are not necessarily "connected".

If $f_E$ and $g_E$ are two indexed collections of real valued functions on a common domain $D$, then we call $f_E$ and $g_E$ order-equivalent iff for all $x \in D$ and for every pair $i,j \in E$

$$\text{sign}(f_i(x) - f_j(x)) = \text{sign}(g_i(x) - g_j(x)).$$

If two function collections $f_E$ and $g_E$ are order-equivalent, then the $f_E$-arrangement and $g_E$-arrangement of $D \times R$ are isomorphic in the following sense:

**Lemma 2.1:**

Let $f_E$ and $g_E$ be two order-equivalent function collections on $D$.

Let $(E^-, E^0, E^+)$ be a partition of $E$. 
(i) There exists a non-empty cell $C_{<x \cdot x + x >}$ in the $f_x$-arrangement if there exists a non-empty cell $C_{<x \cdot x + x >}$ in the $g_x$-arrangement. Moreover, 

\[ \text{proj}(C_{<x \cdot x + x >}) = \text{proj}(C_{<x \cdot x + x >}), \]

where for $C \subseteq D \times R$ the expression \( \text{proj}(C) \) denotes \( \{ x \in D \mid (x, r) \in C \text{ for some } r \in R \} \).

Proof:
To prove both (i) and (ii) it suffices to show that for every \( y = (x, r) \in D \times R \) there exists an \( r' \in R \) such that \( \pi_y((x, r)) = \pi_y((x, r')) \).

Let $<E^-, E^0, E^+>$ = $\pi_y((x, r))$. Because of the order-equivalence between $f_x$ and $g_x$ we know that for $e \in E^-, e^+ \in E^+$ and $e, e' \in E^0$ the relations $g_y(x) < g_y(x) < g_y(x)$ and $g_y(x) = g_y(x)$ hold.

Now let

\[ l = \max \{ g_y(x) \mid e \in E^- \cup E^0 \}, \]

\[ u = \min \{ g_y(x) \mid e \in E^+ \cup E^0 \}, \]

and choose $r' = \frac{1}{2}(l + u)$. Such a choice of $r'$ forces $\pi_y((x, r)) = \pi_y((x, r'))$.

Q.E.D.

In the remainder of this section we consider two important classes of arrangements. Let the domain $D$ be $R^d$ for some $d \geq 1$. For $e \in E$ let $f_e$ be an affine function given by

\[ f_e(x) = <a_e, x > + b_e, \]

where $a_e \in R^d$, $b_e \in R$, and $<, >$ denotes the usual scalar product between vectors. The surface $f_e^0$ of such an affine function is usually called a ("non-vertical") hyperplane in $R^d \times R = R^{d+1}$. The upper and lower hemispaces $f_e^+$ and $f_e^-$ are in this case usually referred to as upper and lower (open) halfspaces. The $f_x$-arrangement of $R^{d+1}$ is usually called an arrangement of hyperplanes, and is, as mentioned in the beginning of this section, a fairly well studied mathematical object [G72],[Z].

Every cell $C_{<x \cdot x + x >}$ in an arrangement of hyperplanes is the intersection of a finite number of halfspaces and hyperplanes and hence a polyhedron. In the literature of hyperplane arrangements one usually considers the topological closure $\overline{C_{<x \cdot x + x >}}$ of a cell $C_{<x \cdot x + x >}$, with

\[ \overline{C_{<x \cdot x + x >}} = \bigcap_{e \in E^-} \bigcup_{e \in E^-} \bigcap_{e \in E^+} \bigcup_{e \in E^+} \bigcap_{e \in E^0} \bigcup_{e \in E^0} \]

The same asymptotic bounds hold for $b_k$ as well.

As finding the 1-level faces is equivalent to
definition conforms with the standard Euclidean Voronoi Diagram as presented, for instance, in [Sh-H].

Let $D$ be $\mathbb{R}^2$, the plane, and let $d(x,y)$ denote the Euclidean distance function. For each $\epsilon \in E$ let $p_\epsilon$ be a point in the plane and define the real valued function $f_\epsilon$ as

$$f_\epsilon(x) = d(p_\epsilon,x).$$

For some $\epsilon \in E$ consider the V-region $V_\epsilon$ in VOD($f_\epsilon$). $V_\epsilon$ contains all points $x \in \mathbb{R}^2$ with the property that

$$\{ \epsilon \in E \mid f_\epsilon(z) = \min_{i \in E} f_i(z) \} = \{ \epsilon \},$$

or, in other words, $V_\epsilon$ contains all $x$ with

$$f_\epsilon(x) < f_i(x) \quad \text{for all } i \in E, i \neq \epsilon,$$

i.e. all $x$ with

$$d(p_\epsilon,x) < d(p_i,x) \quad \text{for all } i \in E, i \neq \epsilon.$$

Thus $V_\epsilon$ is exactly the interior of what is usually considered the Voronoi region of point $p_\epsilon$.

Similarly, if a V-cell $V_{i(j)}$ exists in VOD($f_\epsilon$) for some pair $i,j \in E$, then this V-cell, by the same reasoning as above, contains all $x$ with

$$d(p_i,x) = d(p_\epsilon,x) < d(p_j,x)$$

for all $\epsilon \in E$, $i \neq \epsilon \neq j$.

Thus $V_{i(j)}$ is exactly the relative interior of the edge between the two V-regions $V_{i(i)}$ and $V_{j(j)}$.

Finally, V-cells of the form $V_T$ with $|T| \geq 3$ turn out to be the vertices in the traditional VoD (see Figure 3.1).

The reader may convince himself that with the right choice of functions other kinds of VoDs, such as Voronoi Diagrams with respect to the $L^p$ metric [L80], VoDs of line segments [L-D],[K],[S-L],[Yap], and weighted VoDs [A-E], can be expressed using our functional formalism.

An almost trivial but very important observation is the fact that for a function collection $f_\epsilon$, the Voronoi Diagram of $f_\epsilon$ in $D$ and the $f_\epsilon$-arrangement in $D \times \mathbb{R}$ are intimately related. We have the following

**Theorem 3.1:**

Let $f_\epsilon$ be a collection of real valued functions on a domain $D$, and let $\emptyset \neq T \subseteq E$, and $T' = E - T$.

$V_T$ is a V-cell in VOD($f_\epsilon$) iff $C_{<t,T',r>}$ is a cell in the $f_\epsilon$-arrangement.

Moreover, $V_T = \text{proj}(C_{<t,T',r>})$.

(Note that the cells $C_{<t,T',0>}$ with $S = \emptyset \neq T$ are exactly the 1-level cells.)

**Proof:**

By the definitions of this and the previous section, we have $x \in V_T$ if and only if for some $r \in \mathbb{R}$, $f_{\epsilon'}(x) = r$ for $\epsilon' \in T$, and $f_{\epsilon'}(x) > r$ for $\epsilon' \in T'$, which is the same as saying $\pi_1((x,r)) = <\emptyset,T,T'>$, which in turn holds if and only if $(x,r) \in C_{<t,T',r>}$.

Q.E.D.

**Theorem 3.1** and **Lemma 2.1** immediately yield the important

**Corollary 3.1:**

If $f_\epsilon$ and $g_\epsilon$ are order-equivalent function collections, then VOD($f_\epsilon$) = VOD($g_\epsilon$).

Applying Theorem 3.1 to the cases of collections of affine functions or collections of quadratic functions on $\mathbb{R}^2$ and using the algorithmic facts stated in section 2 we obtain the following important unifying algorithmic result.

**Theorem 3.2:**

Let $D = \mathbb{R}^d$ for some $d \geq 1$.

Let $f_\epsilon$ be a collection of affine functions on $\mathbb{R}^d$ given by

$$f_\epsilon(x) = <a_\epsilon,x> + b_\epsilon, \quad a_\epsilon \in \mathbb{R}^d, b_\epsilon \in \mathbb{R}$$

or let $f_\epsilon$ be a collection of quadratic functions of the form

$$f_\epsilon(x) = (x - p_\epsilon)^T A (x - p_\epsilon) + t_\epsilon,$$

$p_\epsilon \in \mathbb{R}^d, t_\epsilon \in \mathbb{R}$,

where $A$ is a $d \times d$ real symmetric matrix.

VOD($f_\epsilon$) can be constructed in worst case time $O(n \log n)$ for $d = 1, 2$ and $O(n^{(d^2+1)/2})$ for $d \geq 3$.

This is optimal for odd $d$ and for $d = 2$.

**Proof:**

By Theorem 3.1 and Corollary 3.1 it suffices to construct the 1-level cells in an arrangement of hyperplanes in $\mathbb{R}^{d+1}$, which can be done in the given time bounds by Fact 2.4.

Q.E.D.

**Theorem 3.2** is quite important because it unifies a number of algorithmic results in the literature about different kinds of VoDs. Below we give examples of what kind of planar VoDs are generated by collections of quadratic (or, equivalently, affine) functions.

**Example 3.1:** (ordinary Euclidean VoD [Sh-H])

For each $\epsilon \in E$ let $p_\epsilon$ be a point in $\mathbb{R}^2$. Let $l$
constructing the polyhedron formed by the intersection of all lower halfspaces, \([B78],[P-H],[P-M],[S]\) imply

**Fact 2.4:**

All 1-level faces (or equivalently all \(a\)-level faces) of an arrangement of \(a\) hyperplanes in \(R^{d+1}\) can be found in time

\[O(a \log a) \quad \text{for} \quad d = 1, 2\]

\[O(a (k+1)/2) \quad \text{for} \quad d \geq 3.\]

This is worst case optimal for odd \(d\) and \(d = 2\).

Much less is known about complexity bounds for constructing all \(k\)-level faces for arbitrary \(k\).

**Fact 2.5:** \([E]\)

All \(k\)-level faces of an arrangement of \(a\) planes in \(R^3\) can be found in time \(O(\sqrt{k} \log a \log(n,3))\).

This concludes our elaboration on the first class of examples of \(f_k\)-arrangements.

Our second class of examples deals with paraboloids. Let again the domain \(D\) of the functions be \(R^d, d \geq 1\). Let \(A\) be a \(d \times d\) symmetric real matrix. For \(c \in E\) let \(g_c\) be a quadratic function of the form

\[g_c(x) = (x-p_c)^T A (x-p_c) + t_c,\]

where \(p_c \in R^d\) and \(t_c \in R\). The surface \(g_c^0\) is usually called a paraboloid and we thus call such \(g_k\)-arrangement in \(R^{d+1}\) a paraboloid arrangement. To our knowledge paraboloid arrangements per se have not been studied in the literature. However, as the following important lemma shows, there is really no need to do so.

**Lemma 2.2:**

For \(c \in E\) let \(g_c\) be quadratic functions defined by

\[g_c(x) = (x-p_c)^T A (x-p_c) + t_c,\]

\[p_c \in R^d\] and \(t_c \in R\).

For \(c \in E\) let \(f_c\) be affine functions defined by

\[f_c(x) = <x, a_c> + b_c,\]

\[a_c = -2Ap_c, \quad b_c = p_c^T A p_c + t_c.\]

The collection of functions \(f_k\) and \(g_k\) are order-equivalent.

**Proof:**

We have to show that for every \(x \in R^d\) and any pair \(i, j \in E\)

\[\text{sign}(f_i(x) - f_j(x)) = \text{sign}(g_i(x) - g_j(x)).\]

However, it can be checked easily that in our case even

\[f_i(x) - f_j(x) = g_i(x) - g_j(x) = (p_i^T A p_i + t_i) - (p_j^T A p_j + t_j) + 2(p_i - p_j)^T A x\]

holds.

**Q.E.D.**

**Lemma 2.1** and **2.2** imply that the paraboloid arrangement and the hyperplane arrangement generated by \(g_k\) and \(f_k\), respectively, are combinatorially indistinguishable with respect to their cell structure. As a matter of fact, the even stronger relation holds that every cell in the hyperplane arrangement is the homeomorphic image of the corresponding cell in the paraboloid arrangement under the differentiable and invertible mapping \(F_A\) of \(R^{d+1}\) onto itself defined by

\[F_A((x,z)) = (x,x - z^T A x),\]

where \(x \in R^d\) and \(z \in R\).

In other words, \(F_A\) "wraps" \(R^{d+1}\) in such a way that all paraboloids generated by the matrix \(A\) are flattened out into hyperplanes, but all intersection patterns are preserved. Thus all the results stated above about hyperplane arrangements apply to paraboloid arrangements as well.

**III. Voronoi Diagrams**

As in the previous section let \(E\) be a finite index set of \(a\) elements and let \(f_k = \{ f_i \mid i \in E \}\) be an indexed collection of real valued functions on some common domain \(D\). Let \(R_i\) be a function from \(D\) to \(E\) defined by

\[R_i(x) = \{ e \in E \mid f_i(x) = \min_{i \in E} f_i(x) \}.\]

\(R_i\) induces in a canonical way an equivalence relation on \(D\), where for \(x, y \in D\),

\[x \sim y \Leftrightarrow R_i(x) = R_i(y).\]

We call the partition of \(D\) induced by this equivalence relation \(R_i\) the Voronoi Diagram of \(D\) with respect to \(f_k\), for short VOD(\(f_k\)). The equivalence classes of the partition are called Voronoi cells or \(V\)-cells. We denote each \(V\)-cell by \(V_T\), where \(T \subseteq E\) and

\[V_T = \{ z \in D \mid R_T(z) = T \}.\]

A \(V\)-cell \(V_T\) with \(|T| = 1\) is sometimes also called a Voronoi region or \(V\)-region.

This general, functional definition of Voronoi Diagrams might look somewhat startling and unorthodox to a reader only familiar with the usual definition of VoDs. Thus it seems appropriate to show that for the right choice of functions \(f_k\), the Voronoi Diagram of \(f_k\) obtained using our
denote the $2 \times 2$ identity matrix. For each $e \in E$ let $g_e$ be the quadratic function
\[ g_e(z) = (z - p_e)^T f(z - p_e). \]
Note that $g_e(z)$ is the square of the Euclidean distance function between $p_e$ and $z$, i.e.
\[ g_e(z) = (f_e(z))^2, \text{ where } f_e(z) = d(p_e, z). \]
Earlier we argued that VOD($f_E$) is the traditional Euclidean Voronoi Diagram of the points $p_e$. As the distance function is non-negative, the collection $g_E$ is order-equivalent to $f_E$, and hence VOD($g_E$) is the traditional Euclidean Voronoi Diagram as well.

Note 3.1:

The ordinary Euclidean VoD is of such importance that it seems worthwhile to spell out again the geometric intuition that is hidden behind the formalism.

For each $e \in E$ the function $g_e$ describes a paraboloid of rotation $g_e^0$ in $\mathbb{R}^3$ which is tangent to the $x_1x_2$-plane at point $p_e$, has its axis parallel to the $z$-axis, and "opens upward" towards $z = +\infty$. Imagine the set $\{ g_e^0 \mid e \in E \}$ of paraboloids in $\mathbb{R}^3$ penetrating each other. Furthermore, imagine each paraboloid to be opaque and having a unique colour. Finally, imagine an observer standing at $z = -\infty$ looking in the positive $z$-direction. The visible parts of the paraboloids would appear like the VoD of the point set $\{ p_e \mid e \in E \}$ with each V-region coloured differently.

Now imagine that the entire 3-space is "warped" by the function $F_1$, with
\[ F_1(z_1, z_2, z) = (z_1, z_2, z - z_1^2 - z_2^2). \]
Since the mapping $F_1$ leaves the $x$ and $y$ coordinates invariant, and since it also does not change the difference in $z$-coordinate of points with identical $x_1x_2$-coordinates, nothing changes for the observer at $z = -\infty$. He still sees the same coloured tiling of the plane.

The point is, however, that $F_1$ maps every paraboloid $g_e^0$ to the plane $f_e^0$, where
\[ f_e(z) = -2 <p_e, z> + <p_e, p_e>. \]
Thus what the observer at $z = -\infty$ sees, is the projection onto the $x_1x_2$-plane of the boundary of the polyhedron $P$ formed by the intersection of the lower halfspaces $\{ f_e^0 \mid e \in E \}$. Algorithmically, of course, this means that in order to construct the VoD it suffices to construct the polyhedron $P$. As a matter of fact, if one analyzes Hoey and Shamos' Voronoi Diagram algorithm, it is indeed a disguised halfspace intersection algorithm, namely a dual version of Preparata and Hong's convex hull algorithm [P-H].

Finally, we want to point out that the planes $f_e^0$ are not arbitrary planes in $\mathbb{R}^3$, but each $f_e^0$ is the tangent plane to the "upside down" paraboloid of rotation $g_e^0 = \{(x_1, x_2, z) \mid z = -x_1^2 - x_2^2\}$ at point $p_e$ which has the same $x_1$ and $x_2$ coordinates as $p_e$ but has $z$-coordinate $<p_e, p_e>$. 

Example 3.2: (Euclidean furthest point VoD [Sh-H])

Let the set $E$ and the matrix $I$ be as in example 3.1. For $e \in E$ let again $p_e$ be a point in the plane and let $g_e$ be a function defined by
\[ g_e(z) = (z - p_e)^T (-I)(z - p_e). \]
For the collection $g_E$ of functions of this kind, VOD($g_E$) turns out to be what has been traditionally known as the furthest point Voronoi Diagram of the planar point set $\{ p_e \mid e \in E \}$.

Note 3.2:

One of the reasons why the Euclidean closest point and furthest point Voronoi Diagram are of such importance is the usefulness of their geometric dual graphs, the so-called closest point and furthest point Delaunay triangulations [Sh-H]. It seems worthwhile to point out that these triangulations can be naturally derived from the polyhedron $P$ in the paraboloid construction in Note 3.1.

Let $X$ be the set of points $\{ p_e \mid e \in E \}$ on the upside down paraboloid $g_e^0$ (as in Note 3.1). Let $P^*$ be the convex hull of $X$. Then the closest point Delaunay triangulation of the planar point set $\{ p_e \mid e \in E \}$ is exactly the projection of the faces of $P^*$ which are "on top of" $P^*$, i.e., which could be seen by an observer at $z = +\infty$ looking in the negative $z$-direction. Similarly, the furthest point Delaunay triangulation is exactly the projection of the faces of $P^*$ which are "underneath" i.e. the faces visible to an observer at $z = -\infty$.

Example 3.3: (Power Diagrams [A], or VoDs in the "Laguerre geometry" [IM])

Again let $I$ denote the $2 \times 2$ identity matrix. For $e \in E$ let $p_e = (x_e, y_e)$ be a point in the plane and let $t_e$ be a real number. Define functions $g_e$ by
\[ g_e(z) = (z - p_e)^T f(z - p_e) + t_e. \]
The VOD defined by the collection $g_E$ of such functions was first discussed in the context of computational geometry in [A] where it is called "power diagram", and in [IM] where it is called Voronoi Diagram in the Laguerre geometry. Apparently, it is also known as a Dirichlet cell complex [P].
Note 3.3:

We want to point out that VoDs as defined in example 3.3 model a seemingly natural growth process. Imagine a set of non-overlapping circular cells $C_i$ in the plane, each increasing with time in such a way that the growth rate of its radius is inversely proportional to its diameter. Whenever two cells come into contact, they cease to grow in the directions in which overlap would occur. The eventual shape of each cell of such a growth system is given by a VoD as in Example 3.3, where the collection $g$ consists of a function $g_i$ for each cell $C_i$, where $p_i$ is the center of $C_i$, and $t_i$ is the time when it started growing, i.e., the last time when its radius would have been zero.

This can easily be seen as follows: the growth rate of each cell $C_i$ defines a real valued function $g_i$ telling for each point in the plane at what time it would be covered by $C_i$ if it grew uninhibitedly. For each point in the plane one wants to know which cell(s) overgrow it first. Thus the Voronoi Diagram of the functions in $g$ is indeed the desired object. Now it remains to show that the functions $g_i$ have the form given in example 3.3.

We want the radius of an uninhibited circular cell $C_i$ with center $p_i$ to grow at a rate inversely proportional to the diameter of $C_i$, starting at time $t_i$. If $r(t)$ denotes the radius changing with time, then it must satisfy the differential equation

$$\frac{dr}{dt} = -\frac{1}{2r(t)}$$

with initial condition $r(t_i) = 0$.

The solution to this equation is given by

$$r(t) = \sqrt{t - t_i} \quad \text{or} \quad r(t)^2 = t - t_i.$$ Expressing the radius $r(t)$ in terms of Cartesian coordinates yields

$$t = (z - p_i)^T J(z - p_i) + t_i.$$ Example 3.4:

For $e \in E$ let $p_e$ again be a point in the plane, and let $J$ be a 2x2 diagonal matrix with entries 1 and -1.

Let $g_e$ be the collection of functions $g_i$, with

$$g_i(x) = (z - p_i)^T J(z - p_i).$$

The way VO($g$) partitions the plane is rather peculiar. It appears to be a mixture of a Euclidean closest point VoD in the $z$-direction and and Euclidean furthest point VoD in the $y$-direction. An example is given in Figure 3.2. To our knowledge this type of planar VoD has appeared in the literature only once, in a marginal remark in [11M]. It is unclear whether it has any applications.

IV. Order-k and Degree-k Voronoi Diagrams

Recall our definition of a VoD at the beginning of the previous section. We defined it via an equivalence relation $\rho_f$ on a domain $D$ with respect to a set $f$ of real valued functions on $D$, where $\rho_f$ was defined by

$$z \rho_f y \iff R_f(z) = R_f(y),$$

where $R_f(z) = \{ e \in E \mid f_e(z) = \min_{i \in E} f_i(z) \}.

Obviously, there are ways of generalizing this definition by replacing the equivalence relation $\rho_f$ by another one which is not based on the minimum of the function values, but rather on the $k$-th smallest. With this in mind we give the following definition of a $k$-minimum: Let $E$ be an index set, and for $i \in E$ let $x_i \in X$. For an integer $k$, $k$-min $x_i$, is then defined to be the least real number $z$ such that $z \leq z_i$ for $k$ elements $i$ of $E$.

We can now generalize the mapping $R_f$ of section 3 in two interesting ways. For an integer $k$, $1 \leq k \leq n$, define the mappings $R_f^k$ and $S_f^k$ from $D$ to subsets of $E$ by

$$R_f^k(x) = \{ e \in E \mid f_e(x) = k\min_{i \in E} f_i(x) \},$$

$S_f^k(x) = \{ e \in E \mid f_e(x) \leq k\min_{i \in E} f_i(x) \}.$

For every $k$, $R^k$ and $S^k$ induce equivalence relations $\rho^k$ and $\sigma^k$ on $D$, respectively, where for $x, y \in D$

$$x \rho^k y \iff R^k(x) = R^k(y),$$

$$x \sigma^k y \iff S^k(x) = S^k(y).$$

For $1 \leq k \leq n$ we call the partition of $D$ induced by the equivalence relation $\rho^k$ the degree-$k$ Voronoi Diagram on $D$ with respect to $f_E$, or degree-$k$ VOD($f_E$) for short. We call each equivalence class in the degree-$k$ VOD($f_E$) a degree-$k$ V-cell, and we denote such a cell by $k-V_T$, where $T \subseteq E$, and

$$k-V_T = \{ z \in D \mid R^k(z) = T \}.$$ As in the case of the ordinary VoD, as defined in the previous section, we call a degree-$k$ V-cell $k-V_T$ a degree-$k$ Voronoi region if $|T| = 1$.

In the same manner we define the order-$k$ Voronoi Diagram on $D$ with respect to $f_E$, order-$k$ VOD($f_E$) for short, to be the partition of $D$ induced by $\sigma^k$. We call each equivalence class an order-$k$ V-cell and denote it by $k-W_T$, where

$$k-W_T = \{ z \in D \mid S^k(z) = T \}.$$ Order-$k$ Voronoi regions are order-$k$ V-cells $k-W_T$ with $|T| = k$.
Figure 4.1: A degree-2 Voronoi Diagram

Figure 4.2: An order-2 Voronoi Diagram
It should be clear that both order-1-VOD($f_g$) and degree-1-VOD($f_g$) are the same as VOD($f_g$). In the Euclidean case (i.e. $f_g$ is chosen as in Example 3.1) the degree-$k$ Voronoi Diagram partitions the plane by the $k^{th}$ nearest neighbor, whereas the order-$k$ Voronoi Diagram partitions the plane by the $k$ nearest neighbors.

Both the order-$k$ and the degree-$k$ Voronoi Diagram appear to be fairly natural generalizations of the ordinary VDO. So it is somewhat surprising that except for a short remark in [C-Y] so far only the order-$k$ VDO has been considered in the computational geometry literature [Sh-H], [L76], [D], [Bha]. The main reason for this appears to be the fact that in the Euclidean case order-$k$ V-regions are always convex and connected, whereas degree-$k$ V-regions generally consist of several convex components.

As in the case of VOD($f_g$), and perhaps even more so, it turns out to be beneficial to view degree-$k$-VOD($f_g$) and order-$k$-VOD($f_g$) as derived from the $f_g$-arrangement in $D \times R$. To this end we have the following theorems.

**Theorem 4.1:**
For $e \in E$ let $f_e$ be a real valued function on $D$. For some $T \subseteq E$ let $k-W_T$ be a degree-$k$ V-cell.

\[ k-W_T = \text{the union of } \text{proj}(C) \text{ over all } k\text{-level cells } C \text{ in the } f_g\text{-arrangement of the form } C = C_{<f^*,s^*,s^+>} \]

**Proof:**
Let $x \in k-W_T$ and let $r = k-\min f_e(x)$. By definition we have $f_e(x) = r$. Let \[ E^- = \{ e \mid f_e(x) < r \} \]. By the definition of the $k$-min it must be the case that $|E^-| < k$, but $|T| + |E^+| \geq k$. But this is the case if $(x,r) \in C_{<f^*,s^*,s^+>}$ where $E^+ = (E - T) - E^-$ and $C_{<f^*,s^*,s^+>}$ is a $k$-level cell.

Q.E.D.

We state similar theorems for the order-$k$ Voronoi Diagram. The proofs follow directly from the definition and are omitted.

**Theorem 4.2:**
For some $T \subseteq E$ let $k-W_T$ be an order-$k$ V-cell.

\[ k-W_T = \text{the union of } \text{proj}(C) \text{ over all } k\text{-level cells } C \text{ in the } f_g\text{-arrangement with } C = C_{<f^*,s^*,s^+>} \text{ and } T = E^- \cup E^0 \}

Even more useful might be the following

---

1 The reader may convince himself that our definition of order-$k$-VOD($f_g$) actually agrees with the definitions offered in these papers, when $f_g$ is chosen as in Example 3.1 of the previous section.

**Theorem 4.3:**
For some $T \subseteq E$ let $k-W_T$ be an order-$k$ V-cell.

- If $k-W_T$ is an order-$k$ Voronoi region, then
  \[ k-W_T = \text{proj}(C_{<f^*,s^*,s^+>}) \]
  \[ C_{<f^*,s^*,s^+>} \text{ is a } k\text{-belt cell in the } f_g\text{-arrangement.} \]

- Otherwise $k-W_T$ is the union $\text{proj}(C)$ over all cells $C$ that are $k$-level as well as $(k+1)$-level cells and that are of the form $C = C_{<f^*,s^*,s^+>}$ with $E^- \cup E^0 = T$.

Figures 4.1 and 4.2 illustrate the contents of the preceding theorems.

The preceding theorems and Lemma 2.1 have an important corollary.

**Corollary 4.1:**
Let $f_g$ and $g_g$ be two order-equivalent collections of functions on a common domain $D$.

For $1 \leq k \leq n$

\[ \text{degree-$k$-VOD($f_g$)} = \text{degree-$k$-VOD($g_g$)}, \]
\[ \text{order-$k$-VOD($f_g$)} = \text{order-$k$-VOD($g_g$)}. \]

Using the theorems and corollaries of this section along with the facts stated in section 2 we obtain the following algorithmic results.

**Theorem 4.4:**
Let $D = R^d$ for some $d \geq 1$, and let $f_g$ be a collection of affine functions from $R^d$ to $R$, or let $f_g$ be a set of $d$ quadratic functions on $R^d$ generated by one common real symmetric matrix $A$ as in Theorem 3.2.

For $1 \leq k \leq n$ all degree-$k$ and order-$k$ Voronoi Diagrams with respect to $f_g$ can be constructed in time and space $O(n^{d+1})$. This is worst case optimal.

**Proof:**
By the preceding results it suffices to construct all cells in the $f_g$-arrangement. By Fact 2.2 (and in case of the quadratic functions because of Lemma 2.2) this can be done optimally in the given time and space bound.

Q.E.D.

The reader may again consult the examples in section 2 to see that this result covers a rather general class of Voronoi Diagrams, among them, of course, the ordinary Euclidean VoD of point sets. Our result should also be contrasted with the $O(n^d)$ algorithm proposed in [D] to construct all Euclidean order-$k$ VoDs of a planar point set. The optimality claim made there clearly has to be taken with a grain of salt.

Finally there remains the question of how to construct single order-$k$ or degree-$k$ VoDs for affine or quadratic function collections. As a consequence of the main theorems in this section this is
equivalent to constructing all $k$-level or $b$-belt cells in an $f_\mathcal{E}$-arrangement. Thus the time bounds stated in Fact 2.5 apply. However, they appear to be rather weak and we cannot make any claims about optimality.

V. Intersection Problems

In this section we want to show briefly how an $f_\mathcal{E}$-arrangement can be a useful tool for solving certain intersection or union problems for finite sets of objects. Note that using de Morgan's laws, union can be reduced to intersection. We therefore restrict ourselves to intersection problems. At first we describe our method abstractly, later we apply it to solve intersection problems for finite sets of discs and, more generally, finite sets of similar conic sections in $\mathbb{R}^d$.

For $e \in E$ let $B_e$ be some subset of our domain $D$ and assume for each $e$ there is a real-valued function $f_e$ on $D$ with $f_e(x) < 0$ iff $x \in B_e$. We are interested in describing $\bigcap B_e$ in terms of the $f_\mathcal{E}$-arrangement. For this purpose let us identify every subset $B \subseteq D$ with its injection $\{ (x,0) \mid x \in B \}$ into $D \times \mathbb{R}$. This way we can write for every $e \in E$ that $B_e = f_e \cap B$.

The intersection $\bigcap B_e$ can then be written as

$$\bigcap_{e \in \mathcal{E}} (f_e \cap D) = D \cap \bigcap_{e \in \mathcal{E}} f_e$$

which is of course nothing but $D \cap C_{\mathcal{E},\mathcal{M},\mathcal{I}}$.

This means one can construct $\bigcap B_e$ by first constructing the “top” full cell $C_{\mathcal{E},\mathcal{M},\mathcal{I}}$ of the $f_\mathcal{E}$-arrangement and then intersecting this cell with the “base plane” $\{ (x,0) \mid x \in D \}$.

As an example consider the problem of constructing the intersection of a finite set of open discs in the Euclidean plane $\mathbb{R}^2$, where each disc $B_i$ has center $c_i \in \mathbb{R}^2$ and radius $r_i > 0$.

Associate with each disc $B_i$ the real-valued function $f_i$ on $\mathbb{R}^2$, where for $x \in \mathbb{R}^2$

$$f_i(x) = (x - c_i)^T I (x - c_i) - r_i^2,$$

and $I$ is the identity matrix.

Note that as desired the condition $x \in B_i$ iff $f_i(x) < 0$ holds. Therefore the idea outlined above is applicable.

First construct the cell $C_{\mathcal{E},\mathcal{M},\mathcal{I}}$ formed by the intersection of the upper hemispheres $f_i$. As all the functions $f_i$ are quadratic functions generated by the same matrix $I$, the results of section 2 imply that $C_{\mathcal{E},\mathcal{M},\mathcal{I}}$ can be constructed in $O(n \log n)$ time. This cell can then be intersected with the base plane $\{ (x,0) \mid x \in \mathbb{R}^2 \}$ in $O(n)$ time to yield the intersection $\bigcap B_i$ of all the discs in $O(n \log n)$ time overall. The same approach can be used to construct the intersection of a open balls in $\mathbb{R}^d$ in time $O(n^{d+1/2})$ for $d > 2$.

This method of constructing the intersection of discs is very closely related to the one proposed by Brown [BB8]. He uses spheres and spherical inversion to reduce the problem to one of intersecting halfspaces, whereas we, if one analyzes our method in detail, use paraboloids and the “warping” function $F_i$ given at the end of section 2.

We briefly want to mention some more general applications of our method of using the $f_\mathcal{E}$-arrangement formalism to solve intersection problems. One concerns intersecting regions $B_i$ of the form

$$B_i = \{ x \in \mathbb{R}^d \mid f_i(x) < 0 \} ,$$

where

$$f_i(x) = (x - c_i)^T A (x - c_i) + r_i ,$$

$c_i \in \mathbb{R}^d$, $r_i \in \mathbb{R}$, and $A$ a fixed non-singular symmetric matrix.

Such regions $B_i$ are general conic sections. For instance, in the case $d = 2$ and $A$ indefinite, $B_i$ might be the unbounded (non-convex) region between the two branches of a hyperbola. With our formalism the intersection of $n$ such regions $B_i$ can be accomplished in $O(n \log n)$ time in case $d = 2$, and $O(n^{d+1/2})$ time in case $d > 2$. Note that Brown’s method cannot be used for this purpose as it relies on spherical inversion.

The machinery of $f_\mathcal{E}$-arrangements can also be used to solve intersection problems of the form “find all $x$ that lie in exactly $k$ regions $B_i$.” Finding such a set reduces to the problem of intersecting the $k$-belt cells of an $f_\mathcal{E}$-arrangement with the “base plane”. Similarly, the boundary of the set of all $x$ that lie in at least $k$ regions $B_i$ can be constructed via intersecting the $k$-level cells with the “base plane”. We leave the details to the reader.

VI. Conclusions

The initial seed for the ideas in this paper was the observation that Voronoi Diagrams of point sets in the plane are related to 3-dimensional polyhedra whose facets are tangent to a common paraboloid. We discovered this while scrutinizing K.Q. Brown’s use of spherical inversion to relate Voronoi Diagrams in the plane with 3-dimensional polytopes whose vertices all lie on a common sphere.
At first these relationships seemed rather mysterious and inexplicable. Only after we turned our attention to the question of what Voronoi Diagrams really are, did we arrive at the (we think) satisfying explanation of these relationships presented in this paper.

An important concept in this paper is the notion of an $f_\mathcal{K}$-arrangement over a domain $D$, i.e. the partition of $D \times \mathcal{K}$ induced by a finite collection $f_\mathcal{K}$ of real valued functions on $D$. Of course, in its full generality this concept is quite useless. However, with appropriate restrictions $f_\mathcal{K}$-arrangements can provide for interesting geometric-combinatorial research. For instance, if $D = \mathbb{R}^n$ and the functions in $f_\mathcal{K}$ are continuous and satisfy certain simple finite intersection properties, $f_\mathcal{K}$-arrangements have been studied fairly extensively as pseudoplane arrangements [G71],[G72] and also in the context of oriented matroids [M]. From the point of view of computational geometry, pseudoplane arrangements still offer some interesting algorithmic problems. How difficult is it, for instance, to construct all cells in such an arrangement when only $f_\mathcal{K}$ is given? It seems that the $O(n^{n+1})$ algorithm in [EOS] actually generalizes to that case. How difficult is it to construct all $k$-level cells? In particular, for $D = \mathbb{R}^2$ is it possible to construct the $k$-level cells in $O(n \log n)$ time? This would yield a fast construction algorithm for a large class of planar Voronoi Diagrams.

Finally there is the question whether there are other interesting classes of $f_\mathcal{K}$-arrangements. There appear to be several natural and promising ways of arriving at such arrangements. One would be to relax or change the finite intersection properties of functions postulated in the case of pseudoplane arrangements. For instance, in the case $D = \mathbb{R}^2$, allowing two surfaces to intersect and cross either in a line or in a simple closed curve would give rise to a class of arrangements that includes the ones that correspond to weighted Voronoi Diagrams [A-E]. Another interesting way of generalizing would be to change the underlying domain $D$ to, say, a torus, and to try to postulate appropriate intersection properties for that case.

Acknowledgements:

We would like to thank John Gilbert for his careful reading of the manuscript and his many suggestions for improvement. We also want to thank Bennett Battaile, Gianfranco Bilardi, Joseph O'Rourke, and Chee Yap for their comments.

References


