Presburger's Article on Integer Arithmetic: Remarks and Translation

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The participants of the First Congress of Mathematicians of the Slavic Countries held at the Polytechnic Institute in Warsaw in September, 1929. From the conference proceedings.

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An early chapter in the development of decision procedures concerns the theory of Presburger arithmetic. The original article presenting the theory was published in German in 1930 under the title "Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt." My translation of this article appears here. The translation is preceded by remarks about the historical circumstances surrounding the paper and about the paper itself.

The article was written by Mojżesz Presburger, a Polish student of mathematics, and presented at a conference in Warsaw in 1929. In it Presburger showed that the part of number theory which uses only the addition function is complete, that is, every formula or its negation is true. This fragment of number theory has come to be known as Presburger arithmetic. Although the motivation of his paper was to prove the theory complete, the method of proof is constructive and yields a decision procedure or algorithm with which every formula of Presburger arithmetic can be determined to be true or false.

Probably the earliest theorem-proving program ever written for the computer employed Presburger's algorithm to prove theorems of Presburger arithmetic. Martin Davis wrote the program in the summer of 1954 for an electronic digital computer with a memory of only 1024 words (*Davis 1960*). Today algorithms for deciding Presburger arithmetic are important as components of large automatic theorem-proving and program verification efforts (*Bledsoe 1974* and *Shostak 1979*). The ability to deduce facts about some portion of arithmetic is useful, where so many facts are out of reach of any decision procedure. Fortunately, many of the theorems that arise from checking the correctness of computer programs can be established by Presburger's algorithm and other related decision procedures.

The Conference. The First Congress of Mathematicians of the Slavic Countries took place on September 23–27, 1929, at the Polytechnic Institute in Warsaw. Participants came mostly from Poland, but a few came from as far away as Japan and Texas. (Incidentally, transatlantic flights did not begin until ten years later.) Among those attending the conference were Stanisław Ulam, who was then a student at Lwów,† and John von Neumann, who was a Privatdozent‡ at Berlin at that time. Alfred Tarski, whose lectures at the University of Warsaw provided the impetus for Presburger's work, gave two papers. The opening session on Monday included a lecture by Adolf (Abraham) Fraenkel on the life and work of Georg Cantor. For the closing session the participants went by sleeping car to Poznań, where Kazimierz Kuratowski talked about topology. This is reminiscent of the First International Congress of Mathematics in Zürich in 1897 where the participants went by special trains to a mountain retreat for a banquet and the closing session.

Both the program, published in 1929 for the participants, and the proceedings, which did not appear until 1930, were published in Polish and French. Everything possible, title page, headings and the like, appeared in both languages. The welcoming remarks of the mayor of Warsaw, who greeted the participants himself, also appeared in both languages. However, the contributed articles were published in the original language, and papers can be found in German, Polish, Russian, French and Italian; French and German were the most common languages used.

In this brief period between the wars, mathematics flourished in Poland. This was the second international conference of mathematicians in Poland, the first being in Cracow during 1927. The universities located in Cracow and Lwów suffered less under foreign domination than those in Warsaw. As a consequence Cracow and Lwów emerged as centers of Polish mathematics. Both published their own international journals: *Annales de la Société Polonaise de Mathématique* and *Studia Mathematica*. At Lwów the seeds of Polish logic were sown by Kazimierz Twardowski, who was appointed to the chair of philosophy in 1895 (*Jordan 1945*). His students formed the backbone of the school of logic known as the Lwów-Warsaw school.

Despite a hundred years of Prussian and Russian oppression, Warsaw began to emerge as another center of Polish mathematics. The University of Warsaw and the Polytechnic

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† The many names by which this city has been known reflect the turbulence of the time. When Ulam was born in the city in 1909, Lemberg was the capital of Galicia in Austria-Hungry. In the chaos accompanying the end of World War I, Lviv was briefly the capital of a Ukranian republic. At the time of the conference in Warsaw, Lwów was again part of Poland. Today Lvov is found in the Soviet Union.

‡ A *Privatdozent* in those days was a university lecturer who, unlike a professor, was not a salaried employee of the state, but rather was permitted to charge students for lectures in order to make a living.

Institute were organized anew in 1915 after the Tsarist armies left.† As Kuratowski relates (*Kuratowski 1980*), Samuel Dickstein, Jan Lukasiewicz and Stefan Mazurkiewicz were the professors of mathematics at the University. Among the first students were Bronisław Knaster, Stanisław Saks, and Kuratowski. Soon after came Tarski, who received his Ph.D. from the University in 1923 and then remained as an adjunct professor until the outbreak of World War II. Also, at about the same time, Adolf Lindenbaum and Presburger were students as well (*Jordan 1945*). At the time of conference in 1929 Lindenbaum had obtained his doctorate, but apparently Presburger had not.

Because of his important role in the rise of Polish logic, Lukasiewicz deserves a little more attention. He was born in Lwów in 1878 and earned his Ph.D. from the University of Lwów in 1902 under Twardowski. From 1906 he was a Privatdozent of philosophy at Lwów; in 1911 he was made professor. From 1915 to 1939 he was a professor at the University of Warsaw. In July 1944 Lukasiewicz found refuge in Münster, Germany, where he had previously been given an honorary title. From the end of World War II until he died in 1956, he was professor of mathematical logic at the Royal Irish Academy in Dublin. He is perhaps best known today for his invention of a parenthesis-free notation, sometimes called Polish notation. This notation has found application in the organization of some of today's hand calculators. Lukasiewicz also discovered many-valued logics and used them in proofs of the independence of axioms. In addition, he was the first person to suggest a natural deduction style method for logical deduction as opposed to the less natural, formalistic logical systems of Frege, Russell and Hilbert (*Prawitz 1965*, page 98). Lukasiewicz's textbook on mathematical logic, *Elements of Mathematical Logic* was published just after the one by Hilbert and Ackermann. These books are the first textbooks on "modern" or "mathematical" logic, although Lukasiewicz's book is heavily influenced by the philosophical approach to logic dominated by Aristotle and his syllogisms. The book was compiled by Presburger from oral lectures given by Lukasiewicz at the University of Warsaw.

† The expulsion of the Russians by the Germans did not end the military conflicts which had to have an effect on university life. After the defeat of the Germans and the restoration of Poland by the Treaty of Versailles, the Russians threatened Warsaw in 1920. Then internal problems culminated in a military coup d'état which took place in Warsaw in 1926.

This era of Polish mathematical activity is fast receding into history. Kuratowski died in June 1980. Tarski died in California in October 1983. Ulam, who like Tarski emigrated to the United States, died in May 1984.

The Paper. Presburger's paper was scheduled to start Section I at 10:30 on Tuesday, September 24, 1929. In the program the title is listed in Polish: "Zagadnienie zupełności i rozstrzygalności w zastosowanie do pewnego systemu arytmetya liczb całkowitych." The finished article, which was not due until the November after the conference, was published in German and appears on pages 92 through 101. There is an unnumbered "supplement" page appearing after the last numbered page in the proceedings which contains an addition to Presburger's article and errata to another article. This page is cited by several authors as page 395.

The translation here of Presburger's paper uses the phrase "meaningful statements." Today this might be more correctly rendered as "well-formed statements" or "well-formed formulas." Presburger in his paper, and Tarski and Lukasiewicz in their writings use the German term "sinnvolle Ausdrücke." Lukasiewicz, if not the others, uses the equivalent Polish "wyrażenia sensowne." (See *Lukasiewicz 1934* written in Polish and his own translation into German, *Lukasiewicz 1935*.) In the preface of *Elements of Mathematical Logic* Lukasiewicz attributes the phrase to his colleague at the University of Warsaw, Stanisław Leśniewski. Both the translator of this book and Storrs McCall in his translation of Lukasiewicz's 1934 article into English use the the phrase "meaningful statements." Even more recently Kurt Gödel used the phrase in English (*van Heijenoort 1977*, page 616). Apparently the Polish mathematicians were stressing the "meaningful" over the "well-formed" on purpose. Tarski gives the reason: "Instead of 'meaningful statements' one could also say 'regularly constructed statements.' When I use the word 'meaningful,' I do so to express my agreement with the doctrine of intuitionistic formalism." (See *Tarski 1930*, page 363.) Perhaps they were trying not to commit themselves to the precept that they had captured the meaningful statements with their choice of formalism.

The mathematical notation used by Presburger is one that may not be familiar to the modern reader. For one thing Presburger uses the summation symbol "Σ" for the existential quantifier. This notation was first used by the American mathematician Benjamin Osgood Peirce before the turn of the century. At that time logic was studied like a special form of algebra where conjunction acted like multiplication and the exists operator acted like summation. The disjunctive and conjunctive normal forms which Presburger uses date back to these earliest origins of modern logic. Presburger also uses Lukasiewicz's parenthesis-free notation with "A" and "K" as prefix, binary operators for "or" and "and" respectively. This notation is occasionally used today.

The Algorithm. The proof of completeness Presburger gives in his paper contains a decision procedure for formulas of integer arithmetic containing just the plus symbol. The presentation is quite clear and the algorithm is simple enough that one can almost read off a program to implement it directly from the article. The majority of the article is devoted to describing how to eliminate the quantifier "there exists an x" from equations containing x. Here is an example†,

$$\exists x\,(x + x + 1 = y)\ \&\ (x + z = 0).$$

One can replace this formula with another without a quantifier

$$y \equiv_2 1\ \&\ y + z = 1$$

where $y \equiv_2 1$ means that y is congruent to 1 modulo 2, or, in other words, y divided by 2 leaves a remainder of 1. By repeatedly eliminating a quantifier, a formula is obtained with no variables, and it can easily be checked if it is true or false. This is the essence of quantifier elimination.

In preparation for quantifier elimination the formula to be tested must be put in a special form where all quantifiers are at the beginning. The remainder of the formula, called the matrix, must also be put in a special form, the so-called disjunctive normal form, where the matrix is one big disjunction. Every disjunct contains any number of conjuncts. Each conjunct can be either an equation or a congruence and each of these can be negated or not. Thus there are four possibilities. Negated congruences can be eliminated in favor of the other three types, thus simplifying the number of possibilities cases to three.

† Here in this section the vertical bar "|" and the ampersand "&" are used for disjunction and conjunction respectively. This notation is by no means standard today, but has the virtue that these characters appear on the keyboard of most computer terminals. On the other hand, the backwards "E" introduced by Peano is standard notation today for the existential quantifier. This is generally accepted precisely because it distinguishes logic from algebra.

Presburger deals with the six cases formed from a combination of two from the three types of conjuncts: equations, negated equations and congruences. The generalization to more than two conjuncts is easy, and can be found in, for example, *Monk 1970*. Presburger does not explicitly mention the case where a disjunct consists of a single equation, negated equation or congruence. Since this case provides simple examples of the mathematical reasoning Presburger uses, they are given here as a prelude to the complete article. A disjunct consisting of an equation,

$$\exists x(\alpha x + a = b) \quad \longmapsto \quad a \equiv_\alpha b$$

is really the definition of congruence. It appears to be cheating to eliminate a quantifier by forming a congruence, but it is computationally easy to determine if two numbers are congruent, so this approach is justified. A negated equation,

$$\exists x \neg (\alpha x + a = b) \quad \longmapsto \quad 0 = 0$$

can always be satisfied. There is always some integer x that will make the equation false, hence the formula can be replaced by any true formula. A little bit of number theory will convince one that a disjunct consisting of a congruence,

$$\exists x(\alpha x + a \equiv_\beta b) \quad \longmapsto \quad a \equiv_{\gcd(\alpha,\beta)} b$$

can be replaced by a congruence modulo the greatest common divisor of α and β.

The asymptotic running time of Presburger's algorithm is governed, not by the mathematically interesting part of the algorithm, but by putting formulas in disjunctive normal form. In general this increases the length of a formula by a considerable amount with the result that the algorithm can consume much time and space. This combinatorial explosion led Davis in 1954 to despair of implementing more complicated decisions procedures.

The problem with putting formulas in disjunctive normal form is that there are formulas with n literals that for any integer c the disjunctive normal form has more than n^c literals. For example, consider the formula

$$(x_1 \mid x_2) \,\&\, (y_1 \mid y_2) \,\&\, \cdots \,\&\, (z_1 \mid z_2).$$

Let it have a conjuncts, say. It originally has $2a$ literals. In disjunctive normal form it will have 2^{a+1} literals. Since a can be chosen so that $n^c = (2a)^c < 2^{a+1}$, there is no polynomial bound on the increase in size of formulas put in disjunctive normal form.

It is now known that there are more efficient algorithms for deciding formulas of Presburger arithmetic. One is described in *Cooper 1972.* The basic idea is to test the matrix with a small number of integers that cover all possible cases. These integers can be determined without putting the matrix in disjunctive normal form. Thus a quantifier can be eliminated by replacing the matrix by the disjunction of all the different cases. The same idea works for rational numbers as well. See *Hopcroft and Ullman 1979.* Curiously the decision procedure for rationals is even more efficient. A general case for algebraically closed fields was solved by Tarski. The main results in this case were obtained in 1927–1928 in lectures at the University of Warsaw, but nothing was published until after World War II (*Tarksi 1951*).

.Derek Oppen has shown that Cooper's algorithm for deciding Presburger arithmetic increases a formula of length n by no more that $2^{2^{2^{cn}}}$ for some constant c. He has also shown the asymptotic running time is essentially the same function (*Oppen 1978*). This is most probably optimal since it has been shown that Presburger arithmetic requires non-deterministic time $2^{2^{cn}}$ for some constant c. See *Fischer and Rabin 1974.* Any algorithm that would run significantly faster would imply that non-deterministic Turing machines could simulated by deterministic Turing machines with less than an exponential slowdown.

The role of Presburger's algorithm in the development of logic can be appreciated by consulting Beth's copious work, *The Foundations of Mathematics.* Beth actually treats a subtheory with a severe restriction on equations. The place of Presburger's decision procedure as it stands in the edifice of modern logic can be found in *Monk 1970.* This exposition has the algorithm in its most general form with the "less than" predicate in the theory as well as equality. Presburger, as the addendum to the article shows, was aware that this extension was possible. For a broad treatment of quantifier elimination, see *Kreisel and Krivine 1967.*

The translation of Presburger's paper now follows. The footnotes are his own, but pointers to translations and more modern literature are added to the footnotes in brackets. A bibliography is given at the end. It includes all the original references used by Presburger and related literature, historical background material, and contemporary literature concerning Presburger arithmetic.

About the completeness of a certain system of integer arithmetic in which addition is the only operation

by

M. Presburger (Warsaw)

The present note contains a result[1] about the completeness of a set[2] of sentences of integer arithmetic. The proof of completeness sketched below also gives, as a result of its effective character, a process which decides if a given statement in the part of arithmetic under consideration is a true sentence of arithmetic.[3]

We consider a set of statements which we will call *meaningful statements*. The meaningful statements are built with the following symbols:

Individual symbols:	Implication symbol–"C"
	Negation symbol–"N"
	Equals symbol–"="
	Exists symbol–"Σ"
	Plus sign–"+"
	Symbol for zero–"0"
	Symbol for unit–"1".
Symbols for variables:	Predicate variables–"p", "q", "r" ...
	Integer variables–"x", "a", "b", "c" ...

A rigorous definition of a meaningful statement will be omitted here. We mention only that the statement "Cpq" is a meaningful statement and stands for the proposition "if p, then q". Also the statement "Np" is a meaningful statement which stands for the proposition "not p". The meaningful statement "$\Sigma a(a+1=0)$" stands for the proposition that "there is an integer a such that $a+1=0$".

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[1] The result was obtained in May 1928. The problem was posed by Mr. A. Tarski.

[2] See A. Tarski, "Remarques sur les notions fondamentales de la Méthodologie des Mathématiques," *Annales de la Société Polonaise de Mathématique*, volume VII, 1928, pages 270–272. [Since the publication of this article (*Tarski 1928*), Tarski has written several, more comprehensive articles concerning the nature of deductive science. These can be found translated into English in *Tarski 1956*, articles III, V and XII. But here Presburger is only assuming the reader knows Tarski's definition of completeness in a deductive theory which is: every sentence or its negation is in the theory.]

[3] On the topic of completeness and decidability see: D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik*, (Berlin: Springer 1928). [There is an English translation of the second edition of this classic work *Hilbert and Ackermann 1950*.]

The following expressions will be used as abbreviations for meaningful statements:

"Apq" for "$CNpq$", a disjunction of two statements,

"Kpq" for "$NCpNq$" a conjunction of two statements.[4]

Let A be a set of meaningful statements that contains, to start out with, the following sentences:

1. $CCpqCCqrCpr$
2. $CCNppp$
3. $CpCNpq$

These are the three axioms of propositional calculus determined by Mr. J. Lukasiewicz.[5]

Furthermore two sentences about equality belong to A:

4. $a = a$
5. $C(a = b)C(a = c)(b = c)$

as well as the following sentences about arithmetic:

6. $C(a = b)(a + c = b + c)$
7. $C(a + c = b + c)(a = b)$
8. $a + b = b + a$
9. $a + (b + c) = (a + b) + c$[6]
10. $a + 0 = a$
11. $\Sigma b(a + b = c)$

Sentence 11 asserts that the difference of two integers always exists.

Three recursive sets of statements are added to the eleven sentences already mentioned.

[4] The notation we use here originates from Mr. J. Łukasiewicz. See Lukasiewicz, *Elemente der mathematischen Logik*, prepared by M. Presburger, (University lecture notes, Polish), Warsaw, 1929. [Lukasiewicz takes credit for the parenthesis-free notation in the author's preface. See page ix of the English edition *Lukasiewicz 1963.*]

[5] See Lukasiewicz, *loc. cit.*, page 45. [These axioms are on page 28 of the English edition.]

[6] This sentence permits the parentheses to be left out in all expressions of the form "$a + (b + c)$" and "$(a + b) + c$".

12. $C(a+a=b+b)(a=b)$

$C(a+a+a=b+b+b)(a=b)$

$\vdots$

$C(\underbrace{a+a+\ldots+a}_{\alpha}=\underbrace{b+b+\ldots+b}_{\alpha})(a=b)$

$\vdots$

The sentences in 12 could have been expressed as:

$$C(\alpha a=\alpha b)(a=b),$$

but since we do not have the multiplication symbol, a recursive set of statements has been used. When in the future we use an expression of the form "αa", where α is a natural number, it is to be viewed as an abbreviation for the expression "$\underbrace{a+a+\ldots+a}_{\alpha}$".

13. $\Sigma xA(x+x=a)(x+x+1=a)$

$\Sigma xAA(x+x+x=a)(x+x+x+1=a)(x+x+x+1+1=a)$

$\vdots$

$\Sigma xA\ldots A(x+x+x=a)(x+x+x+1=a)(x+x+x+1+1=a)$

$\Sigma x\underbrace{A\ldots A}_{\alpha-1}(\alpha x=a)(\alpha x+1=a)\ldots(\alpha x+\underbrace{1+1+\ldots+1}_{\alpha-1}=a)$

$\vdots$

The statements in 13 could be read as follows: for every natural number α and for every integer a there is always an integer x such that $\alpha x=a$ or $\alpha x+i=a$, where i is a natural number less than a.

14. $N(a+a+1=0)$

$N(a+a+a+1=0)$

$\vdots$

$N(\alpha a+1=0)$

$\vdots$

The sentences in 14 say that for no natural number α larger than 1 does the equality $\alpha a + 1 = 0$ hold.

The statements 1 through 14 which comprise the set A could be understood as an infinite axiom system (if one wanted to introduce such a notion) for a theory of integers in which addition is the only operation.

We now introduce the set of consequences of the set A which we will denote A_x. A_x is the smallest set which contains A and fulfills the following four conditions:[7]

1. If a statement p belongs to A_x, so does every statement which can be obtained from p by substitution.[8]

2. If two statements of the form p and Cpq belong to A_x, then q belongs to A_x as well.

3. If a statement of the form $C\Sigma aqr$ belongs to A_x, then Cqr belongs to A_x as well.

4. If a statement of the form Cqr belongs to A_x, and q contains a free variable of the same form as "a", then $C\Sigma aqr$ belongs to A_x, as long as no such variable occurs in r.

The set A_x defines the system of arithmetic[9] for which we will outline a completeness proof. Without getting into details, we remark that to prove the completeness of the set A_x it is sufficient to show that for all p, a meaningful statement without free variables, either p or Np belongs to A_x. In other words it is sufficient to prove the decidability of meaningful statements without free variables.

Among those meaningful statements present are those that express the congruence of two integers modulo α, where α is a natural number. For example, the expression $\Sigma x(x + x + a = b)$ means the same as $a \equiv b \bmod 2$. Statements of the form $\Sigma x(\alpha x + a = b)$ will be written from now on by $a \equiv_\alpha b$ for the sake of brevity.

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[7] See A. Tarski, *loc. cit.*

[8] It is easy to surmise what the definition of substitution should be in our system. See Hilbert and Ackermann, *loc. cit.*, pages 53–54. [Formulating substitution correctly is apparently not that easy. Hilbert and Ackermann failed to get it right once again in the revised, second edition. See *Church 1956*, page 289. The (incorrect) formulation of substitution by Hilbert and Ackermann can be found on pages 69–70 of the English edition.]

[9] The logical model for the system A_x originates from A. Tarksi, who in his university lectures in the academic year 1927–1928 covered deductive systems without function variables. Among other things, Tarski proved the completeness of the geometry of straight lines which is based on the notions: "b lies between a and c" and "a is the same distance from b as c from d." Tarksi also investigated all the complete systems of the calculus of classes. [See *Beth 1965*, page 584.]

Now we introduce the concept of a *ground statement.* We understand a ground statement to be a meaningful statement in one of the following two forms: $a = b$ or $a \equiv_\alpha b$.

The principle lemma which leads up to our completeness theorem is:

Every meaningful statement without free variables can be transformed into an equivalent statement in disjunctive normal form,[10] *where the members of the disjuncts are either ground statements or their negations without free variables.*

Ground statements without free variables can be converted to one of the following four forms by simple transformations:

$$0 = 0, \quad 1 + 1 + \ldots + 1 = 0, \quad 0 \equiv_\alpha 0, \quad \underbrace{1 + 1 + \ldots + 1}_{\beta} \equiv_\alpha 0$$

Every one of these statements is decidable in our system. In particular it is easy to show that the statement $\underbrace{1 + 1 + \ldots + 1}_{\beta} \equiv_\alpha 0$ is in A_x if and only if β is divisible by α. The expression $N(\underbrace{1 + 1 + \ldots + 1}_{\beta} \equiv_\alpha 0)$ belongs to A_x if and only if β is not divisible by α.

So if an expression is in disjunctive normal form as referred to in the lemma, then our system is decidable. As a consequence, every meaningful statement without free variables is decidable.

The proof of our principle lemma is based on the fact that every meaningful statement can be transformed into an equivalent normal form, so that the "exists" symbol (negated or not) is at the beginning of the expression,[11] and then into an expression in disjunctive normal form in which equations (hence ground statements) or negated equations appear as the members of the disjuncts. The following is an example of an expression in the normal form just described:

$$\Sigma a N \Sigma b \Sigma c N \Sigma d A A K K r_{11} r_{12} r_{13} K r_{21} r_{22} K r_{31} r_{32},$$

where $r_{\vartheta\iota}$ is an equation or the negation of an equation.

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[10] See Hilbert and Ackermann, *loc. cit.*, page 13. [This can be found on page 17 of the English edition.]

[11] See Hilbert and Ackermann, *loc. cit.*, page 63. [First called prenex normal form in the second edition, this can be found on page 83 of the English edition.]

The basic idea of the proof is that one can progressively eliminate the innermost "exists" symbol from the above normal form expression obtaining the equivalent form:

$$\Sigma a N \Sigma b \Sigma c N A A \Sigma d K K r_{11} r_{12} r_{13} \Sigma d K r_{21} r_{22} \Sigma d K r_{31} r_{32}.$$

Suppose then that one could replace every expression of the form

$$\Sigma d K \ldots K r_1 r_2 \ldots r_\alpha,$$

where r_i is a ground statement or the negation of one, by an equivalent expression of the form

$$K \ldots K r'_1 r'_2 \ldots r'_{\alpha'},$$

where r'_i is again either a ground statement or the negation of one, without introducing any new free variables. Then we could obtain an expression in disjunctive normal form by eliminating the "exists" symbols step by step.[12] Because this transformation introduced no free variables, the last expression will have no free variables if the original expression had no such variables.

Hence we have shown that in order to prove the completeness of the system A_x it is sufficient to prove the following lemma.

Every expression of the form:

$$\Sigma x K \ldots K r_1 r_2 \ldots r_\alpha,$$

where each r_i is a ground statement or negated ground statement, can be converted into an equivalent form:

$$K \ldots K r'_1 r'_2 \ldots r'_{\alpha'},$$

where each r_i is again a ground statement or negated ground statement not containing new free variables.

[12] The process described here has already been used by Mr. Th. Skolem and Mr. C. H. Langford. (See Th. Skolem, "Untersuchungen über die Axiome des Klassenkalkuls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen," Videnskapsselskapets Skrifter. I Mat. nat. Klasse, number 3, 1919. See also *Annals of Mathematics*, Second series, volume 28, Princeton, N.J., 1926–27, "Some theorems on deducibility," by C. H. Langford, page 16 and "Theorems on deducibility (Second paper)," by C. H. Langford, page 459.) Mr. Tarski developed it into a systematic and general method. [See section 101 of *Beth 1965* for a description of the quantifier elimination procedure for the theory of dense orderings that Langford developed.]

We will consider the case where each disjunct has only two conjuncts; the generalization to more than two can be easily derived.

We must examine four types of conjuncts: equations, negated equations, congruences, and negated congruences. However, negated congruences can be reduced to congruences, since the statement $N(a \equiv_\alpha b)$ is equivalent to[13]

$$AA\ldots A(a+1 \equiv_\alpha b)(a+1+1 \equiv_\alpha b)\ldots(a+\underbrace{1+1+\ldots+1}_{\alpha-1} \equiv_\alpha b).$$

So there are three types of conjuncts remaining, and we must examine the following combinations:

I. equation, equation
II. congruence, congruence
III. negated equation, negated equation
IV. equation, negated equation
V. equation, congruence
VI. congruence, negated equation.

I. A statement of the form $\Sigma x K r_1 r_2$, where r_1, r_2 are equations, can easily be transformed to the form:

$$\Sigma x K(\alpha x + a = b)(\alpha' x + a' = b').$$

Now we have two linear equations in one unknown x which we can transform so that they have the same coefficient β (the least common multiple of α and α'):

$$\Sigma x K(\beta x + c = d)(\beta x + c' = d').$$

By further transformations we obtain:

$$\Sigma x K(\beta x + c = d)(d + c' = d' + c)$$
$$K \Sigma x(\beta x + c = d)(d + c' = d' + c)$$
$$K(c \equiv_\beta d)(d + c' = d' + c)$$

Thus the lemma is proved for the case of two equations.

[13] This simplification was noticed by Mr. A. Lindenbaum. [Adolf Lindenbaum was killed by the Gestapo in occupied Poland (*Luschei 1962*).]

II. A statement of the form $\Sigma x K r_1 r_2$, where r_1 and r_2 are congruences, can easily be put in the form:

$$\Sigma x K(\alpha x + a \equiv_{\beta} b)(\alpha' x + a' \equiv_{\beta'} b').$$

By the theorem that asserts the equivalence of $a \equiv_{\alpha} b$ and $\beta a \equiv_{\beta\alpha} \beta b$, we can write our two congruences as congruences with the same modulus δ (the least common multiple of β and β'):

$$\Sigma x K(\gamma x + c \equiv_{\delta} d)(\gamma' x + c' \equiv_{\delta} d')$$

We obtain further, when, say, $\gamma > \gamma'$:

$$\Sigma x K(\gamma'' x + c + d' \equiv_{\delta} d + c')(\gamma' x + c' \equiv_{\delta} d'),$$

where $\gamma'' = \gamma - \gamma'$. In this way the coefficient of the unknown in one of the given congruences is reduced from γ to γ'. After repeating this process, we obtain a system of two congruences with the same coefficient of the unknown:

$$\Sigma x K(\eta x + e \equiv_{\delta} f)(\eta x + e' \equiv_{\delta} f').$$

Further transformations give the equivalent forms:

$$\Sigma x K(\eta x + e \equiv_{\delta} f)(f + e' \equiv_{\delta} f' + e)$$

$$K \Sigma x(\eta x + e \equiv_{\delta} f)(f + e' \equiv_{\delta} f' + e)$$

A necessary and sufficient condition for the solution of the congruence $\eta x + e \equiv_{\delta} f$ is that $e \equiv_{\vartheta} f$ holds, where ϑ is the greatest common divisor of η and δ. So finally we get:

$$K(e \equiv_{\vartheta} f)(f + e' \equiv_{\delta} f' + e)$$

Thus the lemma is proved for the case of two congruences.

III. In the case of two negated equations:

$$\Sigma x K N(\alpha x + a = b) N(\alpha' x + a' = b')$$

we have a statement that always holds, so it is equivalent to, say, the statement $0 = 0$.

IV. In the case of an equation and a negated equation we have:

$$\Sigma x K(\alpha x + a = b) N(\alpha' x + a' = b')$$

$$\Sigma x K(\beta x + c = d) N(\beta x + c' = d')$$

$$\Sigma x K(\beta x + c = d) N(d + c' = d' + c)$$

$$K(c \equiv_{\beta} d) N(d + c' = d' + c)$$

V. In the case of an equation and a congruence, we have:

$$\Sigma x K(\alpha x + a = b)(\alpha' x + a' \equiv_{\beta} b')$$

$$\Sigma x K(\gamma x + c = d)(\gamma x + c' \equiv_{\beta'} d')$$

$$\Sigma x K(\gamma x + c = d)(d + c' \equiv_{\beta'} d' + c)$$

$$K(c \equiv_{\gamma} d)(d + c' \equiv_{\beta'} d' + c)$$

VI. Now we consider the case of a congruence and a negated equation:

$$\Sigma x K(\alpha x + a \equiv_{\beta} b) N(\alpha' x + a' = b').$$

It is easy to show that this system is equivalent to the following condition:

$$\Sigma x(\alpha x + a \equiv_{\beta} b)$$

which in turn is equal to the congruence:

$$a \equiv_{\gamma} b,$$

where γ is the greatest common divisor of α and β. Thus the last case of the lemma is confirmed.

In conclusion we want to note that all the arithmetic transformations used above are justified by the definition of the set A_x. This can be checked out in detail.

Thus we have sketched the completeness proof. It is easy to see that the given proof gives us at the same time a process by which one can decide if a given meaningful statement p is in the set A_x in a finite number of steps. That is, *in the area of integer arithmetic restricted to the set A_x there are no more undecidable problems.*

Should we want to introduce the multiplication symbol to our system, we would encounter unsolved problems in the proof of decidability. Since in such an expanded system we could formulate, for instance, the statement:

$$N\Sigma x\Sigma y\Sigma z(\underbrace{x \cdot x \cdot \ldots \cdot x}_{\alpha} + \underbrace{y \cdot y \cdot \ldots \cdot y}_{\alpha} = \underbrace{z \cdot z \cdot \ldots \cdot z}_{\alpha})$$

which is a special case of the Fermat's last theorem. Because α can be an arbitrary number, in order to prove the decidability of the expanded system we would have to be able to decide each special case of Fermat's last theorem.

Additions to the communication by M. Presburger:

1. Sentence 7 in the definition of the set A is superfluous, since it can be derived from the other sentences.

2. The completeness result can be extended to the arithmetic of whole numbers with "0", "1", "+" and ">" as primitive notions.

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