Volume 2: Dynamics

Analytical Elements of Mechanics
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PREFACE

This book is the second of two volumes intended for use in courses in classical mechanics. The first deals with analytical elements of statics; the present one is concerned with dynamics.

As before, the symbolic language used is vector analysis. Proofs and derivations are given in considerable detail, while discursive material is omitted, leaving the teacher free, on the one hand, and obligating him, on the other, to explain the physical significance of mathematically defined quantities and to discuss related topics of practical, philosophical, or historical interest. Initially, physics and mathematics are kept separate, in the belief that this ultimately facilitates the task of relating the two. More than one hundred illustrative examples and problems are worked out in complete detail in the text, and practice problems, closely correlated with matters treated in the text, are provided in the form of Problem Sets, in a number regarded as both necessary and sufficient to insure adequate coverage without expenditure of excessive amounts of time.

As for the "level" of the book, I have used it with undergraduates, both at the University of Pennsylvania and at the Manchester College of Science and Technology, spending approximately ninety lecture hours during a period of thirty weeks, and with graduate students not previously exposed to this material, covering the same ground in fifty to sixty lecture hours.

In the preface to Volume I, I expressed the opinion that one of the principal sources of difficulty in teaching mechanics has been the attempt to deal with forces intuitively. To this I now add the contention that the greatest obstacles standing in the way of effective solution of particle and rigid body dynamics problems are inadequate understanding of kinematics and of the theory of moments and products of inertia of rigid bodies. For this reason, both topics have received very full treatment in the present book, and this treatment differs considerably from those found elsewhere.

Specifically, consideration of kinematics is preceded, in Chapter 1, by an extensive discussion of differentiation of vectors, particular emphasis being placed on the importance of reference frames. Fundamentals of the differential geometry of space curves are introduced in a form rendering the results practically useful, and, incidentally, the opportunity is thus provided to use differentiation theorems developed earlier.

Chapter 2 begins with the definition of "rate of change of orientation of a rigid body," which is a generalization of the concept of angular velocity and makes it unnecessary to discuss finite rotations. The definition of and theorems dealing with angular velocity, which follow, serve to shed light on a number of questions which have not always been answered satisfactorily. For example, the matter of "simultaneous angular velocities" and such related questions as "Does a rigid body possess various angular velocities with respect to various points and lines, or is its angular velocity unique?" can now be disposed of readily (see Sections 2.2.7, 2.2.8, 2.2.1). Next, reversing the usual order, relative velocities and accelerations are defined before absolute velocities and accelerations are mentioned, this making it possible to treat the latter as special cases of the former, which both simplifies and unifies the presentation. The fact that angular velocity and acceleration of a rigid body have been dealt with previously permits immediate consideration of velocity and acceleration relationships applicable to points fixed on a rigid body (see Sections 2.4.5, 2.5.9, and 2.5.10) or free to move in a moving reference frame (see Section 2.5.13).

Two major departures from more conventional presentations of the topic of inertia occur in Chapter 3: Inertia properties of sets of particles and rigid bodies are considered only after similar properties of purely geometric entities (i.e., points, point sets, curves, surfaces, and solids) have been studied, and certain vector quantities, here called "second moments" and having much in common with the vector quantity "traction" of elasticity theory, are taken as the fundamental building blocks of the entire theory. This leads to many relationships applicable both to geometric entities and to material bodies, and permits one without the use of second rank tensors to define quantities and to derive expressions independent of particular coordinate systems. Furthermore, it is precisely these second moment vectors which later quite naturally turn out to characterize the inertia properties of rigid bodies in expressions such as those for inertia torques, angular momenta, and kinetic energy (see Sections **4.1.3**, **4.4.11**, and **4.5.4**).

The study of kinetics is undertaken in Chapter 4, where force concepts previously discussed in Volume I are brought together with ideas developed in the preceding chapters, in a rather broad statement of D'Alembert's principle, which then becomes the point of departure for the examination of various other principles. In this portion of the text and of the related Problem Sets, work done earlier in the course is presumed to justify immediate consideration of topics generally considered abstruse, and the attempt is made not only to state and illustrate the use of each principle, but also to examine it from the point of view of its suitability for the solution of specific classes of problems, the intention being both to increase the student's effectiveness as an analyst and to set the stage for later study of advanced mechanics.

T. R. KANE

Philadelphia, Pennsylvania
March, 1961

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PROBLEM SETS

APPENDIX

1 DIFFERENTIATION OF VECTORS

1.1 Vector functions of a scalar variable

1.1.1 If one or more of the characteristics (magnitude, orientation, sense) of a vector $\mathbf{v}$ in a reference frame R depends on a scalar variable z, $\mathbf{v}$ is called a *vector function of z in R*; otherwise, $\mathbf{v}$ is said to be *independent of z in R*. In particular, if $\mathbf{v}$ is independent of every scalar variable in R, one speaks of $\mathbf{v}$ as *fixed in R*.

Example: In Fig. 1.1.1,* R represents a reference frame con-

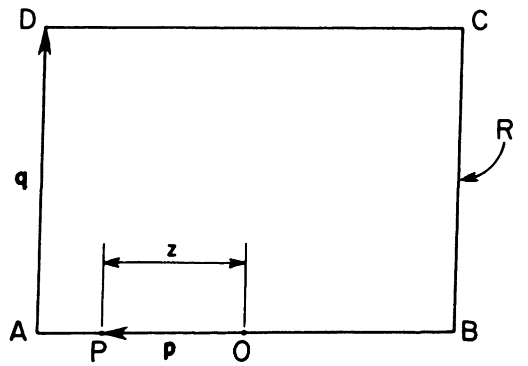


Fig. 1.1.1

sisting of a rigid rectangle $ABCD$. O is the midpoint of line AB, P a point free to move between A and B on line AB, $\mathbf{p}$ the position

*Each figure has the number of the section in which it is first cited. When more than one figure appears in a section, the letters a, b, c, etc. are appended to the section number as in, for example, Fig. 1.1.3a, Fig. 1.1.3b.

vector of P relative to O, and $\mathbf{q}$ the position vector of D relative to A.

Let z be the displacement of P relative to O, regarding z as positive when P is between O and A and as negative when P is between O and B. Then the magnitude of $\mathbf{p}$ depends on the scalar variable z; the orientation of $\mathbf{p}$ in R is independent of z; and the sense of $\mathbf{p}$ in R depends on z. Thus, as two of the characteristics of $\mathbf{p}$ in R depend on z, $\mathbf{p}$ is a vector function of z in R. On the other hand, $\mathbf{q}$ is independent of z in R. In fact, $\mathbf{q}$ is fixed in R.

1.1.2 A vector may be a vector function of a given scalar variable in one reference frame while being independent of this variable in another reference frame.

Example: Fig. 1.1.2 shows two reference frames, R and R',

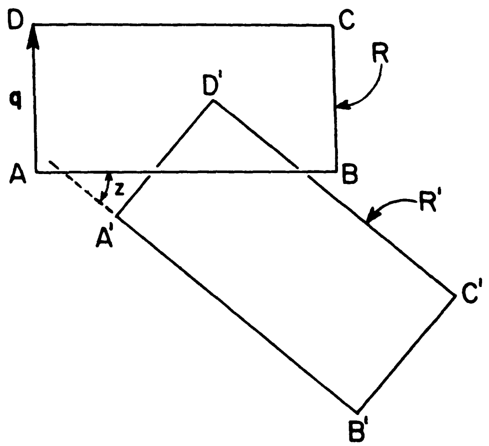


FIG. 1.1.2

consisting of rigid rectangles, $ABCD$ and $A'B'C'D'$, respectively, the two rectangles being free to move in their common plane. Let z be the angular displacement of line $A'B'$ relative to line AB, regarding z as positive when the displacement is generated by a clockwise rotation of $A'B'$ relative to AB. Then $\mathbf{q}$, the position vector of D relative to A, is independent of z in R (see Example 1.1.1), but is a function of z in R', as the orientation and sense of

q in R' depend on z. For example, when $z = \pi/2$, **q** has the sense $B'A'$; while when $z = -\pi/2$, **q** has the sense $A'B'$.

1.1.3 If **v** is a vector function of a scalar variable z in a reference frame R and $\mathbf{n}_i$, $i = 1, 2, 3$, are unit vectors (not parallel to the same plane) fixed in R, the $\mathbf{n}_i$, $i = 1, 2, 3$, measure numbers of **v** (see Vol. I, Sec. 1.10.2) are scalar functions of z which characterize the behavior of **v** in R. The unit vectors may be vector functions of z in a second reference frame R'. If this is the case, it does not affect the behavior of **v** in R, but it does affect **v**'s behavior in R'.

Problem: Referring to Example 1.1.2,* let $\mathbf{n}_1$ and $\mathbf{n}_2$ be unit

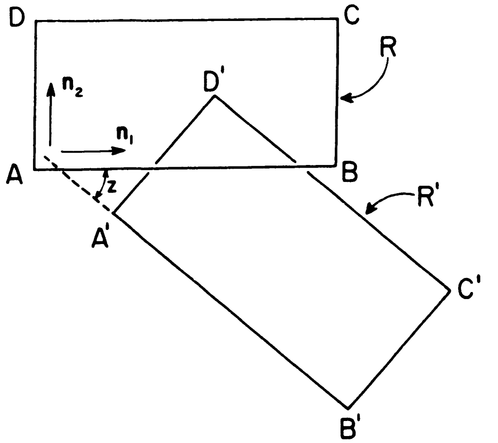


FIG. 1.1.3a

vectors fixed in R and having the directions AB and AD, respectively, as shown in Fig. 1.1.3a. A force **F** is defined as

$$\mathbf{F} = 2\left(1 - \frac{2z}{\pi}\right)\mathbf{n}_1 + 3\sin(2z)\mathbf{n}_2 \text{ lb}$$

Draw sketches showing **F** in R and R', for $z = 0$, $z = \pi/2$, and $z = -\pi/4$.

---

* Illustrative examples and problems are designated by the number of the section in which they appear. For example, Problem 1.1.3 means "the problem appearing in Section 1.1.3."

Solution: From the given expression for $\mathbf{F}$,

$$\mathbf{F}|_{z=0} = 2\mathbf{n}_1|_{z=0} \text{ lb}$$
$$\mathbf{F}|_{z=\pi/2} = 0$$
$$\mathbf{F}|_{z=-\pi/4} = 3\mathbf{n}_1|_{z=-\pi/4} - 3\mathbf{n}_2|_{z=-\pi/4} \text{ lb}$$

Fig. 1.1.3b shows $\mathbf{n}_1$ and $\mathbf{n}_2$ in R for $z = 0$ and $z = -\pi/4$. (As $\mathbf{n}_1$ and $\mathbf{n}_2$ are fixed in R, $\mathbf{n}_1|_{z=0}$ and $\mathbf{n}_1|_{z=-\pi/4}$ are identical; similarly, $\mathbf{n}_2|_{z=0}$ and $\mathbf{n}_2|_{z=-\pi/4}$.) $\mathbf{F}|_{z=0}$ and $\mathbf{F}|_{z=-\pi/4}$ thus appear in R as shown in Fig. 1.1.3c. $\mathbf{F}|_{z=\pi/2}$, being a zero vector, is omitted from the sketch.

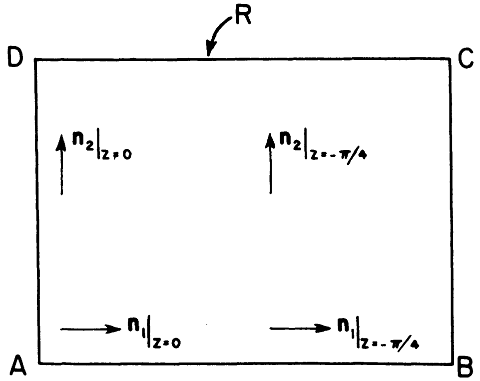


FIG. 1.1.3b

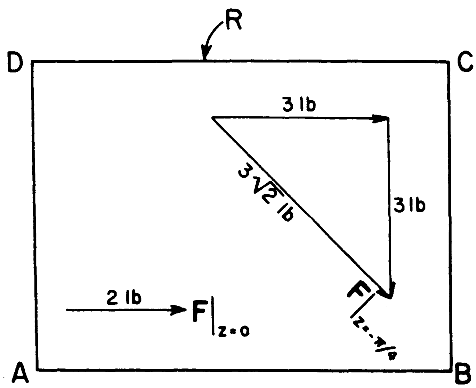


FIG. 1.1.3c

Figure 1.1.3d shows $\mathbf{n}_1$ and $\mathbf{n}_2$ in R' for $z = 0$ and $z = -\pi/4$. (Note that $\mathbf{n}_1|_{z=0}$ now differs from $\mathbf{n}_1|_{z=-\pi/4}$; similarly, $\mathbf{n}_2|_{z=0}$ differs from $\mathbf{n}_2|_{z=-\pi/4}$.) In R', $\mathbf{F}|_{z=0}$ and $\mathbf{F}|_{z=-\pi/4}$ thus appear as shown in Fig. 1.1.3e.

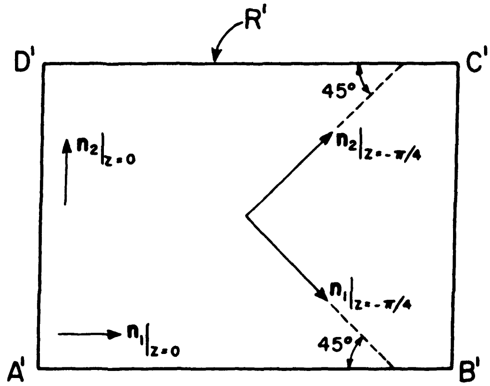


FIG. 1.1.3d

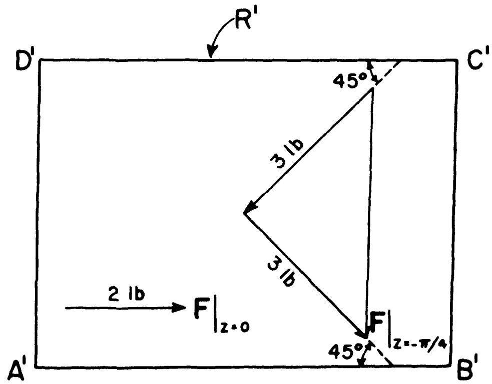


FIG. 1.1.3e

1.1.4 If $\mathbf{v}$ is independent of a variable z in a reference frame R (see 1.1.1), the measure numbers of $\mathbf{v}$, when $\mathbf{v}$ is expressed in terms of unit vectors fixed in R, are independent of z. Conversely, if the measure numbers of $\mathbf{v}$, when $\mathbf{v}$ is expressed in terms of unit vectors fixed in a reference frame R, are constants, then $\mathbf{v}$ is independent of z in R. This follows from the fact that the characteristics of $\mathbf{v}$ in R determine these measure numbers, and vice versa.

1.1.5 When $\mathbf{v}$ is a vector function of a scalar variable z in a reference frame R, the vector obtained by assigning to z a particular value, say z^*, is called the *value of* $\mathbf{v}$ *at* z^*. For example, Figs. 1.1.3c and 1.1.3e each show the values of the vector function $\mathbf{F}$ at $z = 0$ and $z = -\pi/4$.

1.1.6 Two vector functions $\mathbf{v}_1$ and $\mathbf{v}_2$ of the same variable z are said to be *equal in the interval* $z_1 \leqslant z \leqslant z_2$ if and only if the values of $\mathbf{v}_1$ and $\mathbf{v}_2$ have identical characteristics for every z in this interval.

When two equal vector functions are each expressed in terms of unit vectors fixed in any reference frame whatsoever, the measure numbers of corresponding components are equal. Conversely, when the measure numbers of corresponding components of two vector functions are equal for all z in the interval $z_1 \leqslant z \leqslant z_2$, the two vector functions are equal in this interval.

1.1.7 The result of an operation involving two or more values of a vector function—or the values of several vector functions of the same variable, corresponding to distinct values of this variable—depends on the reference frame in which the operation is performed.

Example: For the force $\mathbf{F}$ defined in Problem 1.1.3, the scalar product of $\mathbf{F}|_{z=0}$ and $\mathbf{F}|_{z=-\pi/4}$ in R is given by (see Fig. 1.1.3c)

$$2(3\sqrt{2}) \cos\frac{\pi}{4} = 6 \text{ lb}^2$$

while in R' the scalar product of these two values of $\mathbf{F}$ is equal to zero, $\mathbf{F}|_{z=0}$ and $\mathbf{F}|_{z=-\pi/4}$ being perpendicular to each other in R' (see Fig. 1.1.3e).

1.1.8 The result of an operation involving only one value of a vector function—or the values of several vector functions of the same variable, corresponding to one value of this variable—is independent of the reference frame in which the operation is performed.

Example: Referring to Problem 1.1.3, and letting $\mathbf{E}$ be the vector function

$$\mathbf{E} = -\frac{4z}{\pi}\mathbf{n}_1 \text{ lb}$$

the value of $\mathbf{E}$ at $z = -\pi/4$ is seen to be

$$\mathbf{E}|_{z=-\pi/4} = \mathbf{n}_1|_{z=-\pi/4} \text{ lb}$$

Figures 1.1.3b and 1.1.3c thus show $\mathbf{E}|_{z=-\pi/4}$ and $\mathbf{F}|_{z=-\pi/4}$ in R, while Figs. 1.1.3c and 1.1.3d show these two vectors in R'. It appears that in both reference frames the scalar product of the values of $\mathbf{E}$ and $\mathbf{F}$ at $z = -\pi/4$ is equal to 3 lb².

1.1.9 In order to be unambiguous, any combination of mathematical symbols which indicates an operation involving two or more values of a vector function—or values of several vector functions of the same variable, corresponding to distinct values of this variable—must include specific mention of the reference frame in which the operation is performed. For example, the two scalar products evaluated in Example 1.1.7 can be denoted by

$$\mathbf{F}|_{z=0} \overset{R}{\cdot} \mathbf{F}|_{z=-\pi/4} = 6 \text{ lb}^2$$

and by

$$\mathbf{F}|_{z=0} \overset{R'}{\cdot} \mathbf{F}|_{z=-\pi/4} = 0$$

1.1.10 Although the result of an operation involving vector functions may depend on the reference frame in which the operation is performed (see 1.1.7 and 1.1.8), this result, if it is a vector, can be expressed in terms of unit vectors fixed in any reference frame whatsoever.

Problem: In Fig 1.1.10, R and R' represent two reference frames (coplanar rectangles). $\mathbf{n}_1$ and $\mathbf{n}_2$ represent unit vectors fixed in R'.

Two vector functions, $\mathbf{a}$ and $\mathbf{b}$, are defined as

$$\mathbf{a} = \frac{6}{\pi} z\mathbf{n}_1, \quad \mathbf{b} = \frac{8}{\pi} z\mathbf{n}_2'$$

where z is the angular displacement of R relative to R', measured in radians, z being positive for the configuration shown in Fig. 1.1.10.

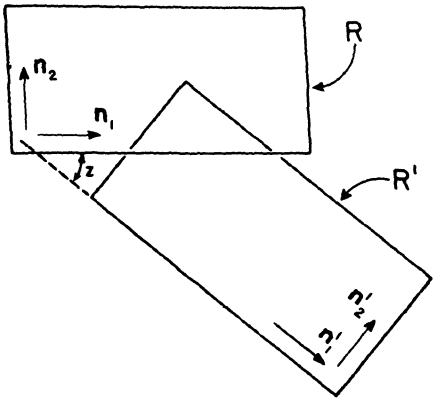


FIG. 1.1.10

Express each of the vectors

$$\mathbf{a}|_{z=\pi/2} \overset{R}{+} \mathbf{b}|_{z=3\pi/2} \quad \text{and} \quad \mathbf{a}|_{z=\pi/2} \overset{R'}{+} \mathbf{b}|_{z=3\pi/2}$$

in terms of (a) $\mathbf{n}_1$ and $\mathbf{n}_2$, (b) $\mathbf{n}_1'$ and $\mathbf{n}_2'$.

Solution (a):*

$$\mathbf{a}|_{z=\pi/2} = 3\mathbf{n}_1|_{z=\pi/2} \underset{(\text{F } 1.1.10)}{=} 3\mathbf{n}_2'$$

$$\mathbf{b}|_{z=3\pi/2} = 12\mathbf{n}_2'|_{z=3\pi/2} \underset{(\text{F } 1.1.10)}{=} -12\mathbf{n}_1$$

$$\mathbf{a}|_{z=\pi/2} \overset{R}{+} \mathbf{b}|_{z=3\pi/2} = 3\mathbf{n}_1 - 12\mathbf{n}_1 = -9\mathbf{n}_1$$

$$\mathbf{a}|_{z=\pi/2} \overset{R'}{+} \mathbf{b}|_{z=3\pi/2} = 3\mathbf{n}_2' + 12\mathbf{n}_2' = 15\mathbf{n}_2'$$

From Fig. 1.1.10,

$$\mathbf{n}_2' = \sin z\, \mathbf{n}_1 + \cos z\, \mathbf{n}_2$$

Hence

$$\mathbf{a}|_{z=\pi/2} \overset{R'}{+} \mathbf{b}|_{z=3\pi/2} = 15(\sin z\, \mathbf{n}_1 + \cos z\, \mathbf{n}_2)$$

Results (a): $-9\mathbf{n}_1$, $15(\sin z\, \mathbf{n}_1 + \cos z\, \mathbf{n}_2)$.

* Numbers beneath equal signs refer either to corresponding sections of the text or to equation numbers appearing in the section under consideration. When a section number is preceded by P, E, or F, this indicates a reference to a specific problem, example, or figure, respectively. For example, (P 1.2.1) is to be read "see the problem discussed in Section 1.2.1."

Solution (b): From Fig. 1.1.10,

$$\mathbf{n}_1 = \cos z\, \mathbf{n}_1' + \sin z\, \mathbf{n}_2'$$

From Solution (a),

$$\mathbf{a}|_{z=\pi/2} \overset{R}{+} \mathbf{b}|_{z=3\pi/2} = -9\mathbf{n}_1$$

Hence

$$\mathbf{a}|_{z=\pi/2} \overset{R'}{+} \mathbf{b}|_{z=3\pi/2} = -9(\cos z\, \mathbf{n}_1' + \sin z\, \mathbf{n}_2')$$

Results (b): $-9(\cos z\, \mathbf{n}_1' + \sin z\, \mathbf{n}_2')$, $15\, \mathbf{n}_2'$.

1.1.11 The vectors and/or scalars resulting from operations performed with vector and/or scalar functions of a variable z are, in general, vector or scalar functions of z (see, for example, Problem 1.1.10.)

1.2 The first derivative of a vector function

1.2.1 Given a vector function $\mathbf{v}$ of a scalar variable z in a reference frame R (see 1.1.1), the *first derivative of* $\mathbf{v}$ *with respect to z in R* is denoted by

$$\frac{^R d\mathbf{v}}{dz} \quad \text{or} \quad {}^R d\mathbf{v}/dz$$

and is defined as

$$\frac{^R d\mathbf{v}}{dz} = \sum_{i=1}^{3} \frac{dv_i}{dz}\, \mathbf{n}_i$$

where (see Fig. 1.2.1) $\mathbf{n}_i$, $i = 1, 2, 3$, are unit vectors (not parallel

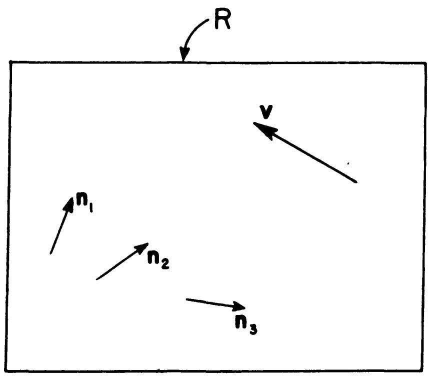


FIG. 1.2.1

to the same plane) fixed in R and v_i is the $\mathbf{n}_i$ measure number of $\mathbf{v}$.

Problem: Referring to Problem 1.1.10, evaluate the first derivative of $\mathbf{a}$ with respect to z in (a) R and (b) R'.

Solution (a):

$$\mathbf{a} = \frac{6}{\pi} z \, \mathbf{n}_1$$

$$\frac{{}^R d\mathbf{a}}{dz} = \frac{d}{dz}\left(\frac{6}{\pi} z\right) \mathbf{n}_1 = \frac{6}{\pi} \mathbf{n}_1$$

Solution (b):

$$\mathbf{n}_1 = \cos z \, \mathbf{n}_1' + \sin z \, \mathbf{n}_2'$$

Hence

$$\mathbf{a} = \frac{6}{\pi} z \cos z \, \mathbf{n}_1' + \frac{6}{\pi} z \sin z \, \mathbf{n}_2'$$

$$\frac{{}^{R'} d\mathbf{a}}{dz} = \frac{d}{dz}\left(\frac{6}{\pi} z \cos z\right) \mathbf{n}_1' + \frac{d}{dz}\left(\frac{6}{\pi} z \sin z\right) \mathbf{n}_2'$$

$$\frac{{}^{R'} d\mathbf{a}}{dz} = \frac{6}{\pi} (\cos z - z \sin z)\mathbf{n}_1' + \frac{6}{\pi} (\sin z + z \cos z)\mathbf{n}_2'$$

1.2.2 If a vector $\mathbf{v}$ is independent of a variable z in a reference frame R (see 1.1.1), the first derivative of $\mathbf{v}$ with respect to z in R (see 1.2.1) is equal to zero. In particular, the first derivative of a zero vector (see Vol. 1, Secs. 1.5 and 1.10.3) with respect to any scalar variable in any reference frame is equal to zero. Conversely, if ${}^R d\mathbf{v}/dz = 0$, $\mathbf{v}$ is independent of z in R. This follows from Secs. 1.1.4 and 1.2.1.

1.2.3 As the derivative of $\mathbf{v}$ with respect to z in R (see 1.2.1) is a vector, the dimensions of the derivative are those of the magnitude of this vector (see Vol. 1, Sec. 1.1.5) and are, therefore, found by determining the quotient of the dimensions of $\mathbf{v}$ and of z.

Problem: The position vector $\mathbf{p}$ (see Vol. 1, Sec. 2.1) of a point P relative to a point O which is fixed in the reference frame R shown in Fig. 1.1.10 is given by

$$\mathbf{p} = 5t^2 \, \mathbf{n}_1 \text{ ft}$$

where t is the time in seconds. Determine the first derivative of $\mathbf{p}$ with respect to t in R at $t = 2$ sec.

Solution:

$$\frac{{}^R d\mathbf{p}}{dt} = 10t\,\mathbf{n}_1$$

$$\left.\frac{{}^R d\mathbf{p}}{dt}\right|_{t=2} = 20\mathbf{n}_1 \text{ ft sec}^{-1}$$

1.2.4 The derivative of a vector function in a reference frame R can be expressed in terms of unit vectors fixed in any other reference frame R' (see 1.1.10).

Problem: Referring to Problem 1.1.10, express the first derivative of $\mathbf{a}$ with respect to z in R in terms of $\mathbf{n}_1'$ and $\mathbf{n}_2'$.

Solution:

$$\frac{{}^R d\mathbf{a}}{dz} \underset{(\text{P } 1.2.1)}{=} \frac{6}{\pi}\mathbf{n}_1$$

$$\mathbf{n}_1 \underset{(\text{F } 1.1.10)}{=} \cos z\,\mathbf{n}_1' + \sin z\,\mathbf{n}_2'$$

Hence

$$\frac{{}^R d\mathbf{a}}{dz} = \frac{6}{\pi}(\cos z\,\mathbf{n}_1' + \sin z\,\mathbf{n}_2')$$

(Compare this result with that given in Solution (b), Problem 1.2.1.)

1.2.5 If two vector functions of a variable z are equal in the interval $z_1 \leqslant z \leqslant z_2$ (see 1.1.6), their derivatives with respect to z (in any reference frame) are equal to each other in this interval.

1.2.6 From the fact that the derivative of a vector function $\mathbf{v}$ with respect to a variable z depends on the reference frame in which the differentiation is performed it follows that a notation such as $d\mathbf{v}/dz$, that is, one which contains no explicit mention of a reference frame, is meaningful only when the context in which it appears clearly requires the choice of a particular reference frame (see 1.1.9). By the same token, if v is a scalar function of z, the letter R should be omitted from any symbol denoting differentiation of v, because the meaning of the symbol is in no way enhanced by the presence of this letter. For example, as $\mathbf{v}\cdot\mathbf{v}$ is a scalar function of z, ${}^R d(\mathbf{v}\cdot\mathbf{v})/dz$ has the same meaning as $d(\mathbf{v}\cdot\mathbf{v})/dz$.

1.2.7 By analogy with definition of the derivative df/dz of a scalar function f of a variable z,

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \tag{A}$$

the derivative of a vector function $\mathbf{v}$ of a variable z in a reference frame R may be defined as

$$\frac{{}^R d\mathbf{v}}{dz} = \lim_{\Delta z \to 0} \frac{{}^R \Delta \mathbf{v}}{\Delta z} \tag{B}$$

where ${}^R\Delta\mathbf{v}$ denotes the difference in R of $\mathbf{v}$ at $z + \Delta z$ and $\mathbf{v}$ at z. Indeed, the validity of (B) follows from the definition given in Sec. 1.2.1, and conversely. In the scalar calculus, knowledge of certain theorems of the theory of limits of scalar functions is prerequisite to effective use of (A) as a point of departure for the study of the properties of derivatives. Similarly, (B) can be regarded as a satisfactory basis for the study of properties of the derivatives of vector functions only if knowledge of the theory of limits of vector functions can be presupposed. This is the reason for not using (B) in place of the definition given in Sec. 1.2.1.

1.2.8 The derivatives of vector functions have the following properties in common with the derivatives of scalar functions: (a) Two functions of a variable z may have the same value for a given value z^* of z while their derivatives at z^* have different values (compare with 1.2.5). (b) Two functions of a variable z may have different values for a given value z^* of z while their derivatives at z^* have the same value. (c) A function of a variable z may be equal to zero for a given value z^* of z while the derivative of the function at z^* is not equal to zero (compare with 1.2.2). (d) The derivative of a function of a variable z may be equal to zero for a given value z^* of z while the function is not equal to zero at z^*.

1.3 The second and higher derivatives of vector functions

1.3.1 The first derivative of a vector function with respect to a variable z in a reference frame R (see 1.2.1) is, in general, a vector function of z, both in R and in any other reference frame R' (see 1.1.11), and can therefore be differentiated with respect to z in R

or in R'. The result of such a differentiation, called a *second derivative,* is denoted by

$$\frac{^{R'}d}{dz}\left(\frac{^{R}d\mathbf{v}}{dz}\right)$$

or, if both differentiations are performed in the same reference frame, by

$$\frac{^{R}d^2\mathbf{v}}{dz^2}$$

Repetition of this process leads to *higher derivatives,* such as, for instance,

$$\frac{^{R}d^2}{dz^2}\left[\frac{^{R'}d}{dz}\left(\frac{^{R}d^3\mathbf{v}}{dz^3}\right)\right]$$

1.4 Derivatives of sums

1.4.1 The first derivative with respect to a variable z in a reference frame R of the resultant (see Vol. 1, Sec. 1.9.1) of the vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$, is equal to the resultant of the first derivatives with respect to z in R of these vectors.

Proof: Let $\mathbf{n}_j$, $j = 1, 2, 3$, be unit vectors fixed in R (and not parallel to the same plane), v_{ij} the $\mathbf{n}_j$ measure number of $\mathbf{v}_i$. Then

$$\mathbf{v}_i = \sum_{j=1}^{3} v_{ij}\mathbf{n}_j$$

and the resultant of the vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$, is given by

$$\sum_{i=1}^{n} \mathbf{v}_i = \sum_{i=1}^{n}\left(\sum_{j=1}^{3} v_{ij}\mathbf{n}_j\right) = \sum_{j=1}^{3}\left(\sum_{i=1}^{n} v_{ij}\right)\mathbf{n}_j$$

The first derivative of the resultant thus becomes (see 1.2.1)

$$\frac{^{R}d}{dz}\left(\sum_{i=1}^{n} \mathbf{v}_i\right) = \sum_{j=1}^{3}\left(\frac{d}{dz}\sum_{i=1}^{n} v_{ij}\right)\mathbf{n}_j$$

But, from the calculus of scalar functions,

$$\frac{d}{dz}\sum_{i=1}^{n} v_{ij} = \sum_{i=1}^{n}\frac{dv_{ij}}{dz}$$

Hence

$$\frac{^R d}{dz}\left(\sum_{i=1}^{n} \mathbf{v}_i\right) = \sum_{j=1}^{3}\left(\sum_{i=1}^{n} \frac{d}{dz} v_{ij}\right)\mathbf{n}_j$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{3} \frac{d}{dz} v_{ij}\mathbf{n}_j\right)$$

$$\underset{(1.2.1)}{=} \sum_{i=1}^{n} \frac{^R d\mathbf{v}_i}{dz}$$

and this is the resultant of the first derivatives with respect to z in R of the vectors $\mathbf{v}_i$, $i = 1, 2, \ldots, n$.

Problem: For a certain value z^* of a variable z the first derivatives with respect to z in a reference frame R of two vector functions $\mathbf{a}$ and $\mathbf{b}$ have the values $\mathbf{n}_1 + 2\mathbf{n}_3$ and $3\mathbf{n}_1 - \mathbf{n}_2 + \mathbf{n}_3$. Determine the first derivative of $\mathbf{a} + \mathbf{b}$ with respect to z in R, at $z = z^*$.

Solution:

$$\frac{^R d}{dz}(\mathbf{a} + \mathbf{b}) = \frac{^R d\mathbf{a}}{dz} + \frac{^R d\mathbf{b}}{dz}$$

$$\left[\frac{^R d}{dz}(\mathbf{a} + \mathbf{b})\right]_{z=z^*} = \left.\frac{^R d\mathbf{a}}{dz}\right|_{z=z^*} + \left.\frac{^R d\mathbf{b}}{dz}\right|_{z=z^*}$$

$$= \mathbf{n}_1 + 2\mathbf{n}_3 + 3\mathbf{n}_1 - \mathbf{n}_2 + \mathbf{n}_3$$

$$= 4\mathbf{n}_1 + 3\mathbf{n}_3 - \mathbf{n}_2$$

1.4.2 The statement made in Sec. 1.4.1 applies also to second and higher derivatives of vector functions.

1.5 Derivatives of products

1.5.1 Given a variable z, a vector function $\mathbf{v}$ of z in a reference frame R, and a scalar function s of z,

$$\frac{^R d}{dz}(s\mathbf{v}) = \frac{ds}{dz}\mathbf{v} + s\,\frac{^R d\mathbf{v}}{dz}$$

Proof: Let $\mathbf{n}_i$, $i = 1, 2, 3$, be unit vectors fixed in R (and not parallel to the same plane), v_i the $\mathbf{n}_i$ measure number of $\mathbf{v}$. Then

$$s\mathbf{v} = s\sum_{i=1}^{3} v_i\mathbf{n}_i = \sum_{i=1}^{3} (s v_i \mathbf{n}_i)$$

and

$$
\begin{aligned}
\frac{{}^R d}{dz}(s\mathbf{v}) &\underset{(1.2.1)}{=} \sum_{i=1}^{3} \frac{d}{dz}(sv_i)\mathbf{n}_i \\
&= \sum_{i=1}^{3}\left(\frac{ds}{dz}v_i + s\frac{dv_i}{dz}\right)\mathbf{n}_i \\
&= \frac{ds}{dz}\sum_{i=1}^{3} v_i\mathbf{n}_i + s\sum_{i=1}^{3}\frac{dv_i}{dz}\mathbf{n}_i \\
&= \frac{ds}{dz}\mathbf{v} + s\,\frac{{}^R d\mathbf{v}}{dz}
\end{aligned}
$$

Problem: A vector $\mathbf{p}$, referred to mutually perpendicular unit vectors $\mathbf{n}_i$, $i = 1, 2, 3$, fixed in a reference frame R, is given by

$$\mathbf{p} = -e^{-3t}\mathbf{n}_2 + \mathbf{n}_3 \text{ ft}$$

where t is the time in seconds. Evaluate the magnitude of the first time-derivative of $\mathbf{p}$ in a reference frame R' for $t = 0$ if, at this time, the first time-derivatives of the unit vectors in R' have the values $4\mathbf{n}_2 \text{ sec}^{-1}, -4\mathbf{n}_1 \text{ sec}^{-1}$, and zero.

Solution:

$$
\begin{aligned}
\frac{{}^{R'} d\mathbf{p}}{dt} &\underset{(1.2.5)}{=} \frac{{}^{R'} d}{dt}(-e^{-3t}\mathbf{n}_2 + \mathbf{n}_3) \underset{(1.4.1)}{=} \frac{{}^{R'} d}{dt}(-e^{-3t}\mathbf{n}_2) + \frac{{}^{R'} d\mathbf{n}_3}{dt} \\
&= \left[\frac{{}^{R'} d}{dt}(-e^{-3t})\right]\mathbf{n}_2 + (-e^{-3t})\,\frac{{}^{R'} d\mathbf{n}_2}{dt} + \frac{{}^{R'} d\mathbf{n}_3}{dt} \\
&= 3e^{-3t}\mathbf{n}_2 - e^{-3t}\,\frac{{}^{R'} d\mathbf{n}_2}{dt} + \frac{{}^{R'} d\mathbf{n}_3}{dt}
\end{aligned}
$$

$$\left.\frac{{}^{R'} d\mathbf{p}}{dt}\right|_{t=0} = 3\mathbf{n}_2 + 4\mathbf{n}_1 \text{ ft sec}^{-1}$$

Result: 5 ft sec^{-1}

1.5.2 Given two vector functions $\mathbf{v}_1$ and $\mathbf{v}_2$ of a variable z in a reference frame R,

$$\frac{d}{dz}(\mathbf{v}_1 \cdot \mathbf{v}_2) = \frac{{}^{R'} d\mathbf{v}_1}{dz} \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \frac{{}^{R'} d\mathbf{v}_2}{dz}$$

where R' is any reference frame whatsoever.

Proof: Resolve $\mathbf{v}_1$ and $\mathbf{v}_2$ into components parallel to mutually perpendicular unit vectors fixed in some reference frame R', then proceed as in Sec. 1.5.1.

Problem: The derivative of a vector $\mathbf{v}$ with respect to a variable z in a reference frame R is perpendicular to $\mathbf{v}$. Show that the derivative with respect to z of the magnitude of $\mathbf{v}$ is equal to zero.

Solution:

$$|\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$$

$$\begin{aligned}\frac{d|\mathbf{v}|}{dz} &= \frac{d}{dz}(\mathbf{v} \cdot \mathbf{v})^{1/2} = \frac{1}{2}(\mathbf{v} \cdot \mathbf{v})^{-1/2} \frac{d}{dz}(\mathbf{v} \cdot \mathbf{v}) \\ &= \frac{1}{2}|\mathbf{v}|^{-1}\left(\frac{{}^R d\mathbf{v}}{dz} \cdot \mathbf{v} + \mathbf{v} \cdot \frac{{}^R d\mathbf{v}}{dz}\right) \\ &= \frac{1}{2}|\mathbf{v}|^{-1}\left(2\mathbf{v} \cdot \frac{{}^R d\mathbf{v}}{dz}\right)\end{aligned}$$

But

$$\mathbf{v} \cdot \frac{{}^R d\mathbf{v}}{dz} = 0$$

if $\mathbf{v}$ is perpendicular to ${}^R d\mathbf{v}/dz$.

1.5.3 Given two vector functions $\mathbf{v}_1$ and $\mathbf{v}_2$ of a variable z in a reference frame R,

$$\frac{{}^R d}{dz}(\mathbf{v}_1 \times \mathbf{v}_2) = \frac{{}^R d\mathbf{v}_1}{dz} \times \mathbf{v}_2 + \mathbf{v}_1 \times \frac{{}^R d\mathbf{v}_2}{dz}$$

Proof: The proof can be carried out by resolving $\mathbf{v}_1$ and $\mathbf{v}_2$ into components parallel to mutually perpendicular unit vectors fixed in R, etc. However, a shorter proof, based on the alternative definition of the derivative discussed in Sec. 1.2.7, is given below, to illustrate the use of this definition.

$$\begin{aligned}\frac{{}^R d}{dz}(\mathbf{v}_1 \times \mathbf{v}_2) &\underset{(1.2.7)}{=} \lim_{\Delta z \to 0} \frac{{}^R \Delta(\mathbf{v}_1 \times \mathbf{v}_2)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{(\mathbf{v}_1 + {}^R\Delta\mathbf{v}_1) \times (\mathbf{v}_2 + {}^R\Delta\mathbf{v}_2) - \mathbf{v}_1 \times \mathbf{v}_2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{{}^R\Delta\mathbf{v}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times {}^R\Delta\mathbf{v}_2 + {}^R\Delta\mathbf{v}_1 \times {}^R\Delta\mathbf{v}_2}{\Delta z}\end{aligned}$$

$$= \left(\lim_{\Delta z \to 0} \frac{{}^R\Delta \mathbf{v}_1}{\Delta z}\right) \times \mathbf{v}_2 + \mathbf{v}_1 \times \left(\lim_{\Delta z \to 0} \frac{{}^R\Delta \mathbf{v}_2}{\Delta z}\right)$$

$$+ \left(\lim_{\Delta z \to 0} \frac{{}^R\Delta \mathbf{v}_1}{\Delta z}\right) \times \left(\lim_{\Delta z \to 0} {}^R\Delta \mathbf{v}_2\right)$$

$$= \frac{{}^Rd\mathbf{v}_1}{dz} \times \mathbf{v}_2 + \mathbf{v}_1 \times \frac{{}^Rd\mathbf{v}_2}{dz}$$

1.5.4 The product P_n of n scalar and/or vector functions F_i, $i = 1, 2, \ldots, n$, of a scalar variable z can always be regarded as the product of only two functions, say P_{n-1} and F_n, where P_{n-1} is the product of the $n - 1$ functions F_i, $i = 1, 2, \ldots, n - 1$. That is,

$$P_n = P_{n-1}F_n$$

From the laws governing the differentiation of scalar functions, together with the theorems in Secs. 1.5.1–1.5.3, it follows that

$$\frac{{}^RdP_n}{dz} = \frac{{}^Rd}{dz}(P_{n-1}F_n) = \frac{{}^RdP_{n-1}}{dz}F_n + P_{n-1}\frac{{}^RdF_n}{dz}$$

After replacing P_{n-1} with $P_{n-2}F_{n-1}$, etc., one obtains

$$\frac{{}^RdP_n}{dz} = \left(\frac{{}^RdF_1}{dz}F_2F_3 \ldots F_n\right) + \left(F_1\frac{{}^RdF_2}{dz}F_3 \ldots F_n\right)$$

$$+ \ldots + \left(F_1F_2F_3 \ldots \frac{{}^RdF_n}{dz}\right)$$

where the letter R may be omitted whenever the quantity being differentiated is a scalar. The order in which the terms in any one product occur is the same as that in which these terms appear in P_n; and all symbols of operation, such as dots, crosses, parentheses, brackets, etc., must be kept in place.

Problem: The line of action of a 10 lb force $\mathbf{F}$ passes through two corners of a rectangular parallelepiped, as shown in Fig. 1.5.4. A line L, lying in the top face of the parallelepiped, passes through one corner of this face and makes an angle θ with one edge.

Letting R be a reference frame fixed in the parallelepiped, determine the magnitude of the derivative in R with respect to θ of the moment $\mathbf{M}^{\mathbf{F}/L}$ of $\mathbf{F}$ about L, for $\theta = 0$.

Solution: Let $\mathbf{n}_i$, $i = 1, 2, 3$, be unit vectors parallel to the edges of the parallelepiped; $\mathbf{p}$ the position vector of a point on the

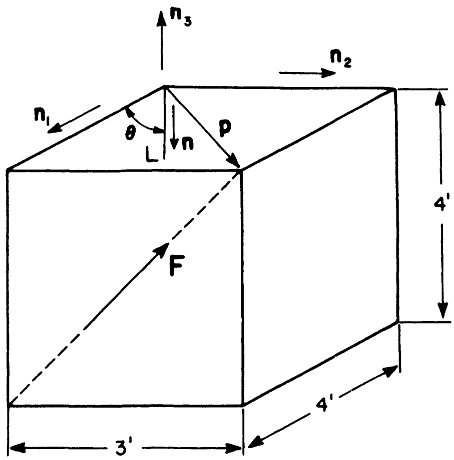


FIG. 1.5.4

line of action of $\mathbf{F}$ relative to a point on L; $\mathbf{n}$ a unit vector parallel to L, as shown in Fig. 1.5.4. Then

$$\mathbf{n} = \cos\theta\,\mathbf{n}_1 + \sin\theta\,\mathbf{n}_2$$

$$\mathbf{p} = 4\mathbf{n}_1 + 3\mathbf{n}_2 \text{ ft}$$

$$\mathbf{F} = 6\mathbf{n}_2 + 8\mathbf{n}_3 \text{ lb}$$

In accordance with the definition of the moment of a vector about a line (see Vol. 1, Sec. 3.2),

$$\mathbf{M}^{\mathbf{F}/L} = [\mathbf{n}, \mathbf{p}, \mathbf{F}]\mathbf{n}$$

Differentiate:

$$\frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{d\theta} = \left[\frac{{}^R d\mathbf{n}}{d\theta}, \mathbf{p}, \mathbf{F}\right]\mathbf{n} + \left[\mathbf{n}, \frac{{}^R d\mathbf{p}}{d\theta}, \mathbf{n}\right]\mathbf{n} + \left[\mathbf{n}, \mathbf{p}, \frac{{}^R d\mathbf{F}}{d\theta}\right]\mathbf{n} + [\mathbf{n}, \mathbf{p}, \mathbf{F}]\frac{{}^R d\mathbf{n}}{d\theta}$$

Now

$$\frac{{}^R d\mathbf{n}}{d\theta} \underset{(1.2.1)}{=} -\sin\theta\,\mathbf{n}_1 + \cos\theta\,\mathbf{n}_2$$

$$\frac{{}^R d\mathbf{p}}{d\theta} = \frac{{}^R d\mathbf{F}}{d\theta} \underset{(1.2.2)}{=} 0$$

Thus

$$\left.\frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{d\theta}\right|_{\theta=0} = [\mathbf{n}_2, 4\mathbf{n}_1 + 3\mathbf{n}_2, 6\mathbf{n}_2 + 8\mathbf{n}_3]\mathbf{n}_1 + [\mathbf{n}_1, 4\mathbf{n}_1 + 3\mathbf{n}_2, 6\mathbf{n}_2 + 8\mathbf{n}_3]\mathbf{n}_2$$

$$= -32\mathbf{n}_1 + 24\mathbf{n}_2 \text{ ft lb rad}^{-1}$$

and

$$\left|\left.\frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{d\theta}\right|_{\theta=0}\right| = 40 \text{ ft lb rad}^{-1}$$

1.6 Derivatives of implicit functions

1.6.1 If $\mathbf{v}$ is a vector function of a scalar variable y in a reference frame R, and y is a function of the scalar variable z, then

$$\frac{{}^R d\mathbf{v}}{dz} = \frac{{}^R d\mathbf{v}}{dy}\frac{dy}{dz}$$

This follows from the definition of the derivative of a vector function (see 1.2.1), together with the laws governing the differentiation of scalar implicit functions.

Problem: Referring to Problem 1.5.4, and supposing that line L revolves uniformly, performing one revolution per second, with θ increasing, determine the first time-derivative in R of the moment of $\mathbf{F}$ about line L, for $\theta = 0$.

Solution: Let t be the time in seconds. Then

$$\theta = 2\pi t \text{ rad}$$

and

$$\frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{dt} = \frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{d\theta}\frac{d\theta}{dt}$$

so that

$$\left.\frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{dt}\right|_{\theta=0} = \left.\frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{d\theta}\right|_{\theta=0} \left.\frac{d\theta}{dt}\right|_{\theta=0}$$

Now

$$\left.\frac{{}^R d\mathbf{M}^{\mathbf{F}/L}}{d\theta}\right|_{\theta=0} \underset{(\text{P } 1.5.4)}{=} -32\mathbf{n}_1 + 24\mathbf{n}_2 \text{ ft lb rad}^{-1}$$

and

$$\left.\frac{d\theta}{dt}\right|_{\theta=0} = 2\pi \text{ rad sec}^{-1}$$

Hence

$$\left.\frac{{}^{R}d\mathbf{M}^{F/L}}{dt}\right|_{\theta=0} = 2\pi(-32\mathbf{n}_1 + 24\mathbf{n}_2) \text{ ft lb sec}^{-1}$$

1.7 The first derivative of a unit vector which remains perpendicular to a line fixed in a reference frame

1.7.1 If a unit vector $\mathbf{n}$ is a function of a scalar variable z in a reference frame R', and $\mathbf{n}$ remains perpendicular to a line K fixed in R' as z varies, then

$$\frac{{}^{R'}d\mathbf{n}}{dz} = \mathbf{k} \times \mathbf{n}\frac{d\theta}{dz}$$

where $\mathbf{k}$ and θ are defined as follows (see Fig. 1.7.1a): $\mathbf{k}$ is a unit vector parallel to line K. θ is the angular displacement of a line L

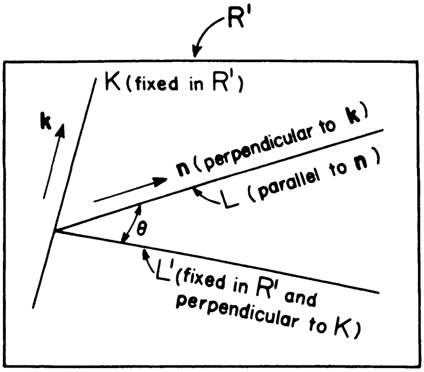


FIG. 1.7.1a

which is parallel to $\mathbf{n}$, relative to a line L' which is fixed in R' and perpendicular to line K, θ being regarded as positive when this displacement is generated by a $\mathbf{k}$ rotation of L relative to L', a $\mathbf{k}$ rotation being one during which a right-handed screw whose axis is parallel to $\mathbf{k}$ advances in the $\mathbf{k}$ direction.

Proof: Let $\mathbf{n}_1$ be a unit vector parallel to line L', $\mathbf{n}_2$ a unit vector perpendicular to both $\mathbf{k}$ and $\mathbf{n}_1$, choosing $\mathbf{n}_1$ and $\mathbf{n}_2$ such that

$$\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{k}$$

Then $\mathbf{n}$, referred to the mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{k}$ (all of which are fixed in R'), is given by (see Fig. 1.7.1b)

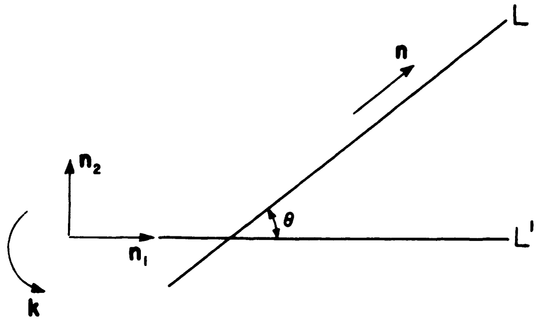


FIG. 1.7.1b

$$\mathbf{n} = \cos\theta\,\mathbf{n}_1 + \sin\theta\,\mathbf{n}_2 \tag{A}$$

and

$$\frac{{}^{R'}d\mathbf{n}}{dz} \underset{(1.6.1)}{=} \frac{{}^{R'}d\mathbf{n}}{d\theta}\frac{d\theta}{dz} \underset{(1.2.1)}{=} (-\sin\theta\,\mathbf{n}_1 + \cos\theta\,\mathbf{n}_2)\frac{d\theta}{dz} \tag{B}$$

But

$$\begin{aligned}\mathbf{k} \times \mathbf{n} &\underset{(A)}{=} \mathbf{k} \times (\cos\theta\,\mathbf{n}_1 + \sin\theta\,\mathbf{n}_2)\\ &= \cos\theta\,\mathbf{k} \times \mathbf{n}_1 + \sin\theta\,\mathbf{k} \times \mathbf{n}_2\\ &\underset{(F1.7.1b)}{=} \cos\theta\,\mathbf{n}_2 + \sin\theta\,(-\mathbf{n}_1)\end{aligned}$$

That is

$$\mathbf{k} \times \mathbf{n} = -\sin\theta\,\mathbf{n}_1 + \cos\theta\,\mathbf{n}_2 \tag{C}$$

Hence

$$\frac{{}^{R'}d\mathbf{n}}{dz} \underset{(B,C)}{=} \mathbf{k} \times \mathbf{n}\,\frac{d\theta}{dz}$$

(Note that $\mathbf{k} \times \mathbf{n}$ is a unit vector perpendicular to both $\mathbf{k}$ and $\mathbf{n}$. It follows that ${}^{R'}d\mathbf{n}/dz$ is perpendicular to both $\mathbf{k}$ and $\mathbf{n}_1$ but is not, in general, a unit vector.)

Problem: In Fig. 1.7.1c, A and B represent two rectangular

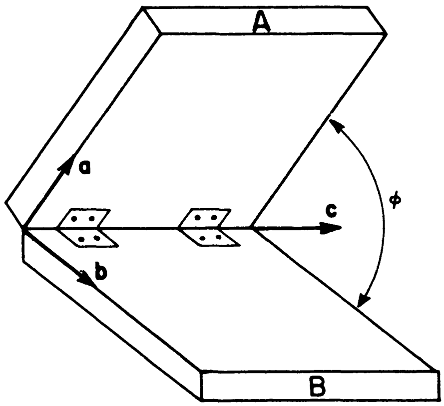


FIG. 1.7.1c

plates connected by hinges. ϕ, the angular displacement of A relative to B, regarded as positive for the configuration shown, is given by

$$\phi = \frac{\pi}{2}\left[1 + \sin\left(\frac{\pi}{2}t\right)\right] \text{ rad}$$

where t is the time in seconds. $\mathbf{a}$ is a unit vector fixed in A, $\mathbf{b}$ a unit vector fixed in B, $\mathbf{c}$ a unit vector fixed in both A and B.

For $t = 2$ sec, determine the time-derivative (a) of $\mathbf{a}$ in B, and (b) of $\mathbf{b}$ in A.

Solution (a): As ϕ is positive when the angular displacement of A relative to B is generated by a $-\mathbf{c}$ rotation of A relative to B,

$$\frac{{}^B d\mathbf{a}}{dt} = -\mathbf{c} \times \mathbf{a}\frac{d\phi}{dt} = -\mathbf{c} \times \mathbf{a}\frac{\pi^2}{4}\cos\frac{\pi}{2}t$$

and

$$\left.\frac{{}^B d\mathbf{a}}{dt}\right|_{t=2} = (-\mathbf{c} \times \mathbf{a})_{\phi=\pi/2}\left(-\frac{\pi^2}{4}\right) = \frac{\pi^2}{4}\mathbf{b} \text{ sec}^{-1}$$

Solution (b): If ϕ is regarded as the angular displacement of B relative to A, then ϕ is positive when this displacement is generated by a $\mathbf{c}$ rotation of B relative to A. Hence

$$\frac{{}^{A}d\mathbf{b}}{dt} = \mathbf{c} \times \mathbf{b}\frac{d\phi}{dt} = \mathbf{c} \times \mathbf{b}\frac{\pi^2}{4}\cos\frac{\pi}{2}t$$

and

$$\left.\frac{{}^{A}d\mathbf{b}}{dt}\right|_{t=2} = (\mathbf{c} \times \mathbf{b})_{\phi=\pi/2}\left(-\frac{\pi^2}{4}\right) = \frac{\pi^2}{4}\mathbf{a}\ \text{sec}^{-1}$$

1.7.2 When a function $f(x)$ has a positive first derivative at $x = x^*$, then $f(x^* + \Delta x)$ is algebraically larger than $f(x^*)$ for every positive and sufficiently small Δx; or, in other words, $f(x)$ is increasing algebraically at x^*; and df/dx measures the rate at which $f(x)$ is increasing. Applying this to the angular displacement $\theta(z)$ defined in Sec. 1.7.1, one concludes that $d\theta/dz$ measures the rate at which L is performing a $\mathbf{k}$ rotation relative to L'.

Problem: In Fig. 1.7.2a, OA represents a line which is perpen-

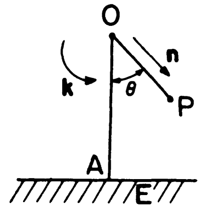


FIG. 1.7.2a

dicular to the surface of the earth E. OP is a rigid rod, attached to a support at O in such a way that it can rotate in a vertical plane fixed in E. $\mathbf{n}$ is a unit vector parallel to OP and having the sense OP.

Supposing that the rod rotates clockwise (as seen by the reader), completing 10 revolutions per second, and that the rotation is uniform, that is, that the angle through which OP turns during any time interval is proportional to this time interval, draw sketches showing the first time-derivative of $\mathbf{n}$ in E for the instants at which the rod passes through its two vertical and two horizontal positions.

Solution: Let $\mathbf{k}$ be a unit vector perpendicular to the plane determined by the points A, O, P, and θ the angular displacement of OP relative to OA, regarded as positive for the configuration shown in Fig. 1.7.2a. Then

$$\frac{{}^E d\mathbf{n}}{dt} \underset{(1.7.1)}{=} \mathbf{k} \times \mathbf{n} \frac{d\theta}{dt}$$

and $d\theta/dt$ is the time-rate at which OP is rotating counterclockwise. Accordingly, $-d\theta/dt$ is the rate at which OP is rotating clockwise and, as θ changes by an amount of $10 \times 2\pi$ radians during each second of the motion,

$$-\frac{d\theta}{dt} = 20\pi \text{ rad sec}^{-1}$$

Hence

$$\frac{{}^E d\mathbf{n}}{dt} = -20\pi\, \mathbf{k} \times \mathbf{n}$$

As $\mathbf{k}$ is perpendicular to $\mathbf{n}$, $\mathbf{k} \times \mathbf{n}$ is a unit vector. Its direction can be determined by inspection. See Fig. 1.7.2b.

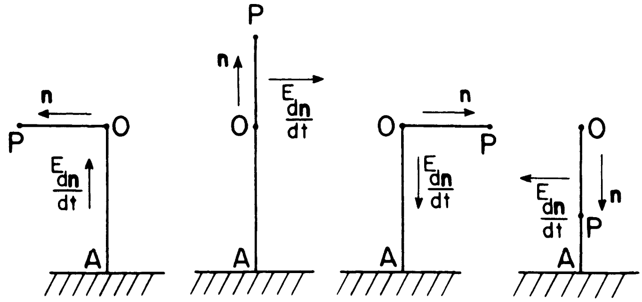


FIG. 1.7.2b

1.7.3 The use of the expression (see 1.7.1)

$$\frac{{}^{R'} d\mathbf{n}}{dz} = \mathbf{k} \times \mathbf{n} \frac{d\theta}{dz}$$

for the evaluation of ${}^{R'}d\mathbf{n}/dz$ is particularly convenient when information regarding the manner in which θ depends on z is confined to knowledge of $d\theta/dz$ for a specific value z^* of z. To determine ${}^{R'}d\mathbf{n}/dz$ at z^* it is then only necessary to know $\mathbf{n}$ at z^*.

Problem: In Fig. 1.7.3a, D represents a circular disc which rolls on a circular track C in such a way that at a certain instant t^*

$$\left.\frac{d\theta}{dt}\right|_{t^*} = -5 \text{ rad sec}^{-1}$$

where θ is the angular displacement of a line d fixed in D, relative to a line c fixed in C, θ being regarded as positive when generated

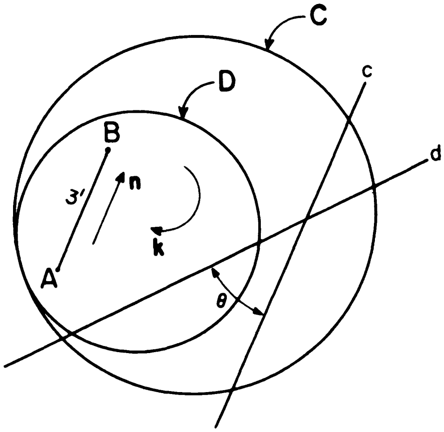


FIG. 1.7.3a

by a clockwise (as seen by the reader) rotation of d (or D) relative to c (or C). A and B are points fixed in D.

Draw a sketch showing the time-derivative in C at t^* of the position vector $\mathbf{p}$ of B relative to A.

Solution: Let $\mathbf{n}$ be a unit vector parallel to line AB. Then $\mathbf{p}$ is given by

$$\mathbf{p} = 3\mathbf{n} \text{ ft}$$

and

$$\left.\frac{{}^C d\mathbf{p}}{dt}\right|_{t^*} \underset{(1.2.5)}{=} \left.\frac{{}^C d}{dt}(3\mathbf{n})\right|_{t^*} = 3 \left.\frac{{}^C d\mathbf{n}}{dt}\right|_{t^*}$$

Let $\mathbf{k}$ be a unit vector perpendicular to the plane of the disc D, choosing the sense of $\mathbf{k}$ such that D rotates clockwise during a $\mathbf{k}$ rotation of D relative to C (see Fig. 1.7.3a). Then

$$\left.\frac{{}^C d\mathbf{n}}{dt}\right|_{t^*} \underset{(1.7.1)}{=} (\mathbf{k} \times \mathbf{n})_{t^*} \left.\frac{d\theta}{dt}\right|_{t^*} = -5(\mathbf{k} \times \mathbf{n})_{t^*} \text{ sec}^{-1}$$

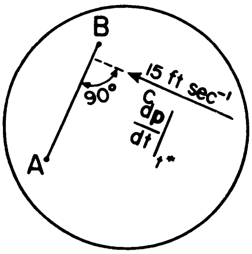
so that

$$\left.\frac{{}^C d\mathbf{p}}{dt}\right|_{t^*} = -15(\mathbf{k} \times \mathbf{n})_{t^*} \text{ ft sec}^{-1}$$

(Note that line d is not parallel to $\mathbf{n}$, but that 1.7.1 is nevertheless used to evaluate ${}^{C}d\mathbf{n}/dt$. The justification for this is that any angle between line c and a line parallel to $\mathbf{n}$ can be expressed as the sum of θ and an angle which is independent of t, so that the time-derivative of such an angle is equal to $d\theta/dt$.)

Result: See Fig. 1.7.3b.

FIG. 1.7.3b



1.8 Taylor's theorem for vector functions

1.8.1 If $\mathbf{v}$ is a vector function of a scalar variable z in a reference frame R (see 1.1.1), the difference in R between the values (see 1.1.5) of $\mathbf{v}$ at $z^* + h$ and at z^*, where z^* is a particular value of z and h a scalar having the same dimensions as z, can be expressed in terms of values of derivatives of $\mathbf{v}$ with respect to z in R at z^*, as follows:

$$\mathbf{v}\big|_{z^*+h} \overset{R}{-} \mathbf{v}\big|_{z^*} = \frac{h}{1!}\left.\frac{{}^{R}d\mathbf{v}}{dz}\right|_{z^*} + \frac{h^2}{2!}\left.\frac{{}^{R}d^2\mathbf{v}}{dz^2}\right|_{z^*} + \cdots$$

Proof: Let $\mathbf{n}_i$, $i = 1, 2, 3$, be unit vectors (not parallel to the same plane) fixed in R. Then v_i, the $\mathbf{n}_i$ measure number of $\mathbf{v}$, is a scalar function of z (see 1.1.3) and, by Taylor's theorem for scalar functions,

$$v_i\big|_{z^*+h} - v_i\big|_{z^*} = \frac{h}{1!}\left.\frac{dv_i}{dz}\right|_{z^*} + \frac{h^2}{2!}\left.\frac{d^2v_i}{dz^2}\right|_{z^*} + \cdots$$

The values of $\mathbf{v}$ at z^* and at $z^* + h$, expressed in terms of their $\mathbf{n}_i$, $i = 1, 2, 3$, components, are

$$\mathbf{v}\big|_{z^*} = \sum_{i=1}^{3} v_i\big|_{z^*}\mathbf{n}_i, \qquad \mathbf{v}\big|_{z^*+h} = \sum_{i=1}^{3} v_i\big|_{z^*+h}\mathbf{n}_i$$

Subtract the first of these from the second, in R:

$$\mathbf{v}|_{z^*+h} \overset{R}{-} \mathbf{v}|_{z^*} = \sum_{i=1}^{3} (v_i|_{z^*+h} - v_i|_{z^*})\mathbf{n}_i$$

Substitute:

$$\mathbf{v}|_{z^*+h} \overset{R}{-} \mathbf{v}|_{z^*} = \sum_{i=1}^{3} \left(\frac{h}{1!} \frac{dv_i}{dz}\bigg|_{z^*} + \frac{h^2}{2!} \frac{d^2v_i}{dz^2}\bigg|_{z^*} + \cdots \right) \mathbf{n}_i$$

$$= \frac{h}{1!} \sum_{i=1}^{3} \frac{dv_i}{dz}\bigg|_{z^*} \mathbf{n}_i + \frac{h^2}{2!} \sum_{i=1}^{3} \frac{d^2v_i}{dz^2}\bigg|_{z^*} \mathbf{n}_i + \cdots$$

$$\underset{(1.2.1, 1.3.1)}{=} \frac{h}{1!} \frac{{}^R d\mathbf{v}}{dz}\bigg|_{z^*} + \frac{h^2}{2!} \frac{{}^R d^2\mathbf{v}}{dz^2}\bigg|_{z^*} + \cdots$$

Problem: In Fig. 1.8.1, R and R' represent two reference frames (coplanar rectangles). $\mathbf{n}_1$ and $\mathbf{n}_2$ represent unit vectors fixed

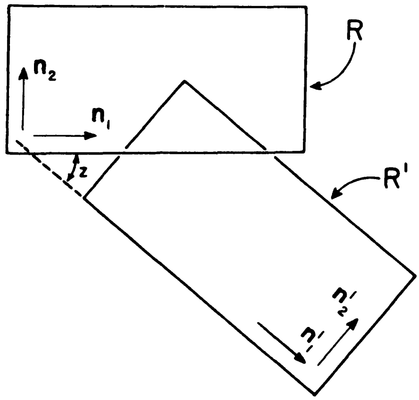


FIG. 1.8.1

in R, and $\mathbf{n}_1'$ and $\mathbf{n}_2'$ are unit vectors fixed in R'. A vector function $\mathbf{v}$ is defined as

$$\mathbf{v} = z^2\, \mathbf{n}_1'$$

where z is the angular displacement of R relative to R'.

Determine approximately the difference between the values of $\mathbf{v}$ at $z = 1.00$ rad and $z = 1.01$ rad in (a) R and (b) R'.

Solution (a): Using only two terms of the Taylor series,

$$\mathbf{v}|_{1.01} \overset{R}{-} \mathbf{v}|_{1.00} = (0.01) \frac{{}^R d\mathbf{v}}{dz}\bigg|_{1.00} + \frac{(0.01)^2}{2} \frac{{}^R d^2\mathbf{v}}{dt^2}\bigg|_{1.00}$$

Next

$$\frac{{}^{R}d\mathbf{v}}{dz} \underset{(1.5.1)}{=} 2z\mathbf{n}_1' + z^2\,\frac{{}^{R}d\mathbf{n}_1'}{dz} = 2z\mathbf{n}_1' + z^2\,\underset{(1.7.1)}{(-\mathbf{n}_2')}$$

and

$$\frac{{}^{R}d^2\mathbf{v}}{dz^2} = 2\mathbf{n}_1' - 2z\mathbf{n}_2' - 2z\mathbf{n}_2' - z^2\mathbf{n}_1' = (2 - z^2)\mathbf{n}_1' - 4z\mathbf{n}_2'$$

Hence

$$\left.\frac{{}^{R}d\mathbf{v}}{dz}\right|_{1.00} = 2\mathbf{n}_1' - \mathbf{n}_2'$$

$$\left.\frac{{}^{R}d^2\mathbf{v}}{dz^2}\right|_{1.00} = \mathbf{n}_1' - 4\mathbf{n}_2'$$

Substitute (and note that the contribution of the second term of the series is much smaller than that of the first):

$$\mathbf{v}|_{1.01} \overset{R}{-} \mathbf{v}|_{1.00} \approx 0.02005\mathbf{n}_1' - 0.01020\mathbf{n}_2'$$

Solution (b): In the present case, use of only the first two terms of the series gives an exact result, because the third and all higher derivatives of $\mathbf{v}$ in R' are equal to zero:

$$\frac{{}^{R'}d\mathbf{v}}{dz} = 2z\mathbf{n}_1', \quad \frac{{}^{R'}d^2\mathbf{v}}{dz^2} = 2\mathbf{n}_1'$$

$$\begin{aligned}\mathbf{v}|_{1.01} \overset{R'}{-} \mathbf{v}|_{1.00} &= (0.01)\left.\frac{{}^{R'}d\mathbf{v}}{dz}\right|_{1.00} + \frac{(0.01)^2}{2}\left.\frac{{}^{R'}d^2\mathbf{v}}{dz^2}\right|_{1.00} + \cdots \\ &= 0.02\mathbf{n}_1' + 0.0001\mathbf{n}_1'\end{aligned}$$

That is,

$$\mathbf{v}|_{1.01} \overset{R'}{-} \mathbf{v}|_{1.00} = 0.0201\mathbf{n}_1'$$

1.8.2 Problem 1.8.1 illustrates how Taylor's theorem may be used for purposes of computation. It should be noted, however, that the same results can be obtained, sometimes more conveniently, without the use of this theorem. For example, as $\mathbf{v}$ is given explicitly in terms of z, Solution (b) of Problem 1.8.1 can be replaced with

$$\mathbf{v}|_{1.01} \overset{R'}{-} \mathbf{v}|_{1.00} = (1.01)^2\mathbf{n}_1' - \mathbf{n}_1' = 0.0201\mathbf{n}_1'$$

The full power of Taylor's theorem becomes apparent in situations involving vector functions which are not specified explicitly.

1.9 Vector tangents of a space curve

1.9.1 If a curve C is fixed in a reference frame R (see Fig. 1.9.1) and T is the tangent to C at a point P of C, a unit vector $\boldsymbol{\tau}$ parallel to T is called a *vector tangent* of C at P and is given by

$$\boldsymbol{\tau} = \frac{\mathbf{p}'}{|\mathbf{p}'|}$$

where $\mathbf{p}$ is the position vector of P relative to a point O fixed in R and primes denote differentiation with respect to z in R, z being

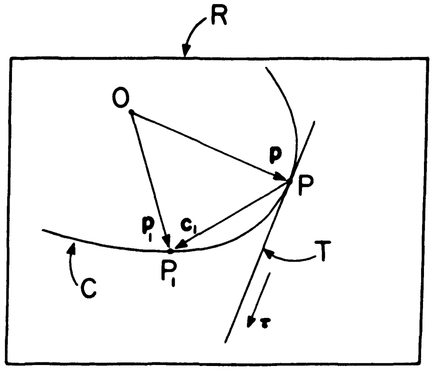


FIG. 1.9.1

any scalar variable such that the position of P on C depends on z.

Proof: Let h be a scalar having the same dimensions as z, P_1 the point of C with which P coincides when z is increased by an amount h, and $\mathbf{p}_1$ the position vector of P_1 relative to O.

By definition, the tangent T is the line which a line passing through P_1 and P approaches as P_1 approaches P. Hence, if $\mathbf{c}_1$ is the position vector of P_1 relative to P, $\mathbf{c}_1$ tends to become parallel to T when h approaches zero. $\boldsymbol{\tau}$ is thus the limit approached by a unit vector parallel to $\mathbf{c}_1$, that is,

$$\boldsymbol{\tau} = \lim_{h\to 0} \frac{\mathbf{c}_1}{|\mathbf{c}_1|}$$

Now

$$\mathbf{c}_1 = \mathbf{p}_1 \overset{R}{-} \mathbf{p}$$

Hence

$$\mathbf{c}_1 \underset{(1.8.1)}{=} \frac{h}{1!}\mathbf{p}' + \frac{h^2}{2!}\mathbf{p}'' + \ldots$$

and (see Vol. 1, Sec. 1.14.11)

$$|\mathbf{c}_1| = (\mathbf{c}_1{}^2)^{1/2} = (h^2\mathbf{p}'^2 + h^3\mathbf{p}' \cdot \mathbf{p}'' + \ldots)^{1/2}$$

so that

$$\frac{\mathbf{c}_1}{|\mathbf{c}_1|} = \frac{\mathbf{p}' + \frac{1}{2}h\mathbf{p}'' + \ldots}{(\mathbf{p}'^2 + h\mathbf{p}' \cdot \mathbf{p}'' + \ldots)^{1/2}}$$

Substitute:

$$\boldsymbol{\tau} = \lim_{h\to 0} \frac{\mathbf{p}' + \frac{1}{2}h\mathbf{p}'' + \ldots}{(\mathbf{p}'^2 + h\mathbf{p}' \cdot \mathbf{p}'' + \ldots)^{1/2}} = \frac{\mathbf{p}'}{(\mathbf{p}'^2)^{1/2}} = \frac{\mathbf{p}'}{|\mathbf{p}'|}$$

Problem: Referring to Problem 1.5.1, suppose that there exists a point O which is fixed in both R and R' and that $\mathbf{p}$ is the position vector of a point P relative to O. Then P traces out a curve C in R and curve C' in R'. Determine the cosine of the angle α between the tangents to C and C' at point P, for $t = 0$.

Solution: Let $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ be vector tangents of C and C' at P. Then

$$\cos\alpha = \boldsymbol{\tau}\big|_{t=0} \cdot \boldsymbol{\tau}'\big|_{t=0}$$

and

$$\boldsymbol{\tau} = \frac{{}^R d\mathbf{p}}{dt} \Big/ \left|\frac{{}^R d\mathbf{p}}{dt}\right|, \quad \boldsymbol{\tau}' = \frac{{}^{R'} d\mathbf{p}}{dt} \Big/ \left|\frac{{}^{R'} d\mathbf{p}}{dt}\right|$$

Now

$$\frac{{}^R d\mathbf{p}}{dt} = 3e^{-3t}\mathbf{n}_2$$

so that

$$\left.\frac{{}^R d\mathbf{p}}{dt}\right|_{t=0} = 3\mathbf{n}_2$$

and, from Problem 1.5.1,

$$\left.\frac{{}^{R'} d\mathbf{p}}{dt}\right|_{t=0} = 3\mathbf{n}_2 + 4\mathbf{n}_1$$

Hence

$$\boldsymbol{\tau}|_{t=0} = \mathbf{n}_2, \quad \boldsymbol{\tau}'|_{t=0} = \tfrac{1}{5}(3\mathbf{n}_2 + 4\mathbf{n}_1)$$

and

$$\cos \alpha = \tfrac{3}{5}$$

1.9.2 The sense of a vector tangent $\boldsymbol{\tau}$, when $\boldsymbol{\tau}$ is obtained by using the expression (see 1.9.1)

$$\boldsymbol{\tau} = \frac{\mathbf{p}'}{|\mathbf{p}'|}$$

depends on the choice of the scalar variable which governs the position of P on C: $\boldsymbol{\tau}$ points in the direction in which P moves when the variable increases algebraically. This is the reason for speaking of "a," rather than "the," vector tangent of a curve.

Proof: Let z and z' be two scalar variables, each of which can be used to describe the position of P on C, and let $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ be the corresponding vector tangents, that is,

$$\boldsymbol{\tau} = \frac{{}^R d\mathbf{p}}{dz} \Big/ \left|\frac{{}^R d\mathbf{p}}{dz}\right|, \quad \boldsymbol{\tau}' = \frac{{}^R d\mathbf{p}}{dz'} \Big/ \left|\frac{{}^R d\mathbf{p}}{dz'}\right|$$

z may be regarded as a function of z', so that

$$\frac{{}^R d\mathbf{p}}{dz'} \underset{(1.6.1)}{=} \frac{{}^R d\mathbf{p}}{dz}\frac{dz}{dz'}$$

Thus

$$\boldsymbol{\tau}' = \frac{\dfrac{{}^R d\mathbf{p}}{dz}\dfrac{dz}{dz'}}{\left|\dfrac{{}^R d\mathbf{p}}{dz}\dfrac{dz}{dz'}\right|} = \frac{\dfrac{{}^R d\mathbf{p}}{dz}\;\dfrac{dz}{dz'}}{\left|\dfrac{{}^R d\mathbf{p}}{dz}\right|\left|\dfrac{dz}{dz'}\right|} \underset{(1.9.1)}{=} \boldsymbol{\tau}\,\frac{\dfrac{dz}{dz'}}{\left|\dfrac{dz}{dz'}\right|}$$

But $(dz/dz')/|dz/dz'|$ is equal to plus or minus one, according as dz/dz' is positive or negative. Hence

$$\boldsymbol{\tau}' = \pm\boldsymbol{\tau}$$

1.9.3 If the scalar variable governing the position of P on C is s, the arc-length displacement of P relative to a point P_0 fixed on C (that is, s is equal to plus or minus the length of the arc of C joining P_0 to P, the sign depending on the initial direction of motion of a point proceeding from P_0 to P), a vector tangent $\boldsymbol{\tau}$ of

C at P (see 1.9.1), which points in the direction in which P moves when s increases algebraically, is given by

$$\boldsymbol{\tau} = \frac{{}^{R}d\mathbf{p}}{ds}$$

Proof: If P_1 is the point to which P moves (see Fig. 1.9.3) when s is increased algebraically by an amount h, the arc PP_1 has

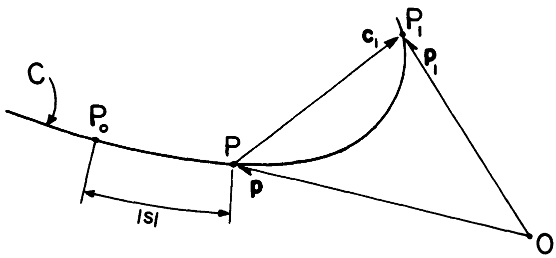


FIG. 1.9.3

the length h and, when h approaches zero, the magnitude of the vector $\mathbf{c}_1$ joining P to P_1 approaches h. This follows from the definition of arc-length. Thus

$$\lim_{h \to 0} \frac{|\mathbf{c}_1|}{h} = 1 \tag{A}$$

Now

$$\mathbf{c}_1 = \mathbf{p}_1 \overset{R}{-} \mathbf{p} \underset{(1.8.1)}{=} \frac{h}{1!} \frac{{}^{R}d\mathbf{p}}{ds} + \frac{h^2}{2!} \frac{{}^{R}d^2\mathbf{p}}{ds^2} + \cdots$$

Hence

$$|\mathbf{c}_1| = (\mathbf{c}_1{}^2)^{1/2} = \left[h^2 \left(\frac{{}^{R}d\mathbf{p}}{ds}\right)^2 + h^3 \frac{{}^{R}d\mathbf{p}}{ds} \cdot \frac{{}^{R}d^2\mathbf{p}}{ds^2} + \cdots\right]^{1/2}$$

and

$$\frac{|\mathbf{c}_1|}{h} = \left[\left(\frac{{}^{R}d\mathbf{p}}{ds}\right)^2 + h \frac{{}^{R}d\mathbf{p}}{ds} \cdot \frac{{}^{R}d^2\mathbf{p}}{ds^2} + \cdots\right]^{1/2}$$

Thus

$$\lim_{h \to 0} \frac{|\mathbf{c}_1|}{h} = \left[\left(\frac{{}^{R}d\mathbf{p}}{ds}\right)^2\right]^{1/2} = \left|\frac{d\mathbf{p}}{ds}\right| \tag{B}$$

Substitute:

$$\left|\frac{{}^{R}d\mathbf{p}}{ds}\right| \underset{(A,B)}{=} 1$$

Next

$$\boldsymbol{\tau} \underset{(1.9.1)}{=} \frac{{}^R d\mathbf{p}}{ds} \Big/ \left|\frac{{}^R d\mathbf{p}}{ds}\right|$$

Hence

$$\boldsymbol{\tau} = \frac{{}^R d\mathbf{p}}{ds}$$

While this expression for $\boldsymbol{\tau}$ appears to be simpler than the one given in Sec. 1.9.1, it is frequently less convenient, because the functional dependence of $\mathbf{p}$ on s is often more complicated than $\mathbf{p}$'s dependence on some other variable.

1.9.4 The plane passing through P and normal to $\boldsymbol{\tau}$ is called the *normal plane* of C at P.

1.10 Vector binormals of a space curve

1.10.1 If a curve C is fixed in a reference frame R (see Fig. 1.10.1a) and B is the binormal to C at a point P of C, a unit vector $\boldsymbol{\beta}$ parallel to B is called a *vector binormal* of C at P and is given by

$$\boldsymbol{\beta} = \frac{\mathbf{p}' \times \mathbf{p}''}{|\mathbf{p}' \times \mathbf{p}''|}$$

where $\mathbf{p}$ is the position vector of P relative to a point O fixed in R and primes denote differentiation with respect to z in R, z being any scalar variable such that the position of P on C depends on z.

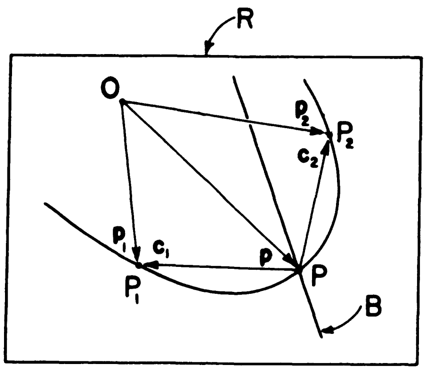


FIG. 1.10.1a

Proof: Let h be a scalar having the same dimensions as z, P_1 the point of C with which P coincides when z is increased by an amount h, P_2 the point of C with which P coincides when z is decreased by an amount h, $\mathbf{p}_1$ the position vector of P_1 relative of O, and $\mathbf{p}_2$ the position vector of P_2 relative to O.

By definition, the binormal B is the line which a line passing through P and perpendicular to both PP_1 and PP_2 approaches as P_1 and P_2 approach P. Hence, if $\mathbf{c}_1$ and $\mathbf{c}_2$ are the position vectors of P_1 and P_2 relative to P, $\mathbf{c}_1 \times \mathbf{c}_2$ tends to become parallel to B when h approaches zero. $\boldsymbol{\beta}$ is thus the limit approached by a unit vector parallel to $\mathbf{c}_1 \times \mathbf{c}_2$, that is,

$$\boldsymbol{\beta} = \lim_{h \to 0} \frac{\mathbf{c}_1 \times \mathbf{c}_2}{|\mathbf{c}_1 \times \mathbf{c}_2|}$$

Now

$$\mathbf{c}_1 \underset{(1.8.1)}{=} \mathbf{p}_1 \overset{R}{-} \mathbf{p} = \frac{h}{1!}\mathbf{p}' + \frac{h^2}{2!}\mathbf{p}'' + \ldots$$

and

$$\mathbf{c}_2 \underset{(1.8.1)}{=} \mathbf{p}_2 \overset{R}{-} \mathbf{p} = \frac{-h}{1!}\mathbf{p}' + \frac{(-h)^2}{2!}\mathbf{p}'' + \ldots$$

Evaluate $\mathbf{c}_1 \times \mathbf{c}_2$ and $|\mathbf{c}_1 \times \mathbf{c}_2|$:

$$\mathbf{c}_1 \times \mathbf{c}_2 = h^3\mathbf{p}' \times \mathbf{p}'' + \ldots$$

$$|\mathbf{c}_1 \times \mathbf{c}_2| = |h^3|\ |\mathbf{p}' \times \mathbf{p}''| + \ldots$$

Substitute and let h approach zero:

$$\boldsymbol{\beta} = \frac{\mathbf{p}' \times \mathbf{p}''}{|\mathbf{p}' \times \mathbf{p}''|}$$

Problem: A straight line AC is drawn on a rectangular sheet of paper $ABCD$ having the dimensions shown in Fig. 1.10.1b. The paper is then folded to form a right-circular cylinder of radius a/π, the line AC thereby being transformed into the (right-wound) circular helix H shown in Fig. 1.10.1c.

Determine the cosine of the angle ψ between a unit vector $\mathbf{k}$ parallel to the axis of the cylinder and a vector binormal $\boldsymbol{\beta}$ of H at a point P of line AC.

Solution: Let P' be a point on AB, such that line PP' is parallel to AD. Let z be the distance between P and P', O the intersection of the axis and base of the cylinder, $\mathbf{n}$ a unit vector parallel to line OP', θ the radian measure of angle AOP', $\mathbf{p}$ the position

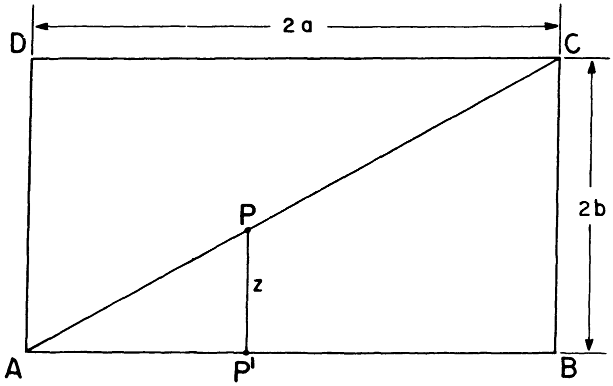


FIG. 1.10.1b

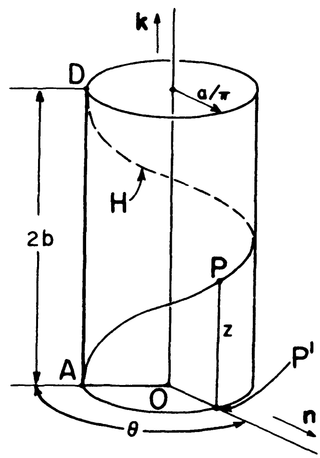


FIG. 1.10.1c

vector of P relative to O, and $\mathbf{p}'$ and $\mathbf{p}''$ the first and second derivatives of $\mathbf{p}$ with respect to z in a reference frame in which H is fixed. Then

$$\cos \psi = \mathbf{k} \cdot \boldsymbol{\beta}$$

where

$$\boldsymbol{\beta} = \frac{\mathbf{p}' \times \mathbf{p}''}{|\mathbf{p}' \times \mathbf{p}''|}$$

Now

$$\mathbf{p} = \frac{a}{\pi}\mathbf{n} + z\mathbf{k}$$

Differentiate with respect to z:

$$\mathbf{p}' = \frac{a}{\pi}\mathbf{k} \underset{(1.7.1)}{\times} \mathbf{n}\frac{d\theta}{dz} + \mathbf{k}$$

From Fig. 1.10.1c

$$\theta = \frac{\text{arc } \widehat{AP'}}{a/\pi}$$

and from Fig. 1.10.1b

$$\text{arc } \widehat{AP'} = za/b$$

Hence

$$\theta = \pi z/b$$

and

$$\frac{d\theta}{dz} = \frac{\pi}{b}$$

Thus

$$\mathbf{p}' = \frac{a}{b}\mathbf{k} \times \mathbf{n} + \mathbf{k}$$

Differentiate again:

$$\mathbf{p}'' = \frac{a}{b}\mathbf{k} \times (\mathbf{k} \times \mathbf{n})\frac{d\theta}{dz}$$

But (see Fig. 1.10.1c)

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{n}) = -\mathbf{n}$$

Hence

$$\mathbf{p}'' = -\frac{\pi a}{b^2}\mathbf{n}$$

Next

$$\mathbf{p}' \times \mathbf{p}'' = -\frac{\pi a}{b^2}\left[\frac{a}{b}(\mathbf{k} \times \mathbf{n}) \times \mathbf{n} + \mathbf{k} \times \mathbf{n}\right]$$

$$= \frac{\pi a}{b^2}\left(\frac{a}{b}\mathbf{k} + \mathbf{n} \times \mathbf{k}\right)$$

and (as $\mathbf{k}$ is perpendicular to $\mathbf{n} \times \mathbf{k}$ and both are unit vectors)

$$|\mathbf{p}' \times \mathbf{p}''| = \frac{\pi a}{b^2}\left(1 + \frac{a^2}{b^2}\right)^{1/2}$$

Thus

$$\boldsymbol{\beta} = \left(\frac{a}{b}\mathbf{k} + \mathbf{n} \times \mathbf{k}\right)\left(1 + \frac{a^2}{b^2}\right)^{-1/2}$$

and

$$\cos\psi = \mathbf{k} \cdot \boldsymbol{\beta} = \left(1 + \frac{b^2}{a^2}\right)^{-1/2}$$

1.10.2 The vectors $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ (see 1.9.1 and 1.10.1) are perpendicular to each other, as can be seen by evaluating $\boldsymbol{\tau} \cdot \boldsymbol{\beta}$.

1.10.3 The sense of a vector binormal $\boldsymbol{\beta}$, when $\boldsymbol{\beta}$ is obtained by using the expression given in Sec. 1.10.1, depends on the choice of the scalar variable which governs the position of P on C. To prove this, proceed as in Sec. 1.9.2.

1.10.4 Once again (see 1.9.3) a simplification is introduced by choosing for the scalar variable governing the position of P on C the arc-length displacement s of point P relative to a point P_0 fixed on C (see Fig. 1.9.3). $\boldsymbol{\beta}$ is then given by

$$\boldsymbol{\beta} = \frac{\dfrac{{}^R d\mathbf{p}}{ds} \times \dfrac{{}^R d^2\mathbf{p}}{ds^2}}{\left|\dfrac{{}^R d^2\mathbf{p}}{ds^2}\right|}$$

Proof: In Sec. 1.9.3 it was shown that

$$\left|\frac{{}^R d\mathbf{p}}{ds}\right| = 1$$

Squaring,

$$\left|\frac{{}^R d\mathbf{p}}{ds}\right|^2 = \left(\frac{{}^R d\mathbf{p}}{ds}\right)^2 = 1$$

Differentiating with respect to s,

$$2\,\frac{{}^R d\mathbf{p}}{ds} \cdot \frac{{}^R d^2\mathbf{p}}{ds^2} = 0$$

Hence, when ${}^R d^2\mathbf{p}/ds^2$ is not equal to zero, ${}^R d\mathbf{p}/ds$ is perpendicular to ${}^R d^2\mathbf{p}/ds^2$. (${}^R d\mathbf{p}/ds$ is never equal to zero, for it is equal to $\boldsymbol{\tau}$, a unit vector, as was shown in Sec. 1.9.3.) Thus, whether or not ${}^R d^2\mathbf{p}/ds^2$ is equal to zero,

$$\left|\frac{^R d\mathbf{p}}{ds} \times \frac{^R d^2\mathbf{p}}{ds^2}\right| = \left|\frac{^R d\mathbf{p}}{ds}\right| \left|\frac{^R d^2\mathbf{p}}{ds^2}\right| = \left|\frac{^R d^2\mathbf{p}}{ds^2}\right|$$

and

$$\underset{(1.10.1)}{\boldsymbol{\beta}} = \frac{\dfrac{^R d\mathbf{p}}{ds} \times \dfrac{^R d^2\mathbf{p}}{ds^2}}{\left|\dfrac{^R d\mathbf{p}}{ds} \times \dfrac{^R d^2\mathbf{p}}{ds^2}\right|} = \frac{\dfrac{^R d\mathbf{p}}{ds} \times \dfrac{^R d^2\mathbf{p}}{ds^2}}{\left|\dfrac{^R d^2\mathbf{p}}{ds^2}\right|}$$

1.10.5 The plane passing through P and normal to $\boldsymbol{\beta}$ is called (for reasons which will appear in subsequent sections) the *plane of curvature* or the *osculating plane* of C at P.

1.11 The vector principal normal of a space curve

1.11.1 If $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ are a vector tangent (see 1.9.1) and a vector binormal (see 1.10.1) of a curve C at a point P, the unit vector $\boldsymbol{\nu}$ given by

$$\boldsymbol{\nu} = \boldsymbol{\beta} \times \boldsymbol{\tau}$$

is called *the vector principal normal* of C at P, provided $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ have the senses obtained by using the same variable z in the expressions given in Secs. 1.9.1 and 1.10.1. The sense of $\boldsymbol{\nu}$, as contrasted with those of $\boldsymbol{\tau}$ and $\boldsymbol{\beta}$ (see 1.9.2 and 1.10.3), is unique.

1.11.2 As the vector $\mathbf{p}' \times \mathbf{p}''$ is perpendicular to $\mathbf{p}'$, the product of the magnitudes of $\mathbf{p}' \times \mathbf{p}''$ and $\mathbf{p}'$ is equal to the magnitude of $(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{p}'$. From Secs. 1.9.1, 1.10.1, and 1.11.1 the following is thus an alternative expression for the vector principal normal:

$$\boldsymbol{\nu} = \frac{(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{p}'}{|(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{p}'|}$$

1.11.3 In terms of derivatives of $\mathbf{p}$ with respect to s, $\boldsymbol{\nu}$ is given by

$$\boldsymbol{\nu} = \frac{\dfrac{^R d^2\mathbf{p}}{ds^2}}{\left|\dfrac{^R d^2\mathbf{p}}{ds^2}\right|}$$

This follows from Secs. 1.11.1, 1.9.3, and 1.10.4, together with the facts that $^R d^2\mathbf{p}/ds^2$ is perpendicular to $^R d\mathbf{p}/ds$ and $^R d\mathbf{p}/ds$ is a unit vector.

1.11.4 The plane passing through P and normal to $\boldsymbol{\nu}$ is called the *rectifying plane* of C at P.

1.12 The vector radius of curvature of a space curve

1.12.1 The position vector $\boldsymbol{\rho}$ of the center of curvature of a space curve C at a point P of C, relative to P, is called the *vector radius of curvature* of C at P, and is given by

$$\boldsymbol{\rho} = \frac{(\mathbf{p}')^2}{(\mathbf{p}' \times \mathbf{p}'')^2} (\mathbf{p}' \times \mathbf{p}'') \times \mathbf{p}'$$

where $\mathbf{p}$ is the position vector of P relative to a point O fixed in a reference frame R in which C is fixed and primes denote differentiation with respect to z in R, z being any scalar variable such that the position of P on C depends on z.

Proof: Let h be a scalar having the same dimensions as z, P_1 the point of C with which P coincides when z is increased by an

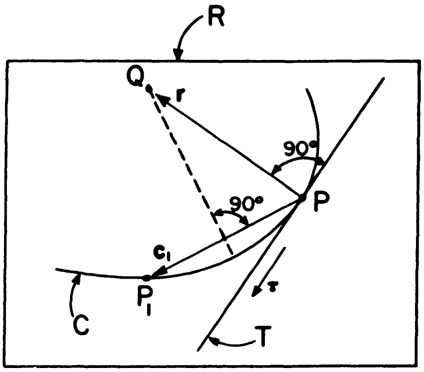


FIG. 1.12.1

amount h (see Fig. 1.12.1), $\mathbf{p}_1$ the position vector of P_1 relative to O, $\boldsymbol{\tau}$ a vector tangent of C at P, and T the tangent to C at P.

By definition, the center of curvature of C at P is the point approached, as P_1 approaches P, by the center Q of a circle which is tangent to T at P and passes through P_1. Letting $\mathbf{r}$ be the position vector of Q relative to P, $\boldsymbol{\rho}$ is thus given by

$$\boldsymbol{\rho} = \lim_{h \to 0} \mathbf{r} \tag{A}$$

Now $\mathbf{r}$ is (by construction) perpendicular to both $\boldsymbol{\tau}$ and $\boldsymbol{\tau} \times \mathbf{c}_1$, where $\mathbf{c}_1$ is the position vector of P_1 relative to P. Hence there exists some scalar, say λ, such that

$$\mathbf{r} = \lambda \boldsymbol{\tau} \times (\boldsymbol{\tau} \times \mathbf{c}_1) \tag{B}$$

On the other hand, as Q lies on the perpendicular bisector of line PP_1, $\mathbf{r}$ may be regarded as the sum of $\mathbf{c}_1/2$ and a vector perpendicular to both $\mathbf{c}_1$ and $\boldsymbol{\tau} \times \mathbf{c}_1$. Hence there exists some scalar, say μ, such that

$$\mathbf{r} = \frac{\mathbf{c}_1}{2} + \mu \mathbf{c}_1 \times (\boldsymbol{\tau} \times \mathbf{c}_1) \tag{C}$$

Dot-multiply each of Eqs. (B) and (C) with $\mathbf{c}_1$ (in order to eliminate μ), equate the resulting expressions for $\mathbf{r} \cdot \mathbf{c}_1$, solve for λ, and substitute into Eq. (B):

$$\mathbf{r} = \frac{\mathbf{c}_1{}^2 \boldsymbol{\tau} \times (\boldsymbol{\tau} \times \mathbf{c}_1)}{2\mathbf{c}_1 \cdot \boldsymbol{\tau} \times (\boldsymbol{\tau} \times \mathbf{c}_1)}$$

Use the relationships

$$\boldsymbol{\tau} \underset{(1.9.1)}{=} \frac{\mathbf{p}'}{|\mathbf{p}'|}, \quad \mathbf{c}_1 \underset{(1.8.1)}{=} \frac{h}{1!}\mathbf{p}' + \frac{h^2}{2!}\mathbf{p}'' + \cdots$$

and let h approach zero:

$$\lim_{h \to 0} \mathbf{r} = \frac{(\mathbf{p}')^2 \mathbf{p}' \times (\mathbf{p}' \times \mathbf{p}'')}{\mathbf{p}'' \cdot \mathbf{p}' \times (\mathbf{p}' \times \mathbf{p}'')}$$

Note that

$$\mathbf{p}' \times (\mathbf{p}' \times \mathbf{p}'') = -(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{p}'$$

while

$$\mathbf{p}'' \cdot \mathbf{p}' \times (\mathbf{p}' \times \mathbf{p}'') = -(\mathbf{p}' \times \mathbf{p}'')^2$$

and use Eq. (A).

1.12.2 The vector radius of curvature $\boldsymbol{\rho}$ of a curve C at a point P (see 1.12.1) has the same direction as the vector principal normal $\boldsymbol{\nu}$ of C at P (see 1.11.1) and can therefore be expressed in the form

$$\boldsymbol{\rho} = \rho \boldsymbol{\nu} \tag{1}$$

where ρ is an intrinsically positive quantity. ρ is called *the radius of curvature* of C at P, and is given by

$$\rho = \frac{|\mathbf{p}'|^3}{|\mathbf{p}' \times \mathbf{p}''|} \tag{2}$$

Proof: Multiply the numerator and denominator of the expression for $\boldsymbol{\rho}$ given in Sec. 1.12.1 with $|\mathbf{p}'|$:

$$\boldsymbol{\rho} = \frac{|\mathbf{p}'|^3}{|\mathbf{p}' \times \mathbf{p}''|} \frac{(\mathbf{p}' \times \mathbf{p}'') \times \mathbf{p}'}{|\mathbf{p}' \times \mathbf{p}''|\ |\mathbf{p}'|} \underset{(1.11.2)}{=} \frac{|\mathbf{p}'|^3}{|\mathbf{p}' \times \mathbf{p}''|} \boldsymbol{\nu}$$

Problem: Determine the radius of curvature at point P of the helix H described in Problem 1.10.1.

Solution: From the solution of Problem 1.10.1,

$$\mathbf{p}' = \frac{a}{b}\mathbf{k} \times \mathbf{n} + \mathbf{k}$$

and

$$|\mathbf{p}' \times \mathbf{p}''| = \frac{\pi a}{b^2}\left(1 + \frac{a^2}{b^2}\right)^{1/2}$$

Hence

$$\rho = \frac{[1 + (a^2/b^2)]^{3/2}}{(\pi a/b^2)[1 + (a^2/b^2)]^{1/2}} = \frac{a}{\pi}\left(1 + \frac{b^2}{a^2}\right)$$

1.12.3 In terms of derivatives of $\mathbf{p}$ with respect to s, $\boldsymbol{\rho}$ and ρ (see 1.12.1 and 1.12.2) are given, respectively, by

$$\boldsymbol{\rho} = \frac{\dfrac{{}^R d^2\mathbf{p}}{ds^2}}{\left(\dfrac{{}^R d^2\mathbf{p}}{ds^2}\right)^2} \tag{1}$$

and

$$\rho = \frac{1}{\left|\dfrac{{}^R d^2\mathbf{p}}{ds^2}\right|} \tag{2}$$

Proof: From Section 1.12.1,

$$\boldsymbol{\rho} = \frac{\left(\dfrac{{}^R d\mathbf{p}}{ds}\right)^2}{\left(\dfrac{{}^R d\mathbf{p}}{ds} \times \dfrac{{}^R d^2\mathbf{p}}{ds^2}\right)^2} \left(\frac{{}^R d\mathbf{p}}{ds} \times \frac{{}^R d^2\mathbf{p}}{ds^2}\right) \times \frac{{}^R d\mathbf{p}}{ds}$$

Now

$$\left(\frac{{}^R d\mathbf{p}}{ds}\right)^2 = \left|\frac{{}^R d\mathbf{p}}{ds}\right|^2 \underset{(1.9.3)}{=} 1 \tag{A}$$

and

$$\left(\frac{{}^R d\mathbf{p}}{ds} \times \frac{{}^R d^2\mathbf{p}}{ds^2}\right)^2 = \left|\frac{{}^R d\mathbf{p}}{ds} \times \frac{{}^R d^2\mathbf{p}}{ds^2}\right|^2 \underset{(1.10.4)}{=} \left|\frac{{}^R d^2\mathbf{p}}{ds^2}\right|^2 \tag{B}$$

Furthermore

$$\left(\frac{{}^R d\mathbf{p}}{ds} \times \frac{{}^R d^2\mathbf{p}}{ds^2}\right) \times \frac{{}^R d\mathbf{p}}{ds} = \left(\frac{{}^R d\mathbf{p}}{ds}\right)^2 \frac{{}^R d^2\mathbf{p}}{ds^2} - \frac{{}^R d^2\mathbf{p}}{ds^2} \cdot \frac{{}^R d\mathbf{p}}{ds} \frac{{}^R d\mathbf{p}}{ds}$$

$$= \underset{(1.9.3)}{\frac{{}^R d^2\mathbf{p}}{ds^2}} - \underset{(1.10.4)}{0}$$

Hence

$$\boldsymbol{\rho} = \frac{\dfrac{{}^R d^2\mathbf{p}}{ds^2}}{\left|\dfrac{{}^R d^2\mathbf{p}}{ds^2}\right|^2} = \frac{\dfrac{{}^R d^2\mathbf{p}}{ds^2}}{\left(\dfrac{{}^R d^2\mathbf{p}}{ds^2}\right)^2}$$

Next

$$\rho \underset{(1.12.2)}{=} \frac{\left|\dfrac{{}^R d\mathbf{p}}{ds}\right|^3}{\left|\dfrac{{}^R d\mathbf{p}}{ds} \times \dfrac{{}^R d^2\mathbf{p}}{ds^2}\right|} \underset{(A,B)}{=} \frac{1}{\left|\dfrac{{}^R d^2\mathbf{p}}{ds^2}\right|}$$

1.13 The Serret-Frenet formulas

1.13.1 The derivatives of the vectors $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, and $\boldsymbol{\nu}$ (see 1.9, 1.10, 1.11) with respect to s are given by

$$\frac{{}^R d\boldsymbol{\tau}}{ds} = \frac{\boldsymbol{\nu}}{\rho} \tag{1}$$

$$\frac{{}^R d\boldsymbol{\beta}}{ds} = -\lambda\boldsymbol{\nu} \tag{2}$$

$$\frac{{}^R d\boldsymbol{\nu}}{ds} = \lambda\boldsymbol{\beta} - \frac{\boldsymbol{\tau}}{\rho} \tag{3}$$

where ρ is the radius of curvature (see 1.12.2) and λ, called the *torsion* of the curve C at point P, is given by

$$\lambda = \boldsymbol{\rho}^2 \left[\frac{{}^R d\mathbf{p}}{ds}, \frac{{}^R d^2\mathbf{p}}{ds^2}, \frac{{}^R d^3\mathbf{p}}{ds^3}\right]$$

The expressions (1), (2), and (3) are called the Serret-Frenet formulas.

Proof (1):

$$\boldsymbol{\tau} \underset{(1.9.3)}{=} \frac{{}^R d\mathbf{p}}{ds}$$

Differentiate with respect to s in R:

$$\frac{{}^R d\boldsymbol{\tau}}{ds} \underset{(1.2.5)}{=} \frac{{}^R d^2\mathbf{p}}{ds^2}$$

$$\underset{(1.11.3)}{=} \left|\frac{{}^R d^2\mathbf{p}}{ds^2}\right| \boldsymbol{\nu} \underset{(1.12.3)}{=} \frac{\boldsymbol{\nu}}{\rho}$$

Proof (2): From Sec. 1.10.4, $\boldsymbol{\beta}$ can be expressed in the form

$$\boldsymbol{\beta} = \frac{{}^R d\mathbf{p}}{ds} \times \frac{{}^R d^2\mathbf{p}}{ds^2}\left[\left(\frac{{}^R d^2\mathbf{p}}{ds^2}\right)^2\right]^{-1/2}$$

Differentiate with respect to s in R:

$$\frac{{}^R d\boldsymbol{\beta}}{ds} \underset{(1.5.4)}{=} \frac{{}^R d\mathbf{p}}{ds} \times \frac{{}^R d^3\mathbf{p}}{ds^3}\left[\left(\frac{{}^R d^2\mathbf{p}}{ds^2}\right)^2\right]^{-1/2}$$

$$- \frac{{}^R d\mathbf{p}}{ds} \times \frac{{}^R d^2\mathbf{p}}{ds^2}\left[\left(\frac{{}^R d^2\mathbf{p}}{ds^2}\right)^2\right]^{-3/2} \frac{{}^R d^2\mathbf{p}}{ds^2} \cdot \frac{{}^R d^3\mathbf{p}}{ds^3}$$

Eliminate the first and second derivatives of $\mathbf{p}$ by using

$$\frac{{}^R d\mathbf{p}}{ds} \underset{(1.9.3)}{=} \boldsymbol{\tau}, \qquad \frac{{}^R d^2\mathbf{p}}{ds^2} \underset{(1.11.3,1.12.3)}{=} \frac{\boldsymbol{\nu}}{\rho}$$

This gives

$$\frac{{}^R d\boldsymbol{\beta}}{ds} = \rho\boldsymbol{\tau} \times \left(\frac{{}^R d^3\mathbf{p}}{ds^3} - \boldsymbol{\nu}\boldsymbol{\nu} \cdot \frac{{}^R d^3\mathbf{p}}{ds^3}\right)$$

$$= \rho\boldsymbol{\tau} \times \left[\boldsymbol{\nu} \times \left(\frac{{}^R d^3\mathbf{p}}{ds^3} \times \boldsymbol{\nu}\right)\right]$$

$$= \rho\boldsymbol{\tau} \cdot \left(\frac{{}^R d^3\mathbf{p}}{ds^3} \times \boldsymbol{\nu}\right)\boldsymbol{\nu} - \rho\boldsymbol{\tau} \cdot \boldsymbol{\nu}\left(\frac{{}^R d^3\mathbf{p}}{ds^3} \times \boldsymbol{\nu}\right)$$

or, noting that $\boldsymbol{\tau} \cdot \boldsymbol{\nu}$ is equal to zero (see 1.11.1),

$$\frac{{}^R d\boldsymbol{\beta}}{ds} = \rho\boldsymbol{\tau} \cdot \left(\frac{{}^R d^3\mathbf{p}}{ds^3} \times \boldsymbol{\nu}\right)\boldsymbol{\nu}$$

Hence

$$\frac{{}^R d\boldsymbol{\beta}}{ds} = -\lambda\boldsymbol{\nu}$$

where

$$\lambda = -\rho\boldsymbol{\tau} \cdot \left(\frac{{}^R d^3\mathbf{p}}{ds^3} \times \boldsymbol{\nu}\right)$$

or, expressing $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ in terms of derivatives of $\mathbf{p}$,

$$\lambda = \rho^2 \left[\frac{{}^R d\mathbf{p}}{ds}, \frac{{}^R d^2\mathbf{p}}{ds^2}, \frac{{}^R d^3\mathbf{p}}{ds^3}\right]$$

Proof (3):

$$\boldsymbol{\nu} \underset{(1.11.1)}{=} \boldsymbol{\beta} \times \boldsymbol{\tau}$$

Differentiate with respect to s in R:

$$\frac{{}^R d\boldsymbol{\nu}}{ds} \underset{(1.5.3)}{=} \frac{{}^R d\boldsymbol{\beta}}{ds} \times \boldsymbol{\tau} + \boldsymbol{\beta} \times \frac{{}^R d\boldsymbol{\tau}}{ds}$$

Use (1) and (2), above:

$$\frac{{}^R d\boldsymbol{\nu}}{ds} = -\lambda\boldsymbol{\nu} \times \boldsymbol{\tau} + \boldsymbol{\beta} \times \frac{\boldsymbol{\nu}}{\rho}$$

But

$$\boldsymbol{\nu} \times \boldsymbol{\tau} \underset{(1.11.1)}{=} (\boldsymbol{\beta} \times \boldsymbol{\tau}) \times \boldsymbol{\tau} = -\boldsymbol{\beta}$$

and

$$\boldsymbol{\beta} \times \boldsymbol{\nu} \underset{(1.11.1)}{=} \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \boldsymbol{\tau}) = -\boldsymbol{\tau}$$

Hence

$$\frac{{}^R d\boldsymbol{\nu}}{ds} = \lambda\boldsymbol{\beta} - \frac{\boldsymbol{\tau}}{\rho}$$

Problem: When a vector tangent $\boldsymbol{\tau}$ of a curve C at a point P is multiplied by a constant r having the dimensions of length, and the resulting vector $\mathbf{p}$ is regarded as the position vector relative to a point O of a point $\bar{P}$ on a curve $\bar{C}$, then $\bar{C}$ lies on a sphere of radius r, center at O, and is called a *spherical indicatrix* of C (see Fig. 1.13.1). Letting $\bar{\rho}$ be the radius of curvature of $\bar{C}$ at $\bar{P}$, express $\bar{\rho}$ in terms of the radius of curvature ρ and the torsion λ of C at P.

Solution: The position of $\bar{P}$ on $\bar{C}$ may be regarded as depending on the arc-length displacement s of P relative to a point P_0 fixed on C. (s is *not* the arc-length displacement of $\bar{P}$ relative to a

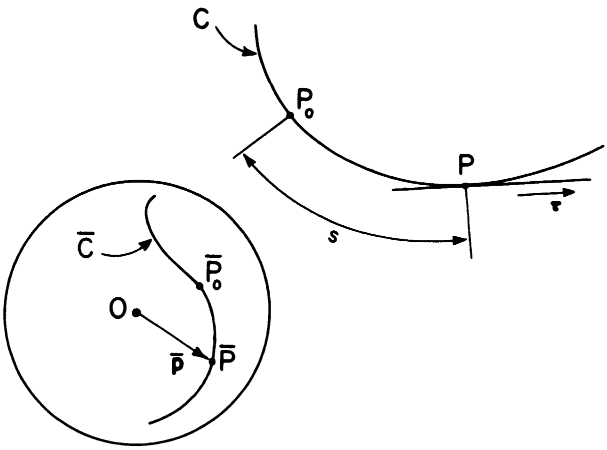


FIG. 1.13.1

point $\bar{P}_0$ fixed on $\bar{C}$.) If primes denote differentiation with respect to s, $\bar{\rho}$ is then given by

$$\bar{\rho} \underset{(1.12.2)}{=} \frac{|\bar{\mathbf{p}}'|^3}{|\bar{\mathbf{p}}' \times \bar{\mathbf{p}}''|}$$

By definition

$$\bar{\mathbf{p}} = r\boldsymbol{\tau}$$

Hence

$$\bar{\mathbf{p}}' = \frac{d}{ds}(r\boldsymbol{\tau}) = r\frac{d\boldsymbol{\tau}}{ds} \underset{(1)}{=} \frac{r}{\rho}\boldsymbol{\nu}$$

where $\boldsymbol{\nu}$ is the vector principal normal of C at P. Next,

$$\bar{\mathbf{p}}'' = \frac{d}{ds}\left(\frac{r}{\rho}\boldsymbol{\nu}\right) = r\left(-\frac{1}{\rho^2}\frac{d\rho}{ds}\boldsymbol{\nu} + \frac{1}{\rho}\frac{d\boldsymbol{\nu}}{ds}\right)$$

$$\underset{(3)}{=} r\left(-\frac{1}{\rho^2}\frac{d\rho}{ds}\boldsymbol{\nu} + \frac{\lambda}{\rho}\boldsymbol{\beta} - \frac{\boldsymbol{\tau}}{\rho^2}\right)$$

Hence

$$\bar{\mathbf{p}}' \times \bar{\mathbf{p}}'' = \frac{r^2}{\rho^2}\left(\lambda\boldsymbol{\nu} \times \boldsymbol{\beta} - \frac{\boldsymbol{\nu} \times \boldsymbol{\tau}}{\rho}\right)$$

or, replacing $\boldsymbol{\nu}$ with $\boldsymbol{\beta} \times \boldsymbol{\tau}$ (see 1.11.1)

$$\bar{\mathbf{p}}' \times \bar{\mathbf{p}}'' = \frac{r^2}{\rho^2}\left(\lambda\boldsymbol{\tau} + \frac{\boldsymbol{\beta}}{\rho}\right)$$

Substitute:

$$\bar{\rho} = \frac{r^3/\rho^3}{(r^2/\rho^2)[\lambda^2 + (1/\rho^2)]^{1/2}} = \frac{r}{(1 + \lambda^2\rho^2)^{1/2}}$$

1.13.2 The torsion λ (see 1.13.1), expressed in terms of derivatives of the position vector $\mathbf{p}$ with respect to a variable z other than s, is given by

$$\lambda = \rho^2[\mathbf{p}', \ \mathbf{p}'', \ \mathbf{p}'''] \ |\mathbf{p}'|^{-6}$$

Proof: Express the first, second, and third derivatives of $\mathbf{p}$ with respect to s in terms of derivatives of $\mathbf{p}$ with respect to z:

$$\frac{{}^R d\mathbf{p}}{ds} \underset{(1.6.1)}{=} \mathbf{p}' \frac{dz}{ds} \tag{A}$$

$$\frac{{}^R d^2\mathbf{p}}{ds^2} \underset{(1.5.1)}{=} \mathbf{p}'' \left(\frac{dz}{ds}\right)^2 + \mathbf{p}' \frac{d^2 z}{ds^2}$$

$$\frac{{}^R d^3\mathbf{p}}{ds^3} \underset{(1.3.1)}{=} \frac{{}^R d}{ds}\left(\frac{{}^R d^2\mathbf{p}}{ds^2}\right) = \mathbf{p}''' \left(\frac{dz}{ds}\right)^3 + 3\mathbf{p}'' \frac{dz}{ds}\frac{d^2 z}{ds^2} + \mathbf{p}' \frac{d^3 z}{ds^3}$$

Evaluate the scalar triple-product of these derivatives:

$$\left[\frac{{}^R d\mathbf{p}}{ds}, \frac{{}^R d^2\mathbf{p}}{ds^2}, \frac{{}^R d^3\mathbf{p}}{ds^3}\right] = [\mathbf{p}', \mathbf{p}'', \mathbf{p}'''] \left(\frac{dz}{ds}\right)^6$$

Dot-multiply each member of Eq. (A) with itself, keeping in mind that (see 1.9.3) ${}^R d\mathbf{p}/ds$ is a unit vector:

$$\left(\frac{dz}{ds}\right)^2 (\mathbf{p}')^2 = 1$$

It follows that

$$\left(\frac{dz}{ds}\right)^6 = [(\mathbf{p}')^2]^{-3} = |\mathbf{p}'|^{-6}$$

and

$$\left[\frac{{}^R d\mathbf{p}}{ds}, \frac{{}^R d^2\mathbf{p}}{ds^2}, \frac{{}^R d^3\mathbf{p}}{ds^3}\right] = [\mathbf{p}', \mathbf{p}'', \mathbf{p}'''] \ |\mathbf{p}'|^{-6}$$

Substitute into the expression for λ given in Sec. 1.13.1.

Problem: Determine the torsion λ of the curve H described in Problem 1.10.1.

Solution: From the solution of Problem 1.10.1,

$$\mathbf{p}' = \frac{a}{b}\mathbf{k} \times \mathbf{n} + \mathbf{k}$$

and

$$\mathbf{p}'' = -\frac{\pi a}{b^2}\,\mathbf{n}$$

Differentiate $\mathbf{p}''$ with respect to z:

$$\mathbf{p}''' = -\frac{\pi a}{b^2}\frac{d\mathbf{n}}{dz} = -\frac{\pi^2 a}{b^3}\,\mathbf{k} \times \mathbf{n}$$

Evaluate $|\mathbf{p}'|$ and $[\mathbf{p}', \mathbf{p}'', \mathbf{p}''']$:

$$|\mathbf{p}'| = \left(1 + \frac{a^2}{b^2}\right)^{1/2}, \quad [\mathbf{p}', \mathbf{p}'', \mathbf{p}'''] = \frac{\pi^3 a^2}{b^5}$$

The radius of curvature ρ was found in Problem 1.12.2:

$$\rho = \frac{a}{\pi}\left(1 + \frac{b^2}{a^2}\right)$$

Hence

$$\lambda = \rho^2[\mathbf{p}', \mathbf{p}'', \mathbf{p}''']\ |\mathbf{p}'|^{-6} = \frac{\pi b}{a^2 + b^2}$$

2 KINEMATICS

2.1 Rates of change of orientation of a rigid body

2.1.1 If the orientation of a rigid body R in a reference frame R' depends on only a single scalar variable z, there exists for each value of z a vector ${}^{R'}\boldsymbol{\omega}_z{}^R$ such that the derivative with respect to z in R' of every vector $\mathbf{c}$ fixed in R (see Fig. 2.1.1) is given by

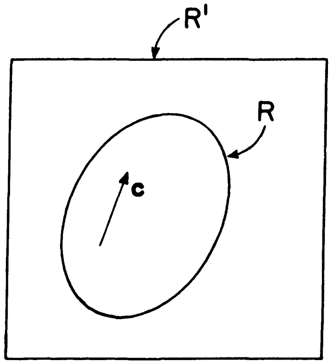


Fig. 2.1.1

$$\frac{{}^{R'}d\mathbf{c}}{dz} = {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{c} \tag{1}$$

The vector ${}^{R'}\boldsymbol{\omega}_z{}^R$ is called the *rate of change of orientation* of R in R' with respect to z, and is given by

$${}^{R'}\boldsymbol{\omega}_z{}^R = \frac{\dfrac{{}^{R'}d\mathbf{a}}{dz} \times \dfrac{{}^{R'}d\mathbf{b}}{dz}}{\dfrac{{}^{R'}d\mathbf{a}}{dz} \cdot \mathbf{b}} \tag{2}$$

where $\mathbf{a}$ and $\mathbf{b}$ are any two nonparallel vectors fixed in R. ${}^{R'}\boldsymbol{\omega}_z{}^R$ is a *free vector*; that is, it is in no way associated with any particular point of either R or R'.

Proof: If $\mathbf{a}$ and $\mathbf{b}$ are fixed in R, their magnitudes and the angle between them are independent of z. Hence

$$\frac{d\mathbf{a}^2}{dz} = 0, \qquad \frac{d\mathbf{b}^2}{dz} = 0, \qquad \frac{d}{dz}(\mathbf{a} \cdot \mathbf{b}) = 0$$

or, carrying out the indicated differentiations and using primes to denote differentiation with respect to z in R',

$$\mathbf{a}' \cdot \mathbf{a} = 0, \qquad \mathbf{b}' \cdot \mathbf{b} = 0 \tag{3}$$

$$\mathbf{a}' \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b}' = 0 \tag{4}$$

From these expressions it follows that the relationships

$$\mathbf{a}' = \frac{\mathbf{a}' \times \mathbf{b}'}{\mathbf{a}' \cdot \mathbf{b}} \times \mathbf{a}$$

and

$$\mathbf{b}' = \frac{\mathbf{a}' \times \mathbf{b}'}{\mathbf{a}' \cdot \mathbf{b}} \times \mathbf{b}$$

are identities. Hence, defining ${}^{R'}\boldsymbol{\omega}_z{}^R$ as

$${}^{R'}\boldsymbol{\omega}_z{}^R = \frac{\mathbf{a}' \times \mathbf{b}'}{\mathbf{a}' \cdot \mathbf{b}}$$

$\mathbf{a}'$ and $\mathbf{b}'$ are seen to be given by

$$\mathbf{a}' = {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{a}, \qquad \mathbf{b}' = {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{b} \tag{A}$$

Next, any vector $\mathbf{c}$ fixed in R can be expressed in the form

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a} \times \mathbf{b} \tag{B}$$

where α, β, and γ are independent of z. Differentiating with respect to z in R', and using Eqs. (A) to eliminate $\mathbf{a}'$ and $\mathbf{b}'$,

$$\begin{aligned}
\mathbf{c}' &= \alpha\, {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{a} + \beta\, {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{b} \\
&\qquad + \gamma[({}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{a}) \times \mathbf{b} + \mathbf{a} \times ({}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{b})] \\
&= \alpha\, {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{a} + \beta\, {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{b} \\
&\qquad + \gamma({}^{R'}\boldsymbol{\omega}_z{}^R \cdot \mathbf{b}\mathbf{a} - \mathbf{a} \cdot \mathbf{b}\, {}^{R'}\boldsymbol{\omega}_z{}^R + \mathbf{a} \cdot \mathbf{b}\, {}^{R'}\boldsymbol{\omega}_z{}^R - \mathbf{a} \cdot {}^{R'}\boldsymbol{\omega}_z{}^R\, \mathbf{b}) \\
&= \alpha\, {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{a} + \beta\, {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{b} + \gamma\, {}^{R'}\boldsymbol{\omega}_z{}^R \times (\mathbf{a} \times \mathbf{b}) \\
&= {}^{R'}\boldsymbol{\omega}_z{}^R \times (\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a} \times \mathbf{b}) \\
&\underset{\text{(B)}}{=} {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{c}
\end{aligned}$$

2.1.2 The importance of rates of change of orientation as analytical tools is due to the fact that they furnish a means of replacing the process of differentiation with that of cross multiplication.

2.1.3 The expression for ${}^{R'}\boldsymbol{\omega}_z{}^{R}$ given in Sec. 2.1.1 appears to be unsymmetrical in $\mathbf{a}$ and $\mathbf{b}$. This lack of symmetry is spurious. For, in view of Eq. (4), Sec. 2.1.1, ${}^{R'}\boldsymbol{\omega}_z{}^{R}$ may be expressed in the form

$${}^{R'}\boldsymbol{\omega}_z{}^{R} = \frac{1}{2}\left(\frac{\mathbf{a}' \times \mathbf{b}'}{\mathbf{a}' \cdot \mathbf{b}} + \frac{\mathbf{b}' \times \mathbf{a}'}{\mathbf{b}' \cdot \mathbf{a}}\right)$$

and this expression remains unaltered when the roles of $\mathbf{a}$ and $\mathbf{b}$ are interchanged.

Problem: A rectangular plate P moves in such a way that the corners A and B remain on two nonintersecting lines while the

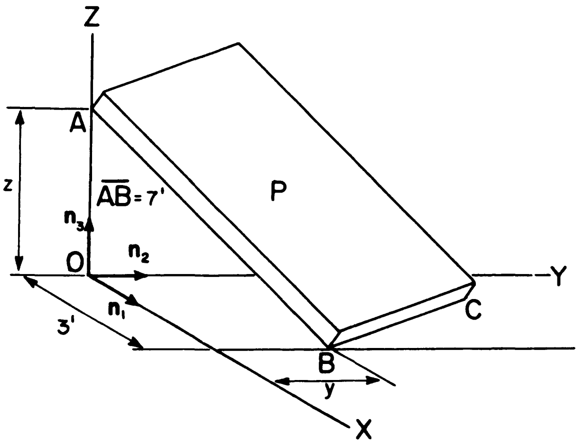


FIG. 2.1.3

edge BC lies at all times in the XOY plane, as shown in Fig. 2.1.3. (OX, OY, and OZ are mutually perpendicular.)

Letting z be the distance between O and A, and R a reference frame in which the axes OX, OY, OZ are fixed, determine ${}^{R}\boldsymbol{\omega}_z{}^{P}$ for the instant when z is equal to 2 ft.

Solution: Let $\mathbf{a}$ be the position vector of A relative to B, and $\mathbf{b}$ a unit vector parallel to edge BC, and use primes to denote differentiation with respect to z in R. Then

$$^R\omega_z{}^P \underset{(2.1.1)}{=} \frac{\mathbf{a}' \times \mathbf{b}'}{\mathbf{a}' \cdot \mathbf{b}} \tag{A}$$

From Fig. 2.1.3,

$$\mathbf{a} = -3\mathbf{n}_1 - y\mathbf{n}_2 + z\mathbf{n}_3$$

where y is the distance from B to the X-axis. As the distance between A and B is fixed at 7 ft,

$$\mathbf{a}^2 = 9 + y^2 + z^2 = 7^2 = 49$$

so that

$$y = (40 - z^2)^{1/2}$$

Hence

$$\mathbf{a} = -3\mathbf{n}_1 - (40 - z^2)^{1/2}\mathbf{n}_2 + z\mathbf{n}_3$$

and

$$\mathbf{a}' = z(40 - z^2)^{-1/2}\mathbf{n}_2 + \mathbf{n}_3$$

Thus

$$\mathbf{a}'|_{z=2} = \frac{1}{3}\,\mathbf{n}_2 + \mathbf{n}_3 \tag{B}$$

As $\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{n}_3$, it can be expressed in terms of the cross product of these vectors:

$$\mathbf{b} = \frac{\mathbf{a} \times \mathbf{n}_3}{|\mathbf{a} \times \mathbf{n}_3|} = \frac{-(40 - z^2)^{1/2}\mathbf{n}_1 + 3\mathbf{n}_2}{(49 - z^2)^{1/2}}$$

Differentiating,

$$\mathbf{b}' = \frac{(49 - z^2)^{1/2}(40 - z^2)^{-1/2}z\mathbf{n}_1 + [-(40 - z^2)^{1/2}\mathbf{n}_1 + 3\mathbf{n}_2](49 - z^2)^{-1/2}z}{49 - z^2}$$

Thus

$$\mathbf{b}|_{z=2} = \frac{-6\mathbf{n}_1 + 3\mathbf{n}_2}{(45)^{1/2}}, \qquad \mathbf{b}'|_{z=2} = \frac{3\mathbf{n}_1 + 6\mathbf{n}_2}{(45)^{3/2}} \tag{C}$$

Substitute from Eq. (B) and (C) into Eq. (A):

$$^R\omega_z{}^P|_{z=2} = \frac{1}{45}\,(-6\mathbf{n}_1 + 3\mathbf{n}_2 - \mathbf{n}_3)\ \mathrm{ft}^{-1}$$

2.1.4 The first derivatives of a vector $\mathbf{v}$ with respect to a scalar variable z in two reference frames R and R' are related to each other as follows:

$$\frac{^{R'}d\mathbf{v}}{dz} = \frac{^{R}d\mathbf{v}}{dz} + {}^{R'}\omega_z{}^R \times \mathbf{v}$$

where ${}^{R'}\boldsymbol{\omega}_z{}^R$ is the rate of change of orientation of R in R' with respect to z (see 2.1.1).

Proof: Let $\mathbf{n}_i$, $i = 1, 2, 3$, be unit vectors (not parallel to the same plane) fixed in R, and v_i the $\mathbf{n}_i$ measure number of $\mathbf{v}$. Then

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{n}_i \tag{A}$$

and, differentiating in R',

$$\frac{{}^{R'}d\mathbf{v}}{dz} = \frac{{}^{R'}d}{dz}\sum_{i=1}^{3} v_i\mathbf{n}_i \underset{(1.4.1)}{=} \sum_{i=1}^{3} \frac{{}^{R'}d}{dz}(v_i\mathbf{n}_i)$$

$$\underset{(1.5.1)}{=} \sum_{i=1}^{3} \frac{dv_i}{dz}\mathbf{n}_i + \sum_{i=1}^{3} v_i \frac{{}^{R'}d\mathbf{n}_i}{dz}$$

$$= \underset{(1.2.1)}{\frac{{}^{R}d\mathbf{v}}{dz}} + \sum_{i=1}^{3} v_i \underset{(2.1.1)}{{}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{n}_i}$$

$$= \frac{{}^{R}d\mathbf{v}}{dz} + {}^{R'}\boldsymbol{\omega}_z{}^R \times \sum_{i=1}^{3} v_i\mathbf{n}_i$$

$$= \frac{{}^{R}d\mathbf{v}}{dz} + {}^{R'}\boldsymbol{\omega}_z{}^R \times \underset{(A)}{\mathbf{v}}$$

Problem: Referring to Problem 2.1.3, and letting $\mathbf{v}$ be a vector given by

$$\mathbf{v} = \frac{360}{z}\mathbf{n}_1 \text{ ft sec}^{-1}$$

determine the derivative of $\mathbf{v}$ with respect to z in P, for $z = 2$ ft.

Solution:

$$\frac{{}^{R}d\mathbf{v}}{dz} = \frac{{}^{P}d\mathbf{v}}{dz} + {}^{R}\boldsymbol{\omega}_z{}^P \times \mathbf{v}$$

Hence

$$\left.\frac{{}^{P}d\mathbf{v}}{dz}\right|_{z=2} = \left.\frac{{}^{R}d\mathbf{v}}{dz}\right|_{z=2} - {}^{R}\boldsymbol{\omega}_z{}^P\big|_{z=2} \times \mathbf{v}\big|_{z=2}$$

Now

$$\mathbf{v}\big|_{z=2} = \frac{360}{2}\mathbf{n}_1 = 180\mathbf{n}_1 \text{ ft sec}^{-1}$$

$$\left.\frac{{}^R d\mathbf{v}}{dz}\right|_{z=2} = \frac{-360}{4}\,\mathbf{n}_1 = -90\mathbf{n}_1 \text{ sec}^{-1}$$

and (see Problem 2.1.3)

$$\left.{}^R\boldsymbol{\omega}_z{}^P\right|_{z=2} = \frac{1}{45}\,(-6\mathbf{n}_1 + 3\mathbf{n}_2 - \mathbf{n}_3) \text{ ft}^{-1}$$

Thus

$$\begin{aligned}\left.\frac{{}^P d\mathbf{v}}{dz}\right|_{z=2} &= -90\mathbf{n}_1 - \frac{180}{45}\,(-3\mathbf{n}_3 - \mathbf{n}_2)\\ &= -90\mathbf{n}_1 + 4\mathbf{n}_2 + 12\mathbf{n}_3 \text{ sec}^{-1}\end{aligned}$$

2.1.5 The derivatives of ${}^{R'}\boldsymbol{\omega}_z{}^R$ (see 2.1.1) with respect to z in R and R' are equal to each other, as may be seen by letting ${}^{R'}\boldsymbol{\omega}_z{}^R$ play the part of $\mathbf{v}$ in the expression given in Sec. 2.1.4.

2.1.6 ${}^{R'}\boldsymbol{\omega}_z{}^R$ and ${}^R\boldsymbol{\omega}_z{}^{R'}$ are related as follows:

$${}^{R'}\boldsymbol{\omega}_z{}^R = -{}^R\boldsymbol{\omega}_z{}^{R'}$$

Proof: From Sec. 2.1.4,

$$\frac{{}^{R'}d\mathbf{v}}{dz} = \frac{{}^R d\mathbf{v}}{dz} + {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{v}$$

Again from Sec. 2.1.4,

$$\frac{{}^R d\mathbf{v}}{dz} = \frac{{}^{R'}d\mathbf{v}}{dz} + {}^R\boldsymbol{\omega}_z{}^{R'} \times \mathbf{v}$$

Thus

$$\frac{{}^{R'}d\mathbf{v}}{dz} = \frac{{}^{R'}d\mathbf{v}}{dz} + {}^R\boldsymbol{\omega}_z{}^{R'} \times \mathbf{v} + {}^{R'}\boldsymbol{\omega}_z{}^R \times \mathbf{v}$$

or

$$({}^R\boldsymbol{\omega}_z{}^{R'} + {}^{R'}\boldsymbol{\omega}_z{}^R) \times \mathbf{v} = 0$$

This equation cannot be satisfied for *all* vectors $\mathbf{v}$ unless

$${}^R\boldsymbol{\omega}_z{}^{R'} + {}^{R'}\boldsymbol{\omega}_z{}^R = 0$$

2.2 Angular velocity

2.2.1 The rate of change of orientation of a rigid body R in a reference frame R' with respect to the time t (see 2.1.1) is called the *angular velocity* of R in R' and is denoted by ${}^{R'}\boldsymbol{\omega}^R$. It is a *free vector* (see 2.1.1).

Problem: Referring to Problem 1.7.3, determine the angular velocity ${}^C\boldsymbol{\omega}^D$ of the disc D in the track C for the instant t^*.

Solution: Let $\mathbf{n}$ and $\mathbf{n}'$ be unit vectors parallel to the plane of D, and $\mathbf{k}$ a unit vector perpendicular to D, all three unit vectors being fixed in D, as shown in Fig. 2 2.1. ($\mathbf{k}$ is fixed also in C.) Then

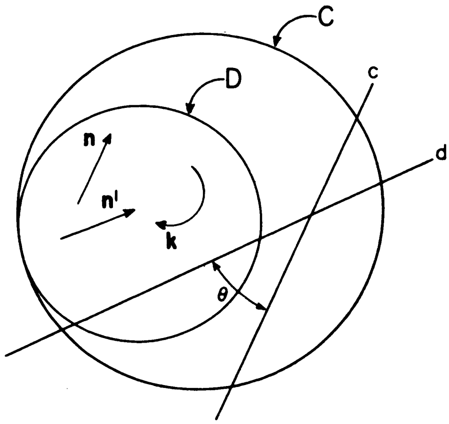


FIG. 2.2.1

$$
{}^C\boldsymbol{\omega}^D \underset{(2.1.1)}{=} \frac{\dfrac{{}^C d\mathbf{n}}{dt} \times \dfrac{{}^C d\mathbf{n}'}{dt}}{\dfrac{{}^C d\mathbf{n}}{dt} \cdot \mathbf{n}'}
$$

Now

$$
\frac{{}^C d\mathbf{n}}{dt} \underset{(1.7.1)}{=} \mathbf{k} \times \mathbf{n}\frac{d\theta}{dt}, \qquad \frac{{}^C d\mathbf{n}'}{dt} \underset{(1.7.1)}{=} \mathbf{k} \times \mathbf{n}'\frac{d\theta}{dt}
$$

Hence

$$
{}^C\boldsymbol{\omega}^D = \frac{(\mathbf{k} \times \mathbf{n}) \times (\mathbf{k} \times \mathbf{n}')}{(\mathbf{k} \times \mathbf{n}) \cdot \mathbf{n}'}\frac{d\theta}{dt} = \frac{(\mathbf{k} \times \mathbf{n}) \cdot \mathbf{n}'\mathbf{k}}{(\mathbf{k} \times \mathbf{n}) \cdot \mathbf{n}'}\frac{d\theta}{dt}
$$

or

$$
{}^C\boldsymbol{\omega}^D = \mathbf{k}\frac{d\theta}{dt}
$$

Thus

$$
{}^C\boldsymbol{\omega}^D|_{t^*} = -5\mathbf{k} \text{ rad sec}^{-1}
$$

2.2.2 The angular velocity ${}^{R'}\boldsymbol{\omega}^{R}$ (see 2.2.1) can always be expressed in the form

$${}^{R'}\boldsymbol{\omega}^{R} = {}^{R'}\omega^{R}\,\mathbf{n}_o$$

where $\mathbf{n}_o$ is a unit vector parallel to ${}^{R'}\boldsymbol{\omega}^{R}$ and ${}^{R'}\omega^{R}$ is a scalar, called the *angular speed* of R in R' for the direction $\mathbf{n}_o$, which is positive when ${}^{R'}\boldsymbol{\omega}^{R}$ and $\mathbf{n}_o$ have the same sense, vanishes when ${}^{R'}\boldsymbol{\omega}^{R}$ is equal to zero, and is negative when the sense of ${}^{R'}\boldsymbol{\omega}^{R}$ is opposite to that of $\mathbf{n}_o$.

Problem: Referring to Problem 1.7.3, determine the angular speed of D in C at time t^* for (a) the $\mathbf{k}$ direction and (b) the $-\mathbf{k}$ direction, $\mathbf{k}$ being the unit vector shown in Fig. 1.7.3a.

Solution: In Problem 2.2.1, the angular velocity ${}^{C}\boldsymbol{\omega}^{D}$ was found to be

$${}^{C}\boldsymbol{\omega}^{D}\big|_{t^*} = -5\mathbf{k} \text{ rad sec}^{-1}$$

which can, alternatively, be expressed as

$${}^{C}\boldsymbol{\omega}^{D}\big|_{t^*} = 5(-\mathbf{k}) \text{ rad sec}^{-1}$$

Result (a): -5 rad sec^{-1}.

Result (b): 5 rad sec^{-1}.

2.2.3 Like other vectors, angular velocities are sometimes depicted most conveniently by means of straight or curved arrows accompanied by measure numbers (see Vol. I, Secs. 1.2.1 and 1.7.5). The angular velocity in question is then regarded as having the direction indicated by the arrow if the measure number is positive, and the opposite direction if it is negative. For example, Fig. 2.2.3a shows four representations of the same angular velocity.

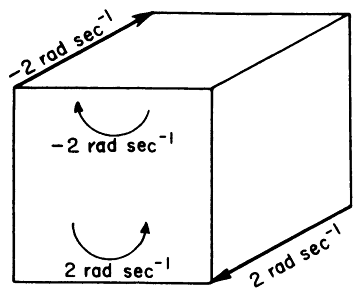


FIG. 2.2.3a

Problem: Referring to Problem 2.2.1, draw a sketch of D showing two representations of ${}^C\boldsymbol{\omega}^D$ for time t^*.

Solution: See Fig. 2.2.3b.

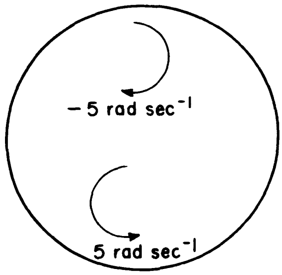


FIG. 2.2.3b

2.2.4 When a rigid body R moves in a reference frame R' in such a way that there exists a unit vector $\mathbf{k}$ whose orientation in both R and R' is independent of time t, R is said to have an *angular velocity of fixed orientation* in R', and this angular velocity is given by

$$ {}^{R'}\boldsymbol{\omega}^{R} = \frac{d\theta}{dt}\mathbf{k} \tag{1}$$

where θ is the angular displacement of a line L fixed in R, relative to a line L' fixed in R', both lines being perpendicular to $\mathbf{k}$, and θ being regarded as positive when the displacement is generated by a $\mathbf{k}$ rotation of L relative to L' (see Fig. 2.2.4). Furthermore,

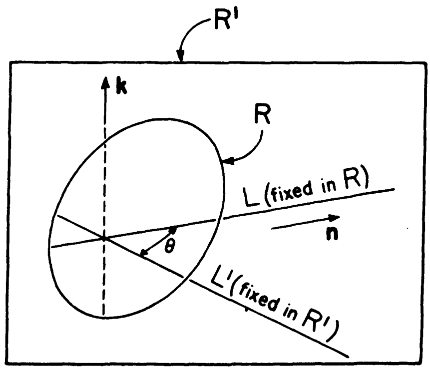


FIG. 2.2.4

$$^{R'}\omega^R = \frac{d\theta}{dt} \tag{2}$$

where $^{R'}\omega^R$ is the angular speed of R in R' for the $\mathbf{k}$ direction (see 2.2.2).

Proof: Let $\mathbf{n}$ be a unit vector parallel to L. Then $\mathbf{n}$ and $\mathbf{k} \times \mathbf{n}$ are vectors fixed in R and (see 2.2.1 and 2.1.1)

$$^{R'}\boldsymbol{\omega}^R = \frac{\dfrac{^{R'}d\mathbf{n}}{dt} \times \dfrac{^{R'}d}{dt}(\mathbf{k} \times \mathbf{n})}{\dfrac{^{R'}d\mathbf{n}}{dt} \cdot (\mathbf{k} \times \mathbf{n})}$$

Now

$$\frac{^{R'}d\mathbf{n}}{dt} \underset{(1.7.1)}{=} \mathbf{k} \times \mathbf{n}\,\frac{d\theta}{dt}$$

and

$$\frac{^{R'}d}{dt}(\mathbf{k} \times \mathbf{n}) \underset{(1.5.3)}{=} \mathbf{k} \times \frac{^{R'}d\mathbf{n}}{dt} = \mathbf{k} \times (\mathbf{k} \times \mathbf{n})\frac{d\theta}{dt} = -\mathbf{n}\,\frac{d\theta}{dt}$$

Hence

$$\frac{^{R'}d\mathbf{n}}{dt} \times \frac{^{R'}d}{dt}(\mathbf{k} \times \mathbf{n}) = -(\mathbf{k} \times \mathbf{n}) \times \mathbf{n}\left(\frac{d\theta}{dt}\right)^2 = \mathbf{k}\left(\frac{d\theta}{dt}\right)^2$$

while

$$\frac{^{R'}d\mathbf{n}}{dt} \cdot (\mathbf{k} \times \mathbf{n}) = (\mathbf{k} \times \mathbf{n}) \cdot (\mathbf{k} \times \mathbf{n})\frac{d\theta}{dt} = \frac{d\theta}{dt}$$

Thus

$$^{R'}\boldsymbol{\omega}^R = \frac{d\theta}{dt}\mathbf{k}$$

It follows from Sec. 2.2.2 that $^{R'}\omega^R = d\theta/dt$.

2.2.5 In descriptions of motions of certain frequently encountered systems it is common practice to omit a number of qualifying phrases used to specify reference frames, axes of rotation, directions, etc. For example, one speaks of "a propeller rotating at 5000 rpm," not stating explicitly that one is referring to rotation of the propeller about an axis fixed in the aircraft to which the propeller is attached and that 5000 rpm is an angular speed of the propeller in the aircraft (see 2.2.2) for a direction parallel to the propeller shaft axis.

Problem: The driver D of a Geneva Stop Mechanism rotates counterclockwise at 3 rpm, thereby causing intermittent rotation of the follower F (see Fig. 2.2.5a).

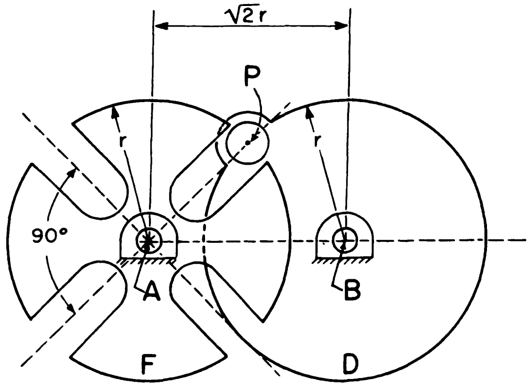


FIG. 2.2.5a

Determine the angular speed of the follower at the instant when point P crosses line AB (between A and B).

Solution: It is implied that F and D rotate about axes which are normal to the middle planes of the discs, pass through points A and B, and are fixed in a reference frame R in which line AB is fixed. In accordance with Sec. 2.2.4, both D and F thus have angular velocities of fixed orientation, and these are given by

$$^R\omega^D = -\frac{d\theta}{dt}\mathbf{k} \tag{A}$$

and

$$^R\omega^F = \frac{d\phi}{dt}\mathbf{k} \tag{B}$$

where $\mathbf{k}$ is a unit vector normal to the plane of the discs, and ϕ and θ are the angular displacements of AP and BP relative to AB, both being regarded as positive when the system is in the configuration indicated in Fig. 2.2.5b. Furthermore, as θ decreases algebraically during counterclockwise rotation of D,

$$\frac{d\theta}{dt} = -3 \times 2\pi = -6\pi \text{ rad min}^{-1} \tag{C}$$

and $d\phi/dt$ (or $-d\phi/dt$) is the angular speed (see 2.2.2) to be determined.

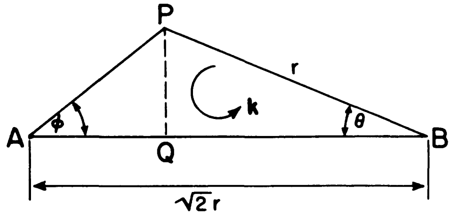


FIG. 2.2.5b

Express ϕ in terms of θ: From Fig. 2.2.5b,

$$\overline{PQ} = r \sin \theta$$

and

$$\overline{PQ} \operatorname{cotan} \phi + r \cos \theta = r\sqrt{2}$$

Hence

$$\sin \theta \operatorname{cotan} \phi + \cos \theta = \sqrt{2}$$

so that

$$\phi = \operatorname{arc\,cotan} \left(\frac{\sqrt{2} - \cos \theta}{\sin \theta} \right)$$

and

$$\frac{d\phi}{dt} = \frac{\sqrt{2} \cos \theta - 1}{3 - 2\sqrt{2} \cos \theta} \frac{d\theta}{dt} \tag{D}$$

When P crosses line AB,

$$\theta = 0$$

Thus, using Eq. (C),

$$\left. \frac{d\phi}{dt} \right|_{\theta=0} = 6\pi \frac{1 - \sqrt{2}}{3 - 2\sqrt{2}} = -45.5 \text{ rad min}^{-1}$$

(The minus sign means that the follower is rotating clockwise at the instant in question, as may be seen either by substitution into Eq. (B) or by noting that an algèbraic decrease in ϕ corresponds to clockwise rotation of line AP relative to line AB.)

2.2.6 The theorem stated in Sec. 2.2.4 applies not only to motions of rotation about axes fixed both in a body and in a reference frame (as in Problem 2.2.5), but also to motions of a body in a reference frame in which no point of the body remains fixed.

Problem: Referring to Problem 1.7.3, determine the angular velocity ${}^C\boldsymbol{\omega}^D$ of the disc D in the track C for the instant t^*.

Solution: Let $\mathbf{k}$ be a unit vector fixed in both C and D, as shown in Fig. 1.7.3a. Then

$$ {}^C\boldsymbol{\omega}^D \underset{(2.2.4)}{=} \frac{d\theta}{dt}\mathbf{k} $$

and

$$ {}^C\boldsymbol{\omega}^D\Big|_{t^*} = \frac{d\theta}{dt}\Big|_{t^*}\mathbf{k} = -5\mathbf{k} \text{ rad sec}^{-1} $$

(Compare this solution with that of Problem 2.2.1.)

2.2.7 Given n reference frames R_i, $i = 1, \ldots, n$, the angular velocity ${}^{R_n}\boldsymbol{\omega}^R$ of a rigid body R in reference frame R_n can be expressed as

$$ {}^{R_n}\boldsymbol{\omega}^R = {}^{R_1}\boldsymbol{\omega}^R + {}^{R_2}\boldsymbol{\omega}^{R_1} + \cdots + {}^{R_n}\boldsymbol{\omega}^{R_{n-1}} $$

Proof: Let $\mathbf{c}$ be any vector fixed in R. Then

$$ \frac{{}^{R_i}d\mathbf{c}}{dt} \underset{(2.1.1)}{=} {}^{R_i}\boldsymbol{\omega}^R \times \mathbf{c} $$

$$ \frac{{}^{R_{i-1}}d\mathbf{c}}{dt} \underset{(2.1.1)}{=} {}^{R_{i-1}}\boldsymbol{\omega}^R \times \mathbf{c} $$

$$ \frac{{}^{R_i}d\mathbf{c}}{dt} \underset{(2.1.4)}{=} \frac{{}^{R_{i-1}}d\mathbf{c}}{dt} + {}^{R_i}\boldsymbol{\omega}^{R_{i-1}} \times \mathbf{c} $$

Hence

$$ {}^{R_i}\boldsymbol{\omega}^R \times \mathbf{c} = {}^{R_{i-1}}\boldsymbol{\omega}^R \times \mathbf{c} + {}^{R_i}\boldsymbol{\omega}^{R_{i-1}} \times \mathbf{c} $$

As this equation is satisfied for *all* $\mathbf{c}$ fixed in R,

$$ {}^{R_i}\boldsymbol{\omega}^R = {}^{R_{i-1}}\boldsymbol{\omega}^R + {}^{R_i}\boldsymbol{\omega}^{R_{i-1}} \tag{A} $$

With $i = n$, Eq. (A) gives

$$ {}^{R_n}\boldsymbol{\omega}^R = {}^{R_{n-1}}\boldsymbol{\omega}^R + {}^{R_n}\boldsymbol{\omega}^{R_{n-1}} \tag{B} $$

With $i = n - 1$, Eq. (A) gives

$$ {}^{R_{n-1}}\boldsymbol{\omega}^R = {}^{R_{n-2}}\boldsymbol{\omega}^R + {}^{R_{n-1}}\boldsymbol{\omega}^{R_{n-2}} $$

Substitute this into Eq. (B):

$$ {}^{R_n}\boldsymbol{\omega}^R = {}^{R_{n-2}}\boldsymbol{\omega}^R + {}^{R_{n-1}}\boldsymbol{\omega}^{R_{n-2}} + {}^{R_n}\boldsymbol{\omega}^{R_{n-1}} $$

Next, use Eq. (A) with $i = n - 2$, then with $i = n - 3$, and so forth.

Problem: Fig. 2.2.7a represents a portion of a helicopter rotor blade system. This portion consists of a blade B, attached at the

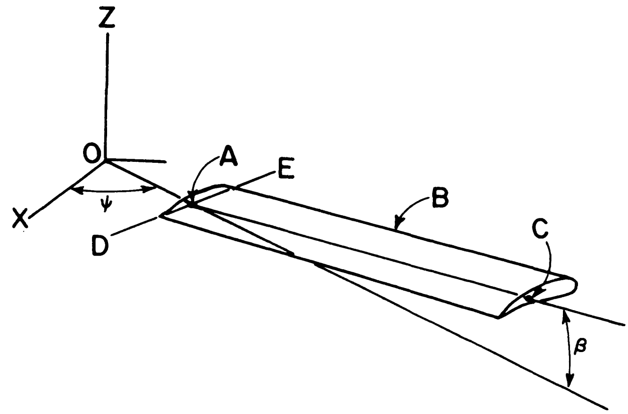


FIG. 2.2.7a

flapping hinge A to an arm OA which is connected to the hub O. During flight, OA remains perpendicular to the rotor shaft axis OZ, which is fixed in the helicopter. The angle (measured by ψ and called the azimuth angle) between OA and a line OX which is fixed in the helicopter and perpendicular to OZ varies, as does the angle (measured by β and called the flapping angle) between the blade axis AC and line OA (extended), the blade rotating about an axis DE (called the flapping-hinge axis) passing through A and perpendicular to both OA and OZ.

Consider a flight during which the orientation of the helicopter relative to the earth remains fixed, the rotor speed $|d\psi/dt|$ remains constant, and the flapping angle is given by

$$\beta = \beta_0 + 0.2 \sin \psi \text{ rad}$$

where β_0 is a constant (called the coning angle). Determine (a) the ratio r of the magnitude of the angular velocity of the blade to the rotor speed, and (b) the (smallest) angle α between the angular velocity of the blade and the rotor shaft axis OZ, both for an instant at which ψ is equal to π rad.

Solution: It is implied that the angular velocity in question is the angular velocity of the blade in a reference frame whose orientation relative to the earth, and hence relative to the helicopter, is fixed; that ψ is to be regarded as positive when the angular displacement of OA relative to OX is generated by a counterclockwise rotation of OA relative to OX, as seen by an observer looking from Z toward O; and that β is to be regarded as positive when the angular displacement of AC relative to the extension of OA is generated by a counterclockwise rotation of AC relative to OA, as seen by an observer looking from D toward E.

Introduce the following (see Fig. 2.2.7b):

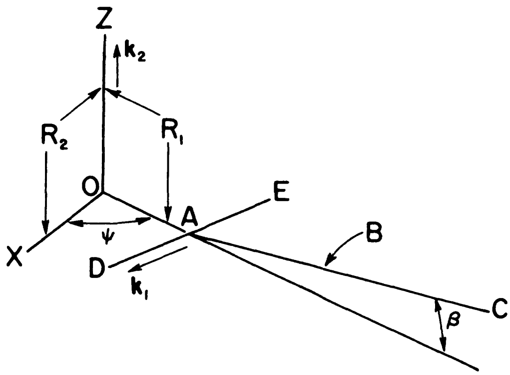


FIG. 2.2.7b

- R_1 a reference frame in which the orientations of OA and OZ are fixed
- R_2 a reference frame in which the orientations of OX and OZ are fixed (and whose orientation relative to the earth is, therefore, fixed)
- $\mathbf{k}_1$ a unit vector parallel to AD (and thus fixed in the blade B and in R_1)
- $\mathbf{k}_2$ a unit vector parallel to OZ (and thus fixed in R_1 and in R_2)

Then

$$
\begin{aligned}
{}^{R_2}\boldsymbol{\omega}^{B} &= {}^{R_1}\boldsymbol{\omega}^{B} + {}^{R_2}\boldsymbol{\omega}^{R_1} \\
&\underset{(2.2.4)}{=} \frac{d\beta}{dt}\,\mathbf{k}_1 + \frac{d\psi}{dt}\,\mathbf{k}_2
\end{aligned}
$$

and

$$\frac{d\beta}{dt} = 0.2 \cos \psi \frac{d\psi}{dt}$$

Hence

$${}^{R_2}\boldsymbol{\omega}^{B} = (0.2 \cos \psi\, \mathbf{k}_1 + \mathbf{k}_2) \frac{d\psi}{dt}$$

and the magnitude of ${}^{R_2}\boldsymbol{\omega}^{B}$ is given by

$$|{}^{R_2}\boldsymbol{\omega}^{B}| = (0.04 \cos^2 \psi + 1)^{1/2} \left|\frac{d\psi}{dt}\right|$$

while the angle α between ${}^{R_2}\boldsymbol{\omega}^{B}$ and line OZ has the value

$$\alpha = \text{arc cos} \left(\frac{\mathbf{k}_2 \cdot {}^{R_2}\boldsymbol{\omega}^{B}}{|{}^{R_2}\boldsymbol{\omega}^{B}|} \right)$$

Hence

$$r = \frac{|{}^{R_2}\boldsymbol{\omega}^{B}|}{\left|\dfrac{d\psi}{dt}\right|} = (0.04 \cos^2 \psi + 1)^{1/2}$$

and

$$\alpha = \text{arc cos}\,(0.04 \cos^2 \psi + 1)^{-1/2}$$

For $\psi = \pi$,

$$r = 1.02, \qquad \alpha = 11.4 \text{ deg}$$

2.2.8 When the angular velocity of a body B in a reference frame R is resolved into components (that is, is expressed as the sum of a number of vectors), these components may always be regarded as angular velocities of certain bodies in certain reference frames (see 2.2.7). In no case, however, are they angular velocities of B in R, for the angular velocity of B in R is unique; that is, B cannot possess simultaneously several angular velocities in R. This follows from the definition of angular velocity (see 2.2.1).

2.2.9 The theorem stated in Sec. 2.2.7 is particularly useful in the analysis of kinematic chains (several rigid bodies connected to each other).

Problem: Fig. 2.2.9a represents schematically a device known as "Hooke's joint," described as follows: Two shafts, S and S', are

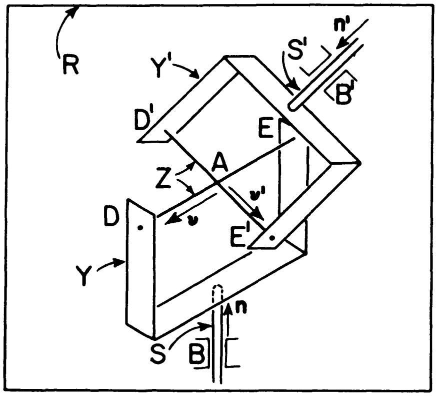


FIG. 2.2.9a

mounted in bearings, B and B', fixed in a reference frame R, the axes of the shafts being parallel to unit vectors $\mathbf{n}$ and $\mathbf{n}'$ and intersecting at a point A. Each shaft terminates in a "yoke," and these yokes, Y and Y', are connected to each other by a rigid cross Z, one of whose arms is supported by bearings D and E fixed in Y, the other by bearings D' and E' fixed in Y'. The arms of Z have equal lengths, form a right angle with each other, and are respectively perpendicular to $\mathbf{n}$ and $\mathbf{n}'$.

Express the ratio of ${}^R\omega^S$ to ${}^R\omega^{S'}$ (the angular speeds of S and S' in R for the directions $\mathbf{n}$ and $\mathbf{n}'$) in terms of $\mathbf{n}$, $\mathbf{n}'$, $\boldsymbol{\nu}$, and $\boldsymbol{\nu}'$, where $\boldsymbol{\nu}$ and $\boldsymbol{\nu}'$ are unit vectors parallel to the arms of Z, as shown in Fig. 2.2.9a.

Solution:

$$
{}^R\boldsymbol{\omega}^S \underset{(2.2.7)}{=} {}^Z\boldsymbol{\omega}^S + {}^{S'}\boldsymbol{\omega}^Z + {}^R\boldsymbol{\omega}^{S'}
$$

From Sec. 2.2.2 and 2.2.4,

$$
{}^R\boldsymbol{\omega}^S = {}^R\omega^S\mathbf{n}, \qquad {}^Z\boldsymbol{\omega}^S = {}^Z\omega^S\boldsymbol{\nu}
$$

$$
{}^{S'}\boldsymbol{\omega}^Z = {}^{S'}\omega^Z\boldsymbol{\nu}', \qquad {}^R\boldsymbol{\omega}^{S'} = {}^R\omega^{S'}\mathbf{n}'
$$

Thus

$$
{}^R\omega^S\mathbf{n} = {}^Z\omega^S\boldsymbol{\nu} + {}^{S'}\omega^Z\boldsymbol{\nu}' + {}^R\omega^{S'}\mathbf{n}'
$$

To eliminate ${}^{Z}\omega^{S}$ and ${}^{S'}\omega^{Z}$, dot-multiply this equation with $\boldsymbol{\nu} \times \boldsymbol{\nu}'$. This gives

$$ {}^{R}\omega^{S}[\mathbf{n}, \boldsymbol{\nu}, \boldsymbol{\nu}'] = {}^{R}\omega^{S'}[\mathbf{n}', \boldsymbol{\nu}, \boldsymbol{\nu}'] $$

so that

$$ \frac{{}^{R}\omega^{S}}{{}^{R}\omega^{S'}} = \frac{[\mathbf{n}', \boldsymbol{\nu}, \boldsymbol{\nu}']}{[\mathbf{n}, \boldsymbol{\nu}, \boldsymbol{\nu}']} \tag{A} $$

To express this relationship in terms of the "angularity" α of the joint and the "angular position" θ of one of the shafts (see Fig. 2.2.9b), note that $\boldsymbol{\nu}'$ can be expressed as

$$ \boldsymbol{\nu}' = \lambda \boldsymbol{\nu} \times \mathbf{n}' $$

where λ is a scalar, because $\boldsymbol{\nu}'$ is perpendicular to both $\boldsymbol{\nu}$ and $\mathbf{n}'$; substitute into Eq. (A) and carry out the indicated operations after resolving $\mathbf{n}$, $\mathbf{n}'$, and $\boldsymbol{\nu}$ into components parallel to the mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ shown in Fig. 2.2.9b. This gives

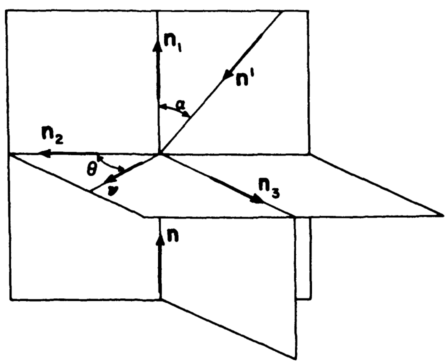


FIG. 2.2.9b

$$ \frac{{}^{R}\omega^{S}}{{}^{R}\omega^{S'}} = \frac{\sin^2 \alpha \cos^2 \theta - 1}{\cos \alpha} $$

2.2.10 The reference frames mentioned in Sec. 2.2.7 need not be fixed in actual bodies comprising a kinematic chain. Frequently these reference frames are introduced as an aid in analysis, but have no physical counterparts.

Example: In Fig. 2.2.10a, D represents a sharp-edged circular disc which moves in such a way that one (but not always the same)

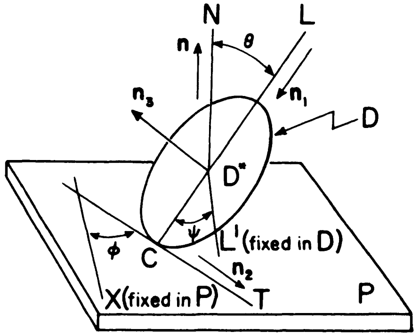


FIG. 2.2.10a

point C of its periphery is at every instant in contact with a plane P. $\mathbf{n}$, $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are unit vectors, $\mathbf{n}$ being parallel to the normal N to plane P, $\mathbf{n}_1$ parallel to the line L passing through the center D^* of D and the point of contact C of D and P, $\mathbf{n}_2$ parallel to the tangent T to D at C (T lies in plane P), and $\mathbf{n}_3$ perpendicular to D and hence to $\mathbf{n}_1$ and $\mathbf{n}_2$. X is a line fixed in P and L' a line fixed in D.

The angular velocity ${}^P\boldsymbol{\omega}^D$ of the disc in a reference frame (again called P) in which P is fixed is to be expressed in terms of θ, ϕ, and

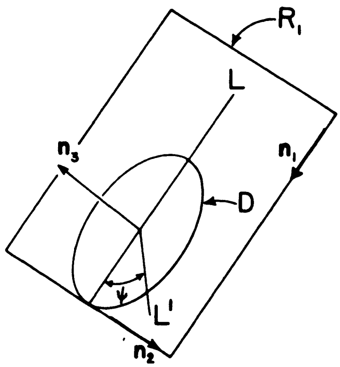


FIG. 2.2.10b

ψ (and their time-derivatives), these being the angles between N and L, X and T, and L and L', respectively.

Let R_1 be a reference frame in which $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are fixed (see Fig. 2.2.10b). Then, as $\mathbf{n}_3$ is fixed in both D and R_1,

$$^{R_1}\boldsymbol{\omega}^D \underset{(2.2.4)}{=} \dot{\psi}\mathbf{n}_3$$

Next, let R_2 be a reference frame in which $\mathbf{n}$ and $\mathbf{n}_2$ are fixed (see Fig. 2.2.10c). As $\mathbf{n}_2$ is fixed in both R_1 and R_2,

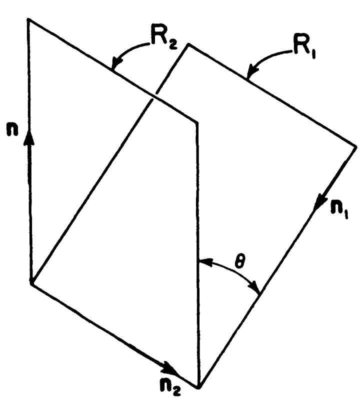


FIG. 2.2.10c

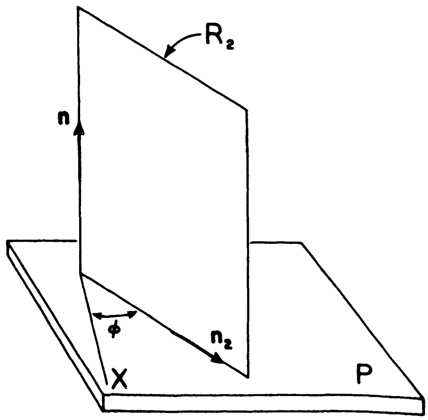


FIG. 2.2.10d

$$^{R_2}\boldsymbol{\omega}^{R_1} \underset{(2.2.4)}{=} -\dot{\theta}\mathbf{n}_2$$

Finally, $\mathbf{n}$ being fixed in both R_2 and P (see Fig. 2.2.10d),

$$^{P}\boldsymbol{\omega}^{R_2} \underset{(2.2.4)}{=} \dot{\phi}\mathbf{n} \underset{(\text{F2.2.10a})}{=} \dot{\phi}(-\cos\theta\,\mathbf{n}_1 + \sin\theta\,\mathbf{n}_3)$$

Now

$$^{P}\boldsymbol{\omega}^D \underset{(2.2.7)}{=} {}^{R_1}\boldsymbol{\omega}^D + {}^{R_2}\boldsymbol{\omega}^{R_1} + {}^{P}\boldsymbol{\omega}^{R_2}$$

Hence

$$^{P}\boldsymbol{\omega}^D = -\dot{\phi}\cos\theta\,\mathbf{n}_1 - \dot{\theta}\mathbf{n}_2 + (\dot{\psi} + \dot{\phi}\sin\theta)\,\mathbf{n}_3$$

2.3 Angular acceleration

2.3.1 The first time-derivative in reference frame R' of the angular velocity $^{R'}\boldsymbol{\omega}^R$ of a rigid body R in R' (see 2.2.1) is called the *angular acceleration* of R in R' and is denoted by $^{R'}\boldsymbol{\alpha}^R$:

$$^{R'}\boldsymbol{\alpha}^R = \frac{^{R'}d}{dt}\, ^{R'}\boldsymbol{\omega}^R$$

Problem: Referring to Problem 2.2.7, determine the angular acceleration of the blade.

Solution: It is implied that the angular acceleration in question is the angular acceleration of the blade in a reference frame whose orientation relative to the earth is fixed. Using the notation introduced in the solution of Problem 2.2.7, this angular acceleration is denoted by $^{R_2}\boldsymbol{\alpha}^B$ and is given by

$$^{R_2}\boldsymbol{\alpha}^B = \frac{^{R_2}d}{dt}\, ^{R_2}\boldsymbol{\omega}^B = \frac{^{R_2}d}{dt}\left[(0.2\cos\psi\,\mathbf{k}_1 + \mathbf{k}_2)\frac{d\psi}{dt}\right]$$

$$\underset{(1.5.1)}{=} \left[\frac{^{R_2}d}{dt}(0.2\cos\psi\,\mathbf{k}_1 + \mathbf{k}_2)\right]\frac{d\psi}{dt} + (0.2\cos\psi\,\mathbf{k}_1 + \mathbf{k}_2)\frac{d^2\psi}{dt^2}$$

or, as

$$\frac{d^2\psi}{dt^2} = 0$$

by

$$^{R_2}\boldsymbol{\alpha}^B \underset{(1.5.1)}{=} \left(-0.2\sin\psi\frac{d\psi}{dt}\mathbf{k}_1 + 0.2\cos\psi\frac{^{R_2}d\mathbf{k}_1}{dt}\right)\frac{d\psi}{dt}$$

$$= \left(-0.2\sin\psi\frac{d\psi}{dt}\mathbf{k}_1 + 0.2\cos\psi\,\mathbf{k}_2 \underset{(1.7.1)}{\times} \mathbf{k}_1\frac{d\psi}{dt}\right)\frac{d\psi}{dt}$$

$$= -0.2\left(\frac{d\psi}{dt}\right)^2(\mathbf{k}_1\sin\psi + \mathbf{k}_1 \times \mathbf{k}_2\cos\psi)$$

2.3.2 Both the angular velocity $^{R'}\boldsymbol{\omega}^R$ of a rigid body R in a reference frame R' (see 2.2.1) and the angular acceleration $^{R'}\boldsymbol{\alpha}^R$ of R in R' (see 2.3.1) can always be resolved into components respectively parallel to unit vectors $\mathbf{n}_i^*$, $i = 1, 2, 3$, fixed in *any* reference frame R^*. When this is done, the three measure numbers of the components of $^{R'}\boldsymbol{\alpha}^R$ are equal to the time-derivatives of the measure numbers of the corresponding components of $^{R'}\boldsymbol{\omega}^R$ if and only if $^{R'}\boldsymbol{\omega}^{R^*} \times {}^{R'}\boldsymbol{\omega}^R$ is equal to zero. In particular, this is the case when $\mathbf{n}_i^*$, $i = 1, 2, 3$, are fixed either in R or in R'.

Proof: $^{R'}\boldsymbol{\omega}^R$ and $^{R'}\boldsymbol{\alpha}^R$, resolved into components respectively parallel to $\mathbf{n}_i^*$, $i = 1, 2, 3$, are given by

$$^{R'}\boldsymbol{\omega}^{R} = \sum_{i=1}^{3} {^{R'}\omega_i}^{R}\mathbf{n}_i{}^{*}$$

$$^{R'}\boldsymbol{\alpha}^{R} = \sum_{i=1}^{3} {^{R'}\alpha_i}^{R}\mathbf{n}_i{}^{*}$$

Now

$$^{R'}\boldsymbol{\alpha}^{R} \underset{(2.3.1)}{=} \frac{^{R'}d}{dt}\, {^{R'}\boldsymbol{\omega}^{R}} \underset{(2.1.4)}{=} \frac{^{R^*}d}{dt}\, {^{R'}\boldsymbol{\omega}^{R}} + {^{R'}\boldsymbol{\omega}^{R^*}} \times {^{R'}\boldsymbol{\omega}^{R}}$$

Hence

$$\sum_{i=1}^{3} {^{R'}\alpha_i}^{R}\mathbf{n}_i{}^{*} = \frac{^{R^*}d}{dt} \sum_{i=1}^{3} {^{R'}\omega_i}^{R}\mathbf{n}_i{}^{*} + {^{R'}\boldsymbol{\omega}^{R^*}} \times {^{R'}\boldsymbol{\omega}^{R}}$$

or

$$\sum_{i=1}^{3} {^{R'}\alpha_i}^{R}\mathbf{n}_i{}^{*} \underset{(1.2.1)}{=} \sum_{i=1}^{3} \frac{d\,{^{R'}\omega_i}^{R}}{dt}\,\mathbf{n}_i{}^{*} + {^{R'}\boldsymbol{\omega}^{R^*}} \times {^{R'}\boldsymbol{\omega}^{R}} \qquad \text{(A)}$$

Thus, if

$$^{R'}\boldsymbol{\omega}^{R^*} \times {^{R'}\boldsymbol{\omega}^{R}} = 0$$

it follows that

$$^{R'}\alpha_i{}^{R} = \frac{d\,{^{R'}\omega_i}^{R}}{dt}, \qquad i = 1, 2, 3 \qquad \text{(B)}$$

If $^{R'}\boldsymbol{\omega}^{R^*} \times {^{R'}\boldsymbol{\omega}^{R}}$ is not equal to zero, it can be expressed in the form

$$^{R'}\boldsymbol{\omega}^{R^*} \times {^{R'}\boldsymbol{\omega}^{R}} = \sum_{i=1}^{3} v_i\mathbf{n}_i{}^{*}$$

where at least one of the scalars v_i, $i = 1, 2, 3$, is not equal to zero. The three scalar equations corresponding to Eq. (A) are then

$$^{R'}\alpha_i{}^{R} = \frac{d\,{^{R'}\omega_i}^{R}}{dt} + v_i, \qquad i = 1, 2, 3$$

and at least one of Eqs. (B) cannot be satisfied.

Example: Referring to Example 2.2.10, the derivatives of the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of $^{P}\boldsymbol{\omega}^{D}$ are

$$\frac{d}{dt}(-\dot{\phi}\cos\theta) = -\ddot{\phi}\cos\theta + \dot{\phi}\dot{\theta}\sin\theta$$

$$\frac{d}{dt}(-\dot{\theta}) = -\ddot{\theta}$$

$$\frac{d}{dt}(\dot{\psi} + \dot{\phi}\sin\theta) = \ddot{\psi} + \ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta$$

The angular acceleration ${}^P\boldsymbol{\alpha}^D$ is given by

$$\begin{aligned}
{}^P\boldsymbol{\alpha}^D \underset{(2.3.1)}{=} \frac{{}^Pd}{dt}\,{}^P\boldsymbol{\omega}^D &\underset{(2.1.4)}{=} \frac{{}^{R_1}d}{dt}\,{}^P\boldsymbol{\omega}^D + {}^P\boldsymbol{\omega}^{R_1} \times {}^P\boldsymbol{\omega}^D \\
&= \frac{{}^{R_1}d}{dt}\,{}^P\boldsymbol{\omega}^D + ({}^{R_2}\boldsymbol{\omega}^{R_1} \underset{(2.2.7)}{+} {}^P\boldsymbol{\omega}^{R_2}) \times {}^P\boldsymbol{\omega}^D \\
&= \mathbf{n}_1\frac{d}{dt}(-\dot{\phi}\cos\theta) + \mathbf{n}_2\frac{d}{dt}(-\dot{\theta}) + \mathbf{n}_3\frac{d}{dt}(\dot{\psi} + \dot{\phi}\sin\theta) \\
&\qquad + [-\dot{\theta}\mathbf{n}_2 + \dot{\phi}(-\cos\theta\mathbf{n}_1 + \sin\theta\mathbf{n}_3)] \times {}^P\boldsymbol{\omega}^D
\end{aligned}$$

That is,

$$\begin{aligned}
{}^P\boldsymbol{\alpha}^D = (-\ddot{\phi}\cos\theta + \dot{\phi}\dot{\theta}\sin\theta - \dot{\theta}\dot{\psi})\mathbf{n}_1 \\
+ (-\ddot{\theta} + \dot{\phi}\dot{\psi}\cos\theta)\mathbf{n}_2 + (\ddot{\psi} + \ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta)\mathbf{n}_3
\end{aligned}$$

and only the $\mathbf{n}_3$ measure number of ${}^P\boldsymbol{\alpha}^D$ is equal to the time-derivative of the corresponding measure number of ${}^P\boldsymbol{\omega}^D$.

2.3.3 ${}^{R'}\boldsymbol{\alpha}^R$ and ${}^R\boldsymbol{\alpha}^{R'}$ are related as follows:

$${}^{R'}\boldsymbol{\alpha}^R = -{}^R\boldsymbol{\alpha}^{R'}$$

Proof:

$${}^{R'}\boldsymbol{\alpha}^R \underset{(2.3.1)}{=} \frac{{}^{R'}d}{dt}\,{}^{R'}\boldsymbol{\omega}^R \underset{(2.1.5)}{=} \frac{{}^Rd}{dt}\,{}^{R'}\boldsymbol{\omega}^R \underset{(2.1.6)}{=} -\frac{{}^Rd\,{}^R\boldsymbol{\omega}^{R'}}{dt} \underset{(2.3.1)}{=} -{}^R\boldsymbol{\alpha}^{R'}$$

2.3.4 The angular acceleration ${}^{R'}\boldsymbol{\alpha}^R$ can always be expressed in the form

$${}^{R'}\boldsymbol{\alpha}^R = {}^{R'}\alpha^R\mathbf{n}_a$$

where $\mathbf{n}_a$ is a unit vector parallel to ${}^{R'}\boldsymbol{\alpha}^R$, and ${}^{R'}\alpha^R$ is a scalar, called the *scalar angular acceleration* of R in R' for the direction $\mathbf{n}_a$, which is positive when ${}^{R'}\boldsymbol{\alpha}^R$ and $\mathbf{n}_a$ have the same sense, vanishes when ${}^{R'}\boldsymbol{\alpha}^R$ is equal to zero, and is negative when the sense of ${}^{R'}\boldsymbol{\alpha}^R$ is opposite to that of $\mathbf{n}_a$.

Problem: Referring to Problem 2.2.5, determine the scalar angular acceleration of F for the $\mathbf{k}$ direction (see Fig. 2.2.5b).

Solution: From Eqs. (B), (C), and (D) of the solution of Problem 2.2.5, the angular velocity of F in R is given by

$${}^R\boldsymbol{\omega}^F = 6\pi\,\frac{1 - \sqrt{2}\cos\theta}{3 - 2\sqrt{2}\cos\theta}\,\mathbf{k} \text{ rad min}^{-1}$$

Let ${}^R\boldsymbol{\alpha}^F$ be the angular acceleration of F in R. Then ${}^R\boldsymbol{\alpha}^F$ is given by

$$ {}^R\boldsymbol{\alpha}^F \underset{(2.3.1)}{=} \frac{{}^Rd}{dt}\,{}^R\boldsymbol{\omega}^F = 6\pi \frac{\sqrt{2}\sin\theta}{(3 - 2\sqrt{2}\cos\theta)^2}\frac{d\theta}{dt}\,\mathbf{k} $$

or, as

$$ \frac{d\theta}{dt} = -6\pi \text{ rad min}^{-1} $$

by

$$ {}^R\boldsymbol{\alpha}^F = -36\pi^2 \frac{\sqrt{2}\sin\theta}{(3 - 2\sqrt{2}\cos\theta)^2}\,\mathbf{k} \text{ rad min}^{-2} $$

Thus the scalar angular acceleration of F for the $\mathbf{k}$ direction is equal to

$$ -36\pi^2 \frac{\sqrt{2}\sin\theta}{(3 - 2\sqrt{2}\cos\theta)^2} $$

and is negative when P is above line AB, positive when P is below line AB.

2.3.5 Referring to Secs. 2.2.2 and 2.3.4, note that, in general,

$$ {}^{R'}\alpha^R \neq \frac{d\,{}^{R'}\omega^R}{dt} $$

2.3.6 Once again (see 2.2.3) a pictorial representation employing straight or curved arrows accompanied by measure numbers is sometimes convenient. For example, Fig. 2.3.6 shows four repre-

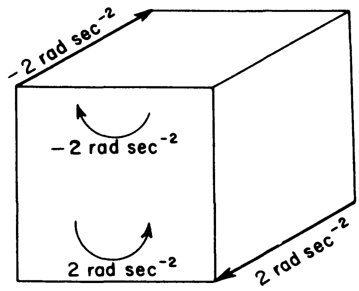


FIG. 2.3.6

sentations of the same angular acceleration.

2.3.7 When a rigid body R moves in a reference frame R' in such a way that there exists a unit vector $\mathbf{k}$ whose orientation in *both* R and R' is independent of time t (see 2.2.4), the angular acceleration ${}^{R'}\boldsymbol{\alpha}^R$ remains parallel to $\mathbf{k}$ and is given by

$$ {}^{R'}\boldsymbol{\alpha}^R = \frac{d^2\theta}{dt^2}\mathbf{k} \tag{1}$$

where θ is defined as in Sec. 2.2.4. Furthermore (compare with 2.3.5),

$$ {}^{R'}\alpha^R = \frac{d\,{}^{R'}\omega^R}{dt} \tag{2}$$

where ${}^{R'}\alpha^R$ and ${}^{R'}\omega^R$ are the scalar angular acceleration (see 2.3.4) and the angular speed (see 2.2.2) of R in R' for the $\mathbf{k}$ direction. And

$$ {}^{R'}\alpha^R = \frac{d^2\theta}{dt^2} \tag{3}$$

Proof:

$$ {}^{R'}\boldsymbol{\alpha}^R \underset{(2.3.1)}{=} \frac{{}^{R'}d\,{}^{R'}\boldsymbol{\omega}^R}{dt} \underset{(2.2.4)}{=} \frac{{}^{R'}d}{dt}\left(\frac{d\theta}{dt}\mathbf{k}\right)$$

$$ \underset{(1.5.1)}{=} \frac{d^2\theta}{dt^2}\mathbf{k} + \frac{d\theta}{dt}\frac{{}^{R'}d\mathbf{k}}{dt} \underset{(1.2.2)}{=} \frac{d^2\theta}{dt^2}\mathbf{k}$$

Next

$$ \frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) \underset{(2.2.4)}{=} \frac{d\,{}^{R'}\omega^R}{dt}$$

Hence

$$ {}^{R'}\boldsymbol{\alpha}^R = \frac{d\,{}^{R'}\omega^R}{dt}\mathbf{k}$$

so that (see 2.3.4)

$$ {}^{R'}\alpha^R = \frac{d\,{}^{R'}\omega^R}{dt} \underset{(2.2.4)}{=} \frac{d^2\theta}{dt^2}$$

Problem: Referring to Problem 2.2.9 and Fig. 2.2.9b, and assuming that the angular speed of S in R for the $\mathbf{n}$ direction is constant, express the scalar angular acceleration (${}^R\alpha^{S'}$) of S' in R for the $\mathbf{n}'$ direction as a function of θ and α.

Solution:

$$ {}^R\alpha^{S'} \underset{(2)}{=} \frac{d\,{}^R\omega^{S'}}{dt} \underset{(P2.2.9)}{=} \frac{d}{dt}\left(\frac{{}^R\omega^S \cos\alpha}{\sin^2\alpha\cos^2\theta - 1}\right)$$

$$ = {}^R\omega^S \sin\alpha \frac{\sin 2\alpha \sin 2\theta}{2(\sin^2\alpha\cos^2\theta - 1)^2}\frac{d\theta}{dt}$$

But

$$\frac{d\theta}{dt} \underset{(2.2.4)}{=} {}^R\omega^S$$

Hence

$${}^R\alpha^S = ({}^R\omega^S)^2 \sin\alpha \frac{\sin 2\alpha \sin 2\theta}{2(\sin^2\alpha\cos^2\theta - 1)^2}$$

2.3.8 The expressions given in Secs. 1.7.1, 2.2.4, and 2.3.7 form the basis of one method for analyzing motions of *plane linkages*. The method consists of equating two expressions for the position vector of a point and repeatedly differentiating the resulting equation, making use of (see 1.7.1 and 2.2.4)

$$\frac{{}^R d\mathbf{n}}{dt} = {}^R\omega^B \mathbf{k} \times \mathbf{n} = {}^R\omega^B \mathbf{n}' \tag{1}$$

$$\frac{{}^R d\mathbf{n}'}{dt} = {}^R\omega^B \mathbf{k} \times \mathbf{n}' = -{}^R\omega^B \mathbf{n} \tag{2}$$

and (see Eq. (2), 2.3.7)

$$\frac{d\,{}^R\omega^B}{dt} = {}^R\alpha^B \tag{3}$$

where (see Fig. 2.3.8a) R is a reference frame in which the plane

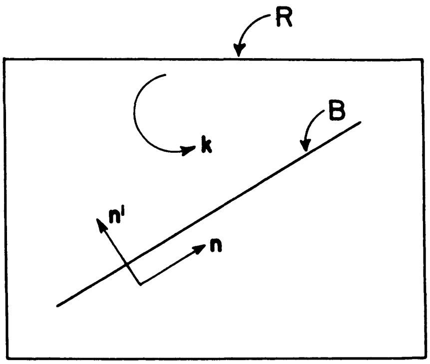


FIG. 2.3.8a

of the linkage is fixed, B is a bar of the linkage, $\mathbf{k}$ is a unit vector perpendicular to the plane of the linkage, $\mathbf{n}$ is a unit vector parallel to B, $\mathbf{n}'$ is the unit vector $\mathbf{k} \times \mathbf{n}$, and ${}^R\omega^B$ and ${}^R\alpha^B$ are the angular speed and scalar angular acceleration (see 2.2.2 and 2.3.4) of B in R for the $\mathbf{k}$ direction.

Example: Figure 2.3.8b shows the configuration at time t^* of a plane four bar linkage, one bar of which, B_4, is fixed. At time t^*

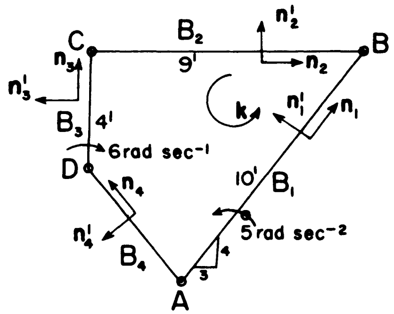


FIG. 2.3.8b

the angular velocity of bar B_3 and the angular acceleration of bar B_1 have the values indicated in the sketch. The angular velocities of bars B_1 and B_2 and the angular accelerations of bars B_2 and B_3 at time t^* are to be determined.

Let $\mathbf{k}$ be a unit vector perpendicular to the plane of the linkage, and introduce unit vectors $\mathbf{n}_i$ and $\mathbf{n}_i'$ respectively parallel and perpendicular to B_i, $i = 1, 2, 3, 4$, choosing $\mathbf{n}_i'$ such that $\mathbf{n}_i' = \mathbf{k} \times \mathbf{n}_i$. Next, note that the position vector of D relative to A is equal both to $\overline{AD}\mathbf{n}_4$ and to $10\mathbf{n}_1 - 9\mathbf{n}_2 - 4\mathbf{n}_3$ ft, so that

$$10\mathbf{n}_1 - 9\mathbf{n}_2 - 4\mathbf{n}_3 = \overline{AD}\mathbf{n}_4$$

Differentiate with respect to t, and let ω_i be the angular speed of bar B_i ($\omega_4 = 0$) for the $\mathbf{k}$ direction:

$$10\omega_1\mathbf{n}_1' - 9\omega_2\mathbf{n}_2' - 4\omega_3\mathbf{n}_3' \underset{(1)}{=} 0 \tag{A}$$

Now, at time t^*,

$$\omega_3 = -6 \text{ rad sec}^{-1}$$

Hence, at time t^*,

$$10\omega_1\mathbf{n}_1' - 9\omega_2\mathbf{n}_2' - 4(-6)\mathbf{n}_3' = 0 \tag{B}$$

This vector equation can be solved for the two scalar unknowns ω_1 and ω_2, either analytically or graphically. One analytical method consists of eliminating one unknown at a time by dot multiplication of the equation with a unit vector perpendicular to the unit vector associated with this unknown. For example, to eliminate ω_2, dot multiply with $\mathbf{n}_2$:

$$10\omega_1\mathbf{n}_1' \cdot \mathbf{n}_2 + 24\mathbf{n}_3' \cdot \mathbf{n}_2 = 0 \tag{C}$$

From Fig. 2.3.8b,

$$\mathbf{n}_1' \cdot \mathbf{n}_2 = \cos(\mathbf{n}_1', \mathbf{n}_2) = -\frac{4}{5}$$

and

$$\mathbf{n}_3' \cdot \mathbf{n}_2 = \cos(\mathbf{n}_3', \mathbf{n}_2) = -1$$

Hence

$$10\omega_1\left(-\frac{4}{5}\right) + 24(-1) = 0$$

and

$$\omega_1 = -3 \text{ rad sec}^{-1}$$

Similarly, dot multiplying Eq. (B) with $\mathbf{n}_1$ to eliminate ω_1,

$$\omega_2 = -2 \text{ rad sec}^{-1}$$

The angular velocities of bars B_1 and B_2 at time t^* are shown in Fig. 2.3.8c. Note that the sense of each of these vectors is op-

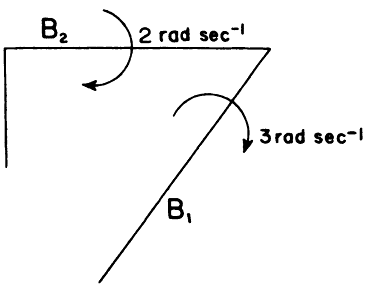


FIG. 2.3.8c

posite to that of $\mathbf{k}$.

Next, differentiate Eq. (A) (*not* Eq. (B), which is valid only at time t^*), letting α_i be the scalar angular acceleration of bar B_i for the $\mathbf{k}$ direction:

$$10\alpha_1\mathbf{n}_1' + 10\omega_1(-\omega_1\mathbf{n}_1) - 9\alpha_2\mathbf{n}_2' \\ -9\omega_2(-\omega_2\mathbf{n}_2) - 4\alpha_3\mathbf{n}_3' - 4\omega_3(-\omega_3\mathbf{n}_3) \underset{(2,3)}{=} 0$$

At time t^* the known values of ω_1, ω_2, ω_3, and α_1 then give

$$50\mathbf{n}_1' - 90\mathbf{n}_1 - 9\alpha_2\mathbf{n}_2' + 36\mathbf{n}_2 - 4\alpha_3\mathbf{n}_3' + 144\mathbf{n}_3 = 0$$

Find α_2 and α_3 by dot multiplication of this equation with $\mathbf{n}_3$ and $\mathbf{n}_2$, respectively:

$$\alpha_2 = 11.33 \text{ rad sec}^{-2}$$

$$\alpha_3 = 14.5 \text{ rad sec}^{-2}$$

The angular accelerations of B_2 and B_3 are shown in Fig. 2.3.8d.

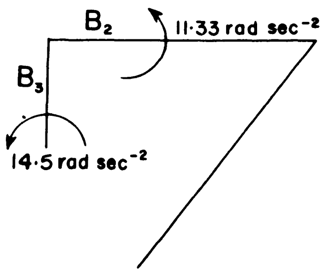


FIG. 2.3.8d

2.3.9 When the geometry of a linkage is complicated, the necessary dot products of unit vectors (see Eqs. (B) and (C), Example 2.3.8) cannot be evaluated so readily as they were in Example 2.3.8. The following graphical procedure yields these dot products with considerable accuracy: Draw a circle, choosing for the radius a length which can readily be divided into small, equal parts. (The larger the number of such parts, the better.) Through the center of the circle draw a line parallel to each unit vector. Taking the sense of each unit vector into account, label one point of intersection of each such line with the circle with a number designating the corresponding unit vector. In Fig. 2.3.9 this is done for the unit vectors used in Example 2.3.8. If the radius of the circle is now regarded as defining a unit of length, the absolute value of the dot product of any two unit vectors is equal to the number of such units contained in the line segment joining the center of the circle to the foot of the perpendicular dropped from the labeled point corresponding to one unit vector on the line corresponding to the other unit vector; and the dot product is positive when the foot of this perpendicular falls on the labeled side of the line, nega-

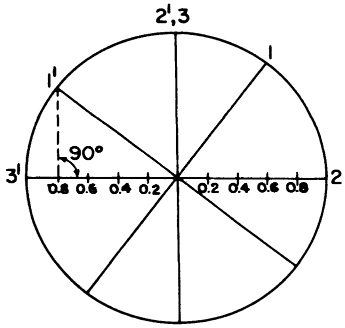


FIG. 2.3.9

tive when it falls on the unlabeled side. For example (see Fig. 2.3.9),

$$\mathbf{n}_1' \cdot \mathbf{n}_2 = -0.8$$

and

$$\mathbf{n}_3' \cdot \mathbf{n}_2 = -1$$

2.3.10 The method discussed in Sec. 2.3.8 applies not only to plane linkages containing bars of fixed lengths, but also to those containing *sliding pairs*.

Problem: Solve Problem 2.2.5 by using the method discussed in Sec. 2.3.8.

Solution: Let z be the (variable) distance between A and P. Then (see Fig. 2.3.10)

$$z\mathbf{n}_F - r\mathbf{n}_D = \sqrt{2}\, r\mathbf{n}$$

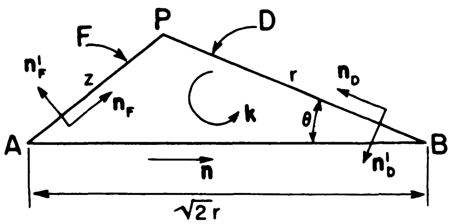


FIG. 2.3.10

and, differentiating,

$$\frac{dz}{dt}\mathbf{n}_F + z\,{}^R\omega^F\,\mathbf{n}_F' - r\,{}^R\omega^D\,\mathbf{n}_D' = 0$$

where

$${}^R\omega^D = 6\pi \text{ rad min}^{-1}$$

Figure 2.3.10 shows that, when $\theta = 0$,

$$\mathbf{n}_F = -\mathbf{n}_D, \qquad \mathbf{n}_F' = -\mathbf{n}_D', \qquad z = (\sqrt{2} - 1)r$$

Hence

$$\left.\frac{dz}{dt}\right|_{\theta=0}(-\mathbf{n}_D) + (\sqrt{2} - 1)r\,{}^R\omega^F|_{\theta=0}(-\mathbf{n}_D') - r(6\pi)\mathbf{n}_D' = 0$$

Thus

$${}^R\omega^F|_{\theta=0} = \frac{6\pi}{1 - \sqrt{2}} = -45.5 \text{ rad min}^{-1}$$

2.3.11 Given n reference frames R_i, $i = 1, 2, \ldots, n$, the angular acceleration ${}^{R_n}\boldsymbol{\alpha}^R$ of a rigid body R in a reference frame R_n is not, in general, equal to the sum

$${}^{R_1}\boldsymbol{\alpha}^R + {}^{R_2}\boldsymbol{\alpha}^{R_1} + \cdots + {}^{R_n}\boldsymbol{\alpha}^{R_{n-1}}$$

(Compare with 2.2.7.)

Problem: Referring to Problem 2.2.7 and Fig. 2.2.7b, determine the angular accelerations ${}^{R_1}\boldsymbol{\alpha}^B$ and ${}^{R_2}\boldsymbol{\alpha}^{R_1}$, then compare their sum with the angular acceleration ${}^{R_2}\boldsymbol{\alpha}^B$ found in Problem 2.3.1.

Solution:

$${}^{R_1}\boldsymbol{\alpha}^B \underset{(2.3.7)}{=} \frac{d^2\beta}{dt^2}\mathbf{k}_1 = -0.2\left(\frac{d\psi}{dt}\right)^2 \sin\psi\,\mathbf{k}_1$$

$${}^{R_2}\boldsymbol{\alpha}^{R_1} \underset{(2.3.7)}{=} \frac{d^2\psi}{dt^2}\mathbf{k}_2 = 0$$

Hence

$${}^{R_1}\boldsymbol{\alpha}^B + {}^{R_2}\boldsymbol{\alpha}^{R_1} = -0.2\left(\frac{d\psi}{dt}\right)^2 \sin\psi\,\mathbf{k}_1$$

while

$${}^{R_2}\boldsymbol{\alpha}^B \underset{(P2.3.1)}{=} -0.2\left(\frac{d\psi}{dt}\right)^2\left(\mathbf{k}_1 \sin\psi + \mathbf{k}_1 \times \mathbf{k}_2 \cos\psi\right)$$

2.4 Relative velocity and acceleration

2.4.1 Given two points P and Q moving in a reference frame R, the first time-derivative in R of the position vector $\mathbf{r}$ of P relative to Q (see Fig. 2.4.1) is called the *velocity of P relative to Q in R* and is denoted by ${}^R\mathbf{v}^{P/Q}$:

$$ {}^R\mathbf{v}^{P/Q} = \frac{{}^Rd\mathbf{r}}{dt} \tag{1} $$

The first time-derivative of ${}^R\mathbf{v}^{P/Q}$ in R is called the *acceleration*

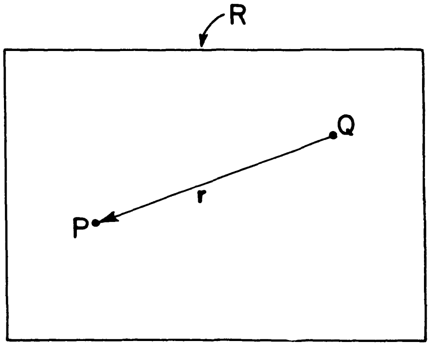


FIG. 2.4.1

of P relative to Q in R and is denoted by ${}^R\mathbf{a}^{P/Q}$:

$$ {}^R\mathbf{a}^{P/Q} = \frac{{}^Rd\,{}^R\mathbf{v}^{P/Q}}{dt} \tag{2} $$

Problem: Two points P and Q move on a rigid body R while R moves in a reference frame R'. At a certain instant t^*, P and Q coincide. Show that at this instant the velocity of P relative to Q in R is equal to the velocity of P relative to Q in R', but that the acceleration of P relative to Q in R is not, in general, equal to the acceleration of P relative to Q in R'.

Solution: With self-explanatory notation,

$$ {}^{R'}\mathbf{v}^{P/Q} \underset{(1)}{=} \frac{{}^{R'}d\mathbf{r}}{dt} \underset{(2.1.4)}{=} \frac{{}^Rd\mathbf{r}}{dt} + {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r} = {}^R\mathbf{v}^{P/Q} + {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r} \tag{A} $$

At time t^*, $\mathbf{r}$ is equal to zero. Hence

$$^{R'}\mathbf{v}^{P/Q}\big|_{t^*} = {}^{R}\mathbf{v}^{P/Q}\big|_{t^*}$$

Next

$$\begin{aligned}
^{R'}\mathbf{a}^{P/Q} &\underset{(2)}{=} \frac{^{R'}d\,^{R'}\mathbf{v}^{P/Q}}{dt} \underset{(A)}{=} \frac{^{R'}d}{dt}\left({}^{R}\mathbf{v}^{P/Q} + {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{r}\right) \\
&= \frac{^{R'}d\,^{R}\mathbf{v}^{P/Q}}{dt} + \frac{^{R'}d\,^{R'}\boldsymbol{\omega}^{R}}{dt} \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^{R} \times \frac{^{R'}d\mathbf{r}}{dt} \\
&= \frac{^{R}d\,^{R}\mathbf{v}^{P/Q}}{dt} \underset{(2.1.4)}{+} {}^{R'}\boldsymbol{\omega}^{R} \times {}^{R}\mathbf{v}^{P/Q} \underset{(2.3.1)}{+} {}^{R'}\boldsymbol{\alpha}^{R} \times \mathbf{r} \\
&\qquad + {}^{R'}\boldsymbol{\omega}^{R} \times \left({}^{R}\mathbf{v}^{P/Q} \underset{(A)}{+} {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{r}\right) \\
&= {}^{R}\mathbf{a}^{P/Q} + 2\,{}^{R'}\boldsymbol{\omega}^{R} \times {}^{R}\mathbf{v}^{P/Q} + {}^{R'}\boldsymbol{\alpha}^{R} \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^{R} \times \left({}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{r}\right)
\end{aligned}$$

Hence, at time t^*,

$$^{R'}\mathbf{a}^{P/Q}\big|_{t^*} = {}^{R}\mathbf{a}^{P/Q}\big|_{t^*} + 2\,{}^{R'}\boldsymbol{\omega}^{R} \times {}^{R}\mathbf{v}^{P/Q}\big|_{t^*}$$

2.4.2 The velocity of a point P relative to a point Q in a reference frame R differs from the velocity of Q relative to P in R only in sense; that is,

$$^{R}\mathbf{v}^{P/Q} = -{}^{R}\mathbf{v}^{Q/P} \tag{1}$$

Similarly, the accelerations $^{R}\mathbf{a}^{P/Q}$ and $^{R}\mathbf{a}^{Q/P}$ are related by

$$^{R}\mathbf{a}^{P/Q} = -{}^{R}\mathbf{a}^{Q/P} \tag{2}$$

This follows from Sec. 2.4.1 and the fact that the position vector of P relative to Q differs from that of Q relative to P only in sense.

2.4.3 The velocity and acceleration of a point P relative to a point Q in a reference frame R (see 2.4.1) are both equal to zero if P and Q are fixed in R, because the position vector of P relative to Q is then fixed in R (see 1.1.1) and all of its time-derivatives are equal to zero (see 1.2.2). The converse is not necessarily true. For example, if P and Q move on opposite sides of a rectangle fixed in R, in such a way that line PQ remains parallel to one side of the rectangle, the velocity and acceleration of P relative to Q in R are equal to zero although P and Q are not fixed in R.

2.4.4 Given n points P_i, $i = 1, 2, \ldots, n$, the velocity and acceleration of the first point relative to the nth in a reference frame R can be expressed as

$$^R\mathbf{v}^{P_1/P_n} = {}^R\mathbf{v}^{P_1/P_2} + {}^R\mathbf{v}^{P_2/P_3} + \ldots + {}^R\mathbf{v}^{P_{n-1}/P_n} \tag{1}$$

and

$$^R\mathbf{a}^{P_1/P_n} = {}^R\mathbf{a}^{P_1/P_2} + {}^R\mathbf{a}^{P_2/P_3} + \ldots + {}^R\mathbf{a}^{P_{n-1}/P_n} \tag{2}$$

Proof: The position vector $\mathbf{r}^{P_1/P_n}$ of P_1 relative to P_n (see Fig. 2.4.4) can be expressed as

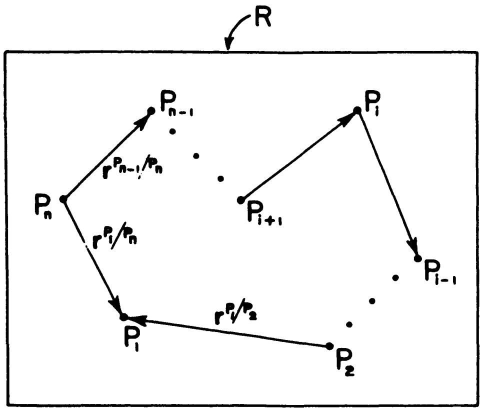


FIG. 2.4.4

$$\mathbf{r}^{P_1/P_n} = \mathbf{r}^{P_1/P_2} + \mathbf{r}^{P_2/P_3} + \ldots + \mathbf{r}^{P_{n-1}/P_n}$$

Hence

$$^R\mathbf{v}^{P_1/P_n} \underset{(2.4.1)}{=} \frac{^Rd\mathbf{r}^{P_1/P_n}}{dt} \underset{(1.4.1)}{=} \frac{^Rd\mathbf{r}^{P_1/P_2}}{dt} + \ldots + \frac{^Rd\mathbf{r}^{P_{n-1}/P_n}}{dt}$$

$$\underset{(2.4.1)}{=} {}^R\mathbf{v}^{P_1/P_2} + \ldots + {}^R\mathbf{v}^{P_{n-1}/P_n}$$

and

$$^R\mathbf{a}^{P_1/P_n} \underset{(2.4.1)}{=} \frac{^Rd\,{}^R\mathbf{v}^{P_1/P_n}}{dt} \underset{(1.4.1)}{=} \frac{^Rd\,{}^R\mathbf{v}^{P_1/P_2}}{dt} + \ldots + \frac{^Rd\,{}^R\mathbf{v}^{P_{n-1}/P_n}}{dt}$$

$$\underset{(2.4.1)}{=} {}^R\mathbf{a}^{P_1/P_2} + \ldots + {}^R\mathbf{a}^{P_{n-1}/P_n}$$

2.4.5 If two points P and Q are fixed on a rigid body R, the velocity $^{R'}\mathbf{v}^{P/Q}$ and acceleration $^{R'}\mathbf{a}^{P/Q}$ of P relative to Q in a reference frame R' are given by (see Fig. 2.4.5a)

$$^{R'}\mathbf{v}^{P/Q} = {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r} \tag{1}$$

$$^{R'}\mathbf{a}^{P/Q} = {}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{v}^{P/Q} \tag{2}$$

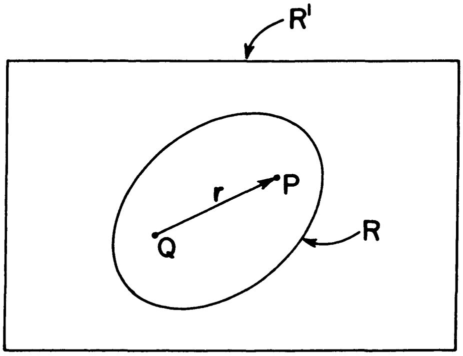


FIG. 2.4.5a

Proof:

$$^{R'}\mathbf{v}^{P/Q} \underset{(2.4.1)}{=} \frac{^{R'}d\mathbf{r}}{dt} \underset{(2.1.1,2.2.1)}{=} {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{r}$$

$$^{R'}\mathbf{a}^{P/Q} \underset{(2.4.1)}{=} \frac{^{R'}d\,^{R'}\mathbf{v}^{P/Q}}{dt} = \frac{^{R'}d}{dt}\left({}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{r}\right)$$

$$\underset{(1.5.3)}{=} \left(\frac{^{R'}d\,^{R'}\boldsymbol{\omega}^{R}}{dt}\right) \times \mathbf{r} + {}^{R}\boldsymbol{\omega}^{R} \times \frac{^{R'}d\mathbf{r}}{dt}$$

$$= \underset{(2.3.1)}{^{R'}\boldsymbol{\alpha}^{R}} \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^{R} \times \underset{(2.4.1)}{^{R'}\mathbf{v}^{P/Q}}$$

Problem: Regarding the earth as a rigid sphere R whose angular velocity $^{R}\boldsymbol{\omega}^{R}$ in a certain reference frame R' is given by

$$^{R'}\boldsymbol{\omega}^{R} = \frac{\pi}{12}\mathbf{k} \text{ rad hr}^{-1}$$

where $\mathbf{k}$ (see Fig. 2.4.5b) is a unit vector fixed in both R (parallel to the earth's north–south axis) and R', and letting P and Q be points fixed on the earth at opposite ends of an equatorial diameter, determine the velocity and acceleration of P relative to Q in R'.

Solution: Let $\mathbf{n}$ be a unit vector directed upward at P, $\mathbf{n}'$ a unit vector directed toward the east at P, as shown in Fig. 2.4.5b, and let $\mathbf{r}$ be the position vector of P relative to Q. Then, taking the earth's radius equal to 3960 miles,

$$\mathbf{r} = 2(3960)\mathbf{n} = 7920\mathbf{n} \text{ mile}$$

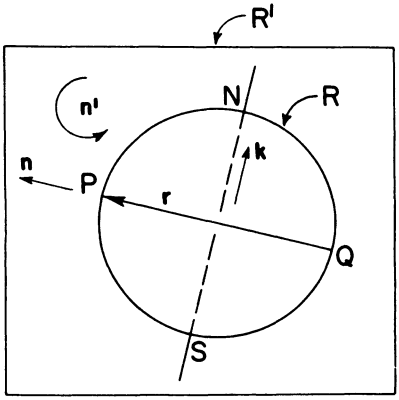


FIG. 2.4.5b

and

$$ {}^{R'}\mathbf{v}^{P/Q} \underset{(1)}{=} {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r} = \frac{\pi}{12}\mathbf{k} \times (7920\mathbf{n}) = 2070\mathbf{n}' \text{ mile hr}^{-1} $$

Next

$$ {}^{R'}\boldsymbol{\alpha}^R \underset{(2.3.1)}{=} \frac{{}^{R'}d\,{}^{R'}\boldsymbol{\omega}^R}{dt} = \frac{{}^{R'}d}{dt}\left(\frac{\pi}{12}\mathbf{k}\right) = 0 $$

Hence

$$ {}^{R'}\mathbf{a}^{P/Q} \underset{(2)}{=} {}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{v}^{P/Q} $$

$$ = \left(\frac{\pi}{12}\mathbf{k}\right) \times (2070\mathbf{n}') = -547\mathbf{n} \text{ mile hr}^{-2} $$

2.5 Absolute velocity and acceleration

2.5.1 Given a point P moving in a reference frame R, and a point O fixed in R, the velocity and acceleration of P relative to O in R (see 2.4.1) are independent of the position of O. This relative velocity and acceleration are called, respectively, the *absolute velocity* of P in R and the *absolute acceleration* of P in R, or, for short, *the* velocity of P in R and *the* acceleration of P in R, and are denoted by ${}^R\mathbf{v}^P$ and ${}^R\mathbf{a}^P$. It follows from Sec. 2.4.1 that (see Fig. 2.5.1)

$$ {}^R\mathbf{v}^P = \frac{{}^Rd\mathbf{p}}{dt} \tag{1} $$

$$^R\mathbf{a}^P = \frac{^Rd\,^R\mathbf{v}^P}{dt} \tag{2}$$

Proof: Let O and O' be two points fixed in R; $\mathbf{r}$ the position vector of O' relative to O; $\mathbf{p}$ the position vector of P relative to O; $\mathbf{p}'$ the position vector of P relative to O', as shown in Fig. 2.5.1.

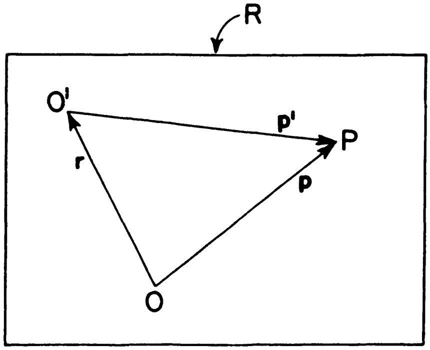


FIG. 2.5.1

It is to be shown that

$$^R\mathbf{v}^{P/O} = {}^R\mathbf{v}^{P/O'}$$

and

$$^R\mathbf{a}^{P/O} = {}^R\mathbf{a}^{P/O'}$$

From Fig. 2.5.1,

$$\mathbf{p} = \mathbf{r} + \mathbf{p}'$$

Differentiate with respect to time t in R:

$$\frac{^Rd\mathbf{p}}{dt} = \frac{^Rd\mathbf{r}}{dt} + \frac{^Rd\mathbf{p}'}{dt}$$

Now

$$\frac{^Rd\mathbf{p}}{dt} \underset{(2.4.1)}{=} {}^R\mathbf{v}^{P/O}, \qquad \frac{^Rd\mathbf{r}}{dt} \underset{(2.4.4)}{=} 0, \qquad \frac{^Rd\mathbf{p}'}{dt} \underset{(2.4.1)}{=} {}^R\mathbf{v}^{P/O'}$$

Hence

$$^R\mathbf{v}^{P/O} = {}^R\mathbf{v}^{P/O'}$$

Differentiate:

$$\frac{^Rd\,^R\mathbf{v}^{P/O}}{dt} = \frac{^Rd\,^R\mathbf{v}^{P/O'}}{dt}$$

But

$$\frac{{}^{R}d\,{}^{R}\mathbf{v}^{P/O}}{dt} \underset{(2.4.1)}{=} {}^{R}\mathbf{a}^{P/O}, \qquad \frac{{}^{R}d\,{}^{R}\mathbf{v}^{P/O'}}{dt} \underset{(2.4.1)}{=} {}^{R}\mathbf{a}^{P/O'}$$

Hence

$${}^{R}\mathbf{a}^{P/O} = {}^{R}\mathbf{a}^{P/O'}$$

2.5.2 If a point P is fixed in a reference frame R, the velocity and the acceleration of P in R (see 2.5.1) are equal to zero. The converse applies to velocities, but not necessarily to accelerations. That is, if the velocity of P in R is equal to zero during a certain time interval, P is fixed in R during this time interval; but the acceleration of P in R may be equal to zero while P is moving in R. This occurs, in fact, whenever P moves in such a way that ${}^{R}\mathbf{v}^{P}$ is independent of t in R (see 1.2.8).

2.5.3 The velocity (${}^{R}\mathbf{v}^{P}$) of a point P in a reference frame R (see 2.5.1) is parallel to the tangent at P to the curve C on which P moves in R. It can, therefore, be expressed in the form

$${}^{R}\mathbf{v}^{P} = v_{\tau}\boldsymbol{\tau} \tag{1}$$

where $\boldsymbol{\tau}$ is a vector tangent of C at P (see 1.9) and v_{τ} is a scalar, called the *speed* of P in R for the $\boldsymbol{\tau}$ direction. v_{τ} is positive when ${}^{R}\mathbf{v}^{P}$ and $\boldsymbol{\tau}$ have the same sense, vanishes when ${}^{R}\mathbf{v}^{P}$ is equal to zero, and is negative when the sense of ${}^{R}\mathbf{v}^{P}$ is opposite to that of $\boldsymbol{\tau}$. Furthermore, if s is the arc-length displacement of P relative to a

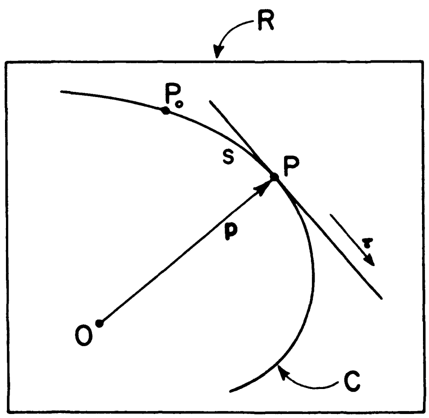


Fig. 2.5.3

point P_0 fixed on C, and $\boldsymbol{\tau}$ points in the direction in which P moves when s increases algebraically, then

$$v_\tau = \frac{ds}{dt} \tag{2}$$

Proof: Let $\mathbf{p}$ be the position vector of P relative to a point O fixed in R (see Fig. 2.5.3). Then

$$^R\mathbf{v}^P \underset{(2.5.1)}{=} \frac{^R d\mathbf{p}}{dt} \underset{(1.6.1)}{=} \frac{^R d\mathbf{p}}{ds}\frac{ds}{dt} \underset{(1.9.3)}{=} \boldsymbol{\tau}\frac{ds}{dt}$$

This shows that $^R\mathbf{v}^P$ is parallel to the tangent to C at P. $^R\mathbf{v}^P$ can, therefore, be expressed in the form

$$^R\mathbf{v}^P = v_\tau \boldsymbol{\tau}$$

where $\boldsymbol{\tau}$ may be either of the two vector tangents of C at P (see 1.9.2). Furthermore,

$$v_\tau = \frac{ds}{dt}$$

if (see 1.9.3) $\boldsymbol{\tau}$ points in the direction in which P moves when s increases algebraically.

Problem: Referring to Problem 1.10.1 and Fig. 1.10.1b, and supposing that point P oscillates on line AC in such a way that the displacement x of P relative to point C, regarded as positive when P is between A and C, is given by

$$x = 1 + \sin 2\pi t \text{ ft}$$

where t is the time in seconds, determine the speed v of P for the direction AC at $t = \frac{1}{2}$ sec and $t = 1$ sec, noting that P occupies the same position at these two instants. Then, supposing that the sheet $ABCD$ has been folded to form the helix H (see Fig. 1.10.1c), determine the velocity $\mathbf{v}$ of P at $t = \frac{1}{2}$ sec, expressing it in terms of the unit vectors $\mathbf{n}$ and $\mathbf{k}$.

Solution: As the curve on which P moves is a straight line, x is equal to the arc-length displacement of P relative to C; and, as x decreases algebraically when P moves in the direction AC, v is given by

$$v \underset{(2)}{=} -\frac{dx}{dt} = -2\pi \cos 2\pi t$$

Hence

$$v|_{t=\frac{1}{4}} = 2\pi \text{ ft sec}^{-1}, \qquad v|_{t=1} = -2\pi \text{ ft sec}^{-1}$$

When the sheet has been folded, x is still equal to the arc-length displacement of P relative to C. The velocity $\mathbf{v}$ of P is, therefore, given by

$$\mathbf{v} \underset{(1.2)}{=} \frac{dx}{dt}\boldsymbol{\tau}$$

where $\boldsymbol{\tau}$ must be the vector tangent which points in the direction in which P moves when x increases algebraically. Using the notation introduced in the solution of Problem 1.10.1, this vector tangent is given by

$$\boldsymbol{\tau} = -\frac{\mathbf{p}'}{|\mathbf{p}'|} = -\frac{(a/b)\mathbf{k} \times \mathbf{n} + \mathbf{k}}{[(a^2/b^2) + 1]^{1/2}}$$

Hence

$$\mathbf{v} = -2\pi \frac{(a/b)\mathbf{k} \times \mathbf{n} + \mathbf{k}}{[(a^2/b^2) + 1]^{1/2}} \cos 2\pi t$$

and

$$\mathbf{v}|_{t=\frac{1}{2}} = 2\pi \frac{(a/b)\mathbf{k} \times \mathbf{n} + \mathbf{k}}{[(a^2/b^2) + 1]^{1/2}}$$

2.5.4 The acceleration (${}^R\mathbf{a}^P$) of a point P in a reference frame R (see 2.5.1) is parallel to the osculating plane at P of the curve C on which P moves in R (see 1.10.5). It can, therefore, be expressed in the form

$$^R\mathbf{a}^P = a_\tau\boldsymbol{\tau} + a_\nu\boldsymbol{\nu} \tag{1}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are, respectively, a vector tangent and the vector principal normal (see 1.9 and 1.11) of C at P, and a_τ and a_ν are scalars. The vectors $a_\tau\boldsymbol{\tau}$ and $a_\nu\boldsymbol{\nu}$ are called, respectively, the *tangential acceleration* and the *normal acceleration* of P in R. a_τ is called the *scalar tangential acceleration* of P in R for the $\boldsymbol{\tau}$ direction, and may be positive, negative, or zero. a_ν is called the *scalar normal acceleration* of P in R, and is never negative. Furthermore, if s is the arc-length displacement of P relative to a point P_0 fixed on C, and $\boldsymbol{\tau}$ points in the direction on which P moves when s increases algebraically, then

$$a_\tau = \frac{dv_\tau}{dt} = \frac{d^2s}{dt^2} \tag{2}$$

and

$$a_\nu = \frac{v_\tau^2}{\rho} = \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 \tag{3}$$

where v_τ is the speed of P in R for the $\boldsymbol{\tau}$ direction (see 2.5.3) and ρ is the radius of curvature of C at P (see 1.12.2).

Proof:

$$\begin{aligned}
{}^R\mathbf{a}^P &\underset{(2.5.1)}{=} \frac{{}^Rd\,{}^R\mathbf{v}^P}{dt} \underset{(2.5.3)}{=} \frac{{}^Rd}{dt}(v_\tau\boldsymbol{\tau}) \\
&\underset{(1.5.1)}{=} \frac{dv_\tau}{dt}\boldsymbol{\tau} + v_\tau \frac{{}^Rd\boldsymbol{\tau}}{dt} \\
&= \frac{dv_\tau}{dt}\boldsymbol{\tau} + v_\tau \underset{(1.6.1)}{\frac{{}^Rd\boldsymbol{\tau}}{ds}\frac{ds}{dt}} \\
&= \frac{dv_\tau}{dt}\boldsymbol{\tau} + v_\tau \underset{(1.13.1),(2.5.3)}{\frac{\boldsymbol{\nu}}{\rho}\, v_\tau}
\end{aligned}$$

or

$${}^R\mathbf{a}^P = \frac{dv_\tau}{dt}\boldsymbol{\tau} + \frac{v_\tau^2}{\rho}\boldsymbol{\nu}$$

This shows that ${}^R\mathbf{a}^P$ is parallel to the osculating plane of C at P (see 1.10.5). ${}^R\mathbf{a}^P$ can, therefore, be expressed in the form

$${}^R\mathbf{a}^P = a_\tau\boldsymbol{\tau} + a_\nu\boldsymbol{\nu}$$

and it follows that

$$a_\tau = \frac{dv_\tau}{dt} \underset{(2.5.3)}{=} \frac{d^2s}{dt^2}$$

$$a_\nu = \frac{v_\tau^2}{\rho} \underset{(2.5.3)}{=} \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2$$

2.5.5 Whether or not it is convenient to work with tangential and normal accelerations depends, in part, on the ease with which the radius of curvature of the curve on which the motion takes place can be determined. One curve for which this determination presents no difficulties is the circle: The radius of curvature of a circle is equal to the circle's radius.

Problem: Starting from rest, an automobile reaches a speed of 90 miles per hour in one minute while travelling with uniformly increasing speed on a circular track having a radius of 500 ft.

Determine the magnitude of the acceleration of a point P on the automobile (a) at the initial instant of the motion, (b) 30 seconds later, and (c) one minute after starting.

Solution: Let r be the radius of the circle C on which P moves, $\boldsymbol{\tau}$ a vector tangent of C at P, v the speed of P for the $\boldsymbol{\tau}$ direction, $\mathbf{a}$ the acceleration of P. Then

$$|\mathbf{a}| \underset{(2.5.4)}{=} (a_\tau^2 + a_\nu^2)^{1/2} \underset{(2.5.4)}{=} \left[\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{r^2}\right]^{1/2}$$

The phrase "with uniformly increasing speed" is ambiguous, because it does not contain any mention of the variable as a function of which the speed is increasing. However, this phrase is generally regarded as referring to the time-rate of change of the speed. That is, it means that

$$\frac{dv}{dt} = c, \text{ a constant}$$

Hence

$$v = ct + c', \qquad c' \text{ a constant} \tag{A}$$

Let $t = 0$ at the initial instant of the motion. Then

$$v|_{t=0} = 0$$

From Eq. (A),

$$v|_{t=0} = c'$$

Thus

$$c' = 0$$

and, again from Eq. (A),

$$v = ct \tag{B}$$

At $t = 1$ min, P has a speed of 90 mile hr^{-1}:

$$v|_{t=1\text{ min}} = 90 \text{ mile hr}^{-1}$$

From Eq. (B),

$$v|_{t=1\text{ min}} = c \text{ mile hr}^{-1}$$

(c expressed in mile hr^{-1} min^{-1}). Thus

$$c = 90 \text{ mile hr}^{-1}\text{ min}^{-1} = 2.2 \text{ ft sec}^{-2}$$

and, again from Eq. (B),

$$v = 2.2t \text{ ft sec}^{-1}$$

(t expressed in sec). Hence

$$\frac{dv}{dt} = 2.2 \text{ ft sec}^{-2}$$

and, for $r = 500$ ft,

$$|\mathbf{a}| = \left[(2.2)^2 + \frac{(2.2t)^4}{(500)^2}\right]^{1/2} \text{ ft sec}^{-2}$$

(t expressed in sec).

Result (a): $|\mathbf{a}|_{t=0} = 2.2$ ft sec^{-2}.

Result (b): $|\mathbf{a}|_{t=30} = 8.98$ ft sec^{-2}.

Result (c): $|\mathbf{a}|_{t=60} = 34.9$ ft sec^{-2}.

2.5.6 When the path of a point P in a reference frame R' is a circle because P is a point of a rigid body R which is rotating about an axis fixed in both R and R', the velocity and acceleration of P in R' may be found by using the expressions given in Sec. 2.4.5, Q now being any point on the axis of rotation. The terms ${}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r}$ and ${}^{R'}\boldsymbol{\omega}^R \times {}^R\mathbf{v}^{P/Q}$ in the expression for the acceleration are then, respectively, the tangential and the normal acceleration of P in R' (see 2.5.4).

Problem: Referring to Example 2.3.8, determine the tangential acceleration $\mathbf{a}_\tau$ and the normal acceleration $\mathbf{a}_\nu$ of point B at time t^*.

Solution: Fig. 2.5.6 shows bar B_1; the angular velocity $\boldsymbol{\omega}_1$ (as found in Example 2.3.8) and angular acceleration $\boldsymbol{\alpha}_1$ of this bar; and the unit vectors $\mathbf{n}_1$, $\mathbf{n}_1'$ and $\mathbf{k}$, previously shown in Fig. 2.3.8b.

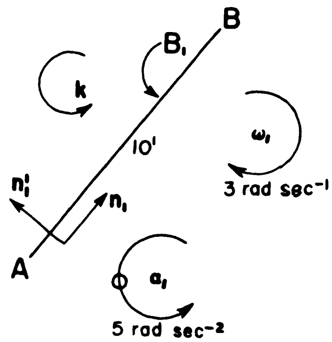


FIG. 2.5.6

Let $\mathbf{r}$ be the position vector of B relative to A. Then, as A is fixed on the axis of rotation of B_1,

$$\mathbf{a}_r = \boldsymbol{\alpha}_1 \times \mathbf{r} = 5\mathbf{k} \times (10\mathbf{n}_1) = 50\mathbf{n}_1' \text{ ft sec}^{-2}$$

and

$$\mathbf{a}_v = \boldsymbol{\omega}_1 \times \mathbf{v}^B = -3\mathbf{k} \times \mathbf{v}^B$$

where

$$\mathbf{v}^B = \boldsymbol{\omega} \times \mathbf{r} = -3\mathbf{k} \times (10\mathbf{n}_1) = -30\mathbf{n}_1' \text{ ft sec}^{-1}$$

Hence

$$\mathbf{a}_v = -3\mathbf{k} \times (-30\mathbf{n}_1') = -90\mathbf{n}_1 \text{ ft sec}^{-2}$$

2.5.7 The motion of a point P is said to be *rectilinear* in a reference frame R when the curve on which P moves in R is a straight line L. The velocity and acceleration of P in R are then parallel to this line, and are given by

$$^R\mathbf{v}^P = \frac{dx}{dt}\mathbf{n} \tag{1}$$

and

$$^R\mathbf{a}^P = \frac{d^2x}{dt^2}\mathbf{n} \tag{2}$$

where (see Fig. 2.5.7) $\mathbf{n}$ is a unit vector parallel to L and x is the

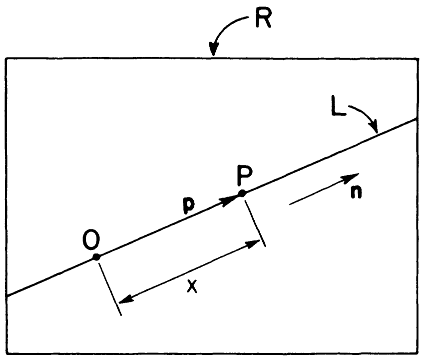


FIG. 2.5.7

displacement of P relative to a point O fixed on L, x being regarded as positive when the position vector $\mathbf{p}$ of P relative to O has the same sense as $\mathbf{n}$.

Proof: As x is positive when $\mathbf{p}$ has the same sense as $\mathbf{n}$,

$$\mathbf{p} = x\mathbf{n}$$

Now

$$^R\mathbf{v}^P \underset{(2.5.1)}{=} \frac{^R d\mathbf{p}}{dt} \underset{(1.2.5)}{=} \frac{d}{dt}(x\mathbf{n}) \underset{(1.5.1,1.2.2)}{=} \frac{dx}{dt}\mathbf{n}$$

Next

$$^R\mathbf{a}^P \underset{(2.5.1)}{=} \frac{^R d\,^R\mathbf{v}^P}{dt} \underset{(1.2.5)}{=} \frac{^R d}{dt}\left(\frac{dx}{dt}\mathbf{n}\right) \underset{(1.5.1,1.2.2)}{=} \frac{d^2x}{dt^2}\mathbf{n}$$

2.5.8 The expressions given in the preceding section can be obtained directly from Secs. 2.5.3 and 2.5.4 by noting that $\mathbf{n}$ is a vector tangent of the curve (L) on which P moves and regarding the radius of curvature of this curve as infinite. Furthermore, Secs. 2.5.3 and 2.5.4 lead to the following expressions, applicable to rectilinear motion:

$$^R\mathbf{v}^P = v\mathbf{n}, \qquad ^R\mathbf{a}^P = a\mathbf{n} \tag{1}$$

where

$$v = \frac{dx}{dt}, \qquad a = \frac{dv}{dt} \tag{2}$$

and v and a are then called, respectively, the speed of P in R for the $\mathbf{n}$ direction and the scalar acceleration of P in R for the $\mathbf{n}$ direction.

Each of the four quantities t, x, v, a may be plotted on rectangular axes as a function of any one of the remaining three; and these plots, called *motion curves*, can be used to solve certain problems by graphical methods. The following properties of motion curves form the bases of such solutions:

1. The slopes of the x–t and v–t curves are proportional to v and a. This follows from the definitions of v and a, Eqs. (2), above.

2. The areas under any portions of the v–t and a–t curves are proportional to the corresponding algebraic increases in x and v, for these areas are proportional to

$$\int_{t_1}^{t_2} v\,dt \quad \text{and} \quad \int_{t_1}^{t_2} a\,dt$$

or, replacing v with dx/dt and a with dv/dt, to

$$\int_{t_1}^{t_2} \frac{dx}{dt}\,dt \quad \text{and} \quad \int_{t_1}^{t_2} \frac{dv}{dt}\,dt$$

so that, integrating, one obtains

$$x|_{t_2} - x|_{t_1} \quad \text{and} \quad v|_{t_2} - v|_{t_1}$$

3. The product of the slope at, and ordinate of, a point on the v–x curve is proportional to a, for this product is given by

$$\frac{dv}{dx}v = \frac{dv}{dt}\frac{dt}{dx}v = \frac{dv}{dt}\frac{v}{dx/dt} = a\frac{v}{v} = a$$

4. The area under any portion of the a–x curve is proportional to one-half of the algebraic increase in v^2, for this area is proportional to

$$\int_{x_1}^{x_2} a\,dx = \int_{x_1}^{x_2} \frac{dv}{dt}\,dx = \int_{x_1}^{x_2} \frac{dv}{dx}\frac{dx}{dt}\,dx = \int_{x_1}^{x_2} \frac{dv}{dx}v\,dx$$

$$= \int_{x_1}^{x_2} \frac{d}{dx}(\tfrac{1}{2}v^2)\,dx = \tfrac{1}{2}[(v|_{x_2})^2 - (v|_{x_1})^2]$$

As the relationship between the quantities t, x, v, a is precisely the same as that between the quantities t, s, v_τ, a_τ in Secs. 2.5.3 and 2.5.4, the above theorems apply to the latter quantities when x is replaced with s, v with v_τ, and a with a_τ.

Problem: Figure 2.5.8 shows the a_τ–s curve for a point P which moves on a helix. It is known that P has a velocity of 12 ft sec^{-1} when $s = 7$ ft. Determine the speed of P for the τ direction when $s = 1$ ft.

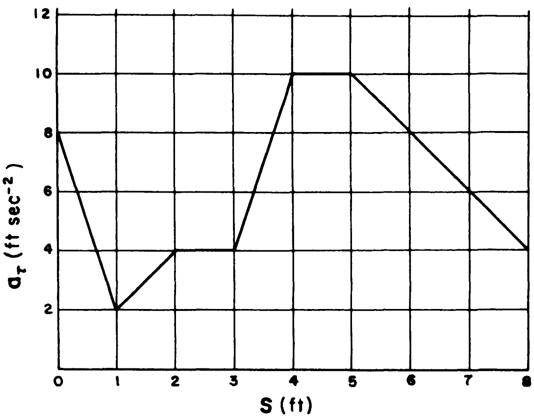


FIG. 2.5.8

Solution: Let $(v_\tau)_1$ and $(v_\tau)_2$ be the speeds of P for the $\boldsymbol{\tau}$ direction when $s = 1$ ft and $s = 7$ ft, respectively, and let A be the area under the a_τ–s curve between $s = 1$ ft and $s = 7$ ft. Then

$$(v_\tau)_2 = \pm 12 \text{ ft sec}^{-1}$$

and

$$A = \tfrac{1}{2}[(v_\tau)_2^2 - (v_\tau)_1^2] = \tfrac{1}{2}[144 - (v_\tau)_1^2]$$

so that

$$(v_\tau)_1 = \pm(144 - 2A)^{1/2}$$

Find A by counting squares in Fig. 2.5.8:

$$A = 20 \text{ squares} = 40 \text{ ft}^2 \text{ sec}^{-2}$$

Then

$$(v_\tau)_1 = \pm(144 - 80)^{1/2} = \pm 8 \text{ ft sec}^{-1}$$

2.5.9 The velocity ${}^{R'}\mathbf{v}^P$ and acceleration ${}^{R'}\mathbf{a}^P$ of every point P of a rigid body R in a reference frame R' can be found as soon as the angular velocity ${}^{R'}\boldsymbol{\omega}^R$, angular acceleration ${}^{R'}\boldsymbol{\alpha}^R$, velocity ${}^{R'}\mathbf{v}^Q$, and acceleration ${}^{R'}\mathbf{a}^Q$ are known, Q being any point fixed on R:

$$^{R'}\mathbf{v}^P = {}^{R'}\mathbf{v}^Q + {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r} \tag{1}$$

$$^{R'}\mathbf{a}^P = {}^{R'}\mathbf{a}^Q + {}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}) \tag{2}$$

where $\mathbf{r}$ is the position vector of P relative to Q. (Q is called a *base-point.*)

Proof: Let O be a point fixed in R' (see Fig. 2.5.9a), $\mathbf{p}$ the position vector of P relative to O, and $\mathbf{q}$ the position vector of Q relative to O. Then

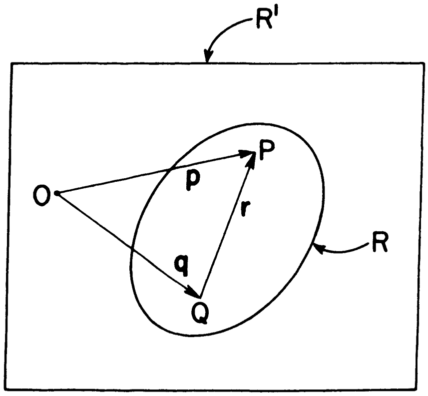


FIG. 2.5.9a

$$ {}^{R'}\mathbf{v}^P \underset{(2.5.1)}{=} \frac{{}^{R'}d\mathbf{p}}{dt} \underset{(1.2.5)}{=} \frac{{}^{R'}d}{dt}(\mathbf{q}+\mathbf{r}) \underset{(1.4.1)}{=} \frac{{}^{R'}d\mathbf{q}}{dt} + \frac{{}^{R'}d\mathbf{r}}{dt} $$

or

$$ {}^{R'}\mathbf{v}^P = \underset{(2.5.1)}{{}^{R'}\mathbf{v}^Q} + \underset{(2.4.1,2.4.5)}{{}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}} \tag{A} $$

Next

$$ {}^{R'}\mathbf{a}^P \underset{(2.5.1)}{=} \frac{{}^{R'}d\,{}^{R'}\mathbf{v}^P}{dt} \underset{(A)}{=} \frac{{}^{R'}d}{dt}({}^{R'}\mathbf{v}^Q + {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}) $$

$$ \underset{(1.4.1)}{=} \frac{{}^{R'}d\,{}^{R'}\mathbf{v}^Q}{dt} + \frac{{}^{R'}d\,{}^{R'}\boldsymbol{\omega}^R}{dt} \times \mathbf{r} \underset{(1.5.3)}{+} {}^{R'}\boldsymbol{\omega}^R \times \frac{{}^{R'}d\mathbf{r}}{dt} $$

$$ = \underset{(2.5.1)}{{}^{R'}\mathbf{a}^Q} + \underset{(2.3.1)}{{}^{R'}\boldsymbol{\alpha}^R} \times \mathbf{r} + \underset{(2.4.1,2.4.5)}{{}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r})} $$

Problem: Referring to Example 2.3.8, and using the results obtained in Problem 2.5.6, determine the acceleration $\mathbf{a}^P$ at time t^*, P being a point fixed on bar B_2, four feet to the left of point B.

Solution: Figure 2.5.9b shows bar B_2, the angular velocity $\boldsymbol{\omega}_2$ and angular acceleration $\boldsymbol{\alpha}_2$ of this bar (previously shown in Figs. 2.3.8c and 2.3.8d), and the acceleration $\mathbf{a}^B$ of point B, resolved

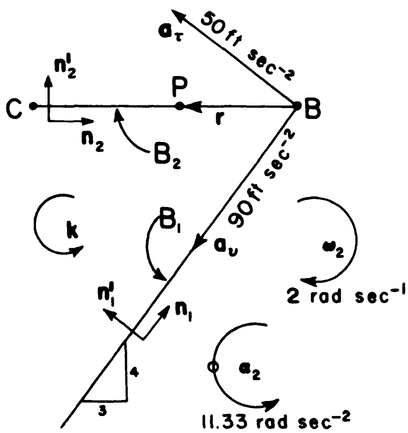


FIG. 2.5.9b

into the two components $\mathbf{a}_\tau$ and $\mathbf{a}_\nu$ found in Problem 2.5.6. (The unit vectors $\mathbf{k}$, $\mathbf{n}_1$, $\mathbf{n}_1'$, etc., are those previously shown in Fig. 2.3.8b.)

The position vector $\mathbf{r}$ of P relative to B is given by

$$\mathbf{r} = -4\mathbf{n}_2 \text{ ft}$$

Hence, using B as base point,

$$\begin{aligned}
\mathbf{a}^P &= \mathbf{a}^B + \boldsymbol{\alpha}_2 \times \mathbf{r} + \boldsymbol{\omega}_2 \times (\boldsymbol{\omega}_2 \times \mathbf{r}) \\
&= -90\mathbf{n}_1 + 50\mathbf{n}_1' + 11.33\mathbf{k} \times (-4\mathbf{n}_2) \\
&\qquad\qquad + (-2\mathbf{k}) \times [(-2\mathbf{k}) \times (-4\mathbf{n}_2)] \\
&= -90\mathbf{n}_1 + 50\mathbf{n}_1' - 45.32\mathbf{n}_2' + 16\mathbf{n}_2 \text{ ft sec}^{-2}
\end{aligned}$$

2.5.10 The velocity relationship of Sec. 2.5.9 furnishes a convenient means for the determination of the velocity of a point P fixed on a rigid body R which is *rolling* on a surface S. For, by definition of rolling, there exists at every instant during such a motion at least one point P_R of R (see Fig. 2.5.10) which is in

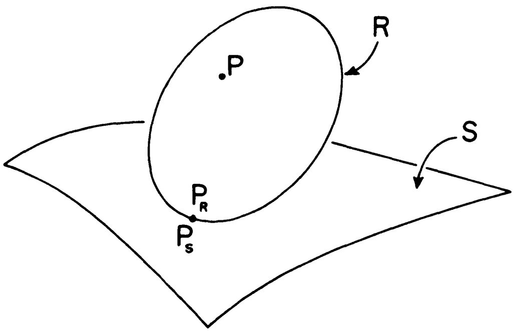


FIG. 2.5.10

contact with a point P_S of S and whose velocity (in any reference frame) is equal to that of P_S. (If ${}^S\boldsymbol{\omega}^R$ is parallel to the tangent plane of S at P_S, R and S are said to be in *pure rolling contact.* If ${}^S\boldsymbol{\omega}^R$ is perpendicular to this plane, R is said to be *pivoting* on S.) P_R is a convenient base point because its velocity is known as soon as the motion of S has been specified.

Problem: Referring to Example 2.2.10, and assuming that D rolls on P, express the velocity ${}^P\mathbf{v}^{D^*}$ and acceleration ${}^P\mathbf{a}^{D^*}$ in terms of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and θ, ϕ, ψ (and their time-derivatives), the radius of D being r.

Solution: Using C as base point,

$$^P\mathbf{v}^C = 0$$

and

$$^P\mathbf{v}^{D^*} \underset{(2.5.9)}{=} {}^P\boldsymbol{\omega}^D \times \mathbf{r}$$

where

$$\mathbf{r} = -r\mathbf{n}_1$$

and

$$^P\boldsymbol{\omega}^D \underset{(E2.2.10)}{=} -\dot{\phi}\cos\theta\mathbf{n}_1 - \dot{\theta}\mathbf{n}_2 + (\dot{\psi} + \dot{\phi}\sin\theta)\mathbf{n}_3$$

Hence

$$^P\mathbf{v}^{D^*} = -r[(\dot{\psi} + \dot{\phi}\sin\theta)\mathbf{n}_2 + \dot{\theta}\mathbf{n}_3]$$

Next

$$^P\mathbf{a}^{D^*} \underset{(2.5.1)}{=} \frac{^Pd\,^P\mathbf{v}^{D^*}}{dt} \underset{(2.1.4)}{=} \frac{^{R_1}d}{dt}\,^P\mathbf{v}^{D^*} + {}^P\boldsymbol{\omega}^{R_1} \times {}^P\mathbf{v}^{D^*}$$

$$= \frac{^{R_1}d}{dt}\,^P\mathbf{v}^{D^*} + ({}^{R_2}\boldsymbol{\omega}^{R_1} \underset{(2.2.7)}{+} {}^P\boldsymbol{\omega}^{R_2}) \times {}^P\mathbf{v}^{D^*}$$

$$= -r[(\ddot{\psi} + \ddot{\phi}\sin\theta + \dot{\phi}\dot{\theta}\cos\theta)\mathbf{n}_2 + \ddot{\theta}\mathbf{n}_3] + [-\dot{\theta}\mathbf{n}_2 + \dot{\phi}(-\cos\theta\mathbf{n}_1 + \sin\theta\mathbf{n}_3)] \times {}^P\mathbf{v}^{D^*}$$

That is

$$\begin{aligned} ^P\mathbf{a}^{D^*} = {} & r[\dot{\theta}^2 + \dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\sin\theta)]\mathbf{n}_1 \\ & - r[\ddot{\psi} + \ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta]\mathbf{n}_2 \\ & + r[-\ddot{\theta} + \dot{\phi}\cos\theta(\dot{\psi} + \dot{\phi}\sin\theta)]\mathbf{n}_3 \end{aligned}$$

2.5.11 At any instant at which the angular velocity $^{R'}\boldsymbol{\omega}^R$ of a rigid body R in a reference frame R' is equal to zero, the velocities of all points of R in R' are equal to each other (see 2.5.9). If $^{R'}\boldsymbol{\omega}^R$ is not equal to zero at a given instant, the points of R (or R extended) having at this instant a minimum velocity ($^{R'}\mathbf{v}^*$) in R' lie on a line (L^*) which is parallel to $^{R'}\boldsymbol{\omega}^R$ and passes through a point whose position vector ($\mathbf{r}^*$) relative to an arbitrarily selected point (Q) of R is given by

$$\mathbf{r}^* = \frac{^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{v}^Q}{(^{R'}\boldsymbol{\omega}^R)^2} \tag{1}$$

The line L^* is called the *instantaneous axis* of R in R'. The *minimum velocity* is given by

$$^{R'}\mathbf{v}^* = \frac{^{R'}\boldsymbol{\omega}^R \cdot {}^{R'}\mathbf{v}^Q}{(^{R'}\boldsymbol{\omega}^R)^2}\,{}^{R'}\boldsymbol{\omega}^R \tag{2}$$

Proof: Let P and Q be two points fixed on R (see Fig. 2.5.11), and let $\mathbf{r}$ be the position vector of P relative to Q. Then the reso-

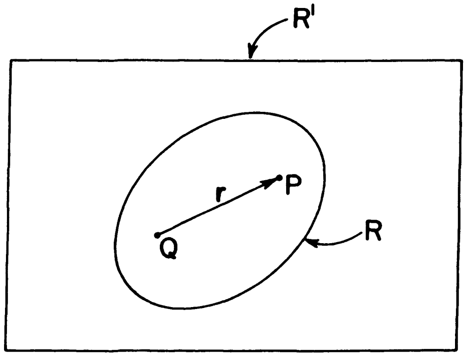


FIG. 2.5.11

lutes parallel to ${}^{R'}\boldsymbol{\omega}^{R}$ of the velocities ${}^{R'}\mathbf{v}^{P}$ and ${}^{R'}\mathbf{v}^{Q}$ are given by (see Vol. I, Sec. 1.14.6)

$$\frac{{}^{R'}\boldsymbol{\omega}^{R} \cdot {}^{R'}\mathbf{v}^{P}}{({}^{R'}\boldsymbol{\omega}^{R})^2}\,{}^{R'}\boldsymbol{\omega}^{R} \quad \text{and} \quad \frac{{}^{R'}\boldsymbol{\omega}^{R} \cdot {}^{R'}\mathbf{v}^{Q}}{({}^{R'}\boldsymbol{\omega}^{R})^2}\,{}^{R'}\boldsymbol{\omega}^{R}$$

and, as

$$^{R'}\boldsymbol{\omega}^{R} \cdot {}^{R'}\mathbf{v}^{P} \underset{(2.5.9)}{=} {}^{R'}\boldsymbol{\omega}^{R} \cdot {}^{R'}\mathbf{v}^{Q} + {}^{R'}\boldsymbol{\omega}^{R} \cdot {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{r} = {}^{R'}\boldsymbol{\omega}^{R} \cdot {}^{R'}\mathbf{v}^{Q} + 0$$

these resolutes are equal to each other. In other words, the velocities of all points of R have equal resolutes parallel to ${}^{R'}\boldsymbol{\omega}^{R}$. Thus, if there exists a point of R whose velocity is parallel to ${}^{R'}\boldsymbol{\omega}^{R}$, this velocity is smaller than that of any point whose velocity is not parallel to ${}^{R'}\boldsymbol{\omega}^{R}$; and letting ${}^{R'}\mathbf{v}^{*}$ be this minimum velocity,

$$^{R'}\mathbf{v}^{*} = \frac{{}^{R'}\boldsymbol{\omega}^{R} \cdot {}^{R'}\mathbf{v}^{Q}}{({}^{R'}\boldsymbol{\omega}^{R})^2}\,{}^{R'}\boldsymbol{\omega}^{R}$$

To show that there always exists at least one point whose velocity is parallel to ${}^{R'}\boldsymbol{\omega}^{R}$ (and hence equal to ${}^{R'}\mathbf{v}^{*}$), it is only necessary to find a vector $\mathbf{r}^{*}$ which satisfies the equation

$$^{R'}\mathbf{v}^{*} \underset{(2.5.9)}{=} {}^{R'}\mathbf{v}^{Q} + {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{r}^{*}$$

that is, the equation

$$\frac{{}^{R'}\boldsymbol{\omega}^R \cdot {}^{R'}\mathbf{v}^Q}{({}^{R'}\boldsymbol{\omega}^R)^2}\,{}^{R'}\boldsymbol{\omega}^R = {}^{R'}\mathbf{v}^Q + {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}^*$$

or, subtracting ${}^{R'}\mathbf{v}^Q$ from both sides,

$${}^{R'}\boldsymbol{\omega}^R \times \frac{{}^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{v}^Q}{({}^{R'}\boldsymbol{\omega}^R)^2} = {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}^* \tag{A}$$

The vector

$$\mathbf{r}^* = \frac{{}^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{v}^Q}{({}^{R'}\boldsymbol{\omega}^R)^2} \tag{B}$$

clearly satisfies Eq. (A). Furthermore, when any vector parallel to ${}^{R'}\boldsymbol{\omega}^R$ is added to $\mathbf{r}^*$, the result continues to satisfy Eq. (A). Thus there exist infinitely many points whose velocities are equal to the minimum velocity ${}^{R'}\mathbf{v}^*$, and they lie on a straight line L^* which is parallel to ${}^{R'}\boldsymbol{\omega}^R$ and passes through the point whose position vector relative to Q is $\mathbf{r}^*$ as given in Eq. (B). It remains to be shown that the points of R lying on L^* are the only points of R whose velocities are equal to ${}^{R'}\mathbf{v}^*$.

Using a point P^* on L^* as base point, the velocity ${}^{R'}\mathbf{v}^A$ of any point A not lying on L^* is given by

$${}^{R'}\mathbf{v}^A \underset{(2.5.9)}{=} {}^{R'}\mathbf{v}^{P^*} + {}^{R'}\boldsymbol{\omega}^R \times \mathbf{a}$$

where the position vector $\mathbf{a}$ of A relative to P^* is not parallel to ${}^{R'}\boldsymbol{\omega}^R$. As

$${}^{R'}\mathbf{v}^{P^*} = {}^{R'}\mathbf{v}^*$$

${}^{R'}\mathbf{v}^A$ is thus given by

$${}^{R'}\mathbf{v}^A = {}^{R'}\mathbf{v}^* + {}^{R'}\boldsymbol{\omega}^R \times \mathbf{a}$$

and cannot be equal to ${}^{R'}\mathbf{v}^*$, because ${}^{R'}\boldsymbol{\omega}^R \times \mathbf{a}$ is not equal to zero.

Problem: Given a rigid body R and a reference frame R', show that (a) when R moves in such a way that one point of R remains fixed in R', the instantaneous axis of R in R' passes through this point, and (b) when R moves in such a way that a line fixed in R remains fixed in R', the instantaneous axis of R in R' coincides with this line.

Solution (a): Let the fixed point be the base point Q. Then

$${}^{R'}\mathbf{v}^Q = 0$$

and the position vector $\mathbf{r}^*$ of one point on the instantaneous axis, relative to the fixed point, is equal to zero, which shows that the instantaneous axis passes through the fixed point.

Solution (b): Let any point of the fixed line be the base point Q. Then, as in (a) above, it follows that the instantaneous axis passes through this point. Hence the instantaneous axis passes through every point of the fixed line, that is, it coincides with it.

2.5.12 If at any instant t^* during the motion of a rigid body R in a reference frame R' the velocity of one point of R (or R extended) is perpendicular to ${}^{R'}\boldsymbol{\omega}^R$, the velocity of any other point of R is either also perpendicular to ${}^{R'}\boldsymbol{\omega}^R$ at this instant, or it is equal to zero. (This follows from the fact that the velocities of all points of R have equal resolutes parallel to ${}^{R'}\boldsymbol{\omega}^R$.) R is then said to be in a state of *plane motion* in R' at this instant.

In accordance with Sec. 2.5.11, the velocity in R' of every point of R lying on the instantaneous axis of R in R' is equal to zero whenever R is in a state of plane motion in R'. For this reason every point of the instantaneous axis is called an *instantaneous center* of R in R'. Furthermore, the velocity ${}^{R'}\mathbf{v}^P$ (but *not*, in general, the acceleration ${}^{R'}\mathbf{a}^P$) of any point P of R has at this instant the value which it would have if the instantaneous axis were fixed both in R and in R' and R were revolving about this axis with an angular velocity equal to the actual angular velocity at this instant; for, at time t^*, ${}^{R'}\mathbf{v}^P$ is given by

$$ {}^{R'}\mathbf{v}^P \underset{(2.5.9)}{=} {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r} $$

where $\mathbf{r}$ is the position vector of P relative to any instantaneous center, and, from Sec. 2.5.6, the velocity of P during the fictitious motion described above is given by the same expression.

To locate the instantaneous axis, and thus all instantaneous centers, of a body in plane motion, it is sufficient to know the orientations of the velocity vectors ${}^{R'}\mathbf{v}^{P_1}$ and ${}^{R'}\mathbf{v}^{P_2}$ of any two points P_1 and P_2 of R, provided these velocities be nonparallel: The instantaneous axis is the intersection of two planes passing through P_1 and P_2, each plane being perpendicular to the velocity vector of the point through which it passes.

Proof: Let P^* be a point on the instantaneous axis. Then the position vector $\mathbf{r}_1^*$ of P^* relative to P_1 is given by

$$\mathbf{r}_1^* \underset{(2.5.11)}{=} \frac{{}^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{v}^{P_1}}{({}^{R'}\boldsymbol{\omega}^R)^2} + \lambda_1 {}^{R'}\boldsymbol{\omega}^R$$

where λ_1 is a certain scalar, and, similarly, the position vector $\mathbf{r}_2^*$ of P^* relative to P_2 is given by

$$\mathbf{r}_2^* = \frac{{}^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{v}^{P_2}}{({}^{R'}\boldsymbol{\omega}^R)^2} + \lambda_2 {}^{R'}\boldsymbol{\omega}^R$$

Now, $\mathbf{r}_1^*$ is perpendicular to ${}^{R'}\mathbf{v}^{P_1}$, and $\mathbf{r}_2^*$ to ${}^{R'}\mathbf{v}^{P_2}$, as

$$\mathbf{r}_1^* \cdot {}^{R'}\mathbf{v}^{P_1} = \lambda_1 {}^{R'}\boldsymbol{\omega}^R \cdot {}^{R'}\mathbf{v}^{P_1}$$

$$\mathbf{r}_2^* \cdot {}^{R'}\mathbf{v}^{P_2} = \lambda_2 {}^{R'}\boldsymbol{\omega}^R \cdot {}^{R'}\mathbf{v}^{P_2}$$

and ${}^{R'}\boldsymbol{\omega}^R$ is perpendicular to both ${}^{R'}\mathbf{v}^{P_1}$ and ${}^{R'}\mathbf{v}^{P_2}$ when R is in a state of plane motion in R'. Thus P^* lies in a plane which passes through P_1 and is perpendicular to ${}^{R'}\mathbf{v}^{P_1}$, and P^* also lies in a plane which passes through P_2 and is perpendicular to ${}^{R'}\mathbf{v}^{P_2}$. Accordingly, P^* lies on the intersection of these two planes.

Problem: Referring to Example 2.3.8, locate an instantaneous center of bar B_2 at time t^*, and use it to determine the angular velocity of this bar at this instant.

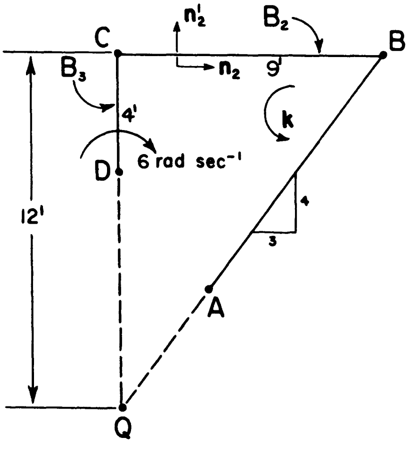


FIG. 2.5.12

Solution: Point C (see Fig. 2.5.12) moves on a circle, center at D. Hence the velocity of C is perpendicular to CD, that is, line CD lies in a plane which passes through C and is perpendicular to the velocity vector of C. Therefore, line CD (or line CD extended) passes through the instantaneous axis of bar B_2. Similarly, point B moves on a circle, center at A, whence line AB (or line AB extended) passes through the instantaneous axis of bar B_2. It follows that point Q, the intersection of lines CD and AB, is an instantaneous center of bar B_2.

The velocity of C, when C is regarded as a point of bar B_3, is given by

$$\mathbf{v}^C \underset{(2.5.6)}{=} 24\mathbf{n}_2 \text{ ft sec}^{-1}$$

Alternatively, when C is regarded as a point of bar B_2—and B_2 as a rigid body rotating about a line which passes through Q, is parallel to $\mathbf{k}$, and is fixed in bar B_2—then $\mathbf{v}^C$ is given by

$$\mathbf{v}^C \underset{(2.5.6)}{=} -12\omega_2\mathbf{n}_2$$

where ω_2 is the angular speed of B_2 for the $\mathbf{k}$ direction. Hence

$$24 = -12\omega_2$$

and the angular velocity of B_2 at time t^* has the value $-2\mathbf{k}$ rad sec^{-1}.

2.5.13 Given two reference frames R and R', the velocities ${}^R\mathbf{v}^P$ and ${}^{R'}\mathbf{v}^P$ of a point P at an instant t^* are related as follows:

$$^{R'}\mathbf{v}^P = {}^R\mathbf{v}^P + {}^{R'}\mathbf{v}^{P^*} \tag{1}$$

where P^* is that point fixed in R which coincides with P at time t^*. The accelerations ${}^R\mathbf{a}^P$ and ${}^{R'}\mathbf{a}^P$ at time t^* are related by

$$^{R'}\mathbf{a}^P = {}^R\mathbf{a}^P + {}^{R'}\mathbf{a}^{P^*} + 2\,{}^{R'}\boldsymbol{\omega}^R \times {}^R\mathbf{v}^P \tag{2}$$

${}^{R'}\mathbf{v}^{P^*}$ and ${}^{R'}\mathbf{a}^{P^*}$ are called, variously, the *drag-*, *vehicle-*, *transport-*, or *coincident point velocity and acceleration for point P in the reference frames R and R'*. The vector $2{}^{R'}\boldsymbol{\omega}^R \times {}^R\mathbf{v}^P$ is called the *Coriolis acceleration of P for the reference frames R and R'*.

Proof: Let O be a point fixed in R, O' a point fixed in R', $\mathbf{p}$ the position vector of P relative to O, $\mathbf{p}'$ the position vector of P relative to O', and $\mathbf{r}$ the position on vector of O relative to O', as shown in Fig. 2.5.13a. Then

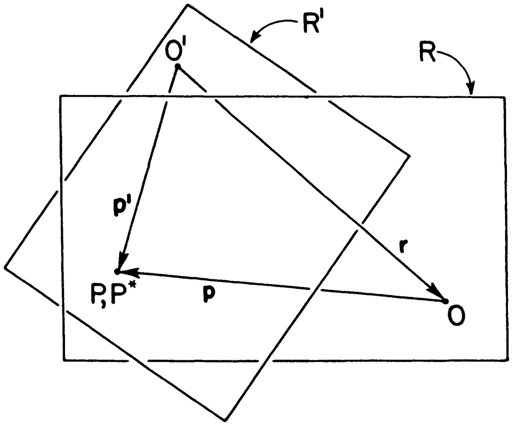


FIG. 2.5.13a

$$^{R'}\mathbf{v}^{P^*} \underset{(2.5.9)}{=} {}^{R'}\mathbf{v}^{O} + {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{p} \tag{A}$$

$$^{R'}\mathbf{a}^{P^*} \underset{(2.5.9)}{=} {}^{R'}\mathbf{a}^{O} + {}^{R'}\boldsymbol{\alpha}^{R} \times \mathbf{p} + {}^{R'}\boldsymbol{\omega}^{R} \times ({}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{p}) \tag{B}$$

$$^{R'}\mathbf{v}^{P} \underset{(2.5.1)}{=} \frac{^{R'}d\mathbf{p}'}{dt} = \frac{^{R'}d}{dt}(\mathbf{r} + \mathbf{p}) \underset{(1.4.1)}{=} \frac{^{R'}d\mathbf{r}}{dt} + \frac{^{R'}d\mathbf{p}}{dt} \tag{C}$$

$$\frac{^{R'}d\mathbf{r}}{dt} \underset{(2.5.1)}{=} {}^{R'}\mathbf{v}^{O} \tag{D}$$

$$\frac{^{R'}d\mathbf{p}}{dt} \underset{(2.1.4)}{=} \frac{^{R}d\mathbf{p}}{dt} + {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{p} \underset{(2.5.1)}{=} {}^{R}\mathbf{v}^{P} + {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{p} \tag{E}$$

$$^{R'}\mathbf{v}^{P} \underset{(C,D,E)}{=} {}^{R'}\mathbf{v}^{O} + {}^{R}\mathbf{v}^{P} + {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{p} \tag{F}$$

$$^{R'}\mathbf{v}^{P} \underset{(F,A)}{=} {}^{R}\mathbf{v}^{P} + {}^{R'}\mathbf{v}^{P^*}$$

Next

$$^{R'}\mathbf{a}^{P} \underset{(2.5.1)}{=} \frac{^{R'}d\,^{R'}\mathbf{v}^{P}}{dt} \underset{(F)}{=} \frac{^{R'}d}{dt}({}^{R'}\mathbf{v}^{O} + {}^{R}\mathbf{v}^{P} + {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{p})$$

$$\underset{(1.4.1)}{=} \frac{^{R'}d\,^{R'}\mathbf{v}^{O}}{dt} + \frac{^{R}d\,^{R}\mathbf{v}^{P}}{dt} \underset{(2.1.4)}{+} {}^{R'}\boldsymbol{\omega}^{R} \times {}^{R}\mathbf{v}^{P}$$

$$+ \frac{^{R'}d\,^{R'}\boldsymbol{\omega}^{R}}{dt} \times \mathbf{p} \underset{(1.5.3)}{+} {}^{R'}\boldsymbol{\omega}^{R} \times \frac{^{R'}d\mathbf{p}}{dt}$$

$$= \underset{(2.5.1)}{{}^{R'}\mathbf{a}^{O}} + \underset{(2.5.1)}{{}^{R}\mathbf{a}^{P}} + {}^{R'}\boldsymbol{\omega}^{R} \times {}^{R}\mathbf{v}^{P} + \underset{(2.3.1)}{{}^{R'}\boldsymbol{\alpha}^{R}} \times \mathbf{p}$$

$$+ {}^{R'}\boldsymbol{\omega}^{R} \times ({}^{R}\mathbf{v}^{P} \underset{(E)}{+} {}^{R'}\boldsymbol{\omega}^{R} \times \mathbf{p})$$

$$\underset{(B)}{=} {}^{R}\mathbf{a}^{P} + {}^{R'}\mathbf{a}^{P^*} + 2\,{}^{R'}\boldsymbol{\omega}^{R} \times {}^{R}\mathbf{v}^{P}$$

Problem: An airplane P flies due south at a constant (low) altitude above a meridian of longitude, with a constant speed of 600 mile hr^{-1}. At a certain instant the line joining P to the center O of the earth makes an angle of 45 degrees with the earth's north–south axis. Letting R be a reference frame fixed in the earth, and R' a reference frame in which the earth's center is fixed and whose motion is such that

$${}^{R'}\boldsymbol{\omega}^{R} = \frac{\pi}{12}\,\mathbf{k} \text{ rad hr}^{-1}$$

where $\mathbf{k}$ (see Fig. 2.5.13b) is a unit vector fixed in both R and R', determine the east–west, north–south, and up–down components of the acceleration of P in R' at this instant.

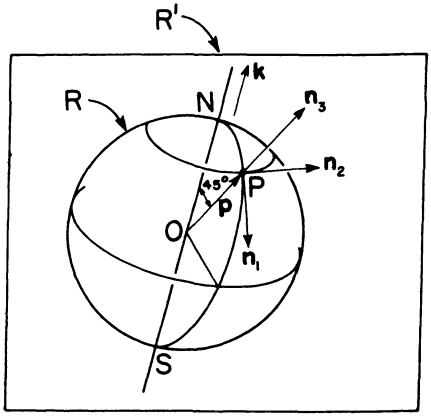


FIG. 2.5.13b

Solution: Let P^* be the point of the earth which coincides with P at the instant under consideration, $\mathbf{p}$ the position vector of P (or P^*) relative to O, $\mathbf{n}_1$ a unit vector perpendicular to line

OP and parallel to the meridian plane passing through P ($\mathbf{n}_1$ points southward at P), $\mathbf{n}_2$ a unit vector perpendicular to the meridian plane passing through P ($\mathbf{n}_2$ points eastward at P), $\mathbf{n}_3$ a unit vector parallel to line OP ($\mathbf{n}_3$ points upward at P).

In R, P moves on a circle, center O, radius 3960 miles. See Secs. 2.5.5 and 2.5.4: The tangential acceleration of P in R is equal to zero, because the speed of P in R for the $\mathbf{n}_1$ direction (see 2.5.3) is constant (600 mile hr^{-1}). The normal acceleration of P in R has the value

$$\frac{(600)^2}{3960}(-\mathbf{n}_3) = -91\mathbf{n}_3 \text{ mile hr}^{-2}$$

$-\mathbf{n}_3$ being the vector principal normal of the curve on which P moves in R. Hence

$${}^R\mathbf{a}^P = -91\mathbf{n}_3 \text{ mile hr}^{-2}$$

Using O as base point (see 2.5.9), find the transport acceleration of P for the reference frames R and R':

$${}^{R'}\mathbf{a}^{P*} \underset{(2.5.9)}{=} {}^{R'}\mathbf{a}^O + {}^{R'}\boldsymbol{\alpha}^R \times \mathbf{p} + {}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{p})$$

where

$${}^{R'}\mathbf{a}^O \underset{(2.5.2)}{=} 0$$

$${}^{R'}\boldsymbol{\alpha}^R \underset{(2.3.1)}{=} \frac{{}^{R'}d}{dt}\,{}^R\boldsymbol{\omega}^{R'} = \frac{\pi}{12}\frac{{}^{R'}d\mathbf{k}}{dt} \underset{(1.2.2)}{=} 0$$

$${}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{p}) = \frac{\pi}{12}\mathbf{k} \times \left[\frac{\pi}{12}\mathbf{k} \times (3960\mathbf{n}_3)\right]$$

$$= -136(\mathbf{n}_1 + \mathbf{n}_3) \text{ mile hr}^{-2}$$

Thus

$${}^{R'}\mathbf{a}^{P*} = -136(\mathbf{n}_1 + \mathbf{n}_3) \text{ mile hr}^{-2}$$

The velocity of P in R is given by

$${}^R\mathbf{v}^P \underset{(2.5.3)}{=} 600\mathbf{n}_1$$

Hence the Coriolis acceleration of P for the reference frames R and R' has the value

$$2\,{}^{R'}\boldsymbol{\omega}^R \times {}^R\mathbf{v}^P = \frac{2\pi}{12}\mathbf{k} \times (600\mathbf{n}_1) = 222\mathbf{n}_2 \text{ mile hr}^{-2}$$

Consequently

$$\begin{aligned} {}^{R'}\mathbf{a}^P &\underset{(2)}{=} {}^{R}\mathbf{a}^P + {}^{R'}\mathbf{a}^{P*} + 2\,{}^{R'}\boldsymbol{\omega}^R \times {}^{R}\mathbf{v}^P \\ &= -91\mathbf{n}_3 - 136(\mathbf{n}_1 + \mathbf{n}_3) + 222\mathbf{n}_2 \\ &= -136\mathbf{n}_1 + 222\mathbf{n}_2 - 227\mathbf{n}_3 \text{ mile hr}^{-2} \end{aligned}$$

Result: The acceleration of P in R' has the following components: 136 mile hr^{-2}, northward; 222 mile hr^{-2}, eastward; 227 mile hr^{-2}, downward.

2.5.14 When a rigid body R rolls on a surface S (see 2.5.10) in such a way that only one point of R is in contact with S at any instant, the points P_R of R which successively come into contact with S form a curve C_R on R, and the points P_S of S which come into contact with R form a curve C_S on S (see Fig. 2.5.14a). These curves have a common tangent at their point of contact; and if P_R^* and P_S^* are the points of the two curves which coincide with each other at an instant t^*, and s_R and s_S are the arc-length displacements of P_R relative to P_R^* and of P_S^* relative to P_S^*, then

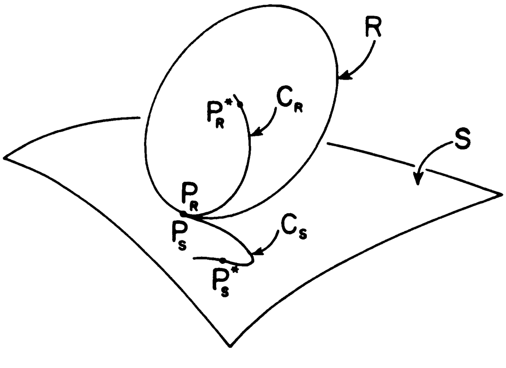


FIG. 2.5.14a

$$s_R = s_S$$

Proof: There exists a point P which remains at all times on both C_R and C_S. (P coincides with P_R and P_S.) In accordance with Sec. 2.5.3,

$${}^{R}\mathbf{v}^P = \frac{ds_R}{dt}\,\boldsymbol{\tau}_R, \qquad {}^{S}\mathbf{v}^P = \frac{ds_S}{dt}\,\boldsymbol{\tau}_S$$

where τ_R and τ_S are vector tangents of C_R and C_S at P_R and P_S. Now

$$^S\mathbf{v}^P \underset{(2.5.13)}{=} {}^R\mathbf{v}^P + {}^S\mathbf{v}^{P_R}$$

and, by definition of rolling (see 2.5.10),

$$^S\mathbf{v}^{P_R} = {}^S\mathbf{v}^{P_S}$$

But, P_S being a point of S,

$$^S\mathbf{v}^{P_S} = 0$$

Hence

$$^S\mathbf{v}^P = {}^R\mathbf{v}^P$$

and

$$\frac{ds_R}{dt}\tau_R = \frac{ds_S}{dt}\tau_S$$

which shows that C_R and C_S have a common tangent at their point of contact and that

$$\frac{ds_R}{dt} = \frac{ds_S}{dt}$$

Consequently

$$s_R = s_S + \text{const.}$$

and, as both s_R and s_S vanish at t^*, it follows that

$$s_R = s_S$$

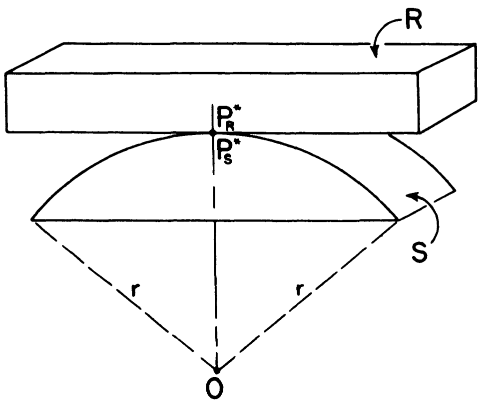


FIG. 2.5.14b

Problem: A block R, placed on a right-cylindrical surface S of radius r (see Fig. 2.5.14b), performs oscillations during which R rolls on S. For an instant at which R occupies the position shown in Fig. 2.5.14c, determine the distance s_R between points P_R^* and P_R.

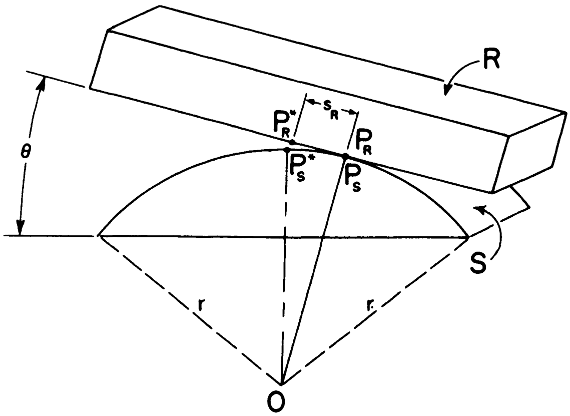


FIG. 2.5.14c

Solution: While R is not a body only one point of which is in contact with S at any instant, the face of R which contains points P_R^* and P_R may be regarded as such a body. Hence

$$s_R = s_S$$

where s_S is the length of the circular arc $P_S^* P_S$ (see Fig. 1.5.14c). As the angle P_S^*-O-P_S is equal to θ,

$$s_S = r\theta$$

Hence

$$s_R = r\theta$$

2.5.15 The definitions of absolute velocity and acceleration (see 2.5.1) are such that the absolute velocity of a point P in a reference frame R is a special case of a relative velocity of P in R (see 2.4.1); similarly for accelerations. Conversely, the velocity ${}^R\mathbf{v}^{P/Q}$ and acceleration ${}^R\mathbf{a}^{P/Q}$ of a point P relative to a point Q in a reference frame R can be expressed in terms of the absolute velocities ${}^R\mathbf{v}^P$ and ${}^R\mathbf{v}^Q$ and absolute accelerations ${}^R\mathbf{a}^P$ and ${}^R\mathbf{a}^Q$ of P and Q in R:

$$ {}^{R}\mathbf{v}^{P/Q} = {}^{R}\mathbf{v}^{P} - {}^{R}\mathbf{v}^{Q} \tag{1}$$

$$ {}^{R}\mathbf{a}^{P/Q} = {}^{R}\mathbf{a}^{P} - {}^{R}\mathbf{a}^{Q} \tag{2}$$

Proof: Let O be a point fixed in R. Take $n = 3$ in Sec. 2.4.4, and let $P_1 = P$, $P_2 = O$, $P_3 = Q$. Then

$$ {}^{R}\mathbf{v}^{P/Q} \underset{(2.4.4)}{=} {}^{R}\mathbf{v}^{P/O} + {}^{R}\mathbf{v}^{O/Q} = \underset{(2.5.1)}{{}^{R}\mathbf{v}^{P}} - \underset{(2.4.2,2.5.1)}{{}^{R}\mathbf{v}^{Q}} $$

and

$$ {}^{R}\mathbf{a}^{P/Q} \underset{(2.4.4)}{=} {}^{R}\mathbf{a}^{P/O} - {}^{R}\mathbf{a}^{O/Q} = \underset{(2.5.1)}{{}^{R}\mathbf{a}^{P}} - \underset{(2.4.2,2.5.1)}{{}^{R}\mathbf{a}^{Q}} $$

Problem: In Fig. 2.5.15a, P and Q represent automobiles ap-

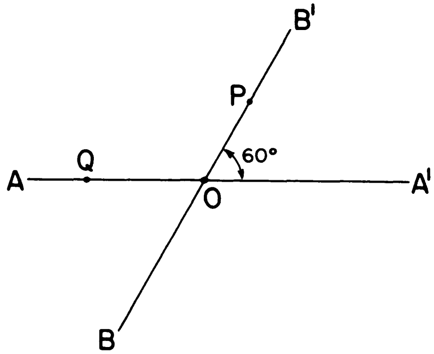


FIG. 2.5.15a

proaching an intersection O at 30 mile hr^{-1} and 40 mile hr^{-1}, respectively. If the automobiles collide at O, what is the magnitude of the velocity of P relative to Q at the instant of collision?

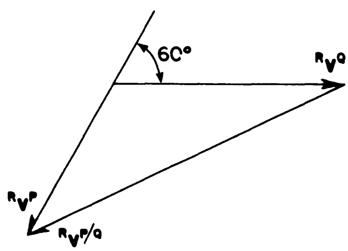


FIG. 2.5.15b

Solution: It is implied that the relative velocity in question is the velocity of P relative to Q in a reference frame R in which the streets AA' and BB' are fixed. The absolute velocities ${}^R\mathbf{v}^P$ and ${}^R\mathbf{v}^Q$ of P and Q in R are shown in Fig. 2.5.15b. The relative velocity ${}^R\mathbf{v}^{P/Q}$ is given by

$$ {}^R\mathbf{v}^{P/Q} = {}^R\mathbf{v}^P - {}^R\mathbf{v}^Q $$

and has the magnitude

$$ \begin{aligned} |{}^R\mathbf{v}^{P/Q}| &= [({}^R\mathbf{v}^P)^2 + ({}^R\mathbf{v}^Q)^2 + 2|{}^R\mathbf{v}^P|\ |{}^R\mathbf{v}^Q| \cos 60°]^{1/2} \\ &= [30^2 + 40^2 + 2(30)(40)(0.5)]^{1/2} \\ &= 60.9 \text{ mile hr}^{-1} \end{aligned} $$

3 SECOND MOMENTS

3.1 Second moments of a point

3.1.1 Given a point O, a point P, a scalar N associated with P (for example, the mass of a particle situated at P), and a unit vector $\mathbf{n}_a$, the vector $\boldsymbol{\Phi}_a^{P/O}$, defined as

$$\boldsymbol{\Phi}_a^{P/O} = N\mathbf{p} \times (\mathbf{n}_a \times \mathbf{p})$$

where $\mathbf{p}$ is the position vector of P relative to O (see Fig. 3.1.1a), is called *the second moment of P with respect to O for the direction* $\mathbf{n}_a$. N is called the *strength* of P.

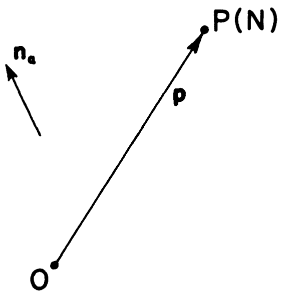


Fig. 3.1.1a

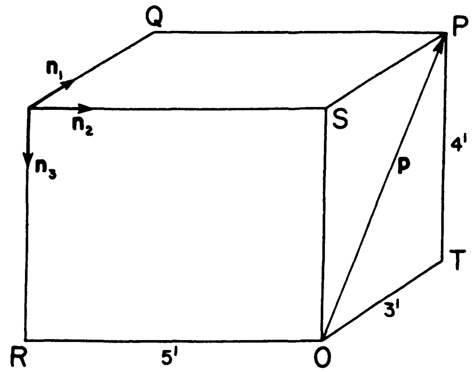


Fig. 3.1.1b

Example: In Fig. 3.1.1b, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. Point P has a strength of 10 slug. $\mathbf{p}$ is the position vector of P relative to O ($\mathbf{p} = 3\mathbf{n}_1 - 4\mathbf{n}_3$ ft). The second moments of P with respect to O for the directions $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ are given by

$$\boldsymbol{\Phi}_1^{P/O} = 10\mathbf{p} \times (\mathbf{n}_1 \times \mathbf{p}) = 160\mathbf{n}_1 + 120\mathbf{n}_3 \text{ slug ft}^2$$
$$\boldsymbol{\Phi}_2^{P/O} = 10\mathbf{p} \times (\mathbf{n}_2 \times \mathbf{p}) = 250\mathbf{n}_2 \text{ slug ft}^2$$
$$\boldsymbol{\Phi}_3^{P/O} = 10\mathbf{p} \times (\mathbf{n}_3 \times \mathbf{p}) = 120\mathbf{n}_1 + 90\mathbf{n}_3 \text{ slug ft}^2$$

3.1.2 Given the second moments $\boldsymbol{\Phi}_i^{P/O}$, $i = 1, 2, 3$, of a point P with respect to a point O for three directions $\mathbf{n}_i$, $i = 1, 2, 3$, (not parallel to the same plane), the second moment $\boldsymbol{\Phi}_a^{P/O}$ of P with respect to O for any direction $\mathbf{n}_a$ can be expressed as

$$\boldsymbol{\Phi}_a^{P/O} = \sum_{i=1}^{3} a_i \boldsymbol{\Phi}_i^{P/O}$$

where a_i is the $\mathbf{n}_i$ measure number of $\mathbf{n}_a$.

Proof: The unit vector $\mathbf{n}_a$, resolved into components parallel to $\mathbf{n}_i$, $i = 1, 2, 3$, is given by

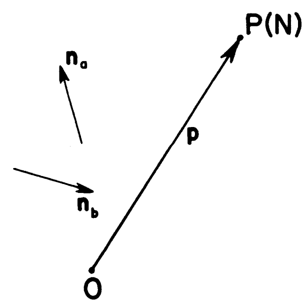
$$\mathbf{n}_a = \sum_{i=1}^{3} a_i \mathbf{n}_i \qquad \text{(A)}$$

Let $\mathbf{p}$ be the position vector of P relative to O, and N the strength of P. Then

$$\boldsymbol{\Phi}_a^{P/O} \underset{(3.1.1)}{=} N\mathbf{p} \times (\mathbf{n}_a \times \mathbf{p}) \underset{(A)}{=} N\mathbf{p} \times \left[\left(\sum_{i=1}^{3} a_i \mathbf{n}_i\right) \times \mathbf{p}\right]$$
$$= \sum_{i=1}^{3} a_i N\mathbf{p} \times (\mathbf{n}_i \times \mathbf{p}) \underset{(3.1.1)}{=} \sum_{i=1}^{3} a_i \boldsymbol{\Phi}_i^{P/O}$$

3.1.3 In general, $\boldsymbol{\Phi}_a^{P/O}$ is not parallel to $\mathbf{n}_a$. (See, for example, $\boldsymbol{\Phi}_1^{P/O}$ and $\boldsymbol{\Phi}_3^{P/O}$ in Example 3.1.1.)

FIG. 3.1.4



3.1.4 Given a point O, a point P of strength N, and two unit vectors $\mathbf{n}_a$ and $\mathbf{n}_b$ (see Fig. 3.1.4), the scalar $\phi_{ab}^{P/O}$, defined as

$$\phi_{ab}^{P/O} = \boldsymbol{\Phi}_a^{P/O} \cdot \mathbf{n}_b$$

where $\boldsymbol{\Phi}_a^{P/O}$ is the second moment of P with respect to O for the direction $\mathbf{n}_a$ (see 3.1.1), is called *the second moment of P with respect to O for the pair of directions* $\mathbf{n}_a$, $\mathbf{n}_b$.

Problem: Referring to Example 3.1.1, determine $\phi_{13}^{P/O}$, the second moment of P with respect to O for the pair of directions $\mathbf{n}_1$, $\mathbf{n}_3$.

Solution:

$$\phi_{13}^{P/O} = \boldsymbol{\Phi}_1^{P/O} \cdot \mathbf{n}_3 = (160\mathbf{n}_1 + 120\mathbf{n}_3) \cdot \mathbf{n}_3 = 120 \text{ slug ft}^2$$

3.1.5 The following is an alternative expression for $\phi_{ab}^{P/O}$ (see 3.1.4):

$$\phi_{ab}^{P/O} = N(\mathbf{n}_a \times \mathbf{p}) \cdot (\mathbf{n}_b \times \mathbf{p})$$

Proof:

$$\phi_{ab}^{P/O} \underset{(3.1.4)}{=} \boldsymbol{\Phi}_a^{P/O} \cdot \mathbf{n}_b \underset{(3.1.1)}{=} N[\mathbf{p}, \mathbf{n}_a \times \mathbf{p}, \mathbf{n}_b]$$

$$= N[\mathbf{n}_a \times \mathbf{p}, \mathbf{n}_b, \mathbf{p}] = N(\mathbf{n}_a \times \mathbf{p}) \cdot (\mathbf{n}_b \times \mathbf{p})$$

3.1.6 The expression for $\phi_{ab}^{P/O}$ (see 3.1.4) given in Sec. 3.1.5 shows that

$$\phi_{ab}^{P/O} = \phi_{ba}^{P/O}$$

because the expression remains unaltered when $\mathbf{n}_a$ and $\mathbf{n}_b$ are interchanged.

3.1.7 When the unit vector $\mathbf{n}_b$ is equal to the unit vector $\mathbf{n}_a$ the expressions given in Secs. 3.1.4 and 3.1.5 become, respectively,

$$\phi_{aa}^{P/O} = \boldsymbol{\Phi}_a^{P/O} \cdot \mathbf{n}_a \tag{1}$$

and

$$\phi_{aa}^{P/O} = N(\mathbf{n}_a \times \mathbf{p})^2 \tag{2}$$

From the second of these it follows that if $l_a^{P/O}$ is the distance from point P (strength N) to a line L_a parallel to $\mathbf{n}_a$, and O is *any* point on L_a (see Fig. 3.1.7), then

$$\phi_{aa}^{P/O} = N(l_a^{P/O})^2 \tag{3}$$

$\phi_{aa}^{P/O}$ is called *the second moment of P with respect to line L_a.*

Problem: Referring to Example 3.1.1, determine the second moment of P with respect to (a) line OR and (b) line OQ.

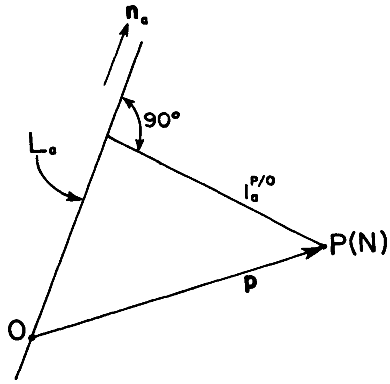


FIG. 3.1.7

Solution (a): Let $\phi_{22}^{P/O}$ be the second moment of P with respect to line OR, and $l_2^{P/O}$ the distance from P to this line. Then

$$\phi_{22}^{P/O} \underset{(1)}{=} \boldsymbol{\Phi}_2^{P/O} \cdot \mathbf{n}_2 = \underset{(\text{E3.1.1})}{(250\mathbf{n}_2)} \cdot \mathbf{n}_2 = 250 \text{ slug ft}^2$$

Alternatively,

$$\phi_{22}^{P/O} \underset{(3)}{=} 10(l_2^{P/O})^2 = 10(25) = 250 \text{ slug ft}^2$$

Solution (b): Let $\mathbf{n}_a$ be a unit vector parallel to line OQ, $\phi_{aa}^{P/O}$ the second moment of P with respect to line OQ, and $\mathbf{r}$ the position vector of P relative to Q. Then

$$\mathbf{n}_a = \pm\frac{3\mathbf{n}_1 - 5\mathbf{n}_2 - 4\mathbf{n}_3}{\sqrt{50}}$$

$$\mathbf{r} = 5\mathbf{n}_2 \quad \text{ft}$$

$$\mathbf{n}_a \times \mathbf{r} = \pm 5\,\frac{3\mathbf{n}_3 - 4\mathbf{n}_1}{\sqrt{50}} \text{ ft}$$

and

$$\phi_{aa}^{P/Q} \underset{(2)}{=} 10(\mathbf{n}_a \times \mathbf{r})^2 = \frac{10(25)(9 + 16)}{50} = 125 \text{ slug ft}^2$$

3.1.8 Given three mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and the six associated second moments $\phi_{ij}^{P/Q}$, $i, j = 1, 2, 3$, of a point P with respect to a point O (see 3.1.4 and 3.1.6), the second moment $\boldsymbol{\Phi}_a^{P/O}$ of P with respect to O can be expressed as

$$\boldsymbol{\Phi}_a^{P/O} = \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{P/O} a_i \mathbf{n}_j$$

where a_i is the $\mathbf{n}_i$ measure number of $\mathbf{n}_a$.

Proof:

$$\boldsymbol{\Phi}_a^{P/O} \underset{(3.1.2)}{=} \sum_{i=1}^{3} a_i \boldsymbol{\Phi}_i^{P/O} \tag{A}$$

If $\mathbf{n}_j$, $j = 1, 2, 3$, are mutually perpendicular unit vectors, the following is an identity (see Vol. I, Sec. 1.14.9):

$$\boldsymbol{\Phi}_i^{P/O} = \sum_{j=1}^{3} \boldsymbol{\Phi}_i^{P/O} \cdot \mathbf{n}_j \mathbf{n}_j \tag{B}$$

But

$$\boldsymbol{\Phi}_i^{P/O} \cdot \mathbf{n}_j \underset{(3.1.4)}{=} \phi_{ij}^{P/O} \tag{C}$$

Hence

$$\boldsymbol{\Phi}_i^{P/O} \underset{(B,C)}{=} \sum_{j=1}^{3} \phi_{ij}^{P/O} \mathbf{n}_j \tag{D}$$

and

$$\boldsymbol{\Phi}_a^{P/O} \underset{(A,D)}{=} \sum_{i=1}^{3} \left(a_i \sum_{j=1}^{3} \phi_{ij}^{P/O} \mathbf{n}_j \right) = \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{P/O} a_i \mathbf{n}_j$$

3.1.9 Given three mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and the six associated second moments $\phi_{ij}^{P/O}$, $i, j = 1, 2, 3$, of a point P with respect to a point O (see 3.1.4 and 3.1.6), the second moment $\phi_{ab}^{P/O}$ of P with respect to O for a pair of directions $\mathbf{n}_a$, $\mathbf{n}_b$ (see 3.1.4) can be expressed as

$$\phi_{ab}^{P/O} = \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{P/O} a_i b_j$$

where a_i and b_i are the $\mathbf{n}_i$ measure numbers of $\mathbf{n}_a$ and $\mathbf{n}_b$.

Proof:

$$\phi_{ab}^{P/O} \underset{(3.1.4)}{=} \boldsymbol{\Phi}_a^{P/O} \cdot \mathbf{n}_b \underset{(3.1.8)}{=} \left(\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{P/O} a_i \mathbf{n}_j \right) \cdot \mathbf{n}_b$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{P/O} a_i \mathbf{n}_j \cdot \mathbf{n}_b$$

$\mathbf{n}_b$, resolved into components parallel to $\mathbf{n}_i$, $i = 1, 2, 3$, is given by

$$\mathbf{n}_b = \sum_{i=1}^{3} b_i \mathbf{n}_i$$

Hence

$$\mathbf{n}_j \cdot \mathbf{n}_b = \sum_{i=1}^{3} b_i \mathbf{n}_j \cdot \mathbf{n}_i$$

But

$$\mathbf{n}_j \cdot \mathbf{n}_i = 0$$

unless $i = j$, in which case

$$\mathbf{n}_j \cdot \mathbf{n}_i = 1$$

Hence

$$\mathbf{n}_j \cdot \mathbf{n}_b = b_j$$

and

$$\phi_{ab}^{P/O} = \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^{P/O} a_i b_j$$

3.1.10 Given three mutually perpendicular unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ and the six associated second moments $\phi_{ij}^{P/O}$, $i, j = 1, 2, 3$, of a point P with respect to a point O (see 3.1.4 and 3.1.6), the second moment of P with respect to a line L_a passing through O (see 3.1.7) can be expressed as

$$\phi_{aa}^{P/O} = \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^{P/O} a_i a_j$$

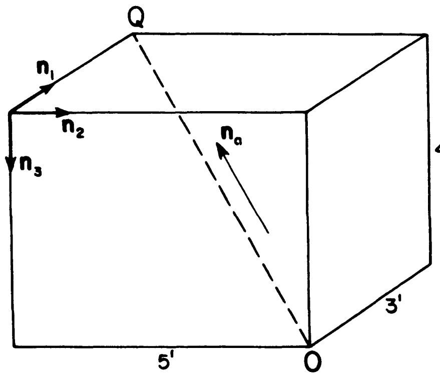


FIG. 3.1.10a

$\phi_{ij}^{P/O}$		j = 1	2	3
i	1	80	0	60
	2	0	125	0
	3	60	0	45

FIG. 3.1.10b

where a_i is the $\mathbf{n}_i$ measure number of a unit vector $\mathbf{n}_a$ parallel to L_a. This is an immediate consequence of Sec. 3.1.9 and the definition of the second moment of a point with respect to a line, Sec. 3.1.7.

Problem: In Fig. 3.1.10a, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. The six second moments $\phi_{ij}^{P/O}$, $i, j = 1, 2, 3$, of a point P (not shown) with respect to point O are recorded in Fig. 3.1.10b.

Determine the second moment of P with respect to line OQ.

Solution: Let $\mathbf{n}_a$ be a unit vector parallel to line OQ (see Fig. 3.1.10a), a_i the $\mathbf{n}_i$ measure number of $\mathbf{n}_a$. Then

$$\mathbf{n}_a = \frac{3\mathbf{n}_1 - 5\mathbf{n}_2 - 4\mathbf{n}_3}{\sqrt{50}}$$

$$a_1 = \frac{3}{\sqrt{50}}, \qquad a_2 = \frac{-5}{\sqrt{50}}, \qquad a_3 = \frac{-4}{\sqrt{50}}$$

and, letting $\phi_{aa}^{P/O}$ denote the second moment of P with respect to line OQ,

$$\begin{aligned}
\phi_{aa}^{P/O} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{P/O} a_i a_j = \sum_{i=1}^{3} (\phi_{i1}^{P/O} a_i a_1 + \phi_{i2}^{P/O} a_i a_2 + \phi_{i3}^{P/O} a_i a_3) \\
&= \left(\sum_{i=1}^{3} \phi_{i1}^{P/O} a_i\right) a_1 + \left(\sum_{i=1}^{3} \phi_{i2}^{P/O} a_i\right) a_2 + \left(\sum_{i=1}^{3} \phi_{i3}^{P/O} a_i\right) a_3 \\
&= (\phi_{11}^{P/O} a_1 + \phi_{21}^{P/O} a_2 + \phi_{31}^{P/O} a_3) a_1 \\
&\qquad + (\phi_{12}^{P/O} a_1 + \phi_{22}^{P/O} a_2 + \phi_{32}^{P/O} a_3) a_2 \\
&\qquad\qquad + (\phi_{13}^{P/O} a_1 + \phi_{23}^{P/O} a_2 + \phi_{33}^{P/O} a_3) a_3 \\
&= \phi_{11}^{P/O} a_1^2 + \phi_{22}^{P/O} a_2^2 + \phi_{33}^{P/O} a_3^2 \\
&\qquad + 2(\phi_{12}^{P/O} a_1 a_2 + \phi_{23}^{P/O} a_2 a_3 + \phi_{31}^{P/O} a_3 a_1) \\
&= 80\left(\frac{9}{50}\right) + 125\left(\frac{25}{50}\right) + 45\left(\frac{16}{50}\right) + 2(60)\left(-\frac{4}{\sqrt{50}}\right)\left(\frac{3}{\sqrt{50}}\right) \\
&= 62.5
\end{aligned}$$

3.2 Second moments of a set of points

3.2.1 Given a point O, a set S of points P_i, $i = 1, 2, \ldots, n$, of (dimensionally homogeneous) strengths N_i, and a unit vector $\mathbf{n}_a$, the vector $\boldsymbol{\Phi}_a^{S/O}$, defined as

$$\boldsymbol{\Phi}_a^{S/O} = \sum_{i=1}^{n} \boldsymbol{\Phi}_a^{P_i/O}$$

where $\boldsymbol{\Phi}_a^{P_i/O}$ is the second moment of P_i with respect to O for the direction $\mathbf{n}_a$ (see 3.1.1), is called *the second moment of S with respect to O for the direction* $\mathbf{n}_a$. (In general, $\boldsymbol{\Phi}_a^{S/O}$ is not parallel to $\mathbf{n}_a$. See 3.1.3.)

3.2.2 Given a point O, a set S of points P_i, $i = 1, 2, \ldots, n$, of (dimensionally homogeneous) strengths N_i, the scalar $\phi_{ab}^{S/O}$, defined as

$$\phi_{ab}^{S/O} = \sum_{i=1}^{n} \phi_{ab}^{P_i/O}$$

where $\phi_{ab}^{P_i/O}$ is the second moment of P_i with respect to O for the pair of direction $\mathbf{n}_a$, $\mathbf{n}_b$ (see 3.1.4), is called *the second moment of S with respect to O for the pair of directions* $\mathbf{n}_a$, $\mathbf{n}_b$.

3.2.3 When the unit vector $\mathbf{n}_b$ is equal to the unit vector $\mathbf{n}_a$, the expression given in Sec. 3.2.3 becomes

$$\phi_{aa}^{S/O} = \sum_{i=1}^{n} \phi_{aa}^{P_i/O}$$

and (see 3.1.7) $\phi_{aa}^{P_i/O}$ is the second moment of P_i with respect to the line L_a which passes through O and is parallel to $\mathbf{n}_a$. Accordingly, $\phi_{aa}^{S/O}$ is called *the second moment of S with respect to line L_a.*

3.2.4 As a consequence of the definitions given in Secs. 3.2.1, 3.2.2, and 3.2.3, the relationships between second moments of a point, discussed in Secs. 3.1.2–3.1.10, are equally valid for second moments of a set of points. That is,

$$\boldsymbol{\Phi}_a^{S/O} \underset{(3.1.2)}{=} \sum_{i=1}^{3} a_i \boldsymbol{\Phi}_i^{S/O} \tag{1}$$

$$\phi_{ab}^{S/O} \underset{(3.1.4)}{=} \boldsymbol{\Phi}_a^{S/O} \cdot \mathbf{n}_b \tag{2}$$

$$\phi_{ab}^{S/O} \underset{(3.1.6)}{=} \phi_{ba}^{S/O} \tag{3}$$

$$\phi_{aa}^{S/O} \underset{(3.1.7)}{=} \boldsymbol{\Phi}_a^{S/O} \cdot \mathbf{n}_a \tag{4}$$

$$\boldsymbol{\Phi}_a^{S/O} \underset{(3.1.8)}{=} \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^{S/O} a_i \mathbf{n}_j \tag{5}$$

$$\phi_{ab}^{S/O} \underset{(3.1.9)}{=} \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^{S/O} a_i b_j \tag{6}$$

$$\phi_{aa}^{S/O} \underset{(3.1.10)}{=} \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^{S/O} a_i a_j \tag{7}$$

3.2.5 When the second moment $\phi_{aa}^{S/O}$ of a set S with respect to a line L_a (see 3.2.3) can be expressed as the product of a quantity N, called the *strength* of S, and defined as

$$N = \sum_{i=1}^{n} N_i \tag{1}$$

and the square of a positive, real quantity $k_a^{S/O}$ (which has the dimensions of length), that is, $k_a^{S/O}$ exists such that

$$\phi_{aa}^{S/O} = N(k_a^{S/O})^2 \tag{2}$$

then $k_a^{S/O}$ is called the *radius of gyration* of S with respect to line L_a, and is equal to the distance between line L_a and any point of strength N whose second moment with respect to L_a (see 3.1.7) has the value $\phi_{aa}^{S/O}$.

Problem: In Fig. 3.2.5a, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. S is a set of points which has a strength of 10 slug and whose second moments $\phi_{ij}^{S/O}$, $i, j = 1, 2, 3$, with respect to point O are tabulated in Fig. 3.2.5b.

Determine the radius of gyration of S with respect to line OQ.

Solution: Let $\mathbf{n}_a$ be a unit vector parallel to line OQ (see Fig. 3.2.5a), $k_a^{S/O}$ the desired radius of gyration, $N = 10$ slug the strength of S, and $\phi_{aa}^{S/O}$ the second moment of S with respect to line OQ. Then

$$k_a^{S/O} \underset{(2)}{=} \left(\frac{\phi_{aa}^{S/O}}{N}\right)^{1/2}$$

and, letting a_i be the $\mathbf{n}_i$ measure number of $\mathbf{n}_a$,

$$\phi_{aa}^{S/O} \underset{(3.2.4)}{=} \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^{S/O} a_i a_j \underset{(P3.1.10)}{=} 62.5 \text{ slug ft}^2$$

Hence

$$k_a^{S/O} = (6.25)^{1/2} = 2.5 \text{ ft}$$

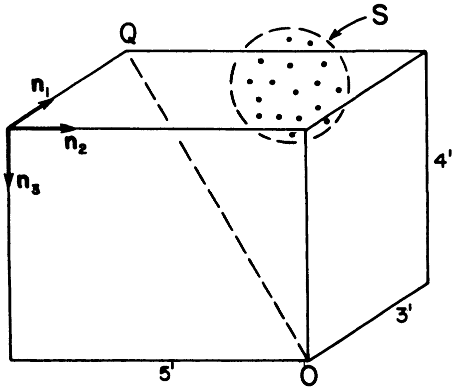


FIG. 3.2.5a

$\phi_{ij}^{S/O}$ slug ft²		j = 1	2	3
i	1	80	0	60
	2	0	125	0
	3	60	0	45

FIG. 3.2.5b

3.2.6 Given a set S of n points, a point O, and two directions $\mathbf{n}_a$ and $\mathbf{n}_b$, the second moments of S with respect to O and with

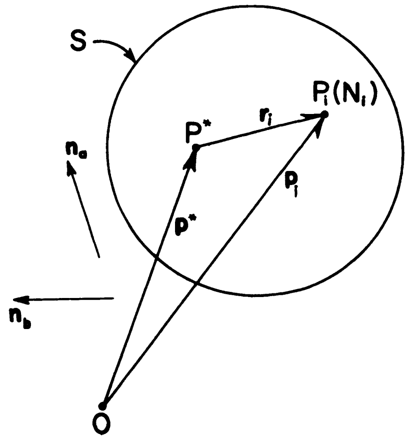


FIG. 3.2.6

respect to the centroid P^* of S (see 3.2.1 and 3.2.2) are related as follows:

$$\mathbf{\Phi}_a^{S/O} = \mathbf{\Phi}_a^{S/P^*} + \mathbf{\Phi}_a^{P^*/O} \tag{1}$$

$$\phi_{ab}^{S/O} = \phi_{ab}^{S/P^*} + \phi_{ab}^{P^*/O} \tag{2}$$

where $\mathbf{\Phi}_a^{P^*/O}$ and $\phi_{ab}^{P^*/O}$ are second moments of the point P^* (see 3.1.1 and 3.1.4) regarded as having a strength equal to the strength N of S (see 3.2.5).

Proof: Let P_i, $i = 1, 2, \ldots, n$, be the points of S, N_i the strength of P_i, $\mathbf{p}_i$ the position vector of P_i relative to O (see Fig. 3.2.6), $\mathbf{p}^*$ the position vector of P^* relative to O, and $\mathbf{r}_i$ the position vector of P_i relative to P^*.

Then

$$\mathbf{p}_i = \mathbf{p}^* + \mathbf{r}_i \tag{A}$$

$$\mathbf{\Phi}_a^{P^*/O} \underset{(3.1.1)}{=} N\mathbf{p}^* \times (\mathbf{n}_a \times \mathbf{p}^*) \tag{B}$$

$$N \underset{(3.2.5)}{=} \sum_{i=1}^{n} N_i \tag{C}$$

and (see Vol. I, Sec. 2.4)

$$\sum_{i=1}^{n} N_i \mathbf{r}_i = 0 \tag{D}$$

Hence

$$\begin{aligned}
\mathbf{\Phi}_a^{S/O} &\underset{(3.2.1)}{=} \sum_{i=1}^{n} \mathbf{\Phi}_a^{P_i/O} \underset{(3.1.1)}{=} \sum_{i=1}^{n} N_i \mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) \\
&\underset{(A)}{=} \sum_{i=1}^{n} N_i (\mathbf{p}^* + \mathbf{r}_i) \times [\mathbf{n}_a \times (\mathbf{p}^* + \mathbf{r}_i)] \\
&= \left(\sum_{i=1}^{n} N_i\right) \mathbf{p}^* \times (\mathbf{n}_a \times \mathbf{p}^*) + \mathbf{p}^* \times \left(\mathbf{n}_a \times \sum_{i=1}^{n} N_i \mathbf{r}_i\right) \\
&\qquad + \left(\sum_{i=1}^{n} N_i \mathbf{r}_i\right) \times (\mathbf{n}_a \times \mathbf{p}^*) + \sum_{i=1}^{n} N_i \mathbf{r}_i \times (\mathbf{n}_a \times \mathbf{r}_i) \\
&= \underset{(B,C)}{\mathbf{\Phi}_a^{P^*/O}} + \underset{(D)}{0} + \underset{(D)}{0} + \underset{(3.1.1,3.2.1)}{\mathbf{\Phi}_a^{S/P^*}}
\end{aligned}$$

Next

$$\begin{aligned}
\phi_{ab}^{S/O} &\underset{(3.2.4)}{=} \mathbf{\Phi}_a^{S/O} \cdot \mathbf{n}_b \underset{(1)}{=} \mathbf{\Phi}_a^{P^*/O} \cdot \mathbf{n}_b + \mathbf{\Phi}_a^{S/P^*} \cdot \mathbf{n}_b \\
&= \underset{(3.1.4)}{\phi_{ab}^{P^*/O}} + \underset{(3.2.4)}{\phi_{ab}^{S/P^*}}
\end{aligned}$$

3.2.7 Given a set S of n points, the strength N of S (see 3.2.5), a line L_a parallel to a unit vector $\mathbf{n}_a$ and passing through a point O, a line L_a^* parallel to $\mathbf{n}_a$ and passing through the centroid P^* of S, and the distance $l_a^{P^*/O}$ between the lines L_a and L_a^* (see Fig. 3.2.7), the second moments of S with respect to the lines L_a and L_a^* (see 3.2.2) are related to each other as follows:

$$\phi_{aa}^{S/O} = \phi_{aa}^{S/P^*} + N(l_a^{P^*/O})^2$$

where O is any point on line L_a.

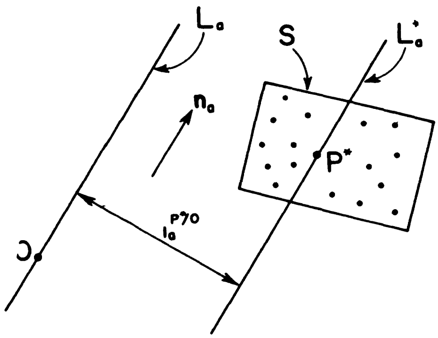


FIG. 3.2.7

Proof: Let $\mathbf{n}_b$ be a unit vector equal to $\mathbf{n}_a$. Then

$$\phi_{aa}^{S/O} \underset{(3.2.6)}{=} \phi_{aa}^{S/P^*} + \phi_{aa}^{P^*/O}$$

The distance from P^* to line L_a is equal to $l_a^{P^*/O}$. Hence, when P^* is regarded as having the strength N,

$$\phi_{aa}^{P^*/O} \underset{(3.1.7)}{=} N(l_a^{P^*/O})^2$$

Thus

$$\phi_{aa}^{S/O} = \phi_{aa}^{S/P^*} + N(l_a^{P^*/O})^2$$

3.2.8 If $k_a^{S/O}$ is the radius of gyration of a set S with respect to a line L_a passing through a point O (see 3.2.5), k_a^{S/P^*} the radius of gyration of S with respect to a line L_a^* which is parallel to L_a and passes through the centroid P^* of S, and $l_a^{P^*/O}$ the perpendicular distance between the lines L_a and L_a^* (see Fig. 3.2.7), then

$$(k_a^{S/O})^2 = (k_a^{S/P^*})^2 + (l_a^{P^*/O})^2$$

Proof: Using the notation of Secs. 3.2.5 and 3.2.7,

$$(k_a^{S/O})^2 \underset{(3.2.5)}{=} \frac{\phi_{aa}^{S/O}}{N} \underset{(3.2.7)}{=} \frac{\phi_{aa}^{S/P^*}}{N} + (l_a^{P^*/O})^2 \underset{(3.2.5)}{=} (k_a^{S/P^*})^2 + (l_a^{P^*/O})^2$$

3.2.9 By successive use of expressions given in Secs. 3.2.4, 3.2.6, and 3:2.7 all second moments of a set S of points can be found whenever the following are known:

(a) The strength of S.
(b) The location of the centroid of S.
(c) The six second moments of S with respect to one point for three mutually perpendicular directions.

Problem: In Fig. 3.2.9a, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. P^* is the centroid of a set S of points. S has a strength N of 25 ft², and the second moments of S with respect to point O have the values shown in Fig. 3.2.9b.

Determine the second moment of S with respect to line AB.

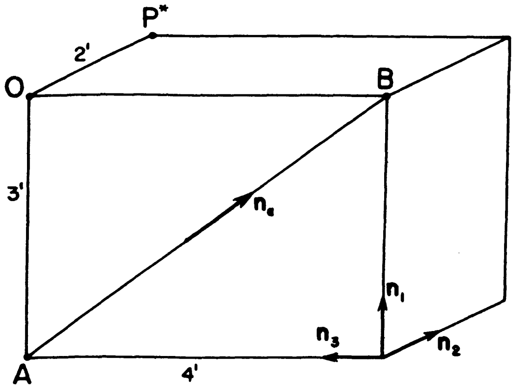


FIG. 3.2.9a

$\phi_{ij}^{S/O}$ (ft⁴)		j = 1	2	3
i	1	25	−8	−75
	2	−8	10	−9
	3	−75	−9	50

FIG. 3.2.9b

Solution: Let $\mathbf{n}_a$ be a unit vector parallel to line AB, and $\phi_{aa}^{S/A}$ the desired second moment. Then

$$\phi_{aa}^{S/A} \underset{(3.2.6)}{=} \phi_{aa}^{S/P^*} + \phi_{aa}^{P^*/A}$$

$$\phi_{aa}^{S/P^*} \underset{(3.2.7)}{=} \phi_{aa}^{S/O} - N(\overline{OP}^*)^2$$

$$= \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{S/O} a_i a_j \underset{(3.2.4)}{} - N(\overline{OP}^*)^2$$

$$\phi_{aa}^{P^*/A} \underset{(3.1.7)}{=} N(\mathbf{n}_a \times \overrightarrow{AP^*})^2$$

Hence

$$\phi_{aa}^{S/A} = \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{S/O} a_i a_j - N(\overline{OP}^*)^2 + N(\mathbf{n}_a \times \overrightarrow{AP^*})^2$$

where

$$a_1 = \tfrac{3}{5}, \qquad a_2 = 0, \qquad a_3 = -\tfrac{4}{5}$$

$$\overline{OP}^* = 2 \text{ ft}, \qquad \overrightarrow{AP^*} = 3\mathbf{n}_1 + 2\mathbf{n}_2 \text{ ft}$$

so that

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{S/O} a_i a_j = 25(\tfrac{9}{25}) + 50(\tfrac{16}{25}) + 2(75)(\tfrac{12}{25}) = 113 \text{ ft}^4$$

$$N(\overline{OP}^*)^2 = 25(4) = 100 \text{ ft}^4$$

$$N(\mathbf{n}_a \times \overrightarrow{AP^*})^2 = 25[(\tfrac{3}{5}\mathbf{n}_1 - \tfrac{4}{5}\mathbf{n}_3) \times (3\mathbf{n}_1 + 2\mathbf{n}_2)]^2 = 244 \text{ ft}^4$$

and

$$\phi_{aa}^{S/A} = 113 - 100 + 244 = 257 \text{ ft}^4$$

3.3 Principal directions, axes, planes, second moments, and radii of gyration of a set of points

3.3.1 Given a set S of points, a point O, and a unit vector $\mathbf{n}_z$, $\mathbf{n}_z$ is called a *principal direction* of S for O if and only if $\mathbf{\Phi}_z^{S/O}$ (see 3.2.1) is parallel to $\mathbf{n}_z$ or equal to zero. When $\mathbf{n}_z$ is a principal direction of S for O, the line L_z parallel to $\mathbf{n}_z$ and passing through O is called a *principal axis* of S for O, the plane P_z passing through O and perpendicular to $\mathbf{n}_z$ a *principal plane* of S for O, the second moment $\phi_{zz}^{S/O}$ a *principal second moment* of S for O, and the radius of gyration $k_z^{S/O}$ a *principal radius of gyration* of S for O.

In the above definitions the point O plays an important part: Only with reference to a specific point is it meaningful to speak of principal direction, axes, etc.

Problem: Fig. 3.3.1 shows a set S of two points P_1 and P_2 of equal strength N. Determine whether or not the second moment of S with respect to line AB is a principal second moment of S for

(a) point A and (b) point B, and show that line P_1P_2 is a principal axis of S for every point on this line.

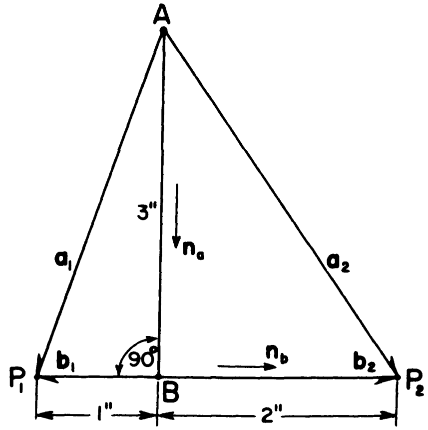


FIG. 3.3.1

Solution: Let $\mathbf{n}_a$ and $\mathbf{n}_b$ be unit vectors parallel to AB and P_1P_2, respectively, and let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}_1$, $\mathbf{b}_2$ be the position vectors of P_1 and P_2 relative to A and B, as shown in Fig. 3.3.1. Then

$$\boldsymbol{\Phi}_a^{S/A} \underset{(3.2.1,3.1.1)}{=} N\mathbf{a}_1 \times (\mathbf{n}_a \times \mathbf{a}_1) + N\mathbf{a}_2 \times (\mathbf{n}_a \times \mathbf{a}_2)$$

$$= N(5\mathbf{n}_a - 3\mathbf{n}_b)$$

This shows that $\mathbf{n}_a$ is not a principal direction of S for point A. Consequently, line AB is not a principal axis of S for point A, and the second moment of S with respect to line AB is not a principal second moment of S for point A.

Next,

$$\boldsymbol{\Phi}_a^{S/B} = N\mathbf{b}_1 \times (\mathbf{n}_a \times \mathbf{b}_1) + N\mathbf{b}_2 \times (\mathbf{n}_a \times \mathbf{b}_2) = 5N\mathbf{n}_a$$

Accordingly, $\mathbf{n}_a$ is a principal direction of S for point B, line AB is a principal axis of S for point B, and the second moment of S with respect to line AB is a principal second moment of S for point B.

Finally, the second moment of S with respect to every point of line P_1P_2 for the direction $\mathbf{n}_b$ is equal to zero. $\mathbf{n}_b$ is thus a principal direction of S for every point of this line, and the line is a principal axis of S for every point on the line.

3.3.2 A unit vector $\mathbf{n}_z$ is a principal direction of a set S for a point O if and only if

$$\boldsymbol{\Phi}_z^{S/O} = \phi_{zz}^{S/O}\mathbf{n}_z$$

Proof: If

$$\boldsymbol{\Phi}_z^{S/O} = \phi_{zz}^{S/O}\mathbf{n}_z$$

then $\boldsymbol{\Phi}_z^{S/O}$ is either parallel to $\mathbf{n}_z$ or equal to zero. Hence (see 3.3.1) $\mathbf{n}_z$ is a principal direction of S for O.

Next, suppose that

$$\boldsymbol{\Phi}_z^{S/O} \neq \phi_{zz}^{S/O}\mathbf{n}_z \tag{A}$$

Then $\mathbf{n}_z$ cannot be a principal direction of S for O. For, $\boldsymbol{\Phi}_z^{S/O}$ cannot be parallel to $\mathbf{n}_z$, because any $\boldsymbol{\Phi}_z^{S/O}$ parallel to $\mathbf{n}_z$ can be expressed as

$$\boldsymbol{\Phi}_z^{S/O} = \lambda\mathbf{n}_z \tag{B}$$

which, after dot-multiplication with $\mathbf{n}_z$, gives

$$\boldsymbol{\Phi}_z^{S/O} \cdot \mathbf{n}_z = \lambda\mathbf{n}_z \cdot \mathbf{n}_z = \lambda \tag{C}$$

But

$$\boldsymbol{\Phi}_z^{S/O} \cdot \mathbf{n}_z \underset{(3.2.4)}{=} \phi_{zz}^{S/O} \tag{D}$$

so that

$$\lambda \underset{(\mathrm{C,D})}{=} \phi_{zz}^{S/O} \tag{E}$$

and

$$\boldsymbol{\Phi}_z^{S/O} \underset{(\mathrm{B,E})}{=} \phi_{zz}^{S/O}\mathbf{n}_z$$

which contradicts the hypothesis, Eq. (A). Nor can $\boldsymbol{\Phi}_z^{S/O}$ be equal to zero, for it would then follow from Eq. (4), Sec. 3.2.4 that

$$\phi_{zz}^{S/O} = 0$$

which means that

$$\boldsymbol{\Phi}_z^{S/O} = \phi_{zz}^{S/O}\mathbf{n}_z$$

and this again contradicts the hypothesis, Eq. (A).

3.3.3 If $\mathbf{n}_z$ is a principal direction of a set S for a point O (see 3.3.1), the second moment $\phi_{yz}^{S/O}$ is equal to zero for all directions $\mathbf{n}_y$ perpendicular to $\mathbf{n}_z$.

Proof:

$$\phi_{yz}^{S/O} \underset{(3.2.4)}{=} \boldsymbol{\Phi}_z^{S/O} \cdot \mathbf{n}_y = (\phi_{zz}^{S/O}\mathbf{n}_z) \underset{(3.3.2)}{\cdot} \mathbf{n}_y = \phi_{zz}^{S/O}(\mathbf{n}_z \cdot \mathbf{n}_y) = 0$$

3.3.4 If the second moments $\phi_{xz}^{S/O}$ and $\phi_{yz}^{S/O}$ of a set S with respect to a point O for the two pairs of directions $\mathbf{n}_x$, $\mathbf{n}_z$ and $\mathbf{n}_y$, $\mathbf{n}_z$ are equal to zero, and $\mathbf{n}_x$ and $\mathbf{n}_y$ are each perpendicular to $\mathbf{n}_z$ (and not parallel to each other), then $\mathbf{n}_z$ is a principal direction of S for point O.

Proof:

$$\phi_{xz}^{S/O} \underset{(3.2.4)}{=} \boldsymbol{\Phi}_z^{S/O} \cdot \mathbf{n}_x, \qquad \phi_{yz}^{S/O} \underset{(3.2.4)}{=} \boldsymbol{\Phi}_z^{S/O} \cdot \mathbf{n}_y$$

Hence, by hypothesis,

$$\boldsymbol{\Phi}_z^{S/O} \cdot \mathbf{n}_x = 0, \qquad \boldsymbol{\Phi}_z^{S/O} \cdot \mathbf{n}_y = 0$$

and $\boldsymbol{\Phi}_z^{S/O}$ is either equal to zero or perpendicular to both $\mathbf{n}_x$ and $\mathbf{n}_y$, that is, parallel to their common normal, $\mathbf{n}_z$. In either case, $\mathbf{n}_z$ is a principal direction of S for point O (see 3.3.1).

3.3.5 If $\mathbf{n}_z$ is a principal direction of S for a point O, and $\mathbf{n}_x$ and $\mathbf{n}_y$ are unit vectors perpendicular to $\mathbf{n}_z$, the second moment $\phi_{xy}^{S/O}$ is not, in general, equal to zero. (Compare with 3.3.3 and 3.3.4.)

3.3.6 In certain situations, principal directions, axes, and planes can be located by inspection. For example, if the points of a set S are placed such that corresponding to every point on one side of a plane P_z there exists a point of the same strength on the other side, the two points lying on the same normal to, and being equidistant from, P_z, then P_z is a principal plane of S for every point of P_z. P_z is then called a *plane of symmetry* of S, and one may say, in short, that a plane of symmetry is a principal plane for each of its points.

Proof: S may be divided into three sets of points: S_a, consisting of points lying in P_z; S_b, consisting of points lying on one side of P_z; and S_c, containing the points lying on the opposite side of P_z.

Consider the second moments of S_a, S_b, and S_c with respect to a point O lying in plane P_z, for the direction $\mathbf{n}_z$ perpendicular to P_z.

S_a: Let $\mathbf{a}$ be the position vector relative to O of a typical point P of strength N. Then $N\mathbf{a} \times (\mathbf{n}_z \times \mathbf{a})$ is parallel to $\mathbf{n}_z$. Hence $\boldsymbol{\Phi}_z^{S_a/O}$ is parallel to $\mathbf{n}_z$.

S_b and S_c: Let $\mathbf{b}$ be the position vector relative to O of a typical point of S_b, and $\mathbf{c}$ the position vector of the corresponding point of S_c (see Fig. 3.3.6). Then $\mathbf{b}$ and $\mathbf{c}$ can be expressed as

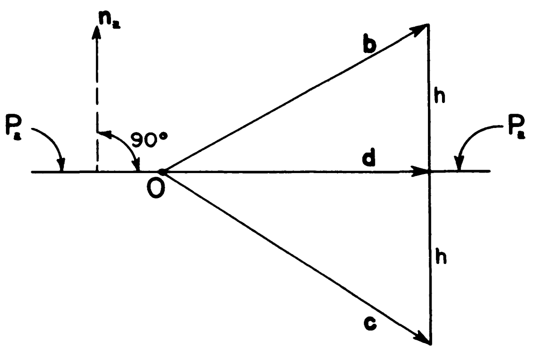


FIG. 3.3.6

$$\mathbf{b} = \mathbf{d} + h\mathbf{n}_z$$
$$\mathbf{c} = \mathbf{d} - h\mathbf{n}_z$$

and, as the two points have the same strength, say N, the contribution of this pair of points to $\boldsymbol{\Phi}_z^{S_b/O} + \boldsymbol{\Phi}_z^{S_c/O}$ is

$$N\mathbf{b} \times (\mathbf{n}_z \times \mathbf{b}) + N\mathbf{c} \times (\mathbf{n}_z \times \mathbf{c}) = 2N\mathbf{d} \times (\mathbf{n}_z \times \mathbf{d})$$

which is parallel to $\mathbf{n}_z$. Thus $\boldsymbol{\Phi}_z^{S_b/O} + \boldsymbol{\Phi}_z^{S_c/O}$ is parallel to $\mathbf{n}_z$. But

$$\boldsymbol{\Phi}_z^{S/O} = \boldsymbol{\Phi}_z^{S_a/O} + \boldsymbol{\Phi}_z^{S_b/O} + \boldsymbol{\Phi}_z^{S_c/O}$$

Hence $\boldsymbol{\Phi}_z^{S/O}$ is parallel to $\mathbf{n}_z$; $\mathbf{n}_z$ is a principal direction of S for point O; and P_z is a principal plane of S for point O. Finally, as O is any point of P_z, P_z is a principal plane of S for every point of P_z.

3.3.7 When all points of a set S lie in a plane, this plane is a principal plane of S for every point in the plane, the plane being a plane of symmetry (see 3.3.6).

Problem: Referring to Fig. 3.2.9a, and letting S' be the set of three points A, B, O, show that (regardless of the strengths of these points) $\phi_{12}^{S'/O}$ is equal to zero.

Solution: The plane determined by the points is a principal plane of S' for O. Hence $\mathbf{n}_2$ is a principal direction of S' for O. As $\mathbf{n}_1$ is perpendicular to $\mathbf{n}_2$,

$$\phi_{12}^{S'/O} \underset{(3.3.3)}{=} 0$$

3.3.8 When a plane A is a principal plane of a set S for a point O (see Fig. 3.3.8a), there exist at least two principal directions of S for O which are parallel to A and perpendicular to each other. If $\mathbf{n}_z$ is one of these principal directions, and $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors perpendicular to each other and parallel to A, the corresponding principal second moment of S for O is given by

$$\phi_{zz}^{S/O} = \frac{\phi_{11}^{S/O} + \phi_{22}^{S/O}}{2} + \left[\left(\frac{\phi_{11}^{S/O} - \phi_{22}^{S/O}}{2}\right)^2 + (\phi_{12}^{S/O})^2\right]^{1/2} \tag{1}$$

and the ratio of the $\mathbf{n}_1$ and $\mathbf{n}_2$ measure numbers of $\mathbf{n}_z$ has the value

$$\frac{z_1}{z_2} = \frac{\phi_{zz}^{S/O} - \phi_{22}^{S/O}}{\phi_{12}^{S/O}} \tag{2}$$

The principal second moment corresponding to the principal direction which is perpendicular to $\mathbf{n}_z$ and parallel to A is given by

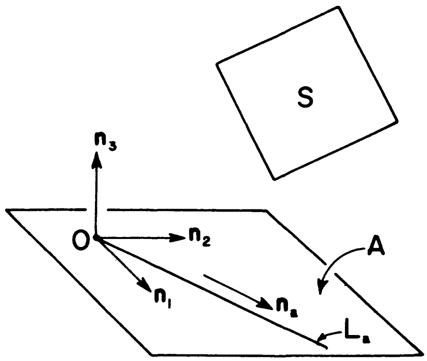


FIG. 3.3.8a

$$\phi_{yy}^{S/O} = \phi_{11}^{S/O} + \phi_{22}^{S/O} - \phi_{zz}^{S/O} \tag{3}$$

Proof: Let $\mathbf{n}_3$ be a unit vector perpendicular to A (and thus a principal direction of S for O). Then, for any direction $\mathbf{n}_a$,

$$\boldsymbol{\Phi}_a^{S/O} \underset{(3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{S/O} a_i \mathbf{n}_j$$

where a_i is the $\mathbf{n}_i$ measure number of $\mathbf{n}_a$. Let $\mathbf{n}_a$ be perpendicular to $\mathbf{n}_3$, so that $a_3 = 0$, and note that (see 3.3.3)

$$\phi_{23}^{S/O} = \phi_{31}^{S/O} = 0$$

It follows that

$$\boldsymbol{\Phi}_a^{S/O} = (\phi_{11}^{S/O}a_1 + \phi_{12}^{S/O}a_2)\mathbf{n}_1 + (\phi_{12}^{S/O}a_1 + \phi_{22}^{S/O}a_2)\mathbf{n}_2 \tag{A}$$

$\mathbf{n}_a$ is a principal direction of S for O if and only if

$$\boldsymbol{\Phi}_a^{S/O} \underset{(3.3.2)}{=} \phi_{aa}^{S/O}\mathbf{n}_a = \phi_{aa}^{S/O}(a_1\mathbf{n}_1 + a_2\mathbf{n}_2)$$

that is, using Eq. (A) to eliminate $\boldsymbol{\Phi}_a^{S/O}$,

$$a_1(\phi_{11}^{S/O} - \phi_{aa}^{S/O}) + a_2\phi_{12}^{S/O} = 0 \tag{B}$$

and

$$a_1\phi_{12}^{S/O} + a_2(\phi_{22}^{S/O} - \phi_{aa}^{S/O}) = 0 \tag{C}$$

As $\mathbf{n}_a$ is a unit vector, a_1 and a_2 must satisfy the further condition

$$a_1^2 + a_2^2 = 1 \tag{D}$$

which shows that a_1 and a_2 cannot vanish simultaneously. From this it follows that Eqs. (B) and (C) can, in general, be satisfied simultaneously only if the determinant of the coefficients of a_1 and a_2 is equal to zero:

$$\begin{vmatrix} (\phi_{11}^{S/O} - \phi_{aa}^{S/O}) & \phi_{12}^{S/O} \\ \phi_{12}^{S/O} & (\phi_{22}^{S/O} - \phi_{aa}^{S/O}) \end{vmatrix} = 0$$

or, expanding,

$$(\phi_{aa}^{S/O})^2 - (\phi_{11}^{S/O} + \phi_{22}^{S/O})\phi_{aa}^{S/O} + \phi_{11}^{S/O}\phi_{22}^{S/O} - (\phi_{12}^{S/O})^2 = 0$$

This quadratic equation has two real roots, which may be called $\phi_{yy}^{S/O}$ and $\phi_{zz}^{S/O}$, and which give the values of two principal second moments of S for O:

$$\phi_{yy}^{S/O} = \frac{\phi_{11}^{S/O} + \phi_{22}^{S/O}}{2} - \left[\left(\frac{\phi_{11}^{S/O} - \phi_{22}^{S/O}}{2}\right)^2 + (\phi_{12}^{S/O})^2\right]^{1/2} \tag{E}$$

$$\phi_{zz}^{S/O} = \frac{\phi_{11}^{S/O} + \phi_{22}^{S/O}}{2} + \left[\left(\frac{\phi_{11}^{S/O} - \phi_{22}^{S/O}}{2}\right)^2 + (\phi_{12}^{S/O})^2\right]^{1/2} \tag{F}$$

Note that

$$\phi_{yy}^{S/O} + \phi_{zz}^{S/O} \underset{(E,F)}{=} \phi_{11}^{S/O} + \phi_{22}^{S/O}$$

so that

$$\phi_{yy}^{S/O} = \phi_{11}^{S/O} + \phi_{22}^{S/O} - \phi_{zz}^{S/O}$$

Let y_1 and y_2 be the values of a_1 and a_2 corresponding to $\phi_{aa}^{S/O} = \phi_{yy}^{S/O}$, and z_1 and z_2 the values of a_1 and a_2 corresponding to $\phi_{aa}^{S/O} = \phi_{zz}^{S/O}$. Replace a_1 with y_1, a_2 with y_2, and $\phi_{aa}^{S/O}$ with $\phi_{yy}^{S/O}$ in Eq. (B), and solve for y_1/y_2:

$$\frac{y_1}{y_2} = \frac{\phi_{12}^{S/O}}{\phi_{yy}^{S/O} - \phi_{11}^{S/O}} \tag{G}$$

Similarly, replace a_1 with z_1, a_2 with z_2, and $\phi_{aa}^{S/O}$ with $\phi_{zz}^{S/O}$ in Eq. (C), and solve for z_1/z_2:

$$\frac{z_1}{z_2} = \frac{\phi_{zz}^{S/O} - \phi_{22}^{S/O}}{\phi_{12}^{S/O}} \tag{H}$$

This shows that, provided $\phi_{12}^{S/O} \neq 0$, there exist two and only two principal axes of S for O which lie in plane A. One is parallel to the principal direction $\mathbf{n}_y$ defined as

$$\mathbf{n}_y = y_1\mathbf{n}_1 + y_2\mathbf{n}_2$$

and the other parallel to the principal direction $\mathbf{n}_z$ defined as

$$\mathbf{n}_z = z_1\mathbf{n}_1 + z_2\mathbf{n}_2$$

That these principal axes are perpendicular to each other follows from

$$\mathbf{n}_y \cdot \mathbf{n}_z = y_1z_1 + y_2z_2 = \left(\frac{y_1}{y_2}\frac{z_1}{z_2} + 1\right) y_2z_2$$

and

$$\frac{y_1}{y_2}\frac{z_1}{z_2} \underset{(\mathrm{G,H})}{=} \frac{\phi_{zz}^{S/O} - \phi_{11}^{S/O}}{\phi_{yy}^{S/O} - \phi_{22}^{S/O}} \underset{(\mathrm{E,F})}{=} -1$$

which give

$$\mathbf{n}_y \cdot \mathbf{n}_z = 0$$

If $\phi_{12}^{S/O} = 0$, Eqs. (B) and (C) reduce to

$$\begin{aligned} a_1(\phi_{11}^{S/O} - \phi_{aa}^{S/O}) &= 0 \\ a_2(\phi_{22}^{S/O} - \phi_{aa}^{S/O}) &= 0 \end{aligned} \tag{I}$$

and, provided $\phi_{11}^{S/O} \neq \phi_{22}^{S/O}$, these two equations, as well as Eq. (D), are satisfied only by either

$$a_1 = \pm 1, \qquad a_2 = 0, \qquad \phi_{aa}^{S/O} = \phi_{11}^{S/O}$$

or

$$a_1 = 0, \qquad a_2 = \pm 1, \qquad \phi_{aa}^{S/O} = \phi_{22}^{S/O}$$

Accordingly, $\pm\mathbf{n}_1$ and $\pm\mathbf{n}_2$ are principal directions, and the only principal directions parallel to A, when $\phi_{12}^{S/O} = 0$ and $\phi_{11}^{S/O} \neq \phi_{22}^{S/O}$.

Finally, if $\phi_{12}^{S/O} = 0$ and $\phi_{11}^{S/O} = \phi_{22}^{S/O}$, Eqs. (D) and (I) can be satisfied by taking

$$\phi_{aa}^{S/O} = \phi_{11}^{S/O} = \phi_{22}^{S/O}$$

and letting a_1 and a_2 have any values consistent with Eq. (D). Consequently, all lines passing through O and lying in plane A are then principal axes of S for O, and the second moment of S with respect to any such line is a principal second moment of S for O.

Problem: The strengths of the four points O, P, Q, R of the rectangle shown in Fig. 3.3.8b are 1, 2, 10, and 20 lb.

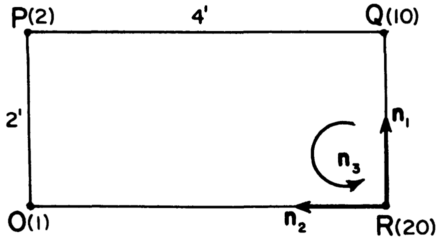


FIG. 3.3.8b

Locate three principal axes of this set S of points, and evaluate the corresponding principal second moments of S for point R.

Solution: Let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be the unit vectors shown in Fig. 3.3.8b. Then (see 3.3.7) $\mathbf{n}_3$ is a principal direction of S for R; a line passing through R and parallel to $\mathbf{n}_3$ is a principal axis of S for R; and the corresponding principal second moment of S for R is given by

$$\phi_{33}^{S/R} \underset{(3.2.3)}{=} \phi_{33}^{O/R} + \phi_{33}^{P/R} + \phi_{33}^{Q/R} + \phi_{33}^{R/R}$$

$$\underset{(3.1.7)}{=} 1(16) + 2(20) + 10(4) + 20(0) = 96 \text{ lb ft}^2$$

Next, to locate two principal axes in the plane of the rectangle, evaluate $\phi_{11}^{S/R}$, $\phi_{22}^{S/R}$, and $\phi_{12}^{S/R}$:

$$\phi_{11}^{S/R} \underset{(3.2.3,3.1.7)}{=} 48 \text{ lb ft}^2, \qquad \phi_{22}^{S/R} = 48 \text{ lb ft}^2$$

$$\phi_{12}^{S/R} \underset{(3.2.3,3.1.5)}{=} -16 \text{ lb ft}^2$$

Then, letting $\mathbf{n}_z$ be one of the principal directions parallel to the plane of the rectangle, the corresponding principal second moment is given by

$$\phi_{zz}^{S/R} = \frac{48 + 48}{2} + \left[\left(\frac{48 - 48}{2}\right)^2 + (-16)^2\right]^{1/2}$$
$$= 48 + 16 = 64 \text{ lb ft}^2$$

and the ratio z_1/z_2 of the $\mathbf{n}_1$ and $\mathbf{n}_2$ measure numbers of $\mathbf{n}_z$ by

$$\frac{z_1}{z_2} = \frac{64 - 48}{-16} = -1$$

The corresponding principal axis L_z thus has the orientation shown in Fig. 3.3.8c. A second principal axis of S for O, L_y, is perpen-

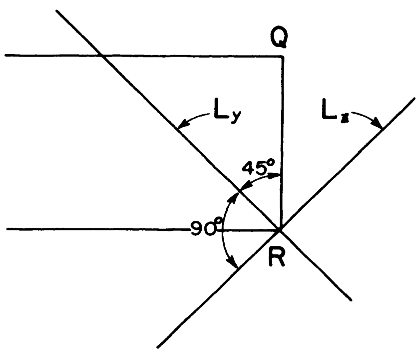


FIG. 3.3.8c

dicular to L_z, and the corresponding principal second moment is given by

$$\phi_{yy}^{S/R} = 48 + 48 - 32 = 64 \text{ lb ft}^2$$

3.3.9 There exist at least three mutually perpendicular principal directions of every set of points for every point in space.

Proof: Let S be a set of points and O a point in space. It suffices to show that there exists *one* principal direction of S for O, since, from Sec. 3.3.8, it then follows that there exist at least two more, perpendicular to each other and to the first.

Let $\mathbf{n}_i$, $i = 1, 2, 3$, be mutually perpendicular unit vectors and a_i the $\mathbf{n}_i$ measure number of a unit vector $\mathbf{n}_a$. Then $\mathbf{n}_a$ is a principal direction of S for O whenever

$$\phi_{aa}^{S/O}\mathbf{n}_a \underset{(3.3.2)}{=} \boldsymbol{\Phi}_a^{S/O} \underset{(3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{S/O} a_i \mathbf{n}_j$$

or, expanding,

$$\left.\begin{aligned} a_1(\phi_{11}^{S/O} - \phi_{aa}^{S/O}) + a_2\phi_{12}^{S/O} + a_3\phi_{13}^{S/O} &= 0 \\ a_1\phi_{12}^{S/O} + a_2(\phi_{22}^{S/O} - \phi_{aa}^{S/O}) + a_3\phi_{23}^{S/O} &= 0 \\ a_1\phi_{13}^{S/O} + a_2\phi_{23}^{S/O} + a_3(\phi_{33}^{S/O} - \phi_{aa}^{S/O}) &= 0 \end{aligned}\right\} \quad (A)$$

where a_1, a_2, a_3 must satisfy the condition

$$a_1^2 + a_2^2 + a_3^2 = 1 \quad (B)$$

which shows that a_1, a_2, a_3 cannot vanish simultaneously. From this it follows that Eqs. (A) can in general be satisfied simultaneously only if the determinant of the coefficients of a_1, a_2, and a_3 is equal to zero; that is,

$$(\phi_{aa}^{S/O})^3 - A_1(\phi_{aa}^{S/O})^2 + A_2(\phi_{aa}^{S/O}) - A_3 = 0 \quad (C)$$

where

$$A_1 = \phi_{11}^{S/O} + \phi_{22}^{S/O} + \phi_{33}^{S/O}$$

$$A_2 = \phi_{11}^{S/O}\phi_{22}^{S/O} + \phi_{22}^{S/O}\phi_{33}^{S/O} + \phi_{33}^{S/O}\phi_{11}^{S/O} - [(\phi_{12}^{S/O})^2 + (\phi_{23}^{S/O})^2 + (\phi_{31}^{S/O})^2]$$

$$A_3 = \phi_{11}^{S/O}\phi_{22}^{S/O}\phi_{33}^{S/O} + 2\phi_{12}^{S/O}\phi_{23}^{S/O}\phi_{31}^{S/O} - [\phi_{11}^{S/O}(\phi_{23}^{S/O})^2 + \phi_{22}^{S/O}(\phi_{31}^{S/O})^2 + \phi_{33}^{S/O}(\phi_{12}^{S/O})^2]$$

Equation (C), being a cubic equation, has at least one real root. (A_1, A_2, and A_3 are real quantities.) This root furnishes the value of one principal second moment of S for O, and, once it has been found, any two of Eqs. (A) can be used together with Eq. (B) to determine the measure numbers defining the corresponding principal direction.

Problem: Figure 3.3.9a shows a set S of points, a point O, and three mutually perpendicular unit vectors $\mathbf{n}_i$, $i = 1, 2, 3$. The six second moments $\phi_{ij}^{S/O}$, $i, j = 1, 2, 3$, are tabulated in Fig. 3.3.9b.

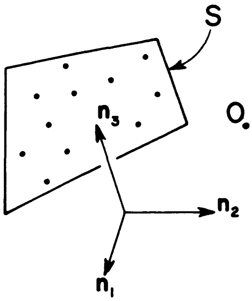


FIG. 3.3.9a

$\phi_{ij}^{S/O}$ (ft^4)		j = 1	2	3
i	1	15	$3\sqrt{3}$	$-2\sqrt{3}$
	2	$3\sqrt{3}$	21	2
	3	$-2\sqrt{3}$	2	12

FIG. 3.3.9b

Determine three principal directions of S for O.

Solution:

$$A_1 = 15 + 21 + 12 = 48 \text{ ft}^4$$

$$A_2 = 315 + 252 + 180 - (27 + 4 + 12) = 704 \text{ ft}^8$$

$$A_3 = 3780 - 72 - (60 + 252 + 324) = 3072 \text{ ft}^{12}$$

Hence

$$(\phi_{aa}^{S/O})^3 - 48(\phi_{aa}^{S/O})^2 + 704\phi_{aa}^{S/O} - 3072 \underset{(C)}{=} 0$$

Solve for $\phi_{aa}^{S/O}$ (by trial and error), calling the roots $\phi_{xx}^{S/O}$, $\phi_{yy}^{S/O}$, $\phi_{zz}^{S/O}$:

$$\phi_{xx}^{S/O} = 8 \text{ ft}^4, \qquad \phi_{yy}^{S/O} = 16 \text{ ft}^4, \qquad \phi_{zz}^{S/O} = 24 \text{ ft}^4$$

Replace a_i with x_i, $i = 1, 2, 3$, $\phi_{aa}^{S/O}$ with $\phi_{xx}^{S/O}$, and use the first two of Eqs. (A):

$$7x_1 + 3\sqrt{3}\, x_2 = 2\sqrt{3}\, x_3$$

$$3\sqrt{3}\, x_1 + 13x_2 = -2x_3$$

Solve for x_1 and x_2 (in terms of x_3):

$$x_1 = \frac{\sqrt{3}}{2}\, x_3, \qquad x_2 = -\frac{1}{2}\, x_3$$

Use Eq. (B):

$$(\tfrac{3}{4} + \tfrac{1}{4} + 1)x_3^2 = 1$$

Thus

$$x_3 = \pm\frac{\sqrt{2}}{2}$$

and

$$x_1 = \pm\frac{\sqrt{6}}{4}, \qquad x_2 = \pm\frac{\sqrt{2}}{4}$$

Let $\mathbf{n}_x$ be the principal direction corresponding to $\phi_{aa}^{S/O} = \phi_{xx}^{S/O}$. Then

$$\mathbf{n}_x = \pm\left(\frac{\sqrt{6}}{4}\mathbf{n}_1 - \frac{\sqrt{2}}{4}\mathbf{n}_2 + \frac{\sqrt{2}}{2}\mathbf{n}_3\right)$$

Similarly, principal directions $\mathbf{n}_y$ and $\mathbf{n}_z$, corresponding to $\phi_{aa}^{S/O} = \phi_{yy}^{S/O}$ and $\phi_{aa}^{S/O} = \phi_{zz}^{S/O}$, are given by

$$\mathbf{n}_y = \pm\left(\frac{\sqrt{6}}{4}\mathbf{n}_1 - \frac{\sqrt{2}}{4}\mathbf{n}_2 - \frac{\sqrt{2}}{2}\mathbf{n}_3\right)$$

$$\mathbf{n}_z = \pm\left(\frac{1}{2}\mathbf{n}_1 + \frac{\sqrt{3}}{2}\mathbf{n}_2\right)$$

3.3.10 Knowledge of three mutually perpendicular principal axes and the corresponding principal second moments (or principal radii of gyration) of a set S for a point O simplifies the determination of second moments of S with respect to O for arbitrary directions $\mathbf{n}_a$ and $\mathbf{n}_b$. For, if $\mathbf{n}_i$, $i = 1, 2, 3$, are mutually perpendicular principal directions of S for O, then $\phi_{12}^{S/O}$, $\phi_{23}^{S/O}$, and $\phi_{31}^{S/O}$ are equal to zero (see 3.3.3), and Eqs. (5), (6), and (7) of Sec. 3.2.4 reduce, respectively, to

$$\boldsymbol{\Phi}_a^{S/O} = \phi_{11}^{S/O}a_1\mathbf{n}_1 + \phi_{22}^{S/O}a_2\mathbf{n}_2 + \phi_{33}^{S/O}a_3\mathbf{n}_3 \tag{1}$$

$$\phi_{ab}^{S/O} = \phi_{11}^{S/O}a_1b_1 + \phi_{22}^{S/O}a_2b_2 + \phi_{33}^{S/O}a_3b_3 \tag{2}$$

and

$$\phi_{aa}^{S/O} = \phi_{11}^{S/O}a_1^2 + \phi_{22}^{S/O}a_2^2 + \phi_{33}^{S/O}a_3^2 \tag{3}$$

3.3.11 By successive use of expressions given in Secs. 3.2.6, 3.2.7, and 3.3.10, all second moments of a set S of points can be found whenever the following are known (compare with 3.2.9):

(a) The strength of S.

(b) The location of the centroid of S.

(c) Three mutually perpendicular principal axes and the corresponding principal second moments (or principal radii of gyration) of S for one point.

3.3.12 Principal directions, axes, etc., for the centroid of a set

S are called *centroidal principal directions, centroidal principal axes,* etc.

Problem: In Fig. 3.3.12a, lines CB, CD, CE represent mutually perpendicular centroidal principal axes of a set S of points. The

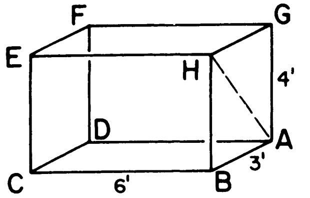


FIG. 3.3.12a

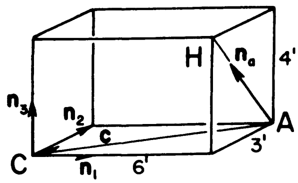


FIG. 3.3.12b

corresponding centroidal principal radii of gyration are equal to 2, 3, and 4 ft.

Determine the radius of gyration of S with respect to line AH.

Solution: Let $\mathbf{n}_a$ and $\mathbf{n}_i$, $i = 1, 2, 3$, be the unit vectors shown in Fig. 3.3.12b, and $k_a^{S/A}$ the desired radius of gyration. Then

$$(k_a^{S/A})^2 \underset{(3.2.8)}{=} (k_a^{S/C})^2 + (l_a^*)^2$$

$$(k_a^{S/C})^2 \underset{(3.2.5)}{=} \frac{\phi_{aa}^{S/C}}{N} \underset{(3.3.10)}{=} \frac{\phi_{11}^{S/C}}{N} a_1^2 + \frac{\phi_{22}^{S/C}}{N} a_2^2 + \frac{\phi_{33}^{S/C}}{N} a_3^2$$

$$\underset{(3.2.5)}{=} (k_1^{S/C})^2 a_1^2 + (k_2^{S/C})^2 a_2^2 + (k_3^{S/C})^2 a_3^2$$

$$(l_a^*)^2 \underset{(F3.3.12b)}{=} (\mathbf{c} \times \mathbf{n}_a)^2$$

Hence

$$(k_a^{S/A})^2 = (k_1^{S/C})^2 a_1^2 + (k_2^{S/C})^2 a_2^2 + (k_3^{S/C})^2 a_3^2 + (\mathbf{c} \times \mathbf{n}_a)^2$$

where

$$\mathbf{n}_a = a_1\mathbf{n}_1 + a_2\mathbf{n}_2 + a_3\mathbf{n}_3$$

and

$$a_1 = 0, \qquad a_2 = -\tfrac{3}{5}, \qquad a_3 = \tfrac{4}{5}$$

$$k_1^{S/C} = 2 \text{ ft}, \qquad k_2^{S/C} = 3 \text{ ft}, \qquad k_3^{S/C} = 4 \text{ ft}$$

$$\mathbf{c} = -6\mathbf{n}_1 - 3\mathbf{n}_2 \text{ ft}$$

$$(\mathbf{c} \times \mathbf{n}_a)^2 = \frac{36(29)}{25} \text{ ft}^2$$

so that

$$(k_a^{S/A})^2 = \frac{9(9)}{25} + \frac{16(16)}{25} + \frac{36(29)}{25} = \frac{1381}{25}$$

and

$$k_a^{S/A} = 7.46 \text{ ft}$$

3.3.13 The locus E of points P whose distance R from a point O is inversely proportional to the square-root of the second moment of a set S of points with respect to line OP is an ellipsoid. It is called a *momental ellipsoid* of S for O.

Proof: Let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be mutually perpendicular principal directions of S for O (see Fig. 3.3.13a), L_1, L_2, L_3 the corresponding principal axes, x_1, x_2, x_3 the coordinates of point P when L_1, L_2, L_3 are regarded as cartesian coordinate axes whose positive senses are defined by $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, and $\mathbf{n}_a$ a unit vector parallel to OP.

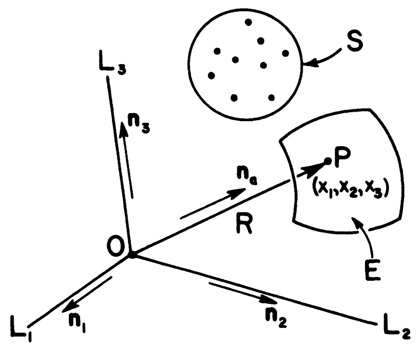


FIG. 3.3.13a

It is to be shown that x_1, x_2, x_3 satisfy the equation of an ellipsoid whenever

$$R = k(\phi_{aa}^{S/O})^{-1/2} \tag{A}$$

where k is a constant. Now

$$\phi_{aa}^{S/O} \underset{(3.3.10)}{=} a_1{}^2\phi_{11}^{S/O} + a_2{}^2\phi_{22}^{S/O} + a_3{}^2\phi_{33}^{S/O} \tag{B}$$

where a_i is the $\mathbf{n}_i$ measure number of $\mathbf{n}_a$ and $\phi_{ii}^{S/O}$ is a principal second moment of S for O. Furthermore, as the position vector of P relative to O can be expressed either as $R\mathbf{n}_a$ or as $x_1\mathbf{n}_1 + x_2\mathbf{n}_2 + x_3\mathbf{n}_3$, a_i can be replaced with x_i/R, so that

$$\phi_{aa}^{S/O} = \frac{1}{R^2}(x_1{}^2\phi_{11}^{S/O} + x_2{}^2\phi_{22}^{S/O} + x_3{}^2\phi_{33}^{S/O})$$

Hence, using Eq. (A) to eliminate $\phi_{aa}^{S/O}$,

$$\frac{k^2}{R^2} = \frac{1}{R^2}(x_1{}^2\phi_{11}^{S/O} + x_2{}^2\phi_{22}^{S/O} + x_3{}^2\phi_{33}^{S/O})$$

or

$$\frac{x_1{}^2}{A_1{}^2} + \frac{x_2{}^2}{A_2{}^2} + \frac{x_3{}^2}{A_3{}^2} = 1 \tag{C}$$

where

$$A_i = k(\phi_{ii}^{S/O})^{-1/2}, \qquad i = 1, 2, 3 \tag{D}$$

Equation (C) is the equation of an ellipsoid whose center is at O, whose axes coincide with the principal axes L_1, L_2, L_3 of S for O, and whose semidiameters are equal to A_i, $i = 1, 2, 3$, as shown in Fig. 3.3.13b.

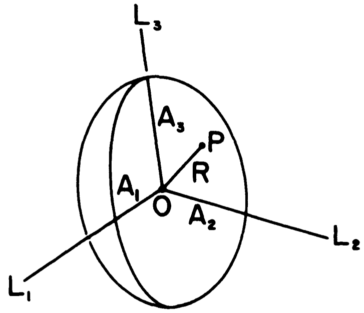


FIG. 3.3.13b

3.3.14 Of all lines passing through a point O, those with respect to which a set S of points has a larger, or smaller, second moment than it has with respect to all other lines passing through O are principal axes of S for O.

Proof: The numbering of three mutually perpendicular principal directions $\mathbf{n}_i$, $i = 1, 2, 3$, of S for O can always be accomplished in such a way that

$$\phi_{11}^{S/O} \geqslant \phi_{22}^{S/O} \geqslant \phi_{33}^{S/O}$$

or, referring to Eq. (D), Sec. 3.3.13, in such a way that the lengths A_1, A_2, A_3 of the semidiameters of a momental ellipsoid of S for O satisfy

$$A_1 \leqslant A_2 \leqslant A_3$$

The distance R from the center O to a point P of the ellipsoid (see Fig. 3.3.13b) is never smaller than the smallest semidiameter, and never larger than the largest. That is,

$$A_1 \leqslant R \leqslant A_3$$

or, using Eqs. (A) and (D), Sec. 3.3.13,

$$k(\phi_{11}^{S/O})^{-1/2} \leqslant k(\phi_{aa}^{S/O})^{-1/2} \leqslant k(\phi_{33}^{S/O})^{-1/2}$$

so that

$$\phi_{11}^{S/O} \geqslant \phi_{aa}^{S/O} \geqslant \phi_{33}^{S/O}$$

which asserts that the second moment of S with respect to a line which passes through O and is not a principal axis of S for O cannot be smaller than the smallest principal second moment of S for O or larger than the largest principal second moment of S for O.

3.3.15 Given a set S of points with positive strengths, no second moment of S with respect to a line (see 3.2.3) is smaller than the smallest centroidal principal second moment of S (see 3.3.12). The smallest centroidal principal second moment is therefore called the *minimum second moment* of S.

Proof: The second moment $\phi_{aa}^{S/O}$ of S with respect to a line L_a passing through a point O is greater than the second moment ϕ_{aa}^{S/P^*} of S with respect to a parallel line passing through the centroid P^* (see 3.2.7); and ϕ_{aa}^{S/P^*} is larger than or equal to the smallest centroidal principal second moment of S (see 3.3.14).

3.4 Second moments of curves, surfaces, and solids

3.4.1 Given a point O, a figure (curve, surface, or solid) F, and a unit vector $\mathbf{n}_a$, a vector $\boldsymbol{\Phi}_a^{F/O}$, called *the second moment of the figure F with respect to point O for the direction* $\mathbf{n}_a$, is obtained by performing the following operations:

(1) Divide F into n elements of arbitrary size and shape.
(2) Select a point in each element.

(3) Assign to each such point a strength equal to the length, area, or volume of the corresponding element.

(4) Evaluate the second moment of this set of points with respect to O for the direction $\mathbf{n}_a$ (see 3.2.1).

(5) Determine the vector approached by this second moment as n tends to infinity and each of the elements chosen in (1) shrinks to a point.

Problem: In Fig. 3.4.1a, O is one endpoint of a straight line S of length L, and $\mathbf{n}_a$ is a unit vector perpendicular to S.

Determine the second moment $\boldsymbol{\Phi}_a^{S/O}$

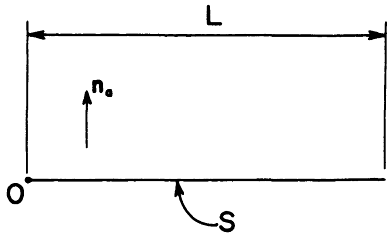


FIG. 3.4.1a

Solution:

(1) Division of S into n elements: Choose elements of equal length, L/n. Number these as shown in Fig. 3.4.1b.

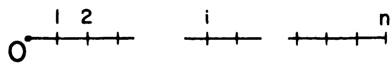


FIG. 3.4.1b

(2) Selection of a point in each element: Use the right-most point in each element, calling these points $P_1, P_2, \ldots, P_n$, as shown in Fig. 3.4.1c. P_i is a typical point of this set of points.

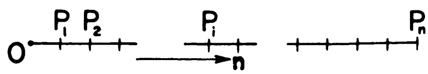


FIG. 3.4.1c

(3) Assigning of strengths to the points P_i, $i = 1, 2, \ldots, n$:

Let N_i be the strength of P_i. As the length of each element is equal to L/n,

$$N_i = L/n, \; i = 1, 2, \ldots, n \tag{A}$$

(4) Evaluation of the second moment of the set of points $P_1, \ldots, P_n$: From Secs. 3.2.1 and 3.1.1 it follows that the second moment is given by

$$\sum_{i=1}^{n} N_i \mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) \tag{B}$$

where $\mathbf{p}_i$ is the position vector of P_i relative to O. Let $\mathbf{n}$ be a unit vector parallel to line S, as shown in Fig. 3.4.1c, and note that the distance from O to P_i is equal to iL/n. Then

$$\mathbf{p}_i = \frac{iL}{n}\mathbf{n}$$

and

$$\mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) = \frac{i^2L^2}{n^2}\mathbf{n} \times (\mathbf{n}_a \times \mathbf{n}) = \frac{i^2L^2}{n^2}\mathbf{n}_a$$

so that

$$N_i\mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) \underset{(A)}{=} \frac{i^2L^3}{n^3}\mathbf{n}_a$$

and (B) can be replaced with

$$\left(\sum_{i=1}^{n} i^2\right)\frac{L^3}{n^3}\mathbf{n}_a$$

The sum appearing in this expression has the value

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{C}$$

Hence the second moment of the set of points $P_1, \ldots, P_n$ with respect to O for the direction $\mathbf{n}_a$ is given by

$$\frac{(n+1)(2n+1)}{6n^2}L^3\mathbf{n}_a$$

(5) Determination of the vector approached by the second moment of the set of point $P_1, \ldots, P_n$: As the elements into which S was divided were chosen in such a way that each element automatically shrinks to a point as n tends to infinity, $\boldsymbol{\Phi}_a^{S/O}$ is given by

$$\begin{aligned}
\boldsymbol{\Phi}_a^{S/O} &= \lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{6n^2}L^3\mathbf{n}_a\right] \\
&= \frac{L^3\mathbf{n}_a}{6}\lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{n^2}\right] \\
&= \frac{L^3\mathbf{n}_a}{6}\lim_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) \\
&= \frac{L^3}{3}\mathbf{n}_a
\end{aligned}$$

3.4.2 In a manner similar to that employed in connection with the location of centroids (see Vol. I, Sec. 2.5.1), the limiting process described in Sec. 3.4.1 can be replaced with the evaluation of an integral:

$$\boldsymbol{\Phi}_a^{F/O} = \int_F \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p})\, d\tau$$

where (see Fig. 3.4.2a) $\mathbf{p}$ is the position vector of a typical point

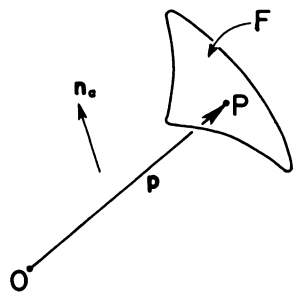


FIG. 3.4.2a

P of the figure F relative to point O, and $d\tau$ is the length, area or volume of a differential element of F.

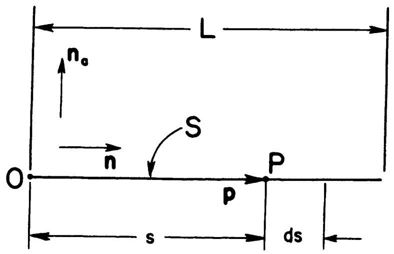


FIG. 3.4.2b

Problem: Solve Problem 3.4.1 by integration.

Solution: Let P be a typical point of S (see Fig. 3.4.2b), $\mathbf{p}$ the position vector of P relative to O, $\mathbf{n}$ a unit vector parallel to S, s the distance from O to P, and $d\tau$ the length of a differential element of S. Then

$$\mathbf{p} = s\mathbf{n}$$

$$\mathbf{p} \times (\mathbf{n}_a \times \mathbf{p}) = s^2 \mathbf{n}_a$$

$$d\tau = ds$$

$$\boldsymbol{\Phi}_a^{S/O} = \int_S \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p})\, d\tau = \int_O^L s^2 \mathbf{n}_a\, ds$$

or, as the characteristics of the unit vector $\mathbf{n}_a$ are independent of s,

$$\boldsymbol{\Phi}_a^{S/O} = \mathbf{n}_a \int_O^L s^2\, ds = \frac{L^3}{3}\mathbf{n}_a$$

3.4.3 Given a point O, a figure F, and two unit vectors $\mathbf{n}_a$ and $\mathbf{n}_b$, the scalar $\phi_{ab}^{F/O}$, defined as

$$\phi_{ab}^{F/O} = \boldsymbol{\Phi}_a^{F/O} \cdot \mathbf{n}_b \tag{1}$$

where $\boldsymbol{\Phi}_a^{F/O}$ is the second moment of F with respect to O for the direction $\mathbf{n}_a$ (see 3.4.1), is called *the second moment of F with respect to O for the pair of directions* $\mathbf{n}_a$, $\mathbf{n}_b$. It follows from Sec. 3.4.2 that $\phi_{ab}^{F/O}$ can be expressed as

$$\phi_{ab}^{F/O} = \int_F (\mathbf{n}_a \times \mathbf{p}) \cdot (\mathbf{n}_b \times \mathbf{p})\, d\tau \tag{2}$$

Problem: $\mathbf{n}_a$ and $\mathbf{n}_b$ in Fig. 3.4.3 are unit vectors parallel and perpendicular to the diameter OQ of a semsicrcular curve C of

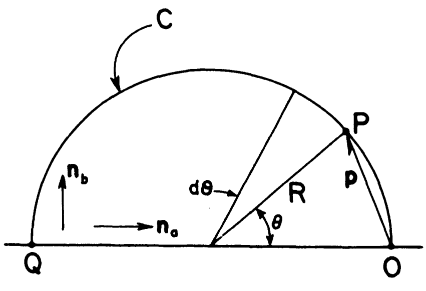


FIG. 3.4.3

radius R. ($\mathbf{n}_b$ is parallel to the plane determined by C.) Evaluate $\phi_{ab}^{C/O}$.

Solution: (see Fig. 3.4.3 for notation):

$$\mathbf{p} = R(\cos\theta - 1)\mathbf{n}_a + R\sin\theta\,\mathbf{n}_a$$

$$(\mathbf{n}_a \times \mathbf{p}) \cdot (\mathbf{n}_b \times \mathbf{p}) = R^2(1 - \cos\theta)\sin\theta$$

$$d\tau = R\,d\theta$$

$$\phi_{ab}^{C/O} = \int_C (\mathbf{n}_a \times \mathbf{p}) \cdot (\mathbf{n}_b \times \mathbf{p})\,d\tau$$

$$= R^3 \int_0^\pi (1 - \cos\theta)\sin\theta\,d\theta = 2R^3$$

3.4.4 When the unit vector $\mathbf{n}_b$ is equal to the unit vector $\mathbf{n}_a$, the expressions given in Sec. 3.4.3 become

$$\phi_{aa}^{F/O} = \mathbf{\Phi}_a^{F/O} \cdot \mathbf{n}_a \tag{1}$$

$$\phi_{aa}^{F/O} = \int_F (\mathbf{n}_a \times \mathbf{p})^2\,d\tau \tag{2}$$

and $\phi_{aa}^{F/O}$ is called *the second moment of F with respect to line L_a*, L_a being the line which passes through O and is parallel to $\mathbf{n}_a$. As $(\mathbf{n}_a \times \mathbf{p})^2$ is equal to the square of the distance $l_a^{P/O}$ from L_a to a typical point P of F, $\phi_{aa}^{F/O}$ can also be expressed as

$$\phi_{aa}^{F/O} = \int_F (l_a^{P/O})^2\,d\tau \tag{3}$$

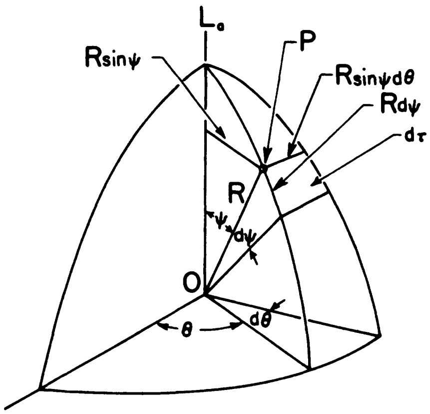


FIG. 3.4.4a

Problem: Determine the second moment of (a) a spherical surface A of radius R and (b) a spherical solid B of radius R, with respect to a line L_a passing through the center O of the figure.

Solution (a) (see Fig. 3.4.4a for notation):

$$l_a^{P/O} = R \sin \psi$$

$$d\tau = (R\, d\psi)(R \sin \psi\, d\theta) = R^2 \sin \psi\, d\theta\, d\psi$$

$$\phi_{aa}^{A/O} = \int_A (l_a^{P/O})^2\, d\tau = R^4 \int_0^{2\pi} d\theta \int_0^{\pi} \sin^3 \psi\, d\psi = \tfrac{8}{3}\pi R^4$$

Solution (b) (see Fig. 3.4.4b for notation):

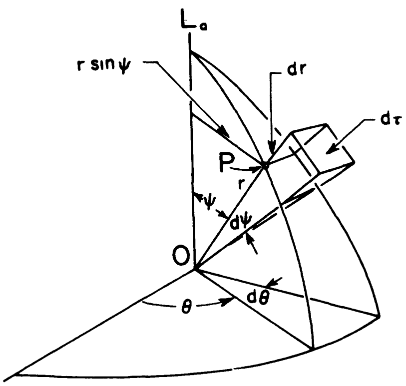


FIG. 3.4.4b

$$l_a^{P/O} = r \sin \psi$$

$$d\tau = (r\, d\psi)(r \sin \psi\, d\theta)\, dr = r^2 \sin \psi\, d\theta\, dr\, d\psi$$

$$\phi_{aa}^{B/O} = \int_B (l_a^{P/O})^2\, d\tau = \int_0^{2\pi} d\theta \int_0^{\pi} d\psi \int_0^R r^4 \sin^3 \psi\, dr = \tfrac{8}{15}\pi R^5$$

3.4.5 The second moment $\phi_{aa}^{F/O}$ (see 3.4.4) can always be expressed as the product of τ, the length, area, or volume of the figure F, and the square of a positive, real quantity $k_a^{F/O}$, called *the radius of gyration of F with respect to line* L_a:

$$\phi_{aa}^{F/O} = \tau(k_a^{F/O})^2$$

Problem: Determine the radius of gyration of (a) a spherical surface A of radius R and (b) a spherical solid B of radius R with respect to a line L_a passing through the center O of the figure.

Solution (a): The area τ of a spherical surface A of radius R is given by (see Solution (a), 3.4.4 for notation)

$$\tau = \int_A d\tau = R^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\psi \, d\psi = 4\pi R^2$$

From Problem 3.4.4,

$$\phi_{aa}^{A/O} = \tfrac{8}{3}\pi R^4$$

Hence

$$k_a^{A/O} = \left(\frac{\phi_{aa}^{A/O}}{\tau}\right)^{1/2} = \left(\frac{2}{3}\right)^{1/2} R$$

Solution (b): The volume τ of a spherical solid B of radius R is given by (see Solution (b), 3.4.4 for notation)

$$\tau = \int_B d\tau = \int_0^{2\pi} d\theta \int_0^{\pi} d\psi \int_0^{R} r^2 \sin\psi \, dr = \tfrac{4}{3}\pi R^3$$

From Problem 3.4.4,

$$\phi_{aa}^{B/O} = \tfrac{8}{15}\pi R^5$$

Hence

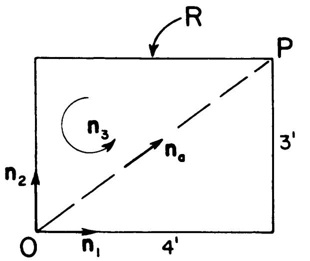
$$k_a^{B/O} = \left(\frac{\phi_{aa}^{B/O}}{\tau}\right)^{1/2} = \left(\frac{2}{5}\right)^{1/2} R$$

3.4.6 Given a figure F, a point O, and a unit vector $\mathbf{n}_z$, $\mathbf{n}_z$ is called a *principal direction* of F for O if and only if $\Phi_z^{F/O}$ (see 3.4.1) is parallel to $\mathbf{n}_z$ or equal to zero. When $\mathbf{n}_z$ is a principal direction of F for O, the line L_z parallel to $\mathbf{n}_z$ and passing through O is called a *principal axis* of F for O; the plane P_z passing through O and perpendicular to $\mathbf{n}_z$ is called a *principal plane* of F for O; the second moment $\phi_{zz}^{F/O}$ is called a *principal second moment* of S for O; and the radius of gyration $k_z^{F/O}$ is called a *principal radius of gyration* of F for O.

Example: Every line which passes through an endpoint O of a straight line S and is perpendicular to S is a principal axis of S for O (see Problem 3.4.1). Also, line S, itself, is a principal axis of S for every point on S.

3.4.7 As a consequence of the definitions given in Secs. 3.4.1–3.4.6, many of the relationships discussed in Parts 3.2 and 3.3 have counterparts in the theory of second moments of curves, surfaces, and solids. After minor changes in wording and notation, these relationships can therefore be used for the solution of problems involving curves, surfaces, and solids.

Problem (a): A rectangular surface R and tabulated values of $\phi_{ij}^{R/O}$, $i, j = 1, 2, 3$, are shown in Fig. 3.4.7a.



$\phi_{ij}^{R/O}$ (ft^4)		j = 1	2	3
i	1	36	-36	0
	2	-36	64	0
	3	0	0	100

FIG. 3.4.7a

Determine the second moment of R with respect to the diagonal OP of R.

Solution: Let

$$\mathbf{n}_a = a_1\mathbf{n}_1 + a_2\mathbf{n}_2 + a_3\mathbf{n}_3$$

be a unit vector parallel to line OP, $\phi_{aa}^{R/O}$ the desired second moment. Then

$$a_1 = \tfrac{4}{5}, \qquad a_2 = \tfrac{3}{5}, \qquad a_3 = 0$$

and

$$\begin{aligned} \phi_{aa}^{R/O} &\underset{(3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/O} a_i a_j \\ &= \phi_{11}^{R/O} a_1^2 + \dots + 2(\phi_{12}^{R/O} a_1 a_2 + \dots) \\ &= 11.52 \text{ ft}^4 \end{aligned}$$

Problem (b): Determine the second moment of a spherical surface of radius R with respect to a tangent to the surface.

Solution: Let A be the surface, L_a a line passing through the center O of A and parallel to the tangent in question, and $\phi_{aa}^{A/P}$ the desired second moment (see Fig. 3.4.7b). Then

$$\phi_{aa}^{A/P} \underset{(3.2.7)}{=} \phi_{aa}^{A/O} + \tau(l_a^{O/P})^2$$

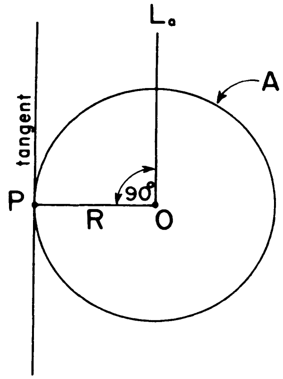


FIG. 3.4.7b

where $\tau = 4\pi R^2$ is the area of A and $l_a^{O/P} = R$ is the distance between the tangent and line L_a. Hence

$$\phi_{aa}^{A/P} \underset{(P3.4.4)}{=} \tfrac{8}{3}\pi R^4 + 4\pi R^4 = \tfrac{20}{3}\pi R^4$$

Problem (c): Determine the radius of gyration of a spherical solid with respect to a tangent to the solid.

Solution: Let B be the solid, L_a a line passing through the center O of B and parallel to the tangent in question, and $k_a^{B/P}$ the desired radius of gyration (see Fig. 3.4.7c). Then

$$(k_a^{B/P})^2 \underset{(3.2.8)}{=} (k_a^{B/O})^2 + (l_a^{O/P})^2$$

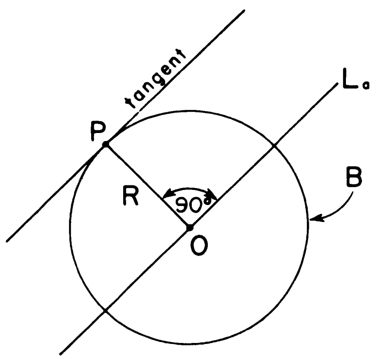


FIG. 3.4.7c

where $l_a^{O/P} = R$ is the distance between the tangent and line L_a. Hence

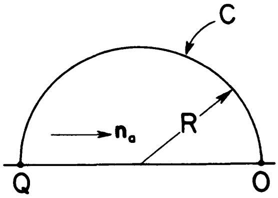
$$(k_a^{B/P})^2 = \underset{(\text{P3.4.5})}{\tfrac{2}{5}R^2} + R^2 = \tfrac{7}{5}R^2$$

and

$$k_a^{B/P} = (\tfrac{7}{5})^{1/2}R$$

Problem (d): In Fig. 3.4.7d, $\mathbf{n}_a$ is a unit vector parallel to the diameter OQ of a semicircular curve C of radius R.

FIG. 3.4.7d



Show that $\mathbf{n}_a$ is not a principal direction of C for point O.

Solution: Let $\mathbf{n}_b$ be any unit vector perpendicular to $\mathbf{n}_a$. If $\mathbf{n}_a$ were a principal direction of C for O, $\phi_{ab}^{C/O}$ would be equal to zero (see 3.3.3). But, for at least one choice of $\mathbf{n}_b$, $\phi_{ab}^{C/O}$ is equal to $2R^3$, as was shown in Problem 3.4.3. Hence $\mathbf{n}_a$ cannot be a principal direction of C for O.

Problem (e): Locate three centroidal principal axes (see 3.3.12) of the rectangular parallelepiped shown in Fig. 3.4.7e.

Solution: The three planes passing through the centroid P^* and parallel to the faces of the parallelepiped are planes of symmetry (see 3.3.6) and, therefore, principal planes for all points in these planes. In particular, they are principal planes for their

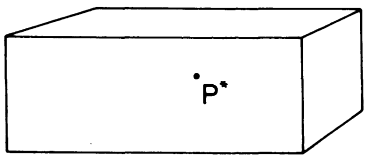


FIG. 3.4.7e

point of intersection, P^*. Consequently, their normals passing through P^* are centroidal principal axes of the parallelepiped.

Problem (f): Figure 3.4.7f shows a plane curve C. O is a point of C.

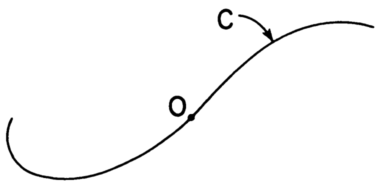


FIG. 3.4.7f

Locate one principal axis of C for point O.

Solution: The plane in which C lies is a principal plane of C for all points in this plane (see 3.3.7). A line passing through O and normal to this plane is thus a principal axis of C for O.

Problem (g): Referring to Problem 3.4.7a, locate three principal axes and determine the corresponding principal second moments of R for point O.

Solution: The plane determined by R is a principal plane of R for all points in this plane (see 3.3.7), hence for point O. A line passing through O and parallel to $\mathbf{n}_3$ is thus a principal axis of R for O, and the corresponding principal second moment of R for O is equal to 100 ft^4.

Let

$$\mathbf{n}_z = z_1\mathbf{n}_1 + z_2\mathbf{n}_2$$

be a unit vector parallel to a principal axis L_z of R for O, this axis lying in R. Then the corresponding principal second moment is given by

$$\phi_{zz}^{R/O} \underset{(3.3.8)}{=} \frac{\phi_{11}^{R/O} + \phi_{22}^{R/O}}{2} + \left[\left(\frac{\phi_{11}^{R/O} - \phi_{22}^{R/O}}{2}\right)^2 + (\phi_{12}^{R/O})^2\right]^{1/2} = 88.6 \text{ ft}^4$$

and

$$\frac{z_1}{z_2} = \frac{\phi_{zz}^{R/O} - \phi_{22}^{R/O}}{\phi_{12}^{R/O}} = -0.682$$

so that L_z has the orientation shown in Fig. 3.4.7g. The remaining principal axis is perpendicular to L_z, and the corresponding principal second moment has the value

$$\phi_{11}^{R/O} + \phi_{22}^{R/O} - \phi_{zz}^{R/O} = 11.4 \text{ ft}^4$$

Note that this second moment is smaller than the second mo-

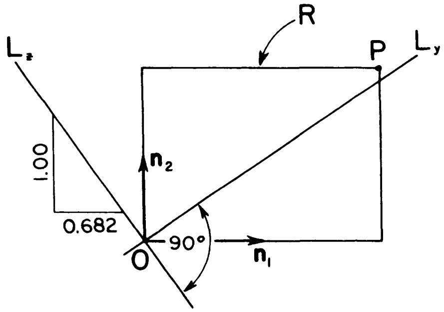


FIG. 3.4.7g

ment of R with respect to line OP (see Problem 3.4.7a), in agreement with Sec. 3.3.14.

Problem (h): The centroidal radius of gyration of a solid cube with respect to a line parallel to an edge of the cube (as found by integration) is equal to $L/(6)^{1/2}$, where L is the length of any edge.

Determine the second moment of a $2 \times 2 \times 2$ ft solid cube C with respect to a diagonal of a face of C.

Solution (see Fig. 3.4.7h for notation): $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are centroidal

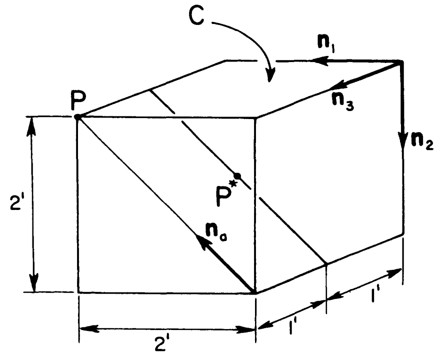


FIG. 3.4.7h

principal directions of C (see 3.3.6). The corresponding centroidal principal radii of gyration are given by

$$k_1^{C/P^*} = k_2^{C/P^*} = k_3^{C/P^*} = 2/\sqrt{6}$$

and, as the volume τ of C is equal to 8 ft³, the centroidal principal second moments of C have the values

$$\phi_{11}^{C/P^*} = \phi_{22}^{C/P^*} = \phi_{33}^{C/P^*} \underset{(3.2.5)}{=} 8(2/\sqrt{6})^2 = \tfrac{16}{3} \text{ ft}^5$$

Hence

$$\phi_{aa}^{C/P^*} \underset{(3.3.10)}{=} a_1{}^2\,\phi_{11}^{C/P^*} + a_2{}^2\,\phi_{22}^{C/P^*} + a_3{}^2\,\phi_{33}^{C/P^*}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 \frac{16}{3} + \left(-\frac{1}{\sqrt{2}}\right)^2 \frac{16}{3} = \frac{16}{3} \text{ ft}^5$$

(This result could have been obtained, alternatively, by noting that the momental ellipsoid (see 3.3.13) of C for P^* must be a sphere, because the three centroidal principal second moments are equal to each other. From this it follows that C has the same second moment with respect to all lines passing through P^*.)

Finally, the desired second moment is given by

$$\phi_{aa}^{C/P} \underset{(3.2.7)}{=} \phi_{aa}^{C/P^*} + \tau(l_a^{P^*/P})^2 = \tfrac{16}{3} + 8(1)^2 = \tfrac{40}{3} \text{ ft}^5$$

3.4.8 The second moment of a plane figure F with respect to a line L_c which is normal to F and intersects the plane of F at a point O is called the *polar second moment of F with respect to O*, and is denoted by $J^{F/O}$. That is,

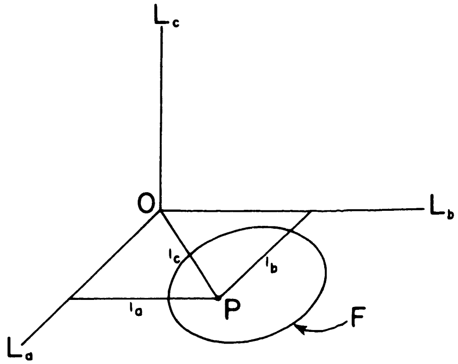


FIG. 3.4.8

$$J^{F/O} = \phi_{cc}^{F/O} \tag{1}$$

If $\phi_{aa}^{F/O}$ and $\phi_{bb}^{F/O}$ are the second moments of F with respect to two orthogonal lines L_a and L_b which lie in the plane of F and intersect at point O, then the polar second moment of F with respect to O is given by

$$J^{F/O} = \phi_{aa}^{F/O} + \phi_{bb}^{F/O} \tag{2}$$

Proof (see Fig. 3.4.8 for notation):

$$\begin{aligned} J^{F/O} = \phi_{cc}^{F/O} &\underset{(3.4.4)}{=} \int_F (l_c)^2\, d\tau \\ &= \int_F (l_a^2 + l_b^2)\, d\tau = \int_F l_a^2\, d\tau + \int_F l_b^2\, d\tau \\ &\underset{(3.4.4.)}{=} \phi_{aa}^{F/O} + \phi_{bb}^{F/O} \end{aligned}$$

3.4.9 The Appendix contains sketches of ten curves, surfaces, and solids. For each figure the centroid and one or more centroidal principal axes (see 3.4.6) are shown, and the squares of the corresponding centroidal principal radii of gyration, as well as the length, area, or volume of the figure, are listed. Where less than three principal axes are shown, the orientation of those not shown can be found either by symmetry considerations (see 3.4.7 and 3.3.6) or by recalling that there exist at least three mutually perpendicular principal axes of every figure for every point in space (see 3.4.7 and 3.3.9). In the case of plane figures, Sec. 3.4.8 can be used to evaluate the third centroidal principal radius of gyration.

The information given in the Appendix, together with the theorems stated in the sections which follow, makes it possible, without integration, to find second moments (or radii of gyration) of a great variety of figures.

3.4.10 Given a point O, two directions $\mathbf{n}_a$ and $\mathbf{n}_b$, and a figure F composed of (dimensionally homogeneous) figures F_i, $i = 1, 2, \ldots, n$, the second moments $\boldsymbol{\Phi}_a^{F/O}$ and $\phi_{ab}^{F/O}$ can be expressed as

$$\boldsymbol{\Phi}_a^{F/O} = \sum_{i=1}^{n} \boldsymbol{\Phi}_a^{F_i/O} \tag{1}$$

$$\phi_{ab}^{F/O} = \sum_{i=1}^{n} \phi_{ab}^{F_i/O} \tag{2}$$

This is a consequence of the fact that $\Phi_a^{F/O}$ (see 3.3.1) is defined in terms of the second moment of a set of points, that this second moment is given by a sum (see 3.2.1), and that this sum obeys the associativity law for vector addition (see Vol. I, Sec. 1.9.2).

Problem: Figure 3.4.10a represents the cross section of a structural steel member.

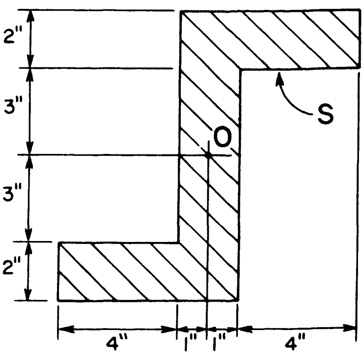


FIG. 3.4.10a

Determine the polar second moment $J^{S/O}$ (see 3.4.8) of this surface S with respect to point O.

Solution: Regard S as being composed of rectangles S_1, S_2, S_3 and let $\mathbf{n}_1$ and $\mathbf{n}_2$ be unit vectors as shown in Fig. 3.4.10b. Next, construct the table shown in Fig. 3.4.10c.

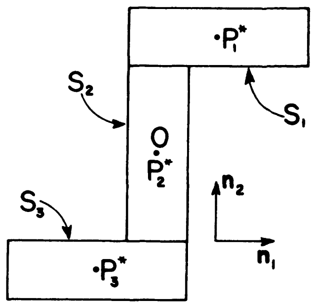


FIG. 3.4.10b

Quantity	Units	$i = 1$	$i = 2$	$i = 3$	Reference
A_i	in.^{2}	12	12	12	App. Fig. 3
$(k_1^{S_i/P_i^*})^2$	in.^{2}	$\frac{1}{3}$	3	$\frac{1}{3}$	App. Fig. 3
$(k_2^{S_i/P_i^*})^2$	in.^{2}	3	$\frac{1}{3}$	3	App. Fig. 3
$(l_1^{P_i^*/O})^2$	in.^{2}	16	0	16	Fig. 3.4.10
$(l_2^{P_i^*/O})^2$	in.^{2}	4	0	4	Fig. 3.4.10
$(k_1^{S_i/O})^2$	in.^{2}	$\frac{49}{3}$	3	$\frac{49}{3}$	3.4.7, 3.2.8
$(k_2^{S_i/O})^2$	in.^{2}	7	$\frac{1}{3}$	7	3.4.7, 3.2.8
$\phi_{11}^{S_i/O}$	in.^{4}	196	36	196	3.4.5
$\phi_{22}^{S_i/O}$	in.^{4}	84	4	84	3.4.5

FIG. 3.4.10c

Then

$$\phi_{11}^{S/O} \underset{(2)}{=} \sum_{i=1}^{3} \phi_{11}^{S_i/O} = 196 + 36 + 196 = 428 \text{ in.}^4$$

$$\phi_{22}^{S/O} \underset{(2)}{=} \sum_{i=1}^{3} \phi_{22}^{S_i/O} = 84 + 4 + 84 = 172 \text{ in.}^4$$

and

$$J^{S/O} \underset{(3.4.8)}{=} \phi_{11}^{S/O} + \phi_{22}^{S/O} = 600 \text{ in.}^4$$

3.4.11 If a figure F can be obtained by removing a figure F_1 from a figure F_2, then, for any point O and unit vectors $\mathbf{n}_a$ and $\mathbf{n}_b$,

$$\boldsymbol{\Phi}_a^{F/O} = \boldsymbol{\Phi}_a^{F_2/O} - \boldsymbol{\Phi}_a^{F_1/O} \tag{1}$$

and

$$\phi_{ab}^{F/O} = \phi_{ab}^{F_2/O} - \phi_{ab}^{F_1/O} \tag{2}$$

as the present situation is a special case of the class of situations described in Sec. 3.4.10.

Problem: Determine the radius of gyration of a right-cylindrical tube T of inner radius R_1, outer radius R_2, with respect to the axis of the tube.

Solution: Regard T as being obtained by removing a solid right-circular cylinder C_1 of radius R_1 from a solid right-circular cylinder C_2 of radius R_2, each of these cylinders being coaxial with T and having a height h. Let $\mathbf{n}_a$ be a unit vector parallel to the common axis, and P^* the common centroid of the three figures T, C_1, C_2. Then, from Fig. 9 of the Appendix,

$$(k_a^{C_1/P^*})^2 = \frac{R_1^2}{2}, \qquad (k_a^{C_2/P^*})^2 = \frac{R_2^2}{2}$$

and the volumes V_1 and V_2 of C_1 and C_2 are given by

$$V_1 = \pi R_1^2 h, \qquad V_2 = \pi R_2^2 h$$

so that (see 3.4.5)

$$\phi_{aa}^{C_1/P^*} = V_1(k_a^{C_1/P^*})^2 = \frac{\pi}{2} hR_1^4$$

$$\phi_{aa}^{C_2/P^*} = V_2(k_a^{C_2/P^*})^2 = \frac{\pi}{2} hR_2^4$$

and

$$\phi_{aa}^{T/P^*} \underset{(2)}{=} \phi_{aa}^{C_2/P^*} - \phi_{aa}^{C_1/P^*} = \frac{\pi}{2} h(R_2^4 - R_1^4)$$

Next, the volume V of T is given by

$$V = V_2 - V_1 = \pi h(R_2^2 - R_1^2)$$

and k_a^{T/P^*}, the desired radius of gyration, by

$$k_a^{T/P^*} \underset{(3.4.5)}{=} \left(\frac{\phi_{aa}^{T/P^*}}{V}\right)^{1/2} = \frac{1}{\sqrt{2}}\left(\frac{R_2^4 - R_1^4}{R_2^2 - R_1^2}\right)^{1/2} = \left(\frac{R_1^2 + R_2^2}{2}\right)^{1/2}$$

3.4.12 Radii of gyration of a surface can sometimes be obtained by regarding the surface as the limiting form of a solid.

Problem: Determine the radius of gyration of a right-circular cylindrical surface S of radius R with respect to the axis of the surface.

Solution: Regard S as the limiting form of the solid T considered in Problem 3.4.11, when $R_1 = R$ and R_2 approaches R:

$$k_a^{S/P^*} = \lim_{R_2 \to R} \left(\frac{R^2 + R_2^2}{2}\right)^{1/2} = R$$

3.5 Second moments of sets of particles and continuous bodies

3.5.1 Given a point O, a direction $\mathbf{n}_a$, and a set S of particles situated at points P_i, $i = 1, 2, \ldots, n$, and having masses m_i, the second moment of the set of points P_i of strengths $N_i = m_i$ with respect to O for the direction $\mathbf{n}_a$ (see 3.2.1) is called *the second moment of the set of particles with respect to O for the direction* $\mathbf{n}_a$. Similarly, second moments of S with respect to O for a pair of directions $\mathbf{n}_a$, $\mathbf{n}_b$, second moments of S with respect to lines, radii of gyration of S, principal directions of S, etc., are defined in terms of the corresponding properties of the set of points P_i of strengths $N_i = m_i$. Consequently, all of the material in Parts 3.2 and 3.3 is directly applicable to sets of particles.

The second moment of a set S of particles with respect to point O for the pair of directions $\mathbf{n}_a$, $\mathbf{n}_b$ is called, alternatively, the *product of inertia* of S with respect to O for the pair of directions $\mathbf{n}_a$, $\mathbf{n}_b$; and the second moment of S with respect to a line L_a is called *the moment of inertia of S about line* L_a.

3.5.2 Given a point O, a unit vector $\mathbf{n}_a$, and a continuous body C regarded as occupying a figure F, a vector $\boldsymbol{\Phi}_a^{C/O}$, called *the second moment of C with respect to O for the direction* $\mathbf{n}_a$, is obtained by performing the operations described in Sec. 3.4.1, after replacing (3) with "Assign to each such point a strength equal to the mass of the material occupying the corresponding element." As in the case of curves, surfaces, and solids (see 3.4.2), the required limiting process can be replaced with the evaluation of an integral:

$$\boldsymbol{\Phi}_a^{C/O} = \int_F \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p})\rho \, d\tau$$

where (see Fig. 3.5.2a) $\mathbf{p}$ is the position vector of a typical point

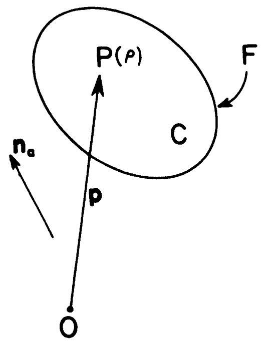


FIG. 3.5.2a

P of the figure F relative to point O, ρ is the mass density of the body C at point P (expressed in units of mass per unit of length, area, or volume, according as F is a curve, surface, or solid), and $d\tau$ is the length, area, or volume of a differential element of F.

Problem: In Fig. 3.5.2b, O represents one end of a thin straight wire W which may be regarded as occupying a straight line S of length L. The mass density ρ of the wire is given by

$$\rho = 10^{-3}\left(1 + \frac{8s}{3L}\right) \text{ slug ft}^{-1}$$

where s is the distance from O to a typical point P of S.

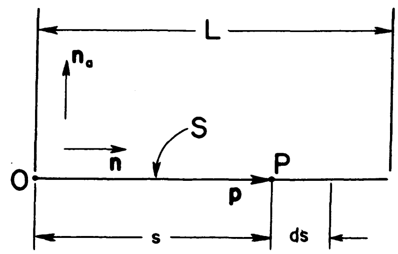


FIG. 3.5.2b

Determine the second moment of W with respect to O for a direction $\mathbf{n}_a$ perpendicular to S.

Solution: Let $\mathbf{n}$ be a unit vector parallel to S (see Fig. 3.5.2b), $\mathbf{p}$ the position vector of P relative to O, and $d\tau$ the length of a differential element of S. Then

$$\begin{aligned} \mathbf{p} &= s\mathbf{n} \\ \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p}) &= s^2\mathbf{n}_a \\ d\tau &= ds \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\Phi}_a^{W/O} &= \int_S \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p})\rho\, d\tau \\ &= 10^{-3}\mathbf{n}_a \int_0^L s^2\left(1 + \frac{8s}{3L}\right) ds \\ &= 10^{-3}L^3\mathbf{n}_a \text{ slug ft}^2 \ (L \text{ expressed in ft}) \end{aligned}$$

3.5.3 Given a point O, two unit vectors $\mathbf{n}_a$ and $\mathbf{n}_b$, and a continuous body C regarded as occupying a figure F, the scalar $\phi_{ab}^{C/O}$, defined as

$$\phi_{ab}^{C/O} = \boldsymbol{\Phi}_a^{C/O} \cdot \mathbf{n}_b \tag{1}$$

where $\boldsymbol{\Phi}_a^{C/O}$ is the second moment of C with respect to O for the direction $\mathbf{n}_a$ (see 3.5.2), is called either *the second moment* or *the product of inertia of C with respect to O for the pair of directions* $\mathbf{n}_a$, $\mathbf{n}_b$. It follows from Sec. 3.5.2 that $\phi_{ab}^{C/O}$ can be expressed as

$$\phi_{ab}^{C/O} = \int_F (\mathbf{n}_a \times \mathbf{p}) \cdot (\mathbf{n}_b \times \mathbf{p})\rho \, d\tau \tag{2}$$

3.5.4 When the unit vector $\mathbf{n}_b$ is equal to the unit vector $\mathbf{n}_a$, the expressions given in Sec. 3.5.3 become

$$\phi_{aa}^{C/O} = \boldsymbol{\Phi}_a^{C/O} \cdot \mathbf{n}_a \tag{1}$$

$$\phi_{aa}^{C/O} = \int_F (\mathbf{n}_a \times \mathbf{p})^2 \rho \, d\tau \tag{2}$$

and $\phi_{aa}^{C/O}$ is called either *the second moment of body C with respect to line* L_a, or *the moment of inertia of body C about line* L_a, L_a being the line which passes through O and is parallel to $\mathbf{n}_a$. As $(\mathbf{n}_a \times \mathbf{p})^2$ is equal to the square of the distance $l_a^{P/O}$ from L_a to a typical point P of F, $\phi_{aa}^{C/O}$ can be expressed as

$$\phi_{aa}^{C/O} = \int_F (l_a^{P/O})^2 \rho \, d\tau \tag{3}$$

Problem (a): Referring to Problem 3.5.2, determine the moment of inertia of W about a line which passes through O and is parallel to $\mathbf{n}_a$.

Solution:

$$\phi_{aa}^{W/O} \underset{(1)}{=} \boldsymbol{\Phi}_a^{W/O} \cdot \mathbf{n}_a \underset{(\text{P3.5.2})}{=} 10^{-3}L^3\, \mathbf{n}_a \cdot \mathbf{n}_a$$
$$= 10^{-3}L^3 \text{ slug ft}^2 \ (L \text{ expressed in ft})$$

Problem (b): Figure 3.5.4 shows the cross section of a steel shell S whose inner surface H is hemispherical and whose wall thickness t varies linearly with the radian measure ψ of the angle between line OP and the axis L_a of the shell. The shell has a mass of m slug.

Determine the moment of inertia of S about line L_a.

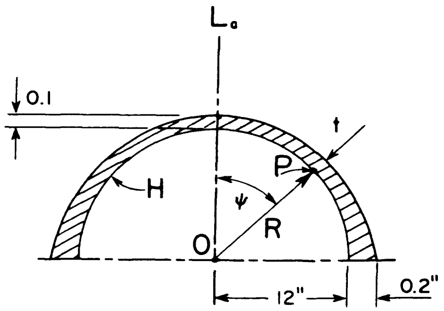


FIG. 3.5.4

Solution: Regard the shell S as matter distributed with variable density on the hemispherical surface H of radius $R = 12$ in. (This is an approximation which may be expected to give good results if the thickness of the shell is sufficiently small in comparison with the radius.)

As t varies linearly with ψ,

$$t = \alpha + \beta\psi$$

where α and β are constants which can be evaluated by noting that $t = 0.1$ for $\psi = 0$ and $t = 0.2$ for $\psi = \pi/2$:

$$t|_{\psi=0} = \alpha = 0.1 \text{ in.}$$

$$t|_{\psi=\pi/2} = \alpha + \beta\pi/2 = 0.2 \text{ in.}$$

$$\beta = \frac{2}{\pi}(0.2 - \alpha) = \frac{2}{\pi}(0.2 - 0.1) = \frac{0.2}{\pi} \text{ in.}$$

Hence

$$t = 0.1 + \frac{0.2}{\pi}\psi \text{ in.}$$

Assume that the steel of which the shell is made has the same properties at all points of the shell, and take ρ, the mass of the shell per unit of area of surface H at point P, proportional to the thickness t at P:

$$\rho = kt = k\left(0.1 + \frac{0.2}{\pi}\psi\right) \tag{A}$$

where k is a constant of proportionality, which can be expressed in terms of the mass m of the shell S (see Fig. 3.4.4a for θ):

$$m = \int_H \rho \, d\tau = \int_0^{2\pi} d\theta \int_0^{\pi/2} k\left(0.1 + \frac{0.2}{\pi}\psi\right) R^2 \sin\psi \, d\psi$$
$$= 2\pi k R^2[0.1 + (0.2/\pi)]$$
$$k = \frac{m}{2\pi R^2[0.1 + (0.2/\pi)]} \tag{B}$$

Let $l_a^{P/O}$ be the distance from line L_a to point P. Then

$$\phi_{aa}^{S/O} \underset{(3)}{=} \int_H (l_a^{P/O})^2 \rho \, d\tau \underset{(A)}{=} \int_0^{2\pi} d\theta \int_0^{\pi/2} kR^4 \left(0.1 + \frac{0.2}{\pi}\psi\right) \sin^3\psi \, d\psi$$
$$= \frac{0.4kR^4(3\pi + 7)}{9} \underset{(B)}{=} \frac{0.4}{18} \frac{3\pi + 7}{0.1\pi + 0.2} mR^2 = 0.71mR^2$$

and, with $R = 12$ in.,

$$\phi_{aa}^{S/O} = 102m \text{ slug in.}^2$$

3.5.5 The second moment $\phi_{aa}^{C/O}$ (see 3.5.4) can always be expressed as the product of the mass m of the body C and the square of a positive, real quantity $k_a^{C/O}$, called the *radius of gyration of C with respect to line L_a*.

In general, if F is the figure which C is regarded as occupying, $k_a^{C/O}$ is not equal to $k_a^{F/O}$ (see 3.4.5).

Problem: Determine the radius of gyration of (a) the wire W and (b) the straight line S described in Problem 3.5.2, with respect to a line passing through O and parallel to $\mathbf{n}_a$.

Solution (a):

$$k_a^{W/O} = \left(\frac{\phi_{aa}^{W/O}}{m}\right)^{1/2}$$
$$\phi_{aa}^{W/O} \underset{(P3.5.4a)}{=} 10^{-3}L^3 \text{ slug ft}^2$$
$$m = \int_S \rho \, d\tau = 10^{-3} \int_0^L \left(1 + \frac{8s}{3L}\right) ds = \frac{7}{3} 10^{-3}L \text{ slug}$$
$$k_a^{W/O} = (\tfrac{3}{7})^{1/2}L$$

Solution (b):

$$k_a^{S/O} \underset{(3.4.5)}{=} \left(\frac{\phi_{aa}^{S/O}}{L}\right)^{1/2} \underset{(3.4.4)}{=} \left(\frac{\mathbf{\Phi}_a^{S/O} \cdot \mathbf{n}_a}{L}\right)^{1/2}$$
$$\underset{(P3.4.2)}{=} \left(\frac{L^2}{3}\right)^{1/2} = \left(\frac{1}{3}\right)^{1/2} L$$

3.5.6 Given a continuous body C, a point O, and a unit vector $\mathbf{n}_z$, $\mathbf{n}_z$ is called a *principal direction* of C for O if and only if $\boldsymbol{\Phi}_z^{C/O}$ (see 3.5.2) is parallel to $\mathbf{n}_z$ or equal to zero. When $\mathbf{n}_z$ is a principal direction of C for O, the line L_z parallel to $\mathbf{n}_z$ and passing through O is called a *principal axis* of C for O; the plane P_z passing through O and perpendicular to $\mathbf{n}_z$ is called a *principal plane* of C for O; the second moment $\phi_{zz}^{C/O}$ is called a *principal second moment* or *principal moment of inertia* of C for O; and the radius of gyration $k_z^{C/O}$ is called a *principal radius of gyration* of C for O.

In general, if F is the figure which C is regarded as occupying, principal directions of C for O are not principal directions of F for O, and vice versa.

3.5.7 As a consequence of the definitions given in Secs. 3.5.2–3.5.6, many of the relationships discussed in Parts 3.2, 3.3, and 3.4 have counterparts in the theory of second moments of continuous bodies. After minor changes in wording and notation these relationships can, therefore, be used for the solution of problems involving continuous bodies.

3.5.8 If the mass density ρ of a continuous body C is the same at all points of the figure F which C is regarded as occupying, both

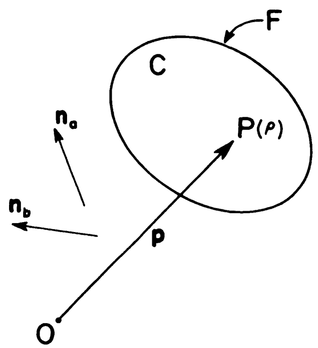


FIG. 3.5.8

the density ρ and the body C are said to be *uniform*. Second moments and radii of gyration of a uniform body C can be expressed in terms of ρ (or the mass m of C and the length, area, or volume

τ of F) and corresponding second moments and radii of gyration of F:

$$\boldsymbol{\Phi}_a^{C/O} = \rho \boldsymbol{\Phi}_a^{F/O} = \frac{m}{\tau} \boldsymbol{\Phi}_a^{F/O} \tag{1}$$

$$\phi_{ab}^{C/O} = \rho \phi_{ab}^{F/O} = \frac{m}{\tau} \phi_{ab}^{F/O} \tag{2}$$

$$k_a^{C/O} = k_a^{F/O} \tag{3}$$

Proof (see Fig. 3.5.8 for notation):

$$\boldsymbol{\Phi}_a^{C/O} \underset{(3.5.2)}{=} \int_F \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p}) \rho \, d\tau = \rho \int_F \mathbf{p} \times (\mathbf{n}_a \times \mathbf{p}) \, d\tau \underset{(3.4.2)}{=} \rho \boldsymbol{\Phi}_a^{F/O}$$

$$\phi_{ab}^{C/O} \underset{(3.5.3)}{=} \boldsymbol{\Phi}_a^{C/O} \cdot \mathbf{n}_b \underset{(1)}{=} \rho \boldsymbol{\Phi}_a^{F/O} \cdot \mathbf{n}_b \underset{(3.4.3)}{=} \rho \phi_{ab}^{F/O}$$

$$k_a^{C/O} \underset{(3.5.5)}{=} \left(\frac{\phi_{aa}^{C/O}}{m} \right)^{1/2} \underset{(2)}{=} \left(\frac{\phi_{aa}^{F/O}}{\tau} \right)^{1/2} \underset{(3.4.5)}{=} k_a^{F/O}$$

Problem: A thin, hemispherical steel shell S has a mass of m slug and a mean radius $R = 12.15$ in. Determine the moment of inertia of S about the axis of S.

Solution: Regard S as matter distributed with uniform density on a hemispherical surface H of radius R. Let $\mathbf{n}_a$ be a unit vector parallel to the axis of S (or H), and O a point on this axis. Then

$$\phi_{aa}^{S/O} \underset{(2)}{=} \frac{m}{A} \phi_{aa}^{H/O}$$

where $A = 2\pi R^2$ is the area of H, and

$$\phi_{aa}^{H/O} \underset{(3.4.11, \mathrm{P}3.4.4)}{=} \tfrac{1}{2}(\tfrac{8}{3}\pi R^4)$$

Hence

$$\phi_{aa}^{S/O} = \tfrac{2}{3} m R^2$$

and, with $R = 12.15$ in.,

$$\phi_{aa}^{S/O} = 98.4m \text{ slug in.}^2$$

3.5.9 Principal directions, axes, and planes of a uniform continuous body C (see 3.5.8) coincide with those of the figure F which C is regarded as occupying.

Proof: It is only necessary to show that a principal direction $\mathbf{n}_z$ of C for a point O is a principal direction of F for O; that is (see 3.4.6 and 3.5.6), that $\boldsymbol{\Phi}_z^{F/O}$ is parallel to $\mathbf{n}_z$ or equal to zero whenever $\boldsymbol{\Phi}_z^{C/O}$ is parallel to $\mathbf{n}_z$ or equal to zero. This follows immediately from Eq. (1), Sec. 3.5.8.

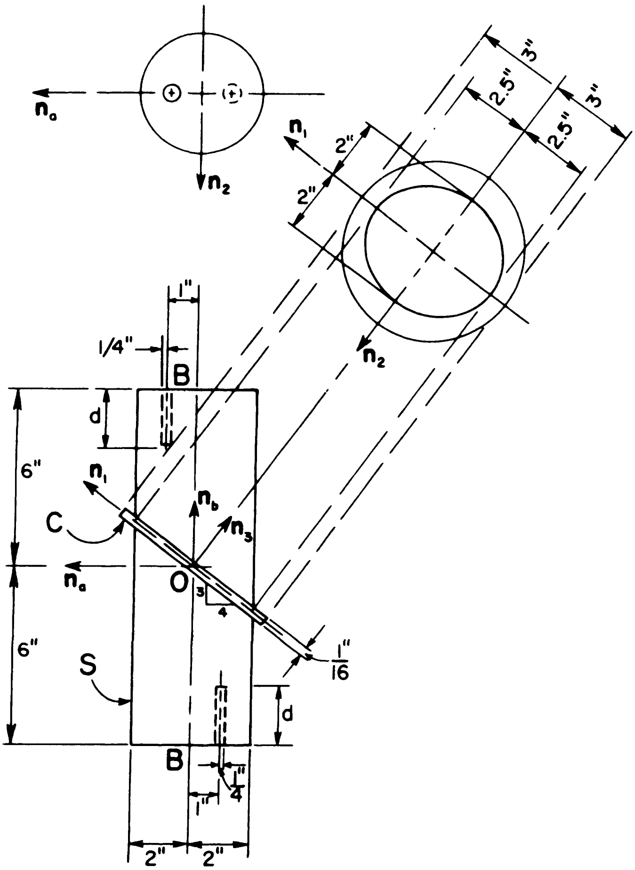


FIG. 3.5.9

Problem: Figure 3.5.9 represents a uniform shaft S which carries a uniform circular disc C, the shaft passing through an elliptical opening in the disc. This assembly A is said to be dynamically balanced for rotation about axis B-B if this axis is a principal axis of the assembly for the mass center O.

Show that balance may be achieved by drilling two $\frac{1}{2}$ in. diameter holes in the shaft, as indicated in Fig. 3.5.9, and determine the (common) depth "d" of the holes, assuming that the ratio of the mass densities of the materials of which C and S are made is equal to 1.2.

Solution: Introduce the unit vectors $\mathbf{n}_a$, $\mathbf{n}_b$, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ shown in Fig. 3.5.9.

The plane passing through point O and perpendicular to $\mathbf{n}_2$ is a principal plane of the assembly A for point O (see 3.5.7 and 3.3.6). Hence $\mathbf{n}_2$ is a principal direction of A for O, and $\phi_{2b}^{A/O}$ is equal to zero (see 3.5.7 and 3.3.3). If $\phi_{ab}^{A/O}$ is also equal to zero, $\mathbf{n}_b$ is a principal direction of A for O (see 3.5.7 and 3.3.4) and axis B-B is a principal axis of A for O (see 3.5.6). Thus

$$\phi_{ab}^{A/O} = 0 \tag{A}$$

guarantees dynamic balance.

Let H_1 and H_2 be the cylindrical bodies removed from S when the holes are drilled. Then

$$\phi_{ab}^{A/O} \underset{(3.5.7,3.4.10,3.4.11)}{=} \phi_{ab}^{S/O} + \phi_{ab}^{C/O} - \phi_{ab}^{H_1/O} - \phi_{ab}^{H_2/O} \tag{B}$$

Both $\mathbf{n}_a$ and $\mathbf{n}_b$ are principal directions of S for O (see 3.5.7 and 3.3.6). Hence

$$\phi_{ab}^{S/O} \underset{(3.5.7,3.3.4)}{=} 0 \tag{C}$$

Regard C as occupying a surface F obtained by removing from a circular surface F_1 an elliptical surface F_2. The principal directions of F for O are $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ (see 3.4.7, 3.3.7 and 3.3.6). Hence

$$\phi_{ab}^{F/O} \underset{(3.4.7,3.3.10)}{=} a_1b_1\phi_{11}^{F/O} + a_2b_2\phi_{22}^{F/O} + a_3b_3\phi_{33}^{F/O}$$

where

$$a_1 = \tfrac{4}{5}, \qquad a_2 = 0, \qquad a_3 = -\tfrac{3}{5}$$

$$b_1 = \tfrac{3}{5}, \qquad b_2 = 0, \qquad b_3 = \tfrac{4}{5}$$

are the $\mathbf{n}_i$, $i = 1, 2, 3$, measure numbers of $\mathbf{n}_a$ and $\mathbf{n}_b$. Thus

$$\phi_{ab}^{F/O} = \tfrac{12}{25}(\phi_{11}^{F/O} - \phi_{33}^{F/O})$$

But

$$\phi_{33}^{F/O} \underset{(3.4.8)}{=} \phi_{11}^{F/O} + \phi_{22}^{F/O}$$

Hence

$$\phi_{ab}^{F/O} = -\tfrac{12}{25}\phi_{22}^{F/O}$$

or, expressing $\phi_{22}^{F/O}$ in terms of second moments of the circular surface F_1 and the elliptical surface F_2,

$$\phi_{ab}^{F/O} \underset{(3.4.11)}{=} -\tfrac{12}{25}(\phi_{22}^{F_1/O} - \phi_{22}^{F_2/O})$$

$$\underset{(3.4.5)}{=} -\frac{12}{25}\left[\underset{(\text{App.F5})}{(\pi \times 3^2)\frac{3^2}{4}} - \underset{(\text{App.F6})}{2(\pi \times 2.5 \times 2/2)\frac{2.5^2}{4}}\right]$$

That is

$$\phi_{ab}^{F/O} = -5.97\pi \text{ in.}^4$$

and, letting ρ^C be the mass of C per unit of area of F,

$$\phi_{ab}^{C/O} \underset{(3.5.8)}{=} \rho^C \phi_{ab}^{F/O} = -5.97\pi\rho^C$$

Now, as C is 1/16 in. thick, and the mass density of the material of which C is made is 1.2 times as great as the mass density ρ^S of the shaft material,

$$\rho^C = \frac{1.2\rho^S}{16}$$

Thus

$$\phi_{ab}^{C/O} = -\frac{7.164}{16}\pi\rho^S \tag{D}$$

Let P_1^* be the mass center of the cylindrical body H_1. Then

$$\phi_{ab}^{H_1/O} \underset{(3.5.7,3.2.6)}{=} \phi_{ab}^{H_1/P_1^*} + \phi_{ab}^{P_1^*/O}$$

$$= \underset{(3.5.7,3.3.6,3.3.4)}{0} + \underset{(3.1.5)}{m_1(\mathbf{n}_a \times \mathbf{p}_1) \cdot (\mathbf{n}_b \times \mathbf{p}_1)}$$

where m_1 is the mass of H_1, and $\mathbf{p}_1$ is the position vector of P_1^* relative to O; that is,

$$m_1 \underset{(\text{App.F9})}{=} \pi(1/4)^2\, d\rho^S$$

and

$$\mathbf{p}_1 = \mathbf{n}_a + \left(6 - \frac{d}{2}\right)\mathbf{n}_b$$

Thus

$$\phi_{ab}^{H_1/O} = -\frac{\pi}{16}\, d\left(6 - \frac{d}{2}\right)\rho^S \tag{E}$$

Similarly

$$\phi_{ab}^{H_2/O} = -\frac{\pi}{16}\, d\left(6 - \frac{d}{2}\right)\rho^S \tag{F}$$

Substitute from Eqs. (C), (D), (E), and (F) into Eq. (B), then use Eq. (A):

$$d^2 - 12d + 7.164 = 0$$

Solve for d:

$$d = 0.63 \text{ in. or } 11.37 \text{ in.}$$

4 LAWS OF MOTION

4.1 Inertia forces and force systems

4.1.1 If ${}^R\mathbf{a}^P$ is the absolute acceleration of a particle P in a reference frame R (see 2.5.1), and m the mass of P, the bound vector ${}^R\mathbf{F}^P$ applied at P and given by

$${}^R\mathbf{F}^P = -m\,{}^R\mathbf{a}^P$$

is called the *inertia force acting on P in R.*

Problem: Referring to Problem 2.5.13, and regarding the airplane P as a particle of mass $m = 3000$ slug, determine the moment about the earth's north-south axis of the inertia force acting on P in (a) R and (b) R'.

Solution (a): As the acceleration of P in R is parallel to $\mathbf{n}_3$ (see Fig. 2.5.13b), the line of action of the inertia force ${}^R\mathbf{F}^P$ intersects the earth's axis. Hence the moment of this force about this line is equal to zero.

Solution (b):

$${}^{R'}\mathbf{F}^P = -m\,{}^{R'}\mathbf{a}^P \underset{(P2.5.13)}{=} 3000(136\mathbf{n}_1 - 222\mathbf{n}_2 + 227\mathbf{n}_3) \text{ slug mile hr}^{-2}$$

The distance s from line NS to the point of application of ${}^{R'}\mathbf{F}^P$ is given by

$$s = 3960 \sin 45° = 2800 \text{ miles}$$

The moment of ${}^{R'}\mathbf{F}^P$ about line NS thus has the value

$$\begin{aligned} sm(-222)\mathbf{k} &= -2800(3000)(222)\mathbf{k} \text{ slug mile}^2\text{ hr}^{-2} \\ &= -\frac{2800(3000)(222)(5280)^2}{(3600)^2}\mathbf{k} \text{ slug ft}^2\text{ sec}^{-2} \\ &= -40{,}000\,\mathbf{k} \text{ ft lb} \end{aligned}$$

4.1.2 Given a continuous body C regarded as occupying a figure F, and a reference frame R, divide F into n elements of arbitrary size and shape; select a point of C in each element, and, regarding this point as a particle whose mass is equal to that of the material in the element, determine the inertia force (see 4.1.1) acting on this particle in R. The system of n forces thus obtained can be replaced with (see Vol. I, Sec. 3.5) a force applied at the mass center P^* of C, together with a couple. When n tends to infinity and each of the elements shrinks to a point, this force and couple approach limits called *the inertia force acting on C in R* and *the inertia couple acting on C in R*.

The inertia force ${}^R\mathbf{F}^C$ (applied at the mass center P^* of C) and the torque ${}^R\mathbf{T}^C$ of the inertia couple acting on C in R are given by

$$ {}^R\mathbf{F}^C = -\int_F {}^R\mathbf{a}^P \rho \, d\tau \tag{1}$$

and

$$ {}^R\mathbf{T}^C = -\int_F \mathbf{r} \times {}^R\mathbf{a}^P \rho \, d\tau \tag{2}$$

where ${}^R\mathbf{a}^P$ is the acceleration in R of a generic point P of C (see Fig. 4.1.2a), ρ is the mass density of C at P, $\mathbf{r}$ is the position vector of P relative to P^*, and $d\tau$ is the length, area, or volume of a differential element of F.

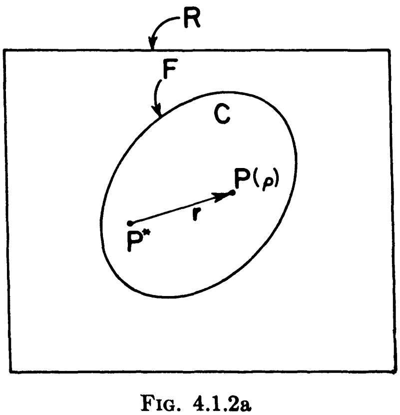


FIG. 4.1.2a

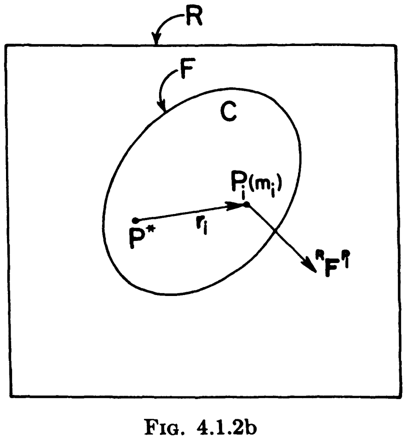


FIG. 4.1.2b

Proof: Let P_i, $i = 1, 2, \ldots, n$, be the points selected in the n elements into which F is divided (see Fig. 4.1.2b). The mass

m_i of a typical element can be expressed as the product of a quantity ρ_i and the length, area, or volume τ_i of the element:

$$m_i = \rho_i\tau_i$$

Hence, letting ${}^R\mathbf{a}^{P_i}$ be the acceleration of P_i in R, the inertia force ${}^R\mathbf{F}^{P_i}$ acting on a particle situated at P_i and having the mass m_i is given by

$${}^R\mathbf{F}^{P_i} \underset{(4.1.1)}{=} -m_i\,{}^R\mathbf{a}^{P_i} = -{}^R\mathbf{a}^{P_i}\rho_i\tau_i$$

and is applied at point P_i.

When the system of forces ${}^R\mathbf{F}^{P_i}$, $i = 1, 2, \ldots, n$, is replaced with a force applied at P^*, together with a couple, the force and the torque of the couple are given by (see Vol. I, Sec. 3.5.6)

$$\sum_{i=1}^{n} {}^R\mathbf{F}^{P_i} = -\sum_{i=1}^{n} {}^R\mathbf{a}^{P_i}\rho_i\tau_i$$

and

$$\sum_{i=1}^{n} \mathbf{r}_i \times {}^R\mathbf{F}^{P_i} = -\sum_{i=1}^{n} \mathbf{r}_i \times {}^R\mathbf{a}^{P_i}\rho_i\tau_i$$

where $\mathbf{r}_i$ is the position vector of P_i relative to P^*. Consequently, if $L[Q]$ denotes the limit approached by a quantity Q as n tends to infinity and the elements of F shrink to points (see Vol. I, Sec. 2.5.1),

$${}^R\mathbf{F}^C = L\left[-\sum_{i=1}^{n} {}^R\mathbf{a}^{P_i}\rho_i\tau_i\right] = -\int_F {}^R\mathbf{a}^P\rho\,d\tau$$

and

$${}^R\mathbf{T}^C = L\left[-\sum_{i=1}^{n} \mathbf{r}_i \times {}^R\mathbf{a}^{P_i}\rho_i\tau_i\right] = -\int_F \mathbf{r} \times {}^R\mathbf{a}^P\rho\,d\tau$$

where ${}^R\mathbf{a}^P$ is the acceleration in R of a generic point P of C (see Fig. 3.4.2a), ρ is the mass density of C at P, $\mathbf{r}$ is the position vector of P relative to P^*, and $d\tau$ is the length, area, or volume of a differential element of F.

4.1.3 The inertia force ${}^{R'}\mathbf{F}^R$ and the torque ${}^{R'}\mathbf{T}^R$ of the inertia couple acting on a *rigid body* R in a reference frame R' (see 4.1.2) are given by

$${}^{R'}\mathbf{F}^R = -m\,{}^{R'}\mathbf{a}^{P^*} \tag{1}$$

and

$${}^{R'}\mathbf{T}^R = -(\mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R\,\boldsymbol{\Phi}_a^{R/P^*} + \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R\,{}^{R'}\boldsymbol{\omega}^R \times \boldsymbol{\Phi}_o^{R/P^*}) \tag{2}$$

where m is the mass of R, ${}^{R'}\mathbf{a}^{P^*}$ is the acceleration of the mass center P^* of R in R', $\mathbf{n}_a$ is a unit vector parallel to ${}^{R'}\boldsymbol{\alpha}^R$, $\mathbf{n}_o$ is a unit vector parallel to ${}^{R'}\boldsymbol{\omega}^R$, and $\boldsymbol{\Phi}_a^{R/P^*}$ and $\boldsymbol{\Phi}_o^{R/P^*}$ are the second moments of R with respect to P^* for the directions $\mathbf{n}_a$ and $\mathbf{n}_o$, respectively.

Proof: From Sec. 4.1.2,

$$ {}^{R'}\mathbf{F}^R = -\int_F {}^{R'}\mathbf{a}^P \rho \, d\tau \tag{A}$$

and

$$ {}^{R'}\mathbf{T}^R = -\int_F \mathbf{r} \times {}^{R'}\mathbf{a}^P \rho \, d\tau \tag{B}$$

where, as P is now a point of a rigid body, ${}^{R'}\mathbf{a}^P$ can be expressed as

$$ {}^{R'}\mathbf{a}^P \underset{(2.5.9)}{=} {}^{R'}\mathbf{a}^{P^*} + {}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}) \tag{C}$$

so that

$$ {}^{R'}\mathbf{F}^R \underset{(A,C)}{=} -\left[{}^{R'}\mathbf{a}^{P^*} \int_F \rho \, d\tau + {}^{R'}\boldsymbol{\alpha}^R \times \int_F \mathbf{r}\rho \, d\tau + {}^{R'}\boldsymbol{\omega}^R \times \left({}^{R'}\boldsymbol{\omega}^R \times \int_F \mathbf{r}\rho \, d\tau\right)\right] \tag{D}$$

and

$$ {}^{R'}\mathbf{T}^R \underset{(B,C)}{=} -\left[\left(\int_F \mathbf{r}\rho \, d\tau\right) \times {}^{R'}\mathbf{a}^{P^*} + \int_F \mathbf{r} \times ({}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r})\rho \, d\tau + \int_F \mathbf{r} \times [{}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r})]\rho \, d\tau\right] \tag{E}$$

The integral

$$ \int_F \rho \, d\tau $$

is equal to the mass m of R, and

$$ \int_F \mathbf{r}\rho \, d\tau = 0 $$

as a consequence of the definition of the mass center or a continuous body (see Vol. I, Sec. 2.7). Hence

$$ {}^{R'}\mathbf{F}^R \underset{(D)}{=} -m \, {}^{R'}\mathbf{a}^{P^*} $$

and

$$ {}^{R'}\mathbf{T}^R \underset{(E)}{=} -\left\{\int_F \mathbf{r} \times ({}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r})\rho \, d\tau + \int_F \mathbf{r} \times [{}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r})]\rho \, d\tau\right\} \tag{F}$$

If $\mathbf{n}_a$ is a unit vector parallel to ${}^{R'}\boldsymbol{\alpha}^R$, and $\mathbf{n}_o$ is a unit vector parallel to ${}^{R'}\boldsymbol{\omega}^R$, then the following are identities:

$$ {}^{R'}\boldsymbol{\alpha}^R = \mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R \, \mathbf{n}_a $$

$$ {}^{R'}\boldsymbol{\omega}^R = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \, \mathbf{n}_o $$

Hence

$$ \mathbf{r} \times ({}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r}) = \mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R \, \mathbf{r} \times (\mathbf{n}_a \times \mathbf{r}) \tag{G} $$

and

$$ \mathbf{r} \times [{}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r})] = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \, \mathbf{r} \times [{}^{R'}\boldsymbol{\omega}^R \times (\mathbf{n}_o \times \mathbf{r})] $$

which, after using the identity

$$ \mathbf{r} \times [{}^{R'}\boldsymbol{\omega}^R \times (\mathbf{n}_o \times \mathbf{r})] = {}^{R'}\boldsymbol{\omega}^R \times [\mathbf{r} \times (\mathbf{n}_o \times \mathbf{r})] $$

becomes

$$ \mathbf{r} \times [{}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r})] = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \, {}^{R'}\boldsymbol{\omega}^R \times [\mathbf{r} \times (\mathbf{n}_o \times \mathbf{r})] \tag{H} $$

Thus

$$ \begin{aligned} {}^{R'}\mathbf{T}^R \underset{(\text{F,G,H})}{=} & -\Big[\mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R \int_F \mathbf{r} \times (\mathbf{n}_a \times \mathbf{r})\rho \, d\tau \\ & \qquad + \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \, {}^{R'}\boldsymbol{\omega}^R \times \int_F \mathbf{r} \times (\mathbf{n}_o \times \mathbf{r})\rho \, d\tau\Big] \\ \underset{(3.5.2)}{=} & -[\mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R \, \boldsymbol{\Phi}_a^{R/P^*} + \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \, {}^{R'}\boldsymbol{\omega}^R \times \boldsymbol{\Phi}_o^{R/P^*}] \end{aligned} $$

Problem: A uniform solid sphere S of mass m and radius r has an angular acceleration ${}^R\boldsymbol{\alpha}^S$ in a reference frame R.

Determine the torque ${}^R\mathbf{T}^S$ of the inertia couple acting on S in R.

Solution: It follows from symmetry considerations (see 3.5.7 and 3.3.6) that every direction is a principal direction of S for the mass center P^* of S. Hence, in particular, unit vectors $\mathbf{n}_a$ and $\mathbf{n}_o$, parallel to ${}^R\boldsymbol{\alpha}^S$ and ${}^R\boldsymbol{\omega}^S$ respectively, are principal directions of S for P^*, and

$$ \boldsymbol{\Phi}_a^{S/P^*} \underset{(3.5.7,3.3.2)}{=} \phi_{aa}^{S/P^*} \mathbf{n}_a $$

$$ \boldsymbol{\Phi}_o^{S/P^*} \underset{(3.5.7,3.3.2)}{=} \phi_{oo}^{S/P^*} \mathbf{n}_o $$

Thus

$$ \mathbf{n}_a \cdot {}^R\boldsymbol{\alpha}^S \, \boldsymbol{\Phi}_a^{S/P^*} = \mathbf{n}_a \cdot {}^R\boldsymbol{\alpha}^S \, \mathbf{n}_a \, \phi_{aa}^{S/P^*} = {}^R\boldsymbol{\alpha}^S \, \phi_{aa}^{S/P^*} $$

and

$$ {}^R\boldsymbol{\omega}^S \times \boldsymbol{\Phi}_o^{S/P^*} = 0 $$

Furthermore

$$\phi_{aa}^{S/P^*} \underset{(3.5.5)}{=} m(k_a^{S/P^*})^2 \underset{(3.5.8,\text{App.F8})}{=} \tfrac{2}{5}mr^2$$

Hence

$$^R\mathbf{T}^S \underset{(2)}{=} -\tfrac{2}{5}mr^2\ ^R\boldsymbol{\alpha}^S$$

4.1.4 If $\mathbf{n}_i$, $i = 1, 2, 3$, are *any mutually perpendicular* unit vectors, and $^{R'}\alpha_i{}^R$ and $^{R'}\omega_i{}^R$ are the $\mathbf{n}_i$ measure numbers of the angular acceleration $^{R'}\boldsymbol{\alpha}^R$ and angular velocity $^{R'}\boldsymbol{\omega}^R$ of a rigid body R in a reference frame R', the torque $^{R'}\mathbf{T}^R$ of the inertia couple acting on R in R' (see 4.1.3) is given by

$$^{R'}\mathbf{T}^R = -\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*}(^{R'}\alpha_i{}^R\,\mathbf{n}_j + {}^{R'}\omega_i{}^R\,{}^{R'}\boldsymbol{\omega}^R \times \mathbf{n}_j)$$

Proof: If $\mathbf{n}_a$ is a unit vector parallel to $^{R'}\boldsymbol{\alpha}^R$, $\mathbf{n}_o$ is a unit vector parallel to $^{R'}\boldsymbol{\omega}^R$, and a_i and o_i are the $\mathbf{n}_i$ measure numbers of $\mathbf{n}_a$ and $\mathbf{n}_o$, then the identities

$$^{R'}\boldsymbol{\alpha}^R = \mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R\,\mathbf{n}_a, \qquad {}^{R'}\boldsymbol{\omega}^R = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R\,\mathbf{n}_o$$

lead immediately to

$$a_i = \frac{^{R'}\alpha_i{}^R}{\mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R}, \qquad o_i = \frac{^{R'}\omega_i{}^R}{\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R}$$

so that

$$\boldsymbol{\Phi}_a^{R/P^*} \underset{(3.5.7,3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*} a_i\mathbf{n}_j = \frac{1}{\mathbf{n}_a \cdot {}^{R'}\boldsymbol{\alpha}^R}\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*}\,{}^{R'}\alpha_i{}^R\,\mathbf{n}_j$$

and

$$\boldsymbol{\Phi}_o^{R/P^*} \underset{(3.5.7,3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*} o_i\mathbf{n}_j = \frac{1}{\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R}\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*}\,{}^{R'}\omega_i{}^R\,\mathbf{n}_j$$

Thus,

$$\begin{aligned} ^{R'}\mathbf{T}^R &\underset{(4.1.3)}{=} -\left(\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*}\,{}^{R'}\alpha_i{}^R\,\mathbf{n}_j + {}^{R'}\boldsymbol{\omega}^R \times \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*}\,{}^{R'}\omega_i{}^R\,\mathbf{n}_j\right) \\ &= -\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*}(^{R'}\alpha_i{}^R\,\mathbf{n}_j + {}^{R'}\omega_i{}^R\,{}^{R'}\boldsymbol{\omega}^R \times \mathbf{n}_j) \end{aligned}$$

(Note that it was not assumed that $\mathbf{n}_i$, $i = 1, 2, 3$, are fixed in either R or R'. Accordingly, the word "any" is to be taken quite literally.)

4.1.5 If $\mathbf{n}_i$, $i = 1, 2, 3$, is a *right-handed set of mutually perpendicular principal directions* of a rigid body R *for the mass center* P^* of R (see 3.5.6), and ${}^{R'}\alpha_i{}^R$ and ${}^{R'}\omega_i{}^R$ are the $\mathbf{n}_i$ measure numbers of the angular acceleration ${}^{R'}\boldsymbol{\alpha}^R$ and angular velocity ${}^{R'}\boldsymbol{\omega}^R$ of R in a reference frame R', the torque ${}^{R'}\mathbf{T}^R$ of the inertia couple acting on R in R' (see 4.1.3) is given by

$$\begin{aligned} {}^{R'}\mathbf{T}^R = &-[\phi_{11}^{R/P^*}\,{}^{R'}\alpha_1{}^R - (\phi_{22}^{R/P^*} - \phi_{33}^{R/P^*})\,{}^{R'}\omega_2{}^R\,{}^{R'}\omega_3{}^R]\mathbf{n}_1 \\ &- [\phi_{22}^{R/P^*}\,{}^{R'}\alpha_2{}^R - (\phi_{33}^{R/P^*} - \phi_{11}^{R/P^*})\,{}^{R'}\omega_3{}^R\,{}^{R'}\omega_1{}^R]\mathbf{n}_2 \\ &- [\phi_{33}^{R/P^*}\,{}^{R'}\alpha_3{}^R - (\phi_{11}^{R/P^*} - \phi_{22}^{R/P^*})\,{}^{R'}\omega_1{}^R\,{}^{R'}\omega_2{}^R]\mathbf{n}_3 \end{aligned}$$

Proof: For $i \neq j$, ϕ_{ij}^{R/P^*} is equal to zero (see 3.5.7 and 3.3.3). Hence the expression given in Sec. 4.1.4 reduces to

$${}^{R'}\mathbf{T}^R = -\sum_{i=1}^{3} \phi_{ii}^{R/P^*}({}^{R'}\alpha_i{}^R\,\mathbf{n}_i + {}^{R'}\omega_i{}^R\,{}^{R'}\boldsymbol{\omega}^R \times \mathbf{n}_i) \tag{A}$$

Now, if $\mathbf{n}_i$, $i = 1, 2, 3$, is a right-handed set of unit vectors,

$$\begin{aligned} {}^{R'}\boldsymbol{\omega}^R \times \mathbf{n}_1 &= {}^{R'}\omega_3{}^R\,\mathbf{n}_2 - {}^{R'}\omega_2{}^R\,\mathbf{n}_3 \\ {}^{R'}\boldsymbol{\omega}^R \times \mathbf{n}_2 &= {}^{R'}\omega_1{}^R\,\mathbf{n}_3 - {}^{R'}\omega_3{}^R\,\mathbf{n}_1 \\ {}^{R'}\boldsymbol{\omega}^R \times \mathbf{n}_3 &= {}^{R'}\omega_2{}^R\,\mathbf{n}_1 - {}^{R'}\omega_1{}^R\,\mathbf{n}_2 \end{aligned}$$

Expand the right-hand member of Eq. (A) and substitute.

(Note that it was not assumed that $\mathbf{n}_i$, $i = 1, 2, 3$, are fixed in either R or R'. In general, there exist only three mutually perpendicular principal directions of R for P^*, and these are fixed in R. However, certain rigid bodies possess infinitely many principal directions for their mass center. It is in the study of the motions of such bodies that the expression for ${}^{R'}\mathbf{T}^R$ given above is most useful.)

Problem: Referring to Example 2.2.10, and supposing that the disc D is uniform (see 3.5.8), has a mass m and radius r, and is rolling on plane P (see 2.5.10), determine the inertia force ${}^P\mathbf{F}^D$ and the torque ${}^P\mathbf{T}^D$ of the inertia couple acting on D in P.

Solution (see Fig. 2.2.10a for notation):

$$\begin{aligned} {}^P\mathbf{F}^D \underset{(4.1.3)}{=} -m\,{}^P\mathbf{a}^{D^*} \underset{(\text{P}2.5.10)}{=} &-mr[\dot\theta^2 + \dot\phi \sin\theta(\dot\psi + \dot\phi \sin\theta)]\mathbf{n}_1 \\ &+ mr[\ddot\psi + \ddot\phi \sin\theta + 2\dot\phi\dot\theta \cos\theta]\mathbf{n}_2 \\ &+ mr[\ddot\theta - \dot\phi \cos\theta(\dot\psi + \dot\phi \sin\theta)]\mathbf{n}_3 \end{aligned}$$

Next (see Example 2.2.10)

$$^P\omega_1{}^D = -\dot\phi\cos\theta, \qquad ^P\omega_2{}^D = -\dot\theta, \qquad ^P\omega_3{}^D = \dot\psi + \dot\phi\sin\theta$$

and (see Example 2.3.2)

$$\begin{aligned} ^P\alpha_1{}^D &= -\ddot\phi\cos\theta + \dot\phi\dot\theta\sin\theta - \dot\theta\dot\psi \\ ^P\alpha_2{}^D &= -\ddot\theta + \dot\phi\dot\psi\cos\theta \\ ^P\alpha_3{}^D &= \ddot\psi + \ddot\phi\sin\theta + \dot\phi\dot\theta\cos\theta \end{aligned}$$

Furthermore

$$\phi_{11}^{D/D^*} = \phi_{22}^{D/D^*} \underset{(3.5.5)}{=} m(k_1^{D/D^*})^2 \underset{(3.5.8,\text{App.F5})}{=} \frac{mr^2}{4}$$

and

$$\phi_{33}^{D/D^*} \underset{(3.4.8)}{=} \phi_{11}^{D/D^*} + \phi_{22}^{D/D^*} = 2\phi_{11}^{D/D^*} = \frac{mr^2}{2}$$

Hence, as $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are principal directions of D for D^* (although they are not fixed in D),

$$\begin{aligned} ^P\mathbf{T}^D = \frac{mr^2}{4}[(\ddot\phi\cos\theta + 2\dot\theta\dot\psi)\mathbf{n}_1 \\ + (\ddot\theta - 2\dot\phi\dot\psi\cos\theta - \dot\phi^2\sin\theta\cos\theta)\mathbf{n}_2 \\ -2(\ddot\psi + \ddot\phi\sin\theta + \dot\phi\dot\theta\cos\theta)\mathbf{n}_3] \end{aligned}$$

4.1.6 If $\mathbf{n}_i$, $i = 1, 2, 3$, is a *right-handed set of mutually perpendicular principal directions* of a rigid body R *for the mass center* P^* *of* R, and these unit vectors are *fixed in* R, then the torque $^{R'}\mathbf{T}^R$ of the inertia couple acting on R in a reference frame R' (see 4.1.3) is given by

$$\begin{aligned} ^{R'}\mathbf{T}^R = &-\left[\phi_{11}^{R/P^*}\frac{d\,^{R'}\omega_1{}^R}{dt} - (\phi_{22}^{R/P^*} - \phi_{33}^{R/P^*})\,^{R'}\omega_2{}^R\,^{R'}\omega_3{}^R\right]\mathbf{n}_1 \\ &-\left[\phi_{22}^{R/P^*}\frac{d\,^{R'}\omega_2{}^R}{dt} - (\phi_{33}^{R/P^*} - \phi_{11}^{R/P^*})\,^{R'}\omega_3{}^R\,^{R'}\omega_1{}^R\right]\mathbf{n}_2 \\ &-\left[\phi_{33}^{R/P^*}\frac{d\,^{R'}\omega_3{}^R}{dt} - (\phi_{11}^{R/P^*} - \phi_{22}^{R/P^*})\,^{R'}\omega_1{}^R\,^{R'}\omega_2{}^R\right]\mathbf{n}_3 \end{aligned}$$

where t is the time and $^{R'}\omega_i{}^R$ is the $\mathbf{n}_i$ measure number of the angular velocity of R in R'.

Proof: See Secs. 4.1.5 and 2.3.2.

Problem: Referring to Problem 2.2.7, let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be the mutually perpendicular unit vectors shown in Fig. 4.1.6, P^* the mass center of B, m the mass of B, and k_1, k_2, k_3 the radii of gyration of B with respect to lines passing through P^* and parallel to $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. Assuming that these lines are principal axes of B for P^*, determine the torque (${}^{R_2}\mathbf{T}^B$) of the inertia couple acting on B in R_2.

Solution: From the solution of Problem 2.2.7,

$$
\begin{aligned}
{}^{R_2}\boldsymbol{\omega}^B &= (0.2 \cos \psi \, \mathbf{k}_1 + \mathbf{k}_2) \frac{d\psi}{dt} \\
&\underset{(\text{F4.1.6})}{=} (0.2 \cos \psi \, \mathbf{n}_1 + \sin \beta \, \mathbf{n}_2 + \cos \beta \, \mathbf{n}_3) \frac{d\psi}{dt}
\end{aligned}
$$

Hence the $\mathbf{n}_i$, $i = 1, 2, 3$, measure numbers of ${}^{R_2}\boldsymbol{\omega}^B$ are

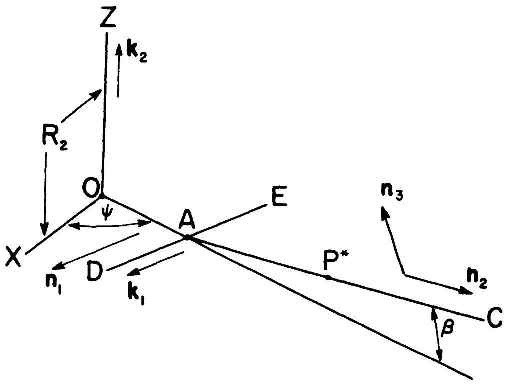


FIG. 4.1.6

$$
{}^{R_2}\omega_1{}^B = 0.2 \cos \psi \frac{d\psi}{dt}, \qquad {}^{R_2}\omega_2{}^B = \sin \beta \frac{d\psi}{dt}, \qquad {}^{R_2}\omega_3{}^B = \cos \beta \frac{d\psi}{dt}
$$

and their time-derivatives (keeping in mind that

$$
\frac{d^2\psi}{dt^2} = 0
$$

and that

$$
\frac{d\beta}{dt} = 0.2 \cos \psi \frac{d\psi}{dt}
$$

as a consequence of the definition of β) are given by

$$\frac{d^{R_2}\omega_1{}^B}{dt} = -0.2 \sin\psi \left(\frac{d\psi}{dt}\right)^2$$

$$\frac{d^{R_2}\omega_2{}^B}{dt} = 0.2 \cos\beta \cos\psi \left(\frac{d\psi}{dt}\right)^2$$

$$\frac{d^{R_2}\omega_3{}^B}{dt} = -0.2 \sin\beta \cos\psi \left(\frac{d\psi}{dt}\right)^2$$

The principal moments of inertia of B for P^* are

$$\phi_{ii}^{B/P^*} \underset{(3.5.5)}{=} mk_i^2, \qquad i = 1, 2, 3$$

Thus

$$\begin{aligned} {}^{R_2}\mathbf{T}^B = -0.2m \left(\frac{d\psi}{dt}\right)^2 \{[-k_1^2 \sin\psi + (k_3^2 - k_2^2) \sin\beta \cos\beta]\mathbf{n}_1 \\ + [(k_1^2 + k_2^2 - k_3^2) \cos\beta \cos\psi]\mathbf{n}_2 \\ + [(-k_1^2 + k_2^2 - k_3^2) \sin\beta \cos\psi]\mathbf{n}_3\} \end{aligned}$$

4.1.7 When a rigid body R has an angular velocity of fixed orientation in a reference frame R' (see 2.2.4), the torque ${}^{R'}\mathbf{T}^R$ of the inertia couple acting on R in R' (see 4.1.3) is given by

$$\begin{aligned} {}^{R'}\mathbf{T}^R = -\{[\phi_{31}^{R/P^*}\, {}^{R'}\alpha^R - \phi_{32}^{R/P^*}({}^{R'}\omega^R)^2]\mathbf{n}_1 \\ + [\phi_{32}^{R/P^*}\, {}^{R'}\alpha^R + \phi_{31}^{R/P^*}({}^{R'}\omega^R)^2]\mathbf{n}_2 + [\phi_{33}^{R/P^*}\, {}^{R'}\alpha^R]\mathbf{n}_3\} \end{aligned}$$

where P^* is the mass center of R, $\mathbf{n}_i$, $i = 1, 2, 3$, is a right-handed set of mutually perpendicular unit vectors with $\mathbf{n}_3$ parallel to the angular velocity of R in R', and ${}^{R'}\omega^R$ and ${}^{R'}\alpha^R$ are the angular speed and the scalar angular acceleration (see 2.2.2 and 2.3.4) of R in R' for the direction $\mathbf{n}_3$.

Proof: The angular velocity ${}^{R'}\boldsymbol{\omega}^R$ and angular acceleration ${}^{R'}\boldsymbol{\alpha}^R$ are given by

$${}^{R'}\boldsymbol{\omega}^R \underset{(2.2.4)}{=} {}^{R'}\omega^R\, \mathbf{n}_3, \qquad {}^{R'}\boldsymbol{\alpha}^R \underset{(2.3.7)}{=} {}^{R'}\alpha^R\, \mathbf{n}_3$$

Hence

$$\mathbf{n}_3 \cdot {}^{R'}\boldsymbol{\omega}^R = {}^{R'}\omega^R, \qquad \mathbf{n}_3 \cdot {}^{R'}\boldsymbol{\alpha}^R = {}^{R'}\alpha^R$$

and

$${}^{R'}\mathbf{T}^R \underset{(4.1.3)}{=} -[{}^{R'}\alpha^R\, \boldsymbol{\Phi}_3^{R/P^*} + ({}^{R'}\omega^R)^2\, \mathbf{n}_3 \times \boldsymbol{\Phi}_3^{R/P^*}] \tag{A}$$

Next, let a_i be the $\mathbf{n}_i$ measure number of a unit vector $\mathbf{n}_a$, and take

$$\mathbf{n}_a = \mathbf{n}_3$$

Then

$$a_1 = a_2 = 0, \qquad a_3 = 1$$

and

$$\boldsymbol{\Phi}_3^{R/P^*} = \boldsymbol{\Phi}_a^{R/P^*} \underset{(3.5.7,3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/P^*} a_i \mathbf{n}_j$$

$$= \sum_{j=1}^{3} \phi_{3j}^{R/P^*}\mathbf{n}_j = \phi_{31}^{R/P^*}\mathbf{n}_1 + \phi_{32}^{R/P^*}\mathbf{n}_2 + \phi_{33}^{R/P^*}\mathbf{n}_3$$

Substitute into Eq. (A). (Note that it was not assumed that $\mathbf{n}_1$ and $\mathbf{n}_2$ are fixed in either R or R'.)

Problem: Show that when a rigid body R has an angular velocity of fixed orientation in a reference frame R', the torque ${}^{R'}\mathbf{T}^R$ of the inertia couple acting on R in R' in general remains parallel to the angular velocity ${}^{R'}\boldsymbol{\omega}^R$ of R in R' if and only if ${}^{R'}\boldsymbol{\omega}^R$ is parallel to a principal axis of R for the mass center P^* of R.

Solution: ${}^{R'}\mathbf{T}^R$ is given by

$$\begin{aligned}{}^{R'}\mathbf{T}^R = -\{&[\phi_{31}^{R/P^*}\,{}^{R'}\alpha^R - \phi_{32}^{R/P^*}({}^{R'}\omega^R)^2]\mathbf{n}_1 \\ &+ [\phi_{32}^{R/P^*}\,{}^{R'}\alpha^R + \phi_{31}^{R/P^*}({}^{R'}\omega^R)^2]\mathbf{n}_2 + [\phi_{33}^{R/P^*}\,{}^{R'}\alpha^R]\mathbf{n}_3\}\end{aligned}$$

where $\mathbf{n}_3$ is parallel to ${}^{R'}\boldsymbol{\omega}^R$. Hence ${}^{R'}\mathbf{T}^R$ is parallel to ${}^{R'}\boldsymbol{\omega}^R$ if and only if

$$\left.\begin{aligned}&\phi_{31}^{R/P^*}\,{}^{R'}\alpha^R - \phi_{32}^{R/P^*}({}^{R'}\omega^R)^2 = 0 \\ \text{and}\qquad &\phi_{32}^{R/P^*}\,{}^{R'}\alpha^R + \phi_{31}^{R/P^*}({}^{R'}\omega^R)^2 = 0\end{aligned}\right\} \quad \text{(A)}$$

If ${}^{R'}\boldsymbol{\omega}^R$ is parallel to a principal axis of R for P^*, then $\mathbf{n}_3$ is a principal direction of R for P^*, $\phi_{31}^{R/P^*} = \phi_{32}^{R/P^*} = 0$ (see 3.5.7 and 3.3.3), and it follows that Eqs. (A) are satisfied for all values of ${}^{R'}\alpha^R$ and ${}^{R'}\omega^R$. Conversely, if Eqs. (A) are to be satisfied for all values of ${}^{R'}\alpha^R$ and ${}^{R'}\omega^R$, then ϕ_{31}^{R/P^*} and ϕ_{32}^{R/P^*} must be equal to zero, and it follows that $\mathbf{n}_3$ is a principal direction of R for P (see 3.5.7 and 3.3.4) and that ${}^{R'}\boldsymbol{\omega}^R$ is parallel to a principal axis of R for P^*.

4.2 D'Alembert's principle

4.2.1 There exist reference frames R, called Newtonian reference frames, such that, in the absence of magnetic and electric effects, the force system consisting of (a) all gravitational and contact forces exerted on any body B (that is, any collection of matter) by other bodies in the universe (see Vol. I, Chapter 4) and (b) all inertia forces acting on B in R (see 4.1.1–4.1.7) is a zero system (see Vol. I, Sec. 3.6) for all motions of B.

This proposition is known as *D'Alembert's principle.*

4.2.2 Given a reference frame R and a Newtonian reference frame R' (see 4.2.1), R is a Newtonian reference frame if and only if the motion of R in R' is one during which the angular velocity of R in R' and the acceleration in R' of at least one point Q fixed in R remain equal to zero.

Proof: Given any body B, let S_a be the system of all gravitational and contact forces acting on B, and S_b and S_b' the systems of inertia forces acting on B in R and in R', respectively. Then the fact that R' is a Newtonian reference frame can be expressed as

$$S_a + S_b' \underset{(4.2.1)}{=} 0 \tag{A}$$

S_b and S_b' differ from each other only because the accelerations ${}^R\mathbf{a}^P$ and ${}^{R'}\mathbf{a}^P$ of points P of B differ from each other as a consequence of the motion of R in R'. Now, at any instant,

$${}^{R'}\mathbf{a}^P \underset{(2.5.13)}{=} {}^R\mathbf{a}^P + {}^{R'}\mathbf{a}^{P^*} + 2\,{}^{R'}\boldsymbol{\omega}^R \times {}^R\mathbf{v}^P \tag{B}$$

where P^* is that point of R which coincides with P at the instant under consideration. Furthermore,

$${}^{R'}\mathbf{a}^{P^*} \underset{(2.5.9)}{=} {}^{R'}\mathbf{a}^Q + {}^{R'}\boldsymbol{\alpha}^R \times \mathbf{r} + {}^{R'}\boldsymbol{\omega}^R \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}) \tag{C}$$

where Q is any point fixed in R and $\mathbf{r}$ is the position vector of P relative to Q. Hence, if ${}^{R'}\boldsymbol{\omega}^R = 0$ (so that ${}^{R'}\boldsymbol{\alpha}^R = 0$) and there exists one point Q such that ${}^{R'}\mathbf{a}^Q = 0$, then

$${}^{R'}\mathbf{a}^{P^*} \underset{(C)}{=} 0$$

and

$${}^{R'}\mathbf{a}^P \underset{(B)}{=} {}^R\mathbf{a}^P$$

from which it follows that

$$S_b' = S_b$$

so that

$$S_a + S_b \underset{(A)}{=} 0$$

and R is a Newtonian reference frame.

Next, let B be a particle P of mass m. Then, if ${}^{R'}\boldsymbol{\omega}^R \neq 0$, it is always possible to find a motion of P during which ${}^R\mathbf{v}^P$ is such that

$${}^{R'}\mathbf{a}^{P*} + 2\,{}^{R'}\boldsymbol{\omega}^R \times {}^R\mathbf{v}^P \neq 0$$

and, consequently,

$${}^{R'}\mathbf{a}^P \underset{(B)}{\neq} {}^R\mathbf{a}^P$$

Similarly, if there exists no point Q fixed in R such that ${}^{R'}\mathbf{a}^Q = 0$, then any motion of P during which ${}^{R'}\boldsymbol{\omega}^R \times {}^R\mathbf{v}^P = 0$ is one for which

$${}^{R'}\mathbf{a}^P \underset{(B)}{=} {}^R\mathbf{a}^P + {}^{R'}\mathbf{a}^{P*}$$

where

$${}^{R'}\mathbf{a}^{P*} \neq 0$$

so that

$${}^{R'}\mathbf{a}^P \neq {}^R\mathbf{a}^P$$

In either case, therefore, there exist motions during which

$$-m\,{}^{R'}\mathbf{a}^P \neq -m\,{}^R\mathbf{a}^P$$

But $-m\,{}^R\mathbf{a}^P$ and $-m\,{}^{R'}\mathbf{a}^P$, applied at P, are now identical with S_b and S_b', respectively. Hence

$$S_b \neq S_b'$$

and

$$S_a + S_b \underset{(A)}{\neq} 0$$

Accordingly, R is not a Newtonian reference frame.

4.2.3 Equations governing gravitational, contact, and inertia forces acting on a body B in a Newtonian reference frame (see 4.2.1), called *equations of motion*, are obtained by using various properties of zero systems (see Vol. I, Sec. 3.6). Before writing such equations, it is always advisable to make a sketch representing the body B and some system of forces equivalent to the system of *all* gravitational, contact, and inertia forces acting on B. (The presence of couples may be indicated by arrows representing the torques of the couples.) Such a sketch is called a *free-body diagram* of B.

Example: In Fig. 4.2.3a, E represents the earth, which is regarded as a sphere having a radius R (3960 miles), a mass m' (4.11×10^{23} slug), and a mass density which at any point of E depends only on the distance from the point to the center C of E. A is a reference frame in which C and the earth's axis, line NS, are fixed, and such that the angular velocity ${}^A\boldsymbol{\omega}^E$ is given by

$$^A\boldsymbol{\omega}^E = \omega\mathbf{k}$$

where $\omega = 2\pi$ rad day^{-1} and $\mathbf{k}$ is a unit vector parallel to line NS; Q a point fixed relative to E; B a rigid body (small in comparison with E) of mass m, which is held at rest relative to E near the surface of E(but not in contact with E) by a light, flexible string QP; P^* the mass center of B; $\mathbf{n}^*$ a unit vector parallel to line CP^*; $\mathbf{n}$ a unit vector parallel to line PQ; and ϕ the angle between $\mathbf{n}^*$ and $\mathbf{k}$.

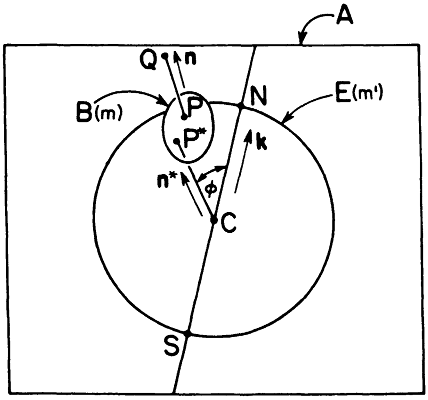


FIG. 4.2.3a

The string tension W (see Vol. I, Sec. 4.8.1), which is called the *weight* of B (see Vol. I, Sec. 4.4.4), and the angle θ between $\mathbf{n}^*$ and $\mathbf{n}$ are to be expressed in terms of ϕ, m, g, and ϵ, where

$$g = \frac{Gm'}{R^2}, \qquad \epsilon = \frac{R\omega^2}{g}$$

and G (3.42×10^{-8} lb ft^2 slug^{-2}) is the gravitational constant (see Vol. I, Sec. 4.2), on the assumption that (a) E and (b) A is a Newtonian reference frame.

Solution (a): Figure 4.2.3b shows a free-body diagram of a body B' consisting of B and a portion of the string, drawn on the basis of the following considerations:

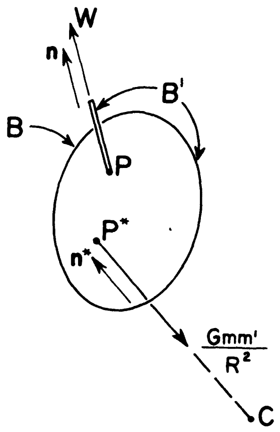


FIG. 4.2.3b

(1) The system of all gravitational forces acting on B' may be regarded as equivalent to the single force of magnitude Gmm'/R^2 (see Vol. I, Secs. 4.4.2 and 4.4.3) shown in Fig. 4.2.3b.

(2) The system of all contact forces acting on B' is represented by the single force of magnitude W (see Vol. I, Sec. 4.8.1). (Contact forces exerted by the air surrounding B' are neglected.)

(3) As every point of B' is at rest in E, no inertia forces act on B' in E (see 4.1.1).

Assuming that E is a Newtonian reference frame, the following equation of motion, called a *force equation*, is justified (see 4.2.1):

$$W\mathbf{n} - \frac{Gmm'}{R^2}\mathbf{n}^* = 0$$

It follows that

$$W = \frac{Gmm'}{R^2}$$

and

$$\mathbf{n} = \mathbf{n}^*$$

Hence

$$W = mg$$

and

$$\theta = (\mathbf{n}, \mathbf{n}^*) = 0$$

Solution (b): In addition to the forces shown in Fig. 4.2.3b, the free-body diagram shown in Fig. 4.2.3c includes two vectors which, together, represent the system of inertia forces acting on B' in reference frame A. These are the inertia force ${}^A\mathbf{F}^B$, given by

$$\begin{aligned} {}^A\mathbf{F}^B &\underset{(4.1.3)}{=} -m\,{}^A\mathbf{a}^{P^*} \\ &\underset{(2.5.9)}{=} -m(\omega\mathbf{k}) \times [(\omega\mathbf{k}) \times (R\mathbf{n}^*)] \\ &= mR\omega^2 \sin\phi\,\mathbf{n}' \end{aligned}$$

where $\mathbf{n}'$ is a unit vector perpendicular to $\mathbf{k}$ and parallel to the

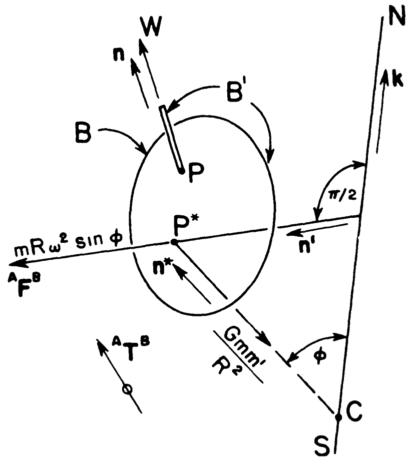


FIG. 4.2.3c

plane determined by lines NS and CP^*, and the torque ${}^A\mathbf{T}^B$ of the inertia couple acting on B in A. (As the string is regarded as "light," no inertia forces acting on the string are shown.)

The following force equation is justified by the assumption that A is a Newtonian reference frame:

$$W\mathbf{n} - \frac{Gmm'}{R^2}\mathbf{n}^* + mR\omega^2 \sin\phi\,\mathbf{n}' = 0 \tag{A}$$

This equation shows that $\mathbf{n}$ is parallel to the plane determined by lines NS and CP^*. Noting that the angle between $\mathbf{n}$ and $\mathbf{n}'$ (see Fig. 4.2.3d) is given by

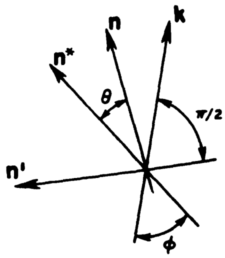


FIG. 4.2.3d

$$(\mathbf{n}, \mathbf{n}') = \theta + \frac{\pi}{2} - \phi \tag{B}$$

cross-multiply Eq. (A) with $\mathbf{n}$ (in order to eliminate W), then solve for θ:

$$\theta = \arctan\left(\frac{\epsilon \sin\phi \cos\phi}{1 - \epsilon \sin^2\phi}\right) \tag{C}$$

where

$$\epsilon = \frac{R\omega^2}{Gm'/R^2} = \frac{R\omega^2}{g} \tag{D}$$

To find W, dot-multiply Eq. (A) with $\mathbf{n}$:

$$W \underset{(\mathrm{A,B})}{=} \frac{Gmm'}{R^2}\cos\theta - mR\omega^2 \sin\phi(\cos\theta\sin\phi - \sin\theta\cos\phi)$$

or

$$W \underset{(\mathrm{D})}{=} mg[(1 - \epsilon\sin^2\phi)\cos\theta + \epsilon\sin\phi\cos\phi\sin\theta] \tag{E}$$

But, from Eq. (C),

$$\sin\theta = \frac{\epsilon\sin\phi\cos\phi}{(1 - 2\epsilon\sin^2\phi + \epsilon^2\sin^2\phi)^{1/2}}$$

and

$$\cos\theta = \frac{1 - \epsilon\sin^2\phi}{(1 - 2\epsilon\sin^2\phi + \epsilon^2\sin^2\phi)^{1/2}}$$

Hence

$$W \underset{(\mathrm{E})}{=} mg(1 - 2\epsilon\sin^2\phi + \epsilon^2\sin^2\phi)^{1/2} \tag{F}$$

(Note that the results of Solution (a) can be obtained from Eqs. (C) and (F) by setting ϵ equal to zero, that is, by setting ω equal to zero (see Eq. (D)).

4.2.4 The results obtained in Example 4.2.3 do not furnish a satisfactory basis for an experiment intended to reveal whether E or A (if either) is a Newtonian reference frame. For with $R = 3960$ miles, $\omega = 2\pi$ rad day^{-1}, and $g = 32.2$ lb slug^{-1}, ϵ has the value

$$\epsilon \underset{(D)}{=} 0.0035$$

and the maximum angle between the plumb line PQ and line CP^*, obtained by taking $\phi = \pi/4$ rad (and regarding A as a Newtonian reference frame) has the very small value

$$\theta_{\max} \underset{(C)}{=} 0.0017 \text{ rad}$$

while the minimum weight of B, corresponding to $\phi = \pi/2$ rad, is

$$W_{\min} \underset{(F)}{=} 0.997 \text{ mg}$$

which differs by less than one part in one hundred from the value obtained by regarding E as a Newtonian reference frame. What the results do show is that for a body at rest relative to the earth the difference between regarding A, on the one hand, and E, on the other, as a Newtonian reference frame is small.

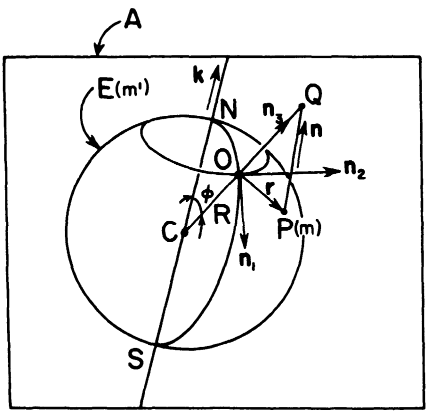


FIG. 4.2. 5a

4.2.5 The hypothesis that A (see Example 4.2.3) is a good approximation to a Newtonian reference frame is supported by an analysis of the motion of *Foucault's pendulum*, described as follows:

In Fig. 4.2.5a, P represents a particle of mass m suspended from a point Q by means of a light string of length L, Q being fixed relative to E. (E and A are defined as in Example 4.2.3.) O is the point of intersection of line CQ and the surface of E, ϕ the angle between lines OC and NS, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ unit vectors pointing southward, eastward, and upward at O, $\mathbf{r}$ the position vector of P relative to O, and $\mathbf{n}$ a unit vector parallel to the string PQ.

The gravitational force $\mathbf{G}$ exerted on P by E is given by (see Vol. 1, Sec. 4.4.2)

$$\mathbf{G} = -Gmm'[(R\mathbf{n}_3 + \mathbf{r})^2]^{-3/2}(R\mathbf{n}_3 + \mathbf{r})$$

For motions during which $|\mathbf{r}|$ is sufficiently small in comparison with R, this reduces to

$$\mathbf{G} \approx -Gmm'[(R\mathbf{n}_3)^2]^{-3/2}(R\mathbf{n}_3) = \frac{Gmm'}{R^2}\,\mathbf{n}_3$$

or, letting

$$g = \frac{Gm'}{R^2}$$

to

$$\mathbf{G} \approx -mg\mathbf{n}_3 \tag{A}$$

The inertia force ${}^A\mathbf{F}^P$ acting on P in A is given by

$${}^A\mathbf{F}^P \underset{(4.1.1)}{=} -m\,{}^A\mathbf{a}^P \underset{(2.5.13)}{=} -m[{}^E\mathbf{a}^P + {}^A\mathbf{a}^{P^*} + 2(\omega\mathbf{k}) \times {}^E\mathbf{v}^P]$$

where P^* is the point of E which coincides with P, so that

$${}^A\mathbf{a}^{P^*} \underset{(2.5.9)}{=} (\omega\mathbf{k}) \times [(\omega\mathbf{k}) \times (R\mathbf{n}_3 + \mathbf{r})]$$

or, again confining attention to motions during which $|\mathbf{r}|$ is much smaller than R,

$${}^A\mathbf{a}^{P^*} \approx (\omega\mathbf{k}) \times [(\omega\mathbf{k}) \times (R\mathbf{n}_3)] = R\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3)$$

Hence ${}^A\mathbf{F}^P$ is given by

$${}^A\mathbf{F}^P \approx -m[{}^E\mathbf{a}^P + R\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3) + 2\omega\mathbf{k} \times {}^E\mathbf{v}^P] \tag{B}$$

The system of all contact forces acting on the body consisting of P and a portion of the string is regarded as equivalent to the

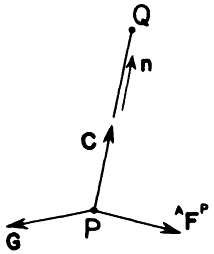


FIG. 4.2.5b

force $\mathbf{C}$ shown in the free-body diagram of this body, Fig. 4.2.5b.

Assuming that A is a Newtonian reference frame, the sum of the moments of the forces $\mathbf{C}$, ${}^A\mathbf{F}^P$, and $\mathbf{G}$ about point Q may be set equal to zero, which yields the following *moment equation:*

$$-(L\mathbf{n}) \times (\mathbf{G} + {}^A\mathbf{F}^P) = 0 \tag{C}$$

or, using Eqs. (A) and (B),

$$\mathbf{n} \times [g\mathbf{n}_3 + {}^E\mathbf{a}^P + R\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3) + 2\omega\mathbf{k} \times {}^E\mathbf{v}^P] \approx 0$$

Now

$$R\omega^2 \underset{(E4.2.3)}{=} g\epsilon \underset{(4.2.4)}{=} 0.0035\, g$$

and

$$|\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3)| \leqslant 1$$

Hence $R\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3)$ can be dropped in comparison with $g\mathbf{n}_3$, and

$$\mathbf{n} \times (g\mathbf{n}_3 + {}^E\mathbf{a}^P + 2\omega\mathbf{k} \times {}^E\mathbf{v}^P) \approx 0 \tag{D}$$

Next, let θ be the angle between $\mathbf{n}$ and $\mathbf{n}_3$ (see Fig. 4.2.5c), $\mathbf{n}'$ a unit vector perpendicular to $\mathbf{n}_3$ and parallel to the plane determined by lines OQ and OP, $\mathbf{n}''$ a unit vector perpendicular to both $\mathbf{n}_3$ and $\mathbf{n}'$, and ψ the angle between $\mathbf{n}_1$ and $\mathbf{n}'$. Then the position vector $\mathbf{r}$ of P relative to O is given by

$$\mathbf{r} = L \sin\theta\, \mathbf{n}' + (h - L\cos\theta)\mathbf{n}_3$$

where h is the distance between O and Q, and

$$\mathbf{n} = -\sin\theta\, \mathbf{n}' + \cos\theta\, \mathbf{n}_3$$

Hence, restricting the discussion to motions during which θ remains small,

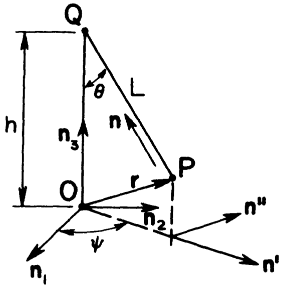


FIG. 4.2.5c

$$\mathbf{r} \approx L\theta\mathbf{n}' + (h - L)\mathbf{n}_3 \tag{E}$$

and

$$\mathbf{n} \approx -\theta\mathbf{n}' + \mathbf{n}_3 \tag{F}$$

Now

$$^E\mathbf{v}^P \underset{(2.5.1)}{=} \frac{^Ed\mathbf{r}}{dt} \underset{(E)}{\approx} L(\dot\theta\mathbf{n}' + \theta\dot\psi\mathbf{n}'') \quad (1.5.1, 1.7.1) \tag{G}$$

and

$$^E\mathbf{a}^P \underset{(2.5.1)}{=} \frac{^Ed\,^E\mathbf{v}^P}{dt} \underset{(G)}{\approx} L[(\ddot\theta - \theta\dot\psi^2)\mathbf{n}' + (\theta\ddot\psi + 2\dot\theta\dot\psi)\mathbf{n}''] \quad (1.5.4, 1.7.1) \tag{H}$$

Furthermore

$$\mathbf{k} \underset{(F4.2.5a)}{=} -\sin\phi\,\mathbf{n}_1 + \cos\phi\,\mathbf{n}_3 \tag{I}$$

Thus, dropping all terms containing θ^2 and $\theta\dot\theta$,

$$\mathbf{n} \times \mathbf{n}_3 \underset{(F)}{\approx} \theta\mathbf{n}''$$

$$\mathbf{n} \times {}^E\mathbf{a}^P \underset{(F,H)}{\approx} L[(\ddot\theta - \theta\dot\psi^2)\mathbf{n}'' - (\theta\ddot\psi + 2\dot\theta\dot\psi)\mathbf{n}']$$

and

$$\mathbf{n} \times (\mathbf{k} \times {}^E\mathbf{v}^P) \underset{(F,G,I)}{\approx} -L\cos\phi(\dot\theta\mathbf{n}' + \theta\dot\psi\mathbf{n}'')$$

Equation (D) therefore leads to the following scalar equations for motions of P during which θ remains small:

$$\theta\ddot\psi + 2\dot\theta(\dot\psi + \omega\cos\phi) \approx 0 \tag{J}$$

$$\ddot\theta + \left(\frac{g}{L} - \dot\psi^2 - 2\omega\dot\psi\cos\phi\right)\theta \approx 0 \tag{K}$$

One solution of these equations is obtained by taking

$$\dot\psi + \omega\cos\phi = 0$$

that is,

$$\dot{\psi} = -\omega \cos \phi \tag{L}$$

which satisfies Eq. (J) for all θ and reduces Eq. (K) to

$$\ddot{\theta} + \frac{g}{L}\theta \approx 0 \tag{M}$$

where $\omega^2 \cos^2 \phi$ has been dropped in comparison with g/L in order to preserve consistency with the approximation made in the derivation of Eq. (D). The general solution of Eq. (M) is

$$\theta = C_1 \sin\left(\sqrt{\frac{g}{L}}\, t\right) + C_2 \cos\left(\sqrt{\frac{g}{L}}\, t\right) \tag{N}$$

where C_1 and C_2 are constants.

Equations (L) and (N) may be interpreted as follows: The assumption that A is a Newtonian reference frame leads to the prediction that there exists a motion of P during which the string QP oscillates in a plane passing through line OQ and rotating through $|2\pi \cos \phi|$ rad per day, the rotation being clockwise as seen by an observer looking from Q toward O when Q is in the northern hemisphere, counterclockwise when Q is in the southern hemisphere. The oscillations have a period $T = 2\pi(L/g)^{1/2}$.

The above predictions are in close agreement with observations of actual motions. Consequently they strengthen the hypothesis that A is a good approximation to a Newtonian reference frame.

The motion of P predicted by assuming that E is a Newtonian reference frame (and retaining all of the approximations made so far) can be found by setting ω equal to zero in Eqs. (J) and (K):

$$\theta\ddot{\psi} + 2\dot{\theta}\dot{\psi} = 0$$

$$\ddot{\theta} + \left(\frac{g}{L} - \dot{\psi}^2\right)\theta = 0$$

The oscillatory motion described by Eq. (N) corresponds to a solution of the second of these if $\dot{\psi} = 0$, which also satisfies the first. Hence the assumption that E is a Newtonian reference frame leads to a prediction which is incorrect as regards $\dot{\psi}$ but correct as regards θ. Note, however, that the error in $\dot{\psi}$ is proportional to time (see Eq. (L)), and that the constant of proportionality has the value $|\omega \cos \phi|$, which suggests that for motions taking place during sufficiently short time intervals, particularly near the equator, E may be a satisfactory approximation to a Newtonian reference frame.

4.2.6 When a particle P moves freely near the surface of E (see 4.2.5), the time interval during which the motion takes place, the locality in which P moves, and the initial velocity of P determine whether or not one may regard E as a Newtonian reference frame when analyzing the motion of P. This is shown as follows:

In Fig. 4.2.6a, P represents a particle of mass m, free to move subject to two restrictions: (1) P does not come into contact with E and (2) the magnitude of the position vector $\mathbf{r}$ of P relative to a point O fixed on E's surface remains small in comparison with the radius R of the earth. (All other symbols appearing in Fig. 4.2.6a have the same meaning as in Example 4.2.3 and Sec. 4.2.5.)

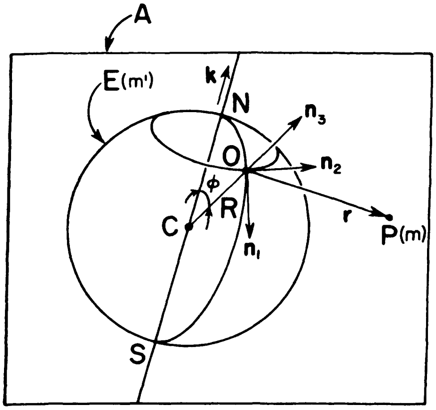


FIG. 4.2.6a

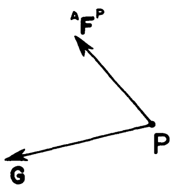


FIG. 4.2.6b

The free-body diagram shown in Fig. 4.2.6b is constructed by proceeding as in Sec. 4.2.5, the gravitational force $\mathbf{G}$ and the inertia force ${}^A\mathbf{F}^P$ acting on P in A again being given by

$$\mathbf{G} \approx -mg\mathbf{n}_3$$

and

$${}^A\mathbf{F}^P \approx -m[{}^E\mathbf{a}^P + R\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3) + 2\omega\mathbf{k} \times {}^E\mathbf{v}^P]$$

Set the sum of ${}^A\mathbf{F}^P$ and $\mathbf{G}$ equal to zero (see 4.2.1), dropping $R\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3)$ in comparison with $g\mathbf{n}_3$ (see 4.2.5) and expressing

${}^E\mathbf{v}^P$ and ${}^E\mathbf{a}^P$ in terms of time-derivatives of $\mathbf{r}$ (see 2.5.1). The assumption that A is a Newtonian reference frame then leads to the following equation of motion:

$$\frac{{}^E d^2\mathbf{r}}{dt^2} = -\left(g\mathbf{n}_3 + 2\omega\mathbf{k} \times \frac{{}^E d\mathbf{r}}{dt}\right) \tag{A}$$

This equation may be solved by noting that

$$\mathbf{r} \underset{(1.8.1)}{=} \mathbf{r}\big|_{t=0} + t\,\frac{{}^E d\mathbf{r}}{dt}\bigg|_{t=0} + \frac{t^2}{2!}\,\frac{{}^E d^2\mathbf{r}}{dt^2}\bigg|_{t=0} + \cdots \tag{B}$$

and using Eq. (A) to obtain the third and all higher time-derivatives of $\mathbf{r}$. That is,

$$\begin{aligned}
\frac{{}^E d^3\mathbf{r}}{dt^3} &\underset{(A)}{=} -2\omega\mathbf{k} \times \frac{{}^E d^2\mathbf{r}}{dt^2} \\
&\underset{(A)}{=} -2\omega\mathbf{k} \times \left[-\left(g\mathbf{n}_3 + 2\omega\mathbf{k} \times \frac{{}^E d\mathbf{r}}{dt}\right)\right] \\
&= 2g\omega\mathbf{k} \times \mathbf{n}_3 + 4\omega^2\mathbf{k} \times \left(\mathbf{k} \times \frac{{}^E d\mathbf{r}}{dt}\right)
\end{aligned}$$

and, similarly,

$$\frac{{}^E d^4\mathbf{r}}{dt^4} = -4g\omega^2\mathbf{k} \times (\mathbf{k} \times \mathbf{n}_3) - 8\omega^3\mathbf{k} \times \left[\mathbf{k} \times \left(\mathbf{k} \times \frac{{}^E d\mathbf{r}}{dt}\right)\right]$$

and so forth. Equation (B) then gives

$$\begin{aligned}
\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t - \frac{gt^2}{2}\mathbf{n}_3 - (\omega t)\mathbf{k} \times \left(\mathbf{v}_0 t - \frac{gt^2}{3}\mathbf{n}_3\right) \\
+ (\omega t)^2\mathbf{k} \times \left[\mathbf{k} \times \left(\frac{2}{3}\mathbf{v}_0 t - \frac{gt^2}{6}\mathbf{n}_3\right)\right] + \cdots
\end{aligned} \tag{C}$$

where $\mathbf{r}_0$ and $\mathbf{v}_0$ are, respectively, the position vector of P relative to O and the velocity of P in E at time $t = 0$.

The corresponding equation based on the assumption that E is a Newtonian reference frame (obtained by setting $\omega = 0$ in Eq. (C)) is

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t - \frac{gt^2}{2}\mathbf{n}_3 \tag{D}$$

The difference between the descriptions of the motion of P obtained by using Eqs. (C) and (D) depends on the time interval during which the motion takes place, because the (dimensionless) factor ωt is proportional to this time interval. Furthermore, for a given value of t, this difference is seen to depend on the locality in which the motion occurs, which determines $\mathbf{n}_3$, and on $\mathbf{v}_0$, the initial velocity of P in E.

4.2.7 As Eq. (C), Sec. 4.2.6, applies only to motions of P during which $|\mathbf{r}|$ remains small in comparison with R, it furnishes no information about certain motions of interest, for example, those of ballistic missiles and earth satellites. An approximate description of such motions is obtained as follows. (Approximations are introduced by neglecting the oblateness of the earth, gravitational forces exerted by bodies other than the earth, and all contact forces.)

In Fig. 4.2.7a, E represents the earth, regarded as a sphere

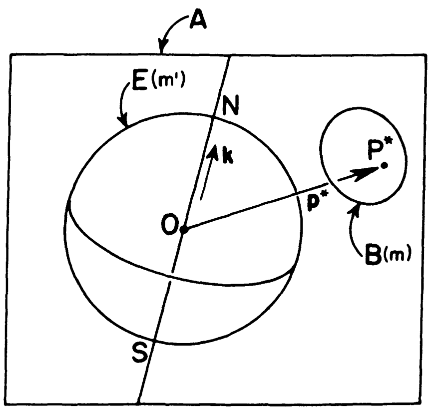


FIG. 4.2.7a

having a radius R, a mass m', and a mass density which at any point of E depends only on the distance from the point to the center O of E. A is a reference frame in which O and the earth's north-south axis, line NS, are fixed, and such that

$$^A\boldsymbol{\omega}^E = \omega\mathbf{k}$$

where

$$\omega = 2\pi \text{ rad day}^{-1}$$

and $\mathbf{k}$ is a unit vector parallel to line NS. B is a rigid body of mass m, whose largest dimension is small in comparison with R, P^* is the mass center of B, and $\mathbf{p}^*$ is the position vector of P^* relative to O.

Assume that gravitational forces exerted on B by bodies other than E are negligible in comparison with those exerted by E. Then the system of all gravitational forces acting on B is equivalent to the force $\mathbf{G}$ shown in the free-body diagram Fig. 4.2.7b, and (see Vol. I, Sec. 4.4.2)

$$\mathbf{G} = -Gmm'(\mathbf{p}^{*2})^{-3/2}\mathbf{p}^*$$

The vectors ${}^A\mathbf{T}^B$ and ${}^A\mathbf{F}^B$ appearing in Fig. 4.2.7b represent the torque of the inertia couple and the inertia force acting on B in A

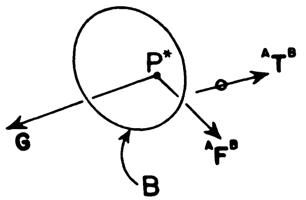


FIG. 4.2.7b

(see 4.1.3). ${}^A\mathbf{F}^B$ is given by

$${}^A\mathbf{F}^B = -m\,{}^A\mathbf{a}^{P^*}$$

Assume that all contact forces acting on B are negligible and that A is a satisfactory approximation to a Newtonian reference frame. Then

$${}^A\mathbf{F}^B + \mathbf{G} \underset{(4.2.3)}{=} 0$$

from which it follows that

$${}^A\mathbf{a}^{P^*} = -Gm'(\mathbf{p}^{*2})^{-3/2}\mathbf{p}^*$$

Now

$${}^A\mathbf{a}^{P^*} \underset{(2.5.1)}{=} \frac{{}^Ad\,{}^A\mathbf{v}^{P^*}}{dt}$$

Hence

$$\frac{{}^Ad\,{}^A\mathbf{v}^{P^*}}{dt} = -k^2(\mathbf{p}^{*2})^{-3/2}\mathbf{p}^* \tag{A}$$

where

$$k^2 = gR^2 \tag{B}$$

and

$$g = \frac{Gm'}{R^2} \tag{C}$$

Cross multiplication of Eq. (A) with $\mathbf{p}^*$ shows that

$$\mathbf{p}^* \times \frac{{}^A d\,{}^A\mathbf{v}^{P^*}}{dt} = 0$$

Now

$$\frac{{}^A d}{dt}(\mathbf{p}^* \times {}^A\mathbf{v}^{P^*}) \underset{(1.5.3)}{=} \left(\frac{{}^A d\mathbf{p}^*}{dt}\right) \times {}^A\mathbf{v}^{P^*} + \mathbf{p}^* \times \frac{{}^A d\,{}^A\mathbf{v}^{P^*}}{dt}$$

$$\underset{(2.5.1)}{=} {}^A\mathbf{v}^{P^*} \times {}^A\mathbf{v}^{P^*} + \mathbf{p}^* \times \frac{{}^A d\,{}^A\mathbf{v}^{P^*}}{dt}$$

$$= \mathbf{p}^* \times \frac{{}^A d\,{}^A\mathbf{v}^{P^*}}{dt}$$

Hence

$$\frac{{}^A d}{dt}(\mathbf{p}^* \times {}^A\mathbf{v}^{P^*}) = 0$$

and (see 1.2.2) the vector $\mathbf{a}$, defined as

$$\mathbf{a} = \mathbf{p}^* \times {}^A\mathbf{v}^{P^*} \tag{D}$$

is independent of time t in reference frame A (see 1.1.1). As $\mathbf{a}$ is at all times perpendicular to $\mathbf{p}^*$, it follows that point P^* moves in a plane (or on a straight line) which passes through the earth's center and whose orientation in A is independent of time. Furthermore, Eq. (D) is valid for all values of t, so that the vector $\mathbf{a}$ can be found as soon as any pair of values $(\mathbf{p}_0^*, {}^A\mathbf{v}_0^{P^*})$ of $\mathbf{p}^*$ and ${}^A\mathbf{v}^{P^*}$ corresponding to the same value of t is known:

$$\mathbf{a} = \mathbf{p}_0^* \times {}^A\mathbf{v}_0^{P^*} \tag{E}$$

Next, cross-multiply Eq. (A) with $\mathbf{a}$:

$$\mathbf{a} \times \frac{{}^A d\,{}^A\mathbf{v}^{P^*}}{dt} = -k^2(\mathbf{p}^{*2})^{-3/2}\mathbf{a} \times \mathbf{p}^*$$

or, as $\mathbf{a}$ is independent of t in A,

$$\frac{{}^A d}{dt}(\mathbf{a} \times {}^A\mathbf{v}^{P^*}) = -k^2(\mathbf{p}^{*2})^{-3/2}\mathbf{a} \times \mathbf{p}^*$$

$$\underset{(D)}{=} -k^2(\mathbf{p}^{*2})^{-3/2}(\mathbf{p}^* \times {}^A\mathbf{v}^{P^*}) \times \mathbf{p}^*$$

$$= -\frac{{}^A d}{dt}\underset{(1.5.1,2.5.1)}{[k^2(\mathbf{p}^{*2})^{-1/2}\mathbf{p}^*]}$$

That is,

$$\frac{^A d}{dt}\left[\mathbf{a} \times {}^A\mathbf{v}^{P^*} + k^2(\mathbf{p}^{*2})^{-1/2}\mathbf{p}^*\right] = 0$$

from which it follows that the vector $\mathbf{b}$, defined as

$$\mathbf{b} = \mathbf{a} \times {}^A\mathbf{v}^{P^*} + k^2(\mathbf{p}^{*2})^{-1/2}\mathbf{p}^* \tag{F}$$

is independent of t in A. $\mathbf{b}$ can thus be found as soon as any pair of values $(\mathbf{p}_0^*, {}^A\mathbf{v}_0^{P^*})$ of $\mathbf{p}^*$ and ${}^A\mathbf{v}^{P^*}$ corresponding to the same value of t is known:

$$\mathbf{b} = \mathbf{a} \times {}^A\mathbf{v}_0^{P^*} + k^2(\mathbf{p}_0{}^{*2})^{-1/2}\mathbf{p}_0{}^* \tag{G}$$

Suppose, temporarily, that $\mathbf{a} \neq 0$. Then, as

$$\mathbf{a} \cdot {}^A\mathbf{v}^{P^*} \underset{(\mathrm{D})}{=} 0$$

the following is an identity:

$${}^A\mathbf{v}^{P^*} = \frac{(\mathbf{a} \times {}^A\mathbf{v}^{P^*}) \times \mathbf{a}}{\mathbf{a}}$$

Solve Eq. (F) for $\mathbf{a} \times {}^A\mathbf{v}^{P^*}$ and substitute:

$${}^A\mathbf{v}^{P^*} = \frac{\mathbf{b} - k^2(\mathbf{p}^{*2})^{-1/2}\mathbf{p}^*}{\mathbf{a}^2} \times \mathbf{a} \qquad (\mathbf{a} \neq 0) \tag{H}$$

Cross-multiply with $\mathbf{p}^*$:

$$\mathbf{p}^* \times {}^A\mathbf{v}^{P^*} \underset{(\mathrm{D})}{=} \mathbf{a} \underset{(\mathrm{H})}{=} \frac{\mathbf{p}^* \times (\mathbf{b} \times \mathbf{a}) - k^2(\mathbf{p}^{*2})^{-1/2}\mathbf{p}^* \times (\mathbf{p}^* \times \mathbf{a})}{\mathbf{a}^2}$$

Expand the right-hand member, keeping in mind that $\mathbf{p}^*$ is perpendicular to $\mathbf{a}$:

$$\mathbf{a} = \frac{-\mathbf{p}^* \cdot \mathbf{b}\mathbf{a} + k^2(\mathbf{p}^{*2})^{1/2}\mathbf{a}}{\mathbf{a}^2}$$

or

$$\mathbf{a}[k^2(\mathbf{p}^{*2})^{1/2} - \mathbf{p}^* \cdot \mathbf{b} - \mathbf{a}^2] = 0$$

As $\mathbf{a} \neq 0$, this implies that

$$k^2(\mathbf{p}^{*2})^{1/2} - \mathbf{p}^* \cdot \mathbf{b} - \mathbf{a}^2 = 0 \tag{I}$$

If $\mathbf{a} = 0$, Eq. (F) gives

$$\mathbf{b} = k^2(\mathbf{p}^{*2})^{-1/2}\mathbf{p}^*$$

which, after dot multiplication with $\mathbf{p}^*$, becomes

$$\mathbf{p}^* \cdot \mathbf{b} = k^2(\mathbf{p}^{*2})^{1/2}$$

As this is precisely the relationship to which Eq. (I) reduces when $\mathbf{a} = 0$, Eq. (I) is, in fact, valid for all values of $\mathbf{a}$. Equation (H), on the other hand, leads to an indeterminate expression for ${}^A\mathbf{v}^{P^*}$ when $\mathbf{a} = 0$. Hence, an alternative expression, applicable when $\mathbf{a} = 0$, is required.

Equation (F) shows that $\mathbf{p}^*$ is parallel to $\mathbf{b}$ for all values of t when $\mathbf{a} = 0$. As the orientation of $\mathbf{b}$ in A is independent of t, this means that P^* moves on a straight line fixed in A and passing through O. $\mathbf{p}^*$ can therefore be expressed as

$$\mathbf{p}^* = x\mathbf{n}^*$$

where $\mathbf{n}^*$ (see Fig. 4.2.7c) is a unit vector fixed in A and pointing from O toward P^*, and x is an intrinsically positive scalar, because

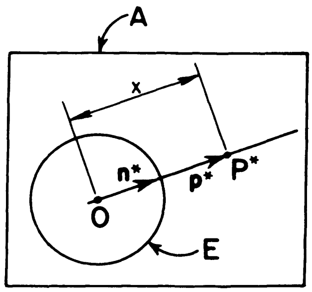


FIG. 4.2.7c

P^* is prevented by E from ever reaching O.

${}^A\mathbf{v}^{P^*}$ is now given by

$$ {}^A\mathbf{v}^{P^*} \underset{(2.5.1)}{=} \frac{dx}{dt}\mathbf{n}^* $$

and

$$ \frac{{}^Ad\,{}^A\mathbf{v}^{P^*}}{dt} = \frac{d^2x}{dt^2}\mathbf{n}^* $$

Hence, with

$$\mathbf{p}^* = x\mathbf{n}^*$$

Eq. (A) gives

$$ \frac{d^2x}{dt^2} = -\frac{k^2}{x^2} $$

and, multiplying with dx/dt,

$$ \frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{dx}{dt}\frac{k^2}{x^2} $$

But

$$\frac{dx}{dt}\frac{d^2x}{dt^2} = \frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2\right]$$

and

$$-\frac{dx}{dt}\frac{k^2}{x^2} = \frac{d}{dt}\left(\frac{k^2}{x}\right)$$

Thus

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{k^2}{x}\right] = 0$$

and

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{k^2}{x} = \frac{1}{2}\left(\frac{dx}{dt}\right)_0^2 - \frac{k^2}{x_0}$$

where $(dx/dt)_0$ and x_0 denote the values of dx/dt and x for a particular value of t.

Solve for dx/dt and multiply with $\mathbf{n}^*$:

$$^{A}\mathbf{v}^{P^*} = \pm\left[\left(\frac{dx}{dt}\right)_0^2 + 2k^2\left(\frac{1}{x} - \frac{1}{x_0}\right)\right]^{1/2}\mathbf{n}^* \tag{J}$$

This relationship replaces Eq. (H) when $\mathbf{a} = 0$.

Equation (I) (together with Eqs. (E) and (G)) can be used to study the curves on which P^* moves in reference frame A. For this purpose it is convenient to rewrite the equation as

$$|\mathbf{p}^*| - \mathbf{p}^* \cdot \boldsymbol{\epsilon} - L = 0 \tag{K}$$

where L and $\boldsymbol{\epsilon}$ are defined as

$$L = \frac{\mathbf{a}^2}{k^2} \tag{L}$$

and

$$\boldsymbol{\epsilon} = \frac{\mathbf{b}}{k^2} \tag{M}$$

L is called the *semi-latus-rectum*, and $\boldsymbol{\epsilon}$ the *vector eccentricity*, of any curve defined by Eq. (K). The magnitude of $\boldsymbol{\epsilon}$ will henceforth be denoted by the symbol ϵ.

Note that L and ϵ may have any values greater than or equal to zero, and that, with one exception, the values of L and ϵ are independent of each other. The exception is that (see Eqs. (E) and (G))

$$L = 0 \text{ implies } \epsilon = 1 \tag{N}$$

(but not vice versa).

With $\epsilon = 0$, Eq. (K) reduces to

$$|\mathbf{p}^*| = L$$

which shows that P^* moves on a circle of radius L, center at O.

When $L = 0$,

$$|\mathbf{p}^*| \underset{(K)}{=} \mathbf{p}^* \cdot \boldsymbol{\epsilon}$$

where (see Eq. (N)) $\boldsymbol{\epsilon}$ is a unit vector. Hence $\mathbf{p}^*$ and $\boldsymbol{\epsilon}$ have the same direction; that is, P^* moves on the semi-infinite straight line shown in Fig. 4.2.7d.

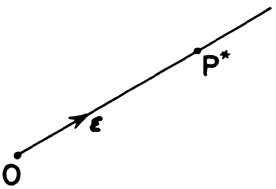


FIG. 4.2.7d

When both L and ϵ are greater than zero, a line perpendicular to $\boldsymbol{\epsilon}$ and passing through the point whose position vector relative to O is the vector $-L\boldsymbol{\epsilon}/\epsilon^2$ can be constructed, and the distances $\overline{OP^*}$ and $\overline{P^*Q}$ (see Fig. 4.2.7e) are then given by

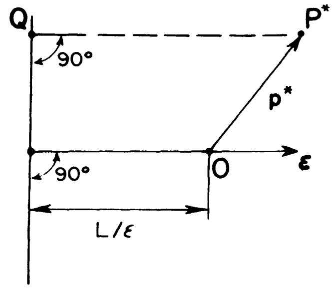


FIG. 4.2.7e

$$\overline{OP^*} = |\mathbf{p}^*|, \qquad \overline{P^*Q} = \frac{L}{\epsilon} + \mathbf{p}^* \cdot \frac{\boldsymbol{\epsilon}}{\epsilon}$$

But

$$|\mathbf{p}^*| \underset{(K)}{=} L + \mathbf{p}^* \cdot \boldsymbol{\epsilon}$$

Hence

$$\overline{OP^*} = \epsilon\overline{P^*Q}$$

and the curve on which P moves in A is an ellipse when $0 < \epsilon < 1$, a parabola when $\epsilon = 1$, and a hyperbola when $\epsilon > 1$. These curves and their orientations relative to O and ϵ are shown in Figs. 4.2.7f, g, h, and a summary of results is given in Table 4.2.7.

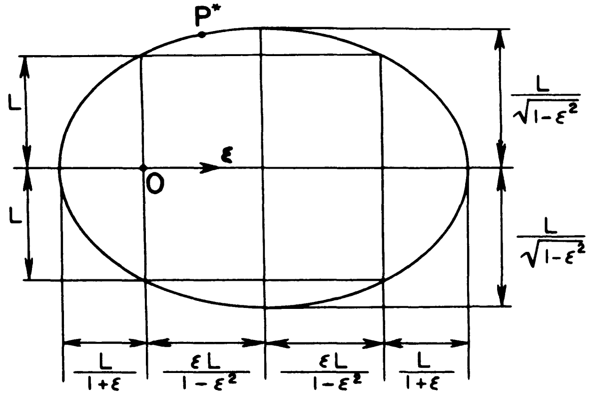


FIG. 4.2.7f

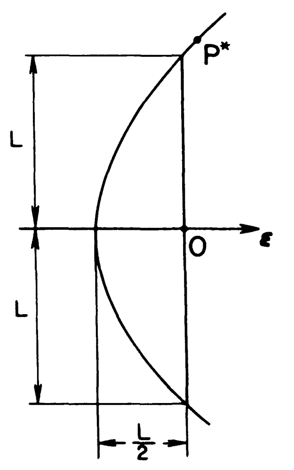


FIG. 4.2.7g

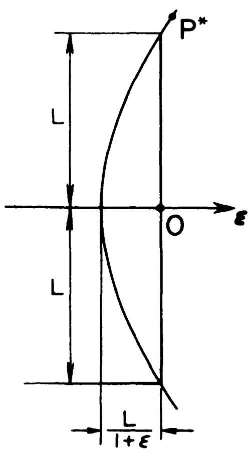


FIG. 4.2.7h

A complete description of the motion of P^* cannot be given unless one knows not only the curve C on which P^* moves, but

TABLE 4.2.7

ϵ	L	Curve	Figure
$\epsilon = 0$	$L > 0$	Circle	—
$0 < \epsilon < 1$	$L > 0$	Ellipse	4.2.7f
$\epsilon = 1$	$L > 0$	Parabola	4.2.7g
$\epsilon = 1$	$L = 0$	Straight line	4.2.7d
$\epsilon > 1$	$L > 0$	Hyperbola	4.2.7h

also the relationship between the time t and the position of P^* on C. This is obtained as follows:

Given a pair of values $(\mathbf{p}_0^*, {}^A\mathbf{v}_0^{P^*})$ of $\mathbf{p}$ and ${}^A\mathbf{v}^{P^*}$, corresponding to the same value of t, find $\mathbf{a}$, $\mathbf{b}$, L, and $\boldsymbol{\epsilon}$ (see Eqs. (E), (G), (L), (M)). Then, provided $L \neq 0$, $\mathbf{p}^*$ remains in a plane perpendicular to $\mathbf{a}$, and it is possible to describe the motion of P^* in terms of polar coordinates p^* and θ, where p^* is the distance between O and P^* (see Fig. 4.2.7i) and θ is the angular displacement of line OP^* relative to a line which passes through O and is parallel to $\boldsymbol{\epsilon}$, θ being regarded as positive when the displacement is generated by an $\mathbf{a}$ rotation of $\mathbf{p}^*$ relative to this line. If $\mathbf{n}^*$ is a unit vector directed from O toward $\mathbf{p}^*$, then

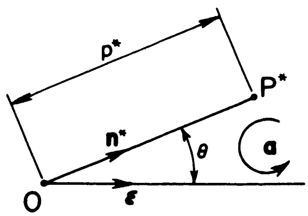


FIG. 4.2.7i

$$\mathbf{p}^* = p^*\mathbf{n}^* \tag{O}$$

and

$${}^A\mathbf{v}^{P^*} \underset{(2.5.1)}{=} \frac{{}^A d\mathbf{p}^*}{dt} \underset{(1.5.1)}{=} \frac{dp^*}{dt}\,\mathbf{n}^* + p^*\,\frac{\mathbf{a}}{|\mathbf{a}|} \underset{(1.7.1)}{\times} \mathbf{n}^*\,\frac{d\theta}{dt}$$

so that

$$\mathbf{a} \underset{(D)}{=} \mathbf{p}^* \times {}^A\mathbf{v}^{P^*} = \frac{dp^*}{dt}\,\mathbf{p}^* \times \mathbf{n}^* + \frac{p^*\mathbf{p}^* \times (\mathbf{a} \times \mathbf{n}^*)}{|\mathbf{a}|}\frac{d\theta}{dt}$$

$$= 0 + \frac{p^{*2}}{|\mathbf{a}|}\frac{d\theta}{dt}\,\mathbf{a}$$

or (as $\mathbf{a} \neq 0$)

$$\frac{p^{*2}}{|\mathbf{a}|}\frac{d\theta}{dt} = 1 \tag{P}$$

Now, it is always possible to choose θ such that

$$p^* - p^*\epsilon\cos\theta - L \underset{(K,O)}{=} 0$$

or

$$p^* = \frac{L}{1 - \epsilon\cos\theta}$$

And

$$|\mathbf{a}| \underset{(L)}{=} k\sqrt{L}$$

Hence

$$\frac{dt}{d\theta} \underset{(P)}{=} \frac{L^{3/2}}{k(1 - \epsilon\cos\theta)^2}$$

and

$$t_2 - t_1 = \frac{L^{3/2}}{k}\int_{\theta_1}^{\theta_2} \frac{d\theta}{(1 - \epsilon\cos\theta)^2} = \frac{L^{3/2}}{k}[F(\theta_2) - F(\theta_1)] \tag{Q}$$

where θ_1 and θ_2 are the values of θ when t is equal to t_1 and t_2, respectively, and, as may be verified by differentiation, the function $F(\theta)$ is described as follows:

For $0 \leqslant \epsilon < 1$,

$$F(\theta) = \frac{\epsilon S(\theta) + \arcsin S(\theta)}{(1 - \epsilon^2)^{3/2}} \tag{R}$$

where

$$S(\theta) = \frac{(1 - \epsilon^2)^{1/2}\sin\theta}{1 - \epsilon\cos\theta} \tag{S}$$

For $\epsilon = 1$, $L > 0$,

$$F(\theta) = \frac{1 + \cos\theta + \sin^2\theta}{3\sin\theta(\cos\theta - 1)}$$

For $\epsilon > 1$,

$$F(\theta) = \frac{i\epsilon S(\theta) - \log\left[\dfrac{\epsilon - \cos\theta}{1 - \epsilon\cos\theta} + iS(\theta)\right]}{(\epsilon^2 - 1)^{3/2}}, \quad i = \sqrt{-1}$$

To obtain the corresponding expressions for the case $L = 0$, note that

$$\frac{dx}{dt} \underset{(J)}{=} \pm\left[\frac{2k^2}{x} + \left(\frac{dx}{dt}\right)_0^2 - \frac{2k^2}{x_0}\right]^{1/2}$$

so that

$$t_2 - t_1 = \pm \int_{x_1}^{x_2} \frac{dx}{[(2k^2/x) + (dx/dt)_0^2 - (2k^2/x_0)]^{1/2}}$$
$$= \pm[G(x_2) - G(x_1)]$$

where x_1 and x_2 are the values of x when t is equal to t_1 and t_2, respectively, and the function $G(x)$ is defined as

$$G(x) = \frac{\sqrt{2k^2x + cx^2}}{c} - \frac{k^2}{c^{3/2}} \log \left(\sqrt{2k^2x + cx^2} + x\sqrt{c} + \frac{k^2}{\sqrt{c}} \right)$$

with

$$c = \left(\frac{dx}{dt}\right)_0^2 - \frac{2k^2}{x_0}$$

Problem: The perigee and apogee of an earth satellite's orbit are the points nearest to and furthest from the earth's center. The corresponding distances d and D between the center of the earth and the satellite's mass center are called the *perigee distance* and the *apogee distance*.

Express the time T required for one passage of a satellite from apogee to perigee in terms of d, D, and k (see Eq. (B)).

Solution: In Fig. 4.2.7j, constructed by reference to Fig. 4.2.7f, the sense of ${}^A\mathbf{v}^{P^*}$ has been chosen arbitrarily. For this choice of

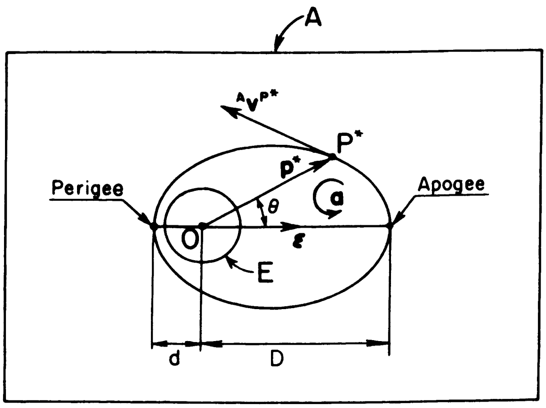


FIG. 4.2.7j

${}^{A}\mathbf{v}^{P^*}$, $\mathbf{a}$ must, in accordance with Eq. (D), have the sense shown in the figure, and θ must be regarded as positive.

Let θ_1 and θ_2 be the values of θ corresponding to $|\mathbf{p}^*| = D$ and $|\mathbf{p}^*| = d$. Then

$$\theta_1 = 0, \qquad \theta_2 = \pi$$

and

$$F(\theta_1) = F(0) \underset{(\mathrm{R})}{=} 0$$

while

$$F(\theta_2) = F(\pi)$$

From Eq. (R)

$$F(\pi) = \frac{\epsilon S(\pi) + \arcsin S(\pi)}{(1 - \epsilon^2)^{3/2}}$$

and from Eq. (S)

$$S(\pi) = \frac{(1 - \epsilon^2)^{1/2} \sin \pi}{1 - \epsilon \cos \pi} = 0$$

Hence

$$F(\pi) = \frac{\pi}{(1 - \epsilon^2)^{3/2}}$$

and, letting t_1 and t_2 be the values of t corresponding to θ_1 and θ_2,

$$T = t_2 - t_1 \underset{(\mathrm{Q})}{=} \frac{\pi}{k} \left(\frac{L}{1 - \epsilon^2} \right)^{3/2}$$

From Figs. 4.2.7f and 4.2.7j,

$$d = \frac{L}{1 + \epsilon}$$

and

$$D = \frac{2\epsilon L}{1 - \epsilon^2} + \frac{L}{1 + \epsilon} = \frac{L}{1 - \epsilon}$$

or, solving for L and ϵ,

$$L = \frac{2dD}{D + d}, \qquad \epsilon = \frac{D - d}{D + d}$$

Consequently

$$T = \frac{\pi}{k}\left(\frac{d + D}{2}\right)^{3/2}$$

4.2.8 Astronomical observations show that there exist bodies (for example, the planets) whose motions in reference frame A (see 4.2.3–4.2.7) conflict with the hypothesis that A is a Newtonian reference frame. To obtain descriptions of these motions analytically, one regards as Newtonian a reference frame A' in which the sun is fixed, the earth's center (that is, a point fixed in A) moves on a certain elliptical path (described once per year), and the angular velocity of A (${}^{A'}\omega^{A}$) is equal to zero. Re-examination of the motions discussed in Secs. 4.2.3–4.2.7 then reveals that the results there obtained remain essentially unaltered because the inertia forces which must now be added are small in comparison with forces already considered. A' is therefore a better approximation to a Newtonian reference frame than is either A or E.

These considerations lead to two conclusions: (1) The search for a truly Newtonian reference frame requires exploration of motions taking place at ever greater distances from the earth. (2) Whether or not a given reference frame may be regarded as Newtonian for the purpose of solving a particular problem depends on the accuracy required of the solution.

Except where indicated otherwise, the earth E is regarded as a Newtonian reference frame throughout the sections which follow. Other reference frames are regarded as Newtonian if and only if their motions relative to E satisfy the requirements set forth in Sec. 4.2.2, and the word "fixed" means "at rest in a Newtonian reference frame."

4.3 Motions of rigid bodies

4.3.1 Infinitely many equations of motion (see 4.2.3) can be written in order to describe any motion of a rigid body. However, at most six such (scalar) equations are both independent of each other and nontrivial (see Vol. I, Sec. 3.6.10). To obtain these six (or, in many cases, a smaller number) with a minimum amount of labor, it is necessary to use those properties of a zero system which permit one to take advantage of specific features of the problem under consideration.

Problem: Referring to Example 2.2.10, and supposing that P is fixed and horizontal, that the disc D is uniform, has a mass m and radius r, and that D is rolling on P, find three equations governing θ, ϕ, and ψ, and use these to discuss the stability of a spinning disc.

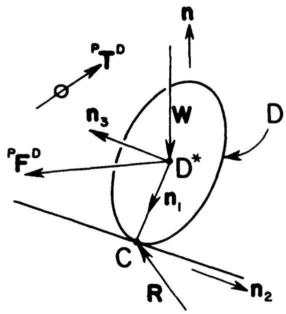


FIG. 4.3.1a

Solution: The vectors appearing in the free-body diagram of D, Fig. 4.3.1a, are described as follows:

$\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, and $\mathbf{n}$ are unit vectors which were described in Example 2.2.10. $\mathbf{W}$ represents the system of all gravitational forces acting on D, and has a magnitude mg. $\mathbf{R}$ represents the reaction (see Vol. I, Sec. 4.6.1) of P on D, and has a line of action passing through point C. (The basis for representation of this reaction by means of a single force is discussed in Vol. I, Sec. 4.8.3.) ${}^P\mathbf{F}^D$ and ${}^P\mathbf{T}^D$ are the inertia force and the torque of the inertia couple acting on D in reference frame P. (Expressions for these in terms of m, r, θ, ϕ, ψ were obtained in Problem 4.1.5.)

To find the desired equations, set the moment about C of all forces acting on D equal to zero, noting that the unknown force $\mathbf{R}$ is thus eliminated from further consideration:

$$(\mathbf{W} + {}^P\mathbf{F}^D) \times (r\mathbf{n}_1) + {}^P\mathbf{T}^D = 0$$

or, writing the corresponding scalar equations of motion,

$$\left.\begin{aligned} \ddot{\phi}\cos\theta + 2\dot{\theta}\dot{\psi} &= 0 \\ 5\ddot{\theta} - 6\dot{\phi}\dot{\psi}\cos\theta - 5\dot{\phi}^2\sin\theta\cos\theta - \frac{4g}{r}\sin\theta &= 0 \\ 3(\ddot{\psi} + \ddot{\phi}\sin\theta) + 5\dot{\phi}\dot{\theta}\cos\theta &= 0 \end{aligned}\right\} \quad \text{(A)}$$

The disc is said to be "spinning" when

$$\left.\begin{aligned} \theta &= 0 \\ \phi &= \phi_0 + \dot{\phi}_0 t \\ \psi &= \psi_0 \end{aligned}\right\} \qquad \text{(B)}$$

where all symbols with the subscript zero denote constants. In particular, $\dot{\phi}_0$ gives the angular speed (for the direction **n**) with which D is spinning. That Eqs. (B) describe a motion which satisfies Eqs. (A) may be verified by substitution.

A motion described by

$$\left.\begin{aligned} \theta &= \theta^* \\ \phi &= \phi_0 + \dot{\phi}_0 t + \phi^* \\ \psi &= \psi_0 + \psi^* \end{aligned}\right\} \qquad \text{(C)}$$

where θ^*, ϕ^*, ψ^* are functions of t, differs only slightly from that of spinning, provided θ^*, ϕ^*, ψ^* remain sufficiently small. Hence the motion of spinning described by Eqs. (B) is said to be *stable* if and only if there exist nonzero functions θ^*, ϕ^*, ψ^* which remain arbitrarily small and which, when θ, ϕ, ψ as given in Eqs. (C) are substituted into Eqs. (A), permit Eqs. (A) to be satisfied, at least approximately.

Substitute from Eqs. (C) into Eqs. (A), expressing $\sin\theta^*$ and $\cos\theta^*$ in terms of powers of θ^*:

$$\ddot{\phi}^*\left(1 - \frac{\theta^{*2}}{2} + \ldots\right) + 2\dot{\theta}^*\dot{\psi}^* = 0$$

$$\begin{aligned} 5\ddot{\theta}^* - 6(\dot{\phi}_0 + \dot{\phi}^*)\dot{\psi}^*\left(1 - \frac{\theta^{*2}}{2} + \ldots\right) \\ - 5(\dot{\phi}_0 + \dot{\phi}^*)^2\left(\theta^* - \frac{\theta^{*3}}{3!} + \ldots\right)\left(1 - \frac{\theta^{*2}}{2} + \ldots\right) \\ - \frac{4g}{r}\left(\theta^* - \frac{\theta^*}{3!} + \ldots\right) = 0 \end{aligned}$$

$$\begin{aligned} 3\left[\ddot{\psi}^* + \ddot{\phi}^*\left(\theta^* - \frac{\theta^{*3}}{3!} + \ldots\right)\right] \\ + 5(\dot{\phi}_0 + \dot{\phi}^*)\dot{\theta}^*\left(1 - \frac{\theta^{*2}}{2} + \ldots\right) = 0 \end{aligned}$$

Drop all terms of second or higher degree in the starred quantities:

$$\ddot{\phi}^* = 0 \tag{D}$$

$$5\ddot{\theta}^* - 6\dot{\phi}_0\dot{\psi}^* - \left(5\dot{\phi}_0^2 + \frac{4g}{r}\right)\theta^* = 0 \tag{E}$$

$$3\ddot{\psi}^* + 5\dot{\phi}_0\dot{\theta}^* = 0 \tag{F}$$

Equation (D) is satisfied by

$$\phi^* = \phi_0^* + \dot{\phi}_0^* t$$

where ϕ_0^* and $\dot{\phi}_0^*$ are constants. Hence ϕ^* can be kept arbitrarily small by taking $\dot{\phi}_0^* = 0$ and assigning a sufficiently small value to ϕ_0^*. Equations (E) and (F) may be solved simultaneously. The latter is satisfied whenever

$$3\dot{\psi}^* + 5\dot{\phi}_0\theta^* = C, \text{ const.}$$

and, solving for $\dot{\psi}^*$ and substituting into Eq. (E), θ^* is seen to be governed by

$$5\ddot{\theta}^* + \left(5\dot{\phi}_0^2 - \frac{4g}{r}\right)\theta^* = 2C\dot{\phi}_0$$

Now, unless

$$5\dot{\phi}_0^2 - \frac{4g}{r} > 0$$

the solution of this equation contains functions which increase either linearly or exponentially with time. It follows that a motion of spinning is stable if and only if

$$\dot{\phi}_0 > 2\sqrt{\frac{g}{5r}}$$

Note that the solution of this problem requires the use of only three (scalar) equations of motion, although five quantities govern the "configuration" of the body under consideration, that is, the position and orientation of the disc. The reason for this is that the analysis is restricted to motions of rolling, and during such motions the acceleration of the center of the disc depends solely on the three functions θ, ϕ, and ψ (see Problem 2.5.10), as a consequence of which the inertia force ${}^{P}\mathbf{F}^{D}$ (see Problem 4.1.5) also depends on only these three quantities. When this restriction is removed, additional unknown functions enter the discussion and more equations of motion are required. This is illustrated by the example which follows.

Example: A spherical body S of mass m and radius r, whose mass density at any point depends only on the distance from the point to the center S^* of S, moves in such a way that it remains in contact with a fixed horizontal plane P. S has a radius of gyration k with respect to any line passing through S^*, and the coefficient of friction for S and P has the value μ. A complete description of all possible motions of S is to be obtained.†

In the free-body diagram of S shown in Fig. 4.3.1b, $\mathbf{n}$ is a unit vector pointing vertically upward. $\mathbf{R}$ represents the reaction of P on S and has a line of action passing through point C, and $\mathbf{F}^S$ and $\mathbf{T}^S$ are the inertia force and the torque of the inertia couple acting on S.

In accordance with Sec. 4.2.3, the resultant of the system of forces represented in Fig. 4.3.1b, and the moment of this system

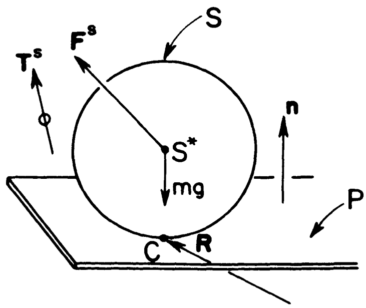


FIG. 4.3.1b

of forces about point C, are each equal to zero:

$$\mathbf{F}^S + \mathbf{R} - mg\mathbf{n} = 0 \tag{A}$$

$$r\mathbf{n} \times \mathbf{F}^S + \mathbf{T}^S = 0 \tag{B}$$

† The analysis which follows is similar to that given in "Principles of Mechanics," by John L. Synge and Byron A. Griffith, McGraw-Hill Book Co. Inc., pp. 447–450, 1949.

$\mathbf{F}^S$ may be expressed as

$$\mathbf{F}^S \underset{(4.1.3)}{=} -m\mathbf{a}^{S^*} \underset{(2.5.1)}{=} -m\frac{d\mathbf{v}^{S^*}}{dt} \tag{C}$$

and $\mathbf{T}^S$ as

$$\mathbf{T}^S \underset{(4.1.3)}{=} -(\mathbf{n}_a \cdot \boldsymbol{\alpha}^S \boldsymbol{\Phi}_a^{S/S^*} + \mathbf{n}_o \cdot \boldsymbol{\omega}^S \boldsymbol{\omega}^S \times \boldsymbol{\Phi}_o^{S/S^*}) \tag{D}$$

As every direction is a principal direction of S for S^*, $\boldsymbol{\Phi}_o^{S/S^*}$ is parallel to $\mathbf{n}_o$ (see 3.5.6), that is, to $\boldsymbol{\omega}^S$, so that

$$\boldsymbol{\omega}^S \times \boldsymbol{\Phi}_o^{S/S^*} = 0 \tag{E}$$

and $\boldsymbol{\Phi}_a^{S/S^*}$ is parallel to $\mathbf{n}_a$ and can be expressed as

$$\boldsymbol{\Phi}_a^{S/S^*} \underset{(3.5.5,3.3.2)}{=} mk^2\mathbf{n}_a \tag{F}$$

Thus

$$\mathbf{T}^S \underset{(D,E,F)}{=} -\mathbf{n}_a \cdot \boldsymbol{\alpha}^S mk^2\mathbf{n}_a$$

or, as $\mathbf{n}_a \cdot \boldsymbol{\alpha}^S\, \mathbf{n}_a$ is equal to $\boldsymbol{\alpha}^S$ and

$$\boldsymbol{\alpha}^S \underset{(2.3.1)}{=} \frac{d\boldsymbol{\omega}^S}{dt}$$

$\mathbf{T}^S$ is given by

$$\mathbf{T}^S = -mk^2\frac{d\boldsymbol{\omega}^S}{dt} \tag{G}$$

Hence

$$-m\frac{d\mathbf{v}^{S^*}}{dt} + \mathbf{R} - mg\mathbf{n} \underset{(A,C)}{=} 0 \tag{H}$$

and

$$r\mathbf{n} \times \frac{d\mathbf{v}^{S^*}}{dt} + k^2\frac{d\boldsymbol{\omega}^S}{dt} \underset{(B,C,G)}{=} 0$$

are two (vector) equations of motion governing all motions of S. The second of these requires that

$$r\mathbf{n} \times \mathbf{v}^{S^*} + k^2\boldsymbol{\omega}^S = \mathbf{b} \tag{I}$$

where $\mathbf{b}$ is a fixed vector, which can be found as soon as any pair of values of $\mathbf{v}^S$ and $\boldsymbol{\omega}^S$ corresponding to the same value of t is known.

The velocity $\mathbf{v}^C$ of the point C of S which is in contact with P is given by

$$\mathbf{v}^C \underset{(2.5.9)}{=} \mathbf{v}^{S^*} + \boldsymbol{\omega}^S \times (-r\mathbf{n}) \tag{J}$$

or, solving Eq. (I) for $\boldsymbol{\omega}^S$ and substituting, by

$$\mathbf{v}^C = \mathbf{v}^{S*} + \frac{r\mathbf{n}}{k^2} \times (\mathbf{b} - r\mathbf{n} \times \mathbf{v}^{S*})$$

$$= \mathbf{v}^{S*} + \frac{r}{k^2} (\mathbf{n} \times \mathbf{b} - r\mathbf{n} \cdot \mathbf{v}^{S*}\mathbf{n} + r\mathbf{v}^{S*})$$

But $\mathbf{v}^{S*}$ is perpendicular to $\mathbf{n}$ so long as S remains in contact with P. Consequently

$$\mathbf{n} \cdot \mathbf{v}^{S*} = 0$$

and

$$\mathbf{v}^C = \left(1 + \frac{r^2}{k^2}\right) \mathbf{v}^{S*} + \frac{r}{k^2} \mathbf{n} \times \mathbf{b} \tag{K}$$

The reaction force $\mathbf{R}$ can be resolved into two components, one ($\mathbf{N}$) parallel to $\mathbf{n}$, the other ($\mathbf{F}$) perpendicular to $\mathbf{n}$. That is,

$$\mathbf{R} = \mathbf{N} + \mathbf{F}$$

Hence

$$-m \frac{d\mathbf{v}^{S*}}{dt} + \mathbf{N} + \mathbf{F} - mg\mathbf{n} \underset{\text{(H)}}{=} 0$$

As $d\mathbf{v}^{S*}/dt$ is at all times perpendicular to $\mathbf{n}$, this equation implies that

$$\mathbf{N} = mg\mathbf{n} \tag{L}$$

and that

$$\mathbf{F} = m \frac{d\mathbf{v}^{S*}}{dt} \tag{M}$$

Two possibilities now present themselves: At any instant, $\mathbf{v}^C$ is either equal to zero, which means that S is rolling on P, or $\mathbf{v}^C$ is not equal to zero, in which case the laws of friction (see Vol. I, Sec. 4.1.12) require that

$$\mathbf{F} = -\mu|\mathbf{N}|\mathbf{n}^C \tag{N}$$

where $\mathbf{n}^C$ is a unit vector having the same direction as $\mathbf{v}^C$, and

$$\underset{\text{(M)}}{\frac{d\mathbf{v}^{S*}}{dt}} \underset{\text{(N)}}{=} -\underset{\text{(L)}}{\mu g}\mathbf{n}^C \tag{O}$$

Considering first the case of rolling, that is, $\mathbf{v}^C = 0$,

$$\mathbf{v}^{S*} \underset{\text{(K)}}{=} \frac{r}{r^2 + k^2} \mathbf{b} \times \mathbf{n} \tag{P}$$

which shows that the center of S moves with constant speed on a straight line, because $\mathbf{b} \times \mathbf{n}$ is a fixed vector. (It follows from Eq. (I) that $\boldsymbol{\omega}^S$ also remains constant.)

Next, when $\mathbf{v}^C$ is not equal to zero,

$$\mathbf{v}^C = v^C \mathbf{n}^C$$

where v^C is a scalar function of time and $\mathbf{n}^C$ is the unit vector introduced in Eq. (N). Thus

$$v^C \mathbf{n}^C \underset{(\mathrm{K})}{=} \left(1 + \frac{r^2}{k^2}\right) \mathbf{v}^{S*} + \frac{r}{k^2} \mathbf{n} \times \mathbf{b}$$

To eliminate $\mathbf{v}^{S*}$ from this equation, differentiate with respect to t and use Eq. (O):

$$\frac{dv^C}{dt} \mathbf{n}^C + v^C \frac{d\mathbf{n}^C}{dt} = -\mu g \left(1 + \frac{r^2}{k^2}\right) \mathbf{n}^C$$

As $d\mathbf{n}^C/dt$ is always perpendicular to $\mathbf{n}^C$ (because $\mathbf{n}^C$ is a unit vector), it appears that

$$\frac{d\mathbf{n}^C}{dt} = 0$$

which implies

$$\mathbf{n}^C = \mathbf{n}_0^C$$

where $\mathbf{n}_0^C$ is a fixed unit vector. Equation (O) thus becomes

$$\frac{d\mathbf{v}^{S*}}{dt} = -\mu g \mathbf{n}_0^C$$

whence

$$\mathbf{v}^{S*} = \mathbf{v}_0^{S*} - \mu g(t - t_0)\mathbf{n}_0^C \tag{Q}$$

where $\mathbf{v}_0^{S*}$ is the value of $\mathbf{v}^{S*}$ at time t_0. And if $\mathbf{p}$ is the position vector of S^* relative to any fixed point O, it follows from the relationship

$$\mathbf{v}^{S*} \underset{(2.5.1)}{=} \frac{d\mathbf{p}}{dt}$$

together with Eq. (Q) that

$$\mathbf{p} = \mathbf{p}_0 + (t - t_0)\mathbf{v}_0^{S*} - \frac{\mu g}{2} (t - t_0)^2 \mathbf{n}_0^C \tag{R}$$

where $\mathbf{p}_0$ is the value of $\mathbf{p}$ at time t_0. (Note that Eq. (R) is the vector equation of a parabola when $\mathbf{n}_0^C$ is not parallel to $\mathbf{v}_0^{S*}$.)

The following is a complete description of the motion of S^* subsequent to an instant t_0 at which S has an angular velocity $\boldsymbol{\omega}_0^S$, S^* has a velocity $\mathbf{v}_0^{S^*}$, and the position vector of S^* relative to a fixed point O has the value $\mathbf{p}_0$:

If the velocity $\mathbf{v}_0^C$ of C at time t_0, given by

$$\mathbf{v}_0^C \underset{(\mathrm{J})}{=} \mathbf{v}_0^{S^*} + r\mathbf{n} \times \boldsymbol{\omega}_0^S \tag{S}$$

is not equal to zero, S^* begins to move on the parabola (or straight line, if $\mathbf{n}_0^C$ is parallel to $\mathbf{v}_0^{S^*}$) defined by

$$\mathbf{p} \underset{(\mathrm{R})}{=} \mathbf{p}_0 + (t - t_0)\mathbf{v}_0^{S^*} - \frac{\mu g}{2}(t - t_0)^2\,\mathbf{n}_0^C \tag{T}$$

where $\mathbf{n}_0^C$ is a unit vector in the direction of $\mathbf{v}_0^C$. During this motion S^* has a velocity

$$\mathbf{v}^{S^*} \underset{(\mathrm{Q})}{=} \mathbf{v}_0^{S^*} - \mu g(t - t_0)\mathbf{n}_0^C \tag{U}$$

and the motion persists until the instant t_1 at which $\mathbf{v}^{S^*}$ attains the value $\mathbf{v}_1^{S^*}$ defined as

$$\mathbf{v}_1^{S^*} \underset{(\mathrm{P})}{=} \frac{r}{r^2 + k^2}\,\mathbf{b} \times \mathbf{n} \tag{V}$$

where

$$\mathbf{b} \underset{(\mathrm{I})}{=} r\mathbf{n} \times \mathbf{v}_0^{S^*} + k^2\boldsymbol{\omega}_0^S \tag{W}$$

The time interval $t_1 - t_0$ is given by

$$t_1 - t_0 \underset{(\mathrm{U})}{=} \frac{(\mathbf{v}_0^{S^*} - \mathbf{v}_1^{S^*}) \cdot \mathbf{n}_0^C}{\mu g} \tag{X}$$

Subsequent to t_1, S^* moves with velocity $\mathbf{v}_1^{S^*}$ on the straight line parallel to $\mathbf{v}_1^{S^*}$ and passing through the point whose position vector $\mathbf{p}_1$ relative to O is

$$\mathbf{p}_1 \underset{(\mathrm{T})}{=} \mathbf{p}_0 + (t_1 - t_0)\mathbf{v}_0^{S^*} - \frac{\mu g}{2}(t_1 - t_0)^2\,\mathbf{n}_0^C \tag{Y}$$

(During both phases of the motion the angular velocity $\boldsymbol{\omega}^S$ has the value

$$\boldsymbol{\omega}^S \underset{(\mathrm{I,W})}{=} \boldsymbol{\omega}_0^S + r\mathbf{n} \times (\mathbf{v}_0^{S^*} - \mathbf{v}^{S^*}) \tag{Z}$$

where $\mathbf{v}^{S^*}$ is given by Eq. (U) prior to t_1 and is equal to $\mathbf{v}_1^{S^*}$ at and subsequent to t_1.)

If $\mathbf{v}_0^C$ (see Eq. (S)) is equal to zero, S^* moves with velocity $\mathbf{v}_0^{S^*}$ on the straight line parallel to $\mathbf{v}_0^{S^*}$ and passing through the position of S^* at time t_0. ($\boldsymbol{\omega}^S$ then remains equal to $\boldsymbol{\omega}_0^S$ throughout the motion.)

4.3.2 When all gravitational forces acting on a body B (any collection of matter) are known, and the motion of B in a Newtonian reference frame is specified, D'Alembert's principle (see 4.2.1 and 4.2.3) permits one to reduce the system of *all* contact forces acting on B to a known force and a couple of known torque. However, the principle does not furnish detailed descriptions of individual contact forces exerted by one body on another. It leads only to descriptions of force systems (called "reactions") equivalent to systems of such contact forces. Furthermore, this principle, alone, never yields sufficient information for the determination of more than one reaction (see Vol. I, Sec. 4.8).

Example: Figure 4.3.2a shows a rigid body R which consists of a rigid body R' and a portion S of a rigid, cylindrical shaft. The shaft is supported by bearings, and S is that part of it which lies between two planes normal to the shaft's axis and passing through the points A and B.

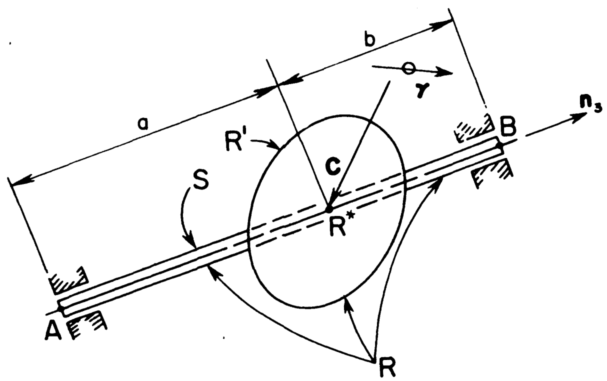


FIG. 4.3.2a

A system of known contact forces is applied to R, and this system is equivalent to a force $\mathbf{C}$, whose line of action passes through the mass center R^* of R, together with a couple of torque $\boldsymbol{\gamma}$.

The reactions of the bearings on the shaft are to be determined for a motion during which R^* remains at rest on line AB while R has an angular speed ω and scalar angular acceleration α for the direction $\mathbf{n}_3$.

In Fig. 4.3.2b, which is a free-body diagram of R, $\mathbf{W}^R$ represents the system of all gravitational forces acting on R. The system of inertia forces acting on R is represented by a torque $\mathbf{T}^R$, given by (see 4.1.7)

$$\mathbf{T}^R = -[(\phi_{31}^{R/R^*}\alpha - \phi_{32}^{R/R^*}\omega^2)\mathbf{n}_1 + (\phi_{32}^{R/R^*}\alpha + \phi_{31}^{R/R^*}\omega^2)\mathbf{n}_2 + (\phi_{33}^{R/R^*}\alpha)\mathbf{n}_3] \quad \text{(A)}$$

Finally, the system of all contact forces acting on R is represented by the known force $\mathbf{C}$ and torque $\boldsymbol{\gamma}$, together with an unknown force $\mathbf{F}$, applied (arbitrarily) at R^*, and the torque $\mathbf{T}$ of an unknown couple.

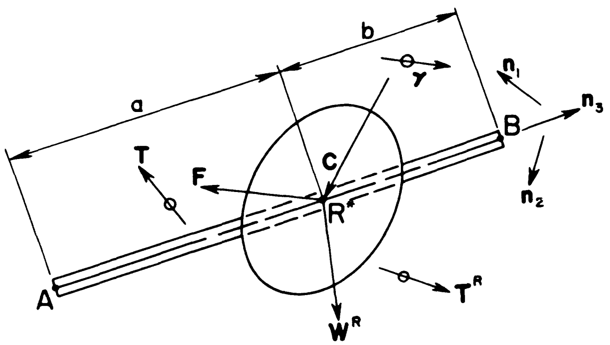


FIG. 4.3.2b

$\mathbf{F}$ and $\mathbf{T}$ may be found by setting the resultant and the moment about R^* of all forces represented in Fig. 4.3.2b equal to zero (see 4.2.3):

$$\mathbf{F} + \mathbf{W}^R + \mathbf{C} = 0$$

$$\mathbf{T} + \mathbf{T}^R + \boldsymbol{\gamma} = 0$$

Thus

$$\mathbf{F} = -(\mathbf{W}^R + \mathbf{C})$$

$$\mathbf{T} = -(\mathbf{T}^R + \boldsymbol{\gamma})$$

The system of all unknown contact forces acting on R has now been reduced to a known force and a couple of known torque. But these do not furnish detailed information about forces exerted on S by the bearings.

The system of unknown contact forces acting on R may be regarded as consisting of four force systems: S_A and S_B, comprised of forces exerted on S across the bearing surfaces at A and B; and S_A' and S_B', containing the forces exerted on S by the portions of the shaft contiguous to S at A and B. In the free-body diagram of R shown in Fig. 4.3.2c, these four force systems are represented by four forces ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}'$, $\mathbf{B}'$) and four torques ($\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}'$) of couples. ($\mathbf{W}^R$, $\mathbf{T}^R$, $\mathbf{C}$, and $\boldsymbol{\gamma}$ have the same meaning as in Fig. 4.3.2b.) Note that the choice of points of application for the forces $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{B}$, $\mathbf{B}'$ is an entirely arbitrary one.

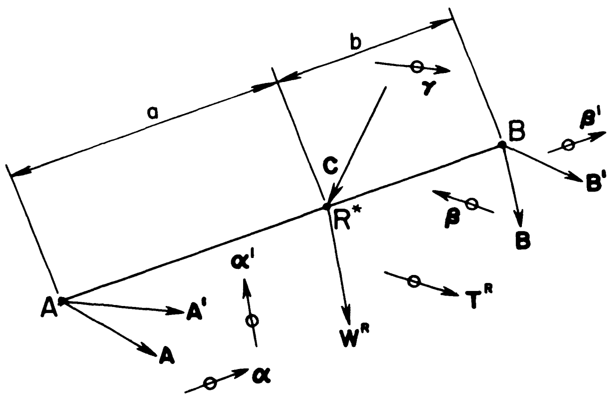


FIG. 4.3.2c

Two equations relating the eight unknown vectors appearing in Fig. 4.3.2c may be obtained by setting moments about points A and B equal to zero:

$$\mathbf{n}_3 \times [a(\mathbf{W}^R + \mathbf{C}) + (a + b)(\mathbf{B} + \mathbf{B}')] + \boldsymbol{\alpha} + \boldsymbol{\alpha}' + \boldsymbol{\beta} + \boldsymbol{\beta}' + \boldsymbol{\gamma} + \mathbf{T}^R = 0 \quad \text{(B)}$$

$$-\mathbf{n}_3 \times [b(\mathbf{W}^R + \mathbf{C}) + (a + b)(\mathbf{B} + \mathbf{B}')] + \boldsymbol{\alpha} + \boldsymbol{\alpha}' + \boldsymbol{\beta} + \boldsymbol{\beta}' + \boldsymbol{\gamma} + \mathbf{T}^R = 0 \quad \text{(C)}$$

In order to extract useful information about the force systems S_A and S_B from these equations, one must make a number of assumptions. If A_i, B_i, etc., are the $\mathbf{n}_i$, $i = 1, 2, 3$, measure numbers of **A**, **B**, etc., these assumptions may take the following form:

The bearing surfaces are so smooth that **A**, **B**, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ are perpendicular to the shaft axis (see Vol. I, Sec. 4.8.2). Thus

$$A_3 = B_3 = \alpha_3 = \beta_3 = 0 \tag{D}$$

The bearings are so short that contact between the shaft and the bearing surfaces takes place only along two circles, one having its center at A, the other at B (see Vol. I, Sec. 4.8.4). Thus

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0 \tag{E}$$

S_A' and S_B' can each be reduced to a force whose line of action coincides with the shaft axis, together with a couple whose torque is parallel to the shaft axis. Thus

$$A_1' = A_2' = B_1' = B_2' = \alpha_1' = \alpha_2' = \beta_1' = \beta_2' = 0 \tag{F}$$

(In order to justify the last assumption, one must consider forces acting on portions of the shaft not shown in Fig. 4.3.2a. As nothing has been said about these forces, the assumption should be regarded as tenuous.)

The scalar equations corresponding to Eqs. (B) and (C), solved for A_1, A_2, B_1, and B_2, now give (after using Eq. (A))

$$\begin{aligned}
A_1 &= -\frac{\phi_{32}^{R/R^*}\alpha + \phi_{31}^{R/R^*}\omega^2}{a+b} - \frac{b(W_1^R + C_1) - \gamma_2}{a+b} \\
A_2 &= +\frac{\phi_{31}^{R/R^*}\alpha - \phi_{32}^{R/R^*}\omega^2}{a+b} - \frac{b(W_2^R + C_2) + \gamma_1}{a+b} \\
B_1 &= +\frac{\phi_{32}^{R/R^*}\alpha + \phi_{31}^{R/R^*}\omega^2}{a+b} - \frac{a(W_1^R + C_1) + \gamma_2}{a+b} \\
B_2 &= -\frac{\phi_{31}^{R/R^*}\alpha - \phi_{32}^{R/R^*}\omega^2}{a+b} - \frac{a(W_2^R + C_2) - \gamma_1}{a+b}
\end{aligned} \tag{G}$$

and these equations, together with Eqs. (D) and (E), constitute a complete description of the reaction of each bearing on the shaft.

Note that each bearing reaction may be replaced with a pair of forces, one force depending on α and ω, the other being independent of these quantities. The two forces (one at each bearing) depending on α and ω are called *dynamic bearing reactions*, and they form a couple whose torque is equal to the negative of the resolute of $\mathbf{T}^R$ perpendicular to $\mathbf{n}_3$. In accordance with Problem 4.1.7, the dynamic bearing reactions thus vanish for all values of ω and α if and only if the axis of rotation of R is a principal axis of R for the mass center R^*. When this requirement is fulfilled, R is said to be *dynamically balanced* for rotation about this axis (see Problem 3.5.9).

4.3.3 A sketch representing the orthogonal projections, on a plane N, of a body B and of all gravitational, contact, and inertia forces acting on B in a Newtonian reference frame is called a *plane free-body diagram* of B. (The presence of couples acting on B is indicated by arrows representing only the $\mathbf{n}$ resolutes of the torques of these couples, $\mathbf{n}$ being a unit vector perpendicular to N.)

Although a plane free-body diagram is *not* a free-body diagram (see 4.2.3), the system of forces represented in it is a zero system (see Vol. I, Sec. 3.6.9). Hence plane free-body diagrams may be used as guides in writing equations which furnish information about both the motion of, and the forces acting on, a body.

Example: Figures 4.3.3a, b, c are plane free-body diagrams of the body R discussed in Example 4.3.2. The assumptions expressed in Eqs. (D), (E), (F) of Example 4.3.2 have been taken into account in drawing these diagrams. Hence these three figures con-

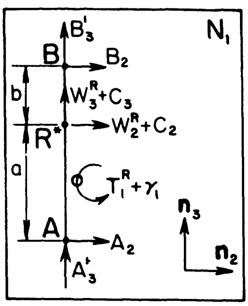


FIG. 4.3.3a

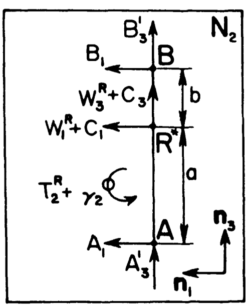


FIG. 4.3.3b

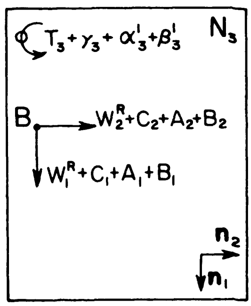


FIG. 4.3.3c

tain the same information as Fig. 4.3.2c together with Eqs. (D), (E), (F), and relationships equivalent to Eqs. (G) may be obtained by setting sums of moments about points A and B equal to zero. For example, using Fig. 4.3.3a,

$$-(a + b)B_2 - a(W_2^R + C_2) + T_1^R + \gamma_1 = 0$$

and

$$(a + b)A_2 + b(W_2^R + C_2) + T_1^R + \gamma_1 = 0$$

4.3.4 The solution of certain problems requires that equations of motion (see 4.2.3) be written for each of several (not necessarily rigid) bodies associated with a given system. To avoid unnecessary labor it is again (see 4.3.1) necessary to take full advantage of specific features of the problem under consideration.

Example: Figure 4.3.4a represents an electrically driven gyroscope, described as follows: The rotor A is rigidly attached to a shaft which is supported by bearings in the inner gimbal ring B. A small electric motor P, fastened to B, drives A by means of a friction-wheel (not shown), whose periphery is in contact with that of A. For balance, a counterweight P' is attached to B, diametrically opposite P. B is supported by bearings in the outer gimbal ring C, and C by bearings in a fixed frame D. If desired, a small block Q can be fastened to the rotor shaft.

Any motion of this system can be described in terms of the three functions $\beta(t)$, $\gamma(t)$, $\delta(t)$. (Lines OX, OY, and OZ are mutually perpendicular, and O is a fixed point.)

In what follows, β is regarded as a prescribed function of time. Differential equations governing γ and δ are obtained, and these are used to discuss motions of steady precession and nutation.

Two sets of mutually perpendicular unit vectors, $\mathbf{b}_i$ and $\mathbf{c}_i$, $i = 1, 2, 3$, are shown in Fig. 4.3.4a. These are fixed in B and C, respectively.

Point O is the mass center of each of the bodies A, B, and C. On the basis of symmetry considerations, $\mathbf{b}_i$, $i = 1, 2, 3$, are regarded as centroidal principal directions of A, and $\mathbf{b}_2$ as a centroidal principal direction of the rigid body R consisting of B, P, and P'. Second moments of A, R, and C with respect to O are denoted by $\phi_{ij}^{A/O}$, $\phi_{ij}^{R/O}$, $\phi_{ij}^{C/O}$, $i, j = 1, 2, 3$, the subscripts referring to the unit vectors $\mathbf{b}_i$, $i = 1, 2, 3$, in the case of A and R, and to $\mathbf{c}_i$, $i = 1, 2, 3$, for C. For example, $\phi_{22}^{A/O}$ is the moment of inertia of A about a line passing through O and parallel to $\mathbf{b}_2$ (and is equal

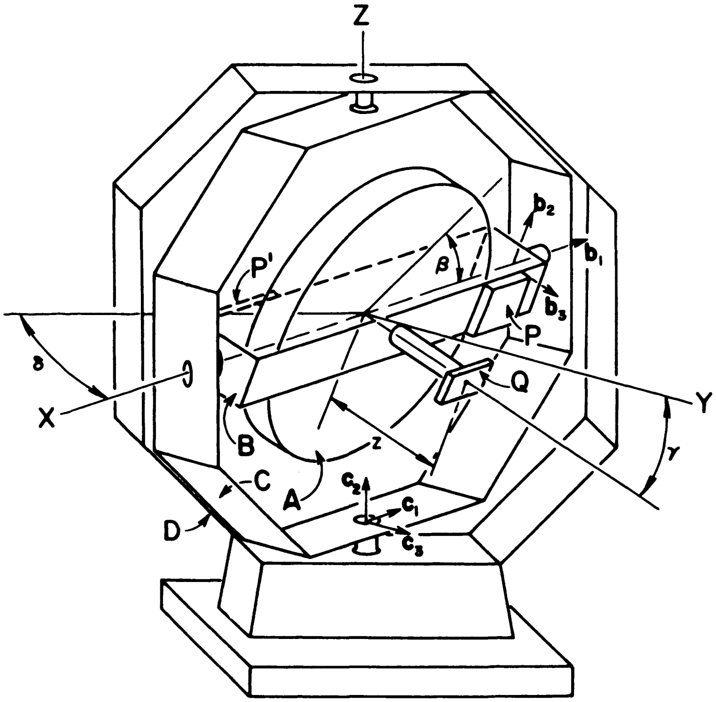


FIG. 4.3.4a

to $\phi_{11}^{A/O}$), while $\phi_{22}^{C/O}$ is the moment of inertia of C about the vertical line passing through O. The mass of Q is denoted by M.

In Fig. 4.3.4b, which is a free-body diagram of the (nonrigid) body consisting of A, R, and Q, the gravitational forces acting on this body are represented by $\mathbf{W}^A$, $\mathbf{W}^R$, and $\mathbf{W}^Q$. $\mathbf{F}^Q$, the inertia force acting on Q, is given by

$$\mathbf{F}^Q \underset{(4.1.1)}{=} -M\mathbf{a}^Q \tag{A}$$

and $\mathbf{T}^A$, the torque of the inertia couple acting on A, by

$$\begin{aligned}\mathbf{T}^A \underset{(4.1.5)}{=} & -[\phi_{11}^{A/O}\alpha_1^A + (\phi_{33}^{A/O} - \phi_{11}^{A/O})\omega_2^A\omega_3^A]\mathbf{b}_1 \\ & - [\phi_{11}^{A/O}\alpha_2^A + (\phi_{11}^{A/O} - \phi_{33}^{A/O})\omega_3^A\omega_1^A]\mathbf{b}_2 - \phi_{33}^{A/O}\alpha_3^A\mathbf{b}_3\end{aligned} \tag{B}$$

The motion of R is identical with that of B. As $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are

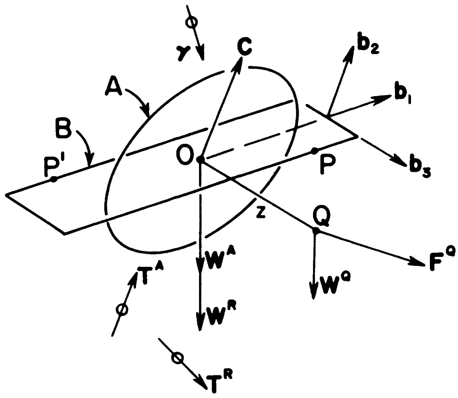


FIG. 4.3.4b

fixed in R, but $\mathbf{b}_1$ and $\mathbf{b}_2$ are not principal directions of R for O,

$$\mathbf{T}^R \underset{(4.1.4,2.3.2)}{=} -\sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/O}(\dot{\omega}_i^B \mathbf{b}_j + \omega_i^B \boldsymbol{\omega}^B \times \mathbf{b}_j) \tag{C}$$

$\mathbf{C}$ and $\boldsymbol{\gamma}$ are a force and the torque of a couple representing the reaction of the gimbal ring C on the body R. The bearings are regarded as frictionless, so that $\boldsymbol{\gamma}$ is perpendicular to $\mathbf{b}_1$ and

$$\boldsymbol{\gamma} \cdot \mathbf{b}_1 = 0 \tag{D}$$

One equation of motion, which does not involve the unknown quantities $\mathbf{C}$ and $\boldsymbol{\gamma}$, is obtained by equating to zero the sum of the moments of all forces represented in Fig. 4.3.4b about a line passing through O and parallel to $\mathbf{b}_1$:

$$[z\mathbf{b}_3 \times (\mathbf{W}^Q + \mathbf{F}^Q) + \mathbf{T}^A + \mathbf{T}^R + \boldsymbol{\gamma}] \cdot \mathbf{b}_1 = 0$$

which, after using Eqs. (A)–(D) and noting that (see Fig. 4.3.4a)

$$\mathbf{W}^Q = -Mg\mathbf{c}_2 = -Mg(\cos\gamma\, \mathbf{b}_2 - \sin\gamma\, \mathbf{b}_3)$$

can be expressed as

$$zM(g\cos\gamma + a_2^Q) - [\phi_{11}^{A/O}\alpha_1^A + (\phi_{33}^{A/O} - \phi_{11}^{A/O})\omega_2^A\omega_3^A] \\ - \sum_{i=1}^{3} [\phi_{i1}^{R/O}\dot{\omega}_i^B + (\phi_{i3}^{R/O}\omega_2^B - \phi_{i2}^{R/O}\omega_3^B)\omega_i^B] = 0$$

where a_2^Q is the $\mathbf{b}_2$ measure number of $\mathbf{a}^Q$. Furthermore, as $\mathbf{b}_2$ is principal direction of R for O,

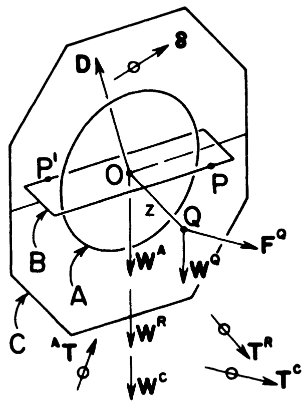
$$\phi_{12}^{R/O} = \phi_{23}^{R/O} \underset{(3.5.7,3.3.3)}{=} 0 \tag{E}$$

Hence the equation of motion becomes

$$zM(g \cos \gamma + a_2^Q) - [\phi_{11}^{A/O}\alpha_1^A + (\phi_{33}^{A/O} - \phi_{11}^{A/O})\omega_2^A\omega_3^A]$$
$$- [\phi_{11}^{R/O}\dot{\omega}_1^B + \phi_{13}^{R/O}(\dot{\omega}_3^B + \omega_1^B\omega_2^B) + (\phi_{33}^{R/O} - \phi_{22}^{R/O})\omega_2^B\omega_3^B] = 0 \tag{F}$$

A second equation is based on a free-body diagram, shown in Fig. 4.3.4c, of the body consisting of A, R, and C. This diagram

FIG. 4.3.4c



differs from that used previously by the omission of $\mathbf{C}$ and $\boldsymbol{\gamma}$ (because these are associated with forces exerted on B by C, that is, with forces not acting on the body presently under consideration), and the addition of the gravitational force $\mathbf{W}^C$, the torque $\mathbf{T}^C$ of the inertia couple acting on C, and the force $\mathbf{D}$ and torque $\boldsymbol{\delta}$, representing the reaction of the frame D on the gimbal ring C.

$\mathbf{T}^C$ is given by

$$\mathbf{T}^C \underset{(4.1.7)}{=} -[\phi_{12}^{C/O}\dot{\omega}_2^C + \phi_{23}^{C/O}(\omega_2^C)^2]\mathbf{c}_1$$
$$- \phi_{22}^{C/O}\dot{\omega}_2^C\mathbf{c}_2 - [\phi_{23}^{C/O}\dot{\omega}_2^C - \phi_{12}^{C/O}(\omega_2^C)^2]\mathbf{c}_3 \tag{G}$$

and, again regarding the bearings as frictionless,

$$\boldsymbol{\delta} \cdot \mathbf{c}_2 = 0 \tag{H}$$

By taking moments about the vertical line passing through O, using Eqs. (A)–(E), (G), and (H), and noting that

$$\mathbf{c}_2 = \cos\gamma\mathbf{b}_2 - \sin\gamma\mathbf{b}_3 \tag{I}$$

the following equation is obtained:

$$\begin{aligned}
&\sin\gamma[\phi_{33}^{A/O}\alpha_3^A + \phi_{33}^{R/O}\dot{\omega}_3^B - \phi_{31}^{R/O}(\omega_2^B\omega_3^B - \dot{\omega}_1^B) \\
&\qquad\qquad + (\phi_{22}^{R/O} - \phi_{11}^{R/O})\omega_1^B\omega_2^B] \\
&- \cos\gamma[zMa_1^Q + \phi_{11}^{A/O}\alpha_2^A + (\phi_{11}^{A/O} - \phi_{33}^{A/O})\omega_3^A\omega_1^A + \phi_{22}^{R/O}\dot{\omega}_2^B \\
&\qquad\qquad + (\phi_{11}^{R/O} - \phi_{33}^{R/O})\omega_1^B\omega_3^B + \phi_{31}^{R/O}(\omega_3^B)^2 - \phi_{31}^{R/O}(\omega_1^B)^2] \\
&\qquad\qquad - \phi_{22}^{C/O}\dot{\omega}_2^C = 0
\end{aligned} \tag{J}$$

The kinematic quantities appearing in Eqs. (F) and (J) are measure numbers of components of $\boldsymbol{\omega}^A$, $\boldsymbol{\omega}^B$, $\boldsymbol{\omega}^C$, $\boldsymbol{\alpha}^A$, and $\mathbf{a}^Q$. These vectors must, therefore, be expressed in terms of β, δ, γ.

First,

$$\boldsymbol{\omega}^A \underset{(2.2.7)}{=} {}^B\boldsymbol{\omega}^A + {}^C\boldsymbol{\omega}^B + {}^D\boldsymbol{\omega}^C$$

$$\underset{(2.2.4)}{=} \dot{\beta}\mathbf{b}_3 + \dot{\gamma}\mathbf{b}_1 + \dot{\delta}\mathbf{c}_2$$

$$\boldsymbol{\omega}^A \underset{(\mathrm{I})}{=} \dot{\gamma}\mathbf{b}_1 + \dot{\delta}\cos\gamma\,\mathbf{b}_2 + (\dot{\beta} - \dot{\delta}\sin\gamma)\mathbf{b}_3 \tag{K}$$

Next,

$$\boldsymbol{\omega}^B \underset{(2.2.7,2.2.4)}{=} \dot{\gamma}\mathbf{b}_1 + \dot{\delta}\cos\gamma\,\mathbf{b}_2 - \dot{\delta}\sin\gamma\,\mathbf{b}_3 \tag{L}$$

and

$$\boldsymbol{\omega}^C \underset{(2.2.4)}{=} \dot{\delta}\mathbf{c}_2 \tag{M}$$

Finally,

$$\boldsymbol{\alpha}^A \underset{(2.3.1)}{=} \frac{d\boldsymbol{\omega}^A}{dt} \underset{(2.1.4)}{=} \frac{{}^Bd\boldsymbol{\omega}^A}{dt} + \boldsymbol{\omega}^B \times \boldsymbol{\omega}^A$$

$$\begin{aligned}
\boldsymbol{\alpha}^A \underset{(\mathrm{K,L})}{=} &(\ddot{\gamma} + \dot{\beta}\dot{\delta}\cos\gamma)\mathbf{b}_1 + (\ddot{\delta}\cos\gamma - \dot{\beta}\dot{\gamma} - \dot{\gamma}\dot{\delta}\sin\gamma)\mathbf{b}_2 \\
&\qquad + (\ddot{\beta} - \ddot{\delta}\sin\gamma - \dot{\gamma}\dot{\delta}\cos\gamma)\mathbf{b}_3
\end{aligned} \tag{N}$$

and

$$\mathbf{a}^Q \underset{(2.5.9)}{=} \boldsymbol{\alpha}^A \times (z\mathbf{b}_3) + \boldsymbol{\omega}^A \times [\boldsymbol{\omega}^A \times (z\mathbf{b}_3)]$$

$$\begin{aligned}
\mathbf{a}^Q \underset{(\mathrm{K,N})}{=} &z[(\ddot{\delta}\cos\gamma - 2\dot{\gamma}\dot{\delta}\sin\gamma)\mathbf{b}_1 - (\ddot{\gamma} + \dot{\delta}^2\sin\gamma\cos\gamma)\mathbf{b}_2 \\
&\qquad - (\dot{\gamma}^2 + \dot{\delta}^2\cos^2\gamma)\mathbf{b}_3]
\end{aligned} \tag{O}$$

The measure numbers required for substitution into Eqs. (F) and (J) are tabulated in Table 4.3.4, and these equations can now

TABLE 4.3.4

	$i = 1$	$i = 2$	$i = 3$
ω_i^A	$\dot{\gamma}$	$\dot{\delta} \cos \gamma$	$\dot{\beta} - \dot{\delta} \sin \gamma$
ω_i^B	$\dot{\gamma}$	$\dot{\delta} \cos \gamma$	$-\dot{\delta} \sin \gamma$
ω_i^C	0	δ	0
α_i^A	$\ddot{\gamma} + \dot{\beta}\dot{\delta} \cos \gamma$	$\ddot{\delta} \cos \gamma - \dot{\beta}\dot{\gamma} - \dot{\gamma}\dot{\delta} \sin \gamma$	$\ddot{\beta} - \ddot{\delta} \sin \gamma - \dot{\gamma}\dot{\delta} \cos \gamma$
$\mathbf{a}_i^Q$	$z(\ddot{\delta} \cos \gamma - 2\dot{\gamma}\dot{\delta} \sin \gamma)$	$-z(\ddot{\gamma} + \dot{\delta}^2 \sin \gamma \cos \gamma)$	$-z(\dot{\gamma}^2 + \dot{\delta}^2 \cos^2 \gamma)$

be expressed as

$$(\phi_1 + Mz^2)\ddot{\gamma} - \phi_{13}^{R/O}\, \ddot{\delta} \sin \gamma$$
$$+ \dot{\delta}[(\phi_2 + Mz^2 - \phi_3)\dot{\delta} \sin \gamma + \phi_{33}^{A/O}\dot{\beta}] \cos \gamma = zMg \cos \gamma \quad \text{(P)}$$

and

$$\frac{d}{dt}\left[(\phi_2 \cos^2 \gamma + \phi_3 \sin^2 \gamma + Mz^2 \cos^2 \gamma + \phi_{22}^{C/O})\dot{\delta}\right.$$
$$\left. - (\phi_{31}^{R/O}\dot{\gamma} + \phi_{33}^{A/O}\dot{\beta}) \sin \gamma\right] = 0 \quad \text{(Q)}$$

where ϕ_1, ϕ_2, ϕ_3 are defined as

$$\phi_1 = \phi_{11}^{A/O} + \phi_{11}^{R/O} \quad \text{(R)}$$

$$\phi_2 = \phi_{22}^{A/O} + \phi_{22}^{R/O} \quad \text{(S)}$$

$$\phi_3 = \phi_{33}^{A/O} + \phi_{33}^{R/O} \quad \text{(T)}$$

and can be recognized as moments of inertia of the body consisting of A and R about lines passing through O and parallel to $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, respectively. (The motivation for expressing the second equation of motion in the form given in Eq. (Q) is explained by the parenthetical comment at the end of Example 4.4.14.)

The gyroscope is said to be in a state of *steady precession* when

$$\dot{\beta} = \omega_s, \qquad \gamma = \gamma_0, \qquad \dot{\delta} = \omega_p$$

where ω_s, γ_0, and ω_p are constants. ω_s is called the *spin velocity*, ω_p the *precession velocity*. Substitution into Eqs. (P) and (Q) shows that this motion is possible only either when

$$\gamma_0 = \pm\frac{\pi}{2}$$

or when ω_s, γ_0, and ω_p are related as follows:

$$\phi_{33}^{A/O}\omega_s\omega_p + (\phi_2 - \phi_3 + Mz^2)\omega_p{}^2 \sin\gamma_0 - zMg = 0 \qquad \text{(U)}$$

Hence for given values of ω_p and γ_0 ($\gamma_0 \neq \pi/2$) there exists a unique value of ω_s; and when ω_s and γ_0 are given, ω_p must have either of the two values

$$\omega_p = \frac{-\omega_s\phi_{33}^{A/O} \pm [(\omega_s\phi_{33}^{A/O})^2 + 4zMg(\phi_2 - \phi_3 + Mz^2)\sin\gamma_0]^{1/2}}{2(\phi_2 - \phi_3 + Mz^2)\sin\gamma_0}$$

The corresponding motions are called a *slow steady precession* and a *fast steady precession.*

Note that there exists only one nonzero precession velocity when $\gamma_0 \neq 0$ and the block Q is removed, that is, $M = 0$; that when $M \neq 0$ one of the values of ω_p approaches infinity as γ_0 approaches zero; and that both values of ω_p vanish when both M and γ_0 are equal to zero.

A motion described by

$$\dot{\beta} = \omega_s, \qquad \gamma = \gamma_0 + \gamma^*, \qquad \dot{\delta} = \omega_p + \dot{\delta}^* \qquad \text{(V)}$$

where ω_s and ω_p are constants, and γ^* and δ^* are periodic functions of time, is called a *periodic nutation.* One such motion takes place subsequent to the application of small disturbing forces when the gyroscope is in one of the states of steady precession described above. An approximate description of this nutation is obtained by substituting from Eqs. (V) into Eqs. (P) and (Q) and dropping all terms of second or higher degree in starred quantities. Restricting the discussion to the case

$$\gamma_0 = 0$$

this gives

$$(\phi_1 + Mz^2)\ddot{\gamma}^* + \omega_p{}^2(\phi_2 + Mz^2 - \phi_3)\gamma^* + \phi_{33}^{A/O}\omega_s(\omega_p + \dot{\delta}^*) \underset{(\mathrm{P,V})}{=} zMg$$

and

$$\frac{d}{dt}[(\phi_2 + Mz^2 + \phi_{22}^{C/O})(\omega_p + \dot{\delta}^*) - \phi_{33}^{A/O}\omega_s\gamma^*] \underset{(\mathrm{Q,V})}{=} 0$$

the second of which requires that

$$\omega_p + \dot{\delta}^* = \frac{\phi_{33}^{A/O}\omega_s}{\phi_2 + Mz^2 + \phi_{22}^{C/O}}\gamma^* + C \qquad \text{(W)}$$

where C is a constant, so that the first becomes

$$(\phi_1 + Mz^2)\ddot{\gamma}^* + \left[\omega_p{}^2\phi(_2 + Mz^2 - \phi_3) + \frac{(\phi_{33}^{A/O}\omega_s)^2}{\phi_2 + Mz^2 + \phi_{22}^{C/O}}\right]\gamma^* = zMg - C\phi_{33}^{A/O}\omega_s \quad \text{(X)}$$

As Eqs. (W) and (X) were obtained by dropping second and higher degree terms in γ^* and $\dot{\delta}^*$, they are valid only when their solutions are functions which can be kept arbitrarily small. In the case of Eq. (X) this is assured by taking

$$C = \frac{zMg}{\phi_{33}^{A/O}\omega_s}$$

in which case this equation has the general solution

$$\gamma^* = A \sin (\omega_n t + \theta) \tag{Y}$$

where A and θ are arbitrary constants, and ω_n, the *circular frequency of nutation*, is given by

$$\omega_n = \left\{\frac{1}{\phi_1 + Mz^2}\left[\omega_p{}^2(\phi_2 + Mz^2 - \phi_3) + \frac{(\phi_{33}^{A/O}\omega_s)^2}{\phi_2 + Mz^2 + \phi_{22}^{C/O}}\right]\right\}^{1/2}$$

Furthermore, when $\gamma_0 = 0$, Eq. (U) reduces to

$$\frac{zMg}{\phi_{33}^{A/O}\omega_s} = \omega_p \tag{Z}$$

so that

$$C = \omega_p$$

and

$$\dot{\delta}^* \underset{\text{(W)}}{=} \frac{\phi_{33}^{A/O}\omega_s}{\phi_2 + Mz^2 + \phi_{22}^{C/O}}\gamma^* \underset{\text{(Y)}}{=} \frac{A\phi_{33}^{A/O}\omega_s}{\phi_2 + Mz^2 + \phi_{22}^{C/O}} \sin (\omega_n t + \theta)$$

which shows that $\dot{\delta}^*$ can be kept arbitrarily small.

Note that Eq. (Z) may be used to express ω_n in the alternative form

$$\omega_n = \left\{\frac{1}{\phi_1 + Mz^2}\left[\frac{(zMg)^2(\phi_2 + Mz^2 - \phi_3)}{(\phi_{33}^{A/O}\omega_s)^2} + \frac{(\phi_{33}^{A/O}\omega_s)^2}{\phi_2 + Mz^2 + \phi_{22}^{C/O}}\right]\right\}^{1/2}$$

which, when $M = 0$, reduces to

$$\omega_n = \frac{\phi_{33}^{A/O}\omega_s}{[\phi_1(\phi_2 + \phi_{22}^{C/O})]^{1/2}}$$

and, when ω_s is sufficiently large, to

$$\omega_n \approx \frac{\phi_{33}^{A/O}\omega_s}{[(\phi_1 + Mz^2)(\phi_2 + Mz^2 + \phi_{22}^{C/O})]^{1/2}}$$

both of which are of the form

$$\omega_n = \frac{\phi_{33}^{A/O}\omega_s}{(I_1I_2)^{1/2}}$$

where (see Eqs. (R) and (S)) I_1 is the sum of the moments of inertia of A, R, and Q about the line passing through O and parallel to $\mathbf{b}_1$, while I_2 is the moment of inertia of all moving parts about the axis of rotation of gimbal ring C when $\gamma = 0$.

4.3.5 The analysis of contact forces exerted *by* a body B on a body B' across a surface σ is frequently facilitated by considering, instead, forces exerted *on* B by B'. This requires use of the following theorem, called the *Law of Action and Reaction:* If the reaction of B' on B across the surface σ is a couple of torque $\mathbf{T}$ and a force $\mathbf{F}$ whose line of action passes through a point P, and the reaction of B on B' across σ is a couple of torque $\mathbf{T}'$ and a force $\mathbf{F}'$ whose line of action passes through the *same* point P, then

$$\mathbf{F}' = -\mathbf{F} \tag{1}$$

and

$$\mathbf{T}' = -\mathbf{T} \tag{2}$$

Proof: Write equations of motion (see 4.2.3) for B, B', and the body consisting of both B and B'. (See Vol. I, Sec. 4.7.)

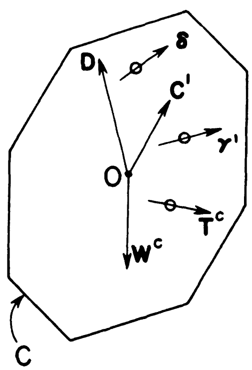


FIG. 4.3.5

Example: Referring to Example 4.3.4, the reaction of the inner gimbal ring B on the outer gimbal ring C is to be determined for a steady precession with $M = 0$ and $\gamma = \gamma_0$, a constant.

The desired reaction can be represented by a force $\mathbf{C}'$ whose line of action passes through point O, together with a couple of torque $\boldsymbol{\gamma}'$ (see Fig. 4.3.5), and can be found by writing equations of motion for C. However, this procedure involves the as yet unknown reaction of D on C ($\mathbf{D}$ and $\boldsymbol{\delta}$) and thus necessitates the prior determination of $\mathbf{D}$ and $\boldsymbol{\delta}$. Alternatively, one may use the fact that

$$\mathbf{C}' = -\mathbf{C}$$

and

$$\boldsymbol{\gamma}' = -\boldsymbol{\gamma}$$

where $\mathbf{C}$ and $\boldsymbol{\gamma}$ are the force and torque previously shown in Fig. 4.3.4b. The force and moment equations

$$\mathbf{C} + \mathbf{W}^A + \mathbf{W}^R = 0$$

and

$$\boldsymbol{\gamma} + \mathbf{T}^A + \mathbf{T}^R = 0$$

then show that

$$\mathbf{C}' = \mathbf{W}^A + \mathbf{W}^R$$

and

$$\boldsymbol{\gamma}' = \mathbf{T}^A + \mathbf{T}^R$$

$\mathbf{T}^A$ and $\mathbf{T}^R$ can be expressed entirely in terms of known quantities by substituting from Table 4.3.4 into Eqs. (B) and (C) after letting

$$\dot{\beta} = \omega_s, \qquad \gamma = \gamma_0, \qquad \dot{\delta} = \omega_p$$

and noting that

$$\omega_s \underset{(\mathrm{U})}{=} \frac{\omega_p(\phi_3 - \phi_2)}{\phi_{33}^{A/O}} \sin \gamma_0$$

This leads to

$$\boldsymbol{\gamma}' = -2\omega_p{}^2\, \phi_{31}^{R/O} \sin \gamma_0\, \mathbf{c}_3$$

4.4 Linear and angular momentum

4.4.1 Given a set S of n particles situated at points P_i, $i = 1, 2, \ldots, n$, having masses m_i, and moving with velocities ${}^R\mathbf{v}^{P_i/Q}$

relative to a point Q in a reference frame R (see 2.4.1), the *linear momentum of S relative to Q in R*, denoted by ${}^R\mathbf{L}^{S/Q}$, is defined as

$$ {}^R\mathbf{L}^{S/Q} = \sum_{i=1}^{n} m_i {}^R\mathbf{v}^{P_i/Q} $$

4.4.2 Given a continuous body C regarded as occupying a figure F, and a point Q moving in a reference frame R, the *linear momentum of C relative to Q in R*, defined as the limit of the corresponding linear momentum of a set of particles constructed in a manner similar to that employed in Sec. 4.1.2, can be expressed as

$$ {}^R\mathbf{L}^{C/Q} = \int_F \rho {}^R\mathbf{v}^{P/Q}\, d\tau $$

where ${}^R\mathbf{v}^{P/Q}$ is the velocity of a generic point P of C relative to Q in R and ρ is the mass density of C at P.

4.4.3 The *linear momentum of a body B* (any collection of matter) *relative to a point Q in a reference frame R* is defined as the sum of the corresponding linear momenta of the particles (see 4.4.1) and continuous bodies (see 4.4.2) comprising B.

Given two points Q and Q' moving in a reference frame R, the linear momenta ${}^R\mathbf{L}^{B/Q}$ and ${}^R\mathbf{L}^{B/Q'}$ are related to each other as follows:

$$ {}^R\mathbf{L}^{B/Q} = {}^R\mathbf{L}^{B/Q'} + {}^R\mathbf{L}^{Q'/Q} \tag{1} $$

where ${}^R\mathbf{L}^{Q'/Q}$ is the linear momentum relative to Q in R of a particle situated at Q' and having a mass m equal to that of the body B; that is,

$$ {}^R\mathbf{L}^{Q'/Q} = m {}^R\mathbf{v}^{Q'/Q} \tag{2} $$

Proof: Use Sec. 2.4.4 and the definitions given in Secs. 4.4.1 and 4.4.2.

4.4.4 The linear momentum ${}^R\mathbf{L}^{B/P^*}$ of a body B relative to the mass center P^* of B is equal to zero. Consequently the linear momentum of B relative to any point Q is given by

$$ {}^R\mathbf{L}^{B/Q} = m {}^R\mathbf{v}^{P^*/Q} $$

where m is the mass of B.

Proof: If B is a set S of n particles, and $\mathbf{r}_i$, $i = 1, 2, \ldots, n$, are the position vectors of the particles relative to P^*, then

$$^R\mathbf{L}^{B/P^*} \underset{(4.4.1)}{=} \sum_{i=1}^{n} m_i{}^R\mathbf{v}^{P_i/P^*} \underset{(2.4.1)}{=} \sum_{i=1}^{n} m_i \frac{^Rd\mathbf{r}_i}{dt} \underset{(1.4.1)}{=} \frac{^Rd}{dt}\sum_{i=1}^{n} m_i\mathbf{r}_i$$

and

$$\sum_{i=1}^{n} m_i\mathbf{r}_i = 0$$

by definition of P^*. Use Sec. 4.4.3. Similarly, if B contains continuous bodies.

4.4.5 If O is a point fixed in a reference frame R, the linear momentum $^R\mathbf{L}^{B/O}$ of a body B (see 4.4.3) is independent of the position of O in R, because the velocity $^R\mathbf{v}^{P/O}$ of any point P of B is independent of the position of O (see 2.5.1). Accordingly, the linear momentum of B in R relative to any point fixed in R is called the *absolute linear momentum* of B in R, or, for short, *the* linear momentum of B in R; it is denoted by $^R\mathbf{L}^B$ and involves only absolute velocities, that is,

$$^R\mathbf{L}^S \underset{(4.4.1)}{=} \sum_{i=1}^{n} m_i{}^R\mathbf{v}^{P_i} \tag{1}$$

$$^R\mathbf{L}^C \underset{(4.4.2)}{=} \int_F \rho{}^R\mathbf{v}^P \, d\tau \tag{2}$$

and can be expressed as

$$^R\mathbf{L}^B \underset{(4.4.4)}{=} m{}^R\mathbf{v}^{P^*} \tag{3}$$

where m is the mass of B and P^* is the mass center of B.

4.4.6 If $\mathbf{F}$ is the resultant of all gravitational and contact forces acting on a body B, and R is a Newtonian reference frame (see 4.2.1), $\mathbf{F}$ is related to the linear momentum $^R\mathbf{L}^B$ of B in R (see 4.4.5) as follows:

$$\mathbf{F} = \frac{^Rd}{dt}\,{}^R\mathbf{L}^B$$

This equality is known as *the linear momentum principle.*

Proof: If B is a set S of particles situated at points P_i, $i = 1, 2, \ldots, n$,

$$\frac{^R d}{dt}\,{}^R\mathbf{L}^B \underset{(4.4.5)}{=} \frac{^R d}{dt}\sum_{i=1}^{n} m_i\,{}^R\mathbf{v}^{P_i} \underset{(1.4.1)}{=} \sum_{i=1}^{n} m_i \frac{^R d\,{}^R\mathbf{v}^{P_i}}{dt}$$

$$\underset{(2.5.1)}{=} \sum_{i=1}^{n} m_i\,{}^R\mathbf{a}^{P_i} \underset{(4.1.1)}{=} -\sum_{i=1}^{n} {}^R\mathbf{F}^{P_i}$$

But

$$\mathbf{F} + \sum_{i=1}^{n} {}^R\mathbf{F}^{P_i} \underset{(4.2.1)}{=} 0$$

Hence

$$-\sum_{i=1}^{n} {}^R\mathbf{F}^{P_i} = \mathbf{F}$$

and

$$\frac{^R d}{dt}\,{}^R\mathbf{L}^B = \mathbf{F}$$

Similarly, if B contains continuous bodies.

4.4.7 If the resultant of all gravitational and contact forces acting on a body B is equal to zero, the linear momentum ${}^R\mathbf{L}^B$ of B in a Newtonian reference frame R (see 4.4.5) is time-independent in R (see 1.1.1). This follows from Secs. 4.4.6 and 1.2.2, and is known as *the principle of conservation of linear momentum.*

4.4.8 Given a set S of n particles situated at points P_i, $i = 1, 2, \ldots, n$, having masses m_i, and moving with velocities ${}^R\mathbf{v}^{P_i/Q}$ relative to a point Q in a reference frame R (see 2.4.1), *the angular momentum of S relative to Q in R,* denoted by ${}^R\mathbf{A}^{S/Q}$, is defined as

$${}^R\mathbf{A}^{S/Q} = \sum_{i=1}^{n} m_i\mathbf{r}_i \times {}^R\mathbf{v}^{P_i/Q}$$

where $\mathbf{r}_i$ is the position vector of P_i relative to Q.

(${}^R\mathbf{A}^{S/Q}$ is also called the *moment of momentum* of S relative to Q in R, because

$$\mathbf{r}_i \times (m_i\,{}^R\mathbf{v}^{P_i/Q})$$

is equal to the moment about Q of the linear momentum $m_i\,{}^R\mathbf{v}^{P_i/Q}$, provided this vector be regarded as having a line of action passing through point P_i.)

4.4.9 Given a continuous body C regarded as occupying a figure F, and a point Q moving in a reference frame R, *the angular* *momentum of C relative to Q in R*, defined as the limit of the corresponding angular momentum of a set of particles constructed in a manner similar to that employed in Sec. 4.1.2, can be expressed as

$$^R\mathbf{A}^{C/O} = \int_F \rho\mathbf{r} \times {}^R\mathbf{v}^{P/Q}\, d\tau$$

where ${}^R\mathbf{v}^{P/Q}$ is the velocity of a generic point P of C relative to Q in R, $\mathbf{r}$ is the position vector of P relative to Q, and ρ is the mass density of C at P.

4.4.10 The angular momentum ${}^R\mathbf{A}^{B/Q}$ of a body B (any collection of matter) relative to a point Q in a reference frame R is defined as the sum of the corresponding angular momenta of the particles (see 4.4.8) and continuous bodies (see 4.4.9) comprising B. It can be expressed in the following form:

$$^R\mathbf{A}^{B/Q} = {}^R\mathbf{A}^{B/P^*} + {}^R\mathbf{A}^{P^*/Q} \tag{1}$$

where P^* is the mass center of B and ${}^R\mathbf{A}^{P^*/Q}$ is the angular momentum relative to Q in R of a particle situated at P^* and having a mass m equal to that of B, that is,

$$^R\mathbf{A}^{P^*/Q} = m\mathbf{r}^* \times {}^R\mathbf{v}^{P^*/Q} \tag{2}$$

where $\mathbf{r}^*$ is the position vector of P^* relative to Q.

Proof (see Fig. 4.4.10 for notation):

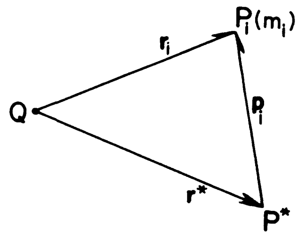


FIG. 4.4.10

Note that by definition of P^*

$$\sum_{i=1}^{n} m_i \mathbf{p}_i = 0 \tag{A}$$

from which it follows (use 1.4.1 and 2.4.1) that

$$\sum_{i=1}^{n} m_i {}^R\mathbf{v}^{P_i/P^*} = 0 \tag{B}$$

Then, if B is a set of n particles,

$$
\begin{aligned}
{}^R\mathbf{A}^{B/Q} &\underset{(4.4.8)}{=} \sum_{i=1}^{n} m_i \mathbf{r}_i \times {}^R\mathbf{v}^{P_i/Q} \\
&= \sum_{i=1}^{n} m_i(\underset{(F4.4.10)}{\mathbf{r}^* + \mathbf{p}_i}) \times (\underset{(2.4.4)}{{}^R\mathbf{v}^{P_i/P^*} + {}^R\mathbf{v}^{P^*/Q}}) \\
&= \mathbf{r}^* \times \sum_{i=1}^{n} m_i {}^R\mathbf{v}^{P_i/P^*} + \sum_{i=1}^{n} m_i \mathbf{p}_i \times {}^R\mathbf{v}^{P_i/P^*} \\
&\qquad + \left(\sum_{i=1}^{n} m_i\right) \mathbf{r}^* \times {}^R\mathbf{v}^{P^*/Q} + \left(\sum_{i=1}^{n} m_i \mathbf{p}_i\right) \times {}^R\mathbf{v}^{P^*/Q} \\
&\underset{(B)}{=} 0 + \underset{(4.4.8)}{{}^R\mathbf{A}^{B/P^*}} + m\mathbf{r}^* \times {}^R\mathbf{v}^{P^*/Q} + \underset{(A)}{0}
\end{aligned}
$$

Similarly,if B contains continuous bodies.

4.4.11 The angular momentum ${}^{R'}\mathbf{A}^{R/Q}$ of a *rigid body* R relative to a *point* Q *fixed on* R can be expressed as

$$ {}^{R'}\mathbf{A}^{R/Q} = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \, \boldsymbol{\Phi}_o^{R/Q} \tag{1} $$

or as

$$ {}^{R'}\mathbf{A}^{R/Q} = \sum_{i=1}^{3} \sum_{j=1}^{3} \phi_{ij}^{R/Q} \, {}^{R'}\omega_i{}^R \mathbf{n}_j \tag{2} $$

where $\mathbf{n}_o$ is a unit vector parallel to the angular velocity (${}^{R'}\boldsymbol{\omega}^R$) of R in the reference frame R' (see 2.2.1) and $\mathbf{n}_j$, $j = 1, 2, 3$, are mutually perpendicular unit vectors.

Proof (see Fig. 4.4.11 for notation):

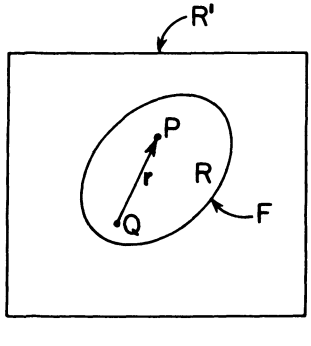


FIG. 4.4.11

$$^{R'}\mathbf{A}^{R/Q} \underset{(4.4.9)}{=} \int_F \rho\mathbf{r} \times {}^{R'}\mathbf{v}^{P/Q}\, d\tau \underset{(2.4.5)}{=} \int_F \rho\mathbf{r} \times ({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r})\, d\tau$$

If $\mathbf{n}_o$ is a unit vector parallel to ${}^{R'}\boldsymbol{\omega}^R$, the following is an identity:

$$^{R'}\boldsymbol{\omega}^R = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R\, \mathbf{n}_o$$

Hence

$$^{R'}\mathbf{A}^{R/Q} = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \int_F \rho\mathbf{r} \times (\mathbf{n}_o \times \mathbf{r})\, d\tau \underset{(3.5.2)}{=} \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R\, \boldsymbol{\Phi}_o^{R/Q}$$

Next

$$\boldsymbol{\Phi}_o^{R/Q} \underset{(3.5.7,3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/Q} o_i \mathbf{n}_j$$

where o_i is the $\mathbf{n}_i$ measure number of $\mathbf{n}_o$, that is,

$$o_i = \frac{{}^{R'}\omega_i{}^R}{\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R}$$

Thus

$$^{R'}\mathbf{A}^{R/Q} = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/Q} \frac{{}^{R'}\omega_i{}^R}{\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R}\, \mathbf{n}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/Q}\, {}^{R'}\omega_i{}^R\, \mathbf{n}_j$$

Problem: Referring to Example 4.3.4, determine the angular momentum $\mathbf{A}^{S/O}$ of the system S of bodies consisting of A, B, P, P', and Q.

Solution:

$$\mathbf{A}^{S/O} \underset{(4.4.10)}{=} \mathbf{A}^{A/O} + \mathbf{A}^{R/O} + \mathbf{A}^{Q/O}$$

$$\begin{aligned}
\mathbf{A}^{A/O} &\underset{(2)}{=} \phi_{11}^{A/O}\omega_1^A\mathbf{b}_1 + \phi_{22}^{A/O}\omega_2^A\mathbf{b}_2 + \phi_{33}^{A/O}\omega_3^A\mathbf{b}_3 \\
&\underset{(F4.3.4d)}{=} \phi_{11}^{A/O}\dot\gamma\mathbf{b}_1 + \phi_{11}^{A/O}\dot\delta\cos\gamma\,\mathbf{b}_2 + \phi_{33}^{A/O}(\dot\beta - \dot\delta\sin\gamma)\mathbf{b}_3
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^{R/O} &\underset{(2)}{=} (\phi_{11}^{R/O}\omega_1^B + \phi_{21}^{R/O}\omega_2^B + \phi_{31}^{R/O}\omega_3^B)\mathbf{b}_1 \\
&\quad + (\phi_{12}^{R/O}\omega_1^B + \phi_{22}^{R/O}\omega_2^B + \phi_{32}^{R/O}\omega_3^B)\mathbf{b}_2 \\
&\quad + (\phi_{13}^{R/O}\omega_1^B + \phi_{23}^{R/O}\omega_2^B + \phi_{33}^{R/O}\omega_3^B)\mathbf{b}_3 \\
&\underset{(F4.3.4d)}{=} (\phi_{11}^{R/O}\dot\gamma - \phi_{31}^{R/O}\dot\delta\sin\gamma)\mathbf{b}_1 \\
&\quad + \phi_{22}^{R/O}\dot\delta\cos\gamma\,\mathbf{b}_2 + (\phi_{13}^{R/O}\dot\gamma - \phi_{33}^{R/O}\dot\delta\sin\gamma)\mathbf{b}_3
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^{Q/O} &\underset{(4.4.8)}{=} M(z\mathbf{b}_3) \times \mathbf{v}^{Q/O} = zM\mathbf{b}_3 \times [\boldsymbol{\omega}^A \underset{(2.4.5)}{\times} (z\mathbf{b}_3)] \\
&= z^2M(\dot\gamma\mathbf{b}_1 + \dot\delta\cos\gamma\,\mathbf{b}_2)
\end{aligned}$$

Thus, using ϕ_1, ϕ_2, ϕ_3 as defined in Eqs. (R), (S), and (T) of Example 4.3.4,

$$\mathbf{A}^{S/O} = [(\phi_1 + Mz^2)\dot{\gamma} - \phi_{31}^{R/O}\dot{\delta} \sin \gamma]\mathbf{b}_1 + (\phi_2 + Mz^2)\dot{\delta} \cos \gamma \, \mathbf{b}_2 + (\phi_{33}^{A/O}\dot{\beta} + \phi_{13}^{R/O}\dot{\gamma} - \phi_3\dot{\delta} \sin \gamma)\mathbf{b}_3$$

4.4.12 The angular momentum ${}^R\mathbf{A}^{B/O}$ (see 4.4.10) depends on the position of the point O, even when this point is fixed in reference frame R (compare with 4.4.5). Consequently no meaning is attributed to the phrase "absolute angular momentum of B in R."

4.4.13 If Q is any point fixed in a Newtonian reference frame R (see 4.2.1), or if Q is the mass center of a body B, then $\mathbf{M}^Q$, the sum of the moments about Q of all gravitational and contact forces acting on B, is related to the angular momentum ${}^R\mathbf{A}^{B/Q}$ (see 4.4.10) as follows:

$$\mathbf{M}^Q = \frac{{}^Rd\,{}^R\mathbf{A}^{B/Q}}{dt}$$

This equality is known as *the angular momentum principle.*

Proof: If B is a set S of particles situated at points P_i, $i = 1, 2, \ldots, n$, and $\mathbf{r}_i$ is the position vector of P_i relative to Q, then

$$\frac{{}^Rd\,{}^R\mathbf{A}^{B/Q}}{dt} \underset{(4.4.8)}{=} \frac{{}^Rd}{dt}\sum_{i=1}^{n} m_i\mathbf{r}_i \times {}^R\mathbf{v}^{P_i/Q}$$

$$\underset{(1.4.1,1.5.3)}{=} \sum_{i=1}^{n} m_i \left(\frac{{}^Rd\mathbf{r}_i}{dt} \times {}^R\mathbf{v}^{P_i/Q} + \mathbf{r}_i \times \frac{{}^Rd\,{}^R\mathbf{v}^{P_i/Q}}{dt}\right)$$

$$\underset{(2.4.1)}{=} 0 + \sum_{i=1}^{n} m_i\mathbf{r}_i \times {}^R\mathbf{a}^{P_i/Q}$$

$$\underset{(2.5.15)}{=} \sum_{i=1}^{n} m_i\mathbf{r}_i \times ({}^R\mathbf{a}^{P_i} - {}^R\mathbf{a}^{Q})$$

$$= \sum_{i=1}^{n} m_i\mathbf{r}_i \times {}^R\mathbf{a}^{P_i} - \left(\sum_{i=1}^{n} m_i\mathbf{r}_i\right) \times {}^R\mathbf{a}^{Q}$$

$$\underset{(4.1.1)}{=} -\sum_{i=1}^{n} \mathbf{r}_i \times {}^R\mathbf{F}^{P_i} - \left(\sum_{i=1}^{n} m_i\mathbf{r}_i\right) \times {}^R\mathbf{a}^{Q}$$

But

$$\mathbf{M}^Q + \sum_{i=1}^{n} \mathbf{r}_i \times {}^R\mathbf{F}^{P_i} \underset{(4.2.1)}{=} 0$$

Hence

$$-\sum_{i=1}^{n} \mathbf{r}_i \times {}^R\mathbf{F}^{P_i} = \mathbf{M}^Q$$

and

$$\frac{{}^Rd}{dt}\, {}^R\mathbf{A}^{B/Q} = \mathbf{M}^Q - \left(\sum_{i=1}^{n} m_i\mathbf{r}_i\right) \times {}^R\mathbf{a}^Q$$

Now

$$\sum_{i=1}^{n} m_i\mathbf{r}_i = 0$$

if Q is the mass center of B, and

$${}^R\mathbf{a}^Q = 0$$

if Q is fixed in R. In either case, therefore,

$$\mathbf{M}^Q = \frac{{}^Rd\,{}^R\mathbf{A}^{B/Q}}{dt}$$

Similarly, if B contains continuous bodies.

4.4.14 A modified form of the angular momentum principle (see 4.4.13), which frequently saves labor in writing scalar equations of motion, is the following:

$$\mathbf{M}^Q \cdot \mathbf{n} = \frac{d}{dt}\left({}^R\mathbf{A}^{B/Q} \cdot \mathbf{n}\right) - {}^R\mathbf{A}^{B/Q} \cdot \frac{{}^Rd\mathbf{n}}{dt}$$

where $\mathbf{n}$ is any unit vector, and $\mathbf{M}^Q$ and ${}^R\mathbf{A}^{B/Q}$ are defined as in Sec. 4.4.13.

Proof:

$$\mathbf{M}^Q \cdot \mathbf{n} \underset{(4.4.13)}{=} \left(\frac{{}^Rd\,{}^R\mathbf{A}^{B/Q}}{dt}\right) \cdot \mathbf{n}$$

Now

$$\frac{d}{dt}\left({}^R\mathbf{A}^{B/Q} \cdot \mathbf{n}\right) \underset{(1.5.2)}{=} \left(\frac{{}^Rd\,{}^R\mathbf{A}^{B/Q}}{dt}\right) \cdot \mathbf{n} + {}^R\mathbf{A}^{B/Q} \cdot \frac{{}^Rd\mathbf{n}}{dt}$$

Hence

$$\left(\frac{{}^Rd\,{}^R\mathbf{A}^{B/Q}}{dt}\right) \cdot \mathbf{n} = \frac{d}{dt}\left({}^R\mathbf{A}^{B/Q} \cdot \mathbf{n}\right) - {}^R\mathbf{A}^{B/Q} \cdot \frac{{}^Rd\mathbf{n}}{dt}$$

and

$$\mathbf{M}^Q \cdot \mathbf{n} = \frac{d}{dt}({}^R\mathbf{A}^{B/Q} \cdot \mathbf{n}) - {}^R\mathbf{A}^{B/Q} \cdot \frac{{}^R d\mathbf{n}}{dt}$$

Example: Referring to Example 4.3.4 and Problem 4.4.11, and letting $\mathbf{M}^O$ be the moment about point O of all gravitational and contact forces acting on S (the body consisting of A, B, P, P', and Q), $\mathbf{M}^O \cdot \mathbf{b}_1$ is seen to be given by

$$\mathbf{M}^O \cdot \mathbf{b}_1 = zMg \cos \gamma$$

Hence

$$zMg \cos \gamma = \frac{d}{dt}(\mathbf{A}^{S/O} \cdot \mathbf{b}_1) - \mathbf{A}^{S/O} \cdot \frac{d\mathbf{b}_1}{dt}$$

Now

$$\mathbf{A}^{S/O} \cdot \mathbf{b}_1 \underset{(\text{P4.4.11})}{=} (\phi_1 + Mz^2)\dot{\gamma} - \phi_{31}^{R/O}\dot{\delta} \sin \gamma$$

so that

$$\frac{d}{dt}(\mathbf{A}^{S/O} \cdot \mathbf{b}_1) = (\phi_1 + Mz^2)\ddot{\gamma} - \phi_{31}^{R/O}(\ddot{\delta} \sin \gamma + \dot{\delta}\dot{\gamma} \cos \gamma)$$

Also

$$\frac{d\mathbf{b}_1}{dt} \underset{(1.7.1)}{=} -\dot{\delta}\mathbf{c}_3 = -\dot{\delta}(\sin \gamma\, \mathbf{b}_2 + \cos \gamma\, \mathbf{b}_3)$$

so that

$$\mathbf{A}^{S/O} \cdot \frac{d\mathbf{b}_1}{dt} \underset{(\text{P4.4.11})}{=} -\dot{\delta}[(\phi_2 + Mz^2)\dot{\delta} \sin \gamma \cos \gamma + (\phi_{33}^{A/O}\dot{\beta} + \phi_{31}^{R/O}\dot{\gamma} - \phi_3\dot{\delta} \sin \gamma) \cos \gamma]$$

Thus

$$zMg \cos \gamma = (\phi_1 + Mz^2)\ddot{\gamma} - \phi_{31}^{R/O}\ddot{\delta} \sin \gamma + \dot{\delta}[(\phi_2 + Mz^2 - \phi_3)\dot{\delta} \sin \gamma + \phi_{33}^{A/O}\dot{\beta}] \cos \gamma$$

in agreement with Eq. (P) of Example 4.3.4. (Equation (Q) of Example 4.3.4 may be obtained in a similar manner, $\mathbf{c}_2$ playing the part of $\mathbf{n}$.)

4.4.15 If Q is any point fixed in a Newtonian reference frame R (see 4.2.1), or if Q is the mass center of a body B, and the sum of the moments about Q of all gravitational and contact forces acting on B is equal to zero, then the angular momentum ${}^R\mathbf{A}^{B/Q}$ (see 4.4.10) is time-independent in R (see 1.1.1). This follows from

Secs. 4.4.13 and 1.2.2, and is known as *the principle of conservation of angular momentum.*

Example: When the system of all gravitational and contact forces acting on a rigid body R is equivalent to a single force whose line of action passes through the mass center P^* of R it is possible for R to move in such a way that the angular velocity ${}^{R'}\boldsymbol{\omega}^R$ of R in a Newtonian reference frame R' remains fixed in both R and R': ${}^{R'}\boldsymbol{\omega}^R$ must be parallel to a principal axis of R for P^*, and the motion is stable if and only if the moment of inertia of R about this axis is either larger or smaller than R's moment of inertia about any other line passing through P^*. This is shown as follows:

Let $\mathbf{n}_i$, $i = 1, 2, 3$, be mutually perpendicular principal directions of R for P^*, and let these be fixed in R (see 3.5.7 and 3.3.9). Then, if ω_i is the $\mathbf{n}_i$ measure number of ${}^{R'}\boldsymbol{\omega}^R$,

$$
{}^{R'}\mathbf{A}^{R/P^*} \underset{(4.4.11)}{=} \phi_{11}^{R/P^*}\omega_1\mathbf{n}_1 + \phi_{22}^{R/P^*}\omega_2\mathbf{n}_2 + \phi_{33}^{R/P^*}\omega_3\mathbf{n}_3 \tag{A}
$$

and

$$
\begin{aligned}
\frac{{}^{R'}d\,{}^{R'}\mathbf{A}^{R/P^*}}{dt} &\underset{(2.1.4)}{=} \frac{{}^{R}d\,{}^{R'}\mathbf{A}^{R/P^*}}{dt} + {}^{R'}\boldsymbol{\omega}^R \times {}^{R'}\mathbf{A}^{R/P^*} \\
&\underset{(A)}{=} [\phi_{11}^{R/P^*}\dot{\omega}_1 - (\phi_{22}^{R/P^*} - \phi_{33}^{R/P^*})\omega_2\omega_3]\mathbf{n}_1 \\
&\qquad + [\phi_{22}^{R/P^*}\dot{\omega}_2 - (\phi_{33}^{R/P^*} - \phi_{11}^{R/P^*})\omega_3\omega_1]\mathbf{n}_2 \\
&\qquad\qquad + [\phi_{33}^{R/P^*}\dot{\omega}_3 - (\phi_{11}^{R/P^*} - \phi_{22}^{R/P^*})\omega_1\omega_2]\mathbf{n}_3
\end{aligned}
$$

so that the principle of conservation of angular momentum gives

$$
\begin{aligned}
\dot{\omega}_1 &= \omega_2\omega_3 C_1 \\
\dot{\omega}_2 &= \omega_3\omega_1 C_2 \\
\dot{\omega}_3 &= \omega_1\omega_2 C_3
\end{aligned} \tag{B}
$$

where the constants C_1, C_2, and C_3 are defined as

$$
\begin{aligned}
C_1 &= (\phi_{22}^{R/P^*} - \phi_{33}^{R/P^*})/\phi_{11}^{R/P^*} \\
C_2 &= (\phi_{33}^{R/P^*} - \phi_{11}^{R/P^*})/\phi_{22}^{R/P^*} \\
C_3 &= (\phi_{11}^{R/P^*} - \phi_{22}^{R/P^*})/\phi_{33}^{R/P^*}
\end{aligned} \tag{C}
$$

Equations (B) show that two of the measure numbers of ${}^{R'}\boldsymbol{\omega}^R$ must vanish if all three are to remain constant, which means that

${}^{R'}\boldsymbol{\omega}^R$, in order to remain fixed in R (and hence in R', see 2.1.5), must be parallel to a principal axis of R for P^*.

To investigate the stability of such a motion, take

$$\omega_1 = \omega_1{}^O + \omega_1{}^*, \qquad \omega_2 = \omega_2{}^*, \qquad \omega_3 = \omega_3{}^*$$

where $\omega_1{}^O$ is a constant and the starred quantities are functions of time. Substitute into Eqs. (B) and drop all terms of the second degree in the starred quantities:

$$\dot{\omega}_1{}^* = 0$$

$$\dot{\omega}_2{}^* = \omega_1{}^O\omega_3{}^*C_2$$

$$\dot{\omega}_3{}^* = \dot{\omega}_1{}^O\dot{\omega}_2{}^*C_3$$

Differentiate the second of these equations and use $\dot{\omega}_3{}^*$ as given by the third; then differentiate the third and use $\dot{\omega}_2{}^*$ as given by the second. The following differential equations are then seen to govern $\omega_2{}^*$ and $\omega_3{}^*$:

$$\ddot{\omega}_2{}^* - (\omega_1{}^O)^2C_2C_3\omega_2{}^* = 0$$

$$\ddot{\omega}_3{}^* - (\omega_1{}^O)^2C_2C_3\omega_3{}^* = 0$$

Hence, in order that the motion

$$\omega_1 = \omega_1{}^O, \qquad \omega_2 = \omega_3 = 0$$

be stable, it is necessary and sufficient that

$$C_2C_3 < 0$$

or, from Eqs. (C), that

$$(\phi_{11}^{R/P^*} - \phi_{22}^{R/P^*})(\phi_{11}^{R/P^*} - \phi_{33}^{R/P^*}) > 0$$

This inequality can be satisfied in only two ways: Either

$$\phi_{11}^{R/P^*} > \phi_{22}^{R/P^*} \quad \text{and} \quad \phi_{11}^{R/P^*} > \phi_{33}^{R/P^*}$$

or

$$\phi_{11}^{R/P^*} < \phi_{22}^{R/P^*} \quad \text{and} \quad \phi_{11}^{R/P^*} < \phi_{33}^{R/P^*}$$

In the first case, the moment of inertia of R about the principal axis passing through P and parallel to ${}^{R'}\boldsymbol{\omega}^R$ is larger, and in the second case it is smaller, than R's moment of inertia about any other line passing through P^*, since R's largest and smallest principal moments of inertia are respectively larger and smaller than R's moment of inertia about any line which passes through P^* and is not a principal axis of R for P^* (see 3.5.7 and 3.3.14).

4.4.16 The angular momentum principle (see 4.4.13 and 4.4.14) furnishes a convenient means for the derivation of differential equations governing the motion of a system when $\mathbf{M}^Q$ does not involve any unknown contact forces (see, for instance, Example 4.4.14). In particular, this is the case whenever the principle of conservation of angular momentum (see 4.4.15) is applicable. When use of the angular momentum principle necessitates the introduction (and subsequent elimination by application of, for example, the linear momentum principle) of contact forces which do not appear in a solution based on D'Alembert's principle (see 4.2.1, 4.2.3, and 4.3.1), the latter principle yields results more readily.

For the purpose of determining contact forces when the motion of a system is known (see 4.3.2), D'Alembert's principle is frequently more convenient than the angular momentum principle, because of the restrictive character of the point Q (see 4.4.13).

4.5 Activity and kinetic energy

4.5.1 Given a set S of n particles situated at points P_i, $i = 1, 2, \ldots, n$, having masses m_i, and moving with velocities ${}^R\mathbf{v}^{P_i/Q}$ relative to a point Q in a reference frame R (see 2.4.1), *the kinetic energy of S relative to Q in R*, denoted by ${}^RK^{S/Q}$, is defined as

$$ {}^RK^{S/Q} = \frac{1}{2} \sum_{i=1}^{n} m_i({}^R\mathbf{v}^{P_i/Q})^2 $$

4.5.2 Given a continuous body C regarded as occupying a figure F, and a point Q moving in a reference frame R, *the kinetic energy of C relative to Q in R*, defined as the limit of the corresponding kinetic energy of a set of particles constructed in a manner similar to that employed in Sec. 4.1.2, can be expressed as

$$ {}^RK^{C/Q} = \frac{1}{2} \int_F \rho({}^R\mathbf{v}^{P/Q})^2 \, d\tau $$

where ${}^R\mathbf{v}^{P/Q}$ is the velocity relative to Q in R of a generic point P of C, and ρ is the mass density of C at P.

4.5.3 *The kinetic energy of a body B* (any collection of matter) *relative to a point Q in a reference frame R* is defined as the sum of the corresponding kinetic energies of the particles and continuous bodies comprising B (see 4.5.1 and 4.5.2). It can be expressed in the following form:

$$^R K^{B/Q} = {^R K^{B/P^*}} + {^R K^{P^*/Q}} \tag{1}$$

where P^* is the mass center of B and $^R K^{P^*/Q}$ is the kinetic energy relative to Q in R of a particle situated at P^* and having a mass m equal to that of B, that is,

$$^R K^{P^*/Q} = \tfrac{1}{2} m ({^R \mathbf{v}^{P^*/Q}})^2 \tag{2}$$

Proof (see Fig. 4.5.3 for notation): If B is a set S of n particles, then by definition of P^*

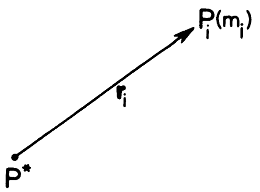


FIG. 4.5.3

$$\sum_{i=1}^{n} m_i \mathbf{r}_i = 0$$

from which it follows (use 1.4.1 and 2.4.1) that

$$\sum_{i=1}^{n} m_i {^R \mathbf{v}^{P_i/P^*}} = 0 \tag{A}$$

Now

$$\begin{aligned}
{^R K^{S/Q}} &\underset{(4.5.1)}{=} \tfrac{1}{2} \sum_{i=1}^{n} m_i ({^R \mathbf{v}^{P_i/Q}})^2 \\
&\underset{(2.4.4)}{=} \tfrac{1}{2} \sum_{i=1}^{n} m_i ({^R \mathbf{v}^{P_i/P^*}} + {^R \mathbf{v}^{P^*/Q}})^2 \\
&= \tfrac{1}{2} \sum_{i=1}^{n} m_i ({^R \mathbf{v}^{P_i/P^*}})^2 + \left(\sum_{i=1}^{n} m_i {^R \mathbf{v}^{P_i/P^*}} \right) \cdot {^R \mathbf{v}^{P^*/Q}} \\
&\qquad + \tfrac{1}{2} \sum_{i=1}^{n} m_i ({^R \mathbf{v}^{P^*/Q}})^2 \\
&= \underset{(4.5.1)}{{^R K^{S/P^*}}} + \underset{(A)}{0} + \tfrac{1}{2} m ({^R \mathbf{v}^{P^*/Q}})^2
\end{aligned}$$

Similarly, if B contains continuous bodies.

4.5.4 *The kinetic energy* ${}^{R'}K^{R/Q}$ *of a rigid body* R *relative to a point* Q *fixed on* R can be expressed as

$$ {}^{R'}K^{R/Q} = \tfrac{1}{2}\phi_{oo}^{R/Q}({}^{R'}\boldsymbol{\omega}^R)^2 \tag{1}$$

or, as

$$ {}^{R'}K^{R/Q} = \tfrac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\phi_{ij}^{R/Q}\,{}^{R'}\omega_i{}^R\,{}^{R'}\omega_j{}^R \tag{2}$$

where ${}^{R'}\boldsymbol{\omega}^R$ is the angular velocity of R in R', $\phi_{oo}^{R/Q}$ is the moment of inertia of R about a line passing through Q and parallel to ${}^{R'}\boldsymbol{\omega}^R$, and ${}^{R'}\omega_i{}^R$, $i = 1, 2, 3$, are the measure numbers of any three mutually perpendicular components of ${}^{R'}\boldsymbol{\omega}^R$.

Proof (see Fig. 4.5.4a for notation):

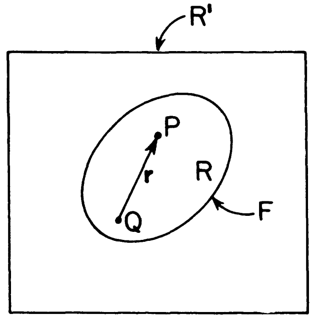


FIG. 4.5.4a

$$ {}^{R'}K^{R/Q} \underset{(4.5.2)}{=} \tfrac{1}{2}\int_F \rho({}^{R'}\mathbf{v}^{P/Q})^2\,d\tau $$

or

$$ {}^{R'}K^{R/Q} \underset{(2.4.5)}{=} \tfrac{1}{2}\int_F \rho({}^{R'}\boldsymbol{\omega}^R \times \mathbf{r})^2\,d\tau \tag{A}$$

Let $\mathbf{n}_o$ be a unit vector parallel to ${}^{R'}\boldsymbol{\omega}^R$, $\mathbf{n}_i$, $i = 1, 2, 3$, mutually perpendicular unit vectors, and o_i the $\mathbf{n}_i$ measure number of $\mathbf{n}_o$. Then

$$ {}^{R'}\boldsymbol{\omega}^R = \mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R\,\mathbf{n}_o \tag{B}$$

and

$$ o_i = \frac{{}^{R'}\omega_i{}^R}{\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R} \tag{C}$$

Hence

$$^{R'}K^{R/Q} \underset{(A,B)}{=} \frac{1}{2}\int_F \rho(\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R\, \mathbf{n}_o \times \mathbf{r})^2\, d\tau$$

$$= \frac{1}{2}(\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R)^2 \int_F \rho(\mathbf{n}_o \times \mathbf{r})^2\, d\tau$$

or

$$^{R'}K^{R/Q} \underset{(3.5.4)}{=} \frac{1}{2}(\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R)^2 \phi_{oo}^{R/Q} \qquad \text{(D)}$$

But

$$(\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R)^2 \underset{(B)}{=} ({}^{R'}\boldsymbol{\omega}^R)^2$$

Thus

$$^{R'}K^{R/Q} = \frac{1}{2}({}^{R'}\boldsymbol{\omega}^R)^2 \phi_{oo}^{R/Q}$$

Next

$$\phi_{oo}^{R/Q} \underset{(3.5.7,3.2.4)}{=} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/Q}\, o_i o_j \underset{(C)}{=} \frac{1}{(\mathbf{n}_o \cdot {}^{R'}\boldsymbol{\omega}^R)^2} \sum_{i=1}^{3}\sum_{j=1}^{3} \phi_{ij}^{R/Q}\, {}^{R'}\omega_i{}^{RR'}\omega_j{}^{R}$$

Substitute into Eq. (D).

Problem: A uniform solid sphere S of mass m performs a pure rolling motion (see 2.5.10) on a surface fixed in a reference frame R. A particle P of the same mass m moves in such a way that its velocity in R is at every instant equal to the velocity in R of the center C of S.

Determine the ratio of the kinetic energy of S relative to O and the kinetic energy of P relative to O, O being a point fixed in R.

Solution: Let r be the radius of S, $\mathbf{r}$ the position vector of C

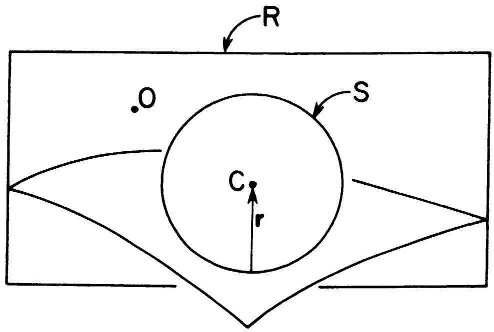


FIG. 4.5.4b

relative to the point of contact between S and the surface on which S rolls (see Fig. 4.5.4b). Then

$$
\begin{aligned}
{}^{R}K^{S/O} &\underset{(4.5.3)}{=} {}^{R}K^{S/C} + \tfrac{1}{2}m({}^{R}\mathbf{v}^{C/O})^2 \\
&= \underset{(1)}{\tfrac{1}{2}\phi_{oo}^{S/C}}({}^{R}\boldsymbol{\omega}^{S})^2 + \tfrac{1}{2}m\underset{(2.5.10)}{({}^{R}\boldsymbol{\omega}^{S} \times \mathbf{r})^2}
\end{aligned}
$$

The moment of inertia of S about all lines passing through C has the value $2mr^2/5$ (see Appendix, Fig. 8); and for pure rolling (see 2.5.10), ${}^{R}\boldsymbol{\omega}^{S}$ is perpendicular to $\mathbf{r}$, so that

$$({}^{R}\boldsymbol{\omega}^{S} \times \mathbf{r})^2 = r^2({}^{R}\boldsymbol{\omega}^{S})^2 \tag{A}$$

Hence

$${}^{R}K^{S/O} = \frac{mr^2}{5}({}^{R}\boldsymbol{\omega}^{S})^2 + \frac{mr^2}{2}({}^{R}\boldsymbol{\omega}^{S})^2 = \frac{7}{10}mr^2({}^{R}\boldsymbol{\omega}^{S})^2$$

Next

$${}^{R}K^{P/O} \underset{(4.5.1)}{=} \tfrac{1}{2}m({}^{R}\mathbf{v}^{P/O})^2$$

By hypothesis

$${}^{R}\mathbf{v}^{P/O} = {}^{R}\mathbf{v}^{C/O} \underset{(2.5.10)}{=} {}^{R}\boldsymbol{\omega}^{S} \times \mathbf{r}$$

Thus

$${}^{R}K^{P/O} = \frac{1}{2}m({}^{R}\boldsymbol{\omega}^{S} \times \mathbf{r})^2 \underset{(A)}{=} \frac{mr^2}{2}({}^{R}\boldsymbol{\omega}^{S})^2$$

and

$$\frac{{}^{R}K^{S/O}}{{}^{R}K^{P/O}} = \frac{7}{5}$$

4.5.5 If O is a point fixed in a reference frame R, the kinetic energy ${}^{R}K^{B/O}$ of a body B (see 4.5.3) is independent of the position of O in R, because the velocity ${}^{R}\mathbf{v}^{P/O}$ of any point P of B is independent of the position of O (see 2.5.1). Accordingly, the kinetic energy of B in R relative to any point fixed in R is called the *absolute kinetic energy of B in R*, or, for short, *the* kinetic energy of B in R. It is denoted by ${}^{R}K^{B}$ and involves only absolute velocities, that is,

$${}^{R}K^{S} \underset{(4.5.1)}{=} \tfrac{1}{2}\sum_{i=1}^{n} m_i({}^{R}\mathbf{v}^{P_i})^2 \tag{1}$$

$${}^{R}K^{C} \underset{(4.5.2)}{=} \tfrac{1}{2}\int_F \rho({}^{R}\mathbf{v}^{P})^2\, d\tau \tag{2}$$

and can be expressed as

$$ {}^{R}K^{B} \underset{(4.5.3)}{=} {}^{R}K^{B/P^*} + {}^{R}K^{P^*} \tag{3} $$

where

$$ {}^{R}K^{P^*} \underset{(4.5.3)}{=} \tfrac{1}{2}m({}^{R}\mathbf{v}^{P^*})^2 \tag{4} $$

and m is the mass of B, while P^* is B's mass center.

Problem: Referring to Example 4.3.4, determine the kinetic energy ${}^{D}K^{S}$ of the system S of bodies consisting of A, B, C, P, P', and Q, in a reference frame in which D is fixed.

Solution:

$$ {}^{D}K^{S} \underset{(4.5.3)}{=} {}^{D}K^{A} + {}^{D}K^{R} + {}^{D}K^{Q} + {}^{D}K^{C} $$

$$ {}^{D}K^{A} \underset{(4.5.4)}{=} \tfrac{1}{2}[\phi_{11}^{A/O}(\omega_1^A)^2 + \phi_{22}^{A/O}(\omega_2^A)^2 + \phi_{33}^{A/O}(\omega_3^A)^2] $$

$$ \underset{(\mathrm{F}4.3.4\mathrm{d})}{=} \tfrac{1}{2}[\phi_{11}^{A/O}\dot{\gamma}^2 + \phi_{11}^{A/O}\dot{\delta}^2 \cos^2\gamma + \phi_{33}^{A/O}(\beta - \dot{\delta}\sin\gamma)^2] $$

$$ \begin{aligned} {}^{D}K^{R} \underset{(4.5.4)}{=} \tfrac{1}{2}[&(\phi_{11}^{R/O}\omega_1^B + \phi_{21}^{R/O}\omega_2^B + \phi_{31}^{R/O}\omega_3^B)\omega_1^B \\ &+ (\phi_{12}^{R/O}\omega_1^B + \phi_{22}^{R/O}\omega_2^B + \phi_{32}^{R/O}\omega_3^B)\omega_2^B \\ &+ (\phi_{13}^{R/O}\omega_1^B + \phi_{23}^{R/O}\omega_2^B + \phi_{33}^{R/O}\omega_3^B)\omega_3^B] \end{aligned} $$

$$ \begin{aligned} = \tfrac{1}{2}\{&[\phi_{11}^{R/O}\dot{\gamma} - \phi_{31}^{R/O}\dot{\delta}\sin\gamma]\dot{\gamma} + \phi_{22}^{R/O}\dot{\delta}^2\cos^2\gamma \\ &+ [-\phi_{31}^{R/O}\dot{\gamma} + \phi_{33}^{R/O}\dot{\delta}\sin\gamma]\dot{\delta}\sin\gamma\} \end{aligned} $$

$$ {}^{D}K^{Q} \underset{(4.5.1)}{=} \tfrac{1}{2}M(\mathbf{v}^Q)^2 \underset{(2.4.5)}{=} \tfrac{1}{2}M(\boldsymbol{\omega}^A \times z\mathbf{b}_3)^2 $$

$$ = \tfrac{1}{2}Mz^2(\dot{\delta}\cos\gamma\,\mathbf{b}_1 - \dot{\gamma}\mathbf{b}_2)^2 = \tfrac{1}{2}Mz^2(\dot{\delta}^2\cos^2\gamma + \dot{\gamma}^2) $$

$$ {}^{D}K^{C} \underset{(4.5.4)}{=} \tfrac{1}{2}\phi_{22}^{C/O}(\omega_2^C)^2 = \tfrac{1}{2}\phi_{22}^{C/O}\dot{\delta}^2 $$

Substitute:

$$ \begin{aligned} {}^{D}K^{S} = \tfrac{1}{2}\{&(\phi_1 + Mz^2)\dot{\gamma}^2 + [(\phi_2 + Mz^2)\cos^2\gamma + \phi_3\sin^2\gamma + \phi_{22}^{C/O}]\dot{\delta}^2 \\ &- 2\phi_{31}^{R/O}\dot{\gamma}\dot{\delta}\sin\gamma + \phi_{33}^{A/O}(\dot{\beta} - 2\dot{\delta}\sin\gamma)\dot{\beta}\} \end{aligned} $$

4.5.6 The kinetic energy ${}^{R}K^{P}$ of a particle P in a Newtonian reference frame R (see 4.5.5), the velocity ${}^{R}\mathbf{v}^{P}$, and the resultant $\mathbf{F}$ of all gravitational and contact forces acting on P are related as follows:

$$\frac{d^R K^P}{dt} = {}^R A^P \tag{1}$$

where ${}^R A^P$, called the *activity in R of the gravitational and contact forces acting on P*, is given by

$$ {}^R A^P = {}^R\mathbf{v}^P \cdot \mathbf{F} \tag{2}$$

Proof: The inertia force ${}^R\mathbf{F}^P$ acting on P in R (see 4.1.1) can be expressed as

$$ {}^R\mathbf{F}^P \underset{(2.5.1)}{=} -m\,\frac{{}^R d\,{}^R\mathbf{v}^P}{dt}$$

Dot-multiply with ${}^R\mathbf{v}^P$:

$$ {}^R\mathbf{v}^P \cdot {}^R\mathbf{F}^P = -m\,{}^R\mathbf{v}^P \cdot \frac{{}^R d\,{}^R\mathbf{v}^P}{dt} \tag{A}$$

In accordance with Sec. 4.2.1,

$$\mathbf{F} + {}^R\mathbf{F}^P = 0$$

or

$$ {}^R\mathbf{F}^P = -\mathbf{F}$$

Hence

$$ {}^R\mathbf{v}^P \cdot \mathbf{F} \underset{(A)}{=} m\,{}^R\mathbf{v}^P \cdot \frac{{}^R d\,{}^R\mathbf{v}^P}{dt}$$

and, defining ${}^R A^P$ as

$$ {}^R A^P = {}^R\mathbf{v}^P \cdot \mathbf{F}$$

it follows that

$$ m\,{}^R\mathbf{v}^P \cdot \frac{{}^R d\,{}^R\mathbf{v}^P}{dt} = {}^R A^P \tag{B}$$

Next

$$\frac{d^R K^P}{dt} \underset{(4.5.5)}{=} \frac{d}{dt}\left[\frac{1}{2} m({}^R\mathbf{v}^P)^2\right] \underset{(1.5.2)}{=} m\,{}^R\mathbf{v}^P \cdot \frac{{}^R d\,{}^R\mathbf{v}^P}{dt} \underset{(B)}{=} {}^R A^P$$

Problem: In Fig. 4.5.6, R represents a Newtonian reference frame, P_1 and P_2 are particles which are fixed in R and have masses m_1 and m_2, and P is a particle of mass m, which moves under the action of gravitational forces exerted on it by P_1 and P_2.

Express the magnitude of the velocity ${}^R\mathbf{v}^P$ in terms of the gravitational constant G, the masses m_1 and m_2, and the distances x_1 and x_2.

Solution: The resultant $\mathbf{F}$ of all gravitational and contact forces acting on P is given by

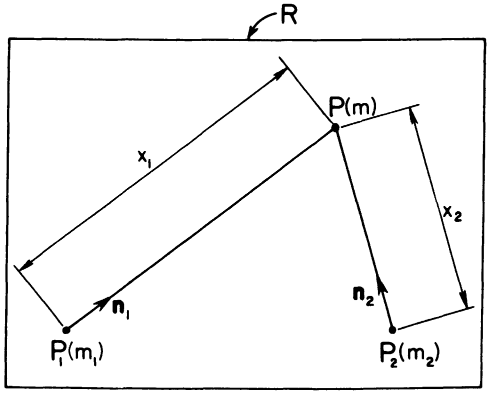


FIG. 4.5.6

$$\mathbf{F} = -\frac{Gmm_1}{x_1^2}\mathbf{n}_1 - \frac{Gmm_2}{x_2^2}\mathbf{n}_2$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the unit vectors shown in Fig. 4.5.6. Hence

$$^R A^P = {^R\mathbf{v}^P} \cdot \mathbf{F} = -Gm\left(\frac{m_1}{x_1^2}\,{^R\mathbf{v}^P} \cdot \mathbf{n}_1 + \frac{m_2}{x_2^2}\,{^R\mathbf{v}^P} \cdot \mathbf{n}_2\right) \qquad \text{(A)}$$

$^R\mathbf{v}^P$ can be expressed either as

$$^R\mathbf{v}^P \underset{(2.5.1)}{=} \frac{^R d}{dt}(x_1\mathbf{n}_1) \underset{(1.5.1)}{=} \dot{x}_1\mathbf{n}_1 + x_1\frac{^R d\mathbf{n}_1}{dt} \qquad \text{(B)}$$

or as

$$^R\mathbf{v}^P \underset{(2.5.1)}{=} \frac{^R d}{dt}(x_2\mathbf{n}_2) \underset{(1.5.1)}{=} \dot{x}_2\mathbf{n}_2 + x_2\frac{^R d\mathbf{n}_2}{dt} \qquad \text{(C)}$$

Thus

$$^R A^P \underset{(A)}{=} -Gm\left[\frac{m_1}{x_1^2}\left(\dot{x}_1 + \underset{(B)}{x_1\mathbf{n}_1} \cdot \frac{^R d\mathbf{n}_1}{dt}\right) + \frac{m_2}{x_2^2}\left(\dot{x}_2 + \underset{(C)}{x_2\mathbf{n}_2} \cdot \frac{^R d\mathbf{n}_2}{dt}\right)\right]$$

But

$$\mathbf{n}_1 \cdot \frac{^R d\mathbf{n}_1}{dt} \underset{(1.5.1)}{=} \frac{1}{2}\frac{d}{dt}(\mathbf{n}_1^2) = \frac{1}{2}\frac{d}{dt}(1) = 0$$

and, similarly,

$$\mathbf{n}_2 \cdot \frac{^R d\mathbf{n}_2}{dt} = 0$$

Consequently

$$^{R}A^{P} = -Gm\left(\frac{m_1\dot{x}_1}{x_1^2} + \frac{m_2\dot{x}_2}{x_2^2}\right) \tag{D}$$

Next

$$^{R}K^{P} = \tfrac{1}{2}m(^{R}\mathbf{v}^{P})^2 \tag{E}$$

Thus

$$\frac{d}{dt}\left[\frac{1}{2}m(^{R}\mathbf{v}^{P})^2\right] \underset{\text{(D,E)}}{=} -Gm\left(\frac{m_1\dot{x}_1}{x_1^2} + \frac{m_2\dot{x}_2}{x_2^2}\right) \tag{F}$$

Note that

$$-Gm\left(\frac{m_1\dot{x}_1}{x_1^2} + \frac{m_2\dot{x}_2}{x_2^2}\right) = \frac{d}{dt}\left[Gm\left(\frac{m_1}{x_1} + \frac{m_2}{x_2}\right)\right] \tag{G}$$

Hence

$$\frac{d}{dt}\left[\frac{1}{2}(^{R}\mathbf{v}^{P})^2 - G\left(\frac{m_1}{x_1} + \frac{m_2}{x_2}\right)\right] \underset{\text{(F,G)}}{=} 0$$

and

$$\frac{1}{2}(^{R}\mathbf{v}^{P})^2 - G\left(\frac{m_1}{x_1} + \frac{m_2}{x_2}\right) = C$$

where C is a constant. As

$$|^{R}\mathbf{v}^{P}| = [(^{R}\mathbf{v}^{P})^2]^{1/2}$$

it follows that

$$|^{R}\mathbf{v}^{P}| = \sqrt{2}\left[C + G\left(\frac{m_1}{x_1} + \frac{m_2}{x_2}\right)\right]^{1/2}$$

4.5.7 The law of motion stated in Sec. 4.5.6 has the following advantages over those discussed previously: It obviates the necessity to consider accelerations and frequently leads to a readily integrable differential equation (see, for example, Eq. (F), Problem 4.5.6). However, as this law furnishes only *one* equation, it must at times be used in conjunction with those considered earlier.

Problem: Figure 4.5.7a represents a parcel chute: A slowly

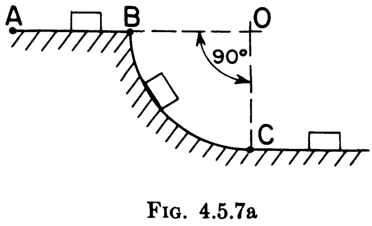


FIG. 4.5.7a

moving belt carries the parcels from A toward B, from which point they slide toward C on a right-circular cylindrical surface.

Regarding the parcels as particles, determine the range of permissible values of the coefficient of friction μ.

Solution: The kinetic energy K^P of a parcel P of mass m is

$$K^P \underset{(4.5.5)}{=} \tfrac{1}{2}m(\mathbf{v}^P)^2$$

where the velocity $(\mathbf{v}^P)$ of P, expressed in terms of the quantities shown in Fig. 4.5.7b, is given by

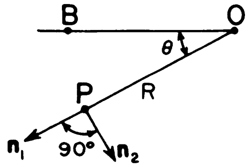


FIG. 4.5.7b

$$\mathbf{v}^P \underset{(2.5.3)}{=} R\dot{\theta}\mathbf{n}_2 \tag{A}$$

Hence

$$K^P = \tfrac{1}{2}mR^2\dot{\theta}^2 \tag{B}$$

The resultant $\mathbf{F}$ of all gravitational and contact forces acting on P is equivalent to the system of three forces shown in Fig. 4.5.7c. Thus

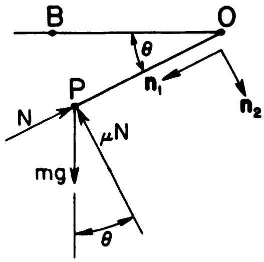


FIG. 4.5.7c

$$\mathbf{F} = (mg\sin\theta - N)\mathbf{n}_1 + (mg\cos\theta - \mu N)\mathbf{n}_2$$

and the activity A^P is given by

$$A^P \underset{(4.5.6)}{=} \mathbf{v}^P \cdot \mathbf{F} \underset{(A)}{=} R\dot{\theta}(mg\cos\theta - \mu N) \tag{C}$$

Consequently, in accordance with Sec. 4.5.6,

$$\frac{d}{dt}\left(\frac{1}{2}mR^2\dot{\theta}^2\right) \underset{(B,C)}{=} R\dot{\theta}(mg\cos\theta - \mu N) \tag{D}$$

Before this equation can be used, the quantity N, which is an as yet unknown function of θ and $\dot{\theta}$, must be eliminated.

Figure 4.5.7d is a plane free-body diagram (see 4.3.3) of P. (It differs from Fig. 4.5.7c only in the addition of the inertia forces $mR\dot{\theta}^2\mathbf{n}_1$ and $-mR\ddot{\theta}\mathbf{n}_2$.)

Set the sum of the $\mathbf{n}_1$ resolutes of all forces equal to zero (see 4.2.3):

$$mR\dot{\theta}^2 + mg\sin\theta - N = 0 \tag{E}$$

Solve for N and substitute into Eq. (D):

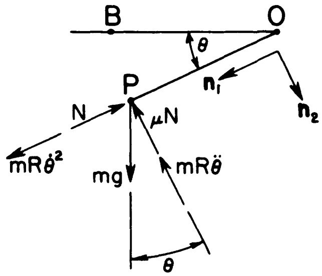


FIG. 4.5.7d

$$\frac{R}{2}\frac{d}{dt}(\dot{\theta}^2) = [g(\cos\theta - \mu\sin\theta) - \mu R\dot{\theta}^2]\dot{\theta} \tag{F}$$

Regard $\dot{\theta}$ as a function of θ. Then

$$\frac{d}{dt}(\dot{\theta}^2) = \left[\frac{d}{d\theta}(\dot{\theta}^2)\right]\frac{d\theta}{dt} = \dot{\theta}\frac{d(\dot{\theta}^2)}{d\theta}$$

and Eq. (F) can be written, alternatively, as

$$\frac{d}{d\theta}(\dot{\theta}^2) + 2\mu\dot{\theta}^2 = \frac{2g}{R}(\cos\theta - \mu\sin\theta)$$

This first-order, linear, nonhomogeneous differential equation has the general solution

$$\dot{\theta}^2 = \frac{2g}{R(1+4\mu^2)}[3\mu\cos\theta + (1-2\mu^2)\sin\theta] + Ce^{-2\mu\theta} \tag{G}$$

where C is a constant whose value is found by taking

$$\mathbf{v}^P = 0$$

when

$$\theta = 0$$

so that

$$\dot{\theta}|_{\theta=0} \underset{(A)}{=} 0$$

which gives

$$C = \frac{-2g}{R(1 + 4\mu^2)} (3\mu)$$

from which it follows that

$$\dot{\theta}^2 \underset{(G)}{=} \frac{2g}{R(1 + 4\mu^2)} [3\mu(\cos\theta - e^{-2\mu\theta}) + (1 - 2\mu^2)\sin\theta]$$

Now, unless

$$\dot{\theta}^2|_{\theta=\pi/2} \geqslant 0$$

P will stop before it reaches point C. Hence μ must be such that

$$-3\mu e^{-\mu\pi} + 1 - 2\mu^2 \geqslant 0$$

In Fig. 4.5.7e, the function $f(\mu)$ defined by

$$f(\mu) = -3\mu e^{-\mu\pi} + 1 - 2\mu^2$$

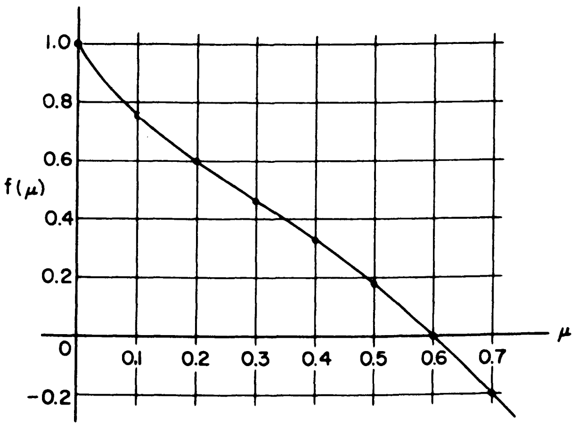


FIG. 4.5.7e

is plotted against μ, for $0 \leqslant \mu \leqslant 0.7$. This graph shows that satisfactory performance may be expected for $0 \leqslant \mu \leqslant 0.6$.

(Note that a second force equation (resolutes parallel to $\mathbf{n}_2$) could have been used in place of Eq. (D):

$$-mR\ddot{\theta} - \mu N + mg \cos\theta = 0$$

which after elimination of N (see Eq. (E)) becomes

$$\ddot{\theta} + \mu\dot{\theta}^2 = \frac{g}{R}(\cos\theta - \mu \sin\theta)$$

This is precisely the equation obtained when the differentiation indicated in the left-hand member of Eq. (F) is carried out.)

4.5.8 When the system of all gravitational and contact forces acting on a rigid body R is equivalent (see Vol. I, Sec. 3.5) to n forces $\mathbf{F}_i$, $i = 1, 2, \ldots, n$, applied at points P_i, $i = 1, 2, \ldots, n$, of R, the kinetic energy (${}^{R'}K^R$) of R in a Newtonian reference frame R' (see 4.2.1) is related to these forces and to the velocities ${}^{R'}\mathbf{v}^{P_i}$, $i = 1, 2, \ldots, n$, as follows:

$$\frac{d\,{}^{R'}K^R}{dt} = {}^{R'}A^R \tag{1}$$

where ${}^{R'}A^R$, called *the activity in R' of the gravitational and contact forces acting on R*, is given by

$${}^{R'}A^R = \sum_{i=1}^{n} {}^{R'}\mathbf{v}^{P_i} \cdot \mathbf{F}_i \tag{2}$$

Proof: In the free-body diagram of R shown in Fig. 4.5.8a, P^*

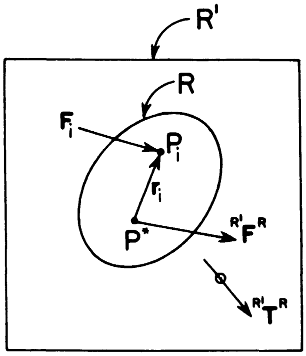


FIG. 4.5.8a

is the mass center of R, and ${}^{R'}\mathbf{F}^R$ and ${}^{R'}\mathbf{T}^R$ represent the system of inertia forces acting on R in R' (see 4.1). These can be expressed as

$$^{R'}\mathbf{F}^R \underset{(4.1.3,2.5.1)}{=} -m \frac{^{R'}d^{R'}\mathbf{v}^{P*}}{dt}$$

and (see 3.5.7 and 3.3.4)

$$^{R'}\mathbf{T}^R \underset{(4.1.4,2.3.1)}{=} -\sum_{j=1}^{3} \phi_{jj}^{R/P*} \left(\frac{d^{R'}\omega_j{}^R}{dt} \mathbf{n}_j + {}^{R'}\omega_j{}^R \, {}^{R'}\boldsymbol{\omega}^R \times \mathbf{n}_j\right)$$

where $\mathbf{n}_j$, $j = 1, 2, 3$, are mutually perpendicular principal directions of R for P^*, fixed in R.

Dot-multiply the first of these equations with ${}^{R'}\mathbf{v}^{P*}$, the second with ${}^{R'}\boldsymbol{\omega}^R$:

$$^{R'}\mathbf{v}^{P*} \cdot {}^{R'}\mathbf{F}^R = -m\,{}^{R'}\mathbf{v}^{P*} \cdot \frac{^{R'}d^{R'}\mathbf{v}^{P*}}{dt} \qquad \text{(A)}$$

$$^{R'}\boldsymbol{\omega}^R \cdot {}^{R'}\mathbf{T}^R = -\sum_{j=1}^{3} \phi_{jj}^{R/P*} \, {}^{R'}\omega_j{}^R \frac{d^{R'}\omega_j{}^R}{dt} \qquad \text{(B)}$$

In accordance with Sec. 4.2.1,

$$^{R'}\mathbf{F}^R + \sum_{i=1}^{n} \mathbf{F}_i = 0$$

and (taking moments about point P^*)

$$^{R'}\mathbf{T}^R + \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_i = 0$$

Hence

$$^{R'}\mathbf{v}^{P*} \cdot {}^{R'}\mathbf{F}^R = -\sum_{i=1}^{n} {}^{R'}\mathbf{v}^{P*} \cdot \mathbf{F}_i \qquad \text{(C)}$$

and

$$^{R'}\boldsymbol{\omega}^R \cdot {}^{R'}\mathbf{T}^R = -\sum_{i=1}^{n} [{}^{R'}\boldsymbol{\omega}^R, \mathbf{r}_i, \mathbf{F}_i] \qquad \text{(D)}$$

But

$$^{R'}\mathbf{v}^{P*} \underset{(2.5.9)}{=} {}^{R'}\mathbf{v}^{P_i} - {}^{R'}\boldsymbol{\omega}^R \times \mathbf{r}_i \qquad \text{(E)}$$

so that

$$
{}^{R'}\mathbf{v}^{P^*} \cdot {}^{R'}\mathbf{F}^R \underset{(\mathrm{C,E})}{=} -\sum_{i=1}^{n} {}^{R'}\mathbf{v}^{P_i} \cdot \mathbf{F}_i + \sum_{i=1}^{n} [{}^{R'}\boldsymbol{\omega}^R, \mathbf{r}_i, \mathbf{F}_i]
$$

$$
= -\sum_{i=1}^{n} {}^{R'}\mathbf{v}^{P_i} \cdot \mathbf{F}_i - {}^{R'}\boldsymbol{\omega}^R \underset{(\mathrm{D})}{\cdot} {}^{R'}\mathbf{T}^R
$$

Thus, defining ${}^{R'}A^R$ as

$$
{}^{R'}A^R = \sum_{i=1}^{n} {}^{R'}\mathbf{v}^{P_i} \cdot \mathbf{F}_i
$$

it follows that

$$
{}^{R'}\mathbf{v}^{P^*} \cdot {}^{R'}\mathbf{F}^R + {}^{R'}\boldsymbol{\omega}^R \cdot {}^{R'}\mathbf{T}^R = -{}^{R'}A^R \tag{F}
$$

Next

$$
{}^{R'}K^R \underset{(4.5.5)}{=} \tfrac{1}{2}m({}^{R'}\mathbf{v}^{P^*})^2 + \tfrac{1}{2} \underset{(4.5.4,3.3.3)}{\sum_{j=1}^{3} \phi_{jj}^{R/P^*}({}^{R'}\omega_j{}^R)^2}
$$

Hence

$$
\frac{d\,{}^{R'}K^R}{dt} = m\,{}^{R'}\mathbf{v}^{P^*} \cdot \frac{{}^{R'}d\,{}^{R'}\mathbf{v}^{P^*}}{dt} + \sum_{j=1}^{3} \phi_{jj}^{R/P^*}\, {}^{R'}\omega_j{}^R \frac{d\,{}^{R'}\omega_j{}^R}{dt}
$$

$$
= -({}^{R'}\mathbf{v}^{P^*} \underset{(\mathrm{A})}{\cdot} {}^{R'}\mathbf{F}^R + {}^{R'}\boldsymbol{\omega}^R \underset{(\mathrm{B})}{\cdot} {}^{R'}\mathbf{T}^R)
$$

$$
\underset{(\mathrm{F})}{=} {}^{R'}A^R
$$

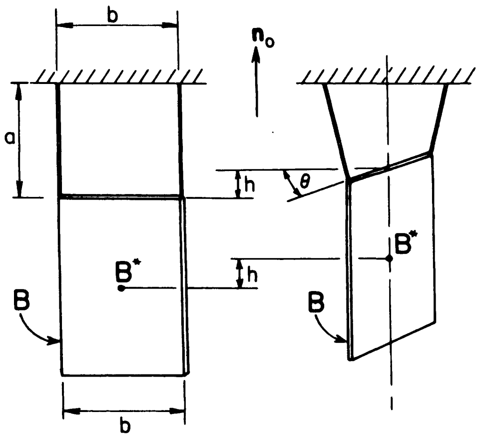


FIG. 4.5.8b

Problem: In Fig. 4.5.8b, B represents a uniform rectangular plate of mass M, which is suspended by two light strings. During one possible motion of this system, both strings remain taut while the plate rotates about a vertical axis passing through the mass center B^* of B, and B^* moves upward and downward on this axis.

Determine the angular velocity of B and the tension T in each of the strings at the instant when $h = 0$ (see Fig. 4.5.8b), assuming that the plate is released from rest when $h = a/6$.

Solution: Let $\mathbf{n}_o$ be a unit vector pointing vertically upward (see Fig. 4.5.8b). Then

$$\mathbf{v}^{B^*} \underset{(2.5.7)}{=} \dot{h}\mathbf{n}_o, \qquad \boldsymbol{\omega}^B \underset{(2.2.4)}{=} \dot{\theta}\mathbf{n}_o \tag{A}$$

$$\begin{aligned} K^B &\underset{(4.5.5)}{=} K^{B/B^*} + K^{B^*} \\ &= \tfrac{1}{2}\underset{(4.5.4)}{\phi_{oo}^{B/B^*}(\boldsymbol{\omega}^B)^2} + \tfrac{1}{2}\underset{(4.5.5)}{M(\mathbf{v}^{B^*})^2} \\ &\underset{(A)}{=} \underset{(3.5.8,3.4.9)}{\frac{1}{2}\frac{Mb^2}{12}}\dot{\theta}^2 + \frac{1}{2}M\dot{h}^2 \end{aligned}$$

or

$$K^B = \frac{M}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) \tag{B}$$

The system of all gravitational and contact forces acting on B is equivalent to the three forces shown in Fig. 4.5.8c, where $\mathbf{F}$ and

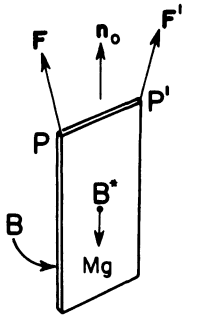


FIG. 4.5.8c

$\mathbf{F}'$, representing forces exerted on B by the strings, are parallel to the strings. In order for the strings to remain taut (but unstretched), the velocities $\mathbf{v}^P$ and $\mathbf{v}^{P'}$ of the corners P and P' must be perpendicular to the strings. Hence

$$\mathbf{v}^P \cdot \mathbf{F} + \mathbf{v}^{P'} \cdot \mathbf{F}' = 0 \tag{C}$$

and the activity A^B is given by

$$A^B = \mathbf{v}^P \cdot \mathbf{F} + \mathbf{v}^{P'} \cdot \mathbf{F}' + \mathbf{v}^{B*} \cdot (-Mg\mathbf{n}_o)$$

That is

$$A^B \underset{(\mathrm{C,A})}{=} -Mg\dot{h} \tag{D}$$

Consequently

$$\frac{d}{dt}\left[\frac{M}{24}(b^2\dot{\theta}^2 \underset{(\mathrm{B})}{+} 12\dot{h}^2)\right] \underset{(1)}{=} \underset{(\mathrm{D})}{-Mg\dot{h}}$$

or

$$\frac{d}{dt}\left[\frac{1}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) + gh\right] = 0$$

from which it follows that

$$\frac{1}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) + gh = C \tag{E}$$

where C is a constant whose value is found by noting that $\dot{\theta}$ and $\dot{h}$ are each equal to zero when $h = a/6$, so that

$$ga/6 \underset{(\mathrm{E})}{=} C \tag{F}$$

and

$$\frac{1}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) \underset{(\mathrm{E,F})}{=} g\left(\frac{a}{6} - h\right) \tag{G}$$

When $h = 0$, h attains its minimum value (see Fig. 4.5.8b). Hence

$$\dot{h}|_{h=0} = 0 \tag{H}$$

and

$$\frac{b^2}{24}\dot{\theta}^2|_{h=0} \underset{(\mathrm{G})}{=} \frac{ga}{6}$$

or

$$\dot{\theta}|_{h=0} = \pm\frac{2}{b}(ga)^{1/2} \tag{I}$$

Thus

$$\boldsymbol{\omega}^B|_{h=0} \underset{(\mathrm{A})}{=} \pm\frac{2}{b}(ga)^{1/2}\mathbf{n}_o$$

As the forces $\mathbf{F}$ and $\mathbf{F}'$ exerted on the plate by the strings have so far entered the solution of the problem only in a form making it impossible to evaluate them (see Eq. (C)), some principle other than the one presently under consideration must be employed for their determination.

The linear momentum $\mathbf{L}^B$ of B is given by

$$\mathbf{L}^B \underset{(4.4.5)}{=} M\mathbf{v}^{B^*} \underset{(A)}{=} M\dot{h}\mathbf{n}_o$$

so that, differentiating,

$$\frac{d\mathbf{L}^B}{dt} = M\ddot{h}\mathbf{n}_o$$

The resultant of all gravitational and contact forces acting on B is

$$\mathbf{F} + \mathbf{F}' - Mg\mathbf{n}_o$$

Consequently

$$\mathbf{F} + \mathbf{F}' - Mg\mathbf{n}_o \underset{(4.4.6)}{=} M\ddot{h}\mathbf{n}_o$$

and, letting

$$\mathbf{F}|_{h=0} = \mathbf{F}'|_{h=0} = T\mathbf{n}_o$$

it follows that

$$2T - Mg = M\ddot{h}|_{h=0} \tag{J}$$

Now, from Fig. 4.5.8d,

$$a^2 = (a - h)^2 + b^2 \sin^2 \frac{\theta}{2}$$

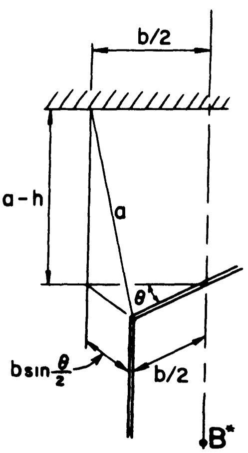


FIG. 4.5.8d

so that, differentiating twice with respect to time,

$$2\dot{h}^2 - 2(a - h)\ddot{h} + \frac{b^2}{2}(\dot{\theta}^2 \cos\theta + \ddot{\theta}\sin\theta) = 0$$

Thus, as $\theta = 0$ when $h = 0$, and from Eqs. (H) and (I),

$$\ddot{h}|_{h=0} = g$$

and

$$2T - Mg \underset{(J)}{=} Mg$$

or

$$T = Mg$$

Note that Eq. (1), Sec. 4.5.8, while not sufficient for the determination of T, provided useful information in the form of Eq. (I).

4.5.9 One of the advantages of the law of motion stated in Sec. 4.5.8 over those discussed previously is that it facilitates elimination of certain contact forces (see, for example, Eq. (C), Problem 4.5.8). In particular, when a rigid body R rolls on a surface S fixed in a Newtonian reference frame R' (see 4.2.1), the system of forces exerted on R by S contributes nothing to the activity ${}^{R'}A^{R}$ (see 4.5.8), because the velocity of every point of R which is in contact with S is equal to zero (see 2.5.10).

Problem: Referring to Problem 2.5.14, and assuming that R is a uniform rectangular parallelepiped having the dimensions shown in Fig. 4.5.9a, obtain a differential equation governing θ, and use it to discuss motions of R during which θ remains small.

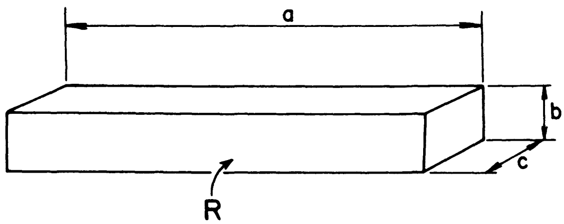


FIG. 4.5.9a

Solution (see Fig. 4.5.9b for notation):

$$K^R \underset{(4.5.5,4.5.4)}{=} \tfrac{1}{2}\phi_{oo}^{R/P^*}(\omega^R)^2 + \tfrac{1}{2}m(\mathbf{v}^{P^*})^2 \tag{A}$$

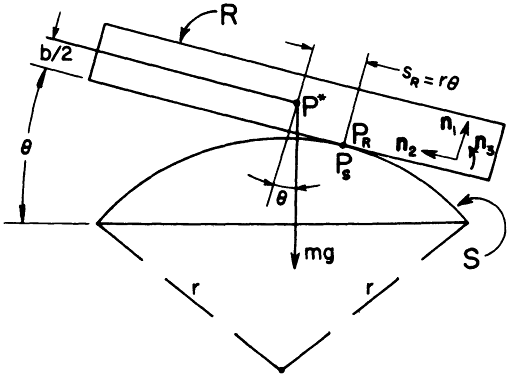


FIG. 4.5.9b

$$\boldsymbol{\omega}^R \underset{(2.2.4)}{=} -\dot{\theta}\mathbf{n}_3 \tag{B}$$

$$\mathbf{v}^{P^*} \underset{(2.5.10)}{=} \boldsymbol{\omega}^R \times \left(s_R\mathbf{n}_2 + \frac{b}{2}\mathbf{n}_1\right)$$

$$= \underset{(B)}{(-\dot{\theta}\mathbf{n}_3)} \times \underset{(P2.5.14)}{\left(r\theta\mathbf{n}_2 + \frac{b}{2}\mathbf{n}_1\right)}$$

$$\mathbf{v}^{P^*} = \dot{\theta}\left(r\theta\mathbf{n}_1 - \frac{b}{2}\mathbf{n}_2\right) \tag{C}$$

$$\phi_{oo}^{R/P^*} \underset{(B)}{=} \phi_{33}^{R/P^*} \underset{(3.5.5)}{=} m(k_3^{R/P^*})^2 \underset{(3.5.8)}{=} m\underset{(App.F7)}{\frac{a^2+b^2}{12}} \tag{D}$$

$$K^R \underset{(A,B,C,D)}{=} \frac{1}{2}m\dot{\theta}^2\left(\frac{a^2}{12} + \frac{b^2}{3} + r^2\theta^2\right) \tag{E}$$

The system of all gravitational forces acting on R is equivalent to the force $\mathbf{F}$ shown in Fig. 4.5.9b, and

$$\mathbf{F} = -mg(\cos\theta\,\mathbf{n}_1 + \sin\theta\,\mathbf{n}_2) \tag{F}$$

As the only contact forces acting on R are those exerted on R by S, the activity A^R is given by

$$A^R = \mathbf{v}^{P^*}\cdot\mathbf{F} \underset{(C,F)}{=} -mg\dot{\theta}\left(r\theta\cos\theta - \frac{b}{2}\sin\theta\right) \tag{G}$$

Hence, in accordance with Sec. 4.5.8,

$$\frac{d}{dt}\left[\frac{\dot\theta^2}{2}\left(\frac{a^2}{12}+\frac{b^2}{3}+r^2\theta^2\right)\right] \underset{(\mathrm{E,G})}{=} -g\dot\theta\left(r\theta\cos\theta-\frac{b}{2}\sin\theta\right) \tag{H}$$

For motions during which θ remains small,

$$\frac{a^2}{12}+\frac{b^2}{3}+r^2\theta^2 \approx \frac{a^2}{12}+\frac{b^2}{3}$$

and

$$r\theta\cos\theta-\frac{b}{2}\sin\theta \approx \left(r-\frac{b}{2}\right)\theta$$

Hence, when θ remains small, Eq. (H) can be replaced with

$$\frac{d}{dt}\left(\frac{\dot\theta^2}{2}\right) = -p^2\theta\dot\theta \tag{I}$$

or, equivalently, with

$$\ddot\theta + p^2\theta = 0 \tag{J}$$

where

$$p^2 = \frac{g(r-\frac{1}{2}b)}{a^2/12+b^2/3} \tag{K}$$

The solution of Eq. (J) assumes one of three forms according as p^2 is positive, negative, or equal to zero:

For $p^2 > 0$ $(r > b/2)$,

$$\theta = \theta_0\cos pt + \frac{\dot\theta_0}{p}\sin pt \tag{L}$$

where θ_0 and $\dot\theta_0$ are the values of θ and $\dot\theta$ at $t = 0$. This describes a harmonic oscillation about the equilibrium position ($\theta = 0$), with circular frequency p and amplitude A,

$$A = [\theta_0{}^2 + (\dot\theta_0/p)^2]^{1/2}$$

and A can be kept arbitrarily small by appropriate choice of θ_0 and $\dot\theta_0$.

For $p^2 < 0$ $(r < b/2)$, let $p^2 = -\bar p^2$. Then

$$\theta = \frac{\theta_0\bar p + \dot\theta_0}{2\bar p}e^{\bar p t} + \frac{\theta_0\bar p - \dot\theta_0}{2\bar p}e^{-\bar p t} \tag{M}$$

The corresponding motion of R is one during which θ increases without limit (so that Eq. (M) must be regarded as describing only the initial stage of the motion), unless

$$\dot{\theta}_0 = -\theta_0 \bar{p}$$

In the latter case, Eq. (M) becomes

$$\theta = \theta_0 e^{-\bar{p}t}$$

which shows that R "drifts" toward the equilibrium position, taking infinitely long to reach it. (When $\theta = \dot{\theta} = 0$, R is then said to be in *unstable equilibrium*, because any disturbance, however small, causes a large departure of R from the equilibrium position.)

For $p^2 = 0$ $(r = b/2)$,

$$\theta = \theta_0 + \dot{\theta}_0 t \tag{N}$$

which again describes a motion during which θ increases without limit, unless $\dot{\theta}_0 = 0$, in which case Eq. (N) asserts that a state of rest in a position other than that corresponding to $\theta = 0$ is possible. However, as Eq. (N) furnishes only an approximate description of motions of R, such states of rest do not, in fact, exist.

4.5.10 When the system of all gravitational and contact forces acting on the body R_j of a set S of N rigid bodies R_j, $j = 1, 2, \ldots, N$, is equivalent to n_j forces $\mathbf{F}_i^j$, $i = 1, 2, \ldots, n_j$, applied at points P_i^j, $i = 1, 2, \ldots, n_j$, of these bodies, the kinetic energy $({}^{R'}K^S)$ of S in a Newtonian reference frame R' is related to these forces and to the velocities ${}^{R'}\mathbf{v}^{P_i^j}$, $i = 1, 2, \ldots, n_j$, as follows:

$$\frac{d\,{}^{R'}K^S}{dt} = {}^{R'}A^S \tag{1}$$

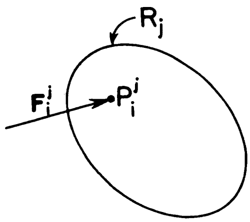
where ${}^{R'}A^S$, called *the activity in R' of the gravitational and contact forces acting on the bodies of S*, is given by

$${}^{R'}A^S = \sum_{j=1}^{N} \sum_{i=1}^{n_j} {}^{R'}\mathbf{v}^{P_i^j} \cdot \mathbf{F}_i^j \tag{2}$$

The contribution to ${}^{R'}A^S$ of all forces exerted by R_j, $j = 1, \ldots, N$, on each other is equal to zero when all gravitational forces exerted by the bodies of S on each other can be neglected (see Vol. I, Sec. 4.4.3) and every contact between the bodies R_j, $j = 1, 2, \ldots, N$, either is a rolling contact (see 2.5.10) or takes place across a smooth surface (see Vol. I, Sec. 4.8.2).

Proof (see Fig. 4.5.10a):

FIG. 4.5.10a



From Sec. 4.5.8,

$$\frac{d^{R'}K^{R_j}}{dt} = {}^{R'}A^{R_j} \tag{A}$$

where

$${}^{R'}A^{R_j} = \sum_{i=1}^{n_j} {}^{R'}\mathbf{v}^{P_i^j} \cdot \mathbf{F}_i^j \tag{B}$$

Let j take on the values $1, 2, \ldots, N$ in Eq. (A), and add the N equations thus obtained:

$$\sum_{j=1}^{N} \frac{d^{R'}K^{R_j}}{dt} = \sum_{j=1}^{N} {}^{R'}A^{R_j} \underset{(B)}{=} \sum_{j=1}^{N} \sum_{i=1}^{n_j} {}^{R'}\mathbf{v}^{P_i^j} \cdot \mathbf{F}_i^j$$

Now

$$\sum_{j=1}^{N} \frac{d^{R'}K^{R_j}}{dt} = \frac{d}{dt} \sum_{j=1}^{N} {}^{R'}K^{R_j} \underset{(4.5.3)}{=} \frac{d^{R'}K^{S}}{dt}$$

and, letting

$$\sum_{j=1}^{N} \sum_{i=1}^{n_j} {}^{R'}\mathbf{v}^{P_i^j} \cdot \mathbf{F}_i^j = {}^{R'}A^{S}$$

it follows that

$$\frac{d^{R'}K^{S}}{dt} = {}^{R'}A^{S}$$

Suppose that all gravitational forces exerted by R_j, $j = 1, 2, \ldots, N$, on each other can be neglected and that $N = 2$, that is, that S consists of only two bodies; further, that these are in contact at only one point, as shown in Fig. 4.5.10b, where $\mathbf{C}_1$ represents the contact force exerted on R_1 by R_2, $\mathbf{C}_1$'s point of application being C_1, and $\mathbf{C}_2$, applied at C_2, represents the contact force exerted on R_2 by R_1. $\mathbf{n}$ is a unit vector perpendicular to the surface of R_1 at C_1.

The contribution of $\mathbf{C}_1$ and $\mathbf{C}_2$ to ${}^{R'}A^S$ is given by

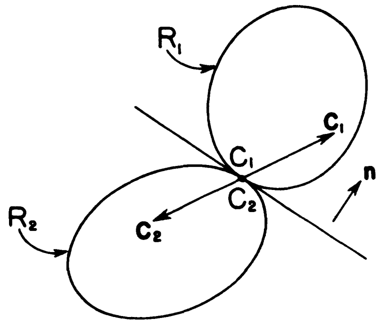


FIG. 4.5.10b

$$ {}^{R'}\mathbf{v}^{C_1} \cdot \mathbf{C}_1 + {}^{R'}\mathbf{v}^{C_2} \cdot \mathbf{C}_2 $$

But

$$ \mathbf{C}_2 \underset{(4.3.5)}{=} -\mathbf{C}_1 $$

Hence

$$ {}^{R'}\mathbf{v}^{C_1} \cdot \mathbf{C}_1 + {}^{R'}\mathbf{v}^{C_2} \cdot \mathbf{C}_2 = ({}^{R'}\mathbf{v}^{C_1} - {}^{R'}\mathbf{v}^{C_2}) \cdot \mathbf{C}_1 \tag{C} $$

If R_1 and R_2 are in rolling contact

$$ {}^{R'}\mathbf{v}^{C_1} \underset{(2.5.10)}{=} {}^{R'}\mathbf{v}^{C_2} $$

Consequently

$$ {}^{R'}\mathbf{v}^{C_1} \cdot \mathbf{C}_1 + {}^{R'}\mathbf{v}^{C_2} \cdot \mathbf{C}_2 \underset{(C)}{=} 0 $$

If R_1 and R_2 are in contact across a smooth surface, $\mathbf{C}_1$ is parallel to $\mathbf{n}$, and the right-hand member of Eq. (C) vanishes for one of two reasons: Either ${}^{R'}\mathbf{v}^{C_1} - {}^{R'}\mathbf{v}^{C_2}$ is perpendicular to $\mathbf{n}$, and hence to $\mathbf{C}_1$, or ${}^{R'}\mathbf{v}^{C_1} - {}^{R'}\mathbf{v}^{C_2}$ is not perpendicular to $\mathbf{n}$, in which case R_1 and R_2 either penetrate each other, which is impossible in view of their rigidity, or they lose contact with each other, which leads to $\mathbf{C}_1 = 0$. Thus, when two bodies are in rolling contact at one point, or when they are in contact across a smooth surface at one point, the contact forces exerted on the bodies by each other contribute nothing to ${}^{R'}A^S$.

The restriction to the case $N = 2$ and a single point of contact is removed by observing that all contact forces exerted on the bodies R_j, $j = 1, 2, \ldots, N$, by each other can be grouped into pairs such as the pair $\mathbf{C}_1$, $\mathbf{C}_2$ considered above.

Problem: In Fig. 4.5.10c, B represents a uniform rectangular plate of mass M, which is suspended by two thin rods A and A', each of mass m. The rods are attached to B and to their support by means of smooth ball and socket connections.

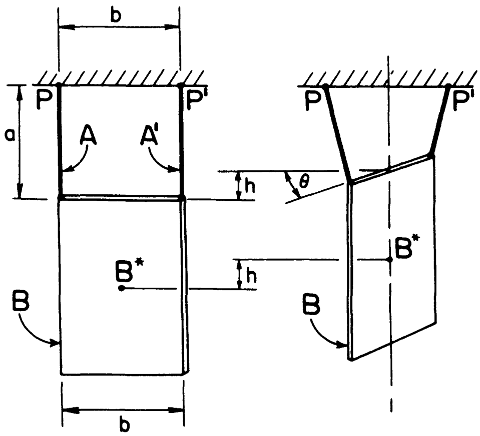


FIG. 4.5.10c

During one possible motion of this system the plate rotates about a vertical axis passing through the mass center B^* of B, while B^* moves upward and downward on this axis.

Determine the angular velocity of B at the instant when $h = 0$ (see Fig. 4.5.10c), assuming that the plate is released from rest when $h = a/6$.

Solution: In Fig. 4.5.10d, which shows rod A at a typical instant during the motion, $\mathbf{n}_0$, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are unit vectors, $\mathbf{n}_0$ being vertical, $\mathbf{n}_1$ parallel to A, $\mathbf{n}_2$ perpendicular to $\mathbf{n}_0$ and $\mathbf{n}_1$, and $\mathbf{n}_3$ perpendicular to $\mathbf{n}_1$ and $\mathbf{n}_2$. Note that

$$\mathbf{n}_0 = -\cos\psi\,\mathbf{n}_1 + \sin\psi\,\mathbf{n}_3 \tag{A}$$

$$\boldsymbol{\omega}^A \underset{(2.2.4,2.2.7)}{=} -\dot{\phi}\mathbf{n}_0 - \dot{\psi}\mathbf{n}_2$$

$$\underset{(A)}{=} \dot{\phi}\cos\psi\,\mathbf{n}_1 - \dot{\psi}\mathbf{n}_2 - \dot{\phi}\sin\psi\,\mathbf{n}_3 \tag{B}$$

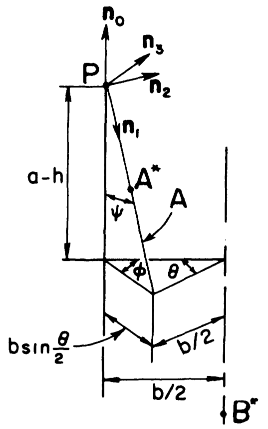


FIG. 4.5.10d

$$\mathbf{v}^{A^*} \underset{(2.5.9)}{=} \boldsymbol{\omega}^A \times \left(\frac{a}{2}\,\mathbf{n}_1\right) \underset{(B)}{=} \frac{a}{2}\,(-\dot{\phi}\sin\psi\,\mathbf{n}_2 + \dot{\psi}\mathbf{n}_3) \tag{C}$$

$\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are principal directions of A for A^* (see 3.5.9 and Appendix, Fig. 1), and

$$\phi_{11}^{A/A^*} = 0, \qquad \phi_{22}^{A/A^*} = \phi_{33}^{A/A^*} = \frac{ma^2}{12} \tag{D}$$

The kinetic energy K^A of A is given by

$$\begin{aligned} K^A &\underset{(4.5.5)}{=} K^{A/A^*} + K^{A^*} \\ &= \tfrac{1}{2}(\phi_{11}^{A/A^*}\omega_1^2 + \underset{(4.5.4)}{\phi_{22}^{A/A^*}\omega_2^2} + \phi_{33}^{A/A^*}\omega_3^2) + \underset{(4.5.5)}{\tfrac{1}{2}m(\mathbf{v}^{A^*})^2} \\ &= \frac{1}{2}\,m\,\frac{a^2}{12}\,(\dot{\psi}^2 + \underset{(B,D)}{\dot{\phi}^2\sin^2\psi}) + \frac{1}{2}\,m\,\frac{a^2}{4}\,(\dot{\phi}^2\underset{(C)}{\sin^2\psi} + \dot{\psi}^2) \end{aligned}$$

or

$$K^A = \frac{ma^2}{6}\,(\dot{\psi}^2 + \dot{\phi}^2\sin^2\psi) \tag{E}$$

Similarly

$$K^{A'} = \frac{ma^2}{6}\,(\dot{\psi}^2 + \dot{\phi}^2\sin^2\psi) \tag{F}$$

From Problem 4.5.8,

$$K^B = \frac{M}{24}\,(b^2\dot{\theta}^2 + 12\dot{h}^2) \tag{G}$$

Hence, letting S be the system of rigid bodies A, A', B,

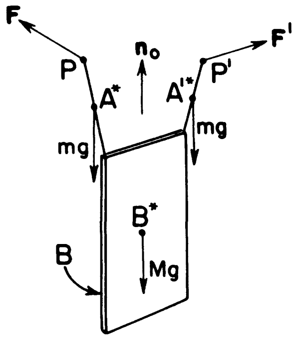
$$K^S \underset{(4.5.3)}{=} K^A + K^{A'} + K^B \tag{H}$$

or

$$K^S \underset{(\mathrm{E,F,G,H})}{=} \frac{ma^2}{3}(\dot{\psi}^2 + \dot{\phi}^2 \sin^2 \psi) + \frac{M}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) \tag{I}$$

The activity (A^S) of all gravitational and contact forces acting on the bodies of S is found by considering the five forces shown in Fig. 4.5.10e, where $\mathbf{F}$ and $\mathbf{F}'$ represent forces exerted on the rods

FIG. 4.5.10e



by the support (see Vol. I, Problem 4.8.2a). Note that the velocities ($\mathbf{v}^P$) and ($\mathbf{v}^{P'}$) of the points P and P' are equal to zero, so that

$$\mathbf{v}^P \cdot \mathbf{F} + \mathbf{v}^{P'} \cdot \mathbf{F}' = 0 \tag{J}$$

A^S is given by

$$A^S \underset{(2)}{=} \mathbf{v}^P \cdot \mathbf{F} + \mathbf{v}^{P'} \cdot \mathbf{F}' + \mathbf{v}^{A^*} \cdot (-mg\mathbf{n}_0) + \mathbf{v}^{A'^*} \cdot (-mg\mathbf{n}_0) + \mathbf{v}^{B^*} \cdot (-Mg\mathbf{n}_0)$$

That is

$$A^S = \underset{(\mathrm{J})}{0} - \underset{(\mathrm{A,C})}{mga\dot{\psi}\sin\psi} - \underset{(\mathrm{P4.5.8})}{Mg\dot{h}} \tag{K}$$

Consequently,

$$\frac{d}{dt}\left[\frac{ma^2}{3}(\dot{\psi}^2 + \dot{\phi}^2 \sin^2 \psi) + \frac{M}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2)\right] = -g(ma\dot{\psi}\sin\psi + M\dot{h}) \text{ or}$$

$$\frac{d}{dt}\left[\frac{ma^2}{3}(\dot{\psi}^2 + \dot{\phi}^2 \sin^2 \psi) + \frac{M}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) + g(-ma\cos\psi + Mh)\right] = 0$$

from which it follows that

$$\frac{ma^2}{3}(\dot{\psi}^2 + \dot{\phi}^2 \sin^2 \psi) + \frac{M}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) + g(-ma\cos\psi + Mh) = C \quad \text{(L)}$$

where C is a constant, whose value is found by noting that, when $h = a/6$,

$$\dot{\theta} = \dot{\phi} = \dot{\psi} = \dot{h} = 0 \quad \text{(M)}$$

and (see Fig. 4.5.10d and recall that A has a length "a")

$$\cos\psi\Big|_{h=a/6} = \frac{a-h}{a}\Big|_{h=a/b} = \frac{5}{6} \quad \text{(N)}$$

so that

$$g\left(-\frac{5}{6}ma + \frac{Ma}{6}\right) \underset{\text{(L,M,N)}}{=} C \quad \text{(O)}$$

and

$$\frac{ma^2}{3}(\dot{\psi}^2 + \dot{\phi}^2 \sin^2 \psi) + \frac{M}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) = g\left[ma\left(\cos\psi - \frac{5}{6}\right) + M\left(\frac{a}{6} - h\right)\right] \quad \text{(P)}$$

From Fig. 4.5.10d,

$$\sin\psi = \frac{b}{a}\sin\frac{\theta}{2}$$

Differentiating,

$$\dot{\psi}\cos\psi = \frac{b\dot{\theta}}{2a}\cos\frac{\theta}{2} \quad \text{(Q)}$$

Now, when $h = 0$, h attains its minimum value. Hence

$$\dot{h}|_{h=0} = 0 \quad \text{(R)}$$

and, as

$$\psi = \theta = 0 \quad \text{(S)}$$

at this instant, it follows that

$$\dot{\psi}|_{h=0} \underset{\text{(Q,S)}}{=} \frac{b}{2a}\dot{\theta}|_{h=0} \quad \text{(T)}$$

Thus when $h = 0$,

$$\frac{ma^2}{3}\frac{b^2\dot{\theta}^2}{4a^2} + \frac{M}{24}b^2\dot{\theta}^2 \underset{\text{(P,R,S,T)}}{=} g\left(\frac{ma}{6} + \frac{Ma}{6}\right)$$

and

$$\dot{\theta}|_{h=0} = \pm\frac{2}{b}\left[ga \cdot \frac{1 + (m/M)}{1 + (2m/M)}\right]^{1/2}$$

Finally,

$$\boldsymbol{\omega}^B \underset{(2.2.4)}{=} \dot{\theta}\mathbf{n}_0$$

Hence

$$\boldsymbol{\omega}^B|_{h=0} = \pm\frac{2}{b}\left[ga \cdot \frac{1 + (m/M)}{1 + (2m/M)}\right]^{1/2}\mathbf{n}_0$$

(When m/M approaches zero, the present result approaches that of Problem 4.5.8.)

4.5.11 When some (or all) of the rigid bodies of the set S considered in Sec. 4.5.10 are connected to each other by light, helical springs (see Vol. I, Sec. 4.8.5), the forces exerted on the bodies by the springs contribute to ${}^{R'}A^S$. If x is the deformation of such a spring ($x > 0$ when the spring is stretched), and $f(x)$ is the function which defines the character of the spring (for example, for a linear spring of modulus k, $f(x) = kx$), the contribution to ${}^{R'}A^S$ of the forces exerted on the bodies of S by this spring is

$$-f(x)\,\frac{dx}{dt}$$

Proof: Figure 4.5.11a shows two of the bodies of S and the

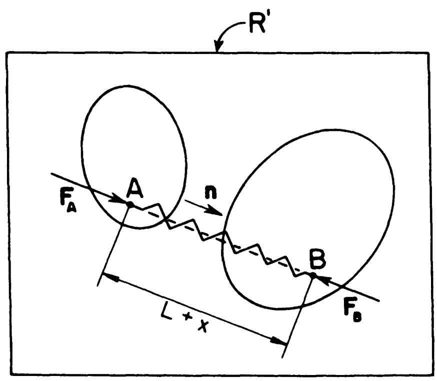


FIG. 4.5.11a

forces $\mathbf{F}_A$ and $\mathbf{F}_B$ exerted on these bodies by a spring of natural length L. $\mathbf{n}$ is a unit vector parallel to the axis of the spring.

$\mathbf{F}_A$ and $\mathbf{F}_B$ are given by

$$\mathbf{F}_A = f(x)\mathbf{n}, \qquad \mathbf{F}_B = -f(x)\mathbf{n} \tag{A}$$

and if ${}^{R'}\mathbf{v}^A$ and ${}^{R'}\mathbf{v}^B$ are the velocities of A and B, the contribution of $\mathbf{F}_A$ and $\mathbf{F}_B$ to ${}^{R'}A^S$ is

$${}^{R'}\mathbf{v}^A \cdot \mathbf{F}_A + {}^{R'}\mathbf{v}^B \cdot \mathbf{F}^B \underset{(A)}{=} f(x)\mathbf{n} \cdot ({}^{R'}\mathbf{v}^A - {}^{R'}\mathbf{v}^B) \tag{B}$$

Now

$${}^{R'}\mathbf{v}^A - {}^{R'}\mathbf{v}^B \underset{(2.5.15)}{=} {}^{R'}\mathbf{v}^{A/B} \underset{(2.4.1)}{=} \frac{{}^{R'}d}{dt}[-(\mathrm{L}+x)\mathbf{n}]$$

$$\underset{(1.5.1)}{=} -\frac{dx}{dt}\mathbf{n} - (\mathrm{L}+x)\frac{{}^{R'}d\mathbf{n}}{dt}$$

so that

$$\mathbf{n} \cdot ({}^{R'}\mathbf{v}^A - {}^{R'}\mathbf{v}^B) = -\frac{dx}{dt} - (\mathrm{L}+x)\mathbf{n} \cdot \frac{{}^{R'}d\mathbf{n}}{dt} \tag{C}$$

But

$$\mathbf{n} \cdot \frac{{}^{R'}d\mathbf{n}}{dt} \underset{(1.5.1)}{=} \frac{1}{2}\frac{d}{dt}(\mathbf{n}^2) = \frac{1}{2}\frac{d}{dt}(1) = 0 \tag{D}$$

Hence

$${}^{R'}\mathbf{v}^A \cdot \mathbf{F}_A + {}^{R'}\mathbf{v}^B \cdot \mathbf{F}_B \underset{(B,C,D)}{=} -f(x)\frac{dx}{dt}$$

Problem: Referring to Problem 4.5.10, determine the circular

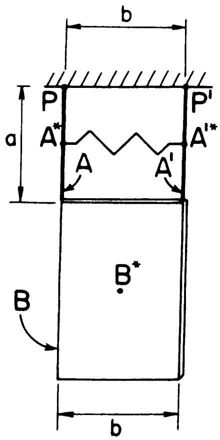


FIG. 4.5.11b

frequency of small oscillations of the system about the vertical line passing through B^*, assuming that the midpoints A^* and A'^* of the rods A and A' are connected by a light, helical spring of natural length L, where

$$L = 0.9\,b$$

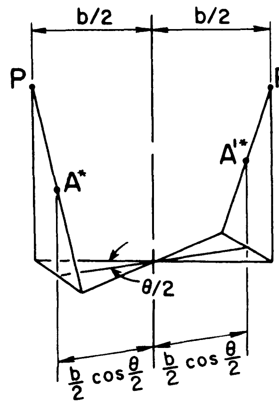
(see Fig. 4.5.11b). Assume that for small deformations the spring is linear and has a spring constant k.

Solution: Let S be the set of rigid bodies A, A', B. Then the kinetic energy K^S of S is given by (see Eq. (I), Problem 4.5.10)

$$K^S = \frac{ma^2}{3}(\dot{\psi}^2 + \dot{\phi}^2 \sin^2 \psi) + \frac{M}{24}(b^2\dot{\theta}^2 + 12\dot{h}^2) \tag{A}$$

At a typical instant during the motion, the distance d between A^* and A'^* is (see Fig. 4.5.11c)

FIG. 4.5.11c



$$d = 2\left(\frac{b}{2}\cos\frac{\theta}{2}\right)$$

The deformation x at this instant is thus given by

$$x = d - L = b\cos\frac{\theta}{2} - 0.9b = b\left(\cos\frac{\theta}{2} - 0.9\right) \tag{B}$$

and, letting $f(x)$ be the function defining the character of the spring, the activity A^S of the gravitational and contact forces acting on the bodies of S is (see Eq. (K), Problem 4.5.10)

$$A^S = -mga\dot{\psi}\sin\psi - Mg\dot{h} - f(x)\frac{dx}{dt} \tag{C}$$

That is

$$A^S \underset{(C)}{=} -mga\dot{\psi}\sin\psi - Mg\dot{h} + f(x)\underset{(B)}{\frac{b}{2}\dot{\theta}}\sin\frac{\theta}{2} \tag{D}$$

Next (see Fig. 4.5.10d),

$$\sin\psi = \frac{b}{a}\sin\frac{\theta}{2} \tag{E}$$

$$\phi = \frac{\pi - \theta}{2} \tag{F}$$

$$a - h = a\cos\psi \tag{G}$$

Hence for motions during which θ remains small,

$$\psi \underset{(E)}{\approx} \frac{b\theta}{2a} \tag{H}$$

and

$$a - h \underset{(G)}{\approx} a\left(1 - \frac{\psi^2}{2}\right) \underset{(H)}{=} a\left(1 - \frac{b^2\theta^2}{8a^2}\right)$$

so that

$$h \approx \frac{b^2\theta^2}{8a} \tag{I}$$

and

$$x \underset{(B)}{\approx} b\left(0.1 - \frac{\theta^2}{8}\right) \tag{J}$$

while

$$f(x) \approx kx \underset{(J)}{=} kb\left(0.1 - \frac{\theta^2}{8}\right) \tag{K}$$

K^S and A^S are thus given, approximately, by

$$K^S \underset{(A)}{\approx} \underset{(H)}{\frac{mb^2\dot{\theta}^2}{12}} + \underset{(F,E,H)}{0} + \frac{Mb^2\dot{\theta}^2}{24} + \underset{(I)}{0}$$

or

$$K^S \approx \frac{b^2\dot{\theta}^2}{24}(M + 2m) \tag{L}$$

and

$$A^S \underset{(D)}{\approx} -\underset{(H,E)}{\frac{mgb^2\theta\dot{\theta}}{4a}} - \underset{(I)}{\frac{Mgb^2\theta\dot{\theta}}{4a}} + \underset{(K)}{\frac{0.1kb^2\theta\dot{\theta}}{4}}$$

or

$$A^S \approx -\left[\frac{gb^2}{4a}(M+m) - \frac{0.1kb^2}{4}\right]\theta\dot{\theta} \tag{M}$$

Now

$$\frac{dK^S}{dt} \underset{(4.5.10,4.5.11)}{=} A^S \tag{N}$$

Hence

$$\frac{d}{dt}\left[\frac{b^2\dot{\theta}^2}{24}(M+2m)\right] \underset{(\mathrm{L,M,N})}{=} -\left[\frac{gb^2}{4a}(M+m) - \frac{0.1kb^2}{4}\right]\theta\dot{\theta}$$

or

$$\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) = -p^2\theta\dot{\theta} \tag{O}$$

where

$$p^2 = \frac{(g/a)(M+m) - 0.1k}{\frac{1}{3}(M+2m)} \tag{P}$$

Equation (O) is identical with Eq. (I) of Problem 4.5.9. In accordance with the discussion given there, harmonic oscillations with circular frequency p can occur if and only if

$$\frac{g}{a}(M+m) - 0.1k > 0 \tag{Q}$$

and the system is in unstable equilibrium, that is, it tends to "buckle," when $\theta = 0$ and Eq. (Q) is not satisfied. Equation (Q) may thus be regarded as furnishing a stability criterion. Note that this equation is independent of the kinetic energy K^S; that is, it depends solely on the activity A^S, given in Eq. (M).

4.5.12 The law of motion stated in Sec. 4.5.10 is particularly useful when the number N of rigid bodies comprising S is large and a single scalar function of time describes the configuration of S at any instant.

Problem: Figure 4.5.12a represents a train S of N gears R_j, $j = 1, 2, \ldots, N$, each gear having a radius one-half as large as that of the one on its left. $\mathbf{n}$ is a unit vector parallel to the axes of the gears, and $\mathbf{T}$ is the torque of a couple applied to R_1, $\mathbf{T}$ being given by

$$\mathbf{T} = T\mathbf{n}$$

where T is an unspecified function of time t.

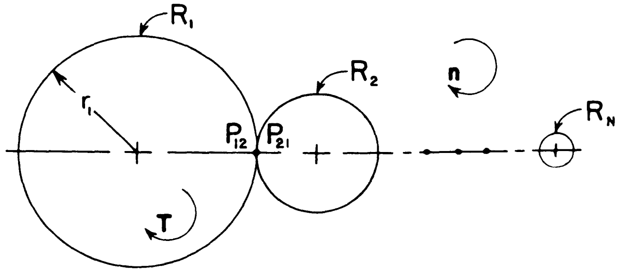


FIG. 4.5.12a

Letting I_1 be the moment of inertia of R_1 about the axis of rotation of R_1, determine the scalar angular acceleration α_1 of R_1 for the **n** direction (see 2.3.7), assuming that all gears are made of the same material and have the same thickness.

Solution: The radius r_j of gear R_j is given by

$$r_j = \frac{r_1}{2^{j-1}}$$

Let ω_j be the angular speed of R_j for the **n** direction. Then the requirement that the points of contact P_{12} and P_{21} of R_1 and R_2 (see Fig. 4.5.12a) have the same velocity takes the form

$$r_1\omega_1 = -r_2\omega_2$$

so that

$$\omega_2 = -\frac{r_1\omega_1}{r_2} \tag{B}$$

Similarly

$$\omega_3 = -\frac{r_2\omega_2}{r_3} \underset{(B)}{=} \frac{r_1\omega_1}{r_3}$$

and, in general,

$$\omega_j = (-1)^{j-1}\frac{r_1\omega_1}{r_j} \underset{(A)}{=} (-1)^{j-1}2^{j-1}\omega_1$$

or

$$\omega_j = (-2)^{j-1}\omega_1 \tag{C}$$

The moment of inertia I_j of R_j about the axis of rotation of R_j is proportional to $r_j{}^4$. Hence,

$$I_j = I_1\frac{r_j{}^4}{r_1{}^4} \underset{(A)}{=} \frac{I_1}{2^{4(j-1)}} \tag{D}$$

and the kinetic energy K^{R_j} of R_j is given by

$$K^{R_j} \underset{(4.5.4,4.5.5)}{=} \frac{1}{2} I_j \omega_j^2 \underset{(C,D)}{=} \frac{1}{2} \frac{I_1}{2^{4(j-1)}} (-2)^{2(j-1)} \omega_1^2$$

or

$$K^{R_j} = \frac{I_1 \omega_1^2}{2^{(2j-1)}} \tag{E}$$

and the kinetic energy K^S of the gear train by

$$K^S \underset{(4.5.3)}{=} \sum_{j=1}^{N} K^{R_j} \underset{(E)}{=} I_1 \omega_1^2 \sum_{i=1}^{N} 2^{(1-2j)}$$

But

$$\sum_{j=1}^{N} 2^{(1-2j)} = \tfrac{2}{3}(1 - 2^{-2N})$$

Hence

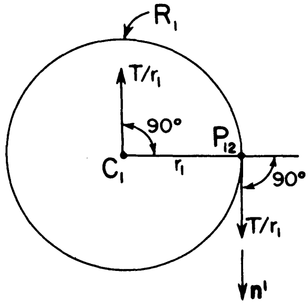
$$K^S = \frac{2 I_1 \omega_1^2}{3} (1 - 2^{-2N}) \tag{F}$$

Of the gravitational and contact forces acting on the bodies of S, the only ones which contribute to the activity A^S (see 4.5.10) are the contact forces exerted on R_1 by the couple of torque $\mathbf{T}$. As this couple is equivalent to the simple couple (see Vol. I, Secs. 3.4.2 and 3.5.7) shown in Fig. 4.5.12b, and

$$\mathbf{v}^{C_1} \underset{(2.5.2)}{=} 0, \qquad \mathbf{v}^{P_{12}} \underset{(2.5.6)}{=} r_1 \omega_1 \mathbf{n}' \tag{G}$$

where $\mathbf{n}'$ is a unit vector, A^S is given by

FIG. 4.5.12b



$$A^S \underset{(4.5.10)}{=} \mathbf{v}^{C_1} \cdot \left(\frac{-T}{r_1} \mathbf{n}'\right) + \mathbf{v}^{P_{12}} \cdot \left(\frac{T}{r_1} \mathbf{n}'\right) \underset{(G)}{=} \omega_1 T \tag{H}$$

Next

$$\frac{dK^S}{dt} \underset{(4.5.10)}{=} A^S$$

Hence

$$\frac{d}{dt}\left[\underset{(F)}{\frac{2I_1\omega_1^2}{3}(1 - 2^{-2N})}\right] = \underset{(H)}{\omega_1 T}$$

or

$$\frac{4}{3}(1 - 2^{-2N})I_1\frac{d\omega_1}{dt} = T$$

But

$$\frac{d\omega_1}{dt} \underset{(2.3.7)}{=} \alpha_1$$

Consequently

$$\alpha_1 = \frac{3T}{4(1 - 2^{-2N})I_1}$$

PROBLEM SETS

PROBLEM SET 1

(See Sections 1.1.1–1.7.3)

(a)* Four rectangular parallelepipeds, R_i, $i = 1, 2, 3, 4$, are arranged as shown in Fig. 1a. The unit vectors $\mathbf{n}_{ij}$, $j = 1, 2, 3$, are respectively parallel to the edges of R_i.

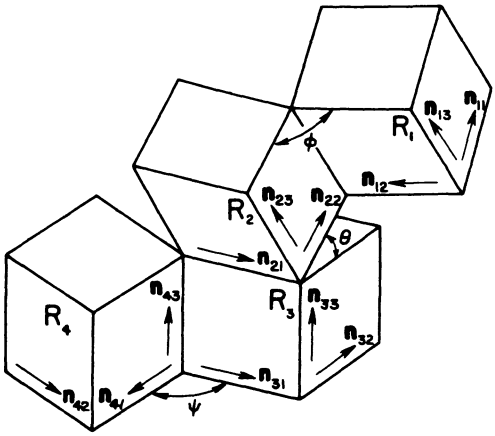


FIG. 1a

The angles ϕ, θ, and ψ, regarded as positive for the configuration shown, depend on a dimensionless variable z, as follows:

$$\phi = -2\pi(z + 2), \qquad \theta = 2\pi z^2, \qquad \psi = 2\pi(1 - 2z^3)$$

For

$$\mathbf{u} = 3\mathbf{n}_{11} - 2z\mathbf{n}_{12} + 4z\mathbf{n}_{13}, \qquad \mathbf{v} = 4z^2\mathbf{n}_{13}$$

evaluate $\mathbf{u}|_{z=\frac{1}{2}} \overset{R_1}{\cdot} \mathbf{v}|_{z=\frac{1}{2}}$, $\mathbf{u}|_{z=\frac{1}{2}} \overset{R_4}{\cdot} \mathbf{v}|_{z=\frac{1}{2}}$, $\mathbf{u}|_{z=0} \overset{R_1}{\cdot} \mathbf{v}|_{z=\frac{1}{2}}$, $\mathbf{u}|_{z=0} \overset{R_4}{\cdot} \mathbf{v}|_{z=\frac{1}{2}}$.

Results: 2, 2, 0, -3.

* Problems containing information relevant to the solution of problems appearing later are marked with an asterisk.

(b) Referring to Problem 1(a), determine the measure numbers of the $\mathbf{n}_{11}$ components of

$$\mathbf{u}|_{z=0} \overset{R_4}{+} \mathbf{v}|_{z=\frac{1}{2}} \quad \text{and} \quad \mathbf{u}|_{z=0} \overset{R_1}{+} \mathbf{v}|_{z=\frac{1}{2}}$$

Results: $2(\cos\psi \cos\phi - \sin\psi \cos\theta \sin\phi)$, 3.

(c) Referring to Problem 1(a), determine the measure numbers, for $z = \frac{1}{2}$, of the $\mathbf{n}_{12}$ and $\mathbf{n}_{42}$ components of ${}^{R_4}d\mathbf{v}/dz$ and ${}^{R_1}d\mathbf{v}/dz$ at $z = \frac{1}{2}$.

Results: 2π, 3π; 0, 0.

(d) Referring to Problem 1(a), determine the measure number, for $z = 0$, of the $\mathbf{n}_{13}$ component of

$$\frac{{}^{R_4}d}{dz}\left(\frac{{}^{R_1}d\mathbf{v}}{dz}\right)$$

at $z = 0$.

Result: 8.

(e) $\mathbf{a}$ and $\mathbf{b}$ are vector functions of a variable z in a reference frame R; $\mathbf{n}_i$, $i = 1, 2, 3$, are mutually perpendicular unit vectors fixed in R; R' is any reference frame other than R; and

$$\frac{{}^{R'}d}{dz}(\mathbf{a} + \mathbf{b}) = \frac{z^3}{3}\mathbf{n}_1 - 3z^3\mathbf{n}_2 + 2z\mathbf{n}_3$$

$$\frac{{}^{R}d}{dz}\left[\frac{{}^{R'}d}{dz}(\mathbf{a} - 2\mathbf{b})\right] = 4z^2\mathbf{n}_1 - \mathbf{n}_2 + 18z\mathbf{n}_3$$

Find the magnitude of

$$\frac{{}^{R}d^2}{dz^2}\left(\frac{{}^{R'}d\mathbf{a}}{dz}\right)$$

at $z = 1$.

Result: 14.

(f)* $\mathbf{v}$ is a vector function of time t in a reference frame R, and $\mathbf{n}_i$, $i = 1, 2, 3$, is a right-handed set of mutually perpendicular unit vectors fixed in R. At time t',

$$\mathbf{v} = \mathbf{n}_1, \qquad \frac{{}^{R}d\mathbf{v}}{dt} = \mathbf{n}_2, \qquad \frac{{}^{R}d^2\mathbf{v}}{dt^2} = \mathbf{n}_3$$

If $\mathbf{n}$ is a unit vector whose direction is at all times the same as that of $\mathbf{v}$, what is the magnitude of ${}^{R}d^2\mathbf{n}/dt^2$ at time t'?

Answer: $(2)^{1/2}$.

(g) A scalar quantity z is defined as

$$z = 3(t - t') - 2 \sin 3(t - t')$$

Referring to Problem 1(f), determine the magnitude of ${}^{R}d^2\mathbf{n}/dz^2$ at time t'.

Result: $(2)^{1/2}/9$.

(h) Referring to Problem 1(a), suppose that z depends on time t in such a way that at a certain instant

$$\phi = 0, \qquad \frac{d\phi}{dt} = -2 \text{ rad sec}^{-1}$$

For this instant, determine the $\mathbf{n}_{11}$, $\mathbf{n}_{12}$, and $\mathbf{n}_{13}$ measure numbers of ${}^{R_2}d\mathbf{n}_{12}/dt$, first using the definition given in Sec. 1.2.1, then the theorem of Sec. 1.7.1.

(i) Referring to Problem 1(a), suppose that at a certain instant $\theta = 0$ and R_2 is rotating at a rate of 6 revolutions per minute relative to R_3, clockwise as seen by an observer to whom $\mathbf{n}_{32}$ appears to point to the right; further, suppose that the rate at which R_2 is rotating clockwise relative to R_3 is decreasing at a rate of 96π rpm/min.

Determine the $\mathbf{n}_{31}$, $\mathbf{n}_{32}$, $\mathbf{n}_{33}$ measure numbers of ${}^{R_1}d^2\mathbf{n}_{33}/dt^2$ and ${}^{R_1}d^2\mathbf{n}_{23}/dt^2$ for this instant.

Results: $0,\ 192\pi^2,\ -144\pi^2;\ 0,\ -192\pi^2,\ -144\pi^2 \text{ min}^{-2}$.

PROBLEM SET 2

(See Sections 1.8.1–1.13.2)

(a)* A circle of radius r is drawn on a sheet of paper which is then folded to form a cylinder of radius R, a space curve C being thus produced. Letting T_1 and T_2 be two tangents to C, each making an angle of sixty degrees with the axis of the cylinder, show that when the angle between T_1 and T_2 is independent of r/R it is equal to sixty degrees.

(b) The tangent T at a point P of a plane curve C makes an angle ϕ with the tangent T_0 at a point P_0 of C. Letting s be the arc-length displacement of P relative to P_0, show that

$$\frac{d\phi}{ds} = \pm\frac{1}{\rho}$$

where ρ is the radius of curvature of C at P. Also, give an example which shows that this relationship is not necessarily valid if C is a space curve.

(c)* Referring to Problem 2(a), and letting $r = 3$ in. and $R = 4$ in., determine the minimum radius of curvature of C.

Result: 2.4 in.

(d) Letting $r = 3$ in. and $R = 4$ in. in Problem 2(a), find the cosine of the angle between the axis of the cylinder and the binormal of C at the points where the radius of curvature of C has a minimum value.

Result: $\pm\frac{3}{5}$.

(e) Referring to Problem 2(a), and letting $r = 3$ in. and $R = 4$ in., determine the torsion of C at the points where the tangent makes an angle of 45 deg with the axis of the cylinder.

Result: $\pm 201/584$ rad in.$^{-1}$.

(f) At a certain point P of a curve C drawn on the surface of the earth (regarded as a sphere of radius 3960 miles) the principal normal of C makes an angle of sixty degrees with the line joining P to the earth's center.

Determine the radius of curvature of C at P.

Result: 1980 miles.

(g) The radius of curvature at a point P of a curve C is equal to 12 ft. C' is the orthogonal projection of C on a plane which is inclined at sixty degrees to the osculating plane and is normal to the rectifying plane of C at P. If P' is the projection of P (P' is a point of C'), what is the radius of curvature of C' at P'?

Answer: 3 ft.

PROBLEM SET 3

(See Sections 2.1.1–2.3.11)

(a)* Figure 3a shows a rigid body R which moves in a reference

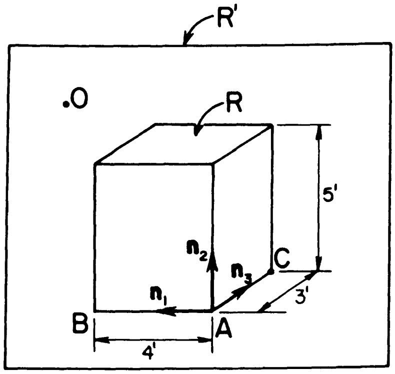


FIG. 3a

frame R' in such a way that at a certain time t' the derivatives in R' of the position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of the points A, B, C relative to a point O fixed in R' have the values $-12\mathbf{n}_3$, 0, $-9\mathbf{n}_1 - 12\mathbf{n}_2 - 12\mathbf{n}_3$ ft sec^{-1}. $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors, and A, B, C are fixed on R.

Determine the $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ measure numbers of the angular velocity of R in R' at time t', and explain why it is not necessary to know whether or not $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are fixed in R, R', or any other reference frame.

Result: 4, -3, 0 rad sec^{-1}.

(b)* For the purpose of analyzing the motion of an airplane it is sometimes convenient to resolve all vector quantities (for example, forces) into components parallel to the vectors $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, $\boldsymbol{\nu}$ (see Parts 1.9, 1.10, 1.11 of the text) associated with the curve C on which some point P of the airplane moves. It then becomes necessary to know the angular velocity of a reference frame R in which these unit vectors are fixed, in the reference frame in which C is fixed. Letting s be the arc-length displacement of P relative to a point P_0 fixed on C, and $\boldsymbol{\lambda}$ and ρ the torsion and radius of curvature to C at P, determine this angular velocity.

Result: $(\lambda\tau + \boldsymbol{\beta}/\rho)ds/dt$.

(c) A point P oscillates on line AB (extended) of the body R of Problem 3(a), in such a way that the displacement x of P relative to A is given by

$$x = 4 + \sin 5(t - t') \text{ ft}$$

Determine the magnitude of the first time-derivative in R' of the position vector of P relative to A, for time t'.

Result: 13 ft sec^{-1}.

(d)* The crank AB of the mechanism shown in Fig. 3d rotates clockwise, performing 15 revolutions per second, thereby causing

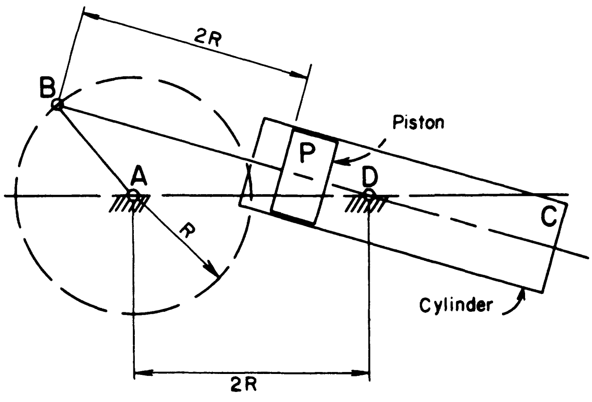


FIG. 3d

the cylinder to oscillate.

Determine the absolute value of the angular speed of the piston P for the instants when B is in its leftmost, rightmost, and upmost positions, and state in each case whether the cylinder is rotating clockwise or counterclockwise. Also, find the smallest angle between AB and AD for which the angular velocity of the cylinder is equal to zero.

Results: 10π rad sec^{-1}, c; 30π rad sec^{-1}, cc; 6π rad sec^{-1}, c; 60 deg.

(e) One method for studying in the laboratory certain forces acting on a body B attached to a space vehicle is to simulate a part of B's motion by mounting B in a manner schematically represented by Fig. 1a, B playing the part of R_1 while the earth (that is, the laboratory) is represented by R_4. By appropriate choice of the functions $\theta(t)$, $\phi(t)$, $\psi(t)$, any desired angular velocity of B can then be obtained.

Letting $\Omega_i(t)$, $i = 1, 2, 3$, be the $\mathbf{n}_{4i}$ measure number of the angular velocity of B in the laboratory, show that the differential equations governing θ, ϕ, and ψ, known as "Euler's kinematical equations," can be put into the form

$$\frac{d\theta}{dt} = \Omega_1 \cos\psi + \Omega_2 \sin\psi$$

$$\frac{d\phi}{dt} = (\Omega_1 \sin\psi - \Omega_2 \cos\psi) \operatorname{cosec}\theta$$

$$\frac{d\psi}{dt} = (-\Omega_1 \sin\psi + \Omega_2 \cos\psi) \operatorname{cotan}\theta + \Omega_3$$

(f) At time t' the second time-derivatives in R' of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of Problem 3(a) have the values 0, $-36\mathbf{n}_1 - 48\mathbf{n}_2$, $-6\mathbf{n}_2 - 75\mathbf{n}_3$ ft sec^{-2}.

Determine the magnitude of the angular acceleration of R in R' at time t'.

Result: 2 rad sec^{-2}.

(g) Regarding the earth as a sphere whose angular velocity in a reference frame R is equal to $(\pi/12)\mathbf{k}$ rad hr^{-1}, where $\mathbf{k}$ is a unit vector parallel to the earth's north–south axis and is fixed in R, determine the magnitude of the angular acceleration in R of a turbine disc which rotates about its own axis at 10,000 revolutions per minute, this axis being normal to the earth's surface at the equator.

Result: 0.076 rad sec^{-2}.

(h)* Losing altitude at a uniform rate of 300 ft sec^{-1}, the mass center of the propeller of an airplane follows the helical path H shown in Fig. 3h. During this motion the propeller's shaft axis remains tangent to H; the inclination of the wings to the principal normal of H is constant; and the propeller rotates clockwise as seen by the pilot, at 1200 revolutions per minute.

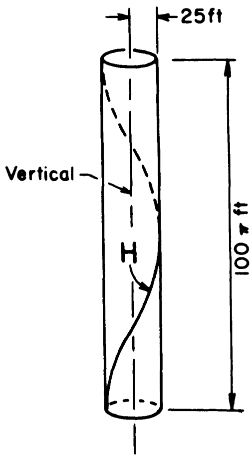


FIG. 3h

Determine the magnitude of the angular acceleration of the propeller.

Result: 337 rad $\sec^{-2}$.

(i) Referring to Problem 3(d), determine the absolute value of the scalar angular acceleration of the piston for the instant when B is vertically above A, and state whether the rate at which the cylinder is rotating counterclockwise is increasing or decreasing at this instant.

Result: $216\pi^2$ rad $\sec^{-2}$, increasing.

(j)* Figure 3j represents Peaucellier's straight line mechanism, so called because point R moves on a straight line perpendicular to AB when the driving arm rotates.

Use a partly graphical method to determine the angular speeds and scalar angular accelerations (for the $\mathbf{k}$ direction) of bars B_2, B_3, and B_4, for an instant at which the configuration of the mechanism is that shown in Fig. 3j and B_1 has the angular velocity and angular acceleration there indicated.

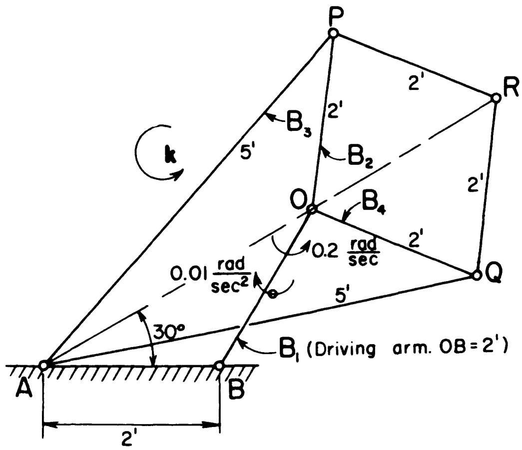


FIG. 3j

Results: $-0.075, 0.054, 0.275$ rad sec^{-1}; $-0.064, -0.030, 0.051$ rad sec^{-2}.

PROBLEM SET 4

(See Sections 2.4.1–2.5.8)

(a) A point P oscillates on one of the edges of a rectangular parallelepiped R, a point P' on the diagonal of one of the faces (see Fig. 4a). The displacement s of P relative to the midpoint O of

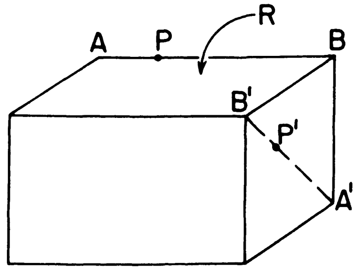


FIG. 4a

AB, regarded as positive when P lies between O and B, is given by

$$s = \frac{12}{\pi^2} \sin \pi t \text{ ft}$$

where t is the time in seconds. s', the displacement of P' relative to the midpoint O' of $A'B'$, is given by

$$s' = \frac{5}{4\pi^2} \cos 2\pi t \text{ ft}$$

and is regarded as positive when P' lies between O' and B'.

Determine the largest value of the magnitude of the acceleration of P' relative to P in R during this motion, and find the cosine of the angle between the acceleration of P relative to P' at $t = \frac{3}{2}$ sec and a unit vector in the direction $A'B'$.

Results: 13 ft sec^{-2}, $-\frac{5}{13}$.

(b)* At a certain instant t' the angular velocity (${}^R\boldsymbol{\omega}^A$) and angular acceleration (${}^R\boldsymbol{\alpha}^A$) of an aircraft A in a reference frame R (see Fig. 4b) are given by

$${}^R\boldsymbol{\omega}^A = 2\mathbf{n}_1 - \mathbf{n}_2 + \mathbf{n}_3 \text{ rad sec}^{-1}$$

$${}^R\boldsymbol{\alpha}^A = \mathbf{n}_1 + \mathbf{n}_2 - 2\mathbf{n}_3 \text{ rad sec}^{-2}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors fixed in A. The distance between two points B and C, which are fixed on A, is 50 ft.

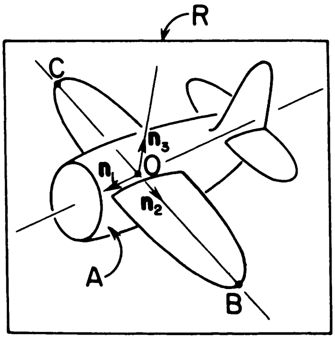


FIG. 4b

Determine the acceleration of B relative to C in R at time t'.

Result: $-250\mathbf{n}_2$ ft sec^{-2}.

(c) In Fig. 4c, r and θ are polar coordinates of a point P which moves in a plane. $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors, parallel and perpendicular to line OP.

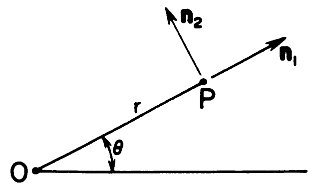


FIG. 4c

Determine the $\mathbf{n}_1$ and $\mathbf{n}_2$ measure numbers of the acceleration of P, assuming that P moves in such a way that (1) θ remains constant; (2) r remains constant; (3) neither θ nor r remains constant. Explain why the results of (3) are not simply the sums of those of (1) and (2).

(d) In Fig. 4d, Q is the point of contact between a sphere S

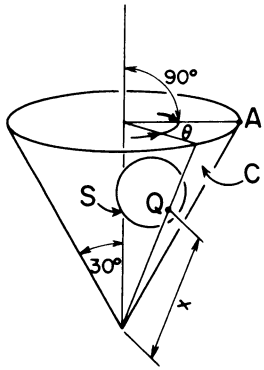


FIG. 4d

and a conical shell C in which S moves in such a way that the distance x decreases uniformly at a rate of 4 in. sec^{-1} and the angle θ increases uniformly at a rate of 5 rad sec^{-1}. (Point A is fixed on C.)

Determine the magnitude of the acceleration of the center of S for an instant at which this point passes through the axis of the cone C.

Result: 20 in. sec^{-2}.

(e) A point P moves with a constant speed of 12 in. sec^{-1} on the curve C described in Problem 2(a). If $r = 3$ in. and $R = 4$ in., what is the maximum magnitude of the acceleration of P during this motion?

Answer: 60 in. sec^{-2}.

(f)* In Fig. 4f, A, B, and C represent points fixed on a ship, and $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors. At a certain instant the velocities of A, B, C are $\mathbf{v}^A = 4\mathbf{n}_2 - 6\mathbf{n}_3$ ft min^{-1},

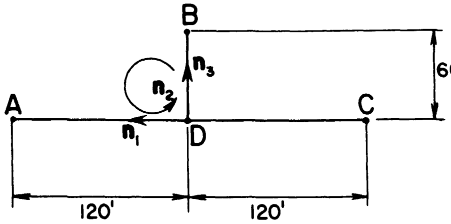


Fig. 4f

$\mathbf{v}^B = 3\mathbf{n}_1 - 6\mathbf{n}_2$ ft min^{-1}, $\mathbf{v}^C = -4\mathbf{n}_2 + 6\mathbf{n}_3$ ft min^{-1}.

Determine the magnitude of the angular velocity of the ship at this instant.

Result: 7 rad hr^{-1}.

(g) Referring to Problem 3(d), let Q be a point on the piston P. For an instant at which B is vertically above A, determine the speed of Q in the cylinder for the direction DB.

Result: $-12(5)^{1/2}\pi R$ units of length per second.

(h) Referring to Problem 3(j), determine the smallest angle between line AB and the acceleration of point O.

Result: 74 deg.

(i) A point P moves on a vertical straight line in such a way that its speed v is given as a function of time t (measured in seconds) by

$$v = 3t^2 - 8 \text{ ft sec}^{-1}$$

If P is moving upward at $t = 1$ sec, is v the speed of P for the upward or downward direction?

At $t = 0$, P is 2 ft above a certain point Q fixed on the line on which P is moving. Determine the position of P at time $t = 3$ sec, and state whether P's acceleration is directed upward or downward at $t = 4$ sec.

Results: downward, 1 ft below Q, downward.

(j) A particle, projected to the left on a horizontal straight line, moves with a time-independent acceleration directed to the right. If it twice passes a point s feet to the left of the point of projection, first at time t_1, next at time t_2, what is the magnitude of the acceleration?

Answer: $2s/t_1t_2$ ft sec^{-2}.

(k) Starting from rest, a point moves with constant acceleration on a straight line. During the first ten minutes of the motion the distance-average speed of the point is equal to 8 ft sec^{-1}. What is the time-average speed of the point during this time interval?

Note: $\bar{f}$, the x-average value of a function $f(x)$ in the interval $x_1 \leqslant x \leqslant x_2$, is defined as

$$\bar{f} = \frac{1}{x_2 - x_1} \int_{q=x_1}^{q=x_2} f(q)\, dq$$

Answer: 6 ft sec^{-1}.

PROBLEM SET 5

(See Sections 2.5.9–2.5.15)

(a) Referring to Problem 4(f), show that the velocity of point D is equal to zero at the instant under consideration.

(b) In Problem 3(h) let A be a point of the propeller, one foot from the propeller's shaft axis, as shown in Fig. 5b.

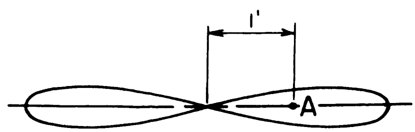


FIG. 5b

Determine the magnitude of the acceleration of A at an instant at which the propeller is horizontal.

Result: 16,200 or 18,000 ft $\sec^{-2}$.

(c)* A circular hoop H of radius R and mass m rolls on a horizontal plane P, the normal to H making a constant angle θ with the normal to P. The center C of H moves with constant speed v on a circle of radius R.

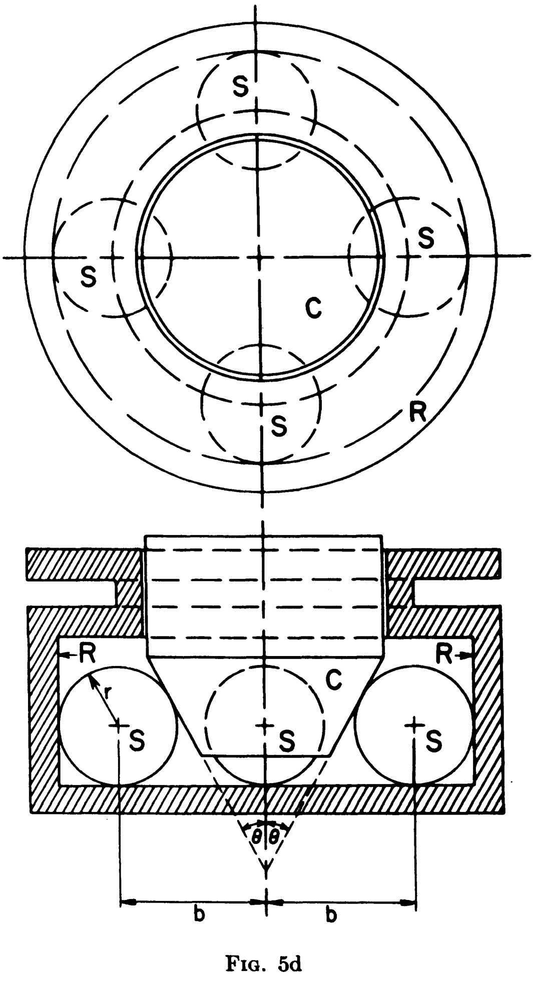


FIG. 5d

Determine the magnitude of the acceleration of that point of H which is in contact with P.

Result: $(v^2/R)(1 \pm \cos\theta)(1 + \sin^2\theta)^{1/2}$.

(d)* A shaft, terminating in a truncated cone C of semivertical angle θ (see Fig. 5d), is supported by a thrust bearing consisting of a fixed race R and four identical spheres S of radius r. When the shaft rotates about its axis, S rolls on R at both of its points of contact with R, and C rolls on S.

Determine the dimension b such that the contact between C and S is one of *pure* rolling.

Result: $r(1 \pm \sin\theta)/(\cos\theta - \sin\theta)$.

(e)* The axes of two shafts S and S' intersect at a point A. Figure 5e illustrates one method of transmitting rotatory motion from S to S' (and vice-versa): Bevel gears B and B', which are, essentially, frusta of cones, are keyed to the shafts, coming into contact with each other along line CD, the angle ϕ being chosen such that B and B' roll on each other at *all* points of contact.

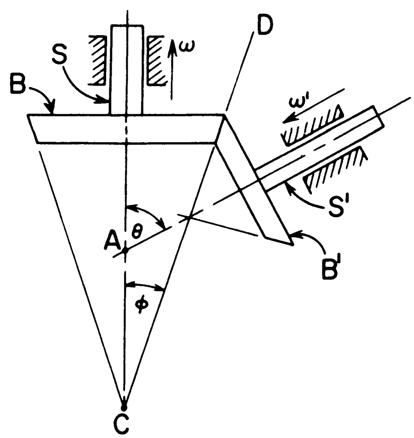


FIG. 5e

Show that points A and C must coincide, and find ϕ as a function of θ and the angular speed ratio ω/ω'.

Result: $\phi = \text{arc cotan}[\cot\text{an}\,\theta + (\omega/\omega')\,\text{cosec}\,\theta]$.

(f) Figure 5f shows schematically how the driving shaft D of an automobile may be connected to the two halves, A_1 and A_2, of the rear axle in such a way as to permit the rear wheels to rotate at different angular speeds in the frame F. This is accomplished as follows: Bevel gears (see Problem 5(e)) B_1 and B_2 are keyed to

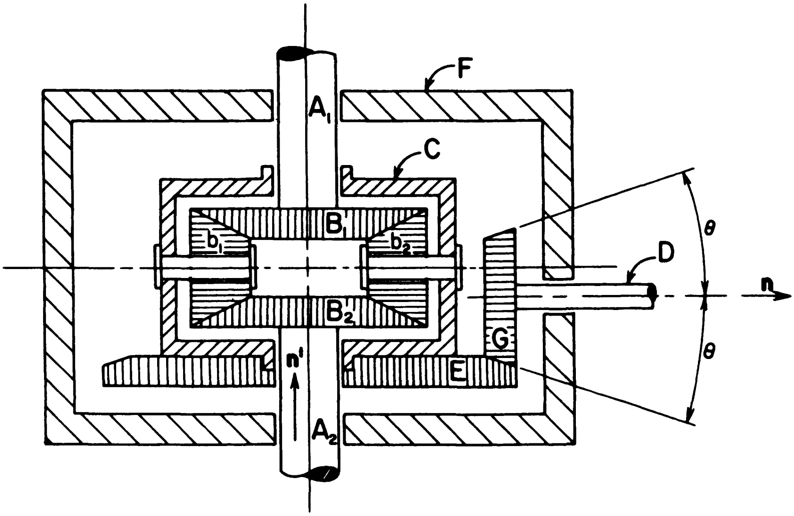


FIG. 5f

A_1 and A_2, and these engage bevel gears b_1 and b_2, the latter being free to rotate on pins fixed in a casing C. C can revolve about the common axis of A_1 and A_2, and a bevel gear E, rigidly attached to C, is driven by bevel gear G, which is keyed to D.

Letting ω be the angular speed of D in F for the direction $\mathbf{n}$, and ω_1, ω_2 the angular speeds of A_1 and A_2 in F for the direction $\mathbf{n}'$, express ω in terms of ω_1, ω_2, and θ.

Result: $\frac{1}{2}(\omega_1 + \omega_2) \cot \theta$.

(g)* A trailer consists of a rigid frame $AB - CD$, to which identical wheels are attached at C and D (see Fig. 5g), the wheels being free to revolve independently about their common axis.

Consider a motion during which the wheels roll on a horizontal plane; the frame remains parallel to this plane; and point A moves

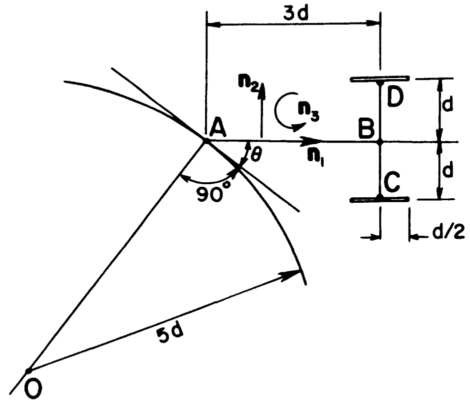


FIG. 5g

with constant speed v on a circle, center at O. Verify that the system approaches a "steady state," that is, that θ approaches a limiting value as time t approaches infinity. Determine the limiting values of the accelerations ($\mathbf{a}^B$, $\mathbf{a}^C$, $\mathbf{a}^D$) of points B, C, D and of the angular accelerations ($\boldsymbol{\alpha}^C$, $\boldsymbol{\alpha}^D$) of the wheels at C and D, expressing all results in terms of the mutually perpendicular unit vectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ shown in Fig. 5g.

Results: $\mathbf{a}^B = -(4v^2/25d)\mathbf{n}_2$, $\mathbf{a}^C = -(3v^2/25d)\mathbf{n}_2$, $\mathbf{a}^D = -(5v^2/25d)\mathbf{n}_2$, $\boldsymbol{\alpha}^C = (6v^2/25d^2)\mathbf{n}_1$, $\boldsymbol{\alpha}^D = (10v^2/25d^2)\mathbf{n}_1$.

(h) Referring to Problem 3(b), and considering a motion during which the orientation of the airplane in reference frame R remains fixed, determine the distance from P to the airplane's instantaneous axis, and find the magnitude of the minimum velocity of any point of the airplane (or airplane extended).

Result: $\rho(1 + \lambda^2\rho^2)^{-1}$, $\rho|\lambda\dot{s}|(1 + \lambda^2\rho^2)^{-1/2}$.

(i) Locate an instantaneous center of bar B_4 of the mechanism described in Problem 3(j), and use it to find the velocity of the midpoint of this bar. Check the result by reference to Sec. 2.5.9.

(j)* A circular tube T revolves at a uniform rate of 4 rad sec^{-1} about an axis A-A which is fixed in a laboratory L and passes through a point B of T. (B is fixed on A-A.) The normal to the plane of T makes an angle of 45 deg with A-A. A particle P, moving with constant speed in the tube, reaches at time t^* the point P^* shown (in two views) in Fig. 5j, and the distance between B and P is increasing at this instant.

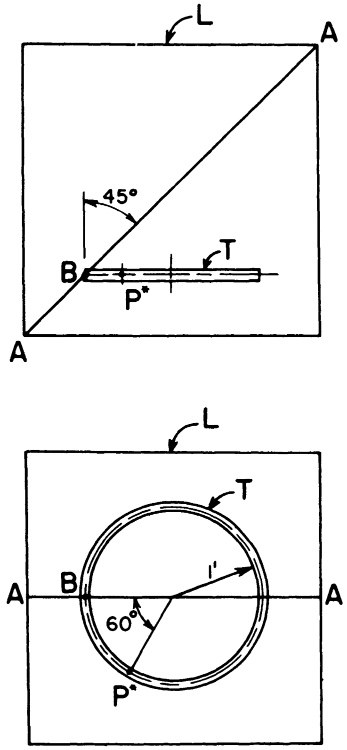


FIG. 5j

If the acceleration of P in L at time t^* is parallel to the plane of T, what is the magnitude of this acceleration?

Answer: $2(127)^{1/2}$ ft sec^{-2}.

(k)* Referring to Problem 4(b), let P and Q be particles moving on line BC with constant speeds (in A) of 40 ft sec^{-1} and 60 ft sec^{-1}, P proceeding from B toward C, Q from C toward B.

If P arrives at C, and Q at B, at time t', what is the magnitude of the acceleration in R of P relative to Q at time t'?

Answer: $50(105)^{1/2}$ ft sec^{-2}.

PROBLEM SET 6

(See Sections 3.1.1–3.2.9)

(a)* In Fig. 6a, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are mutually perpendicular unit vectors; point Q lies in the XOY plane; and point P has a strength of 6 ft^3.

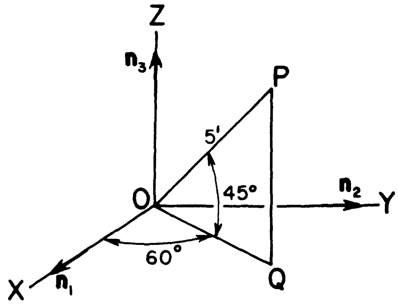


FIG. 6a

Find the second moment of P with respect to line OX by each of the three methods discussed in Sec. 3.1.7, and state which of the three is the most convenient.

(b) Referring to Problem (6a), find the second moment of P with respect to O for the pair of directions $\mathbf{n}_1$, $\mathbf{n}_2$. (Check by using two methods.)

(c) The second moments of a point P with respect to a point Q for three mutually perpendicular directions $\mathbf{n}_i$, $i = 1, 2, 3$, are $\phi_{ij}^{P/Q}$, $i, j = 1, 2, 3$.

Letting $\phi_{aa}^{P/Q}$, $\phi_{bb}^{P/Q}$, $\phi_{cc}^{P/Q}$ be the second moments of P with respect to *any* three mutually perpendicular lines passing through Q, express $\phi_{aa}^{P/Q} + \phi_{bb}^{P/Q} + \phi_{cc}^{P/Q}$ in terms of $\phi_{ij}^{P/Q}$, $i, j = 1, 2, 3$.

Result: $\phi_{11}^{P/Q} + \phi_{22}^{P/Q} + \phi_{33}^{P/Q}$.

(d)* The points P_1, P_2, P_3 shown in Fig. 6d have strengths of 10, 20, 30 in^2.

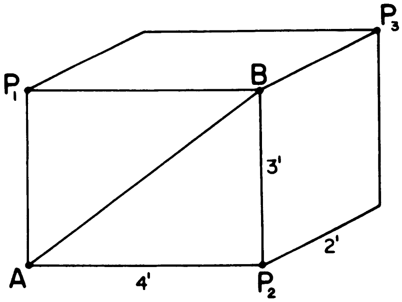


FIG. 6d

Determine the radius of gyration of this set of points with respect to line AB; also, the radius of gyration with respect to the same line of a point which has a strength of $10 + 20 + 30 = 60$ in.^{2} and is situated at the centroid of the set of points P_1, P_2, P_3. Account for the fact that these two radii of gyration are not equal to each other.

Results: 2.21 ft, 1.078 ft.

(e)* The points $P_1, P_2, \ldots, P_8$ shown in Fig. 6e have strengths of 1, 2, 3, 4, 1, 2, 3, 4 tons. Use the most convenient method avail-

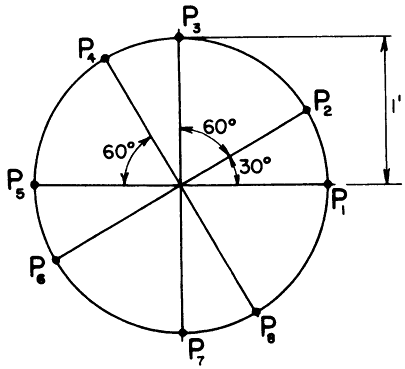


FIG. 6e

able to find the second moment of this set of points with respect to point P_2 for the pair of directions P_5P_1, P_7P_3.

Result: $-4(3)^{1/2}$ ton ft^2.

(f) The points P_i, $i = 1, 2, \ldots, n$, of a set S of n points have strengths N_i. $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are unit vectors, each being parallel to one of the axes OX, OY, OZ of a rectangular cartesian coordinate system and pointing in the positive direction of the axis to which it is parallel. The coordinates of P_i are x_i, y_i, z_i.

Letting $\phi_{ij}^{S/O}$ be the second moments of S with respect to O, express $\phi_{22}^{S/O}$ and $\phi_{23}^{S/O}$ in terms of N_i, x_i, y_i, z_i, and n.

Results:

$$\phi_{22}^{S/O} = \sum_{i=1}^{n} N_i(z_i^2 + x_i^2), \; \phi_{23}^{S/O} = -\sum_{i=1}^{n} N_i y_i z_i.$$

(g) Referring to Fig. 6a, let P be centroid of a set S of points whose second moments $\phi_{ij}^{S/P}$, $i, j = 1, 2, 3$, in units of ft^3, are given in Table 6.

TABLE 6

$\phi_{ij}^{S/P}$		j		
		1	2	3
	1	2	$\sqrt{3}$	3
i	2	$\sqrt{3}$	4	5
	3	3	5	6

FIG. 6g

Assuming that S has a strength of 2 ft, determine the second moment of S with respect to line OQ.

Result: 30 ft^3.

(h) The second moment of a set S of points with respect to a point O for three mutually perpendicular directions $\mathbf{n}_i$, $i = 1, 2, 3$, are ϕ_{ij}, $i, j = 1, 2, 3$. A line L_a passes through O and is perpendicular to $\mathbf{n}_3$. Under these circumstances ϕ_{aa}, the second moment of S with respect to L_a, depends on the orientation of L_a in the plane which passes through O and is perpendicular to $\mathbf{n}_3$.

Determine the maximum value of ϕ_{aa}.

Result:

$$\frac{\phi_{11} + \phi_{22}}{2} + \left[\left(\frac{\phi_{11} - \phi_{22}}{2}\right)^2 + \phi_{12}^2\right]^{1/2}.$$

(i) Determine whether or not the following statement is true: Given any set S of points, a point O, and a plane passing through O, it is always possible to find two directions parallel to this plane and at right angles to each other such that the second moment of S with respect to O for this pair of directions is equal to zero.

(j)* Given a set S of points, a point O, and two lines L_1 and L_2 passing through O and forming a right angle (see Fig. 6j), let $\mathbf{n}_1$ and $\mathbf{n}_2$ be unit vectors respectively parallel to the two lines. Draw a line PQ parallel to $\mathbf{n}_1$, and construct a circle passing through two points A and B and having its center C at the mid-

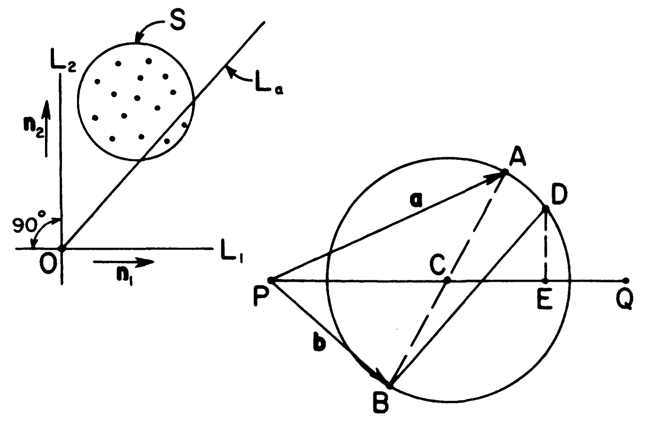


FIG. 6j

point of line AB, A and B being the points whose position vectors, $\mathbf{a}$ and $\mathbf{b}$, relative to P are given by

$$\mathbf{a} = k(\phi_{11}^{S/O}\mathbf{n}_1 + \phi_{12}^{S/O}\mathbf{n}_2)$$
$$\mathbf{b} = k(\phi_{22}^{S/O}\mathbf{n}_1 - \phi_{12}^{S/O}\mathbf{n}_2)$$

where k is a constant of proportionality. (Note that when $\phi_{12}^{S/O} \neq 0$ point C is the intersection of lines PQ and AB.)

Verify that the following graphical method can now be used to find the second moment ($\phi_{aa}^{S/O}$) of S with respect to any line L_a passing through O and coplanar with L_1 and L_2: A line parallel to L_a is drawn through B, intersecting the circle at point D. Then

$$k\phi_{aa}^{S/O} = \overline{PE}$$

where E is the foot of the perpendicular dropped from D on line PQ.

Let S be the set of four points A, B, P_1, P_2 shown in Fig. 6d, these points having strengths of 1, 2, 3, 4 slug, respectively. Determine by graphical methods (1) the second moment of S with respect to line AB, (2) the smallest second moment of S with respect to any line passing through point A and lying in the plane of S, and (3) the angle between line AP_1 and the line passing through A and lying in the plane of S with respect to which S has the largest second moment.

Results: 40.3 slug ft^2, 35.5 slug ft^2, 21.6 deg.

PROBLEM SET 7

(See Sections 3.3.1–3.5.9)

(a) Three points of equal strength N are situated at the corners of an equilateral triangle whose sides have lengths L. Determine the smallest angle between any side of the triangle and any principal axis of the set of points for one vertex of the triangle, and find the principal second moments for one vertex. Also, locate all principal axes for the centroid of the set.

Results: 0 deg; $NL^2/2$, $3NL^2/2$, $2NL^2$.

(b) Four points of equal strength are placed at the points B, P_1, P_2, P_3 of Fig. 6d. Determine the minimum radius of gyration of this set of points.

Result: 1.4. ft.

(e) Referring to Fig. 6a, suppose that P is the centroid of a set S of points, that S has a strength of 2 slug, that $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are centroidal principal directions of S, and that the corresponding centroidal principal second moments of S have the values 40, 20, 30 slug ft^2.

Determine the second moment of S with respect to line OQ.

Result: 50 slug ft^2.

(d)* Verify each of the following statements:

A centroidal principal axis is a principal axis for each of its points.

If a principal axis for a point other than the centroid passes through the centroid, it is a centroidal principal axis.

A line which is a principal axis for two of its points is a centroidal principal axis.

The three principal axes for any point on a centroidal principal axis are parallel to centroidal principal axes.

If two principal second moments for a given point are equal to each other, the second moments with respect to all lines passing through this point and lying in the plane determined by the corresponding principal axes are equal to each other.

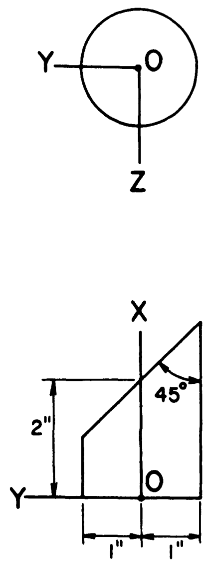


FIG. 7e

(c)* Determine the second moment of the figure shown in Fig. 7e with respect to O for the pair of directions OX, OY, regarding the figure as (1) a surface possessing no plane portions and (2) a solid.

Results: 2π in.^{4}, $\pi/2$ in.^{5}.

(f) Locate one principal axis of the figure shown in Fig. 7e for point O, and show that the X-axis is not a principal axis of the figure for point O.

(g) Determine the radius of gyration of one face of a 1 in. × 1 in. × 1 in. cube with respect to a space diagonal of the cube.

Result: $(5/18)^{1/2}$ in.

(h)* Determine the smallest angle between line OA and any principal axis for point O of the shaded plane surface shown in Fig. 7h.

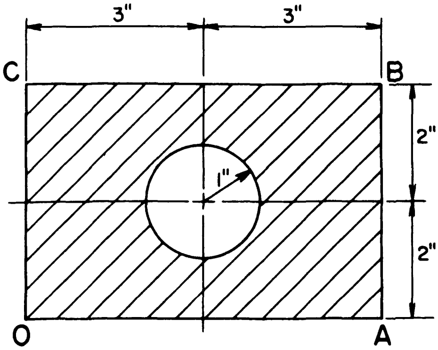


FIG. 7h

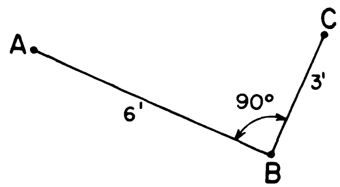


FIG. 7j

Result: 30 deg.

(i) Determine the polar second moment of the shaded plane surface shown in Fig. 7h, with respect to point O.

Result: 373.4 in.^{4}.

(j) Particles of equal mass are placed at the points A, B, C shown in Fig. 7j. Find the minimum radius of gyration of this set of particles.

Result: $(5 - \sqrt{13})^{1/2}$ ft.

(k) The assembly shown in Fig. 7k consists of two brass discs (527 lb ft^{-3}) attached to a steel shaft (489 lb ft^{-3}).

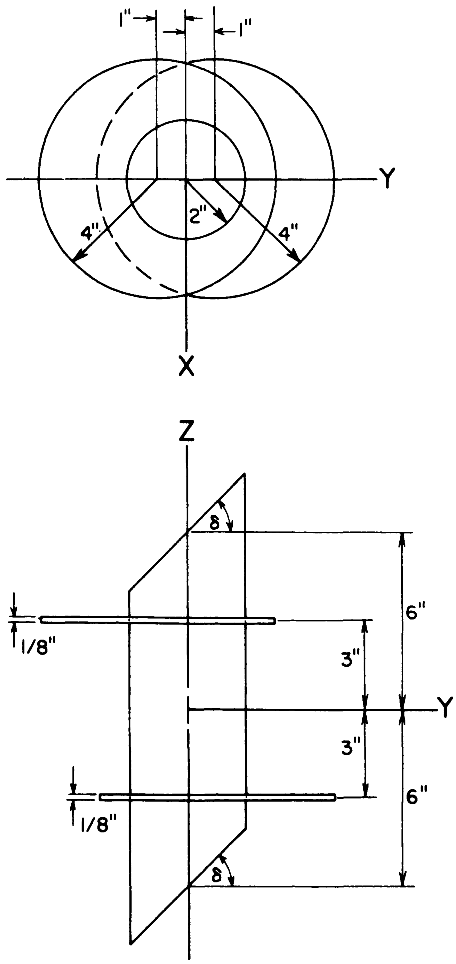


FIG. 7k

Determine the value of the angle δ for which the shaft axis is a principal axis of this assembly.

Result: 15.1 deg.

PROBLEM SET 8

(See Sections 4.1.1–4.1.7)

(a)* A particle P of mass m is free to move between two parallel plates. The plates are attached to each other and can revolve about an axis Y which is fixed in the plates and in a reference frame R, as shown in Fig. 8a.

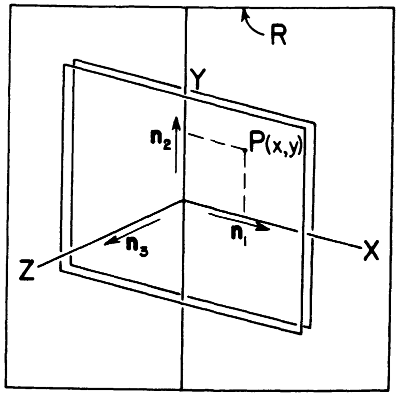


FIG. 8a

Letting X, Y, Z be axes of a rectangular cartesian coordinate system fixed in the plates, and $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ unit vectors respectively parallel to X, Y, Z, express the inertia force acting on P in R in terms of the angular speed ω of the plates in R for the $\mathbf{n}_2$ direction and the coordinates x and y of P.

Result: $m[(x\omega^2 - \ddot{x})\mathbf{n}_1 - \ddot{y}\mathbf{n}_2 + (x\dot{\omega} + 2\omega\dot{x})\mathbf{n}_3]$.

(b)* In Fig. 8b, ABC represents a uniform right-triangular plate of mass m. The plate is supported as follows: Vertex A is fixed and vertex B is attached to a string which is fastened at D, the length of the string being such that line AB is horizontal. (Line AD is vertical.)

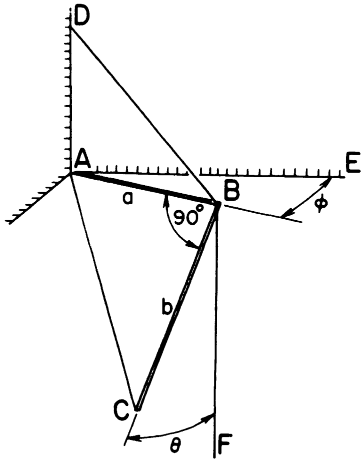


FIG. 8b

Letting AE be a fixed horizontal line, and BF a line parallel to AD, consider motions during which θ, ϕ, and the time-derivatives of θ and ϕ remain "small." Determine the moment of the system of inertia forces acting on the plate (1) about line AB and (2) about line AD.

Results:

$$\frac{mb}{2}\left(\frac{a}{2}\ddot{\phi} + \frac{b}{3}\ddot{\theta}\right), \text{ sense } AB$$

$$\frac{ma}{2}\left(a\ddot{\phi} + \frac{b}{2}\ddot{\theta}\right), \text{ sense } AD$$

(c)* The apex A of a solid right-circular cone of mass m is fixed. The cone moves as follows: The angle θ between the axis of the cone and the vertical remains constant; the plane P determined by the axis of the cone and the vertical passing through A revolves about this vertical at a constant rate of Ω radians per unit of time; and the cone revolves about its axis at a constant rate of ω radians per unit of time in a reference frame in which plane P is fixed.

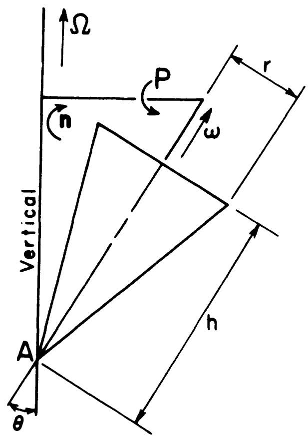


FIG. 8c

Letting **n** be a unit vector perpendicular to plane P, as shown in Fig. 8c, determine the moment of the system of inertia forces acting on the cone, about a line passing through A and parallel to **n**.

Result: $(3/20)m\Omega[(4h^2 - r^2)\Omega \cos \theta - 2r^2\omega] \sin \theta \, \mathbf{n}$.

(d) Referring to Problem 3(h), let $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, $\boldsymbol{\nu}$ be a vector tangent, binormal, and principal normal to H at the mass center P^* of the propeller P, and let $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ be mutually perpendicular unit vectors fixed in the propeller, as shown in Fig. 8d.

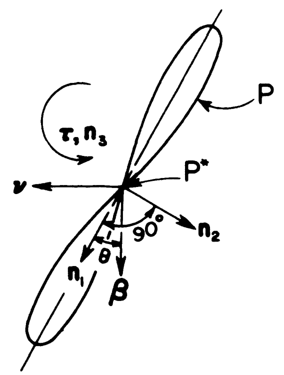


FIG. 8d

Assuming that $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are principal directions of P for P^*, and that

$$\phi_{11}^{P/P^*} \approx 0, \qquad \phi_{22}^{P/P^*} \approx \phi_{33}^{P/P^*} = \tfrac{1}{2} \text{ slug ft}^2$$

determine the $\boldsymbol{\tau}$, $\boldsymbol{\beta}$, $\boldsymbol{\nu}$ measure numbers of the inertia torque acting on P. Repeat after replacing P with a propeller P' consisting of three equally spaced blades, taking

$$\phi_{11}^{P'/P^*} = \phi_{22}^{P'/P^*} = \tfrac{1}{2} \text{ slug ft}^2, \qquad \phi_{33}^{P'/P^*} \approx 1 \text{ slug ft}^2$$

Results: $-1.8 \sin 2\theta$, $172 \sin 2\theta$, $-344 \cos^2 \theta$ ft lb; 0, 0, -344 ft lb.

(e)* Figure 8e represents a reciprocating engine mechanism with a counterweighted crank. P and P' are the mass centers of the connecting rod and the crank, respectively. These parts have masses m and m', and the radius of gyration of the connecting rod with respect to a line passing through P and parallel to the unit vector $\mathbf{n}_3$ is equal to k. The piston has a mass M, and its mass center lies on the axis of the cylinder.

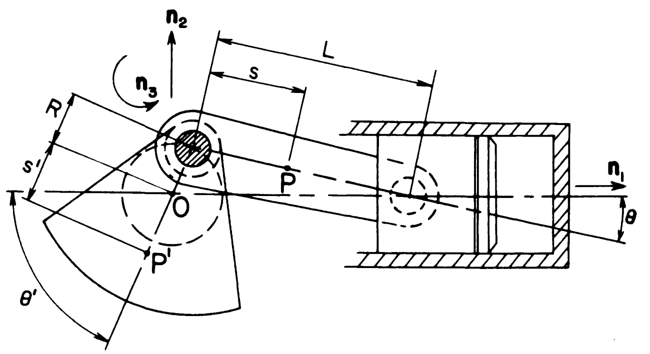


FIG. 8e

Considering motions of the mechanism during which the crank revolves with uniform angular speed, and letting S be the system of all inertia forces acting on the crank, connecting rod, and piston, determine the values of s' for which the resultant of S is perpendicular to $\mathbf{n}_2$, and find the value of k for which the $\mathbf{n}_3$ resolute of the moment of S about point O is equal to zero.

Suggestion: Make use of the fact that

$$R \sin \theta' = L \sin \theta$$

from which it follows by repeated differentiation (and keeping in mind that $d\theta'/dt$ is constant) that

$$\left(\frac{d\theta'}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 - \frac{\cos\theta}{\sin\theta}\frac{d^2\theta}{dt^2}$$

Results: $s' = (m/m')(R/L)(L - s)$, $k = [s(L - s)]^{1/2}$.

(f)* A uniform circular disc of radius R and mass m is rigidly attached to a shaft whose axis passes through the center of the disc. The shaft revolves about its axis, which is fixed, with a uniform angular speed ω. Due to misalignment, the shaft axis makes an angle θ with the normal to the plane of the disc.

Determine the magnitude of the moment of the system of inertia forces acting on the disc, about any point.

Result: $(mR^2\omega^2/8) \sin 2\theta$.

(g) A rigid body B consists of three identical uniform square plates, each of mass m, attached to each other as shown in Fig. 8g.

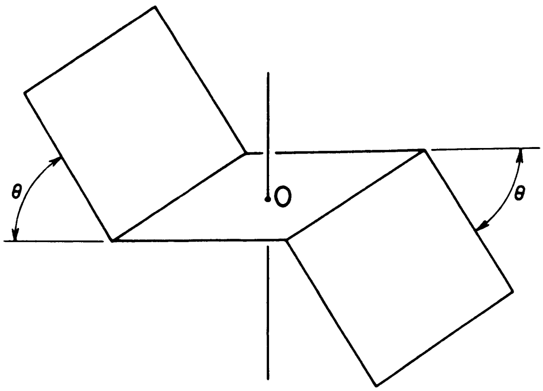


FIG. 8g

B revolves uniformly about an axis which passes through the midpoint O of one of the plates and is perpendicular to this plate.

Letting S be a set of three particles, each having a mass m and placed at the mass center of one of the plates, determine the values of θ for which the system of inertia forces acting on B is equivalent to the system of inertia forces acting on S.

Result: 0, 90 deg.

PROBLEM SET 9

(See Sections 4.2.1–4.2.8)

(a) Referring to Problem 8(a), let R be fixed on the earth, with Y horizontal, and suppose that the plates are made to revolve in such a way that ω is a positive constant.

Considering the plates to be smooth, obtain differential equations governing x and y, and solve these, letting $t = 0$ when $\mathbf{n}_3$ points vertically upward and assuming that at this instant P is at rest relative to the plates, at $x = -g/2\omega^2$, $y = y_0$.

Result: $x = -(g/2\omega^2)(e^{-\omega t} + \sin \omega t)$, $y = y_0$.

(b) Referring to Problem 8(a), let R be fixed on the earth, with Y vertical, and suppose that the plates are made to revolve in such a way that ω is a positive constant.

Considering the plates to be smooth, determine the reaction of the plates on P, assuming that $x = x_0$, $dx/dt = 0$ at $t = 0$.

Result: The force $m\omega^2 x_0(e^{-\omega t} - e^{\omega t})\ \mathbf{n}_3$, line of action through P.

(c) Figure 9c illustrates schematically a mass-spring system which may be used as a component of an inertial navigation device:

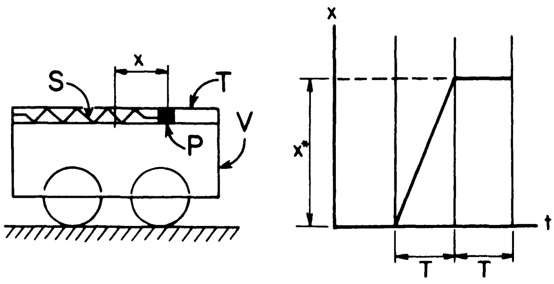


FIG. 9c

A particle P of mass m, attached to a spring S of modulus k, is free to slide in a horizontal tube T, and the tube is rigidly attached to a vehicle V which can move on a straight track parallel to the axis of the tube. When the displacement x of P from P's equilibrium position (x is regarded as positive when S is stretched) is known as a function of time for some time interval, the displacement of V during this time interval can be found.

Determine the displacement of V during the time interval $2T$ for which the displacement-time curve of P is shown in Fig. 9c, assuming that V is at rest at the beginning of this time interval.

Result: $(7/6)(kx^*/m)T^2$, to the left.

(d) A gun at north latitude 45 deg is fired at a target 50 miles to the east of the gun. If the muzzle velocity has a magnitude of 4400 ft sec^{-1}, and the gun is aimed on the basis of computations leaving the earth's rotation out of account, where does the projectile land in relation to the target?

Answer: 859 ft south, 3590 ft east of the target.

(e)* Determine the minimum magnitude of the velocity (in the reference frame A described in Sec. 4.2.7) with which a particle must be projected from the north pole in order to strike the equator.

Result: 4.47 mile sec^{-1}.

(f) A particle is projected from the north pole with the velocity found in Problem 9(e). Find the time of flight and the maximum distance from the center of the earth to the particle.

Results: 32.3 min, 4780 miles.

(g) A particle is projected from the north pole with a velocity having the same direction as that found in Problem 9(e), but having a magnitude $(2)^{1/4}$ times as great.

Determine the angle between a line joining the earth's center to the north pole and a line passing through the earth's center and the point at which the particle strikes the earth.

Result: 180 deg.

(h) An earth satellite whose perigee is 540 miles above the earth's surface traverses its orbit once per day.

Determine the apogee distance and the perigee and apogee speeds of the satellite.

Results: 48,000 miles, 22,400 mile hr^{-1}, 2100 mile hr^{-1}.

(i) Show that a particle projected from the earth's surface in such a way that it moves on a parabolic trajectory does not come into contact with the earth subsequent to the instant of projection, and determine the magnitude of the velocity of projection.

Result: 6.95 mile sec^{-1}.

(j) The mass center of a ballistic rocket crosses the earth's equatorial plane at a height of 540 miles above the surface of the earth, moving parallel to the earth's axis at this instant. The trajectory intersects the earth's axis at a point 6040 miles above the surface of the earth.

Determine the speed of the mass center of the rocket for the instants at which it crosses the equatorial plane and the earth's axis, and find the time which elapses between these instants.

Result: 24,700 mile hr^{-1}, 17,600 mile hr^{-1}, 31.6 min.

PROBLEM SET 10

(See Sections 4.3.1–4.3.5)

(a) The motion described in Problem 8(c) can take place only when ω, Ω, r, h, and θ satisfy a certain equation. Obtain this equation, and use it to find θ if $\omega = 450$ rad sec^{-1}, $\Omega = 10$ rad sec^{-1}, $r = 1$ in., and $h = 3$ in.

Result: 24 deg.

(b) A uniform, sharp-edged, circular disc rolls on a horizontal plane, its center moving on a straight line.

Show that the plane of the disc must be vertical, and determine the minimum speed with which the center must move in order that the motion be stable if the disc has a radius of $\frac{1}{2}$ in.

Result: 8.05 in. sec^{-1}.

(c) A billiard ball is struck in such a way that its center returns to its starting point with one-half of its initial speed. During what fraction of the time required for this motion does the ball roll?

Answer: $\frac{1}{3}$.

(d) A uniform solid sphere is placed at a point O on a horizontal plane and is set in motion with the velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$ (both parallel to the plane) shown in Fig. 10d.

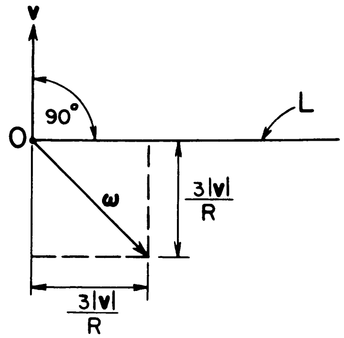


FIG. 10d

Letting d be the distance from point O to the point of line L at which the sphere crosses L, and D the maximum distance between line L and the contact point of the sphere and the plane before the sphere crosses L, determine d/D.

Result: $\frac{48}{7}$.

(e)* When the vertex B of the plate described in Problem 8(b) is held fixed and the plate is permitted to perform small harmonic oscillations, these take place with a frequency f_1. If θ is given some small value while B is held fixed, and the plate is then released from rest, θ again varies harmonically, with a frequency f_2, and line AB oscillates with the same frequency.

Determine f_1/f_2.

Result: $\frac{1}{2}$.

(f)* One end of a uniform, 21 ft long boom is supported in a spherical socket. When the boom is not in use, its upper end is attached to a 5 ft cable and rests on a vertical wall, as shown in Fig. 10f.

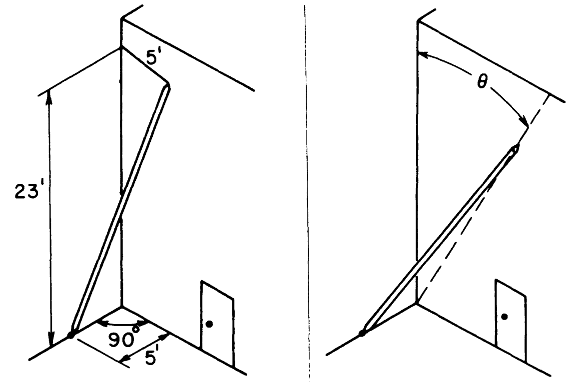


FIG. 10f

If the cable breaks and the boom begins to slide on the wall, there comes an instant t_1 at which the reaction of the wall on the boom vanishes.

Determine the relationship between θ_1 and $\dot{\theta}_1$, these being the values of θ (see Fig. 10f) and $\dot{\theta}$ at time t_1.

Result: $3g \cos \theta_1 = 8(26)^{1/2}\dot{\theta}_1^2$.

(g)* Referring to Problem 8(e), suppose that $\mathbf{n}_3$ is vertical and that k has a value such that the $\mathbf{n}_3$ resolute of the moment about point O of the system of all contact forces acting on the body consisting of the crank, connecting rod, and piston is equal to zero for all motions during which the crank revolves uniformly. Assume that the surface of contact of the cylinder and piston is smooth and that the system of forces exerted on the piston by the contents of the cylinder is equivalent to a force $-p\mathbf{n}_1$ whose line of action passes through point O.

Letting the reaction of the cylinder on the piston be a couple together with a force $\mathbf{R}$ whose line of action passes through the point of intersection of the axes of the cylinder and the crosshead bearing, show that the $\mathbf{n}_3$ resolute of the torque of the couple is equal to zero during all motions of the mechanism, and determine the $\mathbf{n}_2$ resolute of $\mathbf{R}$ for motions during which the crank revolves with constant angular speed ω, assuming that R/L is so small that the second and higher powers of R/L are negligible in comparison with unity.

Result:

$$p(R/L)\sin\theta' - (\omega^2/2)(sm + LM)(R/L)^2 \sin 2\theta'$$

(h) A uniform circular disc of radius 6 in. and mass 0.5 slug is rigidly attached to the end of a cylindrical shaft whose axis passes through the center of the disc. Due to misalignment, the shaft axis makes an angle of 0.5 deg with the normal to the plane of the disc. The shaft is supported by short bearings, placed as shown in Fig. 10h, and the system of forces exerted on the portion of the shaft shown in this sketch—by a contiguous portion—is equivalent to a couple whose torque is parallel to the axis of the shaft.

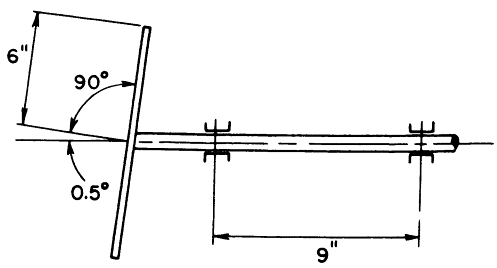


FIG. 10h

Determine the magnitude of each dynamic bearing reaction when the shaft revolves uniformly at 10,000 rpm.

Result: 400 lb.

(i)* Figure 10i represents a device which can be used to demonstrate the stablizing action of a gyroscope. This device consists of a rotor A, a gimbal B, and a frame C. These parts have a common mass center, the point of intersection of the gimbal axis and the rotor axis, and these two axes are perpendicular to each other. The gimbal axis is perpendicular to an edge of C which is sharpened and which can thus serve as a fixed axis of rotation for C when it is placed in contact with a rough support S.

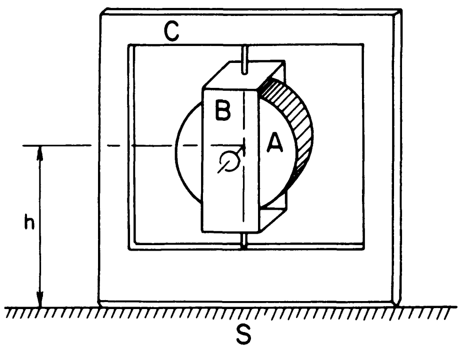


FIG. 10i

During one possible motion of this system, A revolves with constant angular speed ω in B; B remains at rest relative to C, with A's axis perpendicular to C (as indicated in Fig. 10i); and C, supported by a horizontal surface S, remains at rest in a vertical plane. Show that this motion is stable if all bearings are frictionless and ω exceeds a certain critical value ω^*, and determine ω^*, letting m be the sum of the masses of A, B, and C, J the moment of inertia of A about A's axis, and K the sum of the moments of inertia of A and B about the gimbal axis.

Result: $(mghK)^{1/2}/J$.

(j) Figure 10j is a schematic representation of a gyroscopic ship's stabilizer consisting of a cylindrical rotor R, a gimbal G, and two electric motors M and M'. M is attached to G and drives R. M', fastened to the ship S, causes G to rotate relative to S.

Letting E be a reference frame fixed relative to the earth, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ mutually perpendicular unit vectors fixed in G, as shown in Fig. 10j, and t' an instant at which $\mathbf{n}_1$ is parallel to the ship's longitudinal axis, suppose that at t'

$$^E\boldsymbol{\omega}^S = r\mathbf{n}_1, \qquad ^S\boldsymbol{\omega}^G = p\mathbf{n}_2, \qquad ^G\boldsymbol{\omega}^R = s\mathbf{n}_3$$

$$^E\boldsymbol{\alpha}^S = {}^S\boldsymbol{\alpha}^G = {}^G\boldsymbol{\alpha}^R = 0$$

(r is called the "rolling rate" of S, p the "precession rate" of G, s the "spin rate" of R.) Neglecting moments of inertia of G in comparison with those of R, letting J be R's moment of inertia

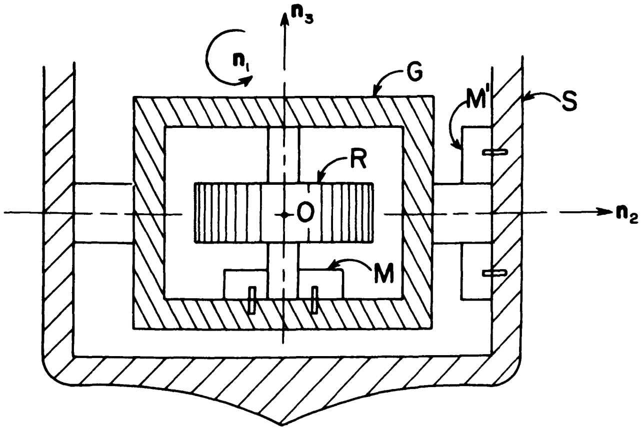


FIG. 10j

about R's axis, and reducing the system of forces exerted at time t' by G on S (and M') to a force applied at point O and a couple, determine the torque $\mathbf{T}$ of this couple. (The $\mathbf{n}_1$ component of $\mathbf{T}$ furnishes a measure of the stabilizing influence of the device.)

Result: $\mathbf{T} = J(-ps\mathbf{n}_1 + sr\mathbf{n}_2 - rp\mathbf{n}_3)$.

(k)* The steady state motion considered in Problem 5(g) can occur only when v does not exceed a certain critical value. When v reaches this value, one of the wheels loses contact with the support (or, equivalently, the reaction of the plane on this wheel vanishes).

Determine the critical value of v, assuming that each wheel has a mass m and moment of inertia $md^2/8$ about line CD, the frame has a mass $2m$, B is the mass center of the frame, and $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ are principal directions of the frame for point B.

Result: $(10gd)^{1/2}$.

PROBLEM SET 11

(See Sections 4.4.1–4.4.16)

(a) Referring to Problem 8(e), determine the value of s' for which the sum of the linear momenta of the crank, connecting rod, and piston is perpendicular to $\mathbf{n}_2$ during all motions of the mechanism.

Result: $(m/m')(R/L)(L - s)$.

(b)* Referring to Problem 5(c), determine the magnitude of the angular momentum of H with respect to the center of the circle on which point C moves.

Result: $(mvR/2)(13 - 8 \cos \theta - 5 \cos^2 \theta)^{1/2}$.

(c) The motion described in Problem 5(c) can occur only when v, θ, and R satisfy a certain equation. Obtain this equation by using (1) D'Alembert's principle and (2) the linear and angular momentum principles, and evaluate the relative advantages of these two methods of solution.

Result: $v^2(\sin \theta + 4 \tan \theta) = 2gR$.

(d) As an alternative to the definition given in Sec. 4.4.8, the angular momentum ${}^R\mathbf{A}^{S/Q}$ of a set S of n particles P_i, $i = 1, 2, \ldots, n$, of masses m_i, $i = 1, 2, \ldots, n$, may be defined as

$$ {}^R\mathbf{A}^{S/Q} = \sum_{i=1}^{n} m_i \mathbf{r}_i \times {}^R\mathbf{v}^{P_i} $$

where $\mathbf{r}_i$ is the position vector of P_i relative to point Q and ${}^R\mathbf{v}^{P_i}$ is the absolute velocity of P_i in reference frame R.

Show that the two definitions lead to identical results when Q is either a point fixed in R or the mass center of S.

(e) Letting Q be a point fixed on a rigid body R, show that the angular momentum (${}^{R'}\mathbf{A}^{R/Q}$) of R relative to Q in a reference frame R' is parallel to the angular velocity (${}^{R'}\boldsymbol{\omega}^R$) of R in R' if and only if ${}^{R'}\boldsymbol{\omega}^R$ is parallel to a principal axis of R for point Q; further, that ${}^{R'}\mathbf{A}^{R/Q}$ is then equal to the product of ${}^{R'}\boldsymbol{\omega}^R$ and the moment of inertia of R about a line passing through Q and parallel to ${}^{R'}\boldsymbol{\omega}^R$.

(f) Use the angular momentum principle to solve Problem 10(e), and compare this method of solution with that used previously.

(g) Use the angular momentum principle to solve Problem 10(g), and compare this method of solution with that used previously.

(h) Use the linear and angular momentum principles to solve Problem 10(k), and compare this method of solution with that used previously.

PROBLEM SET 12

(See Sections 4.5.1–4.5.12)

(a) The particle P whose motion is described in Problem 5(j) has a mass of 0.01 slug. Determine the kinetic energy of P at time t^* in (1) T and (2) L.

Result: 0.01 ft lb, 0.1 ft lb.

(b) Letting S be the set of two particles P and Q of Problem 5(k), and assuming that each particle has a mass of 0.04 slug, evaluate the kinetic energy of S relative to point O in reference frame R at time t'.

Result: 229 ft lb.

(c) Referring to Problem 8(b), and considering motions during which θ remains so small that second and higher degree terms in θ can be dropped, determine the kinetic energy of the plate.

Result: $(m/12)(3a^2\dot{\phi}^2 + 3ab\dot{\phi}\dot{\theta} + b^2\dot{\theta}^2)$.

(d) Determine the kinetic energy of the hoop H whose motion is described in Problem 5(c).

Result: $(mv^2/4)(4 + \sin^2 \theta)$.

(e) A circular tube of radius R is made to revolve with constant angular speed ω about a fixed vertical diameter. A particle, placed in the tube at a point lying on a horizontal diameter, is released from rest (relative to the tube).

Neglecting friction, determine the smallest value of R for which the particle fails to reach the lowest point of the tube during its motion subsequent to being released.

Result: $2g/\omega^2$.

(f) Referring to Problem 10(f), and assuming that friction is negligible, determine (1) the values of θ_1 and $\dot{\theta}_1$ and (2) the distance D between the upper end of the boom and the wall at the instant t_2 at which the boom strikes the ground.

Note: Subsequent to the instant t_1 at which the boom loses contact with the wall, the boom's motion can be described in terms of the two angles θ and ϕ shown in Fig. 12f, where point Q is the foot of the perpendicular dropped on the wall from point P.

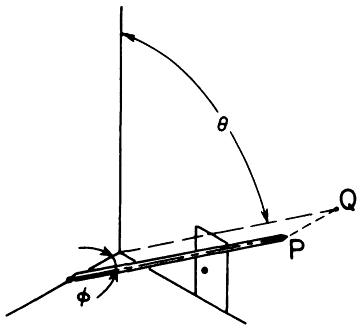


FIG. 12f

Suggestion: Apply D'Alembert's principle (or the angular momentum principle) to find the differential equations governing θ and ϕ during the time interval $t_1 < t < t_2$. Next, noting that ϕ_1 and $\dot{\phi}_1$, the values of ϕ and $\dot{\phi}$ at time t_1, are given by

$$\phi_1 = \cos^{-1}\frac{5}{21} = 1.33 \text{ rad}, \qquad \dot{\phi}_1 = 0$$

use the differential equations and the values of θ_1 and $\dot{\theta}_1$ found in (1), above, to evaluate the second derivatives, $\ddot{\theta}_1$ and $\ddot{\phi}_1$, of θ and ϕ at time t_1; then differentiate the differential equations to find the third and higher derivatives of θ and ϕ at time t_1.

As $\theta = \pi/2$ at the instant t_2 at which the boom strikes the ground, the Taylor series

$$\frac{\pi}{2} = \theta_1 + \frac{\dot{\theta}_1}{1!}(t_2 - t_1) + \frac{\ddot{\theta}_1}{2!}(t_2 - t_1)^2 + \dots$$

can now be used to find $(t_2 - t_1)$ to any degree of accuracy desired, and the series

$$\phi_2 = \phi_1 + \frac{\dot{\phi}_1}{1!}(t_2 - t_1) + \frac{\ddot{\phi}_1}{2!}(t_2 - t_1)^2 + \ldots$$

then yields the value of ϕ_2, in terms of which D is given by

$$D = 5 - 21 \cos \phi_2 \text{ ft.}$$

Results: $\theta_1 = 0.859$ rad, $\dot{\theta}_1 = 1.24$ rad sec^{-1}, $D = 0.38$ ft.

(g)* The system of all gravitational and contact forces acting on a rigid body R is equivalent to a force $\mathbf{F}$, applied at a point P of R, together with a couple of torque $\mathbf{T}$.

Express ${}^{R'}A^R$, the activity of this system of forces in a Newtonian reference frame R', in terms of $\mathbf{F}$, $\mathbf{T}$, ${}^{R'}\mathbf{v}^P$, and ${}^{R'}\boldsymbol{\omega}^R$.

Result: ${}^{R'}A^R = {}^{R'}\mathbf{v}^P \cdot \mathbf{F} + {}^{R'}\boldsymbol{\omega}^R \cdot \mathbf{T}$.

(h) The shaft considered in Problem 5(d) is subjected to the action of a system of contact forces equivalent to a couple whose torque is parallel to the shaft's axis and constant in magnitude. Starting from rest, the shaft acquires an angular speed of N revolutions per unit of time while completing n revolutions.

Determine the magnitude of the torque, assuming that the shaft's axis is vertical, the spheres, each of mass m, are solid, the shaft has a moment of inertia J about its axis, $\theta = 30$ deg, and the contact between C and S is one of pure rolling.

Result: $(\pi N^2/n)[J + 18mr^2(2 + \sqrt{3})/5]$.

(i) Traveling at 30 mile hr^{-1} on a straight, horizontal road, an automobile approaches a traffic light. The automobile has four-wheel brakes, each of which can exert on its wheel a system of forces equivalent to a couple whose torque is parallel to the axis of the wheel and has a magnitude T (the same for all wheels). As the automobile reaches a point 130 ft from the light, the light turns red, and the driver applies the brakes in such a way that throughout the subsequent motion T is proportional to the distance traversed with the brakes engaged. The automobile weighs 1610 lb, and each wheel has a diameter of 2 ft.

Regarding the moments of inertia of the wheels about their axes as negligible, determine the maximum value of T if the automobile comes to rest 20 ft from the traffic light.

Result: 220 ft lb.

(j) In Fig. 12j, A, B, C, D, and E represent identical uniform square plates, each of mass m and side L. These are attached to

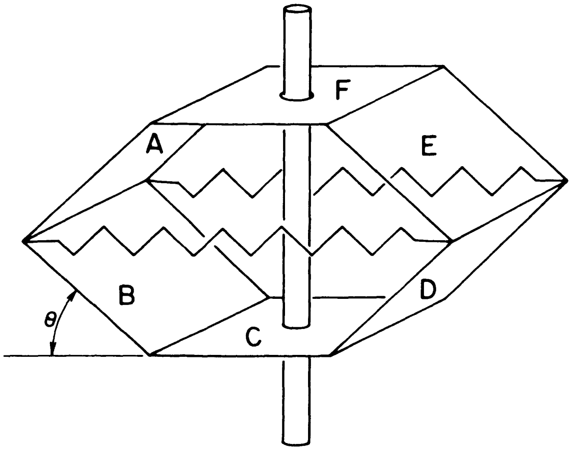


FIG. 12j

each other and to a uniform square plate F of mass $5m$, side L, by means of smooth hinges. A vertical shaft, to which C is rigidly attached, passes through an opening in F, thus leaving F free to slide on the shaft. Two light, linear springs of length L and modulus k connect the plates as shown in Fig. 12j.

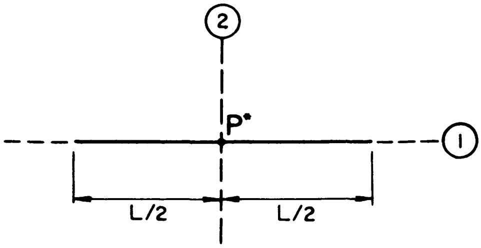
One possible motion of this system is described as follows: The shaft is made to revolve with constant angular speed ω about its axis, and θ remains constant at 45 deg. Determine ω, and find the smallest value of k for which this motion is stable.

Results:

$$\left[6(3 - 2\sqrt{2})\left(2\sqrt{2}\,\frac{k}{m} - 7\,\frac{g}{L}\right)\right]^{1/2}, \qquad \frac{7mg}{\sqrt{2}L}\left(1 + \frac{\sqrt{2}}{3}\right)$$

APPENDIX

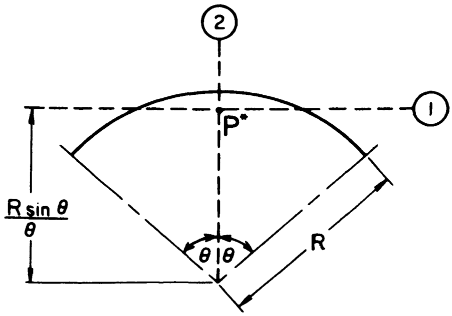
Centroidal Principal Axes
and Squares of
Centroidal Principal Radii of Gyration
of
Curves, Surfaces, and Solids



Length: L

$k_1^2 = 0, \qquad k_2^2 = L^2/12$

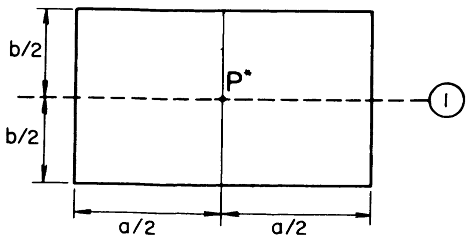
FIG. 1. STRAIGHT LINE



Length: $2R\theta$

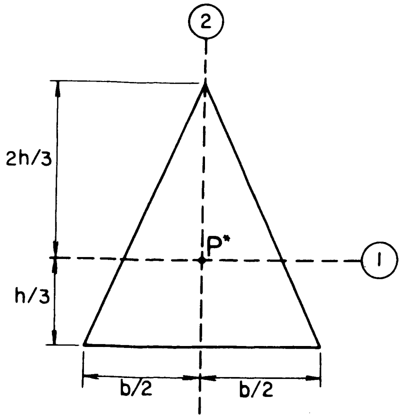
$$k_1^2 = \frac{R^2}{2}\left(1 + \frac{\sin 2\theta}{2\theta} - \frac{2\sin^2\theta}{\theta^2}\right), \qquad k_2^2 = \frac{R^2}{2}\left(1 - \frac{\sin 2\theta}{2\theta}\right)$$

FIG. 2. CIRCULAR ARC



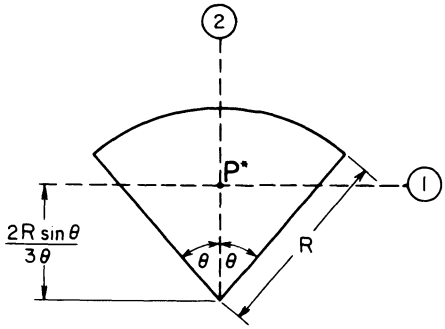
Area: ab
$k_1^2 = b^2/12$

FIG. 3. RECTANGLE



Area: bh/2
$k_1^2 = h^2/18, \qquad k_2^2 = b^2/24$

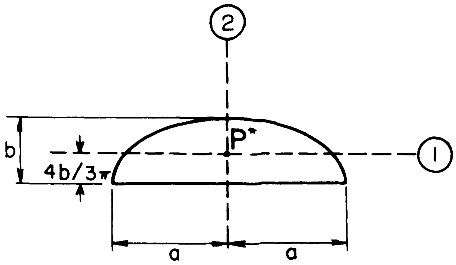
FIG. 4. ISOSCELES TRIANGLE



Area: θR^2

$$k_1^2 = \frac{R^2}{4}\left(1 + \frac{\sin 2\theta}{2\theta} - \frac{16\sin^2\theta}{9\theta^2}\right), \qquad k_2^2 = \frac{R^2}{4}\left(1 - \frac{\sin 2\theta}{2\theta}\right)$$

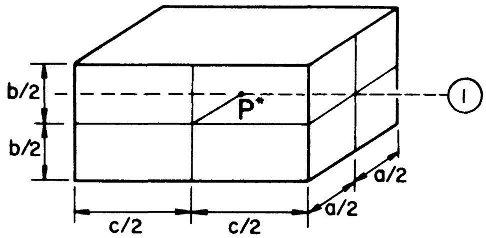
FIG. 5. CIRCULAR SECTOR



Area: $\pi ab/2$

$k_1^2 = b^2(9\pi^2 - 64)/36\pi^2, \qquad k_2^2 = a^2/4$

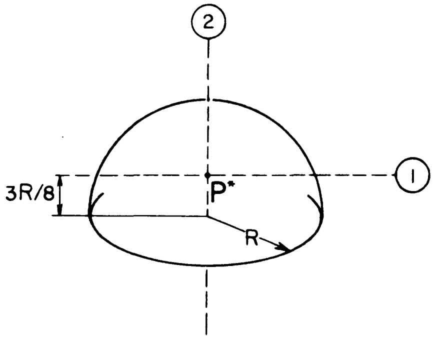
FIG. 6. SEMI-ELLIPSE



Volume: abc

$$k_1^2 = \frac{a^2 + b^2}{12}$$

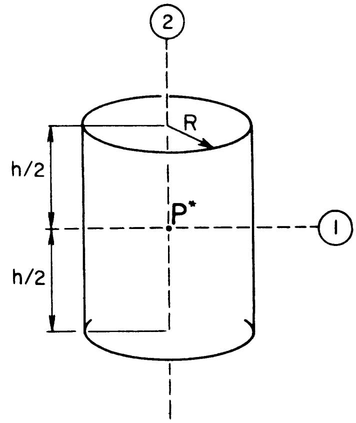
FIG. 7. RECTANGULAR PARALLELEPIPED



Volume: $2\pi R^3/3$

$$k_1^2 = \frac{83}{320} R^2, \qquad k_2^2 = \frac{2}{5} R^2$$

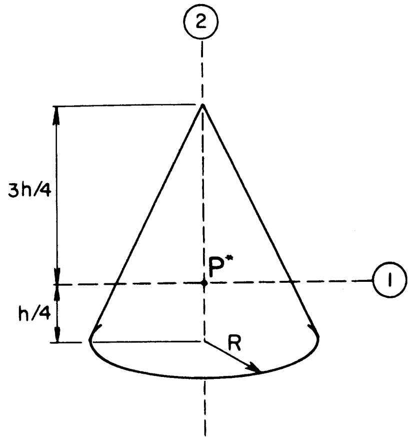
FIG. 8. HEMISPHERE



Volume: $\pi h R^2$

$$k_1^2 = \frac{1}{12}(3R^2 + h^2), \qquad k_2^2 = \frac{R^2}{2}$$

FIG. 9. RIGHT-CIRCULAR CYLINDER



Volume: $\pi h R^2/3$

$k_1^2 = 3(4R^2 + h^2)/80, \qquad k_2^2 = 3R^2/10$

FIG. 10. RIGHT-CIRCULAR CONE

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