ON THE DUALITY OF INTERSECTIONS AND CLOSEST POINTS†

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ABSTRACT

We call the common intersection of k objects a k-intersection. We address questions of the form: Given n objects in the plane,'Does there exist a k-intersection ?', 'How many such k-intersections exist ?' and 'What is the maximum value of k for which there exists a k-intersection ?'. In answering such questions we utilize a *duality* that exists between k-intersections and k-closest points (the k points with the smallest enclosing circle), in that as long as k points are close enough to be enclosed by a circle of radius r, the common intersection of k circles of radius r centered at each of those k points (i.e. their k-intersection) is non - null. A technique, dubbed 'rotation sorting' is then developed to provide efficient solutions to dual questions of the form: Given n points in the plane, 'Does there exist a set of k points which can be covered by a circle of radius r?', 'How many such sets of k points exist ?', and 'What are the maximum number of points that can be enclosed by a circle of radius r ?'.

A similar duality between the intersection of a line with circles and the proximity of this line to the centers of the circles is exploited to obtain efficient algorithms for the transversals of circles in the plane. Extensions of the above questions to three and higher dimensions are also addressed, alongwith problems concerning unequal sized objects. Further certain generalisations of k-intersections and of the number of points enclosed by k objects in two dimensions and higher are shown to be NP-complete and D^P-complete.

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1. Introduction

This paper tries to relate questions about the intersection of objects with that involving the 'closeness' of points in a finite set. Calling the common intersection of k objects a k-intersection we concern ourselves specifically with questions of finding whether there exists a set of k objects, out of a finite set n, having a k-intersection? and the maximum value of k for which there exists such k-intersections? Various geometric applications involve finding the answer to such questions. A dual set of problems consist of finding whether in a finite set of size n, there exist a set of k points close enough to be covered by a fixed sized object? and the maximum number of such points which can be covered by this fixed sized object. Such questions arise in clustering, statistical data analysis and in facility location problems.

Section 2 attemps to relate the above sets of questions and in doing so points out the inherent duality that exists between the intersection and the closeness problems for a class of convex objects. The objects considered here are primarily circles and spheres. Section 3 provides efficient algorithms for these intersection and closest point problems. A new technique called 'rotation sorting' is developed and used. For objects being circles of radius r in the plane, a sorting of the points of a finite set is defined with respect to a single fixed point x of this set, with the other points being ordered in the way they encounter the arc of a circle of radius r as the arc is rotated about x. This technique is also extendable to spheres in arbitrary d-dimensions.

Section 4 looks at some natural question concerning transversals, in particular, whether there exists a circle from a finite set which intersects all other circles – a circular transversal? and whether there exists a line which intersects all circles – a linear transversal. Section 5 generalizes the earlier intersection and closeness problem in two ways. First, the k-intersection problems are discussed for a finite set of unequal radius circles. An interesting side question answered is that of the maximum level of containment (maximum number of circles completely within one another), for a set of circles. Secondly, certain generalisations of k-intersections and of the number of points enclosed by k objects in two dimensions and higher are shown to be NP-complete and D^p-complete.

2. Developing the duality

This section starts with a few introductory facts on 'minimum enclosing circles and common intersections of circles. These are then used in highlighting the duality between the intersections of objects and closest points.

Fact 1: The minimum enclosing circle of a set of points in the plane is *uniquely* determined by either 2 or 3 points.[Rad57]

Fact 2: The minimum enclosing circle of the vertices of an *obtuse angle* (considered $\geq 90°$) triangle is not the circumscribed circle but the circle having the longest side of the triangle as diameter.[Rad57]

Fact 3: The common intersection of 3 circles (i.e. a 3 - intersection) could be null even if the circles pairwise intersect. (This fact also applies to bounded convex and non-convex objects).[Yag61]

Let c_i be the center of the i^{th} circle and let $d(x,y)$ = Euclidean distance between any two points x and y. Let the minimum enclosing circle of the centers $\{c_i\}$ of k circles in the plane be denoted by MEC^k, i.e. if c = center of $MEC^k \Rightarrow \forall$ i=1..k, $d(c,c_i) \leq$ radius of MEC^k.

The first result which approaches the notion of a duality between intersections and closest points, stems directly from the following theorem.

Theorem [Jung's Thm]: Let k points be given in a plane such that each pair of them are at a distance of at most 1 from each other. Then all these points can be enclosed in a circle of radius $1 / \sqrt{3}$. [Yag61]

Considering circles in the plane, two circles of equal radius r intersect iff their centers are at most $2r$ from each other. This then immediately leads to the following Corollary

Corollary : If each of $\binom{k}{2}$ pairs of circles of equal radius r intersect, then the radius of $MEC^k \leq 2r / \sqrt{3}$.

However, pairwise intersection of k circles does not guarantee a non-null k-intersection. Fact 3 affirms this for the case k=3. Nevertheless the following special case holds.

Lemma 1: For 4 circles of equal radius R in the plane, if each of $\binom{4}{2}$ pairs of circles intersect, then $\exists$ a non - null 3-intersection.

Proof : Consider the geometrical figure obtained by joining each pair of the centers of the 4 circles in the plane by straight lines. In this figure there exists at least one set of 3 centers which form an obtuse angle ($\geq 90°$) triangle. As circles pairwise intersect, $r \geq$ radius of MEC^2 for any two centers. From fact 2 it follows that for the vertices of the longest side of the obtuse angle triangle, the radius of MEC^2= radius of MEC^3 for the vertices of the obtuse angle triangle $\Rightarrow$ $r \geq$ radius of MEC^3 for some three center points $\Rightarrow \exists$ a non - null 3-intersection. ■

Further for a general k-intersection we have

Theorem 1: For k circles of equal radius r, the k-intersection is non-null **iff** $r \geq$ radius of MEC^k. Further when $r =$ radius of MEC^k then the r-intersection is a unique point, viz. the center of the MEC^k.

Proof: If the k-intersection is non-null then there exists a point c such that

$$\forall i=1..k,\ d(c,c_i) \leq r \qquad (1)$$

From the definition of the *minimum enclosing circle* of a set of points in the plane it follows that the radius of the MEC^k is the minimum value for which (1) is satisfied $\Rightarrow r \geq$ radius of MEC^k.

If $r \geq$ radius of MEC^k, then for the center c of the MEC^k,

$$\forall i=1..k,\ d(c,c_i) \leq \textit{radius of } MEC^k \leq r$$

This implies that c is in the k-intersection and hence the k-intersection is non-null.

In the limiting case when $r =$ radius of MEC^k, any point c belonging to the k-intersection could be made the center of the MEC^k. However, from fact 1 we know that the MEC^k is unique $\Rightarrow$ the k-intersection is a unique point. ■

This gives rise to a *duality* which exists between k-intersections and k-closest points. As long as k points are close enough to be covered by a circle of radius r, the common intersection of circles of radius r centered at those k points is non-null.

Exploring the common intersection of sets of objects further, we find

Theorem [Helly's Thm]† : For k bounded convex objects in the plane, if each 3 of them [$\binom{k}{3}$ such sets],have a common point, then all these objects have a common point. [Yag61]

For circles of equal radius we have the following case. (We provide a proof because of its simplistic connection to the duality and also because the proof of Helly's theorem is quite complicated).

Corollary : For k circles of equal radius r in the plane if each 3 of them have a non-null 3-intersection then the k-intersection is also non-null.

Proof : From fact 1 it follows that for some set of 3 centers, the MEC^3 encloses all k centers. Thus the radius of some $MEC^3 =$ radius of MEC^k. As every 3-intersection is non-null, Theorem 1 $\Rightarrow r \geq$ radius of $MEC^3 \geq$ radius of $MEC^k \Rightarrow$ the k-intersection is non-null. ■

From the duality also arises the following interesting fact.

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† Helly's Thm extends to n-dimensional space in that if each of n+ 1 bounded convex objects from a certain number of such objects, have a common point then all these objects have a common point.

Corollary : For k points in the plane if each 3 points can be enclosed by a circle of radius r then such a circle encloses all the points.

The above theorem, in particular for the case $k = 4$, does not hold for non - convex objects as can be seen by the example of 3 ovals and a circular annulus.

One can also enumerate the k-intersections as indicated by the following lemma, the proof of which follows from Helly's theorem as well as the definition of a k-intersection.

Lemma 2: For n objects in the plane,

(a) A k-intersection is non-null $\Rightarrow \binom{k}{k-1}$ different $(k\text{-}1)$-intersections are non - null.

(b) For $k > 3$, $\binom{k}{k-1}$ different $(k\text{-}1)$-intersections are non-null $\Rightarrow$ a k-intersection is non - null.

For circles of unequal radii $\{r_i\}$ the following result follows directly from theorem 1

Theorem 2: For k circles of unequal radius r_i, i=1..k, the k-intersection is non-null **if** Minimum $\{r_i\} \geq$ radius of MEC^k.

Proof : Start with k circles each with radius Minimum $\{r_i\}$. From theorem 1, Minimum$\{r_i\} \geq$ radius of $MEC^k \Rightarrow$ the k-intersection is non-null. Clearly for bigger circles with the same centers (and hence same MEC), the k-intersection must be non-null also. ■

3. Intersections and Closest Point Problems

Given n circles of equal radius r in the plane,

(p1) [*Detection of a* k-intersection]
Does there exist a k- intersection ?
(Does there exist a set of k circles whose MEC^k has a radius $\leq r$?)

(p2) [*Counting* k-intersections]
How many k-intersections exist ?
(How many sets of k circles have their MEC^k with a radius $\leq r$?)

(p3) [*Maximum* k-intersection]
Find the maximum value of k for which there exists a k-intersection .
(Find the maximum number k of circles whose MEC^k has a radius $\leq r$)

(p4) [*k-closest points*] †
Find k points amongst n points in the plane, with the smallest enclosing circle.
(Find the k circles with the smallest radius MEC^k).

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† Note that the k-closest points as defined here are quite different to those in [Sha75], wherein the k-closest points are taken to be the k points closest to a new query point x.

Lower Bounds

For $k = 2$, the problems (p1) and (p2), in 1-dimension (and higher) have been shown to have a lower bound of Ω (nlogn) by reduction to the element uniqueness problem. [Sha75]

A Ω (nlogn) lower bound is now shown for the *maximum* k-intersection problem, (p3). In 1-dimension (p3) reduces to finding the maximum number k of intervals having a common intersection, given n intervals of equal length r on a line.

Theorem 3: Given n intervals of equal length r on a line to find the maximum number k of intervals having a common intersection takes Ω (nlogn) time.

Proof : Considering only the left-endpoints of the n intervals, from the earlier duality it is easy to see that the corresponding 1-dimensional dual problem is that of finding the maximum number of points enclosed by a bar of length r given n points on a line.

Assume the multiset of points are randomly chosen from the set of odd natural numbers {1,3,5,...}. Then for a bar of length 1, the set of points are unique iff the maximum number of points enclosed by the bar is 1. This reduction of the element uniqueness problem with a Ω (nlogn) lower bound [Dob75], to our problem proves the theorem. ■

For $k = n$, the problem (p1) is that of detecting the 'common' intersection of n circles of equal radius. This is equivalent to finding the minimum enclosing circle for n points in the plane for which there exists an $O(n)$ and hence optimal solution [Meg82].

Upper Bounds

In 1-dimension where circles of equal radius r degenerate to equal line segments (or intervals) of length r on a line the problems (p1) to (p4) can each be solved in $O(nlogn)$ time by sorting the left-endpoints of the n intervals and scanning the points on the line in linear time. By the lower bounds above this is optimal.

In 2-dimensions we now give solutions for each of the problems (p1) to (p6).

Fact 4: There are at most 2 circles of a given radius r which pass through 2 given points.

Theorem 4: The maximum k-intersection problem can be solved in $O(n^2logn)$ time.

Proof : For each of the centers c_i, i=1..n we do the following

(1) Let M be the set of points within the euclidean distance $\leq 2r$ of c_i. [The size of M is at worst n]. Define the "$\oslash$" order between two points c_j and c_k to be, c_j "$\oslash$" c_k if there is a circle of radius r passing through points c_i and c_k such that the point c_j is 'below' the arc c_i,c_k. Sort the set of points M according to this ordering relation. {This is similar to sorting by polar angle with c_i as

origin, wherein the points c_j and c_k are compared with respect to the line c_i,c_k as opposed to the arc of a circle of radius r as is done above. Thus the definition of c_j 'below' arc c_i,c_k here is : on rotating the arc of a circle of radius r in an anti-clockwise direction about a point c_i, the point c_j intersects the arc before the point c_k}. On the basis of Fact 4 we can test the "$\oslash$" relation in O(1) time giving a worst case O($nlogn$) sorting time for the set M.

(2) Now for each of the points $c_j \in$ M, determine the number of points enclosed by the circle of radius r passing through c_i and c_j. We do this by sweeping in an anti-clockwise direction around the point c_i. Since these points have been sorted the entire operation takes at most O(n) time. {This is done by keeping two lists sorted as above, one with respect to the minor arc c_i, c_j and the other for the corresponding major arc for updating points entering the circle and points leaving the circle respectively, during the anticlockwise sweep }. The maximum number of points that is obtained by this step is the maximum number of circles that have a non-null common intersection, (by Theorem 1). As steps (1) and (2) are repeated for each of the centers of the n circles the total time for the entire operation is O($n*nlogn$) = O(n^2logn). [Note that here one does not obtain a single sorting for all the points as in the 1-dimension case and hence a suitable sorting has to be done for each point in turn giving rise to the above time complexity]. ■

Corollary : The k-intersection detection problem can be solved in O(n^2logn) time.

Proof : There exists a k-intersection iff the maximum value returned by the algorithm of Theorem 3 returns a value $\geq k$. ■

Corollary : The counting k-intersection problem can be solved in O(n^2logn) time.

Proof : For each of the circles obtained in the algorithm of Theorem 3 we determine the number of points it encloses. For each value (number of points) $\geq k$ we can count all the k-intersections by using part (a) of Lemma 2. ■.

For the case of spheres in 3-dimensions, three points are sufficient to specify a sphere of fixed radius r, and rotation sorting can now done with respect to 2 points at a time. For general hyperspheres in d-dimensional space, the sorting is done with respect to d-1 points giving an O($n^d logn$) method for all choices of d-1 points. This proves the following theorem.

Theorem 5: For n hyperspheres in d dimensions, (i) The maximum k-intersection problem, and (ii) The k-intersection detection problem can be solved in O($n^d logn$) time in the worst case.†

† A more exact analysis gives a time bound of O($nm^{d-1}logm$) where m = the maximum number of points close enough to be covered by a hypersphere of radius $2r$.

For the k-closest points problem we use what is known as the order k-nearest neighbor Voronoi diagram, namely the $V_k(S)$ for a set S of points in the plane.[Sha75],[Lee82]. From [Lee82] we know that $V_k(S)$ can be obtained in $O(k^2 nlogn)$ time using $O(k^2(n-k))$ storage. Further the number of regions (polygons) in $V_k(S)$ does not exceed $O(k(n-k))$.

Theorem 6: The k-closest points problem can be solved in $O(k^2 nlogn))$ time.

Proof : Let $S = \{c_1, c_2, .. c_n\}$, be the centers of the circles. Obtain the $V_k(S)$ diagram in $O(k^2 nlogn)$ time. Each of the polygons of the $V_k(S)$ diagram is a region from which a set of points $\{c_{i_1}, .. c_{i_k}\}$ are the k-nearest neighbors. For each such set of points the radius of the minimum enclosing circle is computed. There must exist a region in the $V_k(S)$ diagram from which the k-closest points are the k-nearest neighbors. Hence the points for which the smallest radius is obtained are the k-closest points.

In the worst case, the radius of the minimum enclosing circle needs to be computed $O(k(n-k))$ times for sets of k points at a time, ($O(k)$ time for a set of k points [Meg82]).† Hence the total time is bounded by the time taken to construct the $V_k(S)$ diagram. ■

Theorem 7: The k-intersection detection problem can be solved in $O(\text{minimum}\{O(k^2 nlogn), O(n^2 logn)\}$ time.

Proof: From the corollary to Theorem 4 we know of a solution in $O(n^2 logn)$ time.

For $k < \sqrt{n}$, the $V_k(S)$ diagram of Theorem 6 provides a faster solution. There exists a k-intersection of circles of radius r iff the radius of the minimum enclosing circle of the k-closest points $\leq r$ [Theorem 1]. The assertion follows immediately. ■

4. Transversal problems

Given n circles of equal radius r in the plane,

(p5) [*Circular Transversal*]
Does one of the given circles intersect all the other circles ?

(p6) [*Linear Transversal*]
Does there exist a line which intersects all the circles ?

Theorem 8: The circular transversal problem for n circles of equal radius r in the plane can be solved in $O(nlogn)$ time.

Proof : Let S be the set of the center points c_i, $i=1..n$ of the n circles of equal radius r in the plane. We then construct the farthest point Voronoi

† Actually one can obtain all MEC's in $O(k(n-k+1))$ time, since between adjacent polygons (regions) only the k^{th}-nearest neighbor changes.

diagram $V_f(S)$ for the set S in the plane. This takes O($nlogn$) time [Sha75]. Next for each of the center points c_i we obtain directly from $V_f(S)$ the point c_j farthest from c_i {in constant time} and check if the circle centered at c_i intersects the circle centered at c_j. It is not difficult to see that if the circle C_i centered at c_i intersects the circle centered at c_j (i.e. $d(c_i,c_j) \leq 2r$) $\Rightarrow$ circle C_i intersects all the other circles. Since the farthest point is obtained at most n times, once for each of the c_i, the entire algorithm is bounded by O($nlogn$) time. ■

Corollary : One can find all circular transversals in O($nlogn$) time.

Corollary Given n circles of equal radius r, one can determine whether any circle of radius R with center c is a circular transversal in O($logn$) time using O($nlogn$) preprocessing.

Proof : Obtain the farthest center point c_f for the point c in the farthest point Voronoi diagram $V_f(S)$ as above. This takes O($logn$) time once the $V_f(S)$ is constructed, {O($nlogn$) preprocessing}. If the circle centered at c intersects the farthest circle from it i.e $d(c,c_f) \leq r+R$, then it is a circular transversal. ■

Corollary Given n points in the plane, one can determine whether any circle of radius R with center c encloses all the points in O($logn$) time using O($nlogn$) preprocessing.

Proof : Here, if the farthest point f from the center c is within the circle of radius R i.e., $d(c,f) \leq R$ then the circle centered at c encloses all the points. ■

Theorem 9: The line traversal problem for n circles of equal radius r in the plane can be solved in O($nlogn$) time.

Proof : Let S be the set of the center points of the n circles. Construct the convex hull for this set S of n points in the plane. For each of the parallel support lines of the convex hull polygon find the separating distance. A pair of vertices of the polygon are called antipodal if there exist parallel lines of support of the polygon passing through those vertices. The distance between antipodal vertices are the separating distances for the lines of support. There exists a line transversal for the n circles iff the minimum of the separating distances above is $\leq 2r$. Since there are exactly n antipodal pairs and furthermore can all be found in O(n) time [Sha75a] the complexity of the algorithm is bound by the time to obtain the convex hull of a set of n points in the plane. ■

5. Generalisations

5.1. Unequal sized objects

(p7) [Circle Containment]
Given n circles of unequal radius r_i in the plane, what is the maximum level k of containment (maximum number of circles completely within one another) ? List the k circles in the order in which they are contained within each other.

Lower Bounds

Theorem 10: The circle containment problem takes $\Omega(nlogn)$ time.

Proof: Given a sequence of elements $\{x_i\}$, $i=1..n$, construct n circles with centers at the origin (say) and having radii $\{x_i\}$. The maximum level of containment is n and a listing of the n circles in the order in which they are contained is a sort of the sequence $\{x_i\}$. ■

The problem is $\Omega(nlogn)$ even in 1-dimension, where given n intervals on a line of unequal length l_i we wish to list in order those intervals which enclose each other, conforming to the maximum level of containment. A similar reduction to the one above shows this to be true (construct intervals with their left endpoints all starting at the origin). The containment problem can also be reduced to a directed acyclic graph with vertices $\{v_i\}$ corresponding to circles $\{c_i\}$ and directed edges $<v_i,v_j>$ iff circle c_i is contained in circle c_j. The circle containment problem then reduces to finding the longest directed path in the acyclic graph.

Upper bounds

Theorem 11: The circle containment problem can be solved in $O(n^2)$.

Proof: Given n circles with centres c_i of radius $\{r_i\}$, i=1,2,...,n. Sort the radius $\{r_i\}$ into non-decreasing order. A circle with centre c_i with smaller radius r_i is contained in a circle with centre c_j with larger radius r_j if $d(c_i,c_j) \leq r_j - r_i$. Now consider the circles in increasing order of radii, and for each circle check to see if any if the circles before it (with smaller radius) are contained within it. For each circle one remembers the maximum number and the corresponding list of circles it contains. At the end of the scan of n circles, one obtains the circle with the maximum number of circles contained within it. Clearly the total time taken is $O(n^2)$. ■

Remark: This method is easily generalised to unequal sized spheres

Theorem 12: In 1-dimension the containment problem can be solved in $O(nlogn)$ time and this is optimal.

Proof: Again only a smaller sized interval can be contained in a larger interval. For intervals I_i,I_j with left and right endpoint (l_i,r_i) and (l_j,r_j), respectively, I_i is contained in I_j if $l_j \leq l_i$ and $r_i \leq r_j$. Sort the $2n$ endpoints of the n intervals. Scan the intervals in a left to right order. On encountering a left point l_i of an interval I_i we add this interval name to a separate list L by order of its

left endpoint l_i. On encountering a right endpoint r_j, corresponding to a left endpoint l_j of the interval we determine how many intervals in list L have their left endpoints before l_j. This can be done in O($logn$) time via binary search on list L. The total time taken is bounded by O($nlogn$). ■

A planar intersection graph construction is used to handle the case of the maixmum k-intersection for unequal radius circles [Hop83].

Theorem 13: Given n circles of unequal radius r_i in the plane, (1) the maximum k-closest points problem and (2) the k-intersection detection problem can be solved in O(n^2logn) time.

Proof: We first construct the planar graph formed by the circles in the plane with intersection points conforming to vertices of the graph and intersection free circular arcs as edges. There may be as many as O(n^2) vertices and as many edges. The edge contributed by a circle is directed such that one can differenciate the inside from the outside of the circle. (A circular arc edge $<i,j>$ could be directed from i to j conforming to a clockwise drawing of the circle). The graph can finally be set up with at most a sort of all the edges (in O(n^2logn) time). This planar graph can then be traversed and each of the O(n^2) regions given a count = the number of circles the region is common to, in at worst O(n^2) time. The maimum such count solves the problem. ■

Remark: This intersection approach does not easily extend to spheres in 3 or higher dimensions.

5.2. NP-complete and D^p-complete problems

The *piercing number* for a set of disks is the number of 'needles' required to pierce all the disks of the system [Had64]. For k circles having a common intersection (i.e. a k-intersection), all k circles can be pierced by a single needle.

Theorem 14: [Piercing number]
Given n circles of radius r in the plane, the problem of deciding whether k needles can pierce all n circles is NP-complete.

Proof : Using the duality of Theorem 1, the k-circle covering problem { 'Whether k circles can cover all points in the plane' }, which is known to be NP-complete [Fow81] can easily be reduced to this problem. ■

The problem of deciding 'what the maximum number of circles that k needles can pierce' { the Maximum Piercing problem }, is even harder. It is certainly NP-hard by the above theorem, but it is hardly in NP. The same applies to the dual problem of deciding 'what are the maximum number of points covered by k circles'. Here we shall show that it is D^p-complete [Pap82]. The class of D^p was defined in [Pap82] as follows: L is in D^p iff L is a intersection of L_1 and L_2 such that L_1 is in NP, and L_2 is in co-NP. This class contains both NP and co-NP.

Theorem 15: [k-circle Max-m]
Given n points in the plane and k (relocatable) circles of radius r, the problem of

deciding whether the maximum number of points covered by k circles $= m$ is D^p-complete.

Proof(sketch) :

a) The problem is in D^p since it can be expressed as the predicate $(\exists y_1, \ldots, y_k)$[Circles with centers at y_i, cover m points] $\wedge$ $(\forall z_1, \ldots, z_k)$[Circles with centers at z_i, cover $\leq m$ points].

b) To prove the completeness, we reduce the 3-SAT-UNSAT problem to our problem. { SAT-UNSAT is proved to be D^p-complete in [Pap82] }. By adapting the construction in [FOW81], we construct a separate group of points for each of (F_1,F_2) such that F_i $(i=1,2)$ is satisfiable *iff* all the points of the i^{th} group can be covered by k_i circles. Otherwise (f_i is not satisfiable), k_i circles can cover at least g_i - c_i points, where g_i = the number of points in the *ith* group and c_i = the number of clauses in F_i. We complete the construction by repeating the 1^{st} group of points another c_2 times. { That is there exists a total of c_2+1 groups of g_1 points and a single group of g_2 points with the above properties }. Then we have (F_1,F_2) is in 3-SAT-UNSAT *iff* the maximum numberof points that $k_1*(C+1)+k_2$ circles can cover is such that $(c_2+1)*g_1+g_2-c_2<m<(c_2+1)*g_1+g_2$. Then by another step of a trivial polynomial turing reduction, the proof is complete. ■

Corollary : The Maximum Piercing problem is also D^p-complete.

Theorem 16: [All points Min-k circle]
Given n points in the plane and k (relocatable) circles of radius r the problem of deciding whether the minimum number of circles that cover all points $= k$ is D^p complete.

Proof :

a) The problem is in D^p since it can be expressed as predicate $(\exists y_1, \ldots, y_k)$[Circles with centers at y_i, cover all n points] $\wedge$ $(\forall z_1, \ldots, z_{k-1})$[Circles with centers at z_i, cover $<n$ points].

(b) The completeness construction is similar to above. ■

6. Conclusion

It is not difficult to find practical significance to the above problems. Those dealing with the closeness of points find application in the analysis of clustered data. Facility layout applications are also widely known for problems concerning minimum enclosing circles of points.

The main open problem is Can the maximum k-intersection algorithm be improved ? Similar k-intersection and closest point problems for unequal sized spheres in dimension three and higher also deserve further attention.

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