A Classification of Certain Graphs with Minimal Imperfrection Properties

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ABSTRACT

The family of (α,ω)-graphs are of interest for several reasons. For example, any minimal counter-example to Berge's Strong Perfect Graph Conjecture belongs to this family. This paper accounts for all (4,3)-graphs. One of these is not obtainable by existing techniques for generating $(\alpha+1,\omega)$-graphs from (α,ω)-graphs.

Keywords: perfect graphs, minimal imperfect graphs, (α,ω)-graphs

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1. Introduction

A graph G is said to be <u>perfect</u> if for each induced subgraph G' of G the size of the largest clique of G' is equal to the chromatic number of G'. The Strong Perfect Graph Conjecture of Berge asserts that a graph is perfect if and only if it contains no induced subgraphs which are holes or antiholes, where a <u>hole</u> is a chordless cycle of odd length at least 5, and an <u>antihole</u> is the complement of a hole. Lovász [12], [13] proved a weaker conjecture of Berge: a graph is perfect if and only if its complement is perfect.

The Strong Perfect Graph Conjecture has been established for several classes of graphs including: planar graphs [16], circular arc graphs [17], $K_{1,3}$-free graphs [15], [19], 3-chromatic graphs [18], and graphs with maximum degree at most 6 [7].

Another way to state the Strong Conjecture is to say that any imperfect graph whose proper induced subgraphs are all perfect must be either a hole or an antihole. Padberg [14] showed that such a minimally imperfect, or <u>critical</u>, graph must be an (α,ω)-<u>graph</u>, defined below.

<u>Definition</u>: G is an (α,ω)-graph if, and only if,

(i) its vertex set has size $\alpha\omega + 1$;

(ii) its largest stable set has size α, and its largest clique has size ω;

(iii) each vertex is in precisely α stable sets of size α and ω cliques of size ω;

(iv) each clique of size ω is disjoint from precisely one stable set of size α, and each stable set of size α is disjoint from precisely one clique of size ω.

From now on, we will denote the number $\alpha\omega + 1$ by n.

An (α,ω)-graph is said to be <u>normalized</u> if each of its edges belongs to at least one clique of size ω. Each (α,ω)-graph contains a unique normalized (α,ω)-subgraph, because removing edges which belong to no cliques of size ω does not create any new stable sets of size α ([6], [18]).

The paper of Chvátal, Graham, Perold, and Whitesides [6] establishes two additional contexts in which (α,ω)-graphs arise. First of all, there is a correspondence between normalized (α,ω)-graphs and solutions to the system of equations

$$JX = XJ = \alpha J,\ JY = YJ = \omega J,\ XY = J - I, \tag{1}$$

where X and Y are matrices of 0's and 1's, J has all entries 1, I is the identity matrix, and all these matrices are $n \times n$. Bridges and Ryser [4] call the above matrices X and Y an $(n,0,1)$ system on α,ω. Second, there is a correspondence between normalized (α,ω)-graphs and packings of the complete graph K_n by complete bipartite graphs $K_{\alpha,\omega}$ with each edge of K_n covered exactly twice. C. Huang [9], [10] and C. Huang and Rosa [8] have studied such packings.

The graphs denoted $C_n^{\omega-1}$ are (α,ω)-graphs; they have vertices $v_0,\ldots,v_{\alpha\omega}$, with v_i adjacent to v_j whenever there is a d such that $0 < d < \omega$ and $d \equiv i - j$

or j - i (mod n). Holes and antiholes are of this type. In [6], methods are given for constructing (α,ω)-graphs which are not of this type.

The purpose of this paper is to describe all normalized (4,3)-graphs. One of these is a graph which is neither C_{13}^{2} nor a graph obtainable by the methods of [6]. Of course, none of these graphs is a counterexample to the Strong Perfect Graph Conjecture, as Tucker [18] has shown the conjecture holds for graphs with $\omega \leq 3$.

2. Properties of (α,ω)-graphs

We now list several well known properties of (α,ω)-graphs which we will use frequently throughout this paper. For convenience, we will use the word clique (stable set) to refer to a clique (stable set) of maximum size only.

Lemma 1: If G is an (α,ω)-graph, then G contains exactly $n = \alpha\omega + 1$ cliques (of size ω) and exactly $\alpha\omega + 1$ stable sets (of size α).

Proof: This follows easily from the definition of an (α,ω)-graph. □

Lemma 2: For each vertex y in an (α,ω)-graph G, the vertices of G-y are partitioned by stable sets (cliques) which are paired with the cliques (stable sets) containing y.

Proof: Let X be the matrix whose rows are the incidence vectors of the stable sets $S_1,\dots,S_n$, and let Y be the matrix whose columns are the incidence vectors of the cliques $T_1,T_2,\dots,T_n$. Then we know by [6] that $JX = XJ = \alpha J$, $JY = YJ = \omega J$, and $XY = J-I$, where J is the $n \times n$ matrix of 1's. Also

$$YX = X^{-1}XYX = X^{-1}(J-I)X = X^{-1}JX - I = J - I.$$

The Lemma now follows from the fact that $YX = J - I$. □

Remark 1: A consequence of Lemma 2 is that the pseudo p-critical graphs of Tucker [18] are precisely the (α,ω)-graphs.

Lemma 3: Let G be a graph, and define a graph M(G) by making the vertices of M(G) correspond to the cliques of G and making vertices in M(G) adjacent whenever the corresponding cliques intersect. If G is an (α,ω)-graph, then so is M(G).

Proof: See [18]. □

3. Generation of (4,3)-Graphs from (3,3)-Graphs

We assume throughout the rest of this paper that G is a normalized (4,3)-graph whose cliques are "triangles" $T_1,\dots,T_{13}$ and whose stable sets are $S_1,\dots,S_{13}$, where $S_i \cap T_j = \emptyset$ if and only if $i = j$. By an $i_1,\dots,i_k$-vertex, we mean a vertex which belongs to the stable sets $S_{i_1},\dots,S_{i_k}$. By Δxyz, we mean a triangle whose vertices are x, y, and z.

Remark 2: We emphasize that Lemma 2 says the following: if a vertex y belongs to distinct triangles T_{i_1}, T_{i_2}, and T_{i_3}, then each other vertex of G is exclusively an i_1-vertex, an i_2-vertex, or an i_3-vertex. Also, it says that if distinct T_j and T_k intersect, then there are no j,k-vertices.

Define a graph K(G) from G as follows. Make the vertices of K(G), like the vertices of M(G), correspond to the triangles of G. This time, however, make vertices adjacent whenever the triangles to which they correspond intersect in an edge.

We first show that the maximum degree of K(G) is at most 2. Then we use this fact to prove that K(G) contains a path of length at least 5 if and only if G can be generated from a (3,3)-graph by the first construction method of Chvátal [6].

Lemma 4: The graph K(G) has maximum degree at most 2.

Proof: If a triangle of G met three other triangles in edges, then the four triangles together would give rise to a clique of size 4 in M(G). However, M(G) is a (4,3)-graph according to Lemma 3. □

Lemma 5: No edge of G is in three triangles.

Proof: Suppose triangle T_i with vertices a, b, and c shared an edge ab with triangles T_j and T_k. Then since T_i would intersect S_j and S_k, c would be a j,k-vertex, which is ruled out by Lemma 2. □

Lemma 6: Suppose that H is a (3,3)-graph containing the configuration of four vertices shown in Figure 1(a) and that either $\{v_1,v_2,v_3\}$ is one of three triangles partitioning $H - v_4$ or $\{v_2,v_3,v_4\}$ is one of three triangles partitioning $H - v_1$. Then replacing the configuration shown in Figure 1(a) by the configuration shown in Figure 1(b) generates a (4,3)-graph.

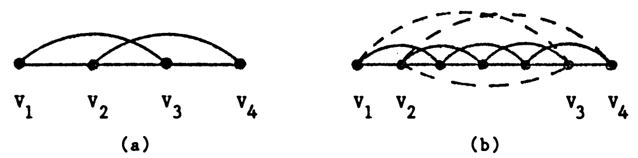

Figure 1: Configuration replacement generating a (4,3)-graph. An edge shown dashed is to be included provided that in H, it belongs to a triangle whose third vertex is in $H - \{v_1,v_2,v_3,v_4\}$.

Proof: This is the construction of [6] specialized to the generation of (4,3)-graphs from (3,3)-graphs. □

Lemma 7: If K(G) contains a path of length at least 5, then G contains the configuration of triangles shown in Figure 2, and

(i) d is the only common neighbor of b and f;

(ii) vertices b and g can have common neighbors only if b and g are adjacent (similarly for a and f).

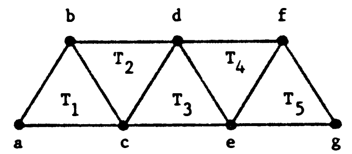

Figure 2: Configuration contained in G, where K(G) contains a path of length 5. Some adjacencies may not be shown.

Proof: If K(G) contains a path of length at least 5, then Lemma 5 implies that G contains the configuration in Figure 2.

(i) Suppose that T_6 is the third triangle containing b and that T_7 is the third triangle containing f. Then each vertex x distinct from b and f must be an i,j-vertex for some i in {1,2,6} and j in {4,5,7}. In particular, if x is a 3-vertex, then by Remark 2 it must be a 3,6,7-vertex, as T_3 intersects T_1, T_2, T_4, and T_5. Hence T_6 and T_7 do not intersect, again by Remark 2. Therefore, b and f have no common neighbors other than d.

(ii) Suppose, by way of contradiction, that b and g are not adjacent but have a common neighbor h. The intersections of $S_1,\dots,S_5$ with a,...,g can be determined by repeated application of Remark 2: $a \in S_2 \cap S_5$, $b \in S_3$, $c \in S_4$,

$d \in S_1 \cap S_5$, $e \in S_2$, $f \in S_3$, and $g \in S_1 \cap S_4$. Consequently, h must be a 2,5-vertex. Hence h is not adjacent to a, as a is also a 2,5-vertex.

Suppose h = a. Let T_6 be a triangle containing edge ag. By part (i), the third vertex of T_6 is neither b nor f. The five vertices of G outside T_1 through T_6 must include one vertex from S_3 and two vertices from each of S_1, S_2, S_4, S_5, and S_6. Remark 2 provides a contradiction: the 4-vertices must be 1-vertices, and the 5-vertices must be 2-vertices, so the 6-vertices must coincide with the 3-vertex. Hence, $h \neq a$.

By part (i), 2,5-vertex h is not adjacent to f. Therefore, since G is assumed normalized, edge bh belongs to some triangle T_6 with a 1,4-vertex i distinct from a,...,h. Edge gh belongs to some triangle T_7 with a 3,6-vertex j distinct from a,...,i. Vertices c and f are 6-vertices. If i were a 7-vertex, then either a or c would be a 7-vertex. Since c is a 6-vertex, it cannot be a 7-vertex, so a would be a 7-vertex. Hence d and g would also be 7-vertices. However, g being a 7-vertex would contradict the fact that g belongs to T_7. Therefore, b must be the 7-vertex of T_6. Then e is also a 7-vertex. Each of $S_1,\ldots,S_6$ contains exactly one of the unlabelled vertices of G. It follows that there is a 1,4-vertex, a 2,5-vertex, and a 3,6-vertex. However, two of these must be 7-vertices, which is impossible. □

Lemma 8: If K(G) contains a path of length at least 5, then G can be obtained by the construction in Lemma 6.

Proof: Assume G contains the configuration shown in Figure 2. Let H be the graph obtained from G by removing c, d, and e, making b adjacent to f and g, and making a adjacent to f. Triangles $T_1,\ldots,T_5$ are destroyed by the removal of c, d, and e.

We need to know that a, b, f, and g do not form a clique in H. If ag were an edge of G, then by Lemma 7, af and bg would also be edges. Then a,...,g would constitute a component of G, as each of these vertices is on three triangles from the set. Also, the remaining six vertices could not possibly have each vertex in exactly three triangles. Hence ag is not an edge of G, and vertices a, b, f, and g do not form a clique in H.

It is easy to check, using Lemma 7, that Δabf and Δbfg are the only triangles which appear in H but not in G. Consequently, H has clique size 3 and contains ten triangles. Also, each vertex of H belongs to three triangles.

Consider how the stable sets of G intersect {a,...,g}. With the exception of S_2, any stable set containing a contains d: if a is an i-vertex, b is a j-vertex, and c is a k-vertex, then d must be an i-vertex again, as long as d is not in T_i. Hence three stable sets contain both a and d. Similarly, three stable sets contain both d and g. Since vertex d belongs to S_1 and S_5, it follows that the two remaining stable sets to which d belongs both contain a and g. Further consideration along these lines shows that G has, in addition to $S_1,\ldots,S_5$ and the two stable sets containing a, d, and g, three stable sets containing b and e and three stable sets containing c and f. This accounts for all thirteen of the stable sets S_i of G. This information, along with other information we are about to produce, is represented in Figure 3.

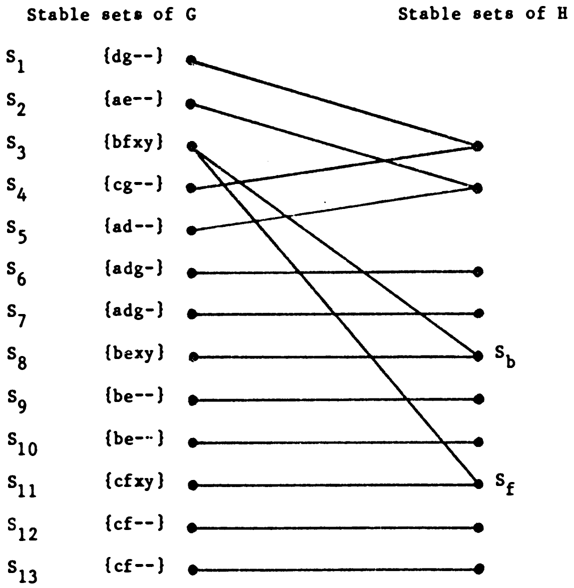

Figure 3: Stable sets of G and H, where K(G) has a path of length at least 5.

With the exception of S_3, each S_i loses a vertex of T_3 in the formation of H from G. The description of the stable sets of G shows that the remaining vertices of S_i, $i \neq 3$, continue to be mutually non-adjacent in H, as no stable set of G contains both b and g or both a and f. We will call the set of remaining vertices of S_i, $i \neq 3$, the *image* of S_i. Since S_3 is the only stable set of G containing both b and f, the joining of b to f produces in H two stable sets S_b and S_f of size 3 containing b and f, respectively. For future reference, let $S_3 = \{b,f,x,y\}$, and note that $S_b = \{b,x,y\}$ and $S_f = \{f,x,y\}$. Also, note that $\{b,e,x,y\}$ is a stable set (one of the $S_6,\ldots,S_{13}$) whose image is S_b and that $\{c,f,x,y\}$ is a stable set (one of the $S_6,\ldots,S_{13}$) whose image

is S_f.

We now check that the stable sets of H have the properties of stable sets in (3,3)-graphs. Clearly, the stability number of H is 3. The vertices of each stable set of H together with one of c, d, e form a stable set of G. By Remark 2, each vertex of G except c is an i-vertex for exactly one i in {1,2,3}. Each vertex except d is a j-vertex for exactly one j in {2,3,4}. Consequently, each 1-vertex except d is also a 4-vertex. It follows that S_1 and S_4 have the same image. Similarly, S_2 and S_5 have the same image. As previously declared, $\{b,e,x,y\}$ has image S_b and $\{c,f,x,y\}$ has image S_f. By referring to the list of stable sets of G given in Figure 3, we can now deduce that there are no additional agreements among the images of these stable sets. This fact is depicted in Figure 3. Hence H contains exactly ten stable sets. Each vertex of H is a 1,4-vertex, a 2,5-vertex, or a 3-vertex of G. Figure 3 shows that each of these vertices lies in a pair of stable sets of G which have a common image in H. Hence, each vertex of H lies on at most three stable sets of of H. Counting vertex-stable set incidences in H shows that each vertex lies in exactly three stable sets.

Next, we check that each of the ten triangles of H is disjoint from exactly one stable set of H, and vice versa. The common image of S_1 and S_4 is the only stable set of H disjoint from Δabf, and the common image of S_2 and S_5 is the only stable set disjoint from Δbfg. For $6 \leq i \leq 13$, T_i belongs to H, and T_i is disjoint from a stable set of H if and only if that stable set is the image of the stable set S_i disjoint from T_i in G. Thus, each triangle of H is disjoint from a unique stable set. Since the images of $S_6,\ldots,S_{13}$ together with the images of S_1 and S_2 account for all the stable sets of H, each stable set of H is disjoint from at least one triangle of H. It follows that the property of being disjoint pairs the triangles and stable sets of H.

We have now shown that H is a (3,3)-graph. By Lemma 2, H - a is partitioned by the triangles of H disjoint from the stable sets of H containing a. Since the common image of S_2 and S_5 contains a and is disjoint from Δbfg, Δbfg is in this partition of H - a by cliques. It follows that we can generate G from H by the construction of Lemma 6 by choosing v_1, v_2, v_3, and v_4 to be a, b, f, and g, respectively. □

4. Short Chains of Triangles

In this section, we show that if G is a (4,3)-graph, then K(G) must contain a path of length at least 4.

Lemma 9: K(G) cannot have maximum path length 1.

Proof: If K(G) had no paths of length greater than 1, then every vertex of G would have degree 6. The union of the vertices of any triangle and their neighbors would contain fifteen vertices, which is impossible. □

Lemma 10: K(G) cannot have maximum path length 2.

Proof: Suppose K(G) had maximum path length 2. Then we could assume G contains the configuration shown in Figure 4. Notice that vertices a, ..., h are distinct. By Remark 2, vertex a, a vertex f of T_3, and a vertex h of T_4 are 2-vertices. Vertices e, d, and g are 1-vertices, b is a 4-vertex, and c is a 3-vertex. The remaining five vertices of G must consist of a 1-vertex i, a 2-vertex j, and three 3,4-vertices k, l, and m. Vertex a lies on T_1 and two additional triangles T_5 and T_6. The 1-vertex of T_5 and T_6 must be i. Vertex d lies on T_2 and two additional triangles T_7 and T_8. The 2-vertex of T_7 and

T_8 must be j.

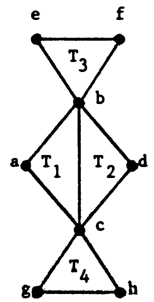

Figure 4: Configuration contained in G, where K(G) has maximum path length 2. Some adjacencies may not be shown.

Some 3,4-vertex k must lie on two of T_5, T_6, T_7, and T_8. Since T_5 and T_6 meet at a and i, and T_7 and T_8 meet at d and j, we may assume that k belongs to T_5 and T_7. Vertex i is the 7-vertex of T_5. The 6-vertex of T_5 must be k, and the 8-vertex of T_7 must be k. Thus T_6 and T_8 are disjoint, by Remark 2. Thus the 3-vertex of T_6 is a 3,4-vertex l not in T_8, and the 3-vertex of T_8 must be m. Note that m must then be the 7-vertex of T_8.

Vertex i is on T_5, T_6, and a third triangle whose 3-vertex cannot be c, k, or l, as K(G) has maximum path length 2. Hence this third triangle contains the 3,4,7-vertex m. But vertices i and m cannot be adjacent as they are both 7-vertices. Thus no G exists whose K(G) has maximum path length 2. □

Lemma 11: K(G) cannot have maximum path length 3.

Proof: Suppose, by way of contradiction, that K(G) has maximum path length 3. Then we may assume that G contains the configuration shown in Figure 5. It follows from Lemma 4 that vertices a, ..., i are distinct. By Remark 2, a, e, a vertex f of T_4, and a vertex i of T_5 are 2-vertices. Furthermore, b is a 3-vertex, d is a 1-vertex, c is a 4,5-vertex, h is a 1,4-vertex, and g is a 3,5-vertex. The remaining vertices of G consist of two 1,4-vertices j and k, and two 3,5-vertices l and m.

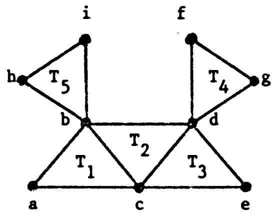

Figure 5: Configuration contained in G, where K(G) has maximum path length 3. Some adjacencies may not be shown.

By Lemma 2, G-h is partitioned by triangles T_1, T_4, and two additional triangles T_6 and T_7. We may assume T_6 = {e,j,l} and T_7 = {i,k,m}. Since h is a 6-vertex, i cannot be a 6-vertex. Vertex m cannot be a 6-vertex because m is a 3-vertex, and T_3 meets T_6. Thus the 6-vertex of T_7 must be k. Since k is a 1,4,6-vertex, G-k is partitioned by T_1, T_4, T_6 and a fourth triangle T_8 which must be {h,i,m}.

The 8-vertex of T_5 must be h, and consequently, e must be the 8-vertex of T_3. Since g is a 5-vertex, and T_8 meets T_5, f must be the 8-vertex of T_4. If c were a 4,5,6-vertex, then T_4,T_5,T_6, and {a,k,m} would partition G-c. However, {a,k,m} cannot be a triangle because it and triangles T_7, T_8, and T_5 would correspond to a path of length 4 in K(G). Hence the 6-vertex of T_1 is not c, but a, as 3-vertex b is not a possibility since T_3 meets T_6.

Consideration of G-f shows that {a,e,g,j,k,l} must be partitioned by two triangles. However, 1,4,6,8-vertex k cannot be adjacent to a 2-vertex in this

set, as a is also a 6-vertex, and e is also an 8-vertex. Thus no G exists whose K(G) has maximum path length 3. □

5. The (4,3)-Graphs

The (4,3)-graphs which are obtainable from (3,3)-graphs by the method of Lemma 6 are listed in [6]. One of these graphs was discovered independently by H.-C. Huang [3], [11]. The following theorem completes the description of all normalized (4,3)-graphs.

Theorem: The normalized (4,3)-graphs consist of the graph shown in Figure 6 together with those graphs which can be constructed from (3,3)-graphs by the method of Lemma 6.

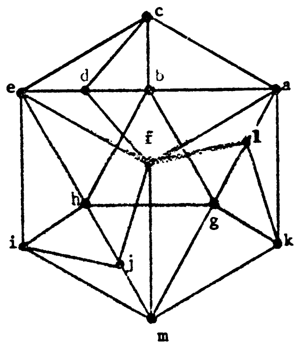

Figure 6: The normalized (4,3)-graph not constructible by the method of [6].

Proof: Suppose G is a normalized (4,3)-graph which cannot be constructed from a (3,3)-graph by the method of Lemma 6. Then by Lemmas 8, 9, 10, and 11, K(G) has maximum path length 4. We may assume the configuration shown in Figure 7 appears in G, where a,...,h are distinct vertices.

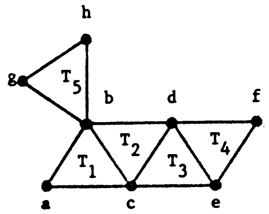

Figure 7: Configuration contained in G, where K(G) has maximum path length 4. Some adjacencies may not be shown.

Vertex e lies on T_3,T_4, and an additional triangle. Since the maximum path length of K(G) is 4, e has degree 5. If e were adjacent to two vertices not in {a,...,h}, then exactly three vertices of G would not be adjacent to either of b or e. Since every stable set which contains b also contains e with the exception of S_3, there are three stable sets containing both b and e. Thus the three vertices adjacent to neither b nor e would be mutually non-adjacent and would form with b and e a stable set of size 5. Hence e is adjacent to a fourth vertex in {a,...,h}. The 1,4-vertex h of T_5 is the only possibility. Since G is normalized, edge eh belongs to some triangle T_6 whose remaining vertex i is a 3,5-vertex.

Consider tbe remaining vertices. There is a 2,6-vertex j, a 3,5-vertex k, and two 1,4-vertices l and m. These vertices are precisely the ones adjacent to neither b nor e, so the subgraph they induce contains exactly three non-edges but no stable set of size 3. Since each triangle containing j has a 1-vertex, j must be adjacent to at least two 1-vertices by Lemma 5. Both these vertices must in fact be 1,4-vertices, as j is not adjacent to d. Similarly, k must be adjacent to two 1,4-vertices. If neither j nor k were adja-

cent to h, then both would be adjacent to l and to m. Then the subgraph induced by {j,k,l,m} would contain at least four edges, which is impossible. Therefore, we may assume that j is adjacent to h (by symmetry).

The 6-vertex of T_5 must be g. Consider the two stable sets containing g other than S_2 or S_6. Each one either contains c, and hence f and i, or contains d, and hence a and i. Hence {g,c,f,i} and {g,d,a,i} are both stable sets. Consequently, g is not adjacent either to i or to f. Also, i and j must be adjacent, or {g,i,a,d,j} would be a stable set: j is not adjacent to d, and since j is a 2-vertex, it is not adjacent to 2-vertices a and g.

Note that at this point, we know all three triangles containing h, namely T_5, T_6, and a triangle consisting of h, i, and j.

By Remark 2, G without 1,4-vertex h is partitioned by T_1, T_4, and two additional triangles. The triangle containing g must have k as its 3,5-vertex and must contain a 1,4-vertex l. The vertices i, j, and m form the remaining triangle.

At the moment, we know only of two of the triangles containing g: T_5 and Δgkl. No additional adjacencies among b, h, l, and k can be made, so g must be adjacent to a fifth vertex. The only possibility is m, and m must be joined to k to form the missing triangle.

We have now found all three of the edges in {j,k,l,m}: namely, km, kl, and jm. Vertices j and k must each have at least one more neighbor. The only possibility is that j is adjacent to f, and k is adjacent to a. Vertex m must be the 1-vertex of the triangle containing edge jf. Vertex i is in T_6, Δjmi, and Δijh. Hence l cannot be adjacent to i. Vertex l must then be adjacent to a and f, and a must be the common neighbor of f and l. Hence G is the

(4,3)-graph shown in Figure 6. □

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