PERMUTATIONS & THE APL GRADE DOWN FUNCTION*

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ABSTRACT

The APL gradeup(gradedown) function (denoted by ⍋{respectively ⍒}) applied to a vector $v \epsilon R^n$ 'grades' the elements of v in ascending (descending) order. (Among equal elements of v the ranking is determined by their position.) For example, if v=(2.3,4.7,6.8,0.6,3.7,4.7) then ⍋v is (4,1,5,2,6,3) and ⍒v is (3,2,6,5,1,4). An immediate consequence of the versatility of this functional form is that the expression v[⍋v] (v[⍒v]) 'sorts' the elements of v in ascending (descending) order.

The question of characterizing all vectors $v \epsilon R^n$ such that ⍋v=v was answered in a paper by Cooper, Best and Kennedy [see Cooper, et al.(1)]. The results there are essentially determined by scrutinizing the selection property of ⍋v; that is, if $v=(x_1, x_2, \ldots, x_n)$ then ⍋v can be visualized as the unique permutation $P \epsilon S_n$ such that $x_{P(1)}, x_{P(2)}, \ldots, x_{P(n)}$ is a list of the elements of v in ascending order. Two straightforward consequences of this interpretation are: 1) if $P \epsilon S_n$, then ⍋P is the inverse of P and 2) the 'fixed' points of the mapping ⍋ are precisely the involutions of S_n.

In this note we continue this investigation for the gradedown function and also resolve the open questions posed in that paper.

First a few remarks about the notation used in this paper. R^n and S_n have their usual meaning. If $v=(x_1,x_2,\ldots,x_n)$ is a vector in R^n then the 'reverse' of v is the vector $R(v)=(x_n,x_{n-1},\ldots,x_1)$. Occassionally when $v=P\epsilon S_n$ is a permutation the vector representation will denote the permutation in standard 'linear' form; this abuse of notation will be clear from the context.

Clearly if $v\epsilon R^n$ then $\downarrow v=\uparrow(-v)$. Observe that if $v=(x_1,x_2,\ldots,x_n)$ and the components of v are distinct then $\downarrow v=R(\uparrow v)$. This characterization fails in general, however, as if $v=(4,3,1,6,4,3)$ then $\uparrow v=(3,2,6,1,5,4)$, $R(\uparrow v)=(4,5,1,6,2,3) \neq \downarrow v=(4,1,5,2,6,3)$. An immediate consequence of these observations and the results of Cooper, et al.[see Cooper, et al.(1)] is the following.

Theorem 1. Let $v\epsilon R^n$. $\downarrow v=v$ iff v is a permutation P of $\{1,2,\ldots,n\}$ and $RP=P^{-1}$.

In words, the 'fixed' points of the mapping $\downarrow$ are exactly the permutations P in S_n satisfying the condition that the 'reverse' of P is the inverse of P.

As an aside it is interesting to note that although the theorem is aesthetically pleasing it yields no insight into the number of solutions of the equation $\downarrow v=v$, if indeed there are any! For example, if n=3 there are no solutions and for n=4,5 there are exactly two. { $P=(3,1,4,2)$, $Q=(2,4,1,3)$ }, { $R=(2,5,3,1,4)$, $S=(4,1,3,5,2)$ }.

Recall that the solutions of $\uparrow v=v$ are precisely the involutions in S_n. If t_n denotes the number of involutions of $\{1,2,\ldots,n\}$ then a variety of formulas have been discovered to compute t_n. We list a few of those here for comparison. [see D. Knuth(2)]

A simple recurrence equation for t_n, discovered by H. Rothe[see H. Rothe(3)], is

$$t_n = t_{n-1} + (n-1)t_{n-2},$$

a series representation,

$$t_n = \sum_{k\geq 0} t_n(k),\ t_n(k) = \frac{n!}{(n-2k)!2^k k!},$$

the generating function,

$$G_n(z) = \sum_n t_n \frac{z^n}{n!} = e^{z+\frac{z^2}{2}},$$

and finally an asymptotic series for t_n,

$$t_n = \frac{1}{\sqrt{2}} n^{\frac{n}{2}} e^{\frac{-n}{2} + \sqrt{n} + \frac{-1}{4}} (1 + \frac{7}{24} n^{\frac{-1}{2}} + O(n^{\frac{-3}{4}})). \text{ [see Knuth(2)]}$$

Noting the complexity of the asymptotic behaviour of t_n a natural question is whether the behaviour of r_n (where r_n dentoes the number of permutations P of $\{1,2,\ldots,n\}$ such that $P=R(P^{-1}))$ is a nontrivial one. We defer the investigation of that problem and similar ones to another paper.

The remarks preceding Theorem 1 trivially imply the following.

Corollary 2. For $v \in R^n$, $\uparrow\uparrow v=v$ iff v is a permutation of $\{1,2,\ldots,n\}$.

In words, the 'fixed' points of the functional form $\uparrow\uparrow$ are precisely the permutations of $\{1,2,\ldots,n\}$. It is not surprising that gradedown behaves analogously. Before we characterize the 'fixed' points of $\downarrow\downarrow$, however, we need some technical lemmas. The next few results are intended to be a representative (certainly not exhaustive) list of some of the identities which are direct consequences of the definitions and also illustrate the interesting interplay between the functional forms $\uparrow$ and $\downarrow$ and the concepts of the reverse and inverse of a permutation.

Lemma 3. If P is a permutation of $\{1,2,\ldots,n\}$ then the following obtain: a) $\downarrow\uparrow P=R(P)$, b) $\downarrow\downarrow P=\uparrow\downarrow(R(P))$, c) $R(\uparrow\downarrow P)=\uparrow\downarrow(RP)$.

Proof.

The first part of the lemma is trivial noting that if P is a permutation of $\{1,2,\ldots,n\}$ then $\downarrow P=R(\uparrow P)=R(P^{-1})$. Hence $\downarrow\uparrow P=\downarrow P^{-1}$ $=R((P^{-1})^{-1})=R(P)$.

For part b) we first observe that if v is a vector in R^n consisting of distinct components and S,T are permutations of $\{1,2,\ldots,n\}$ with $v[S]=v[T]$ then $S=T$. If $P\epsilon S_n$ then $\downarrow P[\downarrow\downarrow P]=(n,n-1,\ldots,1)$. But $\downarrow P[\uparrow\downarrow P]=(1,2,\ldots,n)$. Hence $\downarrow P[\uparrow\downarrow(RP)]=(n,n-1,\ldots,1)$. Therefore the permutations $\downarrow\downarrow P$ and $\uparrow\downarrow RP$ are equivalent.

For part c) since $\downarrow P=R(\uparrow P)$ we have $\downarrow\downarrow P=R\uparrow\downarrow P$. Applying part b) we obtain the result. □

In permutation notation part c)(or part b)) asserts that for an arbitrary permutation P of $\{1,2,\ldots,n\}$, $R((R(P^{-1}))^{-1})=(R(RP)^{-1})^{-1}$. The interested reader is encouraged to give a direct proof of this result as an exercise using only the definitions of the inverse and reverse of a permutation.

Lemma 4. If P is permutation of $\{1,2,\ldots,n\}$ then T.A.E. a) $\downarrow\downarrow P=\uparrow\downarrow(RP)$, b) $\uparrow\downarrow\downarrow P=\downarrow(RP)$, c) $\uparrow\downarrow\downarrow P=\downarrow\downarrow\uparrow P$.

Proof.

a) implies b). If $\downarrow\downarrow P=\uparrow\downarrow(RP)$ then $\uparrow\downarrow\downarrow P=\uparrow\uparrow\downarrow(RP)$ $=\downarrow(RP)$.

b) implies c). Clearly $\uparrow\downarrow\downarrow P=\downarrow(RP)$ implies $\uparrow\downarrow\downarrow P=\downarrow\downarrow\uparrow P$ since $RP=\downarrow\uparrow P$ by Lemma 3, part a).

c) implies a). If $\uparrow\downarrow\downarrow P=\downarrow\downarrow\uparrow P$ then $\uparrow\downarrow\downarrow P=\downarrow(RP)$ which yields $\uparrow\uparrow\downarrow\downarrow P=\uparrow\downarrow(RP)$ and therefore $\downarrow\downarrow P=\uparrow\downarrow(RP)$. The lemma follows by transitivity of implication. □

Theorem 5. If P is a permutation of $\{1,2,\ldots,n\}$ then the following obtain. a) $\downarrow\downarrow\downarrow P=(RP)^{-1}$, b) $\uparrow\downarrow P=(R(P^{-1}))^{-1}$, c) $P=\downarrow((RP)^{-1})$.

Proof.

Parts b) and c) are direct notational translations so we concentrate on part a). By Lemma 3, part b) we have $\downarrow\downarrow P=$ $\uparrow\downarrow(RP)$ therefore $\downarrow\downarrow\downarrow P$ $=\downarrow\uparrow\downarrow(RP)=R(\downarrow(RP))$ by Lemma 3, part a). But $R(\downarrow(RP))=R(R(RP)^{-1})=(RP)^{-1}$ by the definition of $\downarrow(RP)$ and properties of the reverse of a permutation.□

Theorem 6. For $v\epsilon R^n$, $\downarrow\downarrow v=v$ iff $v=P$ is a permutation of $\{1,2,\ldots,n\}$ and $(RP)^{-1}=R(P^{-1})$.

Proof.

Clearly $\downarrow\downarrow v=v$ implies $v=P\epsilon S_n$. Thus suppose $\downarrow\downarrow P=P$. Then $\downarrow\downarrow\downarrow P$ $=\downarrow P$. Hence applying Theorem 5, part a) and the definition of $\downarrow P$ we obtain $(RP)^{-1}=R(P^{-1})$.

Conversely assume $v=P\in S_n$ with $(RP)^{-1}=R(P^{-1})$. Then $\downarrow\downarrow\downarrow P=\downarrow P$ which implies $\downarrow\downarrow\downarrow\downarrow P=\downarrow\downarrow P$. But $\downarrow\downarrow\downarrow P=(RP)^{-1}$ implies $\downarrow\downarrow\downarrow\downarrow P= R(((RP)^{-1})^{-1})=R(RP)=P$. Hence $\downarrow\downarrow v= \downarrow\downarrow P=P=v$. □

Thus the 'fixed' points of the form $\downarrow\downarrow$ are exactly the permutations of $\{1,2,\ldots,n\}$ such that the 'inverse of the reverse' is the 'reverse of the inverse'.

We close this note by proving the last two conjectures in the paper by Cooper, Kennedy and Best. [see Cooper, et al.(1)]

Corollary 7. For $v\in R^n$, $\downarrow\downarrow\downarrow\downarrow\downarrow v=\downarrow\uparrow\downarrow\uparrow\downarrow v= \downarrow v$.

Proof.

First note that if v is a vector in R^n, then $\downarrow v$ is a permutation of $\{1,2,\ldots,n\}$, say P. Thus we show that $\downarrow\downarrow\downarrow\downarrow P=\downarrow\uparrow\downarrow\uparrow P=P$ for any $P\in S_n$. By Theorem 5, part a) $\downarrow\downarrow\downarrow P=(RP)^{-1}$. Hence $\downarrow\downarrow\downarrow\downarrow P=\downarrow(RP)^{-1}= R(RP)=P$. Similarly, $\downarrow\uparrow\downarrow\uparrow P= \downarrow\uparrow(RP)=R(RP)=P$. The theorem follows by transitivity of equality. □

Corollary 7 is the original conjecture stated in the aforementioned paper but it is actually a special case of a more general result which follows directly by inducting on various conclusions in this note.

Corollary 8. For $k\in N$ and P a permutation of $\{1,2,\ldots,n\}$ the following hold.

$$\text{a)}\quad \downarrow^k P = \begin{cases} P & \text{if } k=0(\text{mod}4), \\ R(P^{-1}) & \text{if } k=1(\text{mod}4), \\ (R(RP)^{-1})^{-1} & \text{if } k=2(\text{mod}4), \\ (RP)^{-1} & \text{if } k=3(\text{mod}4). \end{cases}$$

$$\text{b)}\quad \uparrow^k P = \begin{cases} P & \text{if } k=0(\text{mod}2), \\ P^{-1} & \text{if } k=1(\text{mod}2). \end{cases}$$

$$\text{c)}\quad (\downarrow\uparrow)^k P = \begin{cases} P & \text{if } k=0(\text{mod}2), \\ RP & \text{if } k=1(\text{mod}2). \end{cases}$$

$$\text{d)}\quad (\uparrow\downarrow)^k P = \begin{cases} P & \text{if } k=0(\text{mod}2), \\ (R(P^{-1}))^{-1} & \text{if } k=1(\text{mod}2). \end{cases}$$

Finally, combining parts a) and b) of Corollary 8 we record the following.

Corollary 9. Let $k,l,m,n \in N$ and P be a permutation of $\{1,2,\ldots,n\}$.

For l,m even,

$$\text{a)}\quad \downarrow^k\uparrow^l P = \uparrow^m\downarrow^n P = \begin{cases} P & \text{if } k,n=0(\text{mod}4), \\ R(P^{-1}) & \text{if } k,n=1(\text{mod}4), \\ (R(RP)^{-1})^{-1} & \text{if } k,n=2(\text{mod}4), \\ (RP)^{-1} & \text{if } k,n=3(\text{mod}4). \end{cases}$$

For l,m odd,

$$\text{b)}\quad \downarrow^k\uparrow^l P = \uparrow^m\downarrow^n P = \begin{cases} P^{-1} & \text{if } k,n=0(\text{mod}4), \\ RP & \text{if } k=1, n=3(\text{mod}4), \\ R((RP)^{-1}) & \text{if } k,n=2(\text{mod}4), \\ (R(P^{-1}))^{-1} & \text{if } k=3, n=1(\text{mod}4). \end{cases}$$

References

1. Cooper, Kennedy, Best, An Algebraic Investigation of the APL "Grade-up" Function(1979), Central Mo. State University.

2. Knuth, Donald E., The Art of Computer Programming, Vol.3(1973), 65-67.

3. Rothe, Heinrich A., Sammlung combinatorisch-analytischer Abhandlungen 2 (Leipzig, 1800), 263-305.