PROOF THEORETIC MAXIMALITY OF LOGICAL CALCULI

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0. INTRODUCTION

0.1. Proof theoretic maximality

While the completeness of classical first-order logic for validity in structures has been proved by Gödel in 1930, the completeness problem for intuitionistic logic is ramified and not quite settled as yet. Its significance itself is subject to debate. Should the set theoretic notion of a first order structure (domain + relations) be regarded as the definitive intuitionistic notion of semantics? Completeness for validity in structures being, as it is, inconsistent with Church's Thesis (see Tr [77], p. 49; Le [76]), which one of the two is intuitionistically incorrect? Et cetera. (See Tr [77] for review and references.)

One alternative approach to the intuitionistic completeness problem is to veer to semantics that are non-standard, in the sense that the logical constants are themselves subject to a semantic explication. These notions, widely investigated in a classical metamathematical setting, narrow down from pseudo-Boolean valued models, through topological to Beth and Kripke models.

Another approach, attempting to preserve the neutral role of the logical constants, is to consider provability in theories in place of satisfaction in structures. Namely, one is concerned with proof theoretic maximality properties of logical calculi, probing whether each non-tautology has an (arithmetic) instance that is not provable in a given mathematical theory T. The stronger T, the stronger the maximality property. It is not necessary that the theory T be "formal" (r.e.); e.g., for classical logic T may consist of all true formulas of arithmetic (and maximality reverts to Hilbert-Bernays completeness); for intuitionistic logic, T may be IA* = HA* = Intuitionistic (Heyting's) Arithmetic IA augmented with the recursive omega rule = IA augmented with transfinite induction over all recursive well-orderings, a theory to which we return below.

The first explicit maximality theorems were discovered by deJongh [73] and Myhill [72], for intuitionistic propositional logics. (For classical propositional logic maximality is implicit in Kripke [63].)

<u>THEOREM</u> (de Jongh). If a propositional formula $F[P_1,\ldots,P_k]$ is underivable in Intuitionistic (Heyting's) Propositional Logic, then there are arithmetic sentences $A_1,\ldots,A_k$ for which $F[A_1,\ldots,A_k]$ is not a theorem of Intuitionistic (Heyting's) Arithmetic.

0.2 <u>Restricted and uniform maximality</u>.

Let S be a set of arithmetical relations.

<u>DEFINITION</u>. A logical calculus L is (S) maximal for an arithmetic theory T if, when $F[P_1,\ldots,P_k]$ is a formula in the language of L, with all predicate

letters among $P_1,\dots,P_k$, and $\not\vdash_L F[P_1,\dots,P_k]$, then $\not\vdash_T F[A_1,\dots,A_k]$ for some arithmetical relations $A_1,\dots,A_k$ (in S). Here each A_i is of the same arity as P_i, and $F[A_1,\dots,A_k]$ stands for the result of replacing in F each atomic subformula $P_i(t_1,\dots,t_{ri})$ by $A_i[t_1/x_1,\dots,t_{ri}/x_{ri}]$.

The maximality is uniform if there is a sequence $\{A_i\}_i$ (from S), s.t. for any formula F as above, $\not\vdash_T F[A_1,\dots,A_k]$ whenever $\not\vdash_L F[P_1,\dots,P_k]$.

A greater uniformity of substitution is obtained when $A_i[x] \equiv A[\bar{i},x]$ for some formula A (provably in Primitive Recursive Arithmetic, say). All uniform maximality theorems we shall prove are of this, stronger, type.

In §1 below we exhibit a syntactic technique, using self-referential formulas, for deriving uniform maximality from (local) maximality. The technique is based on Kripke [63], where it is proved that classical propositional logic is uniformly Σ_1^0 maximal for consistent r.e. theories.

In analogy to strong semantic completeness, we define L to be strongly (S) maximal for T if, whenever $\Gamma \not\vdash_L F$, there are (S) instances Γ^*, F^* of Γ, F, respectively, s.t. $\Gamma^* \not\vdash_T F^*$. Clearly, uniform ($S$) maximality implies strong uniform (S) maximality.

A maximality property is said to be with respect to a set P of formulas if the property holds for logical forumas F from P.

REMARK. Kreisel's Basis Theorem is a powerful mean for reducing to Δ_2^0 the level in the arithmetical hierarchy of relational instances of Π_1^0 properties. In our context, $\not\vdash F[A]$ is a Π_1^0 property of the relation A, and one may be tempted to infer that maximality automatically implies Δ_2^0

maximality. However, $\{A \mid \not\vdash F[A]\}$ is a set of syntactic objects (or, equivalently, their codes); the corresponding set of characteristic functions is not even arithmetical.

0.3. Maximality in relation to semantic completeness.

Clearly, completeness of a logical calculus L for arithmetic models, relative to a semantic σ, implies maximality of L for any arithmetic theory T that is σ-sound. On the other hand, L may be very "incomplete" while maximal for a sufficiently weak T. For instance, de Jongh's theorem implies that Minimal (i.e., negation free) Propositional Logic is maximal for Minimal Arithmetic.

At the other end of the spectrum, maximality for as powerful a theory as IA^* is of particular interest. The classical analogue of IA^* is complete for (classical) truth in the standard model (Kr,Sh,Wa [60]). Also, assuming Church's Thesis CT_0 (cf. Tr [73]), IA^* is complete for prenex formulas. Finally, IA^* encompasses the arithmetic fragments of virtually all classically sound typed intuitionistic theories, and, in particular, the Intuitionistic Impredicative Theory of Types (Le [75]).

A proof theoretic analogue to semantic completeness was proposed already by Hilbert [30], who conjectured that there exists a suitable (sound) formal theory T disproving some instances of any non-theorem of classical predicate logic. Contrary to maximality, this property does imply the semantic completeness of predicate logic, and for the Finitistic Philosophy of Mathematics advocated by Hilbert, it has the virtue of not implicitly referring to satisfaction in infinite structures.

As it turned out, the conjecture fails for any classically sound r.e. theory T. If true, provability in Predicate Logic could have been decided by a simultaneous enumeration of proofs of Predicate Logic and of proofs of T.

0.4 Maximality properties of propositional logics.

To state our results for intuitionistic logics in full generality, say that a theory T is regular if its arithmetic fragment is an extension of IA with an r.e. set of schemas of transfinite induction over recursive well-orderings. As mentioned, theories as strong as Intuitionistic Impredicative Analysis in all finite types are regular. Clearly, any r.e. subtheory of IA* is a subtheory of some regular theory.

We prove that intuitionistic propositional logic IL_0 is uniformly Σ_1^0 maximal for any regular theory, and therefore for any r.e. subtheory of IA* (theorem II). The substitutions may be replaced by any type of disjunction of one-quantifier sentences (corollary II.1). Independence of Σ_1^0 sentences over IA, although not assured by independence over Classical (Peano's) Arithmeitc CA (proposition 2.2), is guaranteed by another simple condition (theorem III).

For the non r.e. theory IA*, IL_0 is maximal with substitutions of the form $\forall x[C \vee \neg C]$, where C is Σ_1^0 (theorem XIV(b)).

For (non-regular) extensions T of IA by familiar schemas, we find that IL_0 is uniformly Σ_2^0 maximal for IA+M+IP (theorem V), and uniformly Σ_1^0 maximal for IA+IP and for IA+CT_0, (theorems VI,VII). Here M stands for Markov's Schema, and IP for the Schema of Indepdnence of Premiss (closed instances only in both cases).

Classical Propositional Logic CL_0 is characterized as the smallest propositional logic maximal for intuitionistic provability over quantifier free sentences: a propositional formula $F[p_1,\ldots,p_k]$ is a tautology iff all instances of F with $0=0$ and $0=\bar{1}$ substituted for $p_1,\ldots,p_k$ are theorems of IA . By de Jongh's Theorem, IL_0 is characterized as the smallest propositional logic maximal for IA with Σ_1^0 substitutions. Is there any propositional logic intermediary between IL_0 and CL_0 that can be characterized as the smallest propositional logic maximal for some regular theory, with substitutions of a given syntactic structure? We find that, for reasonable classes of substitutions, there is exactly one such logic, namely,

$$IL_0 + \{\neg\neg p \to p \mid p \text{ a propositional variable}\} ,$$

which is maximal for a regular theory with Π_1^0 substitutions (theorem IV).

0.5 Maximality properties of predicate logics.

The local Δ_2^0 maximality of Classical First Order Logic (without equality) for any sound theory is immediate by Hilbert-Bernays' [39] version of the Completeness Theorem. A slight refinement establishes maximality for any Σ_1^0-sound extension of Peano's Arithmetic (theorem VIII). The uniform, and hence strong, maximality for any Σ_1^0-sound r.e. extension of Peano's Arithmetic, with either Σ_2^0 or Π_2^0 substitutions, follows by uniformization (theorem IX).

Intuitionistic Predicate Logic is uniformly, and hence strongly maximal for any regular theory, with either Σ_2^0 or Π_2^0 substitutions (theorems X, XI). Although not Σ_1^0 maximal for Heyting's Arithmtic IA (Le [76]), Intuitionistic Predicate Logic is Σ_1^0 maximal (for regular theories) w.r.t. formulas satisfying certain decidability properties (theorem XIII), and, in particular, w.r.t. prenex formulas (corollaryXIII.1). Intuitionistic Logic is also uniformly maximal for IA*, with substitutions of the form $\forall x[A \vee B]$, where A is Σ_2^0 and B is Π_2^0. For formulas satisfying the forementioned decidability properties A may be Σ_1^0 and B Π_1^0 (theorem XIV).

The main technique used in the proofs is a combinatorial analysis of recursive cut-free infinitary proofs (i.e., proofs using the recursive omega-rule). Similar methods were applied elsewhere to derive other metamathematical properties of Intuitionistic Arithmetic (Le [80], [80a], [A]; Fr [A]).

Consider now predicate logics with Equality. Augmenting Classical Logic with axioms stating the existence of an unbounded number of distinct elements, one obtains uniform Σ_2^0 and Π_2^0 maximality for sound theories. To obtain the maximality properties listed above for Intuitionistic Logic with equality, one has to further augment the logic with an axiom stating the decidability of equality (theorem XVI).

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1. UNIFORMIZATION OF MAXIMALITY

We use $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{A}$ to denote the languages of propositional logic, first-order predicate logic (without equality) and first-order arithmetic, respectively. It is convenient to assume that $\mathcal{A}$ contains symbols for all primitive recursive functions, and that theories in $\mathcal{A}$ (like Peano's or Heyting's Arithmetic) contain the appropriate defining equations as axioms; it is well-known that such extensions are conservative. IL_0 , IL_1 and IA will denote Heyting's calculi for intuitionistic propositional logic, predicate logic and arithmetic, respectively. CL_0 , CL_1 and CA will denote the corresponding classical calculi (so CA is Peano's Arithmetic).

Let A be an arithmetic formula with x as the only free variable. Given a predicate formula $F[P_1^{r1},\ldots,P_k^{rk}]$ we write $F[A]$ for

$$F[A[\langle\bar{1},x_1,\ldots,x_{r1}\rangle/x],A[\langle\bar{2},x_1,\ldots,x_{r2}\rangle/x],\ldots,A[\langle\bar{k},x_1,\ldots,x_{rk}\rangle/x]] .$$

Here $\langle\cdots\rangle$ is a canonical sequence coding.

If S is a set of arithmetic formulas (e.g., Σ_1^0), then a theory T in $\mathcal{A}$ is S-<u>sound</u> if a formula F is true whenever $F \in S$ and $\vdash_T F$.

<u>THEOREM I</u>. (Uniformaization of maximality.) Let T be a Σ_r^0 extension of IA $(r \geq 1)$, complete for Σ_r^0 sentences, and $\neg\neg\Sigma_r^0$-sound . Let $H = \{F_n\}_n$ be a Σ_r^0 set of (codes of) predicate formulas; i.e., the function $\lambda n.\ulcorner F_n\urcorner$ is recursive in Π_{r-1}^0 . Suppose there is a Σ_s^0 $(s \geq r)$ formula $B(n,x)$ of $\mathcal{A}$ satisfying, for each $F_n \in H$, $\nvdash_T F_n[B_n]$, where $B_n :\equiv B(\bar{n},x)$. Then there is a Σ_s^0 formula $C(x)$ s.t. $\nvdash_T F_n[C]$ for all $F_n \in H$.

PROOF. Let $\underline{Prov}$ be a canonical Π^0_{r-1} proof predicate for T, and let $\underline{Pr}(x) :\equiv \exists p\, \underline{Prov}(p,x)$. Using diagonalization (cf. Kr,Lé[68]), define a Σ^0_r formula J for which the following is provable in IA.

$$(1) \qquad J(n) \leftrightarrow \exists p\{\underline{Prov}(p, `\neg\, \underline{Pr}(\overline{\ulcorner F_n[B_n]\urcorner}) \to \neg\, J(\bar{n})`) \\ \& \ \forall\langle q,m\rangle < \langle p,n\rangle \ [\neg\, \underline{Prov}(q, `\neg\, \underline{Pr}(\overline{\ulcorner F_m[B_m]\urcorner}) \to \neg\, J(\bar{m})`)]\}$$

Here $\langle\rangle$ is a canonical pairing function, and the bounded quantifier is a self-explanatory abbreviation. Also, $`\neg\, \underline{Pr}(\overline{\ulcorner F_m[B_m]\urcorner}) \to \neg\, J(\bar{m})`$ stands for an expression $g(m)$, where g is a suitable primitive recursive function.

Since T contains IA,

$$(2) \qquad \vdash_T J(n) \ \& \ J(m) \to n=m \ .$$

Define $\quad C(x) :\equiv \exists n[J(n) \ \& \ B(n,x)]$. Then (2) implies

$$(3) \qquad J(\bar{n}) \ \vdash_T \ \forall x[C(x) \leftrightarrow B(\bar{n},x)] \ .$$

Assume

$$(4) \qquad \vdash_T F_n[C] \ .$$

Since $\nvdash_T F_n[B_n]$ by assumption, (3) implies $\nvdash_T J(\bar{n})$. But T is complete for Σ^0_r sentences, so $J(\bar{n})$ is false.

The argument is formalizable in T. First, if $\vdash_T F_n[C]$, then $\vdash_T \underline{Pr}(\overline{\ulcorner F_n[C]\urcorner})$. Formalizing the rest of the argument yields $\neg\, \underline{Pr}([F_n[B_n]]) \vdash_T \neg$ by a proof with some code p. By (1), enumerating pairs $\langle q,m\rangle < \langle p,n\rangle$ weakly gives some m (i.e.,, $\neg\neg\exists m$) for which $J(\bar{m})$ is true; in particular,

(5) $$\neg\, \underline{Pr}(\overline{'F_m[B_m]'}) \vdash_T \neg\, J(\bar{m}) .$$

Since J is Σ_r^0, we also have $\vdash_T J(\bar{m})$, and so, by (5), $\vdash_T \neg\neg\, \underline{Pr}('F_m[B_m]')$.
By the $\neg\neg\ \Sigma_r^0$-soundness of T this implies $\sim\sim \vdash_T F_m[B_m]$, contradicting the theorem's assumption. Thus (4) fails. □

2. MAXIMALITY OF PROPOSITIONAL LOGICS

2.1 Maximality theorems.

As mentioned above, the uniform Σ_1^0 maximality of classical propositional logic (for any consistent r.e. theory) was proved by Kripke [63]. This implies uniform Π_1^0 maximality, and no room is left for improvement.

Turning to Intuitionistic Propositional Logic IL_0, we note that $\vdash_{IA} \neg\neg F \to F$ for any arithmetic sentence F with no strictly positive occurrences of disjunction or $\exists$. The simplest substitutions for which IL_0 may be maximal (at least for IA) are, therefore, Σ_1^0 sentences, and binary disjunctions of Π_1^0 sentences, that we name β-sentences.

<u>THEOREM II</u>. (Uniform Σ_1^0 maximality of Intuitionistic Propositional Logic) IL_0 is uniformly Σ_1^0 maximal for any regular theory T, and hence - for any r.e. subtheory of IA* . Moreover, given a regular theory T, there is a Σ_1^0 formula B s.t. $\vdash_T F[B]$ implies $\vdash_{IL_0} F$ for any propositional formula F.

We postpone the proof to §2.5. Local Σ_1^0 maximality was first proved by de Jongh and Smoryński (cf. Smoryński [73]).

<u>COROLLARY II.1</u>. IL_0 is uniformly β maximal for any regular T. Moreover, IL_0 is uniformly maximal for T with any type of disjuction of one-quantifier formulas as substitutions. Namely, any substitutions of one of the forms

$$B_{2i} \vee B_{2i-1},\ B_{2i} \vee \neg B_{2i-1},\ B_{2i} \vee \neg\neg B_{2i-1},\ \neg B_{2i} \vee \neg B_{2i-1},$$
$$\neg B_{2i} \vee \neg\neg B_{2i-1},\ \neg\neg B_{2i} \vee \neg\neg B_{2i-1}\ .$$

where B is the formula of the theorem. (Note that $\neg B_i$ is equivalent in II to a purely universal sentence.)

PROOF. Let each H_i be one of the disjunctions above, and assume that $\vdash_T F[H_1,\ldots,H_k]$. Then, by the theorem, $\vdash_{IL_0} F[A_1,\ldots,A_k]$, where each A_i is one of the propositional formulas

$$q_{2i} \vee q_{2i-1},\ q_{2i} \vee \neg q_{2i-1},\ q_{2i} \vee \neg\neg q_{2i-1},\ \neg q_{2i} \vee \neg q_{2i-1},$$
$$\neg q_{2i} \vee \neg\neg q_{2i-1},\ \neg\neg q_{2i} \vee \neg\neg q_{2i-1}\ ,$$

corresponding to the form of H . For each sentence A_i , $\nvdash_{IL_0} \neg\neg A_i$ and $\nvdash_{IL_0} \neg\neg A_i \rightarrow A_i$. So, by Le [80], $\vdash_{IL_0} F$. □

COROLLARY II.2. IL_0 is strongly maximal for any r.e. subtheory of IA^* , with any one of the substitutions above.

2.2 Tests for propositional independence.

At least for Σ_2^0 sentences propositional independence over Intuitionistic Arithmetic implies propositional independence over Classical Arithmetic. Suppose that $A_1,\ldots,A_k$ are intuitionistically independent Σ_2^0 sentences, and $\vdash_{CA} F[A_1,\ldots,A_k]$, where the propositional schema F is not a classical tautology; then $\vdash_{IA} F^+[A_1,\ldots,A_k]$, where F^+ is the double negation translation of F , since for a Σ_2^0 sentence A , $\vdash_{IA} \neg\neg A \leftrightarrow \neg\neg A^+$. This is a contradiction, since F^+ is not an intuitionistic tautology.

The converse is more problematic. H. Friedman [73] showed that Π_2^0 sentences independent in CA over all true Π_1^0 sentences are also independent in IA (see corollary X below). He also conjectured that already for Σ_1^0 sentences classical independence ensures intuitionistic independence. This is true for a single Σ_1^0 sentence (corollary III), but fails already for two sentences.

<u>PROPOSITION 2.2</u>. There are Σ_1^0 sentences A,B that are independent over CA, but not over IA.

<u>PROOF</u>. By Kripke [63] there are Σ_1^0 sentences A,C, each independent of CA, s.t.

(1) $\qquad C \vdash_{CA} \neg A$.

Let D be independent over CA+A, and let $B :\equiv A\&D.\vee C$. Then the set {A,B} is independent over CA:

(i) If $A \vdash B$, i.e. $A \vdash A\&D. \vee C$, then $A \vdash D$ by (1), contradicting the choice of D.

(ii) If $B \vdash A$, then $C \vdash A$, so [illegible] C by (1), contradicting the choice of C.

(iii) If $B \vdash \neg A$, then $A\&D \vdash \neg A$, so $A \vdash \neg D$, contradicting the choice of D.

On the other hand, the set {A,B} is not independent over IA: $A\&D \vdash A$ and $C \vdash \neg A$, so $\vdash B \rightarrow A \vee \neg A$. □

A similar counter-example was discovered independently by D.H.J. de Jongh.

While intuitionistic independence of Σ_1^0 sentences is not guaranteed by classical independence, it is guaranteed by certain relatively simple test cases. Define a sequence $G_n[p_1,\ldots,p_n]$ of propositional formulas inductively, as follows.

$$G_1[p] :\equiv p \vee \neg p \; ; \; G_{k+1}[p_1,\ldots,p_{k+1}] :\equiv \bigvee_{i=1}^{k+1} (p_i \to G_k^i) \, ,$$

where $G_k^i :\equiv G_k[p_1,\ldots,p_{i-1},p_{i+1},\ldots,p_{k+1}]$.

THEOREM III. (Test cases for intuitionistic independence) Let T be a regular theory, and let $A_1,\ldots,A_k$ be Σ_1^0 sentences. Suppose $F[p_1,\ldots,p_k]$ is a propositional formula not derivable in IL_0 , and $\vdash_T F[A_1,\ldots,A_k]$. Then $\vdash_T G_k[A_1,\ldots,A_k]$.

COROLLARY III. (de Jongh, Smoryński; cf. Sm [73], 5.3.18) If A is a sentence independent of T , and $\vdash_T F[A]$ then $\vdash_{IL_0} F[p]$.

Le[80] gives another generalization of the corollary.

For β-sentences as substitutions the situation is less clear. We can show, by a disproportionately long and intricate proof that we skip here, that if U_1, U_1', U_2, U_2' are classically independent Π_1^0 sentences, then the two β-sentences $U_i \vee U_i'$ $(i=1,2)$ are intuitionistically independent. Our proof fails for three β-sentences.

2.3 An infinitary sequential calculus for arithmetic

Subformulas and immediate subformulas of an arithmetic sentence are defined as usual. In particular, $F[\bar{n}/x]$ is an (immediate) subformula of $\forall xF$, for each numeral $\bar{n}$. We use an infinitary sequential calculus IA^∞ for arithmetic sentences (i.e., closed formulas). By using variants of the omega rule, one obtains cut-free proofs for all theorems of IA , that satisfy the subformula property: each formula in the proof is a subformula of the derived formula. No reasonable finitary calculus may be complete for IA and satisfy this property (Le [79], p. 5).

DEFINITIONS. An antecedent is a finite set of sentences. A sequent is a pair $\Gamma:F$, where Γ is an antecedent and F is a sentence; F is the succedent of the sequent. Γ,A will stand for $\Gamma\cup\{A\}$; Γ,θ for $\Gamma\cup\theta$, etc. The inference rules of IA^{ω} are the following schemas.

(a) PROPOSITIONAL LOGIC:

$\top$: $\quad \Gamma,A : A$

$\bot$: $\quad \Gamma,\bot : A$

&L: $\dfrac{\Gamma,A,B : C}{\Gamma,A\&B : C}$ $\qquad$ &R : $\dfrac{\Gamma : A \qquad \Gamma : B}{\Gamma : A\&B}$

$\vee$L: $\dfrac{\Gamma,A : C \quad \Gamma,B : C}{\Gamma,A\vee B : C}$ $\qquad$ $\vee$R : $\dfrac{\Gamma : A}{\Gamma : A\vee B}$; $\dfrac{\Gamma : B}{\Gamma : A\vee B}$

$\to$L: $\dfrac{\Gamma,A\to B : A \quad \Gamma,B : C}{\Gamma,A\to B : C}$. $\qquad$ $\to$R : $\dfrac{\Gamma,A : B}{\Gamma : A\to B}$

(b) QUANTIFICATION:

$\forall$L: $\dfrac{\Gamma,\forall xA,A[\bar{n}/x] : B}{\Gamma,\forall xA : B}$ $\qquad$ $\forall$R : $\dfrac{\{\Gamma : A[\bar{n}/x]\}_{n\geq 0}}{\Gamma : \forall xA}$

$\exists$L: $\dfrac{\{\Gamma,\exists xA,A[\bar{n}/x] : B\}_{n\geq 0}}{\Gamma,\exists xA : B}$ $\qquad$ $\exists$R : $\dfrac{\Gamma : A[\bar{n}/x]}{\Gamma : \exists xA}$

Note that we dispense of all structural rules, in particular - of CUT .

(c) ARITHMETIC:

An equation E between (closed) terms of $\mathcal{A}$ is true if E equates terms with the same numeric value, when each function symbol is interpreted as the primitive recursive function it denotes. It is fairly easy to see that there exists a recursive function <u>eval</u>, provably total in IA, s.t. <u>eval</u>('E') = 0 iff E is true, where 'E' is a canonical code for E (cf. Tr [73], 1.5.3). This shows that coded instances of the following rules can be recursively recognized in IA.

TE: $\Gamma : E$, if E is a true equation

FE: $\Gamma, E : A$, if E is a false equation.

A tree is <u>well</u> <u>founded</u> if every branch terminates. A <u>derivation</u> of IA^{∞} is a well-founded tree of sequents, where at each node the sequents relate according to one of the inference rules above.

Our inference rules are set up to validate the following.

<u>LEMMA 2.3.1</u>. Let Γ and Γ' be antecedents, and suppose Γ occurs above Γ' in a derivation. Then Γ implies in intuitionistic propositional logic each sentence in Γ'.

<u>PROOF</u>. Immediate. □

<u>LEMMA 2.3.2</u>. (subformula property) Let Δ be a derivation of IA^{∞} deriving the sequent $\Gamma : A$. Each sentence occurring in Δ is a subformula of A or of some member of Γ.

<u>PROOF</u>. Let u^+ be a node in Δ other than the root, and let u be the immediate parent-node of u^+. Inspection of the inference rules shows that each sentence occurring in the sequent at node u^+ is a subformula of some sentence occurring at u. The proof is concluded by induction on the height of u^+ in Δ. □

Fix a canonical Gödel coding for the syntax of $\mathcal{A}$ and for sequents. Each proof Δ in IA^{∞} can be represented by a function ϕ_{Δ} that assigns codes of sequents to sequence-numbers

$$\langle\rangle\ ,\ \langle 0\rangle\ ,\ \langle 1\rangle\ ,\ldots,;\ \langle 0,0\rangle,\ \langle 0,1\rangle\ ,\ldots;\ \langle 1,0\rangle\ \ldots;;\ \langle 0,0,0\rangle\ ,\ldots$$

in the universal spread U . W.l.o.g., no code is 0 , and ϕ_{Δ} may be assumed total, with $\phi_{\Delta}(x) = 0$ for each x other than codes of those nodes in U where Δ has a sequent. The statement that a function ϕ represents a proof is formalized as the conjunction of the Π_1^0 predicate $LC(\phi)$, stating that ϕ is locall correct, i.e., that the non-zero values of ϕ are codes of sequents relating according to the inference rules, and of the Π_1^1 predicate $WF(\phi)$, stating that the non-zero values of ϕ form a well-founded tree.

A <u>recursive proof</u> is a number d s.t. the recursive function $\{d\}$ coded by d is a proof of IA^{∞} . Let IA^* denote the recursive part of IA^{∞} . This notation agrees with the definition of IA^* in 0.1 above: it is easy to see that $IA^* = IA$ extended with transfinite induction over all recursive well-orderings. For most applications, it is not the well-foundedness of a proof Δ that one needs, but the schema of Bar Induction over $\phi = \phi_{\Delta}$:

$$BI[\phi,P(x)] :=\ \forall u\{[\forall n.\phi(u*\hat{n})\neq 0 \to P(u*\hat{n})] \to .\ \phi(u)\neq 0 \to P(u)\} \to P(\langle\rangle)$$

Here P is a predicate letter, and $*,\hat{n},\langle\rangle$ stand for the concatenation operation on sequence-numbers, the singleton sequence-number $\langle n\rangle$, and the empty sequence-number, respectively. For a well-founded Δ , this is a special case of the usual schema of Bar Induction, sometimes referred to as "induction over unsecured sequences."

Well-foundedness of ϕ_Δ , $WF(\phi)$, clearly implies $BI[\phi,P(x)]$, but we shall be concerned with the derivability in given theories of the latter, not of the implication. The formal derivability condition we need is the Π^1_1 predicate

$$\underline{Prov}^*(d,\ulcorner F\urcorner) \;:\equiv\; \forall x\, \exists y\; T(d,x,y)\ \&\ \underline{Prov}^{bi}(\{d\},\ulcorner F\urcorner)$$

where

$$\underline{Prov}^{bi}(\phi,\ulcorner F\urcorner) :\equiv LC(\phi)\ \&\ \forall P\, BI[\phi,P(x)]\ \ \&\ \phi\,(\langle\rangle) = \ulcorner{:}F\urcorner\ .$$

Also,

$$\underline{Pr}^*(x) :\equiv \exists\, d\underline{Prov}^*(d,x)\ .$$

Let T be a theory in a language containing $\mathscr{A}$. Write T^P for the second order extension of T , with quantification over predicate letters, but with comprehension restricted to arithmetic formulas. IA^P is well known to be conservative over IA (cf. Tr [73], 1.9.6). Define

$$IA^*[T] := \{F \mid T^P \vdash \exists\, d\underline{Prov}^*(d,\overline{\ulcorner F\urcorner})\}\ .$$

<u>LEMMA 2.3.3</u>. Let T be a regular theory.

(a) T^P is a conservative extension of T .

(b) $T = IA^*[T]$.

<u>SKETCH OF PROOF</u>. $IA \subseteq IA^*[IA]$ is essentially a cut elimination theorem for IA^* (cf. Lo-Es [68]). The extension to $T \subseteq IA^*[T]$ is straightforward. The converse is by transfinite induction using a restricted truth definition, as in Kr [65],3.33. Details for the system considered here are provided in Le [A]. □

It will be convenient to discuss derivations of IA* informally, and then to point out that the notions used may be arithmetized, and the arguments formalized in some formal theory (usually IA^p). In informal discussions u,v,w will range over nodes of the universal spread; when d codes a proof of IA^*, Δ_d will denote that proof, and $\sigma^{d,u}$, $\Gamma^{d,u}$, $A^{d,u}$, $\rho^{d,u}$, will denote the sequent of Δ_d at u (coded by $\{d\}(u)$) , the antecedent and the succedent of $\sigma^{d,u}$ and the (first) rule of inference by which $\sigma^{d,u}$ is correctly derived in Δ_d , respectively. Note that $\rho^{d,u}$ may be found primitive-recursively from $\sigma^{d,u}$ and $\sigma^{d,u*\hat{n}}$, $n = 0,1,2$. The superscript d will be dropped when no ambiguity may arise.

LEMMA 2.3.4. Assume that all quantifier free subformulas of F are equations in the immediate scope of an existential quantifier. If

$\underline{\mathrm{Prov}}^*(d,\ulcorner F\urcorner)$,

then, w.l.o.g., only top nodes of Δ_d have equations as succedents. Moreover, there is a primitive recursive function f s.t., provably in IA ,

$\underline{\mathrm{Prov}}^*(d,\ulcorner F\urcorner) \rightarrow. \underline{\mathrm{Prov}}^*(f(d),\ulcorner F\urcorner)$ & "all succedents in $\{f(d)\}$ that are equations occur at top nodes."

PROOF. We describe a primitive recursive transformation on Δ_d yielding $\Delta_{f(d)}$. Given a node u in Δ_d s.t. A^u is an equation, let $\ell(u)$ be the longest initial segment of u s.t. $A^{\ell(u)}$ is not an equation. By the subformula property and the lemma's condition, $\rho^{\ell(u)}$ is $\exists R$. Perform successively the following transformations.

1. Replace each A^u as above by $A^{\ell(u)}$.

2. If u is a top node, put $\Gamma^u : A^u$ at $u*\hat{0}$.

3. Collapse the resulting tree at each node $\ell(u)$; i.e., put at each $\ell(u)*w$ the sequent previously at $\ell(u)*\hat{0}*w$.

The proof $\Delta_{f(d)}$ is easily seen to satisfy the lemma's requirements. □

An alternative proof of the lemma uses permutations of inferences (as in Kl [52A]), and bar induction on Δ_d , therefrom the primitive recursive f is obtained by the S-m-n Theorem.

2.4. <u>Proof of theorem III</u>

Let $F[p_1,\ldots,p_k]$ be a propositional formula, underivable in IL_0 , and let $E_1,\ldots,E_k$ be purely existential sentences, i.e., of the form $\exists xG$ where G is an equation between primitive recursive terms. We prove in IA^P that

(1) $\underline{Pr}*('F[E_1,\ldots,E_k]')$ implies $G_k[E_1,\ldots,E_k]$.

Hence, for any regular theory T ,

$$\begin{array}{lll} & T \vdash F[E_1,\ldots,E_k] & \\ \Rightarrow & T^P \vdash \underline{Pr}*('F[E_1,\ldots,E_k]') & \text{by } 2.3.3(b) \\ \Rightarrow & T^P \vdash G_k[E_1,\ldots,E_k] & \text{by } (1) \\ \Rightarrow & T \vdash G_k[E_1,\ldots,E_k] & \text{by } 2.3.3(a) . \end{array}$$

To prove (1), assume $\underline{Prov}*(d,'F[E_1,\ldots,E_k]')$. By 2.3.4 we may assume that $A^{d,u}$ is an equation only for top nodes u of Δ_d .

We make a number of additional ad hoc notational conventions. As mentioned above, we omit the superscript d. For a node u in Δ_d, let Θ^u denote the set of those E_i's for which $\Gamma^u \vdash_{IL_0} E_i$, where $\vdash_{IL_0}$ stands for provability using propositional inferences only. Let Ξ^u denote the set of all other E_i's $(i=1,\dots,k)$. For $\Xi^u = \{E_{i1},\dots,E_{im}\}$, let $G[\Xi^u] \quad :\equiv \quad G_m[E_{i1},\dots,E_{im}]$. Note that this definition is independent of the order of the E_{ij}'s, up to provable equivalence in IL_0. Finally, write Γ_0^u for the set of equations in Γ^u.

<u>LEMMA 2.4.1</u>. If A^u is not an equation, and

(2) $\Gamma^u \nvdash_{IL_0} A^u$,

then

(3) $(\Gamma_0^u, \Theta^u) \to G[\Xi^u]$,

where (Γ,Θ) stands for the conjunction of all formulas in Γ,Θ.

<u>THE LEMMA IMPLIES (1)</u>: Take $u = \langle\ \rangle$. □

<u>PROOF OF THE LEMMA</u>. By bar induction on Δ_d.

<u>Basis</u>. If u is a top node in Δ_d, then ρ^u is either one of T, $\bot$, TE, contradicting the lemma's assumptions, or FE, in which case (3) is trivial.

<u>Induction Step</u>. By cases for ρ^u. By the subformula property, the rules for $\vee$ are excluded.

<u>Case (a)</u>: Propositional rules.

Assume $\Gamma^{u} \not\vdash_{IL_\wedge} A^{u}$; then, w.l.o.g., $\Gamma^{u*\hat\wedge} \not\vdash_{IL_\wedge} A^{u*\delta}$. By the subformula property, no formula which is an active premise in the inference is an equation, because such a formula must fall in the immediate scope of a propositional connective. Hence, $A^{u*\hat\wedge}$ is not an equation, for if $A^{u*\hat\wedge}$ is not active in the inference, then $A^{u*\hat\wedge} \equiv A^{u}$, which is not an equation by assumption. By induction assumption applied to $u*\hat\wedge$,

(4) $(\Gamma_\wedge^{u*\delta}, \Delta^{u*\hat\wedge}) \to G[\Xi^{u*\hat\wedge}]$.

By 2.3.1 there is a set $\Sigma = \{P_{j1}, \dots, P_{jr}\}$ s.t.

$$\Delta^{u*\hat\wedge} = \Delta^{u} \cup \Sigma, \quad \Xi^{u} = \Xi^{u*\hat\wedge} \cup \Sigma.$$

Also, since no formula active in the premise may be an equation, $\Gamma_0^{u*\delta} = \Gamma_\wedge^{u}$. So (4) implies

(5) $(\Gamma_\wedge^{u}, \Delta^{u}, \Sigma) \to G[\Xi^{u*\hat\wedge}]$.

By the definition of the formulas G_L , $\Sigma \to G[\Xi^{u*\hat\wedge}]$ implies (in $I?_\wedge$) $G[\Sigma, \Xi^{u*\hat\wedge}]$, i.e., $G[\Xi^{u}]$. So (5) implies (3).

<u>Case (b)</u>: $\exists R$; say $\dfrac{\Gamma^{u} : A^{u*\hat\wedge}}{\Gamma^{u} : A^{u}}$.

By the subformula property A^{u} is one of the $\Sigma_1^\wedge$ sentences P_i , and $A^{u*\delta}$ is an equation. Thus, $u*\hat\wedge$ is a top node in A_d . By (2) $\models \Gamma^{u}$, so either $A^{u*\hat\wedge} \in \Gamma^{u}$ or $A^{u*\hat\wedge}$ is true, and $\Gamma_\wedge^{u} \to A^{u}$ in any case. On the other hand, (2) implies $A^{u} \in \Xi^{u}$, so $A^{u} \to G[\Xi^{u}]$, and (3) follows.

Case (c): $\exists L$; say $\dfrac{\{\Gamma, \exists xB, B[\bar{n}/x] \ : A''\}_n}{\Gamma, \exists xB \ : A''}$ $\qquad \Theta'' = (\Gamma, \exists xB)$

Claim: If A'' is not an equation, and $\Theta, \exists xB \not\vdash_{I\Upsilon_\Lambda} A''$, then $\Gamma, \exists xB, B[\bar{n}/x] \not\vdash_{IL_\Lambda} A''$ for each $\bar{n}$. To see this observe that, by the subformula property, all formulas in Γ are either equations or propositional schemes : $P_1, \ldots, P_k$. If Π is a proof in $I\Upsilon_\Lambda$ of $\Gamma, \exists xB, B[\bar{n}/x] \ : A''$, then the same statement holds also for Π. Deleting the equation $B[\bar{n}/x]$ from all antec of sequents in Π leaves a correct $I\Upsilon_\Lambda$-proof of $\Gamma, \exists xB : A''$. This proves the claim.

By the claim we may apply ind. hyp. to each one of the premises $u * \hat{n}$; s

$$(\Gamma_\Lambda^{u*\hat{n}}, \Delta^{u*\hat{n}}) \rightarrow \rho[\Xi^{u*\hat{n}}] \quad \text{for all } n .$$

But the claim also implies that $\Delta^{u*\hat{n}} = \Delta''$ and $\Xi^{u*\hat{n}} = \Xi''$, while $\Gamma_\Lambda^{u*\hat{n}} = (\Gamma_\Lambda''$, so

$$(\Gamma_\Lambda'' , \Delta'', B[\bar{n}/x]) \rightarrow \rho[\Xi''] \quad \text{for all } n ,$$

i.e.,

$$(\Gamma_\Lambda'' , \Delta'' , \exists xB) \rightarrow \rho[\Xi^u] .$$

Since $\exists xB \in \Delta''$ trivially,

$$(\Gamma_\Lambda'' , \Theta'') \rightarrow G[\Xi''] ,$$

as required. This concludes the proof of the induction step and of the lemma. Γ

2.5. Proof of theorem II

LEMMA 2.5.1. Let T be an r.e. $\neg\neg\Sigma_1^0$ sound extension of IA. Suppose T satisfies the disjunction property: if $\vdash_T A \vee B$ then either $\vdash_T A$ or $\vdash_T B$. Then, for each $k \geq 1$, there are sentences $B_1,\ldots,B_k$ s.t. $\nvdash_T G_k[B_1,\ldots,B_k]$.

First we give an explicit construction that does not use the Uniformization Theorem. This technique seems to be of independent interest. A shorter proof is sketched at the end of this section 2.5.

SUBLEMMA 2.5.2. Disjoint r.e. sets R,S are effectively inseparable iff there is a total recursive function f satisfying $f(i,j) \notin W_i \cup W_j$ whenever $W_i \cap R = \emptyset$ and $W_j \cap S = \emptyset$.

PROOF. Routine. □

We use underlined majuscules to denote occurrences of formulas. $F[H/\underline{G}]$ will denote the result of substituting H for $\underline{G}$ in F.

SUBLEMMA 2.5.3. (a1) If $\underline{G}$ is a positive subformula of F, then

$$G \to H \vdash_{IL_0} F \to F[H/\underline{G}] .$$

(a2) If $\underline{G}$ is negative in F, then $G \to H \vdash_{IL_0} F[H/\underline{G}] \to F$.

(b) Let p,q be fixed propositional letters. Let F^0 (F^-) come from F by simultaneously replacing each occurrence (negative occurrences, respectively) of p by $p \vee q$. Then

(1) $$\neg q \vdash_{IL_0} F^0 \to F$$

PROOF. (a) is straightforward by induction on the length of F. By (a2) we have $F^- \vdash F$, and since $\neg q \vdash p \vee q \to p$ we get by (a1) $\neg q \vdash F^0 \to F^-$, hence (1). □

PROOF OF 2.5.1. Let R,S be effectively inseparable r.e. sets, and let f be given by 2.5.2. Shepherdson [60] exhibits a Σ_1^0 formula $F(x)$ s.t.

(2) $$R = \{n \mid \vdash_T F(\bar{n})\} \; ; \quad S = \{n \mid \vdash_T \neg F(\bar{n})\} \; .$$

The proof is intuitionistic.

We construct, by recursion on k, a sequence B_i^k, $i \geq 0$, of Σ_1^0 sentences, satisfying

(3) $$\nvdash_T G_k[B_{i_1}^k,\ldots,B_{i_k}^k]$$

for any distinct $i_1,\ldots,i_k$.

Basis: Let T be a Σ_1^0 Rosser sentence for T; set $B_i^1 \equiv R$ for all i.

Induction Step. Assume (3) for a given k. We define a sequence C_m^k $(m \geq 1)$ of Σ_1^0 sentences, every Boolean combination C^* of which satisfies

(4) $$C^* \nvdash_T G_k[B_{i_1}^k,\ldots,B_{i_k}^k]$$

for any distinct $i_1,\ldots,i_k$. By a Boolean combination we mean here a list $\{D_m\}_m$ where each D_m is either C_m^k or $\neg C_m^k$.

The sentences C_m^k are defined by recursion on m, simultaneously with two prim. rec. functions g_k,h_k, as follows. Assume C_j^k, $g_k(j)$, $h_k(j)$ are defined for $j < m$, and let $g_k(m)$ be defined by

(5) $$W_{g_k(m)} = \{n \mid F(\bar{n}),\ C^* \vdash_T G_k[B_{i_1}^k,\ldots,B_{i_k}^k] \text{ for some Boolean combination } C^* \text{ of } C_1^k,\ldots,C_{m-1}^k \text{, and some distinct } i_1,\ldots,i_k\} \; .$$

$W_{h_k(m)}$ is defined likewise, but with $\neg F(\bar{n})$ in place of $F(\bar{n})$. Let

(6) $$C^k_m :\equiv F(\overline{f(g_k(m), h_k(m))})$$

We prove (4) by induction on m. The induction assumption for $m - 1$ implies, by (2), (5) and the consistency of T that

$$W_{g_k(m)} \cap R = \emptyset \ ; \ W_{h_k(m)} \cap S = \emptyset .$$

So $f(g_k(m), h_k(m)) \notin W_{g_k(m)} \cup W_{h_k(m)}$, which by (5) and (6) yields (4) for m.

To conclude the induction step define

(7) $$B^{k+1}_i :\equiv \text{ a } \Sigma^0_1 \text{ sentence equivalent to } B^k_i \vee C^k_i .$$

The proof of the lemma is completed by induction on k. The basis is immediate by the disjunction property of T. To prove the induction step, assume

(8) $$\vdash_T G_{k+1}[B^{k+1}_{i_1},\ldots,B^{k+1}_{i_{k+1}}]$$

for some distinct $i_1,\ldots,i_{k+1}$. By the disjunction property of T and the definition of G_{k+1},

$$\vdash_T B^{k+1}_{i_1} \to G_k[B^{k+1}_{i_2},\ldots,B^{k+1}_{i_{k+1}}] \text{ , say.}$$

By (7) this implies

$$C^k_{i_1} \vdash_T G_k[B^k_{i_2} \vee C^k_{i_2},\ldots,B^k_{i_{k+1}} \vee C^k_{i_{k+1}}] .$$

And so, by 2.5.2 (b)

$$C^k_{i_1}, \neg C^k_{i_2}, \ldots, \neg C^k_{i_{k+1}} \vdash_T G_k[B^k_{i_2}, \ldots, B^k_{i_{k+1}}]$$

contradicting (4). □

PROOF OF THEOREM II - CONCLUDED.

Let T be a regular theory. By theorem III and theorem I, there is a Σ^0_1 formula B s.t. $\not\vdash_T G_k[B]$ for all k. Assume that $\vdash_T F[B]$, where F is a propositional formula. Suppose that F is underivable in IL_0. Then, by theorem III, $\vdash_T G_k[B]$, contradicting 2.5.1. This shows that F is not unprovable in IL_0; since IL_0 is decidable, F is provable. □

REMARK. Uniformization may be used to obtain a shorter proof of 2.5.1. Assume $\not\vdash_T G_k[B_1,\ldots,B_k]$. Then $\not\vdash_T B_{k+1} \to G_k[B_1,\ldots,B_k]$ for $B_{k+1} :\equiv (0=0)$. By uniformization, there are Σ^0_1 sentences $B'_1,\ldots,B'_{k+1}$ s.t. $\not\vdash_T B'_i \to G^i_k[B'_1,\ldots,B'_{k+1}]$ for $i=1,\ldots,k+1$. So $\not\vdash_T G_{k+1}[B_1,\ldots,B_{k+1}]$. We use this technique again in 3.4.

2.6. Classification of propositional logics by their maximality properties.

DEFINITION. A frame is a predicate letter formula ϕ in which some predicate letter occurs, but no predicate letter occurs twice. An arithmetic formula F is a case of ϕ if F is the result of substituting equations for the atomic subformulas of ϕ. Write $\psi \subset \phi$ if each case of ψ is equivalent in IA_0 (:= IA with induction restricted to quantifier free formulas) to some case of ϕ. Say that L is ϕ-maximal (for T) if L is maximal (for T) with cases of ϕ as substitutions.

LEMMA 2.6.1. Suppose a logical calculus L is ψ-maximal for an extension T of IA. If $\psi \subset \phi$, then L is ϕ-maximal for T.

PROOF. Immediate. $\square$

We classify the frames into classes Φ_0, Φ_1, Φ_2.

$\phi \in \Phi_0$ if there is a strictly positive non-vacuous occurrence of $\exists$ in ϕ, or if there is a strictly positive occurrence of disjunction in ϕ, where each one of the disjuncts contains a non-vacuous occurrence of a quantifier.

$\phi \in \Phi_1$ if $\phi \notin \Phi_0$, and there is a non-vacuous occurrence of a quantifier in ϕ.

$\phi \in \Phi_2$ in all other cases.

Also, define a set of four frames:

$$\Phi := \{\exists xP(x),\ \forall xP(x) \vee \forall yQ(y),\ \forall xP(x) \vee \neg\forall yQ(y),\ \neg\forall xP(x) \vee \neg\forall yQ(y)\}$$

<u>LEMMA 2.6.2</u>. (i) If $\phi \in \Phi_0$ then $\psi \subset \phi$ for some $\psi \in \Phi$.

(ii) If $\phi \in \Phi_1$ then $\psi \subset \phi$ for $\psi \equiv \neg\exists xP(x)$ or $\neg\neg\exists xP(x)$.

<u>PROOF</u> (i) <u>Case (a)</u>. ϕ has a strictly positive subformula $\exists x\phi_0(x)$ where ϕ_0 has an atomic subformula $R(x,\vec{y})$. Given a Σ_1^0 sentence $A \equiv \exists xA_0$, A is clearly equivalent (in IA_0) to the result of substituting A_0 for $R(x,\vec{0})$, if R is positive in ϕ_0, of substituting an equation equivalent to $\neg A_0$ for R, if R is negative in ϕ_0, and of suitably substituting $\bar{0}=\bar{0}$ or $\bar{0}=\bar{1}$ for other predicate letters in ϕ. Recall that predicate letters may occur only once in ϕ, so no clashes arise. Thus $\exists xP(x) \subset \phi$.

<u>Case (b)</u>: Otherwise, and ϕ has a strictly positive subformula of the form $\phi_0 \vee \phi_1$, where each ϕ_i contains a non-vacuous quantifier. By suitable substitutions, as for case (a), we find that $\phi_i \subset \forall xP(x)$ or $\neg\forall xP(x)$.

The proof of (ii) is similar. $\square$

Let IL_0^+ denote IL_0 extended with all sentences $\neg\neg p \to p$ as new axioms. Note that $\not\vdash_{IL_0^+} p \vee \neg p$.

<u>LEMMA 2.6.3</u>. (i) If $\vdash_{IL_0} F[\neg\neg p_1,\ldots,\neg\neg p_k]$, then $\vdash_{IL_0^+} F[p_1,\ldots,p_k]$.

(ii) If $\vdash_{IL_0} F[\neg p_1,\ldots,\neg p_k]$, then $\vdash_{IL_0^+} F[p_1,\ldots,p_k]$.

PROOF: (i) is trivial, since $\vdash_{IL_0^+} p_i \leftrightarrow \neg\neg p_i$. To prove (ii), assume $\vdash_{IL_0} F[\neg q_1,\ldots,\neg q_k]$. Then $\vdash_{IL_0} F[\neg\neg p_1,\ldots,\neg\neg p_k]$, which by (i) implies $\vdash_{IL_0^+} F[p_1,\ldots,p_k]$. □

THEOREM IV. Let ϕ be a frame.

(i) If $\phi \in \Phi_0$, then IL_0 is ϕ-maximal for any regular theory T .

(ii) If $\phi \in \Phi_1$, then no strict subtheory of IL_0^+ (in particular - IL_0) is ϕ-maximal even for IA_0 ; but IL_0^+ is ϕ-maximal for any regular theory T .

(iii) If $\phi \in \Phi_2$, then no strict subtheory of CL_0 (in particular - IL_0^+) is ϕ-maximal for IA_0 , but CL_0 is ϕ-maximal for any regular T .

PROOF: (i) follows from theorem II, by 2.6.1 and 2.6.2(i).

(ii) If $\phi \in \Phi_1$, then, in IA_0, $\neg\neg A \rightarrow A$ for each case A of ϕ ; so, no strict subtheory of IL_0^+ is ϕ-maximal even for IA_0 . On the other hand, given T and a propositional formula $F[p_1,\ldots,p_k]$, there are (by theorem I and 2.6.3) Σ_1^0 sentences $A_1,\ldots,A_k$ s.t. $\vdash_T F[\neg A_1,\ldots,\neg A_k]$ implies $\vdash_{IL_0^+} F[p_1,\ldots,p_k]$. Thus IL_0^+ is $(\neg \exists x P(x))$ - maximal, and likewise also $(\neg\neg \exists x P(x))$ - maximal. So, by 2.6.2(ii) and 2.6.1 IL_0^+ is ϕ-maximal.

(iii) If $\phi \in \Phi_2$, then each case A of ϕ is equivalent in IA_0 to either $\bar{0} = \bar{0}$ or $\bar{0} = \bar{1}$. If B is a predicate formula, and B^* comes from B by substituting $\bar{0} = \bar{0}$ or $\bar{0} = \bar{1}$ for the predicate-letters, then $\vdash_{IA_0} B^* \vee \neg B^*$. Thus no strict subtheory of CL_0 is ϕ-maximal for IA_0 .

On the other hand, there is a predicate letter in ϕ, so both $\bar{0} = \bar{0}$ and $\bar{0} = \bar{1}$ are cases of ϕ ; hence CL_0 is ϕ-maximal, since CL_0 is complete for truth-table semantics. □

<u>OPEN PROBLEM</u>. Is there a propositional logic other than IL_0, IL_0^+ and CL_0, which is ϕ-maximal for IA (say), where predicate letters are allowed to occur in ϕ more than once?

2.7. <u>Maximality of IL_0 for some non-regular extensions of IA</u>.

We examine maximilaity for some extensions of IA which are not r.e. subtheories of IA*. From the myriad of such extensions the ones that have attracted most attention are those arising from IA by adding one or more of the following schemas.

1. Markov's principle:

M $\qquad \forall x[A \vee \neg A] \;\&\; \neg\forall xA \to \exists x\neg A$

2. The independence of premise principle:

IP $\qquad [\neg B \to \exists xA[x]] \to \exists x[\neg B \to A[x]] \quad$ (x not free in B)

3. Church's thesis:

CT_0 $\qquad \forall x\exists yA[x,y] \to \exists e\forall x\exists z[T(e,x,z) \;\&\; A[x,U(z)]]$

(where T,U are Kleene's calculability predicate and result-extracting function, respectively). The logical schemas M and IP_0 are classically valid while CT_0 is a classically invalid arithmetic schema (cf. Tr [73] for details). We restrict attention to closed instances of these schemas.

LEMMA 2.7. (Le [A]). Let Θ be the set of sentences of $\mathcal{A}$ where no negative occurrence of $\exists$ or $\vee$ falls in the scope of a universal quantifier. Let T be a regular theory, and $\hat{\Pi}_1^0$ denote the set of all true Π_1^0 sentences.

(i) $T + M + IP$ is conservative over $T + \hat{\Pi}_1^0$ w.r.t. Θ .

(ii) $T + CT_0$ is conservative over T w.r.t. Θ .

THEOREM V. Let T be regular. IL_0 is uniformly Σ_2^0 maximal for $T + M + IP$.

PROOF. By theorems II and III, relativized to $\hat{\Pi}_1^0$, there are Σ_2^0 sentences A_i , s.t.

(1) $$\vdash_{T+\hat{\Pi}_1^0} F[A_1,\ldots,A_k] \quad \text{implies} \quad \vdash_{IL_0} F[p_1,\ldots,p_k]$$

for each propositional formula F . But in $F[A_1,\ldots A_k]$ no occurrence of $\exists$ or $\vee$ is in the scope of a universal quantifier, so by 2.7(i)

$$\vdash_{T+M+IP} F[A_1,\ldots,A_k] \quad \text{implies} \quad \vdash_{T+\hat{\Pi}_1^0} F[A_1,\ldots,A_k] \ ,$$

which, combined with (1), concludes the proof. $\square$

Note that IL_0 is not Σ_1^0 maximal for IA+M , since $\vdash_{IA_0+M} \neg\neg A \to A$ for each Σ_1^0 sentence A . It is an open problem whether IL_0 is maximal for IA+M with β substitutions, or with any disjunction of Π_1^0 sentences. The local maximality of IL_0 for IA+M (M with parameters), was proved by Smoryński [73]. The method of Kripke models he used also fails to yield simpler substitutions.

<u>THEOREM VI</u>. Let T be regular. Then IL_0 is uniformly Σ_1^0 and β maximal for T+IP .

The proof of theorem VI is essentially the same as that of theorem II, since the subformula property holds for IA+IP , up to minor details that do not affect the proof. (Compare Tr [73],4.3.)

<u>THEOREM VII</u>. IL_0 is uniformly Σ_1^0 and β maximal for $T+CT_0$.

<u>PROOF</u>. As for theorem V, this follows from theorem I and 2.7(ii). $\sqsubset$

3. MAXIMALITY OF FIRST ORDER PREDICATE LOGICS

3.1. Uniform maximality of classical logic.

THEOREM VIII (Wang[51]). Pure first order classical logic CL_1 is Δ_2^0 maximal for any classically sound theory, and for any Σ_1^0 -sound extension of CA.

PROOF. Assume $\nvdash_{CL_1} F[P_1,\dots,P_k]$.

Case(a) By Hilbert-Bernays [39] proof of Gödel's Completeness Theorem, there are Δ_2^0 predicates $B_1,\dots,B_k$ s.t. $F[B_1,\dots,B_k]$ is false (cf. Kl[52] §72). Thus, if T is a sound theory, $\nvdash_T F[B_1,\dots,B_k]$.

Case(b) Hilbert-Bernay's proof (in a suitably arithmetized form) is derivable in CA. I.e., if $T \supset CA$, then for suitable Δ_2^0 formulas $B_1,\dots,B_k$

$$\neg \underline{Pr}_{CL_1}(\overline{\ulcorner F[\vec{P}]\urcorner}) \vdash_T \neg F[B_1,\dots,B_k]$$

If $\vdash_T F[B_1,\dots,B_k]$, then $\vdash_T \underline{Pr}_{CL_1}(\overline{\ulcorner F[\vec{P}]\urcorner})$. So, if T is Σ_1^0 -sound, then, $\vdash_{CL_1} F[\vec{P}]$. ◻

THEOREM IX. CL_1 is uniformly (and hence strongly) Σ_2^0 maximal and Π_2^0 maximal for any Σ_2^0-sound r.e extension of CA .

PROOF. Hilbert-Bernay's [39] proof is uniform: a Δ_2^0 counter-model to a non-tautology F has a Σ_2^0 index recursive in $\ulcorner F\urcorner$. The theorem, for Σ_2^0 maximaility, follows by the Uniformization Theorem I, with r=2 , since the non-tautologies form a Π_1^0 set.

If B is a uniform Σ_2^0 substitution for CL_1, let C be a Π_2^0 formula equivalent to $\neg B$. Then $\vdash_{CL_1} F[P_1,\ldots,P_k]$ implies $\nvdash_{CL_1} F[\neg P_1,\ldots,\neg P_k]$, and so $\nvdash_T F[\neg B]$, i.e., $\nvdash_T F[C]$. Thus CL_1 is also uniformly Π_2^0 maximal. □

OPEN PROBLEM. Is CL_1 uniformly Δ_2^0 maximal for Peano's Arithmetic CA ?

3.2. Maximality properties of intuitionistic predicate logic.

NOTATIONS. Given a theory T, write (T,n) for T extended with all true Π_n^0 sentences. Write $(T;n)$ for (T,n) extended with all sentences of the form $\forall x(A\vee\neg A)$, where A is Σ_n^0. Clearly, when T is r.e., (T,n) and $(T;n)$ are Σ_{n+1}^0.

LEMMA 3.2.1 Let T be a regular theory. (T,n) and $(T;n)$ satisfy the existential instantiation property: if $\vdash_{T,n} \exists xA$ then $\vdash_{T,n} A[\overline{m}/x]$ for some numeral $\overline{m}$, and similarly for $(T;n)$.

PROOF. The statement is straightforward by Kleene's [62] slash method, appropriately relativized. A different proof uses derivations of IA^*, with a relativized version of 2.3.3. □

THEOREM X. Let T be regular. IL_1 is uniformly Π_{n+1}^0 maximal for (T,n), for each $n \geq 1$. In particular, IL_1 is Π_2^0 maximal for $(T,1)$, and so for T.

As for theorem II, the proof uses a reduction. Let $B[x] \equiv \forall z B_0[x,z]$ be a Π^0_{n+1} formula, with x as the only free variables. Define

$$S_B[y] :\equiv \forall x \neq y B[x] \rightarrow \forall z \neg\neg B_0[y,z]$$

Theorem X is a consequence of the following two lemmas.

LEMMA 3.2.2. Let T be a regular theory, $n \geq 1$. Assume $B[x]$ is a Π^0_{n+1} formula s.t. $\neg\neg\forall xB$ is true. Suppose $\not\vdash_{IL_1} F$ and $\vdash_{T,n} F[B]$. Then there (weakly) exists a k s.t. $\vdash_{T,n} S_B[\bar{k}]$.

LEMMA 3.2.3. Let T be an r.e. theory, $n \geq 0$. Suppose T,n is consistent. Then there exists a Π^0_{n+1} formula $B[x]$ s.t. $\neg\neg\forall xB$ is true, and $\not\vdash_{T,n} S_B[k]$ for all $k \geq 0$.

A special case of 3.2.2 is the following.

COROLLARY X. Let $n \geq 1$. Assume $A_1,\ldots,A_k$ are Π^0_{n+1} sentences, classically independent over all true Π^0_n sentences. Then they are intuitionistically independent over all true Π^0_n sentences.

With $n=1$, the corollary implies the result of H. Friedman [73], that Π^0_2 sentences classically independent over true Π^0_1 sentences are intuitionistically independent.

Theorem X shows that properties of T do not "relativize" to (T,n), but to $(T;n)$. Even a single Π^0_1 sentence A is not "intuitionistically independent," since $\vdash_{IA} \neg\neg A \rightarrow A$, but, by the corollary, $\not\vdash_{T,n} \neg\neg B \rightarrow B$ for an appropriate Π^0_{n+1} sentence B. On the other hand, $\vdash_{T;n} \neg\neg B \rightarrow B$ is demonstrated as for the case $n=0$.

Unlike the classical case, Σ^0_n and Π^0_n maximality properties are not directly related. Therefore, the following result is of independent interest.

<u>THEOREM XI</u>. Let T be regular. IL_1 is uniformly Σ^0_{n+1} maximal for (T;n) for each $n \geq 1$. In particular, IL_1 is uniformly Σ^0_2 maximal for (T;1), and so - for T.

The proof is a generalization of the methods used in proving theorem II.

Improvement on the complexity of substitutions given in theorems X and XI is bounded by the following negative result.

<u>THEOREM XII</u>. (Leivant[76]) IL_1 is not Σ^0_1 maximal for IA.

The counter-example used in proving theorem XII is syntactically complex, leaving open the possibility that for simpler predicate formulas, such as the familiar schemas of CL_1-IL_1, Σ^0_1 substitutions <u>can</u> be found that yield instances underivable in IA. Theorem XIII below shows that this is indeed the case.

A sequential calculus for IL_1 is readily obtained from the calculus in 2.3.: delete TE,FE, replace $\forall$R and $\exists$L by the new inference rules

$$\left.\begin{array}{ll} \forall R: & \dfrac{\Gamma:A}{\Gamma:\forall xA} \\[2ex] \exists L: & \dfrac{\Gamma,A:B}{\Gamma,\exists xA:B} \end{array}\right\} \text{(x not free in the derived sequent)}$$

and let terms of $\mathscr{L}_1$ replace numerals as proper terms of $\forall$L and $\exists$R.

DEFINITION. Let σ, σ' be sequents. σ' is an ancestor of σ if there is a sequence of consecutive inferences of (the calculus above for) IL_1, with σ' as one of the premises of the first inference and σ as the conclusion of the last. $\sigma = (\Gamma : F)$ is hereditarily decidable if there is a decision procedure for the derivability, in IL_1, of all sequents σ' ancestor of σ and of all sequents $\Gamma' : A$ where Γ' is the antecedent of an acestor, and A is an atomic subformula of F. F is hereditarily decidable if $:F$ is.

THEOREM XIII. IL_1 is Σ_1^0 maximal for any r.e. subtheory T of IA^* w.r.t. hereditarily decidable formulas. Moreover, the maximality is uniform, in the following sense. Assume $\Psi = \{F_n\}_n$ is an r.e. set of formulas for which there is a uniform decision procedure, applicable to all sequents $\Gamma : G$ ancestor to $:F_n$ for some $F_n \in \Psi$, as well as to $\Gamma : A$ where A is an atomic subformula of F_n. Then IL_1 is uniformly Σ_1^0 maximal for T w.r.t. Ψ.

COROLLARY XIII.1. IL_1 is uniformly Σ_1^0 maximal for any regular T w.r.t. the set of prenex formulas.

COROLLARY XIII.2. IL_1 is Σ_1^0 maximal for any regular T w.r.t. formulas with no nesting of quantifiers.

Most examples of classical tautologies underivable in IL_1 fall under the last corollary. E.g., $\forall x \neg\neg P(x) \to \forall x \neg\neg P(x)$, $\neg\neg\forall x(P(x) \vee \neg P(x))$, $\forall x(P \vee Q(x)) \to P \vee \forall x Q(x)$, $(\neg P \to \exists x Q(x)) \to \exists x(\neg P \to Q(x))$, $\forall x(P(x) \vee \neg P(x)) \,\&\, \neg\forall x P(x). \to \exists x \neg P(x)$.

OPEN PROBLEM. Is IL_1 (locally or uniformly) Δ_2^0 or β-maximal for IA ? More generally, is there any frame ϕ s.t. $\phi \subset \forall x \exists y P(x,y)$, $\phi \subset \exists x \forall y P(x,y)$, and IL_1 is ϕ-maximal for IA ?

We turn to maximality for the whole of IA^* . IA^* is complete for Σ_3^0 sentences, so not even IL_0 is Δ_3^0 maximal for IA^* . Denote by δ_1 the class of sentences of the form $\forall x[C \vee \bar{C}]$, where C is Σ_1^0 , and $\bar{C}$ is the (classical) prenex form of $\neg C$.

THEOREM XIV. (a) IL_1 is uniformly δ_2-maximal for IA^* .

(b) IL_1 is δ_1-maximal for IA^* w.r.t. hereditarily decidable formulas. This maximality is uniform in the sense of theorem XIII. In particular, IL_0 is uniformly δ_1-maximal for IA^* .

OPEN PROBLEM Is IL_1 Σ_3^0 , Σ_4^0 or δ_1-maximal for IA^* ? Is IL_0 Σ_3^0 -maximal for IA^* ?

The first maximality result for IL_1 is due to D.H.J. de Jongh [73a], who proved that if $\nvdash_{IL_1} F[P_1,\ldots,P_k]$, then $\nvdash_{IA} F^A[B_1,\ldots,B_k]$ for suitable arithmetical predicates $A,B_1,\ldots,B_k$, where the superscript stands for relativization of quantifiers. In particular, IL_1 is maximal for IA w.r.t. predicate formulas that are already relativized. This relativized maximality does not characterize IL_1 as the set of predicate formulas "locally provable in IA." De Jongh's proof simulates Kripke models via a notion of forced realizability, using an infinite sequence of incomparable degrees.

Contents of the rest of Section 3: Theorem X is proved in 3.3. Proofs of theorems XI and XIII are sketched in 3.4 and that of XIV in 3.5. Maximality of predicate logics with equality is discussed in 3.6. Section 3.7 concludes, with remarks on the maximality of CL_1 (with and without equality) for systems of set theory.

3.3 Proof of Π_2^0 maximality (theorem X).

Let T be a regular theory. Let T^c denote T augmented with classical logic.

LEMMA 3.3.1. There exists a (classically) Σ_{n+1}^0 predicate $J(x)$ satisfying

(1) $\qquad \vdash_{IA} J(x) \,\&\, J(y) \to x = y$

and (2) $\qquad \nvdash_{T^c,n} \neg J(\bar{m})$ for each m.

PROOF. Straightforward by Kripke [63], relativized to true Π_n^0 sentences. □

LEMMA 3.3.2. There exists a Π_{n+1}^0 predicate $B(x)$ s.t.

(3) $\qquad \nvdash_{T^c,n} \neg \forall x[B(x) \leftrightarrow A(x)]$

for each Π_n^0 predicate A.

PROOF. Let $E(m,x)$ be a Π_n^0 predicate universal for unary Π_n^0 predicates, and let $B(x) :\equiv \forall m[J(m) \to E(m,x)]$. By (1), $J(\bar{m})$ implies $B(x) \leftrightarrow E(\bar{m},x)$; hence, if (3) fails for $A(x) \equiv E(\bar{m},x)$, then $\vdash_{T^c,n} \neg J(\bar{m})$, contradicting (2). □

PROOF OF LEMMA 3.2.3. Assume $\vdash_{T,n} \forall x \neq \bar{k}\ B[x] \to \forall z \neg\neg B_0[\bar{k},z]$. Then $\vdash_{T^c,n} \neg \forall x\ [x \neq \bar{k} \leftrightarrow B[x]]$, contradicting 3.3.2.

To prove $\neg\neg\forall x B[x]$ assume $\neg \forall x B(x)$. Since $\forall x B[x]$ is Π^0_{n+1} , $\neg \forall x B(x)$ implies $\vdash_{T^c,n} \neg \forall x B[x]$, contradicting again 3.3.2. □

Let F be a predicate formula, and let Γ, G consist of subformulas of $F[B]$. We write $\Gamma \vdash^B_{IL_1} G$ if $\Gamma : G$ can be dervied using only inference rules of IL_1 , where each occurrence $B[<\bar{n},t_1,\ldots,t_k>]$ is treated as an atomic formula $P_n(t_1,\ldots,t_k)$.

LEMMA 3.3.3. Let B be a Π^0_{n+1} predicate with a non-vacuous quantifiers. Let $\Gamma, A, C[x]$ consist of schemas in B , and let θ consist of strict subsentences of B . Then

(a) If $\Gamma, \theta \vdash^B_{IL_1} A$ then $\Gamma \vdash^B_{IL_1} A$

(b) If $\bar{p}$ does not occur in $\Gamma, \theta, C[x]$, then $\Gamma, \theta \vdash^B_{IL_1} C[\bar{p}]$ implies $\Gamma, \theta \vdash^B_{IL_1} \forall x C[x]$, and $\Gamma, \theta, C[\bar{p}] \vdash^B_{IL_1} A$ implies $\Gamma, \theta, \exists x C[x] \vdash^B_{IL_1} A$.

PROOF. (a) is immediate from the definition of $\vdash^B_{IL_1}$. For (b), replace $\bar{p}$ by a parameter not occurring in the given proof. □

To simply the exposition, we assume for the rest of 3.3. that $n = 1$. The proof for $n > 1$ is similar (with IA* extended with true Π^0_n sentences as new axioms).

NOTATIONS. Let B be a Π^0_2 formula.

Assume $\underline{\mathrm{Prov}}^*(d,\ulcorner F[B]\urcorner)$. It will be convenient to distinguish between those instances of $\exists L$, $\exists R$, ($\forall L$, $\forall R$ respectively) in Δ_d with a Σ^0_1 subformula of B as the active conclusion (premise, resp.), and all those instances of $\exists L, \exists R, \forall L, \forall R$ in which all active formulas are schemas in B . We write $\exists^1 L$, $\exists^1 R$, $\forall^2 L$, $\forall^2 R$ for instances of the first kind, $\exists^* L$, $\exists^* R$, $\forall^* L$, $\forall^* R$ for those of the second.

Write $\underline{\mathrm{Start}}(d,u)$ if the following three conditions hold:

(i) $\Gamma^u \overset{B}{\underset{IL_1}{\nVdash}} A^u$;

(ii) All strict subsentences of B in Γ^u are true ;

(iii) A^u is a schema in B .

Write $\underline{\mathrm{Goal}}(d,u)$ if (i), (ii) and the following two conditions hold:

(iv) A^u is $B[t/x]$ for some t , and ρ^u is $\forall^2 R$;

(v) No node $v \succ u$ satisfies (i), (ii), (iv) in place of u .

We omit the first argument d when in no danger of confusion.

LEMMA 3.3.4. Suppose that $\nVdash_{IL_1} F$ and that $\underline{\mathrm{Prov}}^*(d,\ulcorner F[B]\urcorner)$ and $\forall x B$ are true. Then $\neg\neg\exists v\underline{\mathrm{Goal}}(d,v)$.

SUBLEMMA. Assume $\underline{\mathrm{Start}}(w)$, and suppose that ρ^w is not $\forall^2 R$. Then $\neg\neg\exists p\underline{\mathrm{Start}}(w*\hat{p})$.

<u>PROOF OF SUBLEMMA</u>. By cases on ρ^w . $T,\bot$ are excluded by (i), FE is excluded by (ii), TE and $\exists^1 R$ are excluded by (iii). If ρ^w is a propositional rule, then (ii) and (iii) hold for $u=w*\hat{p}$ for any p , and (i) for $u=w$ implies the weak existence of a p s.t. (i) hold for $u=w*\hat{p}$. If ρ^w is $\forall*L$ or $\exists*R$, take $p=0$. If ρ^w is $\forall*R$ or $\exists*L$, let $\bar{p}$ be the first numeral not occurring in σ^w , and use 3.3.3(b). If ρ^w is $\exists^1 L$ then, by (ii), the active conclusion $\exists xE$ (E an equation) is true. Let $p := \mu x.E$, and use 3.3.3(a). If ρ^w is $\forall^2 L$, say

$$\frac{\Gamma,B_0[t,\bar{m}] \ : \ A^w}{\Gamma,\forall zB_0[t,z] \ : \ A^w} ,$$

then $B_0[t,\bar{m}]$ is a true Σ_1^0 sentence, since $\forall xB \equiv \forall x\forall zB_0[x,t]$ is assumed true.

<u>PROOF OF THE LEMMA</u>. Assume <u>Start</u>(u) . We prove, by induction on Δ_d , that $\neg\neg\exists v \succeq u$<u>Goal</u>(v) . In particular, for u=<> this establishes the lemma. Since the statement we prove is negated, we may use individual instances of excluded third. If <u>Goal</u>(u) , we are done. If $\exists u'\succ u$<u>Start</u>(u') then, by ind. ass., $\neg\neg\exists v \succ u'$<u>Goal</u>(v) . Else, then ρ^u is not $\forall^2 R$; so , by the sublemma, $\neg\neg\exists p$<u>Start</u>($u*\hat{p}$) . By ind. ass., <u>Start</u>($u*\hat{p}$) $\rightarrow \neg\neg\exists v \succeq u*\hat{p}$<u>Goal</u>(v) . So $\neg\neg\exists p$<u>Start</u>($u*\hat{p}$) implies $\neg\neg\exists v \succeq u*\hat{p}$<u>Goal</u>(v) . □

Fix a primitive recursive coding of finite sets of naturals (say, $\{n_0,\ldots,n_k\} := \sum_{i=0}^{k} 2^{n_i}$) and of the basic set-theoretic operations on them. Let <u>Bar</u>(d,y,x) be a Σ_1^0 predicate that holds when x codes a non-empty set of nodes in Δ_d above y , satisfying conditions (vi)-(xi) below whenever $y \preceq w \prec v \in x$.

(vi) ρ^w is a left-rule and $A^v \equiv A^w \equiv A^y$;

(so, e.g., when ρ^w is $\rightarrow L$ then $v \succeq w{*}\hat{1}$) ;

(vii) if ρ^w is $\exists{*}L$ then $v \succeq w{*}\hat{p}$, where $\bar{p}$ is the first numeral not occurring in σ^w ;

(viii) if ρ^w is $\exists^1 L$, with $\exists xE$ as the active conclusion, then $v \succeq w{*}\hat{p}$, where $p = \mu x.E$;

(ix) if ρ^w is $\vee L$ and $v \succeq w{*}\hat{b}$ (b=0 or 1), then there is also a $v' \in x$ above the other premise $w{*}\hat{c}$ of w (c=1 or 0, respectively);

(x) if ρ^w is $\forall^2 L$ then the active premise is a true (Σ_1^0) formula;

(xi) ρ^v is $\forall^2 L$ or $\exists^1 R$, and all equations and Σ_1^0 sentences in Γ^v are true.

Let <u>Crit</u>(d,u) be a Δ_2^0 predicate that holds exactly when (i), (ii), (iv) and

(xii) $\forall x,m[$<u>Bar</u>$(d,u{*}\hat{m},x) \rightarrow \exists v \epsilon x L(d,v)]$,

where $L(d,v)$ is a (recursive) predicate that holds iff ρ^v is $\exists^1 R$, or ρ^v is $\forall^2 L$ with an active conclusion $B[s]$ s.t. the term s is syntactically distinct from the term t of (iv). Again, we omit d when in no danger of confusion.

The following lemma is useful, since Crit is Σ^0_2 , while Goal is not.

LEMMA 3.3.5. Assume Prov*(d,'F[B]') . Goal(d,u) implies Crit(d,u) .

PROOF. Assume Goal(u) . We need to prove only (xii). Since $\exists v\varepsilon x L(v)$ is decidable, it suffices to derive a contradiction from

(1) Bar$(u*\hat{m},x)$, and

(2) $\forall v\varepsilon x \neg L(v)$, i.e. for each $v\varepsilon x$ ρ^v is $\forall^2L$ with $B[t]\equiv A^u$ as the active conclusion in Γ^v .

Assume (1) and (2), and let $w \succeq u*\hat{m}$ be s.t. $w \preceq v$ for some $v \varepsilon x$. We prove, by induction on

$$h := \max[\underline{\ell th}(v) - \underline{\ell th}(w) \mid v \varepsilon x]) = \text{the height of } x \text{ above } w$$

that

(3) $\neg\neg\, \Gamma^w \vdash^B_{IL_1} A^u$.

Since $\Gamma^{u*\hat{m}} = \Gamma^u$, taking $w = u*\hat{m}$ yields $\Gamma^u \vdash^B_{IL_1} A^u$, condtradicting (i).

Basis. $h = 0$; then $w \varepsilon x$, and by (2) $A^u \;\varepsilon\, \Gamma^w$.

Induction Step. Consider cases for ρ^w . By (vi) all right-rules are excluded.

(a) ρ^w is $\rightarrow L$; say,

$$\frac{\Gamma, C \rightarrow D : C \qquad\qquad \Gamma, D : B_0[t]}{\Gamma, C \rightarrow D : B_0[t]}$$

By the subformula property, C is a schema in B, so by (v), applied to $w*\hat{\cap}$,

$$(4)\quad \sim\sim[\Gamma, \cap \rightarrow D \vdash^{B}_{IL_1} C] \ .$$

If $w \prec v$ then by (vi) $v \succeq w*\hat{1}$, and by ind. hyp. applied to $w*\hat{1}$, $\sim[\Gamma, D \vdash^{B}_{IL_1} A^u]$, which together with (4) implies (3).

The cases for $\&L$ and $\vee*L$ are similar. The case for $\neg L$ is also similar, (x) permitting the application of (v) to $w*\hat{\cap}$.

(b) ρ^w is $\neg*L$; say

$$\frac{\{\Gamma, \neg x \cap[x], \cap[\bar{n}] : B_{\cap}[t]\}_n}{\Gamma, \forall \cap[x] : B_{\cap}[t]} \ .$$

Then, by (vii), $v \succeq w*\hat{p}$, where $\bar{p}$ is the first numeral not in σ^w . By ind. hyp. applied to $w*\hat{p}$,

$$\sim[\Gamma, \cap[\bar{p}] \vdash^{B}_{IL_1} A^u]$$

and (1$\cap$) follows by 3.3.3(b).

The case for $\neg\vee$ is similar, using (viii).

(d) ρ^w is $\vee L$; say

$$\frac{\Gamma, \cap : B_{\cap}[t] \qquad\qquad \Gamma, D : B_{\cap}[t]}{\Gamma, \cap \vee D : B_{\cap}[t]}$$

Then, w.l.o.g., $v \succeq w*\hat{0}$, and by (ix) $v' \succeq w*\hat{1}$ for some $v' \in x$. By ind. hyp. applied to both premises,

$$\sim\sim[\Gamma, C \vdash^{D}_{IT_1} A^u] \text{ and } \sim\sim[\Gamma, D \vdash^{D}_{IT_1} A^u] \text{ ;}$$

(3) follows.

NOTATION. We write $z \sqsupset x$ if x, z are (codes of) non-identical sets of nodes and $\forall v \in x \exists w \in z\; w \succeq v$.

LEMMA 3.3.6. Assume that y is not a top node in Λ_d, that A^y is Σ^0_1, and that all equations and Σ^0_1 sentences in Γ^y are true. Then

(5) $$\neg\neg\exists x[\underline{\mathrm{Bar}}(y,x) \;\&\; \neg\exists z \sqsupset x\; \underline{\mathrm{Bar}}(y,z)]$$

PROOF. By bar-induction on Λ_d. The induction basis holds vacuously by assumption. For the induction step consider cases for ρ^y.

(a) ρ^y is $\exists^1 D$; then $\underline{\mathrm{Bar}}(y,\{y\})$, and by (xi) $\neg\exists z \sqsupset \{y\}\; \underline{\mathrm{Bar}}(y,z)$.

(b) ρ^y is $\mathrm{v}^2\mathrm{I}$; if the active (Σ^0_1) premise $\exists xP$ of ρ^y is true, then by ind. hyp. we get (5) for $y*\hat{0}$ in place of y, and (5) follows. If $\exists xP \in \Gamma^{y*\hat{0}}$ is false, then by (xi),

$$\underline{\mathrm{Bar}}(y,\{y\}) \;\&\; \neg\exists z \sqsupset \{y\}\; \underline{\mathrm{Bar}}(y,z) \text{ .}$$

Thus (5) is derived from excluded third, and being negated (5) follows by propositional logic.

(c) ρ^y is another (left) rule; (5) follows by a straightforward application of the ind. hyp. . □

<u>LEMMA 3.3.7</u>. Assume $\underline{\text{Prov}}^*(d,'F[B]')$. Then, $\underline{\text{Crit}}(d,u) \to \exists x S_B[x]$.

<u>PROOF</u>. (in IA) Assume $\underline{\text{Crit}}(u)$, $A^u \equiv B[t]$, $t = \langle \bar{i},\bar{p}_1,\ldots,\bar{p}_n\rangle$, ρ^u is $\forall^2 R$. We prove $S_B[t]$ by demonstrating

(6) $\forall m[\forall x \neq t\ B[x] \to \neg\neg B_0[t,m]]$.

Fixing m, assume

(7) $\underline{\text{Bar}}(u*\hat{m},x)\ \&\ \forall z \supset| x\ \neg\ \underline{\text{Bar}}(u*\hat{m},z)$.

We derive from (7), using excluded third,

(8) $\forall x \neq t B[x] \to B_0[t,\bar{m}]$.

So (6) follows from 3.3.6.

<u>Case 1</u>. ρ^v is $\exists^1 R$ for some $v \in x$; say

$$\frac{\Gamma^v : E}{\Gamma^v : B_0[t,\bar{m}]} .$$

By 2.3.4 we may assume, w.l.o.g., that $v*\hat{0}$ is a top-node in Δ_d , while by (xi) all equations in $\Gamma^{v*\hat{0}} = \Gamma^v$ are true. So E is true, $B_0[t,\bar{m}]$ is true, and (8) follows.

<u>Case 2</u>. Otherwise. Since x is non-empty, (7) and (xii) imply that for some $v \in x$ ρ^v is $\forall^2 L$ with an active conclusion $B[s]$, $s \neq t$:

$$\frac{\Gamma,B_0[s,\bar{k}] : B_0[t,\bar{m}]}{\Gamma,B[s] : B_0[t,\bar{m}]} .$$

By the subformula property s has the form $\langle \bar{j}, \bar{q}_1, \ldots, \bar{q}_r \rangle$, so $s \not\geq t$ implies $s \neq t$. If $B_0[s,\bar{k}]$ is true, then by 3.3.6. $\neg\neg\exists x'\,\underline{Bar}(v*\hat{0},x')$. But if $\underline{Bar}(v*\hat{0},x')$ then, letting $z := (x\setminus\{v\}) \cup x'$ we have $\underline{Bar}(u*\hat{m},z)$ & $z \dashv x$, contradicting (7). Finally, if $B_0[s,\bar{k}]$ is false, then $\forall y \not\geq t\, B[y]$ must be false, since $s \not\geq t$, hence (8).

<u>PROOF OF 3.2.2 (case n=1)</u>. Assume $\not\vdash_{IL_1} F, \forall xB$, $\vdash_{T,1} F[B]$. By 2.3.3 (relativiz $\underline{Prov^*}(d, \ulcorner F[B] \urcorner)$ for some d, provably in $(T,1)$. So by 3.3.4 and 3.3.5, $\neg\neg\exists u\underline{Crit}(d,u)$. $\underline{Crit}$ is Σ_2^0, so $\underline{Crit}(d,u) \to (\vdash_{T,1}\underline{Crit}(\bar{d},\bar{u}))$. So by 3.3.7, assuming $\underline{Crit}(d,u)$, $\vdash_{T,1} \underline{Prov^*}(\bar{d}, \overline{\ulcorner F[B] \urcorner})$ implies $\vdash_{T,1} \exists x S_B[x]$, and by 3.2.1, $\vdash_{T,1} S_B[\bar{k}]$ for some k. Thus $\neg\neg\exists u\underline{Crit}(d,u)$ implies the weak existence of a k for which $\vdash_{T,1} S_B[\bar{k}]$. Since the conclusion is negated, assuming $\neg\neg\forall xB$ in place of $\forall xB$ suffices. $\Box$

3.4 Proof of theorems XI, XIII.

The proofs sketched here are refinements of the proof of theorem II. Two difficulties arise. First, for theorem XI, derivability in IL_1 is not decidable. Hence, we relativize the argument to Π^0_1. Second, quantifiers in the logical schema yield infinitely many instances of the substituted arithmetic formula. We replace the schemas Γ_k by transfinite propositional formulas; these can be coded, but not given, in the language $\mathcal{A}$ of first-order arithmetic.

To spare the reader technical details of little relevance to the core of the proof, we shall only prove, for theorem XI, that for a suitable Π^0_2 formula B, if $\vdash_T P[B]$, then $\vdash_{IL_1} P$. When the premise is weakened to $\vdash_{T;1} P[B]$, slight modifications are necessary. This would prove theorem XI for $n=1$. The proof for arbitrary $n \geq 1$ is analogous.

An infinitary language $\mathcal{A}_\omega$ is defined as follows. The <u>alphabet</u> is that of $\mathcal{A}$, plus the symbols $\wedge$ and $\vee$ for <u>countable conjunction</u> and <u>disjunction</u>, of which &, V, $\forall$ and $\exists$ may be viewed as special cases. A <u>formula</u> of $\mathcal{A}_\omega$ is a well-founded countable tree, with equations or $\perp$ at top-nodes (leaves), and logical constants at all other nodes, subject to the obvious restrictions. We use the usual typographic representation of formulas. A formula is <u>recursive</u> if its coding function is recursive. An <u>antecedent</u> is a countable set of formulas. A <u>sequent</u> is a pair $\Gamma{:}A$ where Γ is an antecedent and A is a sentence. The <u>rules of inference</u> of the calculus IA^∞_ω are those of IA^∞, plus the obvious rules for $\bigwedge$ and $\bigvee$:

$$\bigwedge L: \frac{\Gamma, (A_i)_{i\in I} : B}{\Gamma, \bigwedge_{i\in I} A_i : B} \quad ; \quad \bigwedge R: \frac{(\Gamma : A_i)_{i\in I}}{\Gamma : \bigwedge_{i\in I} A_i}$$

$$\vee L: \frac{(\Gamma, A_i : \Delta)_{i\in I}}{\Gamma, \vee_{i\in I} A_i : \Delta} \qquad\qquad \vee R: \frac{\Gamma : A_j}{\Gamma : \vee_{i\in I} A_i} \quad (j\in I)$$

Let <u>sub</u> be a primitive recursive function satisfying $\{\underline{sub}(e,i)\}n \simeq \{e\}\langle i * n$

So, if $\{e\}$ codes $\wedge_i F_i$ then $\{\underline{sub}(e,i)\}$ codes F_i. Indices e_1, e_2 s.t. $\{e_1\}$ and $\{e_2\}$ code sequents are <u>related</u> <u>correctly</u> (by an inference rule ρ if the sequents coded relate as they should by ρ, and e_1, e_2 relate corresp-ingly via <u>sub</u>. E.g., $\langle g,a\rangle$ relate correctly to each $\langle g_i, a_i\rangle$, $i\in I$, via $\wedge R$, if $\{a\}\langle\rangle \simeq \ulcorner\wedge\urcorner$, $g_i = g$ and $a_i = sub(a,i)$. Clearly, being related correctl is a recursive relation.

A <u>proof</u> of IA^{∞}_{ω} $(IA^{(n)}_{\omega})$ is a function (recursive in Π^1_n, respectively) a well-founded tree of codes of sequents of $\mathcal{A}_{\omega}$, related correctly by the inf rules above. Just as for IA^*, we use $IA^{(n)}_{\omega}$ solely as a combinatorial apparatus; no epistemological legitimacy is claimed or used. $IA^{(n)}_{\omega}[T]$ will denote the set of sequents of $\mathcal{A}_{\omega}$ proved in T to have a proof in $IA^{(n)}_{\omega}$. Clearly, when T is Σ^1_k, then so is $IA^*_{\omega}[T] = IA^{(1)}_{\omega}[T]$.

<u>LEMMA 3.4.1.</u> Let $T \supseteq IA$. Then $IA^{(n)}_{\omega}[T]$ satisfies the general disjunction instantiation property: if $\vee_{i\in I} F_i$ is a theorem of $IA_{\omega}[T]$, then so is F_j for some $j\in I$.

<u>PROOF</u>. The disjunction property of IA^{∞}_{ω} is proved in IA trivially. The lemma follows. □

Let $W_n, n \geq 0$, be an enumeration of all r.e. sets. Let s be a primitive recursive function satisfying $W_{s(a,b)} = W_a \cup W_b$. Let F be a Σ^1_2 formula, Δ a recursive well-founded tree of sequence-numbers. The formula $F(a,u) \equiv F_{F,\Delta}(a,u)$ of $\mathcal{A}_{\omega}$ is defined by bar-induction on Δ, as follows.

$G(a,u) :\equiv \exists x \notin W_a (B[x] \vee \neg B[x])$

if u is a top-node of Δ .

$G(a,u) :\equiv \exists p \exists b[\ u*\hat{p} \in \Delta \ \&\ (\forall x \in W_b B[x] \ \rightarrow G(s(a,b), u*\hat{p}))]$

if u is an inner node of Δ .

$G(a,u) :\equiv \perp$ if $u \notin \Delta$.

LEMMA 3.4.2. Suppose $\not\vdash_{IL_1} F$ and $\vdash_T F[B]$. Let $D :\equiv \forall a \forall x[x \in W_a \vee x \notin W_a]$. Then there is a recursive tree Δ , proved in T^P to be well-founded, and there is an a , s.t. $G_{B,\Delta}(a,<>)$ is a theorem of $IA_\omega^{(1)}[T + D]$.

PROOF. Assume $\vdash_T F[B]$. Let Δ be a derivation of IA^* for $F[B]$, provably correct in T^P . As in 2.4.1 one concludes, in $IA_\omega^{(1)}+D$, that $G_{B,\Delta}(p(u),u)$ is true, where p is a primitive recursive function satisfying

$$x \in W_{p(u)} \Leftrightarrow \Gamma^u \vdash^B_{IL_1} B[\bar{x}] \ .$$

Clearly, $(IA_\omega^{(1)} + D)[T] \subseteq IA_\omega^{(1)}[T + D]$. The lemma follows by taking $u = <>$.

□

LEMMA 3.4.3. For each recursive (non-empty) tree Δ and each a there exists a Σ^0_2 formula B , with index recursive in Δ, a, for which $G_{B,\Delta}(a,<>)$ is not a theorem of $IA_\omega^{(1)}[T;1]$.

PROOF. By bar-induction on Δ . If Δ is a singleton the lemma is immediate, with $B[x]$ defined to be the Σ^0_2 Rosser sentence for $(T;1)$. Suppose Δ is not a singleton, and let Δ_p denote the subtree of Δ with $\hat{p}$ as a root. Suppose that for each p,a a Σ^0_2 formula $B_{p,a}$ is given, with an index recursive in p,a, satisfying the lemma for Δ_p, a . Applying the Uniformization Theorem I, whose proof clearly works for $IA_\omega^{(1)}[T;1]$, there is a Σ^0_2 formula B' satisfying the lemma for all a and Δ_p .

Let $B_b[x] :\equiv B'[x] \vee x\varepsilon W_b$. By bar-induction on Δ for u it is easy to see that

$$G_{B',\Delta}(s(b,c),u) \longleftrightarrow G_{B_b,\Delta}(s(b,c),u)$$

for any b and any $u\varepsilon\Delta$. Hence

$$IA_\omega^{(1)}[T;1] \not\vdash \hat{p}\varepsilon\Delta \;\&(\forall x\varepsilon W_b B_b[x] \rightarrow G_{B_b,\Delta}(s(a,b),\hat{p}))$$

for any b and $\hat{p}\varepsilon\Delta$. By the Uniformization Theorem there is a Σ_2^0 formula B s.t.

$$IA_\omega^{(1)}[T;1] \not\vdash \hat{p}\varepsilon\Delta \;\&(\forall x\varepsilon W_b B[x] \rightarrow G_{B,\Delta}(s(a,b),\hat{p}))$$

for all b and $\hat{p}\varepsilon\Delta$. So, by 3.4.1

$$IA_\omega^{(1)}[T;1] \not\vdash G_{B,\Delta}(a,\langle\rangle) \; .$$

The process above is recursive in p,a and in the function giving the indices of the B_b's. The lemma follows by the Recursion Theorem. □

PROOF OF THEOREM XI. (Simplified form of the case $n = 1$.) By 3.4.3 and the Uniformization Theorem there is a Σ_2^0 formula B s.t. $G_{B,\Delta}(a,\langle\rangle)$ is not a theorem of $IA_\omega^{(1)}[T;1]$ for any a and any recursive Δ proved in T^P to be well-founded. Combining this with 3.4.2,

$$\vdash_T F[B] \text{ implies } \sim\vdash_{IL_1} F \; .$$

This implication has been proved in IA , and is classically equivalent to a Π_2^0 statement S . So S is also a theorem of IA (cf, e.g., Friedman[77]). But S implies that $\vdash_T F[B]$ implies $\vdash_{IL_1} F$. □

PROOF OF THEOREM XIII. The proof is the same as for XI, except that the hereditary decidability condition permits the use of $IA_\omega^*[T] = IA_\omega^{(0)}[T]$ in place of $IA_\omega^{(1)}[T;1]$; thence a Σ_1^0 formula B may be constructed. □

3.5 Proof of Theorem XIV.

Recall that IA* = IA augmented with the recursive ω-rule = IA augmented with transfinite induction over all recursive well-orderings. Fix a Σ_2^0 formula $C[x,y]$ and consider the δ_2 forumla $B[x] \equiv \forall y B_0[x,y] \equiv \forall y(C \vee \bar{C})$. Define $P_{C,n} := \{m | C[n,m]\}$ and $N_{C,n} := \{\langle j,m\rangle | C[j,m],\ j \neq n\}$.

Theorem XIV(a) is a consequence of the following two lemmas ((b) is analogous, with a Σ_1^0 formula C).

LEMMA 3.5.1. If $\nvdash_{IL_1} F[\hat{P}]$ and Prov*(d,'F[B]'), then there weakly exists some $n \geq 0$ for which $P_{C,n}$ is weakly recursive in $N_{C,n}$ plus the set $\hat{\Pi}_1^0$ of canonical indexes of true Π_1^0 sentences. If F is hereditarily decidable, then $P_{C,n}$ is weakly recursive in $N_{C,n}$.

LEMMA 3.5.2. There exists a Σ_2^0 formula C s.t. $P_{C,n}$ is not recursive in $N_{C,n} + \hat{\Pi}_1^0$, for any $n \geq 0$.

PROOF. Relativize the proof of Sacks [63] p. 51. □

PROOF OF 3.5.1. For brevity, we shall argue classically. An intuitionistic proof is readily obtained by double-negation translation. Also, we treat only the general case. The modifications needed for an hereditarily decidable F will be obvious.

Assume $\nvdash_{IL_1} F[\vec{P}]$ and Prov*(d,'F[B]'). For a node u in Δ_d , write Γ_0^u for the set of strict subsentences of B_0 in Γ^u , Γ_1^u for $\Gamma^u - \Gamma_0^u$. By the subformula property, all formulas in Γ_1^u are instances of B_0 or schemas in B .

LEMMA 3.5.3. $\exists u Q(u)$, where $Q(u)$ is the conjunction of (i)-(iv) below.

(i) $\Gamma_1^u \not\vdash^B_{IL_1} A^u$

(ii) all sentences in Γ_0^u are true.

(iii) A^u is $B[t]$ for some t, and ρ^u is $\underline{\forall R}$.

(iv) For each node $v \succ u$ in Δ_d, if A^v is a schema in B, then $\Gamma_1^v \vdash^B_{IL_1} A^v$.

PROOF. Analogously to the proofs of theorem III and 3.2.2 one shows, by bar-induction on Δ_d, that if w is a node in Δ_d s.t. A^w is a schema in B and (i), (ii) hold for w in place of u, then $\exists u \geq w Q(u)$. The lemma follows by taking $w = \langle\ \rangle$.

As in 3.3, we write $\exists^* L$ (resp. $\exists^j L$) for an instance of $\exists E$ whose active conclusion is a schema in B (a prenex Σ_j^0 sentence, respectively). Let $R_u := \{k | B_0[t,\bar{k}] \in \Gamma^u\}$.

LEMMA 3.5.4. Assume $Q(u)$, with $A^u \equiv B[t]$. Suppose that $m \notin R_u$. Then there is a sequence $s = (s_1,\dots,s_k)$ satisfying for each $i = 0,\dots,k$ the following properties, where x_i denotes $u * \hat{m} * \langle s_1,\dots,s_i\rangle$.

(i) $\Gamma_1^{x_i} \not\vdash^B_{IL_1} B[t]$;

(ii) $R_{x_i} \subset R_u$;

(iii) all sentences in $\Gamma_0^{x_i}$ are true ;

(iv) A^{x_i} is $B_0[t,\bar{m}]$;

(v) if ρ^{x_i} is $\exists^* L$, then $\bar{s}_{i+1}$ is the first numeral not occurring in σ^{x_i} ; if ρ^{x_i} is $\exists^j L$ ($j = 1$ or 2), with $\exists x E$ say as the active conclusion, then $s_{i+1} = \mu x.E$.

(vi) ρ^{x_k} is $\forall R$.

Moreover, s depends on m recursively, with $N_{C,n}$, and $\hat{\Pi}_1^0$ as oracles (where n is the numeric value of t).

PROOF. We assume known, for $k \in R_u$, whether $C[n,k]$ or $\neg C[n,k]$. Since R_u is finite, this is a finite amount of information. To construct the sequence s, it suffices to prove that for any node $x_i \succeq u*\hat{m}$ that satisfies (i)-(iv), and where ρ^{x_i} is not $\vee R$, there exists an s_{i+1} satisfying (v), and s.t. (i)-(iv) hold for $x_{i+1} := x_i * \hat{s}_{i+1}$. Consider cases for ρ^{x_i}.

The rules T, $\perp$, TE and FE are exluded by (i), (i), (iv) and (iii), respectively.

If ρ^{x_i} is $\rightarrow L$, say

$$\frac{\Gamma, G \rightarrow H: G \qquad \Gamma, H : B_0[t,\bar{m}]}{\Gamma, G \rightarrow H: B_0[t,\bar{m}]}$$

then G is a schema in B, by the subformula property. We assume $Q(u)$; so, by 3.5.3 (iv) applied to $x_i*\hat{0}$, $\Gamma, G \rightarrow H \vdash^{B}_{IL_1} G$. Hence (i) holds for $x_{i+1} := x_i*\hat{1}$, since (i) holds for x_i.

If ρ^{x_i} is $\forall L$ with $B[r]$ as active conclusion, then $r \not\equiv t$ by (i), and so $r \neq t$ by the subformula property (as in 3.3.7). So $x_{i+1} := x_i*\hat{0}$ satisfies (ii).

The case that ρ^{x_i} is another (left) rule, with a schema in B as active conclusion, is straightforward.

If ρ^{x_i} is $\vee L$ with $B_0[t,\bar{k}] \equiv C[t,\bar{k}] \vee \bar{C}[t,\bar{k}]$ as active conclusion, then $k \in R_u$ by (ii). We let $s_{i+1} = 0$ if $C[n,k]$, $s_{i+1} = 1$ if $\neg C[n,k]$. This guarantees (iii) for x_{i+1}.

If ρ^{x_i} is $\vee L$ with $B_0[r,\bar{k}]$ as active conclusion, where r is not t, then $r \neq t$ by the subformula property. We select $s_{i+1} = 0$ or 1 according to whether $C[r,k]$ or $\neg C[r,k]$; this is recursive in $N_{C,n}$.

The sequence s is recursive in $N_{C,n} + \hat{\Pi}_1^0$, since each one of the steps above is. □

<u>LEMMA 3.5.5</u>. Assume $m \notin R_u$, and s a sequence for m given by 3.5.4. If $C[t,\bar{m}]$ is the active premise of ρ^{x_k}, then $C[t,m]$ is true. Similarly for $\bar{C}[t,\bar{m}]$.

<u>PROOF</u>. We prove, by bar-induction on Δ_d, that if $x = x_i \geq u*\hat{m}$ is a node satisfying (i), (iii) of 3.5.4, and s.t. A^x is a strict subformula of $B_0[t]$, then A^x is true.

<u>Basis</u>. By (iii) ρ^x cannot be $\perp$ or FE. If ρ^x is TE, then A^x is trivially true, and if ρ^x is $\underline{T}$, A^x is true by (iii).

<u>Induction Step</u>. The only interesting cases are the following.

(a) ρ^x is $\to$L ; argue as in 3.5.4.

(b) ρ^x is $\vee$L, with $C[r,\bar{k}] \vee \bar{C}[r,\bar{k}]$ as active conclusion. Apply induction assumption to $x*\hat{0}$ or $x*\hat{1}$ accordingly to whether $C[r,\bar{k}]$ is true or false. (Recall our convention to argue classically.) □

<u>PROOF of 3.5.1 - CONCLUDED</u>. Assume $\nvdash_{IL_1} F[\hat{P}]$ and <u>Prov</u>*(d,'F[B]'). Let u be a node in Δ_d s.t. Q(u) (by 3.5.3). Say, A^u is B[t], and n the numeric value of t. For $m \notin R_u$, to decide $m \in P_{C,n}$ use 3.5.4 to construct a sequence s satisfying 3.5.4 (i)-(vi). Then, by 3.5.5 $m \in P_{C,n}$ iff $A^{x_k*\hat{0}}$ is $C[t,\bar{m}]$. Since R_u is finite, this concludes the proof. □

3.6 Predicate logics with equality

Let $IL_1^=$ denote IL_1 with equality, as described in Kl[52]§73, and let $CL_1^=$ denote the corresponding classical system. If $F[\vec{P},=]$ is a predicate formula (with equality), B an arithmetic formula, define $F[B]$ as in 2.1, but with instances of equality left as equality.

PROPOSITION 3.6.1. $CL_1^=$ is not maximal for HA, since none of the sentences

$$(1)_k \qquad \exists v_1 \ldots \exists v_k [v_1 \neq v_2 \;\&\; v_1 \neq v_3 \;\&\; \ldots \;\&\; x_{k-1} \neq x_k]$$

is derived in $CL_1^=$. Then $IL_1^= + (1)$ is not maximal for HA, since

$$(2) \qquad \forall x \forall y [x=y \vee x \neq y]$$

is not a theorem of $IL_1^= + (1)$.

Let $CL_1^{a=}$ stand for $CL_1^= + (1)$, and $IL_1^{a=}$ for $IL_1^= + (1) + (2)$.

THEOREM XV. $CL_1^{a=}$ is $\Delta_2^{\cap}$ and uniformly $\Sigma_2^{\cap}$ and $\Pi_2^{\cap}$ maximal for any arithmetically sound theory.

PROOF. The proofs of theorems VIII and IX clearly apply to $CL_1^{a=}$ (cf. Kl[52] §73 th. 39). □

THEOREM XVI.

(a) $IL_1^{a=}$ is uniformly $\Pi_2^{\cap}$ and $\Sigma_2^{\cap}$ maximal for any regular theory T. $IL_1^{a=}$ is uniformly δ_2-maximal for HA^{Δ}.

(b) $IL_1^{a=}$ is $\Sigma_1^{\cap}$ (δ_1) maximal w.r.t. hereditarily decidable formulas for any regular T (for HA^{Δ}, repsectively). The maximality is uniform in the sense of theorem XIII.

Actually, (b) may be generalized as follows. Let F be a decidable fra[gment] of IL_1 (e.g., $I\Sigma_0$), and let F^{a-} be the (deductive closure of the) un[ion] of F and the purely equational part of $I\Sigma_1^{a-}$. By Lifschitz [67] (lemm[a]) F^{a-} is also decidable. Consequently, the proof of (b) establishes that is uniformly Σ_1^0 maximal for any r.e. subtheory of Π^{Δ}, and uniformly maximal for Π^{*}.

Note that the maximality results for $I\Sigma_1^{a-}$ imply the corresponding r[esults] for $I\Sigma_1$, since $I\Sigma_1^{a-}$ is conservative over $I\Sigma_1$ for equation-free formul[as].

PROOF. We use the notational conventions of 3.3, with some obvious modifications. In amending the proofs for $I\Sigma_1$, we use properties of $IL_1^{a\cdot}$ given in the following three lemmas.

LEMMA 3.6.2. Let $\Gamma, A[x], \Delta$ consist of subformulas of $F[D]$; suppose $\bar{n}_0, \ldots$ are all numerals occurring in Γ, A are among $\bar{n}_0, \ldots, \bar{n}_k$; set $\bar{p} := \max_i[n_i]+1$

(a) If $\Gamma \vdash^{D}_{IL_1^{a-}} A[\bar{n}_i]$ for $i \le k$, and $\Gamma, \{\bar{p} \ne \bar{n}_i\}_{i \le k} \vdash^{D}_{I\Sigma_1^{a-}} A[\bar{p}]$, then $\Gamma \vdash^{D}_{I\Sigma_1^{a-}} \forall x A[x]$.

(b) If $\Gamma, A[\bar{n}_i] \vdash^{D}_{I\Sigma_1^{a-}} \Delta$ for $i \le k$ and $\Gamma, \{\bar{p} \ne \bar{n}_i\}_{i \le k}, A[\bar{p}] \vdash^{D}_{I\Sigma_1^{a-}} \Delta$ then $\Gamma, \exists x A[x] \vdash^{D}_{I\Sigma_1^{a-}} \Delta$.

PROOF. (a) The premises imply that $\Gamma, \{\bar{p} = \bar{n}_i \vee \bar{p} \ne \bar{n}_i\}_{i \le k} \vdash^{D}_{I\Sigma_1^{a-}} A[\bar{p}]$; so $\Gamma \vdash^{D}_{I\Sigma_1^{a-}} A[\bar{p}]$, and hence $\Gamma \vdash^{D}_{I\Sigma_1^{a-}} \forall x A[x]$ (as in 3.3.3.(b)). The proof of (b) is similar. □

LEMMA 3.6.3. For $\Gamma, A[x], C, p$ as above:

(a) If $\Gamma, \{\bar{p} \neq \bar{n}_i\}_{i \leq k} \vdash^{B}_{IL_1^{a=}} A[\bar{p}]$, then $\Gamma \vdash^{B}_{IL_1^{a=}} \exists x A[x]$.

(b) If $\Gamma, \{\bar{p} \neq \bar{n}_i\}_{i \leq k}, A[\bar{p}] \vdash^{B}_{IL_1^{a=}} C$ then $\Gamma, \forall x A[x] \vdash^{B}_{IL_1^{a=}} C$.

PROOF. Use $(1)_{k+1}$. □

LEMMA 3.6.4. Let C be a purely equational Σ_1^0 sentence, with the numerals $\bar{n}_0, \ldots, \bar{n}_k$ as the only terms (other than variables). If C is false, then $D \vdash_{IL_1^{a=}} \neg C$, where $D := \{\bar{n}_i \neq \bar{n}_j\}_{i<j \leq k}$.

PROOF. Straightforward, using axioms (1) and (2). □

We prove theorem XVI for the case of Π_2^0 maximality, by indicating the modifications needed in the proof of theorem X. Other cases of XVI are obtained from the analogous results for IL_1 using similar modifications.

Let $F[\vec{P},=]$ be a predicate formula, B a Π_2^0 formula. We distinguish between the original equality signs in $F[B]$, and the substituted ones. Suppose $\underline{\text{Prov}}^*(d, 'F[B]')$; we similarly distinguish between original and substituted equalities in a sequent σ^u in Δ_d , according to the status of the corresponding ancestor in $F[B]$. Clearly, an original equality equates only variables and/or numerals. An occurrence of a number $\bar{n}$ in σ^u is potent if it is not a descendent of an occurrence of $\bar{n}$ in $F[B]$. The diagram of a node u in Δ_d is the set

$$\mathrm{Diag}(u) := \{\bar{m} \neq \bar{n} \mid m \neq n \text{ and } m,n \text{ are potent at } u \text{ in } \Delta_d\} .$$

Consider now the proof of theorem X in 3.3. The presence of original equations make it possible that propositional rules, $\exists^*L$, $\exists^*R$, $\forall^*L$ and $\forall^*R$ have Σ_1^0 sentences or equations as active premises or conclusions. In the definition of start condition (i) should be amended to

$$\text{(i)}' \quad \Gamma^u , \mathrm{Diag}(u) \not\vdash^{B}_{IL_1^{a=}} A^u .$$

In the proof of the sublemma in 3.3.4, 3.6.2 is to be used in place of 3.3.3 for the cases $\exists^*R$, $\forall^*L$. Lemma 3.6.3 is needed for the cases $\exists^*R$, $\forall^*L$. For the rest of the proof of 3.2.2 the following observation has to be kept in mind. If Start(u) or Goal(u) , then all Σ_1^0 sentences and all equations in Γ^u are true : the substituted ones - by (ii) of Start , and the original ones - by (i)' and 3.6.4. Other aspects of the proof remain unchanged.

Proofs of other maximality theorems are amended in essentially the same way. $\square$

3.7. Maximality of classical logic for set theories.

THEOREM XVII. CL_1 is Δ_2^0 maximal, and uniformly Σ_2^0 and Π_2^0 (whence Δ_0) maximal for any arithmetically sound set theory T.

PROOF. Suppose $F[P_1 \ldots P_k]$ is a sentence not derivable in CL_1. As for theorem VIII, let $B_1,\ldots,B_k$ be Δ_2^0 formulas s.t. $\neg F^\omega[\vec{B}]$,

The idea is to show that $\neg F[\vec{C}]$, where C_i is B_i with each variable $x \notin \omega$ identified with, say, 0.

Let β range over Boolean functions on the subscripts of the variables occurring in F. Given a formula A, write A^β for the result of substituting 0 for free occurrences of x_i, if $\beta(i) = 1$. Write $\vec{x}\varepsilon_\beta\omega$ if $x_i\varepsilon\omega$ when $\beta(i) = 0$. Let $C_i :\equiv \bigwedge\beta[\vec{x}\varepsilon_\beta\omega \to B_i^\beta]$; clearly, each C_i is Δ_2^0. For a subformula G of F, let $G^+ :\equiv \bigwedge\beta[\vec{x}\varepsilon_\beta\omega \to G^\beta[C_i]]$; let G^ω be $\forall\vec{x}\varepsilon\omega$ in front of G, with all quantifiers restricted to ω. It is easy to see, by induction on G, that $G^+ \leftrightarrow G^\omega[B]$. In particular, since F is a sentence, $\neg F[\vec{C}] \leftrightarrow \neg F^\omega[\vec{B}]$. Hence CL_1 is Δ_2^0 maximal for any arithmetically sound set theory. The uniform Σ_2^0 and Π_2^0 maximality follows by uniformization, as in theorem IX. □

When equality is considered a logical constant, the situation is radically different. It seems unlikely that any formula underivable in $CL_1^{a=}$ can be falsified in the universe of sets by definable relations, not to speak of arithmetical relations. However, this does not preclude a proof of maximality (for ZF, say), using a technique different from the above, and possibly closer to the proof-theoretic analysis used in 3.3 for intuitionistic systems.

However, the technique of theorem XVII may be applied to $CL_1^{a=}$, provided the language of the set theory considered is suitable augmented. Let σ be a fixed unary function letter, and let ZF_σ stand for ZF extended with the new axiom $\forall x[x \neq \emptyset \rightarrow \sigma(x) \in x]$ (i.e., σ is a global choice function). U. Felgner [71] has proved that ZF_σ is conservative over ZFC. It is well known that in ZF_σ there is a well-ordering $<_\sigma$ of the whole universe V, definable in σ. E.g., $<_\sigma$ may successively well-order $V_{\alpha+1} - V_\alpha$, for α running through the ordinals. Using $<_\alpha$, the construction used to prove the upwards Skoleur-Löwenheim theorem yields:

<u>LEMMA 3.7.1</u>. If $\vdash_{CL_1^{a=}} F[P_1,\ldots,P_k]$, then there are formulas $B_1,\ldots,B_k$, Δ_2 in σ, s.t. $\neg F[B_1,\ldots,B_k]$.

From 3.7.1 we conclude, as in the proof of theorem XVII:

<u>THEOREM XVIII</u>. $CL_1^{a=}$ is uniformly Δ_2 maximal for ZF_σ. □

(Note that uniformization adds a <u>numeric</u> existential quantifier.)

Actually, the substitutions may be reduced to ones Δ_2 in a linear (not necessarily well-founded) ordering $\prec$ of V, since H. Friedman has proved (unpublished) that a counter-model to an underivable formula may be defined already from such an ordering. Also, we may avoid global choice (which is strictly stronger in GNB than local choice, cf. Easton [64]), but at the cost of a drastic inflation of the language with a choice-function σ_α for V_α, for each ordinal α. The argument proceeds as before, via the Lévy-Montague Reflection Theorem (Lévy [60]).

Note that maximality fails if the language is enriched on the side of logic. Namely, if ϵ is considered a logical constant (as $=$ is in $CL_1^=$), i.e., if models are required to be transitive, then using standard methods one defines a formula $F[P,=,\epsilon]$ formalizing "P is a truth definition for the language of ZF." As noted by Lévy [60], §5, $\vdash_{ZF} \neg F[B,=,\epsilon]$ for each formula B of the language of ZF, while clearly $\nvdash_{CL_1^{a=}} \neg F[P,=,\epsilon]$, since $\exists P\, F[P,=,\epsilon]$ is provable in GNB.

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