PROOF THEORETIC MAXIMALITY

OF LOGICAL CALCULI

by

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0. INTRODUCTION

0.1. Proof theoretic maximality

While the completeness of classical first-order logic for validity in structures has been proved by Gödel in 1930, the completeness problem for intuitionistic logic is ramified and not quite settled as yet. Its significance itself is subject to debate. Should the set theoretic notion of a first order structure (domain + relations) be regarded as the definitive intuitionistic notion of semantics? Completeness for validity in structures being, as it is, inconsistent with Church's Thesis (see Tr [77], p. 49; Le [76]), which one of the two is intuitionistically incorrect? Et cetera. (See Tr [77] for review and references.)

One alternative approach to the intuitionistic completeness problem is to veer to semantics that are non-standard, in the sense that the logical constants are themselves subject to a semantic explication. These notions, widely investigated in a classical metamathematical setting, narrow down from pseudo-Boolean valued models, through topological to Beth and Kripke models.
Another approach, attempting to preserve the neutral role of the logical constants, is to consider provability in theories in place of satisfaction in structures. Namely, one is concerned with proof theoretic maximality properties of logical calculi, probing whether each non-tautology has an (arithmetic) instance that is not provable in a given mathematical theory $T$. The stronger $T$, the stronger the maximality property. It is not necessary that the theory $T$ be "formal" (r.e.); e.g., for classical logic $T$ may consist of all true formulas of arithmetic (and maximality reverts to Hilbert-Bernays completeness); for intuitionistic logic, $T$ may be $\text{IA}^* = \text{HA}^*$. Intuitionistic (Heyting's) Arithmetic $\text{IA}$ augmented with the recursive omega rule = $\text{IA}$ augmented with transfinite induction over all recursive well-orderings, a theory to which we return below.

The first explicit maximality theorems were discovered by de Jongh [73] and Hyhill [72], for intuitionistic propositional logics. (For classical propositional logic maximality is implicit in Kripke [63].)

**Theorem** (de Jongh). If a propositional formula $F[P_1, \ldots, P_k]$ is underivable in Intuitionistic (Heyting's) Propositional Logic, then there are arithmetic sentences $A_1, \ldots, A_k$ for which $F[A_1, \ldots, A_k]$ is not a theorem of Intuitionistic (Heyting's) Arithmetic.

0.2 Restricted and uniform maximality.

Let $S$ be a set of arithmetical relations.

**Definition.** A logical calculus $L$ is $(S)$ maximal for an arithmetic theory $T$ if, when $F[P_1, \ldots, P_k]$ is a formula in the language of $L$, with all predicate
letters among \( P_1, \ldots, P_k \), and \( \forall L F[P_1, \ldots, P_k] \), then \( \forall T F[A_1, \ldots, A_k] \) for some arithmetical relations \( A_1, \ldots, A_k \) (in S). Here each \( A_i \) is of the same arity as \( P_i \), and \( F[A_1, \ldots, A_k] \) stands for the result of replacing in \( F \) each atomic subformula \( P_i(t_1, \ldots, t_{r_i}) \) by \( A_i[t_1/x_1, \ldots, t_{r_i}/x_{r_i}] \).

The maximality is uniform if there is a sequence \( \{A_i\}_1 \) (from S), s.t. for any formula \( F \) as above, \( \forall T F[A_1, \ldots, A_k] \) whenever \( \forall L F[P_1, \ldots, P_k] \).

A greater uniformity of substitution is obtained when \( A_i[x] \equiv A[I, x] \) for some formula \( A \) (provably in Primitive Recursive Arithmetic, say). All uniform maximality theorems we shall prove are of this, stronger, type.

In §1 below we exhibit a syntactic technique, using self-referential formulas, for deriving uniform maximality from (local) maximality. The technique is based on Kripke [63], where it is proved that classical propositional logic is uniformly \( T_1^0 \) maximal for consistent r.e. theories.

In analogy to strong semantic completeness, we define \( L \) to be strongly (S) maximal for \( T \) if, whenever \( T \models_L F \), there are (S) instances \( \Gamma^a, F^a \) of \( \Gamma, F \), respectively, s.t. \( \Gamma^a \models_T F^a \). Clearly, uniform (S) maximality implies strong uniform (S) maximality.

A maximality property is said to be with respect to a set \( P \) of formulas if the property holds for logical formulas \( F \) from \( P \).

REMARK. Kreisel's Basis Theorem is a powerful mean for reducing to \( \Delta_2^0 \) the level in the arithmetical hierarchy of relational instances of \( \Pi_1^0 \) properties. In our context, \( \forall F[A] \) is a \( \Pi_1^0 \) property of the relation \( A \), and one may be tempted to infer that maximality automatically implies \( \Delta_2^0 \).
maximality. However, \( \{ A \mid \not \exists F[A] \} \) is a set of syntactic objects (or, equivalently, their codes); the corresponding set of characteristic functions is not even arithmetical.

### 0.3. Maximaliy in relation to semantic completeness.

Clearly, completeness of a logical calculus \( L \) for arithmetic models, relative to a semantic \( \sigma \), implies maximality of \( L \) for any arithmetic theory \( T \) that is \( \sigma \)-sound. On the other hand, \( L \) may be very "incomplete" while maximal for a sufficiently weak \( T \). For instance, de Jongh's theorem implies that Minimal (i.e., negation free) Propositional Logic is maximal for Minimal Arithmetic.

At the other end of the spectrum, maximality for as powerful a theory as \( \text{IA}^\# \) is of particular interest. The classical analogue of \( \text{IA}^\# \) is complete for (classical) truth in the standard model (Kr,Sh,Wa [60]). Also, assuming Church's Thesis \( \text{CT}_0 \) (cf. Tr [73]), \( \text{IA}^\# \) is complete for prenex formulas. Finally, \( \text{IA}^\# \) encompasses the arithmetic fragments of virtually all classically sound typed intuitionistic theories, and, in particular, the Intuitionistic Impredicative Theory of Types (Le [75]).

A proof theoretic analogue to semantic completeness was proposed already by Hilbert [30], who conjectured that there exists a suitable (sound) formal theory \( T \) disproving some instances of any non-theorem of classical predicate logic. Contrary to maximality, this property does imply the semantic completeness of predicate logic, and for the Finitistic Philosophy of Mathematics advocated by Hilbert, it has the virtue of not implicitly referring to satisfaction in infinite structures.
As it turned out, the conjecture fails for any classically sound r.e. theory T. If true, provability in Predicate Logic could have been decided by a simultaneous enumeration of proofs of Predicate Logic and of proofs of T.

0.4 Maximal properties of propositional logics.

To state our results for intuitionistic logics in full generality, say that a theory T is regular if its arithmetic fragment is an extension of IA with an r.e. set of schemas of transfinite induction over recursive well-orderings. As mentioned, theories as strong as Intuitionistic Impredicative Analysis in all finite types are regular. Clearly, any r.e. subtheory of IA is a subtheory of some regular theory.

We prove that intuitionistic propositional logic IL is uniformly $\Sigma_1^0$ maximal for any regular theory, and therefore for any r.e. subtheory of IA (theorem II). The substitutions may be replaced by any type of disjunction of one-quantifier sentences (corollary II.1). Independence of $\Sigma_1^0$ sentences over IA, although not assured by independence over Classical (Peano's) Arithmetic CA (proposition 2.2), is guaranteed by another simple condition (theorem III).

For the non r.e. theory IA, IL is maximal with substitutions of the form $\forall x [C \lor - C]$, where $C$ is $\Sigma_1^0$ (theorem XIV(b)).

For (non-regular) extensions T of IA by familiar schemas, we find that IL is uniformly $\Sigma_2^0$ maximal for IA+M+IP (theorem V), and uniformly $\Sigma_1^0$ maximal for IA+IP and for IA+CT, (theorems VI, VII). Here M stands for Markov's Schema, and IP for the Schema of Independence of Premiss (closed instances only in both cases).
Classical Propositional Logic $CL_0$ is characterized as the smallest propositional logic maximal for intuitionistic provability over quantifier free sentences: a propositional formula $F[p_1, \ldots, p_k]$ is a tautology iff all instances of $F$ with $0\cdot 0$ and $0\cdot \overline{1}$ substituted for $p_1, \ldots, p_k$ are theorems of IA. By de Jongh's Theorem, $IL_0$ is characterized as the smallest propositional logic maximal for IA with $\Sigma^0_1$ substitutions. Is there any propositional logic intermediary between $IL_0$ and $CL_0$ that can be characterized as the smallest propositional logic maximal for some regular theory, with substitutions of a given syntactic structure? We find that, for reasonable classes of substitutions, there is exactly one such logic, namely,

$$IL_0 + \{ \gamma \rightarrow p \cdot p | p \text{ a propositional variable} \},$$

which is maximal for a regular theory with $\Pi^0_1$ substitutions (theorem IV).

0.5 Maximal properties of predicate logics.

The local $\Delta^0_2$ maximality of Classical First Order Logic (without equality) for any sound theory is immediate by Hilbert-Bernays' [39] version of the Completeness Theorem. A slight refinement establishes maximality for any $\Sigma^0_1$-sound extension of Peano's Arithmetic (theorem VIII). The uniform, and hence strong, maximality for any $\Sigma^0_1$-sound r.e. extension of Peano's Arithmetic, with either $\Sigma^0_2$ or $\Pi^0_2$ substitutions, follows by uniformization (theorem IX).
Intuitionistic Predicate Logic is uniformly, and hence strongly maximal for any regular theory, with either $\Sigma_2^0$ or $\Pi_2^0$ substitutions (theorems X, XI). Although not $\Sigma_1^0$ maximal for Heyting's Arithmetic IA (Le [76]), Intuitionistic Predicate Logic is $\Sigma_1^0$ maximal (for regular theories) w.r.t. formulas satisfying certain decidability properties (theorem XIII), and, in particular, w.r.t. prenex formulas (corollary XIII.1). Intuitionistic Logic is also uniformly maximal for IA*, with substitutions of the form $\forall x[A \lor B]$, where $A$ is $\Sigma_2^0$ and $B$ is $\Pi_2^0$. For formulas satisfying the forementioned decidability properties $A$ may be $\Sigma_1^0$ and $B$ $\Pi_1^0$ (theorem XIV).

The main technique used in the proofs is a combinatorial analysis of recursive cut-free infinitary proofs (i.e., proofs using the recursive omega-rule). Similar methods were applied elsewhere to derive other metamathematical properties of Intuitionistic Arithmetic (Le [80], [80a], [A]; Fr [A]).

Consider now predicate logics with Equality. Augmenting Classical Logic with axioms stating the existence of an unbounded number of distinct elements, one obtains uniform $\Sigma_2^0$ and $\Pi_2^0$ maximality for sound theories. To obtain the maximality properties listed above for Intuitionistic Logic with equality, one has to further augment the logic with an axiom stating the decidability of equality (theorem XVI).
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1. **Uniformization of Maximality**

   We use $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{A}$ to denote the languages of propositional logic, first-order predicate logic (without equality) and first-order arithmetic, respectively. It is convenient to assume that $\mathcal{A}$ contains symbols for all primitive recursive functions, and that theories in $\mathcal{A}$ (like Peano's or Heyting's Arithmetic) contain the appropriate defining equations as axioms; it is well-known that such extensions are conservative. $\mathcal{IL}_0$, $\mathcal{IL}_1$ and $\mathcal{IA}$ will denote Heyting's calculi for intuitionistic propositional logic, predicate logic and arithmetic, respectively. $\mathcal{CL}_0$, $\mathcal{CL}_1$ and $\mathcal{CA}$ will denote the corresponding classical calculi (so $\mathcal{CA}$ is Peano's Arithmetic).

   Let $A$ be an arithmetic formula with $x$ as the only free variable.

   Given a predicate formula $F[r_1, \ldots, r_k]$ we write $F[A]$ for

   $$F[A(<\bar{x}, x_1, \ldots, x_{r_1}>/x), A(<\bar{x}, x_1, \ldots, x_{r_2}>/x), \ldots, A(<\bar{x}, x_1, \ldots, x_{r_k}>/x)].$$ 

   Here $\cdots$ is a canonical sequence coding.

   If $S$ is a set of arithmetic formulas (e.g., $\Sigma_1^0$), then a theory $T$ in $\mathcal{A}$ is **S-sound** if a formula $F$ is true whenever $F \in S$ and $\vdash_T F$.

**Theorem 1.** (Uniformization of maximality.) Let $T$ be a $\Sigma^0_r$ extension of $\mathcal{IA}$ ($r \geq 1$), complete for $\Sigma^0_r$ sentences, and $\neg \gamma \Sigma^0_r$-sound. Let $H = \{F_n\}$ be a $\Sigma^0_r$ set of (codes of) predicate formulas; i.e., the function $\lambda n. F_n$ is recursive in $\Pi^0_{r-1}$. Suppose there is a $\Sigma^0_s$ ($s \geq r$) formula $B(n, x)$ of $\mathcal{A}$ satisfying, for each $F_n \in H$, $\vdash_T F_n[B_n]$, where $B_n \equiv B(n, x)$. Then there is a $\Sigma^0_s$ formula $C(x)$ s.t. $\not\vdash_T F_n[C]$ for all $F_n \in H$. 
PROOF. Let \( \text{Prov} \) be a canonical \( \Sigma^0_{T-1} \) proof predicate for \( T \), and let
\[
\text{Pr}(x) := \exists p \ \text{Prov}(p,x). \quad \text{Using diagonalization (cf. Kr,Le[68]), define a}
\]
\( \Gamma^0 \) formula \( J \) for which the following is provable in \( \text{IA} \).
\[
(1) \quad J(n) \leftrightarrow \exists p(\text{Prov}(p, \neg \text{Pr}(F_n[B_n]^n) \rightarrow \neg J(\tilde{n})))
\]
\[
\land \forall q,m < p,n \ [\neg \text{Prov}(q, \neg \text{Pr}(F_m[B_m]^m) \rightarrow \neg J(\tilde{m}))]
\]
Here \( \leftrightarrow \) is a canonical pairing function, and the bounded quantifier is a self-explanatory abbreviation. Also, \( \neg \text{Pr}(F_m[B_m]^m) \rightarrow \neg J(\tilde{m}) \) stands for an expression \( g(m) \), where \( g \) is a suitable primitive recursive function.

Since \( T \) contains \( \text{IA} \),
\[
(2) \quad \vdash T J(n) \land J(m) \rightarrow n=m.
\]
Define \( C(x) := \exists n(J(n) \land B(n,x)) \). Then (2) implies
\[
(3) \quad J(\tilde{n}) \vdash T \forall x(C(x) \leftrightarrow B(\tilde{n},x)).
\]
Assume
\[
(4) \quad \vdash T F_n[C].
\]
Since \( \vdash T F_n[B_n] \) by assumption, (3) implies \( \vdash T J(\tilde{n}) \). But \( T \) is complete for \( \Sigma^0 \) sentences, so \( J(\tilde{n}) \) is false.

The argument is formalizable in \( T \). First, if \( \vdash T F_n[C] \), then
\[
\vdash T \text{Pr}(F_n[C]^n). \quad \text{Formalizing the rest of the argument yields } \neg \text{Pr}(F_n[B_n]) \vdash T \neg
\]
by a proof with some code \( p \). By (1), enumerating pairs \( \langle q,m \rangle < \langle p,n \rangle \)
weakly gives some \( m \) (i.e., \( \neg \exists m \) for which \( J(\tilde{m}) \) is true; in particular,
(5) \[ \neg \Pr(\overline{F_{m} B_{m}}) \models T \models J(\overline{m}) . \]

Since \( J \) is \( L_{T}^{0} \), we also have \( T \models J(\overline{m}) \), and so, by (5), \( T \models \neg \Pr(\overline{F_{m} B_{m}}) \).

By the \( \gamma \gamma \) \( L_{T}^{0} \)-soundness of \( T \) this implies \( \models T \models \overline{F_{m} B_{m}} \), contradicting the theorem's assumption. Thus (4) fails.
2. MAXIMALITY OF PROPOSITIONAL LOGICS

2.1 Maximaliy theorems.

As mentioned above, the uniform $\Sigma^0_1$ maximality of classical propositional logic (for any consistent r.e. theory) was proved by Kripke [63]. This implies uniform $\Pi^0_1$ maximality, and no room is left for improvement.

Turning to Intuitionistic Propositional Logic $IL_0$, we note that

$$\models_{IA} \neg \neg F \rightarrow F$$

for any arithmetic sentence $F$ with no strictly positive occurrences of disjunction or $\exists$. The simplest substitutions for which $IL_0$ may be maximal (at least for $IA$) are, therefore, $\Sigma^0_1$ sentences, and binary disjunctions of $\Pi^0_1$ sentences, that we name $\beta$-sentences.

**THEOREM II. (Uniform $\Sigma^0_1$ maximality of Intuitionistic Propositional Logic)**

$IL_0$ is uniformly $\Sigma^0_1$ maximal for any regular theory $T$, and hence - for any r.e. subtheory of $IA^*$. Moreover, given a regular theory $T$, there is a $\Sigma^0_1$ formula $B$ s.t. $\models_T F[B]$ implies $\models_{IL_0} F$ for any propositional formula $F$.

We postpone the proof to §2.5. Local $\Sigma^0_1$ maximality was first proved by de Jongh and Smoryński (cf. Smoryński [73]).

**COROLLARY II.1.** $IL_0$ is uniformly $\beta$ maximal for any regular $T$. Moreover, $IL_0$ is uniformly maximal for $T$ with any type of disjunction of one-quantifier formulas as substitutions. Namely, any substitutions of one of the forms

$$B_{21} \lor B_{21-1}, \neg B_{21} \lor B_{21-1}, \neg B_{21} \lor \neg B_{21}, \neg B_{21} \lor \neg B_{21-1},$$

$$\neg B_{21} \lor \neg B_{21-1}, \neg B_{21} \lor B_{21-1},$$

are acceptable.
where $B$ is the formula of the theorem. (Note that $\neg B_1$ is equivalent in IA to a purely universal sentence.)

**Proof.** Let each $H_i$ be one of the disjunctions above, and assume that 
\[ \vdash_{F} F[H_1, \ldots, H_k]. \] Then, by the theorem, $\vdash_{IL_0} F[A_1, \ldots, A_k]$, where each $A_i$ is one of the propositional formulas
\[ q_{2i} \lor q_{2i-1}, \quad q_{2i} \land q_{2i-1}, \quad q_{2i} \land \neg q_{2i-1} \quad \neg q_{2i} \lor \neg q_{2i-1}, \quad \neg q_{2i} \land q_{2i-1}, \quad \neg q_{2i} \land \neg q_{2i-1}, \]
corresponding to the form of $H$. For each sentence $A_i$, $\vdash_{IL_0} \neg \neg A_i$ and $\vdash_{IL_0} \neg \neg A_i + A_i$. So, by Le [80], $\vdash_{IL_0} F$.

**Corollary II.2.** $IL_0$ is strongly maximal for any r.e. subtheory of $IA^*$, with any one of the substitutions above.

### 2.2 Tests for propositional independence.

At least for $\Sigma^0_2$ sentences propositional independence over Intuitionistic Arithmetic implies propositional independence over Classical Arithmetic. Suppose that $A_1, \ldots, A_k$ are intuitionistically independent $\Sigma^0_2$ sentences, and $\vdash_{CA} F[A_1, \ldots, A_k]$, where the propositional schema $F$ is not a classical tautology; then $\vdash_{IA} F^+[A_1, \ldots, A_k]$, where $F^+$ is the double negation translation of $F$, since for a $\Sigma^0_2$ sentence $A$, $\vdash_{IA} \neg \neg A \iff \neg \neg A^+$. This is a contradiction, since $F^+$ is not an intuitionistic tautology.

The converse is more problematic. H. Friedman [73] showed that $\Pi^0_2$ sentences independent in CA over all true $\Pi^0_1$ sentences are also independent
in IA (see corollary X below). He also conjectured that already for $\Sigma^0_1$ sentences classical independence ensures intuitionistic independence. This is true for a single $\Sigma^0_1$ sentence (corollary III), but fails already for two sentences.

**PROPOSITION 2.2.** There are $\Sigma^0_1$ sentences $A, B$ that are independent over $CA$, but not over $IA$.

**PROOF.** By Kripke [63] there are $\Sigma^0_1$ sentences $A, C$, each independent of $CA$, s.t.

$$C \vdash_{CA} \neg A.$$ 

Let $D$ be independent over $CA+A$, and let $B := A\&D \lor C$. Then the set $\{A, B\}$ is independent over $CA$:

(i) If $A \vdash B$, i.e. $A \vdash A\&D \lor C$, then $A \vdash D$ by (1), contradicting the choice of $D$.

(ii) If $B \vdash A$, then $C \vdash A$, so $\vdash \neg C$ by (1), contradicting the choice of $C$.

(iii) If $B \vdash \neg A$, then $A\&D \vdash \neg A$, so $A \vdash \neg D$, contradicting the choice of $D$.

On the other hand, the set $\{A, B\}$ is not independent over $IA$:

$$A\&D \vdash \neg A \text{ and } C \vdash \neg A, \text{ so } \vdash \neg B \lor A \lor \neg A.$$

A similar counter-example was discovered independently by D.H.J. de Jongh.

While intuitionistic independence of $\Sigma^0_1$ sentences is not guaranteed by classical independence, it is guaranteed by certain relatively simple test cases. Define a sequence $G_n[p_1, \ldots, p_n]$ of propositional formulas inductively, as follows.
$G_k[p] := p \lor \neg p$; $G_{k+1}[p_1, \ldots, p_{k+1}] := \bigvee_{i=1}^{k+1} (p_i \lor G_i^k)$.

where $G_i^k := G_k[p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{k+1}]$.

**Theorem III.** (Test cases for intuitionistic independence) Let $T$ be a regular theory, and let $A_1, \ldots, A_k$ be $\Pi^0_1$ sentences. Suppose $F[p_1, \ldots, p_k]$ is a propositional formula not derivable in $IL_0$, and $\vdash_T F[A_1, \ldots, A_k]$.

Then $\vdash_T G_k[A_1, \ldots, A_k]$.

**Corollary III.** (de Jongh, Smoryński; cf. Sm [73], 5.3.18) If $A$ is a sentence independent of $T$, and $\vdash_T F[A]$ then $\vdash_{IL_0} F[p]$.

Le [80] gives another generalization of the corollary.

For $\beta$-sentences as substitutions the situation is less clear. We can show, by a disproportionately long and intricate proof that we skip here, that if $U_1, U_1', U_2, U_2'$ are classically independent $\Pi^0_1$ sentences, then the two $\beta$-sentences $U_1 \lor U_1'$ ($i=1,2$) are intuitionistically independent. Our proof fails for three $\beta$-sentences.

2.3 An infinitary sequential calculus for arithmetic

Subformulas and immediate subformulas of an arithmetic sentence are defined as usual. In particular, $F[w/x]$ is an (immediate) subformula of $\forall x F$, for each numeral $w$. We use an infinitary sequential calculus $IA^\omega$ for arithmetic sentences (i.e., closed formulas). By using variants of the omega rule, one obtains cut-free proofs for all theorems of $IA$, that satisfy the subformula property: each formula in the proof is a subformula of the derived formula. No reasonable finitary calculus may be complete for $IA$ and satisfy this property (Le [79], p. 5).
DEFINITIONS. An antecedent is a finite set of sentences. A sequent is a pair $\Gamma : F$, where $\Gamma$ is an antecedent and $F$ is a sentence; $F$ is the succedent of the sequent. $\Gamma ; A$ will stand for $\Gamma u(A) ; \Gamma ; 0$ for $\Gamma u 0$, etc. The inference rules of IA are the following schemas.

(a) PROPOSITIONAL LOGIC:

- $\top : \Gamma , A : A$
- $\bot : \Gamma , \bot : A$
- $\&L: \Gamma , A, B : C \Gamma , A \& B : C$
- $\&R: \Gamma ; A \Gamma ; B \Gamma ; A \& B$
- $\lor L: \Gamma , A : C \Gamma , B : C \Gamma , A \lor B : C$
- $\lor R: \Gamma ; A \Gamma ; B \Gamma ; A \lor B$
- $\rightarrow L: \Gamma , A + B : A \Gamma , B : C \Gamma , A + B : C$
- $\rightarrow R: \Gamma ; A \Gamma ; B \Gamma ; A \rightarrow B$

(b) QUANTIFICATION:

- $\forall L: \Gamma , \forall x, A[\bar{n}/x] : B \Gamma , \forall x A : B$
- $\forall R: \{\Gamma ; A[\bar{n}/x]\}_{n \geq 0} \Gamma ; \forall x A$
- $\exists L: \Gamma , \exists x, A[\bar{n}/x] : B \Gamma , \exists x A : B$
- $\exists R: \Gamma ; A[\bar{n}/x] \Gamma ; \exists x A$

Note that we dispense of all structural rules, in particular - of CUT.

(c) ARITHMETIC:

An equation $E$ between (closed) terms of $IA$ is true if $E$ equates terms with the same numeric value, when each function symbol is interpreted as the primitive recursive function it denotes. It is fairly easy to see that
there exists a recursive function \( \text{eval} \), provably total in \( \text{IA} \), s.t.
\[
\text{eval}(\text{'E'}) = 0 \text{ iff } \text{E is true, where 'E' is a canonical code for E (cf. Try [73], 1.5.3). This shows that coded instances of the following rules can be recursively recognized in \( \text{IA} \).
\]

\[
\begin{align*}
\text{TE: } & \quad \Gamma : E \quad \text{if } E \text{ is a true equation} \\
\text{FE: } & \quad \Gamma, E : A \quad \text{if } E \text{ is a false equation.}
\end{align*}
\]

A tree is well founded if every branch terminates. A derivation of \( \text{IA}^* \) is a well-founded tree of sequents, where at each node the sequents relate according to one of the inference rules above.

Our inference rules are set up to validate the following.

\begin{lemma} \label{lemma:2.3.1}
Let \( \Gamma \) and \( \Gamma' \) be antecedents, and suppose \( \Gamma \) occurs above \( \Gamma' \) in a derivation. Then \( \Gamma \) implies in intuitionistic propositional logic each sentence in \( \Gamma' \).
\end{lemma}

\begin{proof}
Immediate.
\end{proof}

\begin{lemma} \label{lemma:2.3.2}
(subformula property) Let \( \Delta \) be a derivation of \( \text{IA}^* \) deriving the sequent \( \Gamma : A \). Each sentence occurring in \( \Delta \) is a subformula of \( A \) or of some member of \( \Gamma \).
\end{lemma}

\begin{proof}
Let \( u^+ \) be a node in \( \Delta \) other than the root, and let \( u \) be the immediate parent-node of \( u^+ \). Inspection of the inference rules shows that each sentence occurring in the sequent at node \( u^+ \) is a subformula of some sentence occurring at \( u \). The proof is concluded by induction on the height of \( u^+ \) in \( \Delta \).
\end{proof}
Fix a canonical Gödel coding for the syntax of \( \mathcal{A} \) and for sequents. Each proof \( \Delta \) in \( \text{IA}^\omega \) can be represented by a function \( \phi_\Delta \) that assigns codes of sequents to sequence-numbers

\[
\langle \rangle, \langle 0 \rangle, \langle 1 \rangle, \ldots, \langle 0, 0 \rangle, \langle 0, 1 \rangle, \ldots, \langle 1, 0 \rangle, \ldots, \langle 0, 0, 0 \rangle, \ldots
\]

in the universal spread \( U \). W.l.o.g., no code is \( 0 \), and \( \phi_\Delta \) may be assumed total, with \( \phi_\Delta(x) = 0 \) for each \( x \) other than codes of those nodes in \( U \) where \( \Delta \) has a sequent. The statement that a function \( \phi \) represents a proof is formalized as the conjunction of the \( \Pi_1^0 \) predicate \( \text{LC}(\phi) \), stating that \( \phi \) is locally correct, i.e., that the non-zero values of \( \phi \) are codes of sequents relating according to the inference rules, and of the \( \Pi_1^1 \) predicate \( \text{WF}(\phi) \), stating that the non-zero values of \( \phi \) form a well-founded tree.

A recursive proof is a number \( d \) s.t. the recursive function \( (d) \) coded by \( d \) is a proof of \( \text{IA}^\omega \). Let \( \text{IA}^* \) denote the recursive part of \( \text{IA}^\omega \). This notation agrees with the definition of \( \text{IA}^* \) in 0.1 above:

It is easy to see that \( \text{IA}^* = \text{IA} \) extended with transfinite induction over all recursive well-orderings. For most applications, it is not the well-foundedness of a proof \( \Delta \) that one needs, but the schema of Bar Induction over \( \phi = \phi_\Delta \):

\[
\text{BI}[\phi, P(x)] := \forall u (\forall n. \phi(u^n) \neq 0 \rightarrow P(u^n)) \rightarrow \phi(u) \neq 0 \rightarrow P(u) \rightarrow P(\langle \rangle)
\]

Here \( P \) is a predicate letter, and \( ^*, \hat{n}, \langle \rangle \) stand for the concatenation operation on sequence-numbers, the singleton sequence-number \( \langle n \rangle \), and the empty sequence-number, respectively. For a well-founded \( \Delta \), this is a special case of the usual schema of Bar Induction, sometimes referred to as "induction over unsecured sequences."
Well-foundedness of $\phi$, $WF(\phi)$, clearly implies $BI[\phi, P(x)]$, but we shall be concerned with the derivability in given theories of the latter, not of the implication. The formal derivability condition we need is the $\Pi^1_1$ predicate

$$Prov^*(d,'F') := \forall x \exists y T(d,x,y) \& Prov^bi((d),'F')$$

where

$$Prov^bi(\phi,'F') := LC(\phi) \& \forall P BI[\phi, P(x)] \& \phi(\langle \rangle) = 'F'$$

Also,

$$Pr^*(x) := \exists d Prov^*(d,x)$$

Let $T$ be a theory in a language containing $\mathcal{L}$. Write $T^P$ for the second order extension of $T$, with quantification over predicate letters, but with comprehension restricted to arithmetic formulas. $IA^P$ is well known to be conservative over $IA$ (cf. Tr [73], 1.9.6). Define

$$IA^*[T] := \{ F \mid T^P \vdash \exists d Prov^*(d,'F') \}$$

**Lemma 2.3.3.** Let $T$ be a regular theory.

(a) $T^P$ is a conservative extension of $T$.

(b) $T = IA^*[T]$.

**Sketch of proof.** $IA \subseteq IA^*[IA]$ is essentially a cut elimination theorem for $IA^*$ (cf. Lo-Es [68]). The extension to $T \subseteq IA^*[T]$ is straightforward. The converse is by transfinite induction using a restricted truth definition, as in Kr [65], 3.33. Details for the system considered here are provided in Le [A].
It will be convenient to discuss derivations of $\text{IA}^*$ informally, and then to point out that the notions used may be arithmetized, and the arguments formalized in some formal theory (usually $\text{IA}^P$). In informal discussions $u, v, w$ will range over nodes of the universal spread; when $d$ codes a proof of $\text{IA}^*$, $\Delta_d$ will denote that proof, and $\sigma_{d,u}$, $r_{d,u}$, $A_{d,u}$, $\rho_{d,u}$, will denote the sequent of $\Delta_d$ at $u$ (coded by $(d)(u)$), the antecedent and the succedent of $\sigma_{d,u}$ and the (first) rule of inference by which $\sigma_{d,u}$ is correctly derived in $\Delta_d$, respectively. Note that $\rho_{d,u}$ may be found primitive-recursively from $\sigma_{d,u}$ and $\sigma_{d,u^{*n}}$, $n = 0, 1, 2$. The superscript $d$ will be dropped when no ambiguity may arise.

**Lemma 2.3.4.** Assume that all quantifier free subformulas of $F$ are equations in the immediate scope of an existential quantifier. If

$$\text{Prov}^*(d, 'F'),$$

then, w.l.o.g., only top nodes of $\Delta_d$ have equations as succedents. Moreover, there is a primitive recursive function $f$ s.t., provably in $\text{IA}$,

$$\text{Prov}^*(d, 'F') + \text{Prov}^*(f(d), 'F') \& \text{"all succedents in } (f(d)) \text{ that are equations occur at top nodes."}$$

**Proof.** We describe a primitive recursive transformation on $\Delta_d$ yielding $\Delta_f(d)$. Given a node $u$ in $\Delta_d$ s.t. $A^u$ is an equation, let $f(u)$ be the longest initial segment of $u$ s.t. $A^f(u)$ is not an equation. By the subformula property and the lemma's condition, $\rho^f(u)$ is $\not\in R$. Perform successively the following transformations.
1. Replace each \( A^u \) as above by \( A^f(u) \).

2. If \( u \) is a top node, put \( f^u : A^u \) at \( u^* \).

3. Collapse the resulting tree at each node \( f(u) \); i.e., put at each \( f(u)^*w \) the sequent previously at \( f(u)^*w \).

The proof \( \Delta_f(d) \) is easily seen to satisfy the lemma's requirements. \( \square \)

An alternative proof of the lemma uses permutations of inferences (as in KI [52A]), and bar induction on \( \Delta_d \), therefrom the primitive recursive \( f \) is obtained by the S-m-n Theorem.

2.4. Proof of theorem III

Let \( F[p_1, \ldots, p_k] \) be a propositional formula, underviable in \( L_0 \), and let \( E_1, \ldots, E_k \) be purely existential sentences, i.e., of the form \( \exists x G \) where \( G \) is an equation between primitive recursive terms. We prove in \( L^P \) that

\[
(1) \quad \text{Pr}^*(F[E_1, \ldots, E_k]) \implies G_k[E_1, \ldots, E_k].
\]

Hence, for any regular theory \( T \),

\[
T \vdash F[E_1, \ldots, E_k]
\]

\[
\Rightarrow T^P \vdash \text{Pr}^*(F[E_1, \ldots, E_k]) \quad \text{by 2.3.3(b)}
\]

\[
\Rightarrow T^P \vdash G_k[E_1, \ldots, E_k] \quad \text{by (1)}
\]

\[
\Rightarrow T \vdash G_k[E_1, \ldots, E_k] \quad \text{by 2.3.3(a)}.
\]

To prove (1), assume \( \text{Pr}^*(d, F[E_1, \ldots, E_k]) \). By 2.3.4 we may assume that \( A^{d,u} \) is an equation only for top nodes \( u \) of \( \Delta_d \).
We make a number of additional ad hoc notational conventions. As mentioned above, we omit the superscript \( d \). For a node \( u \) in \( \Delta_d \), let \( \theta^u \) denote the set of those \( E_1 \)'s for which \( I^u \models_{ILO} E_1 \), where \( I^u \models_{ILO} \) stands for provability using propositional inferences only. Let \( Z^u \) denote the set of all other \( E_1 \)'s (\( i=1,\ldots,k \)). For \( \Sigma^u = \{ E_{i1},\ldots,E_{im} \} \), let

\[
G(\Sigma^u) := G(E_{i1},\ldots,E_{im})
\]

Note that this definition is independent of the order of the \( E_{ij} \)'s, up to provable equivalence in \( ILO_0 \). Finally, write \( \Gamma^u \) for the set of equations in \( \Gamma^u \).

**Lemma 2.4.1.** If \( A^u \) is not an equation, and

(2) \( \Gamma^u \not\models_{ILO} A^u \),

then

(3) \( (\Gamma_0^u,\theta^u) \models G(\Sigma^u) \),

where \( (\Gamma,\theta) \) stands for the conjunction of all formulas in \( \Gamma,\theta \).

**The Lemma Implies (1):** Take \( u = \langle \rangle \).

**Proof of the Lemma.** By bar induction on \( \Delta_d \).

**Basis.** If \( u \) is a top node in \( \Delta_d \), then \( \rho^u \) is either one of \( T, i, TE \) contradicting the lemma's assumptions, or \( FE \), in which case (3) is trivial.

**Induction Step.** By cases for \( \rho^u \). By the subformula property, the rules for \( \gamma \) are excluded.

**Case (a):** Propositional rules.
Assume $\Gamma^u \models_{\text{IL}_0} A^u$; then, w.l.o.g., $\Gamma^{u\delta} \models_{\text{IL}_0} A^{u\delta}$. By the subformula property, no formula which is an active premise in the inference is an equation, because such a formula must fall in the immediate scope of a propositional connective. Hence, $A^{u\delta}$ is not an equation, for if $A^{u\delta}$ is not active in the inference, then $A^{u\delta} \equiv A^u$, which is not an equation by assumption. By induction assumption applied to $u\delta$,

(4) \((\gamma^u, \theta^{u\delta}, \varepsilon, \Sigma) \rightarrow G[\varepsilon^{u\delta}]\).

By 2.3.1 there is a set $I = \{E_{j_1}, \ldots, E_{j_k}\}$ s.t.

$$\theta^{u\delta} = \theta^u \cup I, \quad \Sigma = \Sigma^{u\delta} \cup I.$$ 

Also, since no formula active in the premise may be an equation, $\gamma^{u\delta}_0 = \gamma^u_0$.

So (4) implies

(5) \((\gamma^u_0, \theta^u, I) \rightarrow G[\varepsilon^{u\delta}]\).

By the definition of the formulas $G_k$, $I \rightarrow G[\varepsilon^{u\delta}]$ implies (in $\text{IL}_0$) $G[I, \varepsilon^{u\delta}]$, i.e., $G[\varepsilon^u]$. So (5) implies (3).

Case (b): $2R$; say $\frac{\gamma^u : A^{u\delta}}{\Gamma^u : A^u}$.

By the subformula property $A^u$ is one of the $\Sigma^0$ sentences $E_1$, and $A^{u\delta}$ is an equation. Thus, $u\delta$ is a top node in $A_\delta$. By (2) $I \not\in \Gamma^u$, so either $A^{u\delta} \in \Gamma^u$ or $A^{u\delta}$ is true, and $\gamma^u_0 \vdash A^u$ in any case. On the other hand, (2) implies $A^u \in \varepsilon^u$, so $A^u \vdash G[\varepsilon^u]$, and (3) follows.
Case (c): \( \exists L \); say

\[
\frac{\Gamma, \exists x B, B[\bar{n}/x] : A^u \cdot \bar{n}}{\Gamma, \exists x B : A^u}
\]

\( \Gamma^u = (\Gamma, \exists x B) \)

Claim: If \( A^u \) is not an equation, and \( \Gamma, \exists x B \models_{IL_0} A^u \), then

\( \Gamma, \exists x B, B[\bar{n}/x] \models_{IL_0} A^u \) for each \( \bar{n} \). To see this observe that, by the subformula property, all formulas in \( \Gamma \) are either equations or propositional schemas: \( E_1, \ldots, E_k \). If \( \Pi \) is a proof in \( IL_0 \) of \( \Gamma, \exists x B, B[\bar{n}/x] : A^u \), then the same statement holds also for \( \Pi \). Deleting the equation \( B[\bar{n}/x] \) from all antecedents in \( \Pi \) leaves a correct \( IL_0 \)-proof of \( \Gamma, \exists x B : A^u \). This proves the claim.

By the claim we may apply induction hypothesis to each one of the premises \( u^n \cdot s \)

\[
(r_0^{u^n}, \theta^{u^n}) \Rightarrow G[z^{u^n}] \quad \text{for all } n .
\]

But the claim also implies that \( \theta^{u^n} = \theta^u \) and \( z^{u^n} = z^u \), while \( r_0^{u^n} = (r_0^u) \), so

\[
(r_0^u, \theta^u, B[\bar{n}/x] \Rightarrow G[z^u]) \quad \text{for all } n ,
\]

i.e.,

\[
(r_0^u, \theta^u, \exists x B) \Rightarrow G[z^u] .
\]

Since \( \exists x B \vDash \theta^u \) trivially,

\[
(r_0^u, \theta^u) \Rightarrow G[z^u] ,
\]

as required. This concludes the proof of the induction step and of the lemma. \( \Box \)
2.5. Proof of theorem II

**Lemma 2.5.1.** Let $T$ be an r.e. $\gamma \vdash T^0$ sound extension of IA. Suppose $T$ satisfies the disjunction property: if $T \vdash A \lor B$ then either $T \vdash A$ or $T \vdash B$. Then, for each $k \geq 1$, there are sentences $B_1, \ldots, B_k$ s.t. $T \not\vdash \gamma C_k[B_1, \ldots, B_k]$.

First we give an explicit construction that does not use the Uniformization Theorem. This technique seems to be of independent interest. A shorter proof is sketched at the end of this section 2.5.

**Sublemma 2.5.2.** Disjoint r.e. sets $R, S$ are effectively inseparable iff there is a total recursive function $f$ satisfying $f(i,j) \neq W_i \cup W_j$ whenever $W_i \cap R = \emptyset$ and $W_j \cap S = \emptyset$.

**Proof.** Routine. □

We use underlined majuscules to denote occurrences of formulas. $F[H/G]$ will denote the result of substituting $H$ for $G$ in $F$.

**Sublemma 2.5.3.** (a) If $G$ is a positive subformula of $F$, then $G \vdash H \vdash F + F[H/G]$.

(a2) If $G$ is negative in $F$, then $G \vdash H \vdash F[H/G] + F$.

(b) Let $p, q$ be fixed propositional letters. Let $F^0 (F^-)$ come from $F$ by simultaneously replacing each occurrence (negative occurrences, respectively) of $p$ by $p \lor q$. Then

\begin{equation}
\gamma q \vdash F^0 \lor F
\end{equation}

**Proof.** (a) is straightforward by induction on the length of $F$. By (a2) we have $F^- \vdash F^0$, and since $\gamma q \vdash p \lor q \lor p$ we get by (a1) $\gamma q \vdash F^0 \lor F^-$, hence (1). □
PROOF OF 2.5.1. Let \( R, S \) be effectively inseparable r.e. sets, and let \( f \) be given by 2.5.2. Shepherdson [60] exhibits a \( \Sigma_1^0 \) formula \( F(x) \) s.t.

\[
R = \{ n \mid \models_T F(\overline{n}) \}; \quad S = \{ n \mid \models_T \neg F(\overline{n}) \}.
\]

The proof is intuitionistic.

We construct, by recursion on \( k \), a sequence \( B_i^k, i \geq 0 \), of \( \Sigma_1^0 \) sentences, satisfying

\[
\models_T G_k[B_{i_1}^k, \ldots, B_{i_k}^k]
\]

for any distinct \( i_1, \ldots, i_k \).

**Basis:** Let \( T \) be a \( \Sigma_1^0 \) Rosser sentence for \( T \); set \( B_i^1 = R \) for all \( i \).

**Induction Step.** Assume (3) for a given \( k \). We define a sequence \( C_m^k (m \geq 1) \) of \( \Sigma_1^0 \) sentences, every Boolean combination \( C^* \) of which satisfies

\[
\models_T C_k[B_{i_1}^k, \ldots, B_{i_k}^k]
\]

for any distinct \( i_1, \ldots, i_k \). By a Boolean combination we mean here a list \( (D_m) \) where each \( D_m \) is either \( C_m^k \) or \( \neg C_m^k \).

The sentences \( C_m^k \) are defined by recursion on \( m \), simultaneously with two prim. rec. functions \( g_k, h_k \), as follows. Assume \( C_j^k, g_k(j), h_k(j) \) are defined for \( j < m \), and let \( g_k(m) \) be defined by

\[
W_{g_k(m)} = \{ n \mid F(\overline{n}), \models_T C_k[B_{i_1}^k, \ldots, B_{i_k}^k] \}
\]

for some Boolean combination \( C^* \) of

\( C_1^k, \ldots, C_{m-1}^k \), and some distinct \( i_1, \ldots, i_k \).

\( W_{h_k(m)} \) is defined likewise, but with \( \neg F(\overline{n}) \) in place of \( F(\overline{n}) \). Let
(6) \( C^k_m \equiv F(f(g_k(m), h_k(m))) \)

We prove (4) by induction on \( m \). The induction assumption for \( m - 1 \) implies, by (2), (5) and the consistency of \( T \) that

\[ W_{g_k(m)} \cap R = \emptyset ; \quad W_{h_k(m)} \cap S = \emptyset. \]

So \( f(g_k(m), h_k(m)) \notin W_{g_k(m)} \cup W_{h_k(m)} \), which by (5) and (6) yields (4) for \( m \).

To conclude the induction step define

(7) \( B^k_{i_1} \equiv a^{0}_{i_1} \) sentence equivalent to \( B^k_{i_1} \lor C^k_{i_1} \).

The proof of the lemma is completed by induction on \( k \). The basis is immediate by the disjunction property of \( T \). To prove the induction step, assume

(8) \( \frac{\vdash}{T} G_{k+1}_{i_1} [B^k_{i_1}, \ldots, B^k_{i_{k+1}}] \)

for some distinct \( i_1, \ldots, i_{k+1} \). By the disjunction property of \( T \) and the definition of \( G_{k+1} \),

\[ \frac{\vdash}{T} B^k_{i_1} + G_{k}[B^k_{i_2}, \ldots, B^k_{i_{k+1}}], \text{ say}. \]

By (7) this implies

\[ C^k_{i_1} \vdash T G_{k}[B^k_{i_2} \lor C^k_{i_2}, \ldots, B^k_{i_{k+1}} \lor C^k_{i_{k+1}}]. \]
And so, by 2.5.2 (b)

\[ C_{i_1}^k \land C_{i_2}^k \land \ldots \land C_{i_{k+1}}^k \models_T G_k[B_{i_2}^k, \ldots, B_{i_{k+1}}^k] \]

contradicting (6).

\[ \square \]

PROOF OF THEOREM II - CONCLUDED.

Let \( T \) be a regular theory. By theorem III and theorem I, there is a \( \Sigma_1^0 \) formula \( B \) s.t. \( \models_T G_k[B] \) for all \( k \). Assume that \( \models_T F[B] \), where \( F \) is a propositional formula. Suppose that \( F \) is undervisible in \( \text{IL}_0 \). Then, by theorem III, \( \models_T G_k[B] \), contradicting 2.5.1. This shows that \( F \) is not unprovable in \( \text{IL}_0 \); since \( \text{IL}_0 \) is decidable, \( F \) is provable.

\[ \square \]

REMARK. Uniformization may be used to obtain a shorter proof of 2.5.1. Assume 
\[ \models_T G_k[B_1, \ldots, B_k] \]  
Then \[ \models_T B_{k+1} \rightarrow G_k[B_1, \ldots, B_k] \] for \( B_{k+1} := (0=0) \). By uniformization, there are \( \Sigma_1^0 \) sentences \( B_1', \ldots, B_{k+1}' \) s.t. \( \models_T B_i' \rightarrow G_k[B_1', \ldots, B_{k+1}'] \) for \( i=1, \ldots, k+1 \). So \[ \models_T G_{k+1}[B_1', \ldots, B_{k+1}'] \]. We use this technique again in 3.4.
2.6. **Classification of propositional logics by their maximality properties.**

**DEFINITION.** A frame is a predicate letter formula \( \phi \) in which some predicate letter occurs, but no predicate letter occurs twice. An arithmetic formula \( F \) is a case of \( \phi \) if \( F \) is the result of substituting equations for the atomic subformulas of \( \phi \). Write \( \psi \subset \phi \) if each case of \( \psi \) is equivalent in \( \text{IA}_0 \) (:= IA with induction restricted to quantifier free formulas) to some case of \( \phi \). Say that \( L \) is \( \phi \)-maximal (for \( T \)) if \( L \) is maximal (for \( T \)) with cases of \( \phi \) as substitutions.

**LEMMA 2.6.1.** Suppose a logical calculus \( L \) is \( \phi \)-maximal for an extension \( T \) of \( \text{IA} \). If \( \psi \subset \phi \), then \( L \) is \( \phi \)-maximal for \( T \).

**PROOF.** Immediate.

We classify the frames into classes \( \phi_0, \phi_1, \phi_2 \).

\( \phi \subset \phi_0 \) if there is a strictly positive non-vacuous occurrence of \( g \) in \( \phi \), or if there is a strictly positive occurrence of disjunction in \( \phi \), where each one of the disjuncts contains a non-vacuous occurrence of a quantifier.

\( \phi \subset \phi_1 \) if \( \phi \not\subset \phi_0 \), and there is a non-vacuous occurrence of a quantifier in \( \phi \).

\( \phi \subset \phi_2 \) in all other cases.

Also, define a set of four frames:

\[ \phi := \{ \exists x P(x), \forall x P(x) \lor \forall y Q(y), \forall x P(x) \lor \forall y Q(y), \forall x P(x) \lor \forall y Q(y) \} \]
**LEMMA 2.6.2.** (i) If \( \phi \in \phi_0 \) then \( \psi \in \phi \) for some \( \psi \in \phi \).

(ii) If \( \phi \in \phi_1 \) then \( \psi \in \phi \) for \( \psi \equiv \neg \exists x P(x) \) or \( \neg \neg \exists x P(x) \).

**PROOF (i) Case (a).** \( \phi \) has a strictly positive subformula \( \exists x \phi_0 (x) \) where \( \phi_0 \) has an atomic subformula \( R(x, y) \). Given a \( \exists_1^0 \) sentence \( A \equiv \exists x A_0 \), \( A \) is clearly equivalent (in \( \mathcal{L}_A^0 \)) to the result of substituting \( A_0 \) for \( R(x, y) \), if \( R \) is positive in \( \phi_0 \), of substituting an equation equivalent to \( \gamma A_0 \) for \( R \), if \( R \) is negative in \( \phi_0 \), and of suitably substituting \( \delta = \delta \) or \( \delta = \bar{1} \) for other predicate letters in \( \phi \). Recall that predicate letters may occur only once in \( \phi \), so no clashes arise. Thus \( \exists x P(x) \in \phi \).

Case (b): Otherwise, and \( \phi \) has a strictly positive subformula of the form \( \phi_0 \vee \phi_1 \), where each \( \phi_1 \) contains a non-vacuous quantifier. By suitable substitutions, as for case (a), we find that \( \phi_1 \in \forall x P(x) \) or \( \neg \forall x P(x) \).

The proof of (ii) is similar. \( \square \)

Let \( \mathcal{L}^+_0 \) denote \( \mathcal{L}_0 \) extended with all sentences \( \neg \phi \vee \phi \) as new axioms. Note that \( \models_{\mathcal{L}^+_0} \phi \vee \neg \phi \).

**LEMMA 2.6.3.** (i) If \( \models_{\mathcal{L}^+_0} F[\gamma_1 p_1, \ldots, \gamma_k p_k] \), then \( \models_{\mathcal{L}^+_0} F[p_1, \ldots, p_k] \).

(ii) If \( \models_{\mathcal{L}_0} F[\gamma p_1, \ldots, \gamma p_k] \), then \( \models_{\mathcal{L}_0} F[p_1, \ldots, p_k] \).
PROOF: (i) is trivial, since $\vdash_{\mathcal{L}_0} p_1 \leftrightarrow \neg \neg p_1$. To prove (ii), assume $\vdash_{\mathcal{L}_0} F[q_1, \ldots, q_k]$. Then $\vdash_{\mathcal{L}_0} F[\neg \neg p_1, \ldots, \neg \neg p_k]$, which by (i) implies $\vdash_{\mathcal{L}_0} F[p_1, \ldots, p_k]$. \hfill $\square$

THEOREM IV. Let $\phi$ be a frame.

(i) If $\phi \in \Phi_0$, then $\mathcal{L}_0$ is $\phi$-maximal for any regular theory $T$.

(ii) If $\phi \in \Phi_1$, then no strict subtheory of $\mathcal{L}_0^+$ (in particular $\mathcal{L}_0$) is $\phi$-maximal even for $\mathcal{I}_0$; but $\mathcal{L}_0^+$ is $\phi$-maximal for any regular theory $T$.

(iii) If $\phi \in \Phi_2$, then no strict subtheory of $\mathcal{C}_0$ (in particular $\mathcal{L}_0$) is $\phi$-maximal for $\mathcal{I}_0$, but $\mathcal{C}_0$ is $\phi$-maximal for any regular $T$.

PROOF: (i) follows from theorem II, by 2.6.1 and 2.6.2(i).

(ii) If $\phi \in \Phi_1$, then, in $\mathcal{I}_0$, $\neg \neg A \rightarrow A$ for each case $A$ of $\phi$; so, no strict subtheory of $\mathcal{L}_0^+$ is $\phi$-maximal even for $\mathcal{I}_0$. On the other hand, given $T$ and a propositional formula $F[p_1, \ldots, p_k]$, there are (by theorem I and 2.6.3) $\Xi_1$ sentences $A_1, \ldots, A_k$ s.t. $\vdash_T F[\neg A_1, \ldots, \neg A_k]$ implies $\vdash_{\mathcal{L}_0} F[p_1, \ldots, p_k]$. Thus $\mathcal{L}_0^+$ is ($\neg \Xi P(x)$) - maximal, and likewise also $\mathcal{L}_0^+$ ($\neg \Xi P(x)$) - maximal. So, by 2.6.2(ii) and 2.6.1 $\mathcal{L}_0^+$ is $\phi$-maximal.

(iii) If $\phi \in \Phi_2$, then each case $A$ of $\phi$ is equivalent in $\mathcal{I}_0$ to either $\bar{\delta} = \bar{\delta}$ or $\bar{\delta} = \bar{\delta}$. If $B$ is a predicate formula, and $B^\alpha$ comes from $B$ by substituting $\bar{\delta} = \bar{\delta}$ or $\bar{\delta} = \bar{\delta}$ for the predicate-letters, then $\vdash_{\mathcal{I}_0} B^\alpha \lor \neg B^\alpha$. Thus no strict subtheory of $\mathcal{C}_0$ is $\phi$-maximal for $\mathcal{I}_0$. 


On the other hand, there is a predicate letter in ◇, so both 0 = 0 and 0 = 1 are cases of ◇; hence $CL_0$ is ◇-maximal, since $CL_0$ is complete for truth-table semantics.

OPEN PROBLEM. Is there a propositional logic other than $IL_0$, $IL^+$ and $CL_0$, which is ◇-maximal for IA (say), where predicate letters are allowed to occur in ◇ more than once?

2.7. Maximality of $IL_0$ for some non-regular extensions of IA.

We examine maximality for some extensions of IA which are not r.e. subtheories of IA*. From the myriad of such extensions the ones that have attracted most attention are those arising from IA by adding one or more of the following schemas.

1. Markov's principle:

   \[ M \quad \forall x[A \lor \neg A] \& \neg \forall xA \rightarrow \exists x\neg A \]

2. The independence of premise principle:

   \[ IP \quad [\neg B \rightarrow \exists xA[x]] \rightarrow \exists [\neg B \rightarrow A[x]] \quad (x \text{ not free in } B) \]

3. Church's thesis:

   \[ CT_0 \quad \forall x\exists yA[x,y] \rightarrow \exists y\forall z[T(e,x,z) \& A[x,U(z)]] \]

(Where $T, U$ are Kleene's calculability predicate and result-extracting function, respectively). The logical schemas $M$ and $IP_0$ are classically valid while $CT_0$ is a classically invalid arithmetic schema (cf. Tr (73) for details). We restrict attention to closed instances of these schemas.
**Lemma 2.7.** (Le [A]). Let $\Theta$ be the set of sentences of $\mathcal{A}$ where no negative occurrence of $\exists$ or $\forall$ falls in the scope of a universal quantifier.

Let $T$ be a regular theory, and $\mathcal{N}^0_1$ denote the set of all true $\mathcal{N}^0_1$ sentences.

(i) $T + M + IP$ is conservative over $T + \mathcal{N}^0_1$ w.r.t. $\Theta$.

(ii) $T + CT_0$ is conservative over $T$ w.r.t. $\Theta$.

**Theorem V.** Let $T$ be regular. $IL_0$ is uniformly $\Sigma^0_2$ maximal for $T + M + IP$.

**Proof.** By theorems II and III, relativized to $\mathcal{N}^0_1$, there are $\Sigma^0_2$ sentences $A_i$, s.t.

(1) $\vdash_{T + \mathcal{N}^0_1} F[A_1, \ldots , A_k]$ implies $\vdash_{IL_0} F[p_1, \ldots , p_k]$ for each propositional formula $F$. But in $F[A_1, \ldots , A_k]$ no occurrence of $\exists$ or $\forall$ is in the scope of a universal quantifier, so by 2.7(1)

$\vdash_{T + M + IP} F[A_1, \ldots , A_k]$ implies $\vdash_{T + \mathcal{N}^0_1} F[A_1, \ldots , A_k]$,

which, combined with (1), concludes the proof.

Note that $IL_0$ is not $\Sigma^0_1$ maximal for $IA + M$, since $\vdash_{IA_0 + M} \neg \neg A + A$

for each $\Sigma^0_1$ sentence $A$. It is an open problem whether $IL_0$ is maximal for $IA + M$ with $\Theta$ substitutions, or with any disjunction of $\mathcal{N}^0_1$ sentences. The local maximality of $IL_0$ for $IA + M$ (M with parameters), was proved by Smoryński [73]. The method of Kripke models he used also fails to yield simpler substitutions.
THEOREM VI. Let $T$ be regular. Then $IL_0$ is uniformly $\Sigma^0_1$ and $\beta$ maximal for $T+IP$.

The proof of theorem VI is essentially the same as that of theorem II, since the subformula property holds for $IA+IP$, up to minor details that do not affect the proof. (Compare Tr [73], 4.3.)

THEOREM VII. $IL_0$ is uniformly $\Sigma^0_1$ and $\beta$ maximal for $T+CT_0$.

PROOF. As for theorem V, this follows from theorem I and 2.7(ii).
3. MAXIMALITY OF FIRST ORDER PREDICATELOGICS

3.1. Uniform maximality of classical logic.

**THEOREM VIII** (Wang[51]). Pure first order classical logic \( CL_1 \) is \( \Delta^0_2 \) maximal for any classically sound theory, and for any \( \Sigma^0_1 \)-sound extension of \( CA \).

**PROOF.** Assume \( \mathcal{V}_{CL_1}^T F[P_1,\ldots,P_k] \).

**Case(a)** By Hilbert-Bernays [39] proof of Gödel’s Completeness Theorem, there are \( \Delta^0_2 \) predicates \( B_1,\ldots,B_k \) s.t. \( F[B_1,\ldots,B_k] \) is false (cf. KL[52] §72). Thus, if \( T \) is a sound theory, \( \mathcal{V}_T^F[B_1,\ldots,B_k] \).

**Case(b)** Hilbert-Bernays’s proof (in a suitably arithmetized form) is derivable in \( CA \). I.e., if \( T \supset CA \), then for suitable \( \Delta^0_2 \) formulas \( B_1,\ldots,B_k \)

\[
\neg \mathcal{V}_{CL_1}^T F[B_1,\ldots,B_k] \Rightarrow \neg \mathcal{V}_T F[B_1,\ldots,B_k] \]

If \( \mathcal{V}_T^F[B_1,\ldots,B_k] \), then \( \mathcal{V}_{CL_1}^T (\neg F[B_1,\ldots,B_k]) \). So, if \( T \) is \( \Sigma^0_1 \)-sound, then, \( \mathcal{V}_{CL_1}^T F[F] \).

**THEOREM IX.** \( CL_1 \) is uniformly (and hence strongly) \( \Sigma^0_2 \) maximal and \( \Pi^0_2 \) maximal for any \( \Sigma^0_2 \)-sound r.e. extension of \( CA \).

**PROOF.** Hilbert-Bernays’s [39] proof is uniform: a \( \Delta^0_2 \) counter-model to a non-tautology \( F \) has a \( \Delta^0_2 \) index recursive in \( \neg F \). The theorem, for \( \Sigma^0_2 \) maximality, follows by the Uniformization Theorem I, with \( r=2 \), since the non-tautologies form a \( \Pi^0_1 \) set.
If \( B \) is a uniform \( \Sigma_2^0 \) substitution for \( \text{CL}_1 \), let \( C \) be a \( \Pi_2^0 \) formula equivalent to \( \neg B \). Then \( \vdash_{\text{CL}_1} F[P_1, \ldots, P_k] \) implies \( \vdash_{\text{CL}_1} F[\neg P_1, \ldots, \neg P_k] \).

and so \( \vdash_{\text{T}} F[\neg B] \), i.e., \( \vdash_{\text{T}} F[C] \). Thus \( \text{CL}_1 \) is also uniformly \( \Pi_2^0 \) maximal. \( \square \)

OPEN PROBLEM. Is \( \text{CL}_1 \) uniformly \( \Delta^0_2 \) maximal for Peano's Arithmetic \( \text{CA} \)?

3.2. Maximal properties of intuitionistic predicate logic.

NOTATIONS. Given a theory \( T \), write \( (T,n) \) for \( T \) extended with all true \( \Pi_n^0 \) sentences. Write \( (T;n) \) for \( (T,n) \) extended with all sentences of the form \( \forall x(A \land \neg A) \), where \( A \) is \( \Pi_n^0 \). Clearly, when \( T \) is r.e., \( (T,n) \) and \( (T;n) \) are \( \Sigma_{n+1}^0 \).

**Lemma 3.2.1** Let \( T \) be a regular theory. \( (T,n) \) and \( (T;n) \) satisfy the existential instantiation property: if \( \vdash_{T,n} \exists x A \) then \( \vdash_{T,n} A[\bar{m}/x] \) for some numeral \( \bar{m} \), and similarly for \( (T;n) \).

**Proof.** The statement is straightforward by Kleene's [62] slash method, appropriately relativized. A different proof uses derivations of \( \text{IA}^k \), with a relativized version of 2.3.3. \( \square \)

**Theorem X.** Let \( T \) be regular. \( \text{IL}_1 \) is uniformly \( \Pi_n^0 \) maximal for \( (T,n) \), for each \( n \geq 1 \). In particular, \( \text{IL}_1 \) is \( \Pi_2^0 \) maximal for \( (T,1) \), and so for \( T \).
As for theorem II, the proof uses a reduction. Let $B(x) \equiv \forall x B_0[x,z]$ be a $\Pi^0_{n+1}$ formula, with $x$ as the only free variables. Define

$$S_B[y] \equiv \forall x \forall y B[x] \lor \forall z \forall y B_0[y,z]$$

Theorem X is a consequence of the following two lemmas.

**Lemma 3.2.2.** Let $T$ be a regular theory, $n \geq 1$. Assume $B[x]$ is a $\Pi^0_{n+1}$ formula s.t. $\forall x B$ is true. Suppose $|T|_{L_1} F$ and $|T_n| F[B]$. Then there (weakly) exists a $k$ s.t. $|T_n| S_B[k]$.

**Lemma 3.2.3.** Let $T$ be an r.e. theory, $n \geq 0$. Suppose $T_n$ is consistent. Then there exists a $\Pi^0_{n+1}$ formula $B[x]$ s.t. $\forall x B$ is true, and

$$|T_{n+1}| S_B[k]$$

for all $k \geq 0$.

A special case of 3.2.2 is the following.

**Corollary X.** Let $n \geq 1$. Assume $A_1, \ldots, A_k$ are $\Pi^0_{n+1}$ sentences, classically independent over all true $\Pi^0_n$ sentences. Then they are intuitionistically independent over all true $\Pi^0_n$ sentences.

With $n=1$, the corollary implies the result of H. Friedman [73], that $\Pi^0_2$ sentences classically independent over true $\Pi^0_1$ sentences are intuitionistically independent.

Theorem X shows that properties of $T$ do not "relativize" to $(T; n)$, but to $(T; n)$. Even a single $\Pi^0_1$ sentence $A$ is not "intuitionistically independent," since $|T_A| \gamma A \lor A$, but, by the corollary, $|T_n| \gamma B \lor B$ for an appropriate $\Pi^0_{n+1}$ sentence $B$. On the other hand, $|T_n| \gamma B \lor B$ is demonstrated as for the case $n=0$.

Unlike the classical case, $\Sigma^0_n$ and $\Pi^0_n$ maximality properties are not directly related. Therefore, the following result is of independent interest.
THEOREM XI. Let $T$ be regular. $IL_1$ is uniformly $\Sigma^0_{n+1}$ maximal for $(T;n)$ for each $n \geq 1$. In particular, $IL_1$ is uniformly $\Sigma^0_2$ maximal for $(T;1)$, and so - for $T$.

The proof is a generalization of the methods used in proving theorem II.

Improvement on the complexity of substitutions given in theorems X and XI is bounded by the following negative result.

THEOREM XII. (Leivant[76]) $IL_1$ is not $\Pi^0_1$ maximal for $IA$.

The counter-example used in proving theorem XII is syntactically complex, leaving open the possibility that for simpler predicate formulas, such as the familiar schemas of $CL_1$-$IL_1$, $\Sigma^0_1$ substitutions can be found that yield instances undervisible in $IA$. Theorem XIII below shows that this is indeed the case.

A sequential calculus for $IL_1$ is readily obtained from the calculus in 2.3.: delete $TE,FE$, replace $VR$ and $2L$ by the new inference rules

$$
\begin{align*}
VR: & \quad \frac{\Gamma;A}{\Gamma;\forall xA} \\
2L: & \quad \frac{\Gamma;A:B}{\Gamma;\exists x A:B}
\end{align*}
$$

(x not free in the derived sequent)

and let terms of $P_1$ replace numerals as proper terms of $VL$ and $2R$. 

DEFINITION. Let \( \sigma, \sigma' \) be sequents. \( \sigma' \) is an ancestor of \( \sigma \) if there is a sequence of consecutive inferences of (the calculus above for) \( IL_1 \), with \( \sigma' \) as one of the premises of the first inference and \( \sigma \) as the conclusion of the last. \( \sigma = (\Gamma:F) \) is hereditarily decidable if there is a decision procedure for the derivability, in \( IL_1 \), of all sequents \( \sigma' \) ancestor of \( \sigma \) and of all sequents \( \Gamma':A \) where \( \Gamma' \) is the antecedent of an ancestor, and \( A \) is an atomic subformula of \( F \). \( F \) is hereditarily decidable if \( :F \) is.

THEOREM XIII. \( IL_1 \) is \( \Sigma^0_1 \) maximal for any r.e. subtheory \( T \) of \( TA^* \) w.r.t. hereditarily decidable formulas. Moreover, the maximality is uniform, in the following sense. Assume \( \Psi = \{F_n\} \) is an r.e. set of formulas for which there is a uniform decision procedure, applicable to all sequents \( \Gamma:G \) ancestor to \( :F_n \) for some \( F_n \in \Psi \), as well as to \( \Gamma:A \) where \( A \) is an atomic subformula of \( F_n \). Then \( IL_1 \) is uniformly \( \Sigma^0_1 \) maximal for \( T \) w.r.t. \( \Psi \).

COROLLARY XIII.1. \( IL_1 \) is uniformly \( \Sigma^0_1 \) maximal for any regular \( T \) w.r.t. the set of prenex formulas.

COROLLARY XIII.2. \( IL_1 \) is \( \Pi^0_1 \) maximal for any regular \( T \) w.r.t. formulas with no nesting of quantifiers.

Most examples of classical tautologies underviavile in \( IL_1 \) fall under the last corollary. E.g., \( \forall x \lor \neg P(x) \lor \forall x \land P(x) \lor \exists x \forall P(x) \lor \forall x\neg P(x)) \), \( \forall x(P \lor Q(x)) \lor \forall x\forall Q(x) \lor \neg \forall x\neg P(x) \lor \forall x Q(x)) \), \( \forall x(P(x) \lor \forall x\exists P(x)) \lor \forall x\exists P(x) \lor \exists x\neg P(x) \).
OPEN PROBLEM. Is $\mathbf{IL}_1$ (locally or uniformly) $\delta^0_2$ or $\beta$-maximal for $\mathbf{IA}$?

More generally, is there any frame $\phi$ s.t. $\phi \in \forall x \exists y P(x,y) \land \exists x \forall y P(x,y)$, and $\mathbf{IL}_1$ is $\psi$-maximal for $\mathbf{IA}$?

We turn to maximality for the whole of $\mathbf{IA}^*$. $\mathbf{IA}^*$ is complete for $\Gamma^0_3$ sentences, so not even $\mathbf{IL}_0$ is $\delta^0_3$ maximal for $\mathbf{IA}^*$. Denote by $\delta_1$ the class of sentences of the form $\forall x [C \lor \bar{C}]$, where $C$ is $\Gamma^0_1$, and $\bar{C}$ is the (classical) prenex form of $\neg C$.

THEOREM XIV. (a) $\mathbf{IL}_1$ is uniformly $\delta_2$-maximal for $\mathbf{IA}^*$.

(b) $\mathbf{IL}_1$ is $\delta_1$-maximal for $\mathbf{IA}^*$ w.r.t. hereditarily decidable formulas. This maximality is uniform in the sense of theorem XIII. In particular, $\mathbf{IL}_0$ is uniformly $\delta_1$-maximal for $\mathbf{IA}^*$.

OPEN PROBLEM. Is $\mathbf{IL}_1 \Sigma^0_3$, $\Sigma^0_4$ or $\delta_1$-maximal for $\mathbf{IA}^*$? Is $\mathbf{IL}_0 \Gamma^0_3$-maximal for $\mathbf{IA}^*$?

The first maximality result for $\mathbf{IL}_1$ is due to D.H.J. de Jongh [73a], who proved that if $\mathbf{IL}_1 F[P_1, \ldots, P_k]$, then $\mathbf{IA} F[A_1, \ldots, A_k]$ for suitable arithmetical predicates $A, B_1, \ldots, B_k$, where the superscript stands for relativization of quantifiers. In particular, $\mathbf{IL}_1$ is maximal for $\mathbf{IA}$ w.r.t. predicate formulas that are already relativized. This relativized maximality does not characterize $\mathbf{IL}_1$ as the set of predicate formulas "locally provable in $\mathbf{IA}$." De Jongh's proof simulates Kripke models via a notion of forced realizability, using an infinite sequence of incomparable degrees.
Contents of the rest of Section 3: Theorem X is proved in 3.3. Proofs of theorems XI and XIII are sketched in 3.4 and that of XIV in 3.5. Maximality of predicate logics with equality is discussed in 3.6. Section 3.7 concludes, with remarks on the maximality of $\text{CL}_1$ (with and without equality) for systems of set theory.

3.3 Proof of $\Pi^0_2$ maximality (theorem X).

Let $T$ be a regular theory. Let $T^c$ denote $T$ augmented with classical logic.

**Lemma 3.3.1.** There exists a (classically) $\Sigma^0_{n+1}$ predicate $J(x)$ satisfying

(1) $T^c, n \vdash J(x) \land J(y) \rightarrow x = y$

and (2) $T^c, n \not\vdash \forall \bar{m} \neg J(\bar{m})$ for each $\bar{m}$.

**Proof.** Straightforward by Kripke [63], relativized to true $\Pi^0_n$ sentences. \[\square\]

**Lemma 3.3.2.** There exists a $\Pi^0_{n+1}$ predicate $B(x)$ s.t.

(3) $T^c, n \vdash \forall \bar{m} [B(x) \leftrightarrow A(x)]$

for each $\Pi^0_n$ predicate $A$.

**Proof.** Let $E(m,x)$ be a $\Pi^0_n$ predicate universal for unary $\Pi^0_n$ predicates, and let $B(x) := \forall m [J(m) \rightarrow E(m,x)]$. By (1), $J(\bar{m})$ implies $B(x) \leftrightarrow E(\bar{m}, x)$; hence, if (3) fails for $A(x) \equiv E(\bar{m}, x)$, then $T^c, n \not\vdash \neg J(\bar{m})$, contradicting (2). \[\square\]
PROOF OF LEMMA 3.2.3. Assume $\models_{T,n} \forall x \forall y \exists z B(x) \rightarrow \forall z \forall z B_0[E,z]$. Then

$\models_{T',n} \forall x \exists y \forall z [x \neq y \leftrightarrow B(x)]$, contradicting 3.3.2.

To prove $\neg \forall y \forall z B[x]$ assume $\neg \forall y \forall z B(x)$. Since $\forall z B[x]$ is $\Pi^0_{n+1}$, $\neg \forall z B[x]$ implies $\models_{T',n} \neg \forall z B[x]$, contradicting again 3.3.2. \qed

Let $F$ be a predicate formula, and let $\Gamma, G$ consist of subformulas of $F[B]$. We write $\Gamma \models^{B}_{\text{IL}_1} G$ if $\Gamma;G$ can be derived using only inference rules of $\text{IL}_1$, where each occurrence $B[\langle n, t_1, \ldots, t_k \rangle]$ is treated as an atomic formula $P_n(t_1, \ldots, t_k)$.

LEMMA 3.3.3. Let $B$ be a $\Pi^0_{n+1}$ predicate with a non-vacuous quantifiers.

Let $\Gamma, A, C[x]$ consist of schemas in $B$, and let $\theta$ consist of strict sub-sentences of $B$. Then

(a) If $\Gamma, \theta \models^{B}_{\text{IL}_1} A$ then $\Gamma \models^{B}_{\text{IL}_1} A$

(b) If $\bar{p}$ does not occur in $\Gamma, \theta, C[x]$, then $\Gamma, \theta \models^{B}_{\text{IL}_1} C[\bar{p}]$ implies

$\Gamma, \theta \models^{B}_{\text{IL}_1} \forall x C[x]$, and $\Gamma, \theta, C[\bar{p}] \models^{B}_{\text{IL}_1} A$ implies $\Gamma, \theta, \exists x C[x] \models^{B}_{\text{IL}_1} A$.

PROOF. (a) is immediate from the definition of $\models^{B}_{\text{IL}_1}$. For (b), replace $\bar{p}$ by a parameter not occurring in the given proof. \qed

To simply the exposition, we assume for the rest of 3.3. that $n = 1$.

The proof for $n > 1$ is similar (with IA* extended with true $\Pi^0_n$ sentences as new axioms).
NOTATIONS. Let $B$ be a $\Pi^0_2$ formula.

Assume $\text{Prov}^*(d,'F(B)'')$. It will be convenient to distinguish between those instances of $\exists L, \exists R, (\forall L, \forall R$ respectively) in $A_d$ with a $\Sigma^0_1$ subformula of $B$ as the active conclusion (premise, resp.), and all those instances of $\exists L, \exists R, \forall L, \forall R$ in which all active formulas are schemas in $B$. We write $\exists^1 L, \exists^1 R, \forall^2 L, \forall^2 R$ for instances of the first kind, $\exists^* L, \exists^* R, \forall^* L, \forall^* R$ for those of the second.

Write $\text{Start}(d,u)$ if the following three conditions hold:

(i) $\Gamma^u B \vdash A^u$ ;

(ii) All strict subsentences of $B$ in $\Gamma^u$ are true ;

(iii) $A^u$ is a schema in $B$ .

Write $\text{Goal}(d,u)$ if (i), (ii) and the following two conditions hold:

(iv) $A^u$ is $B[t/x]$ for some $t$, and $\rho^u$ is $\forall^2 R$ ;

(v) No node $v \succ u$ satisfies (i), (ii), (iv) in place of $u$ .

We omit the first argument $d$ when in no danger of confusion.

LEMMA 3.3.4. Suppose that $\psi_{\Pi^0_1} F$ and that $\text{Prov}^*(d,'F(B)'')$ and $\forall x B$ are true. Then $\forall \exists^* \forall \text{Goal}(d,v)$.

SUBLEMMA. Assume $\text{Start}(\psi)$, and suppose that $\rho^v$ is not $\forall^2 R$. Then $\not\exists^* \psi \text{Start}(\psi^\rho)$.
PROOF OF SUBLEMMA. By cases on $\rho^w$, $T, t$ are excluded by (i), $FE$ is excluded by (ii), $TE$ and $A^1R$ are excluded by (iii). If $\rho^w$ is a propositional rule, then (ii) and (iii) hold for $u=\omega^*p$ for any $p$, and (i) for $u^w$ implies the weak existence of a $p$ s.t. (i) hold for $u=\omega^*p$. If $\rho^w$ is $V^xL$ or $A^\delta R$, take $p=0$. If $\rho^w$ is $V^\delta R$ or $A^\delta L$, let $\tilde{p}$ be the first numeral not occurring in $w^w$, and use 3.3.3(b). If $\rho^w$ is $A^1L$ then, by (ii), the active conclusion $\exists x\vec{E} \quad (E \text{ an equation})$ is true. Let $p := \mu x.E$, and use 3.3.3(a). If $\rho^w$ is $V^2L$, say

$$\Gamma, B_0[t, \bar{m}] : A^w$$

$$\Gamma, \forall x B_0[t, x] : A^w$$

then $B_0[t, \bar{m}]$ is a true $\Sigma^0_1$ sentence, since $\forall x B \equiv \forall x \forall z B_0[x, t]$ is assumed true.

PROOF OF THE LEMMA. Assume $Start(u)$. We prove, by induction on $\Delta_d$, that $\Gamma \vdash_{\forall \exists} u_{\text{Goal}}(v)$. In particular, for $u=\omega$ this establishes the lemma.

Since the statement we prove is negated, we may use individual instances of excluded third. If $\text{Goal}(u)$, we are done. If $\exists u^{\prime} u_{\text{Start}}(u^{\prime})$ then, by ind. ass., $\Gamma \vdash_{\forall \exists} u^{\prime}_{\text{Goal}}(v)$. Else, then $u$ is not $V^2R$; so, by the sublemma, $\Gamma \vdash u_{\text{Start}}(u^*\tilde{p})$. By ind. ass., $Start(u^*\tilde{p}) \equiv \forall \exists v u^*\tilde{p}_{\text{Goal}}(v)$. So $\Gamma \vdash u_{\text{Start}}(u^*\tilde{p})$ implies $\Gamma \vdash_{\forall \exists} u^*\tilde{p}_{\text{Goal}}(v)$. \qed

Fix a primitive recursive coding of finite sets of naturals (say, $\{n_0, \ldots, n_k\} := k \in \omega$) and of the basic set-theoretic operations on them. Let $Bar(d, y, x)$ be a $\Sigma^0_1$ predicate that holds when $x$ codes a non-empty set of nodes in $\Delta_d$ above $y$, satisfying conditions (vi)-(xi) below whenever $y \preceq w < v \in x$. 

\[(vi) \quad Bar(d, y, x) \rightarrow \forall z \in x (z \in Bar(d, y, x))
\]

\[(vii) \quad \forall z \in x (Bar(d, y, x) \rightarrow Bar(d, y, z))
\]

\[(viii) \quad \forall z \in x (z < d \rightarrow \exists z^\prime \in x (z^\prime > z \land Bar(d, y, z^\prime)))
\]

\[(ix) \quad \forall z \in x (z > d \rightarrow \exists z^\prime \in x (z^\prime < z \land Bar(d, y, z^\prime)))
\]

\[(x) \quad Bar(d, y, x) \rightarrow \exists z \in x (z = d)
\]
(vi) $\rho^v$ is a left-rule and $A^v = A^w = A^v$;
    (so, e.g., when $\rho^v$ is $+L$ then $v \succeq w^*1$);

(vii) if $\rho^v$ is $3^zL$ then $v \succeq w^y\bar{p}$, where $\bar{p}$ is the first numeral not occurring in $w^v$;

(viii) if $\rho^v$ is $3^1L$, with $3x E$ as the active conclusion, then $v \succeq w^y\bar{p}$, where $p = ux.E$;

(ix) if $\rho^v$ is $v^1L$ and $v \succeq w^y_b$ ($b=0$ or $1$), then there is also a $v^1 \subset x$ above the other premise $w^z$ of $w = c=1$ or $0$, respectively);

(x) if $\rho^v$ is $v^2L$ then the active premise is a true $\mathcal{L}_1^0$ formula;

(xi) $\rho^v$ is $v^2L$ or $3^1R$, and all equations and $\mathcal{L}_1^0$ sentences in $\Gamma^v$ are true.

Let $\text{Crit}(d,u)$ be a $\Delta_2^0$ predicate that holds exactly when (i), (ii), (iv) and

(xii) $\forall x, y \exists [\text{Bar}(d,u^y_x, x) + 3^y x L(d,v)]$,

where $L(d,v)$ is a (recursive) predicate that holds iff $\rho^v$ is $3^1R$, or $\rho^v$ is $v^2L$ with an active conclusion $B[s]$ s.t. the term $s$ is syntactically distinct from the term $t$ of (iv). Again, we omit $d$ when in no danger of confusion.
The following lemma is useful, since \textit{Crit} is \( \Sigma_2 \), while \textit{Goal} is not.

\textbf{Lemma 3.3.5.} Assume \textit{Prov}*(d,'F(B)') . \textit{Goal}(d,u) implies \textit{Crit}(d,u).

\textbf{Proof.} Assume \textit{Goal}(u). We need to prove only (xii). Since \( \exists v \forall x L(v) \) is decidable, it suffices to derive a contradiction from

1. \( \text{Bar}(u^s \tilde{m}, x) \), and
2. \( \forall v \forall x \rightarrow L(v) \), i.e. for each \( v \forall x \rho^v \) is \( v \forall L \) with \( B[t] \equiv A^u \) as the active conclusion in \( \Gamma^v \).

Assume (1) and (2), and let \( w \supset u^s \tilde{m} \) be s.t. \( w \preceq v \) for some \( v \in x \).

We prove, by induction on

\[ h := \max \{ \text{ith}(v) - \text{ith}(w) \mid v \in x \} \] the height of \( x \) above \( w \)

that

\[ \Gamma^w \vdash B \text{ IL}_{1} A^u. \] (3)

Since \( \Gamma^u \vdash \Gamma^u \), taking \( w = u^s \tilde{m} \) yields \( \Gamma^u \vdash B \text{ IL}_{1} A^u \), contradicting (i).

\textbf{Basis.} \( h = 0 \); then \( w \preceq x \), and by (2) \( A^u \in \Gamma^v \).

\textbf{Induction Step.} Consider cases for \( \rho^v \). By (vi) all right-rules are excluded.

(a) \( \rho^v \) is \( + L \); say,

\[
\begin{align*}
\Gamma, C & \vdash D : C \\
\Gamma, D & : B_0[t] \\
\hline
\Gamma, C & \vdash D : B_0[t]
\end{align*}
\]
By the subformula property, \( C \) is a schema in \( B \), so by (v), applied to \( w \hat{\alpha} \),

\[
(4) \quad \sim \{ \Gamma, C + D \}=^{B}_{II1} C.
\]

If \( w < v \) then by (vi) \( v \geq w \hat{\alpha} \), and by ind. hyp. applied to \( w \hat{\alpha} \),

\[
\sim \{ \Gamma, D \}=^{B}_{II1} A^u
\]

which together with (4) implies (3).

The cases for \( \delta L \) and \( v^* L \) are similar. The case for \( v^2 L \) is also similar, (x) permitting the application of (v) to \( w \hat{\alpha} \).

(b) \( \rho^w \) is \( z^w L \); say

\[
\frac{(\Gamma, 2xC[x], C[\vec{n}] : B_0[t])_n}{\Gamma, 2xC[x] : B_0[t]}
\]

Then, by (vii), \( v \geq w \hat{\rho} \), where \( \hat{\rho} \) is the first numeral not in \( \sigma^w \). By ind. hyp. applied to \( w \hat{\rho} \),

\[
\sim \{ \Gamma, C[\hat{\rho}] \}=^{B}_{II1} A^u
\]

and (10) follows by 3.3.3(b).

The case for \( \delta^1 L \) is similar, using (viii).

(d) \( \rho^w \) is \( v^1 L \); say

\[
\frac{\Gamma, C : B_0[t]}{\Gamma, C \cdots D : B_0[t]}
\]
Then, w.l.o.g., $v \geq w^0$, and by (ix) $v' \geq w^1$ for some $v' e x$. By ind. hyp. applied to both premises,

$$\sim \sim [\Gamma, C | [B] \Gamma_1 A^u] \text{ and } \sim \sim [\Gamma, D | [B] \Gamma_1 A^u];$$

(3) follows.

**NOTATION.** We write $z \models x$ if $x,z$ are (codes of) non-identical sets of node and $\forall z x z w e z v \models v$.

**Lemma 3.3.6.** Assume that $y$ is not a top-node in $\Delta_d$, that $A^y$ is $\xi^0_1$, and that all equations and $\xi^0_1$ sentences in $\Gamma^y$ are true. Then

$$\sim \exists x [\text{Bar}(y, x) \land \sim \exists z [x \text{Bar}(y, z)]]$$

**Proof.** By bar-induction on $\Delta_d$. The induction basis holds vacuously by assumption. For the induction step consider cases for $\rho^y$.

(a) $\rho^y$ is $\xi^1 R$; then $\text{Bar}(y, (y))$, and by (vi) $\sim \exists x \models (y) \text{Bar}(y, z)$.

(b) $\rho^y$ is $\xi^2 L$; if the active ($\xi^0_1$) premise $\exists x \xi^2$ of $\rho^y$ is true, then by ind. hyp. we get (5) for $y^0$ in place of $y$, and (5) follows. If $\exists x \xi^2 \models y^0$ is false, then by (xi),

$$\text{Bar}(y, (y)) \land \sim \exists z [y \text{Bar}(y, z)] .$$

Thus (5) is derived from excluded third, and being negated (5) follows by propositional logic.

(c) $\rho^y$ is another (left) rule; (5) follows by a straightforward application of the ind. hyp. 

C
**Lemma 3.3.7.** Assume $\text{Prov}^*(d, \text{'F}[B])$. Then, $\text{Crit}(d, u) \rightarrow \exists x S_B[x]$.

**Proof.** (in IA) Assume $\text{Crit}(u), A^u \equiv B[t], \ t = \langle I, P_1, \ldots, P_n \rangle$, $\rho^u$ is $V^2 R$. We prove $S_B[t]$ by demonstrating

$$(6) \ \forall m [\forall x \forall t \ B[x] \rightarrow \forall y \forall_B B_0[t, m]]$$

Fixing $m$, assume

$$(7) \ \forall x \forall z \forall \bar{m} \ B[u, \bar{m}, x \rightarrow \forall y \forall_B B_0[t, \bar{m}]]$$

We derive from (7), using excluded third,

$$(8) \ \forall x \forall t \forall z \forall \bar{m} \ B[x] \rightarrow B_0[t, \bar{m}]$$

So (6) follows from 3.3.6.

**Case 1.** $\rho^v$ is $\exists^1 R$ for some $v \in x$; say

$$\Gamma^v : \exists \Gamma^v : B_0[t, \bar{m}]$$

By 2.3.4 we may assume, w.l.o.g., that $v^\hat{\sigma}$ is a top-node in $\Delta_d$, while by (xii) all equations in $\Gamma^{v^\hat{\sigma}} = \Gamma^v$ are true. So $E$ is true, $B_0[t, \bar{m}]$ is true, and (8) follows.

**Case 2.** Otherwise. Since $x$ is non-empty, (7) and (xii) imply that for some $v \in x \rho^v$ is $V^2 L$ with an active conclusion $B[s], \ s \neq t$:

$$\Gamma, B_0[s, \bar{m}] : B_0[t, \bar{m}]$$
By the subformula property $s$ has the form $<j, q_1, \ldots, q_k>$, so $s \neq t$ implies $s \neq t$. If $B_0[s, k]$ is true, then by 3.3.6, $\forall x' \text{ Bar}(v^*o, x')$. But if $\text{Bar}(v^*o, x')$ then, letting $z := (x \setminus (v)) \cup x'$ we have $\text{Bar}(u^*o, z) \& z \models x$, contradicting (7). Finally, if $B_0[s, k]$ is false, then $\forall y \not\models B[y]$ must be false, since $s \neq t$, hence (8).

**Proof of 3.2.2 (case $n=1$).** Assume $\models_{IL_1} F \land \forall x B$, $\models_{T_1} F[B]$. By 2.3.3 (relativized $\text{Prov}^*(d, 'F[B]')$ for some $d$, provably in $(T, 1)$). So by 3.3.4 and 3.3.5,

$\forall u \text{ Crit}(d, u)$. Crit is $\Delta^0_2$, so $\text{Crit}(d, u) \iff (\models_{T_1} \text{Crit}(d, u))$. So by 3.3.7, assuming $\text{Crit}(d, u)$, $\models_{T_1} \text{Prov}^*(d, 'F[B]')$ implies $\models_{T_1} \exists x \text{S}_B[x]$, and by 3.2.1, $\models_{T_1} \exists \text{S}_B[k]$ for some $k$. Thus $\forall u \text{ Crit}(d, u)$ implies the weak existence of a $k$ for which $\models_{T_1} \exists \text{S}_B[k]$. Since the conclusion is negated, assuming $\forall \forall x B$ in place of $\forall x B$ suffices. □
3.4 Proof of theorems XI, XIII.

The proofs sketched here are refinements of the proof of Theorem II. Two difficulties arise. First, for theorem XI, derivability in $IL_1$ is not decidable. Hence, we relativize the argument to $I^0$. Second, quantifiers in the logical schema yield infinitely many instances of the substituted arithmetic formula. We replace the schemata $C_k$ by transfinite propositional formulas; these can be coded, but not given, in the language $\mathcal{A}$ of first-order arithmetic.

To spare the reader technical details of little relevance to the core of the proof, we shall only prove, for theorem XI, that for a suitable $\Sigma^0_2$ formula $B$, if $\models_T F[B]$, then $\models_{IL_1} F$. When the premise is weakened to $\models_T F[B]$, slight modifications are necessary. This would prove theorem XI for $n=1$. The proof for arbitrary $n \geq 1$ is analogous.

An infinitary language $\mathcal{A}_\omega$ is defined as follows. The alphabet is that of $\mathcal{A}$, plus the symbols $\land$ and $\lor$ for countable conjunction and disjunction, of which $\land$, $\lor$, $\neg$, and $\exists$ may be viewed as special cases. A formula of $\mathcal{A}_\omega$ is a well-founded countable tree, with equations or 1 at top-nodes (leaves), and logical constants at all other nodes, subject to the obvious restrictions. We use the usual typographic representation of formulas. A formula is recursive if its coding function is recursive. An antecedent is a countable set of formulas. A sequent is a pair $\Gamma;A$ where $\Gamma$ is an antecedent and $A$ is a sentence. The rules of inference of the calculus $IA_\omega$ are those of $IA^\omega$, plus the obvious rules for $\land$ and $\lor$:

$$\text{AL: } \begin{array}{c} \Gamma, (A_i)_{i \in I} : B \\ \hline \Gamma; \land_{i \in I} A_i : B \end{array} \quad \text{AR: } \begin{array}{c} \Gamma; A_i : A_i \\ \hline \Gamma; A_i \end{array}$$
Let \( \text{sub} \) be a primitive recursive function satisfying \( (\text{sub}(e, i))_{n \leq e} < i \).

So, if \( (e) \) codes \( \Lambda_1^F \) then \( (\text{sub}(e, i)) \) codes \( F_1 \). Indices \( e_1, e_2 \) s.t. \( (e_1) \) and \( (e_2) \) code sequents are related correctly (by an inference rule \( \rho \)) if the sequents coded relate as they should by \( \rho \), and \( e_1, e_2 \) relate correspondingly via \( \text{sub} \). E.g., \( <g, a> \) relate correctly to each \( <g_1, a_1> \), \( i \in I \), via \( AR \), if \( (a) \) \( \Rightarrow \) \( \Lambda_1 \), \( g_1 = g \) and \( a_1 = \text{sub}(a, i) \). Clearly, being related correctly is a recursive relation.

A proof of \( \Lambda^\omega_\omega (\Lambda^\omega_\omega (n)) \) is a function (recursive in \( \eta_\omega^0 \), respectively) a well-founded tree of codes of sequents of \( \varepsilon_\omega \), related correctly by the inference rules above. Just as for \( \Lambda_\omega \), we use \( \Lambda^\omega_\omega (n) \) solely as a combinatorial apparatus; no epistemological legitimacy is claimed or used. \( \Lambda^\omega_\omega (n)[T] \) will denote the set of sequents of \( \varepsilon_\omega \) proved in \( T^P \) to have a proof in \( \Lambda^\omega_\omega \).

Clearly, when \( T \) is \( \Sigma^0_k \), then so is \( \Lambda^\omega_\omega (n)[T] = \Lambda^\omega_\omega (0)[T] \).

**Lemma 3.4.1.** Let \( T \geq \Lambda_\omega \). Then \( \Lambda^\omega_\omega (n)[T] \) satisfies the general disjunction instantiation property: if \( \forall_{i \in I} F_i \) is a theorem of \( \Lambda^\omega_\omega (T) \), then so is \( F_j \) for some \( i \in I \).

**Proof.** The disjunction property of \( \Lambda^\omega_\omega \) is proved in \( \Lambda_\omega \) trivially. The lemma follows.

Let \( W_n, n \geq 0 \), be an enumeration of all r.e. sets. Let \( s \) be a primitive recursive function satisfying \( W_s(a, b) = W_a \cup W_b \). Let \( B \) be a \( \Sigma^0_2 \) formula, \( \Lambda \) a recursive well-founded tree of sequence-numbers. The formula \( G(a, u) = G_B, \Lambda(a, u) \) of \( \varepsilon_\omega \) is defined by bar-induction on \( \Lambda \), as follows.
\[ G(a,u) \equiv 3x \forall W_a (B[x] \lor \neg B[x]) \]

\[ \text{if } u \text{ is a top-node of } \Delta. \]

\[ G(a,u) \equiv 3p \exists b \left( u \not\doteq p \land (\forall x \forall W_b, B[x] \rightarrow G(s(a,b), u \not\doteq p)) \right) \]

\[ \text{if } u \text{ is an inner node of } \Delta. \]

\[ G(a,u) \equiv 1 \text{ if } u \not\doteq \Delta. \]

**Lemma 3.4.2.** Suppose \( \models_{L_1}^\ast F \) and \( \models_{L_1}^\ast F[B] \). Let \( D : = \forall a \forall x \forall W_a x \in W_a \lor x \not\in W_a \). Then there is a recursive tree \( \Delta \), proved in \( T^P \) to be well-founded, and there is an \( a \), s.t. \( G_{B_1,\Delta}(a,<>) \) is a theorem of \( IA_{\omega}^{(1)}[T + D] \).

**Proof.** Assume \( \models_{L_1}^\ast F[B] \). Let \( \Delta \) be a derivation of \( IA_{\omega}^* \) for \( F[B] \), provably correct in \( T^P \). As in 2.4.1 one concludes, in \( IA_{\omega}^{(1)}+D \), that \( G_{B_1,\Delta}(p(u),u) \) is true, where \( p \) is a primitive recursive function satisfying

\[ x \in W_{p(u)} \iff T^P_{L_1} \models_{L_1}^\ast B[x]. \]

Clearly, \( (IA_{\omega}^{(1)} + D)[T] \subseteq IA_{\omega}^{(1)}[T + D] \). The lemma follows by taking \( u = <> \).

**Lemma 3.4.3.** For each recursive (non-empty) tree \( \Delta \) and each \( a \) there exists a \( \Sigma_2 \) formula \( B \), with index recursive in \( \Delta, a \), for which \( G_{B_1,\Delta}(a,<>) \) is not a theorem of \( IA_{\omega}^{(1)}[T;1] \).

**Proof.** By bar-induction on \( \Delta \). If \( \Delta \) is a singleton the lemma is immediate, with \( B[x] \) defined to be the \( \Sigma_2 \) Rosser sentence for \( (T;1) \). Suppose \( \Delta \) is not a singleton, and let \( \Delta_p \) denote the subtree of \( \Delta \) with \( \hat{p} \) as a root. Suppose that for each \( p, a \) a \( \Sigma_2 \) formula \( B_{p,a} \) is given, with an index recursive in \( p, a \), satisfying the lemma for \( \Delta_p, a \). Applying the Uniformization Theorem I, whose proof clearly works for \( IA_{\omega}^{(1)}[T;1] \), there is a \( \Sigma_2 \) formula \( B \) satisfying the lemma for all \( a \) and \( \Delta_p \).
Let $B_b[x] \equiv B'[x] \lor x \varepsilon W_b$. By bar-induction on $\Delta$ for $u$ it is easy to see that
\[ G_{B',\Delta}(s(b,c),u) \leftrightarrow G_{B_b,\Delta}(s(b,c),u) \]
for any $b$ and any $u \in \Delta$. Hence
\[ IA^{(1)}_\omega[T;1] \not\vdash \exists p \in \Delta \exists (x \varepsilon W_b \mid B_b[x] \rightarrow G_{B_b,\Delta}(s(a,b),p)) \]
for any $b$ and $p \in \Delta$. By the Uniformization Theorem there is a $\Sigma^0_2$ formula $B$ s.t.
\[ IA^{(1)}_\omega[T;1] \not\vdash \exists p \in \Delta \exists (x \varepsilon W_b \mid B[x] \rightarrow G_{B,\Delta}(s(a,b),p)) \]
for all $b$ and $p \in \Delta$. So, by 3.4.1
\[ IA^{(1)}_\omega[T;1] \not\vdash G_{B,\Delta}(a,<>) \]
The process above is recursive in $p,a$ and in the function giving the indices of the $B_b$'s. The lemma follows by the Recursion Theorem.

PROOF OF THEOREM XI. (Simplified form of the case $n = 1$.) By 3.4.3 and the Uniformization Theorem there is a $\Sigma^0_2$ formula $B$ s.t. $G_{B,\Delta}(a,<>)$ is not a theorem of $IA^{(1)}_\omega[T;1]$ for any $a$ and any recursive $\Delta$ proved in $T^P$ to be well-founded. Combining this with 3.4.2,
\[ \vdash_T F[B] \implies \not\vdash_{IL_1} F \]
This implication has been proved in $IA$, and is classically equivalent to a $\Pi^0_2$ statement $S$. So $S$ is also a theorem of $IA$ (cf. e.g., Friedman[77]). But $S$ implies that $\vdash_T F[B] \implies \vdash_{IL_1} F$.

PROOF OF THEOREM XIII. The proof is the same as for XI, except that the hereditary decidability condition permits the use of $IA^*_\omega[T] = IA^{(0)}_\omega[T]$ in place of $IA^{(1)}_\omega[T;1]$, thence a $\Sigma^0_1$ formula $B$ may be constructed.
3.5 Proof of Theorem XIV.

Recall that $\text{IA}^* = \text{IA}$ augmented with the recursive $\omega$-rule = $\text{IA}$ augmented with transfinite induction over all recursive well-orderings. Fix a $\Sigma^0_2$ formula $C(x,y)$ and consider the $\delta^0_2$ formula $B(x) = \forall y B_0(x,y) \equiv \forall y (C \lor \neg C)$. Define $P_{C,n} := \{m \mid C(n,m)\}$ and $N_{C,n} := \{<j,m> \mid C(j,m), j \downarrow n\}$.

Theorem XIV(a) is a consequence of the following two lemmas ((b) is analogous, with a $\Pi^0_1$ formula $C$).

**Lemma 3.5.1.** If $\exists_{\Pi^0_1} F(\tilde{F})$ and $\text{Prov}^*(d, 'F[B]')$, then there weakly exists some $n \geq 0$ for which $P_{C,n}$ is weakly recursive in $N_{C,n}$ plus the set $\Pi^0_1$ of canonical indexes of true $\Pi^0_1$ sentences. If $F$ is hereditarily decidable, then $P_{C,n}$ is weakly recursive in $N_{C,n}$.

**Lemma 3.5.2.** There exists a $\Sigma^0_2$ formula $C$ s.t. $P_{C,n}$ is not recursive in $N_{C,n} + \Pi^0_1$, for any $n \geq 0$.

**Proof.** Relativize the proof of Sacks [63] p. 51. □

**Proof of 3.5.1.** For brevity, we shall argue classically. An intuitionistic proof is readily obtained by double-negation translation. Also, we treat only the general case. The modifications needed for an hereditarily decidable $F$ will be obvious.

Assume $\exists_{\Pi^0_1} F(\tilde{F})$ and $\text{Prov}^*(d, 'F[B]')$. For a node $u$ in $A_d$, write $r^u_0$ for the set of strict subsentences of $B_0$ in $r^u$, $r^u_1$ for $r^u - r^u_0$. By the subformula property, all formulas in $r^u_1$ are instances of $B_0$ or schemas in $B$. 
LEMMA 3.5.3. \( \exists u Q(u) \), where \( Q(u) \) is the conjunction of (i)-(iv) below.

(i) \( \Gamma_1^u \models^B A^u \)

(ii) all sentences in \( \Gamma_0^u \) are true.

(iii) \( A^u \) is \( B[t] \) for some \( t \), and \( \rho^u \) is \( \forall R \).

(iv) For each node \( v \models u \) in \( \Delta_d \), if \( A^v \) is a schema in \( B \), then \( \Gamma_1^v \models^B \Gamma_1^u A^v \).

PROOF. Analogously to the proofs of theorem III and 3.2.2 one shows, by bar-induction on \( \Delta_d \), that if \( w \) is a node in \( \Delta_d \) s.t. \( A^w \) is a schema in \( B \) and (i), (ii) hold for \( w \) in place of \( u \), then \( \exists u \models Q(u) \). The lemma follows by taking \( w = \langle \rangle \).

As in 3.3, we write \( \exists^L \) (resp. \( \exists^J \)) for an instance of \( 3E \) whose active conclusion is a schema in \( B \) (a prenex \( \exists^0_j \) sentence, respectively). Let

\[ R_u := \{ k | B_0^0[t, \bar{w}] \in \Gamma^u \} \] .

LEMMA 3.5.4. Assume \( Q(u) \), with \( A^u \models B[t] \). Suppose that \( \bar{m} \not\in R_u \). Then there is a sequence \( s = (s_1, \ldots, s_k) \) satisfying for each \( i = 0, \ldots, k \) the following properties, where \( x_i \) denotes \( u^{\bar{m}^*} s_1^{\ldots} s_i^{\ldots} \).

(i) \( \Gamma_1^{x_i} \models^B B[t] \);

(ii) \( R_{x_i} \subset R_u \);

(iii) all sentences in \( \Gamma_0^{x_i} \) are true;

(iv) \( A^{x_i} \) is \( B_0^0[t, \bar{m}] \);

(v) if \( \rho^{x_i} \) is \( \exists^L \), then \( \bar{s}_{i+1} \) is the first numeral not occurring in \( \sigma^{x_i} \); if \( \rho^{x_i} \) is \( \exists^J \) (\( j = 1 \) or 2), with \( \exists x \ E \) say as the active conclusion, then \( s_{i+1} = \mu x. E \)

(vi) \( \rho^k \) is \( \forall R \).
Moreover, $s$ depends on $m$ recursively, with $N_{C,n}$, and $n^0_1$ as oracles (where $n$ is the numeric value of $t$).

**PROOF.** We assume known, for $k \in R_u$, whether $C[n,k]$ or $\neg C[n,k]$. Since $R_u$ is finite, this is a finite amount of information. To construct the sequence $s$, it suffices to prove that for any node $x_i \geq u^*\alpha$ that satisfies (i)-(iv), and where $\rho^1$ is not $\forall \forall R$, there exists an $s_{i+1}$ satisfying (v), and s.t. (i)-(iv) hold for $x_{i+1} := x_i^1 \ast \bar{s}_{i+1}$. Consider cases for $\rho^1$.

The rules $T$, $I$, TE and FE are excluded by (i), (ii), (iv) and (iii), respectively.

If $\rho^1$ is $\forall L$, say

$$
\Gamma, G \vdash \Gamma; H : B_0[t, \bar{w}] \\
\Gamma, G \vdash H : B_0[t, \bar{w}]
$$

then $G$ is a schema in $B$, by the subformula property. We assume $Q(u)$; so, by 3.5.3 (iv) applied to $x_1^{\alpha^0}$, $\Gamma, G \vdash H \succeq_{1L_1} B$. Hence (i) holds for $x_{i+1} := x_i^{\alpha^1}$, since (i) holds for $x_i$.

If $\rho^1$ is $\forall L$ with $B[r]$ as active conclusion, then $r \neq t$ by (i), and so $r \neq t$ by the subformula property (as in 3.3.7). So $x_{i+1}^{\alpha^0}$ satisfies (ii).

The case that $\rho^1$ is another (left) rule, with a schema in $B$ as active conclusion, is straightforward.

If $\rho^1$ is $\forall L$ with $B_0[t, \bar{k}] \equiv C[t, \bar{k}] \lor \bar{C}[t, \bar{k}]$ as active conclusion, then $k \in R_u$ by (ii). We let $s_{i+1} = 0$ if $C[n,k]$, $s_{i+1} = 1$ if $\neg C[n,k]$. This guarantees (iii) for $x_{i+1}$.

If $\rho^1$ is $\forall L$ with $B_0[r, \bar{k}]$ as active conclusion, where $r$ is not $t$, then $r \neq t$ by the subformula property. We select $s_{i+1} = 0$ or $1$ according to whether $C[r, k]$ or $\neg C[r, k]$; this is recursive in $N_{C,n}$.
The sequence $s$ is recursive in $N_{C,n} + \overline{n}_1^0$, since each one of the steps above is.

**Lemma 3.5.5.** Assume $m \notin R_u$, and $s$ a sequence for $m$ given by 3.5.4. If $C(t, m)$ is the active premise of $\varrho^x_k$, then $C(t, m)$ is true. Similarly for $\overline{C}(t, m)$.

**Proof.** We prove, by bar-induction on $\Delta_d$, that if $x = x_1 \triangleright u^s \bar{m}$ is a node satisfying (i), (iii) of 3.5.4, and s.t. $A^x$ is a strict subformula of $B_0[t]$, then $A^x$ is true.

**Basis.** By (iii) $\varrho^x$ cannot be $1$ or $\bar{E}$. If $\varrho^x$ is $\bar{E}$, then $A^x$ is trivially true, and if $\varrho^x$ is $\bar{I}$, $A^x$ is true by (iii).

**Induction Step.** The only interesting cases are the following.

(a) $\varrho^x$ is $\bar{L}$; argue as in 3.5.4.

(b) $\varrho^x$ is $\bar{V}L$, with $C(r, \bar{k}) \lor \overline{C}(r, \bar{k})$ as active conclusion. Apply induction assumption to $x^\bar{0}$ or $x^\bar{1}$ accordingly to whether $C(r, \bar{k})$ is true or false. (Recall our convention to argue classically.)

**Proof of 3.5.1 - Concluded.** Assume $\varphi_i^T \in P[\bar{P}]$ and $\text{Prova}(d, 'P[B]' )$. Let $u$ be a node in $\Delta_d$ s.t. $Q(u)$ (by 3.5.3). Say, $A^u$ is $B[t]$, and $n$ the numeric value of $t$. For $m \notin R_u$, to decide $m \in P_{C,n}$ use 3.5.4 to construct a sequence $s$ satisfying 3.5.4 (i)-(vi). Then, by 3.5.5 $m \in P_{C,n}$ iff $x^\bar{0}$ is $C(t, m)$. Since $R_u$ is finite, this concludes the proof.
3.6 Predicate logics with equality

Let $\mathcal{IL}^e_1$ denote $\mathcal{IL}_1$ with equality, as described in KL[52]173, and let $\mathcal{CL}^e_1$ denote the corresponding classical system. If $F[\mathcal{P}_e]_1$ is a predicate formula (with equality), $B$ an arithmetic formula, define $F[B]$ as in 2.1, but with instances of equality left as equality.

**PROPOSITION 3.6.1.** $\mathcal{CL}^e_1$ is not maximal for $CA$, since none of the sentences

$$(1) \quad \exists x_1 \ldots \exists x_k [x_1 \neq x_2 \land x_1 \neq x_3 \land \ldots \land x_{k-1} \neq x_k]$$

is derived in $\mathcal{CL}^e_1$. Even $\mathcal{IL}^e_1 + (1)$ is not maximal for $IA$, since

$$(2) \quad \forall x \forall y [x = y \lor x \neq y]$$

is not a theorem of $\mathcal{IL}^e_1 + (1)$.

Let $\mathcal{CL}^e_1$ stand for $\mathcal{IL}^e_1 + (1)$, and $\mathcal{IL}^e_1$ for $\mathcal{IL}^e_1 + (1) + (2)$.

**THEOREM XV.** $\mathcal{CL}^e_1$ is $\Delta^0_2$ and uniformly $\Pi^0_2$ and $\Pi^0_2$ maximal for any arithmetically sound theory.

**PROOF.** The proofs of theorems VIII and IX clearly apply to $\mathcal{CL}^e_1$ (cf. Ke[52] 173 thm. 39).

**THEOREM XVI.**

(a) $\mathcal{IL}^e_1$ is uniformly $\Pi^0_2$ and $\Pi^0_2$ maximal for any regular theory $T$.

$\mathcal{IL}^e_1$ is uniformly $\delta^0_2$-maximal for $IA^*$. 

(b) $\mathcal{IL}^e_1$ is $\Pi^0_1$ ($\delta^0_1$) maximal w.r.t. hereditarily decidable formulas for any regular $T$ (for $IA^*$, respectively). The maximality is uniform in the sense of theorem XIII.
Actually, (b) may be generalized as follows. Let $F$ be a decidable fra. of $\mathbb{IL}_1$ (e.g., $\mathbb{IL}_0$), and let $F^a$ be the (deductive closure of the) union of $F$ and the purely equational part of $\mathbb{IL}_1^a$. By Lifschits [67] (Lemma 8) $F^a$ is also decidable. Consequently, the proof of (b) establishes that $\mathbb{IL}_1^a$ is uniformly maximal for any r.e. subtheory of $\mathbb{IA}^*$, and uniformly maximal for $\mathbb{IL}_1^a$.

Note that the maximality results for $\mathbb{IL}_1^a$ imply the corresponding result for $\mathbb{IL}_1$, since $\mathbb{IL}_1^a$ is conservative over $\mathbb{IL}_1$ for equation-free formulæ.

**PROOF.** We use the notational conventions of 3.3, with some obvious modifications. In amending the proofs for $\mathbb{IL}_1$, we use properties of $\mathbb{IL}_1^a$ given in the following three lemmas.

**Lemma 3.6.2.** Let $\Gamma, A[x], C$ consist of subformulas of $F[B]$; suppose $\bar{n}_0, \ldots, \bar{n}_k$ are all numerals occurring in $\Gamma, A$ are among $\bar{n}_0, \ldots, \bar{n}_k$; set $\bar{p} = \max[\{\bar{n}_i\}]_{i=1}^k$.

(a) If $\Gamma \vdash_{\mathbb{IL}_1^a} A[\bar{n}_i]$ for $i \leq k$, and $\Gamma, (\bar{p} \neq \bar{n}_i)_{i \leq k} \vdash_{\mathbb{IL}_1^a} A[\bar{p}]$, then $\Gamma \vdash_{\mathbb{IL}_1} \forall x A[x]$.

(b) If $\Gamma, A[\bar{n}_i] \vdash_{\mathbb{IL}_1^a} C$ for $i \leq k$ and $\Gamma, (\bar{p} \neq \bar{n}_i)_{i \leq k} \vdash_{\mathbb{IL}_1} A[\bar{p}]$, then $\Gamma, \exists x A[x] \vdash_{\mathbb{IL}_1^a} C$.

**PROOF.** (a) The premises imply that $\Gamma, (\bar{p} = \bar{n}_i \lor \bar{p} \neq \bar{n}_i)_{i \leq k} \vdash_{\mathbb{IL}_1^a} A[\bar{p}]$, and hence $\Gamma \vdash_{\mathbb{IL}_1} \forall x A[x]$ (as in 3.3.3.(b)). The proof of (b) is similar.
Lemma 3.6.3. For \( \Gamma, A[x], C, p \) as above:

(a) If \( \Gamma, (p \not= \bar{n})_{i \leq k} \vdash^B_{IL_1} A[\bar{p}], \) then \( \Gamma \vdash^B_{IL_1} \exists x A[x]. \)

(b) If \( \Gamma, (p \not= \bar{n})_{i \leq k}, A[\bar{p}] \vdash^B_{IL_1} C \) then \( \Gamma, \forall x A[x] \vdash^B_{IL_1} C. \)

Proof. Use (1)_{k+1}.

Lemma 3.6.4. Let \( C \) be a purely equational \( \Sigma^0_1 \) sentence, with the numerals \( \bar{n}_0, \ldots, \bar{n}_k \) as the only terms (other than variables). If \( C \) is false, then

\( D \vdash^B_{IL_1} \neg C \), where \( D := (\bar{n}_i \not= \bar{n}_j)_{i < j \leq k} \).

Proof. Straightforward, using axioms (1) and (2).

We prove theorem XVI for the case of \( \Pi^0_2 \) maximality, by indicating the modifications needed in the proof of theorem X. Other cases of XVI are obtained from the analogous results for \( IL_1 \) using similar modifications.

Let \( F[\bar{P}, \bar{=}] \) be a predicate formula, \( B \) a \( \Pi^0_2 \) formula. We distinguish between the original equality signs in \( F[B] \), and the substituted ones. Suppose \( Prov^*(d, 'F[B]' ) \); we similarly distinguish between original and substituted equalities in a sequent \( o^u \) in \( \Delta_d \), according to the status of the corresponding ancestor in \( F[B] \). Clearly, an original equality equates only variables and/or numerals. An occurrence of a number \( \bar{n} \) in \( o^u \) is potent if it is not
a descendent of an occurrence of $\vec{n}$ in $F[B]$. The diagram of a node $u$ in $\Delta_d$ is the set

$$\text{Diag}(u) := \{ \vec{m} \neq \vec{n} \mid m \neq n \text{ and } m, n \text{ are potent at } u \text{ in } \Delta_d \}.$$ 

Consider now the proof of theorem X in 3.3. The presence of original equations make it possible that propositional rules, $\mathcal{R}_L$, $\mathcal{R}_R$, $\mathcal{W}_L$ and $\mathcal{W}_R$ have $\Gamma^{0}_1$ sentences or equations as active premises or conclusions. In the definition of start condition (i) should be amended to

$$(i)'' \Gamma^u, \text{Diag}(u) \vdash^{B}_{\Gamma^{0}_1} A^u\Gamma^u.$$ 

In the proof of the sublemma in 3.3.4, 3.6.2 is to be used in place of 3.3.3 for the cases $\mathcal{R}_R$, $\mathcal{W}_L$. Lemma 3.6.3 is needed for the cases $\mathcal{R}_R$, $\mathcal{W}_L$. For the rest of the proof of 3.2.2 the following observation has to be kept in mind. If $\text{Start}(u)$ or $\text{Goal}(u)$, then all $\Gamma^{0}_1$ sentences and all equations in $\Gamma^u$ are true: the substituted ones - by (ii) of $\text{Start}$, and the original ones - by (i)' and 3.6.4. Other aspects of the proof remain unchanged.

Proofs of other maximality theorems are amended in essentially the same way.
3.7. Maximaly of classical logic for set theories.

**Theorem XVII.** $\text{CL}_1$ is $\Delta^0_2$ maximal, and uniformly $\Sigma^0_2$ and $\Pi^0_2$ (whence $\Delta^0_0$) maximal for any arithmetically sound set theory $T$.

**Proof.** Suppose $F[P_1,\ldots,P_k]$ is a sentence not derivable in $\text{CL}_1$. As for theorem VIII, let $B_1,\ldots,B_k$ be $\Delta^0_2$ formulas s.t. $\neg F[\vec{B}]$.

The idea is to show that $\neg F[\vec{C}]$, where $C_1$ is $B_1$ with each variable $x \notin \omega$ identified with, say, 0.

Let $\beta$ range over Boolean functions on the subscripts of the variables occurring in $F$. Given a formula $A$, write $A^\beta$ for the result of substituting 0 for free occurrences of $x_1$, if $\beta(i) = 1$. Write $\check{x^\beta}_B \omega$ if $x_1 \epsilon \omega$ when $\beta(i) = 0$. Let $C_1 := \Lambda \beta [x^\beta_B \omega \rightarrow B_1^\beta]$; clearly, each $C_1$ is $\Delta^0_2$. For a subformula $G$ of $F$, let $G^+ := \Lambda \beta [x^\beta_B \omega \rightarrow G^\beta[C_1]]$; let $G^\omega$ be $\forall x \epsilon \omega$ in front of $G$, with all quantifiers restricted to $\omega$. It is easy to see, by induction on $G$, that $G^+ \leftrightarrow G^\omega[B]$. In particular, since $F$ is a sentence, $\neg F[\vec{C}] \leftrightarrow \neg F[\vec{B}]$. Hence $\text{CL}_1$ is $\Delta^0_2$ maximal for any arithmetically sound set theory. The uniform $\Sigma^0_2$ and $\Pi^0_2$ maximality follows by uniformization, as in theorem IX.

When equality is considered a logical constant, the situation is radically different. It seems unlikely that any formula underyivable in $\text{CL}_1$ can be falsified in the universe of sets by definable relations, not to speak of arithmetical relations. However, this does not preclude a proof of maximality (for $\text{ZF}$, say), using a technique different from the above, and possibly closer to the proof-theoretic analysis used in 3.3 for intuitionistic systems.

However, the technique of theorem XVII may be applied to $\text{CL}_1$, provided the language of the set theory considered is suitable augmented. Let $\sigma$ be a fixed unary function letter, and let $\text{ZF}_\sigma$ stand for $\text{ZF}$ extended with
the new axiom $\forall x [x \notin \emptyset + \sigma(x) \epsilon x]$ (i.e., $\sigma$ is a global choice function).

U. Felgner [71] has proved that $\text{ZF}_0$ is conservative over $\text{ZFC}$. It is well known that in $\text{ZF}_0$ there is a well-ordering $<_0$ of the whole universe $V$, definable in $\sigma$. E.g., $<_0$ may successively well-order $V_{\alpha+1} - V_\alpha$, for $\alpha$ running through the ordinals. Using $<_0$, the construction used to prove the upwards Skolem-Löwenheim theorem yields:

**Lemma 3.7.1.** If $\vdash_{\text{CL}_1}^{\text{ZF}} \forall x [F(P_1, \ldots, P_k)]$, then there are formulas $B_1, \ldots, B_k$ such that $\Delta_2$ in $\sigma$, s.t. $\neg F[B_1, \ldots, B_k]$.

From 3.7.1 we conclude, as in the proof of theorem XVII:

**Theorem XVIII.** $\text{CL}_1^{\text{ZF}}$ is uniformly $\Delta_2$ maximal for $\text{ZF}_0$.

(Note that uniformization adds a numeric existential quantifier.)

Actually, the substitutions may be reduced to ones $\Delta_2$ in a linear (not necessarily well-founded) ordering $<$ of $V$, since H. Friedman has proved (unpublished) that a counter-model to an undervisible formula may be defined already from such an ordering. Also, we may avoid global choice (which is strictly stronger in $\text{GNB}$ than local choice, cf. Easton [64]), but at the cost of a drastic inflation of the language with a choice-function $\sigma_\alpha$ for $V_\alpha$, for each ordinal $\alpha$. The argument proceeds as before, via the Lévy-Montague Reflection Theorem (Lévy [60]).

Note that maximality fails if the language is enriched on the side of logic. Namely, if $\epsilon$ is considered a logical constant (as it is in $\text{CL}_1^{\text{ZF}}$), i.e., if models are required to be transitive, then using standard methods one defines a formula $F[P, \epsilon]$ formalizing "$P$ is a truth definition for the language of $\text{ZF}$." As noted by Lévy [60], §5, $\vdash_{\text{ZF}} \neg F[B, \epsilon]$ for each formula $B$ of the language of $\text{ZF}$, while clearly $\vdash_{\text{CL}_1^{\text{ZF}}} F[P, \epsilon]$, since $\text{ZF} F[P, \epsilon]$ is provable in $\text{GNB}$. 
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