THE 'ALMOST ALL' THEORY OF SUBRECURSIVE DEGREES IS DECIDABLE

Kurt Mehlhorn*

TR 73-170

May 1973

Department of Computer Science
Cornell University
Ithaca, New York 14850

*This research was supported by NSF Grant GJ-579.

THE 'ALMOST ALL' THEORY OF SUBRECURSIVE DEGREES IS DECIDABLE

Kurt Mehlhorn*

Abstract:

We use constructive measure theory to show the decidability of the 'almost all' theory of subrecursive degrees. The formulas of this theory are built up using the constant 0 standing for the minimum degree, the functions $\cup, \cap$ standing for the join and meet of two degrees respectively, the relation $\leq$ standing for the reducibility-ordering, the logical connectives &, $\neg$ and the quantifier (almost $\forall a$). An efficient decision procedure is described.

---

*This research was supported by NSF Grant GJ-579.

THE 'ALMOST ALL' THEORY OF SUBRECURSIVE DEGREES IS DECIDABLE

Kurt Mehlhorn

I. Introduction

In [3] we investigated the size of sets of computable functions using category-theoretic methods (in the sense of the Baire Category theorem). In this paper we use measure-theoretic concepts.

Recursive function theorists used measure theoretic methods very successfully (see [7] and [8]). John Stillwell has recently shown that the 'almost all' theory of Turing degrees is decidable ([9]). Turing degrees are a classification of the set of all 0-1 valued functions. Computer scientists are more concerned with the class of computable functions. Classical measure theory is not an appropriate tool for an investigation of the set of computable functions, but constructive measure theory is. Constructive measure theory was developed by Erret Bishop ([1]).

Let R be the set of 0-1 valued computable functions. In section II we introduce the basic notions of constructive mathematics as far as they are relevant to this paper. In section III we define a constructive measure on R. We do this by first defining an integral on the set of continuous functions from R into $\mathbb{R}$ (the set of constructive reals) and then approximating characteristic functions by continuous functions. Most theorems of classical measure theory are also true in constructive measure theory. In particular Fubini's theorem is true.

In section IV we compare two different concepts of smallness: meagre sets and sets of measure 0. We show that they capture different aspects of smallness: There are sets A,B such that $A \cap B = \phi$, $A \cup B = R$, A is meagre and B has measure zero.

Section VI is the heart of the paper. We prove that the almost all theory of subrecursive (elementary recursive, primitive recursive) degrees is decidable. The almost all theory arises from the first order theory by replacing the quantifier $\forall a$ by the quantifier $\overset{\infty}{\forall} a$ meaning "for almost all a"; i.e. we consider the theory with the constant 0 standing for the minimum degree (the class of primitive recursive 0-1 valued functions), the functions $\cup$ and $\cap$ standing for the join and meet of two degrees respectively, the relation $\leq$ standing for 'less than or equal' in the reducibility ordering (elementary recursive in, primitive recursive in), the logical connectives &, $\neg$ and the quantifier $\overset{\infty}{\forall}$. The proof of the decidability follows closely Stillwell's proof for the decidability of the almost all theory of Turing degrees. We then give some examples of theorems which can be proved using the decision procedure. Finally we derive a bound for the complexity of the decision procedure. It is shown that already a crude implementation runs in order n^2 where n is the length of the formula.

II. Preliminaries

This paper is an exercise in constructive mathematics. In this section we introduce the basic notions: real numbers, continuous functions and complemented sets. Readers discouraged by the brevity of our treatment should consult Erret Bishop's book "Foundations of Constructive Analysis" ([1]).

Two sets are fundamental for our discussion: the set $\mathbb{R}$ of real numbers and the set R of 0-1 valued computable functions. Real numbers are introduced as particular Cauchy sequences of rational numbers.

(2.1) Definition (Bishop) A real number is a sequence $\{x_n\}$ of rational numbers such that

$$|x_m - x_n| \leq m^{-1} + n^{-1}$$

for all m and n. Two real numbers $x \equiv \{x_n\}$ and $y \equiv \{y_n\}$ are equal if

$$|x_n - y_n| \leq 2n^{-1} \quad \text{for all } n \in \mathbb{N}.$$

All the usual arithmetic operations - addition, multiplication, exponentiation, etc. - are well defined on $\mathbb{R}$.

Next we define a metric on R. The metric d is a function from R x R into $\mathbb{R}$ defined as follows

$$\text{(2.2)} \qquad d(f,g) = \begin{cases} 0 & \text{if } f = g \\ 1/(n+1) & \text{if } n = \mu x[f(x) \neq g(x)] \end{cases}$$

This definition does not look constructive; after all $f = g$ is undecidable. But note that d maps $R \times R$ into the real numbers and not into the rational numbers. Thus d produces a sequence of rationals for every pair f,g of computable functions. d might proceed as follows. In order to produce the n-th element of the sequence d checks if there is a $x < n$ such that $f(x) \neq g(x)$. If there is such an x, let x_0 be the least such x. Then d generates $1/(x_0+1)$ as the n-element, otherwise it generates $1/n$.

The metric d defines a neighborhood structure on R. The open sphere of radius $r > 0$ about $f \in R$ is the subset

$$S(f,r) \equiv \{g \in R;\ d(f,g) < r\}$$

of R. Let $n \in \mathbb{N}$ be the least x such that $1/x \leq r$. Then

$$S(f,r) = S(f,1/n) = \{g \in R;\ f(x) = g(x) \text{ for } x < n\}$$

R can be represented as the infinite binary tree (pruned by a all non-recursively enumerable branches); every computable function corresponds to a branch of this tree, open spheres correspond to subtrees. We also note that R is a compact metric space under d (see [1] for an exact definition).

We now introduce continuous functions from R into $\mathbb{R}$.

(2.3) Definition (Bishop) $F: R \to \mathbb{R}$ is continuous if for each $\varepsilon > 0$ there is a $\omega(\varepsilon) > 0$ such that

$$|F(f) - F(g)| < \varepsilon$$

whenever $d(f,g) < \omega(\varepsilon)$. The operation $\varepsilon \to \omega(\varepsilon)$ is called a modulus of continuity for F.

We denote the set of continuous function from R into $\mathbb{R}$ by C(R).

Characteristic functions of open spheres are simple examples of continuous functions. Let f be an element of R and n be a natural number. Let C be the characteristic function of $S(f,1/n)$. A modulus of continuity for C is given by the constant function $\omega(\varepsilon) = 1/n$. If $d(g_1,g_2) < 1/n = \omega(\varepsilon)$ then either $g_1, g_2 \in S(f,1/n)$ or $g_1,g_2 \notin S(f,1/n)$. Therefore $|C(g_1) - C(g_2)| = 0$ whenever $d(g_1,g_2) < 1/n = \omega(\varepsilon)$.

The next concept is one which differs drastically from the 'equivalent concept' of classical mathematics: complemented set.

(2.4) <u>Definition</u> (Bishop) A complemented set in R (relative to C(R)) consists of an ordered pair (A,B) of subsets of R such that for every $f \in A$ and $g \in B$ there exists $F \in C(R)$ with $F(f) \neq F(g)$. The union $(A,B) \equiv \bigcup_{t\in T} (A_t,B_t)$ of a family $\{(A_t,B_t)\}_{t\in T}$ of complemented sets is defined by

$$A \equiv \bigcup_{t\in T} A_t \qquad B \equiv \bigcap_{t\in T} B_t$$

The intersection $(A,B) \equiv \bigcap_{t\in T} (A_t,B_t)$ is defined by

$$A \equiv \bigcap_{t\in T} A_t \qquad B \equiv \bigcup_{t\in T} B_t$$

The complement $-(A,B)$ of a complemented set (A,B) is defined by $-(A,B) = (B,A)$.

Unions and intersections of families of complemented sets and complements of complemented sets are again complemented sets.

<u>Example</u>: Let $f \in R$ and n be a natural number. The classical complement of the open sphere $S(f,1/n)$ can be written as union of open spheres $\bigcup_{\substack{g(x)\neq f(x) \\ \text{for some } x<n-1}} S(g,1/x+1)$. It is worth noting that this union is really only a finite union. We will usually denote it by $\overline{S(f,1/n)}$. Since the characteristic function of $S(f,1/n)$ is continuous the pair $(S(f,1/n), \overline{S(f,1/n)})$ is a complemented set (relative to $C(R)$).

We will write $B_0(R)$ for the family $\{(S(f,1/n), \overline{S(f,1/n)})\}_{f\in R, n\in \mathbb{N}}$ of complemented subsets of R. Since we can replace '$f \in R$' by '$f \in R$ and f has finite support' in this definition this is really a family of sets. The family $B_0(R)$ generates the class $B(R)$ of Borel sets.

(2.5) <u>Definition</u> (Bishop)

(a) Each complemented set $(S(f,1/n), \overline{S(f,1/n)})$ is a Borel set.

(b) Each countable union of Borel sets already constructed in a Borel set.

(c) Each countable intersection of Borel sets already constructed is a Borel set.

(d) That's all.

It is easy to show that B(R) is closed under complement. It follows from the example preceeding the definition (2.5) for the basis sets.

$$-\left(S(f,1/n),\ \overline{S(f,1/n)}\right) = -\left(S(f,1/n),\ \bigcup_{\substack{g(x)\neq f(x) \\ \text{for some } x\leqslant n-1}} S(g,1/x+1)\right) =$$

$$= \left(\bigcup_{\substack{g(x)\neq f(x) \\ \text{for some } x\leqslant n-1}} S(g,1/x+1),\ S(f,1/n)\right)$$

$$= \bigcup_{\substack{g(x)\neq f(x) \\ \text{for some } x\leqslant n-1}} \left(S(g,1/x+1),\ \overline{S(g,1/x+1)}\right)$$

We now proceed by induction.

III. A constructive measure on R

In this section we define a constructive measure on R. In order to do so we first solve a more general task: we define an integral (measure) on the set C(R) of continuous functions. The measure of sets is then defined by approximation.

Though continuous functions are mathematically appealing, they are still a rather elusive concept. The next theorem shows more clearly what continuous functions really are.

Let T be the set of all finite initial segments. If $g \in R$ then let g_n denote the initial segment $g|\{0,1,\ldots,n-1\}$ of length n.

(3.1) Definition: Let f_1: $T \to Q$ and f_2: $\mathbb{N} \to \mathbb{N}$ be functions such that for all $g \in R$

$$\{f_1(g_{f_2(n)})\}_n$$

is a regular sequence; i.e. for all $n, m \in \mathbb{N}$

$$|f_1(g_{f_2(m)}) - f_1(g_{f_2(n)})| < 1/m + 1/n \quad .$$

Then the pair (f_1,f_2) defines a function F from R to $\mathbb{R}$.

$$F(g) = \{f_1(g_{f_2(n)})\}_n$$

(3.2) Representation theorem: F: $R \to \mathbb{R}$ is continuous if and only if it is defined by a pair (f_1,f_2).

Proof:

$\Leftarrow$: Let F be induced by (f_1,f_2); f_1: $T \to Q$ and f_2: $\mathbb{N} \to \mathbb{N}$. We define a modulus of continuity for F.

$$\omega(\epsilon) = f_2(\mu n[n > 2 \cdot \epsilon^{-1}])$$

Let g^1, $g^2 \in R$ such that $d(g^1, g^2) < \omega(\epsilon)$ and let $n_o = \mu n[n > 2 \cdot \epsilon^{-1}]$. Then

$$d(g^1, g^2) < 1/f_2(n_o)$$

and therefore $g^1_{f_2(n_o)} = g^2_{f_2(n_o)}$. F maps g_1, g_2 into a sequence of rational $\{x_n\}$, $\{y_n\}$ respectively. We have to show

$$|F(g^1) - F(g^2)| < \epsilon \quad .$$

This is done by giving a bound for the sequence of rationals defining $F(g^1) - F(g^2)$.

$$\epsilon - |(F(g^1) - F(g^2))_n| =$$

$$\epsilon - |f_1(g^1{}_{f_2(2n)}) - f_1(g^2{}_{f_2(2n)})|$$

$$\geq \epsilon - (|f_1(g^1{}_{f_2(2n)}) - f_1(g^1{}_{f_2(2n_o)})| +$$

$$|f_1({}^1{}_{f_2(2n_o)}) - f_1(g^2{}_{f_2(2n_o)})| +$$

$$|f_1(g^2{}_{f_2(2n_o)}) - f_1(g^2{}_{f_2(2n)})|)$$

$$\geq \epsilon - [(1/2n + 1/2n_o) + 0 + (1/2n + 1/2n_o)]$$

$$\geq \epsilon - 1/n - 1/n_o \geq (\epsilon - 2/n_o) + (1/n_o - 1/n) \geq 1/2n_o$$

for $n \geq 2n_o$.

"$\Rightarrow$": Let F be continuous with modulus ω. If $t \in T$ then $t^o \in R$ with $t^o(x) = t(x)$ for $x < \ell h(t)$ and $t^o(x) = 0$ otherwise. We define

$$f_1(t) = F(t^o) \qquad \text{for } t \in T$$

$$f_2(n) = \mu x[x \in \mathbb{N} \text{ and } x > \omega(1/2n)^{-1}]$$

We first show that the pair (f_1, f_2) defines a function $\hat{F}$: $R \to \mathbb{R}$. Let n, $m \in \mathbb{N}$, $n \leq m$ and let $g \in R$. Then

$$d(g^o{}_{f_2(n)}, g^o_{f_2(n)}) < 1/f_2(n) < \omega(1/2n)$$

and therefore

$$|f_1(g_{f_2(n)}) - f_1(g_{f_2(m)})| =$$

$$|F(g^o_{f_2(n)}) - F(g^o_{f_2(m)})| < 1/2n \leq 1/n + 1/m \ .$$

It remains to show that $\hat{F} = F$. Let $g \in R$. Then

$$d(g, g^o{}_{f_2(n)}) < 1/f_2(n) < \omega\,(1/2n)$$

and therefore

$$|F(g) - (\hat{F}(g)_n| = |F(g) - F(g^o{}_{f_2(n)})| < 1/2n \qquad \text{for all } n.$$

This implies $F = \hat{F}$. Q.E.D.

The representation theorem (4.1) allows us to define an integral on C(R). If F: $R \to \mathbb{R}$ is continuous then there is a pair (f_1, f_2) inducing F. The sequence

$$\{2^{-f_2(n)} \sum_{\ell h(t) = f_2(n)} f_1(t)\}_n$$

is regular. The real number defined by the sequence does not depend on the particular pair (f_1, f_2) chosen to represent F. We prove only the first assertion; the second one is left to the reader.

$n, m \in N$, n m.

$$\left| 2^{-f_2(n)} \sum_{\ell h(t)=f_2(n)} f_1(t) - 2^{-f_2(m)} \sum_{\ell h(t)=f_2(m)} f_1(t) \right|$$

$$\leq 2^{-f_2(m)} \left| 2^{f_2(m)-f_2(n)} \sum_{\ell h(t)=f_2(n)} f_1(t) - \sum_{\ell h(t)=f_2(m)} f_1(t) \right|$$

$$\leq 2^{-f_2(m)} \sum_{\ell h(t)=f_2(n)} \sum_{\ell h(s)=f_2(m)} |f_1(t)-f_1(s)|$$

$$\leq 2^{-f_2(m)} \sum_{\ell h(s)=f_2(m)} \left(1/n + 1/m \right) \leq 1/n + 1/m$$

(3.3) Definition: Let $F \in C(R)$ and let (f_1, f_2) define F. The integral $\int F d\mu$ is given by

$$\{2^{-f_2(n)} \sum_{\ell h(t)=f_2(n)} f_1(t)\}$$

We have already shown that $\int F d\mu$ is well-defined. It is trivial to show that the mapping $F \mapsto \int F d\mu$ is linear and that for all $F \in C(R)$ there exists a constant $C_F \geq 0$ such that $|\int G d\mu| \leq C_F$ whenever $G \in C(R)$ and $|G| \leq |F|$. It is also clear that $F \geq 0$ implies $\int F d\mu \geq 0$. Therefore the mapping $F \mapsto \int F d\mu$ defined on $C(R)$ is a positive integral (measure) in the sense of ([1]).

(3.4) Theorem: The mapping $F \mapsto \int F d\mu$ defined in (4.3) is a positive measure on $C(R)$.

We now turn to the measure of complemented sets.

(3.5) Definition: Let A be a complemented subset of R, let ϵ be a positive real number. A continuous function F: $R \to \mathbb{R}$, with $0 \leq F \leq 1$ approximates A to within ϵ if there exists a sequence $\{F_j\}$ of non-negative continuous functions such that

(a) $\sum_{j=1}^{\infty} \int F_j d\mu$ exists and is less then ϵ

(b) $f \in -A$ whenever $F(f) - \sum_{j=1}^{N} F_j(f) \leq 1 - \delta$

(c) $f \in A$ whenever $1 - F(f) - \sum_{j=1}^{N} F_j(f) \leq 1 - \delta$

for all $N \in \mathbb{N}$ and some $\delta > 0$.

The interpretation of definition (4.5) is that A is "nearly" the set where F is positive and -A is "nearly" the set where F is less than 1. A complemented set which for each $\epsilon > 0$ has an approximation to within ϵ is called integrable.

Example: We have shown above that the characteristic functions of open spheres $S(f, 1/n)$ are continuous. Therefore open spheres are integrable. Let f_1: $T \to \mathbb{Q}$ and f_2: $\mathbb{N} \to \mathbb{N}$ be defined by:

$$f_1(t) = \begin{cases} 1 \text{ if } t \text{ extends } f_{n-1} \text{ or } f_{n-1} \text{ extends } t \\ 0 \text{ otherwise} \end{cases}$$

and

$$f_2(t) = n \quad \text{for all } x \quad .$$

It is easy to see that the pair (f_1, f_2) defines the characteristic function of $S(f, 1/n)$. Since

$$2^{-f_2(x)} \sum_{\ell h(t) = f_2(x)} f_1(t) = 2^{-n} \sum_{\ell h(t)=n} f_1(t)$$

$$= 2^{-n} \cdot 1$$

$$= 2^{-n}$$

$\mu S(f, 1/n) = 2^{-n}$

As a second example we consider singleton sets $\{f\}$. Let F: $R \to R$ be the zero-function; i.e. $F(f) = 0$ for all $f \in R$. Let $\epsilon > 0$. Find a $n \in \mathbb{N}$ such that $2^{-n} < \epsilon$. The characteristic function of $S(f, 1/n)$ is an errorsequence for the approximation of $\{f\}$ by F. We verify conditions a) b) and c) of definition (4.5).

(a) $\int C_{S(f,\ 1/n)} d\mu \;=\; 2^{-n} \leq \epsilon$

(b) for $g \notin \{f\}$ $\quad F(g) - C_{S(f,\ 1/n)}(g) \leq 0$

(c) for $g \in \{f\}$ $\quad 1 - F(g) - C_{S(f,\ 1/n)}(g) = 0$

Bishop ([1]) has shown that every Borel set $\in B(R)$ is integrable, that product measures can be defined and that the theorem of Fubini is true for product measures. We will use these facts implicitly in the next sections.

At this point we should rationalize why we did not introduce directly a measure on the Borel sets $B(R)$. We think there are two good reasons. First it is a very tedious task to prove that the measure assigned to a Borel set is independent of its particular representation. Second it would not permit to use as much of Bishop's work as we will in the next sections.

Finally we emphasize that the measure defined above is a natural one. If we assume that it is equiprobable for a function value to be 0 or 1, then the probability that an arbitrary characteristic function is an element of the sphere $S(g, 1/n)$ is exactly 2^{-n}; this equals $\mu S(g, 1/n)$.

IV. Meagre sets and sets of measure zero

In [3] we introduced the concept of (effectively) nowhere dense and meagre set. For the convenience of the reader we include the definitions here.

(4.1) Definition A subset $A \subseteq R$ is nowhere dense if there is a computable function $v:T \to T$ such that

$$\forall t \in T \qquad v(t) \sqsupset t$$

$$\exists n \in N \quad \forall t \in T \qquad \ell h(t) \geq n \Rightarrow U_{v(t)} \cap A = \phi$$

(here $U_{v(t)} = \{f \in R;\ v(t) \sqsubseteq f\}$ v is called a verifying function for A.

$M \subseteq R$ is meagre if there is a (computable function)

$v:\ N \times T \to T$ and a decomposition $M = \bigcup_{i=0}^{\infty} M_i$ such that M_i is nowhere dense with verifying function $\lambda t[v(i,t)]$ for all i.

A nowhere dense set is small in a geometric sense: it is perforated with holes (namely the $U_{v(t)}$'s) and one can effectively find these holes. A set is meagre if it can be approximated by nowhere dense sets.

A set has measure zero if it is small in a metric sense. It can be covered by a sequence of open spheres of arbitrary small total length.

So both concepts capture an aspect of smallness. Nevertheless they can be diametrically opposed in some cases. This fact is well-known in the classical theory ([5]); almost the same proof works in the constructive case.

(4.2) Theorem: There is a complemented set (A,B) such that $A \cup B = R$, A has measure 0 and B is meagre.

Proof: Let $\{t_i\}$ be a listing of T. We define a new list $\{t_{i,j}\}$

$$t_{i,j}(x) = \begin{cases} t_i(x) & \text{if } x \in \text{dom } t_i \text{ and } x \leq i+j \\ 0 & \text{if } x \notin \text{dom } t_i \text{ and } x \leq i+j \\ \Omega & \text{if } x > i+j \end{cases}$$

We set $G_j = \bigcup_{i=0}^{\infty} U_{t_{i,j}}$

and $A = \bigcap_{j=0}^{\infty} G_j$

Let $\varepsilon > 0$. We choose a j such that $2^{-j} < \varepsilon$. Since $A \subset G_j$

$$\mu A \leq \mu G_j \leq \sum_{i=0}^{\infty} \mu U_{t_{i,j}}$$

$$= \sum_{i=0}^{\infty} 2^{-(i+j+1)} = 2^{-j} < \varepsilon$$

This shows that A is a set of measure 0.

Since $B = \bar{A} = \bigcup_{j=0} \bar{G}_j$ we only have to show that $\bar{G}_j$ is nowhere dense.

$$\bar{G}_j = \bigcap_{i=0}^{\infty} \overline{U_{t_{i,j}}}$$

The following simple program computes a verifying function for $\bar{G}_j$.

$v_j(s)$: find an i such that $s \sqsubseteq t_{i,j}$;
output $t_{i,j}$.

Then $U_{v_j(s)} \subset U_{t_{i,j}}$ for some i and therefore $U_{v_j(s)} \cap \bar{G}_j = \phi$.

V. Decidability of the 'Almost All' Theory of Subrecursive degrees

This section is the heart of the paper. We give a decision procedure for the almost all theory of subrecursive degrees. The result is valid for all subrecursive reducibility orderings studied so far, i.e. for elementary degrees ([3]) and ([5]) and primitive recursive, doubly recursive, ... degrees ([2]) and ([3]).

Subrecursive degree structures are induced by a list of total recursive operators $\{Op_i\}_{i=0}^{\infty}$; e.g. the class of primitive recursive operators. This list of operators defines a partial ordering on R. Let f, g$\in$ R . We write $f \leq g$ if $f = Op_i [g]$ for some i. If $\{Op_i\}$ is the class of primitive recursive operators $f \leq g$ reads as "f is primitive recursive in g." Every element f$\in$ R generates a degree

$$d(f) = \{g \in R;\ f \leq g \text{ and } g \leq f\}$$

There is a minimum degree

$$0 = \{g;\quad g \leq f \quad \text{for all } f\}$$

in the example 0 is the class of primitive recursive functions. The partial ordering $\leq$ on R defines in a natural way an ordering on the degrees. The class of degrees is an upper semi-lattice under this ordering; the least upper bound of two degrees a=d(f) and b=d(g) is denoted by $a \cup b$ and is generated by f join g; here f join g (2x) = f(x) and f join g (2x+1) = g(x) for all x . It is not known whether the degree structure is a lattice, i.e. if two degrees a, b always have a greatest lower bound. If the meet of two degrees a, b does exist we will denote it by $a \cap b$.

We prove that meets almost always exist. So far we described the nonlogical symbols of the theory: the constant O, the functions $\cup$ and $\cap$ and the relation $\leq$.

The logical symbols are the connectives & and $\neg$ and the quantifier (almost $\forall$). We will mostly write $\overset{\infty}{\forall}$ instead of (almost $\forall a$). The interpretations of $\overset{\infty}{\forall} a P(a)$ is: the set $\{f;\ P(d(f))\}$ has measure 1.

The proof of the decision procedure follows Stillwell's proof of the decidability of the almost all theory of Turing degrees ([9]). First we show a normal form theorem for terms; i.e. every term is equivalent to a term containing only the function symbol $\cup$. Then we show that $t_1(a_1,\ldots a_n) \leq t_2(a_2,\ldots,a_n)$ is either true for almost all $a_1,\ldots,a_n$ or false for almost all $a_1,\ldots,a_n$. Then we proceed by induction on the rank of a formula. Finally we give some examples of theorems which can be proved using the decision procedure and we derive a bound on the complexity of the decision procedure.

We need a few lemmas:

<u>(5.1) Lemma</u>: Let F be a formula with the free variables $a_1,\ldots,a_n$. Then the set $\{(f_1,\ldots,f_n);F[d(f_1),\ldots,d(f_n)]\}$ is a Borel set; in particular it is integrable.

<u>Proof</u>: If we prove the assertion for atomic formula then a simple induction yields the assertion for all formula. In order to prove it for atomic formula $t_1(a_1,\ldots a_n) \leq t_2(a_1,\ldots,a_n)$ where t_1, t_2 are terms we proceed by induction on the rank of the terms. Again the induction is simple. The basic atomic

formula are: $a_1 \leq a_2$, $a_1 \leq a_2 \cup a_3$, $a_1 \cup a_2 \leq a_3$, $a_1 \cap a_2 \leq a_3$, $a_1 \leq a_2 \cap a_3$. We have to show that the set of instances satisfying any of these formula is a Borel set. E.g. we consider $a_1 \leq a_2$. Let $t \in T$, and let U_t stand for the open sphere $\{f;\ t \sqsubset f\}$. If $Op[t](w)$ is defined then $Op[f]\ (w) = Op\ [t](w)$ for all $f \in U_t$. Therefore

$$\{(f,g);\ d(f) \leq d(g)\} = \{(f,g);\ f = Op_i\ [g] \text{ for some } i\}$$

$$= \bigcup_{i \in \mathbb{N}} \{\{f,g);\ f = Op_i\ [g]\}$$

$$= \bigcup_{i \in \mathbb{N}} \bigcap_{s \in T} \left[(U_{Op_i[s]} x U_s) \cup \bigcup_{\substack{s \not\sqsubseteq t \\ t \not\sqsubseteq s}} (R \times U_t) \right]$$

The only other nontrivial case is $a_1 \cap a_2 \leq a_3$.

$$\{(f,g,h);\ d(f) \cap d(g) \leq d(h)\} =$$

$$= \{(f,g,h);\ \forall i \forall j \exists \ell\ [Op_i[f] = Op_j[g] \Rightarrow Op_i[f] = Op_i[h]]\}$$

$$= \bigcap_{i \in N} \bigcap_{j \in N} \bigcup_{\ell \in N} \{(f,g,h);\ Op_i[f] \neq Op_j[g] \vee Op_i[f]=Op_i[h]\}.$$

$$\{(f,g,h);\ Op_i[f] \neq Op_j[g] \vee Op_i[f] = Op_i[h]\}$$

$$= \{(f,g,h);\ Op_i[f] \neq Op_j[g]\} \cup \{(f,g,h);\ Op_i[f] = Op_i[h]\}$$

We show that the first set is Borel.

$$\{(f,g,h);\ Op_i[f] \neq Op_j[g]\} =$$

$$\bigcup_{\substack{s,t,v \in T \\ Op_i[s] \not\subset Op_j[t] \vee \\ Op_j[t] \not\subset Op_i[s]}} U_s \times U_t \times U_v$$

The induction step is left to the reader. Q.E.D.

The next lemma states that $a \cup b$ is a 'random' degree.

(5.2) Lemma: Let F be a formula. Then

$$\mu\{f; F(d(f))\} = \mu\{(f,g); F(d(f) \cup d(g))\}$$

Proof: The set of instances d(f) satisfying F is a Borel set by lemma (5.1). It is therefore sufficient to show that the mapping $G: B(R) \times B(R) \to B(R)$ induced by $(f,g) \mapsto f$ join g is measure preserving. This is obvious for open spheres; induction yields the assertion that for all Borel sets. Q.E.D.

Lemma (5.2) together with the next lemma and its corollary yields the normal form theorem for terms.

(5.3) Lemma: Given any $f, g, \in R$. If $f \nleq g$ then

$$\mu\{h; d(f) \leq d(g) \cup d(h)\} = 0$$

Proof: Since $\mu\{h; d(f) \leq d(g) \cup d(h\} =$

$$= \mu\Big(\bigcup_{i \in N} \{h; f = Op_i\ [g \text{ join } h]\}\Big)$$

$$\leq \sum_{i \in N} \mu\{h; f = Op_i\ [g \text{ join } h]\}$$

we have only to prove $\mu\{h; f = Op_i\ [g \text{ join } h]\} = 0$ for all i. Assume $\mu\{h; f = Op_i\ [g \text{ join } h]\} > 0$. Since $\{h; f = Op_i[g \text{ join } h]\} = \bigcap_{n \in \mathbb{N}} \{h; f_n \sqsubset Op_i[g \text{ join } h]\}$ the sequence $\{\mu\{h; f_n \sqsubset Op_i[g \text{ join } h]\}\}$ is a monotonically decreasing sequence of positive numbers. Therefore

$$\mu\{h; f=Op_i[f \text{ join } h]\}/\mu\{h; f_n \sqsubset Op_i[g \text{ join } h]\}$$

is defined for all n and converges against 1. Hence we can find

$\in N$ (by searching) such that

$$(*) \quad n \geq n_o \Rightarrow \mu\{h;\ f = Op_i[g \text{ join } h]\}/\mu\{h; f_n \sqsubset Op_i[g \text{ join } h]\} > 3/4$$

S be the set of initial segments t such that

$$f_{n_o} \sqsubseteq Op_i\ [g \text{ join } t\ ] \text{ for } t \in S$$

but not

$$f_{n_o} \sqsubseteq Op_i\ [g \text{ join } t'] \text{ for any } t' \sqsubset t.$$

S is a finite set. We can rewrite (*) as:

$$(**) \quad \frac{\displaystyle\sum_{t \in S} \mu(U_t) \frac{\mu\{\ h;\ f = Op_i\ [g \text{ join } h]\ \&\ t \sqsubset h\}}{\mu(U_t)}}{\displaystyle\sum_{t \in S} \mu\ (U_t)}$$

We can interpret (**) as a weighted average. Therefore there exists some $t_o \in S$ such that

$$3/4 \leq \mu\{h;\ f = Op_i\ [g \text{ join } h]\ \&\ t_o \sqsubset h\}\ /\ \mu(U_{t_o})$$

We can now give a simple program for computing f from g. It is easy to show that there is an operator Op_i' [ ] such that Op_i [g join h](n) converges at Op_i'[g](n) for all $h \in R$; i.e. that the arguments of all calls of h are less than Op_i'[g](n). (see [3]). The following program computes f.

<u>if</u> $x < n_o$ <u>then</u> Op_i[g join t_o] (x)

<u>else</u>

$\ell \leftarrow Op_i'$ [g](x) .

compute $Op_i[g \text{ join } t](x)$ for
all t such that
$t_0 \sqsubseteq t$ and $\ell h(t) = \ell$.
Output the value which is given
more than 3/4 of the times.

It is obvious from the remarks preceding the program that this program computes f. It is again an operator in the class $\{Op_i\}$ since all t's such as $\ell h(t) = \ell$ can be described by integers $\leq 2^{\ell}$ and exponentiation is already elementary recursive.

(5.4) Corollary: Given any f, $g \in R$. Then for almost all h $(d(f) \cup d(g)) \cap (d(f) \cup d(h))$ is defined and equal to $d(f)$.

Proof: The set of h such that

$$(d(f) \cup d(g) \cap (d(f) \cup d(h))$$

is either undefined or not equal to $d(f)$ can be written as follows:

$$\bigcup_{d(i) \leq d(f) \cup d(g)} \{h;\ d(f) < d(i) \leq d(f) \cup d(h)\}$$

By lemma (5.3) all these sets have measure 0. QED (5.4) enables us to compute the meet of any two terms $t_1 = a_1 \cup \ldots \cup a_n \cup b_1 \cup \ldots \cup b_m$ and $t_2 = a_1 \cup \ldots \cup a_n \cup c_1 \cup \ldots \cup c_k$; namely $t_1 \cap t_2 = a_1 \cup \ldots \cup a_n$ for almost all instances of the variables. We used here that $b_1 \cup \ldots \cup b_m$ and $c_1 \cup \ldots \cup c_k$ are random degrees. (lemma (6.2)). Applying this technique repeatedly we get the normal form theorem.

(5.5) Theorem: For every term t there is a term $t' = a_1 \cup \ldots \cup a_n$ such that $t = t'$ for almost all instances of the variables.

We turn now to the relation $\leq$. Let t_1, t_2 be terms. Then $t_1 \leq t_2$ if and only if $t_1 \cap t_2$ exists and $t_1 \cap t_2 = t_1$. We can assume w. l. o. g. that t_1, t_2 are in normal form, i.e. $t_1 = a_1 \cup \ldots \cup a_n \cup b_1 \cup \ldots \cup b_k$ and $t_2 = a_1 \cup \ldots \cup a_n \cup c_1 \cup \ldots \cup c_m$. Then $t_1 \cap t_2 = a_1 \cup \ldots \cup a_n$ for almost all instances of the variables. Therefore $t_1 \leq t_2$ if and only if $k = 0$. Hence either $t \leq t_2$ for almost all instances of the variables or $t_1 \not\leq t_2$ for almost all instances of the variables. So every atomic formula is satisfied by a set of instances whose measure is either 0 or 1. We say that atomic formulas are 0-1 valued.

We show this fact for all formulas by induction.

If $\phi = \phi_1 \,\&\, \phi_2$ with ϕ_1 and ϕ_2 0-1 valued then the 0-1 valuedness of ϕ follows from the following property of measures:

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

The same property yields the assertion for $\phi = \neg\phi_1$.

Assume now: $\phi = (\overset{\infty}{\forall} a_1)\,\phi_1(a_1, a_2, \ldots, a_n)$ with 0-1 valued ϕ_1. If $n = 1$ then ϕ is either true or false and hence 0-1 valued, if $n > 1$ then the assertion follows from the theorem of Fubini.

<u>(5.6) Theorem</u>: Every formula is 0-1 valued.

<u>(5.7) Corollary</u>: The 'almost all' theory of subrecursive degrees is decidable.

We give now two examples of theorems which can be proved using the decision procedure.

$$\underbrace{\overset{\infty}{\forall} a \; \overset{\infty}{\forall} b \; \underbrace{\left(\underbrace{\underbrace{\neg \underbrace{(a \leq b)}_{\text{0-valued}}}_{\text{1-valued}} \;\; \& \;\; \underbrace{\neg \underbrace{(b \leq a)}_{\text{0-valued}}}_{\text{1-valued}}}_{\text{1-valued}} \right)}_{\text{1-valued}}}_{\text{true}}$$

This theorem states that almost all pairs of degrees are incomparable. The next theorem strengthens this result. It states that almost all pairs of degrees are incomparable in a very strong sense.

$$\overset{\infty}{\forall} a \; \overset{\infty}{\forall} b \quad (a \cap b = 0)$$

i.e. their meet is the minimum degree. Lemma (5.2) states that $a \cup b$ is a 'random degree'. Therefore we can infer from the second example.

$$\overset{\infty}{\forall} c \, \exists a \; \exists b \; (c = a \cup b \;\&\; a \neq 0 \;\&\; b \neq 0 \;\&\; a \cap b = 0)$$

Finally we estimate the complexity of the decision procedure. A straightforward implementation runs in time $0(n^2)$ where n is the length of the input. A formula of length n contains at most n different variables. It is therefore possible to describe a term by a binary vector of length n. The number of operations is also bounded by n. Each operation costs at most n binary operations. This shows that the decision procedure can be implemented with runtine $0(n^2)$.

<u>(5.8) Theorem</u>: The time-complexity of the decision procedure has order less than $0(n^2)$.

References

[1] Bishop, E., Foundations of Constructive Analysis, McGraw Hill, 1967.

[2] Machtey, M., "Augmented Loop Languages and Classes of Computable Functions", JCSS, 1972, pp. 603-624.

[3] Machtey, M., "The Honest Subrecursive Classes are a Lattice", Purdue University, Comp. Sci. Tech. Rpt. 82.

[4] Mehlhorn, K., "On the Size of Sets of Computable Functions", Cornell University, Tech. Rpt., 72-164.

[5] Meyer, A.R., and D.M. Richie, "A Classification of the Recursive Functions", Zeitschrift fur math. Logik, Vol. 18, 1972.

[6] Oxtoby, Category and Measure.

[7] Rogers, H., Jr., Theory of Recursive Functions and Effective Computability, McGraw Hill, 1967.

[8] Sacks, Degrees of Unsolvability , Annals of Math. Studies, No. 55, Princeton University Press, Princeton, N.J., 1967.

[9] Stillwell, J., "Decidability of the 'Almost All' Theory of Degrees", JSL, Sept. 1972.