DECIDABLE PAIRING FUNCTIONS

by

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BIOGRAPHICAL SKETCH

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INTRODUCTION

In Chapter I of this paper we show that the usual, text-book pairing functions have decidable first-order theories. This will be done by exhibiting an infinite axiomatization of certain pairing functions which we characterize as "acyclic except for Δ". This condition is satisfied by the usual pairing functions. We then use a technique of Ehrenfeucht and Fraissé to show that the decision problem for such a pairing function is effectively reduced to the decision problem for the function restricted to Δ.

In contrast to the decidability of the first-order theories of certain pairing functions, we show that the weak second-order and monadic second-order theories of these pairing functions are undecidable.

In Chapter II we show how to extend the first-order Ehrenfeucht game to a second-order game. Using this extended game, we show that the monadic second-order theory of an equivalence relation is decidable.

Some of the results of Chapter I were announced in [16].

CHAPTER I

SOME DECIDABLE PAIRING FUNCTIONS

§1. Preliminaries

We denote by ω the non-negative integers, formed in the usual way. Thus $0 = \emptyset$, $n+1 = n \cup \{n\}$, and the usual linear ordering, $<$, is the same as $\in$; we shall write whichever seems more appropriate. We always assume that $i,j,k,\ell,m,n \in \omega$, and frequently that $p,q,r,s,t \in \omega$.

If Σ is a finite alphabet, then Σ^* denotes the set of all finite strings over Σ, and Σ^+ denotes the set of all non-null, finite strings over Σ. Unless otherwise stated, we assume $\Sigma = \{K, L\}$, and that σ, τ are elements of $\{K,L\}^*$. $\|\sigma\|$ denotes the length of σ, which is the number of symbols in σ. For $s \in \Sigma$, $s^k = s_0 \ldots s_{k-1}$, where $s_i = s$ for $i < k$. It will be necessary to establish a linear ordering, $\preccurlyeq$, on Σ^* by ordering first by length, and within equal lengths by the usual lexicographic ordering on the symbols of the strings involved, considered in reverse order. Specifically, for $\sigma = s_0 s_1 \ldots s_{\ell-1} \in \Sigma^*$, where $s_i \in \Sigma$, define $\sigma^\dagger = s_{\ell-1} \ldots s_1 s_0$, and let $\preccurlyeq_{lex}$ be the usual lexicographic ordering on $\{K,L\}^*$. Define $\preccurlyeq$ to be the linear ordering on Σ^* obtained by setting $\sigma \preccurlyeq \tau$ if and only if $\|\sigma\| < \|\tau\|$ or $(\|\sigma\| = \|\tau\|$ and $\sigma^\dagger \preccurlyeq_{lex} \tau^\dagger)$. Let $\sigma_0, \sigma_1, \ldots$ be the enumeration of Σ^* in which $\sigma_0 = \Lambda$ (= the empty string) and $i \leq j$ if and only if $\sigma_i \preccurlyeq \sigma_j$.

We shall also need the partial ordering, $\leq$, of Σ^* obtained by setting $\sigma \leq \tau$ if and only if there is a $\sigma' \in \Sigma^*$ such that $\sigma'\sigma = \tau$. As usual, we write $\sigma < \tau$ for $\sigma \leq \tau$ & $\sigma \neq \tau$. $Pd(\sigma)$ denotes the predecessor of σ with respect to $\leq$; i.e., $Pd(\sigma) = \tau$ if and only if $\tau < \sigma$ & $\forall \tau' (\tau' < \sigma \rightarrow \tau' \leq \tau)$.

For any set A, $\#(A)$ = cardinality of A.

By a relational system $\mathfrak{M} = \langle \Omega, R_i \rangle_{i \in I}$ we mean a non-empty set Ω together with a sequence of relations indexed by a set I, in which $R_i \subseteq \Omega^{n(i)}$, where $n: I \rightarrow \omega$. $\mathfrak{M}$ is finitary if and only if $\#(I) < \aleph_0$. Two relational systems, $\mathfrak{M}_0 = \langle \Omega_0, R_i \rangle_{i \in I_0}$ and $\mathfrak{M}_1 = \langle \Omega_1, S_i \rangle_{i \in I_1}$, are similar if and only if $I_0 = I_1$ and for $i \in I_0$, $R_i \subseteq \Omega_0^{n(i)}$ if and only if $S_i \subseteq \Omega_1^{n(i)}$.

Associated with any relational system in an obvious fashion is a first-order language with equality. Thus when we specify a relational system, we also specify a first-order language, and we assume without further indication that the formulas mentioned are formulas of that language. We shall concern ourselves with two languages: $\mathcal{L}_0$, which has binary relational symbols for the relations that are the graphs of the projections, K and L, of a pairing function J; and $\mathcal{L}_1$, which is obtained from $\mathcal{L}_0$ by adding symbols for two unary relations Γ, Δ. It should be noted that we shall frequently write formulas as though the language had functions and as though unary relations were sets; it is to be understood that these formulas are abbreviations for formulas properly in the language. If $\mathfrak{M} = \langle \Omega, R_0, \ldots \rangle$, we write

$a \in \mathfrak{M}$ for $a \in \Omega$, and $A \subseteq \mathfrak{M}$ for $A \subseteq \Omega$.

We shall use the notation $\varphi(v_0,\ldots,v_{k-1})$ to indicate that φ is a formula with free variables among $v_0,\ldots,v_{k-1}$; if $\varphi(v_0,\ldots,v_k)$ is such a formula, then we shall use the notation $\mathfrak{M} \models \varphi[a_0,\ldots,a_{k-1}]$ to mean that $a_j \in \mathfrak{M}$ $(j < k)$ and that φ is satisfied in $\mathfrak{M}$ when v_j is replaced by a_j $(j < k)$ in φ.

For any formula φ, $(\varphi)^0 = \varphi$ and $(\varphi)^1 = \neg\varphi$.

The framework for proving our results includes some work of Ehrenfeucht [4] which we shall sketch here. Assume that $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are two similar relational systems; then the Ehrenfeucht game $G_n(\mathfrak{M}_0,\mathfrak{M}_1)$ is played as follows. There are two players, designated I and II. At move j $(j < n)$, player I selects one of the relational systems ($\mathfrak{M}_0$ or $\mathfrak{M}_1$) and an element from that relational system. Player II then selects an element from the other relational system. Designate by x_j the element chosen from $\mathfrak{M}_0$ at move j, regardless of which player chose it; similarly, $y_j \in \mathfrak{M}_1$.

Player II wins $G_n(\mathfrak{M}_0,\mathfrak{M}_1)$ if and only if, at the end of move $n-1$, the map $x_0 \longleftrightarrow y_0,\ldots,x_{n-1} \longleftrightarrow y_{n-1}$ is an isomorphism between $\{x_j \mid j < n\}$ and $\{y_j \mid j < n\}$ with respect to the relations of the relational systems, restricted to these sets. Otherwise player I wins.

Player II is said to have a winning strategy if there is a rule by which he can always win, regardless of player I's choices. Specifically, a strategy for player II is a sequence of functions, $f_0,f_1,\ldots,f_{n-1}$, such that

$f_j : (\mathfrak{M}_0 \times \mathfrak{M}_1)^j \times \mathfrak{M}_k \times 2 \to \mathfrak{M}_{1-k}$. Assume that at the beginning of move j, for $j < n$, $x_0 \longleftrightarrow y_0, \ldots, x_{j-1} \longleftrightarrow y_{j-1}$ have been chosen, and assume further that player I selects $z \in \mathfrak{M}_k$ $(k < 2)$ as his choice. Player II is using the strategy $f_0, \ldots, f_{n-1}$ if at move j he selects $f_j(x_0, y_0, \ldots, x_{j-1}, y_{j-1}, z, k) \in \mathfrak{M}_{1-k}$. The strategy is a winning strategy for $G_n(\mathfrak{M}_0, \mathfrak{M}_1)$ if player II can always win $G_n(\mathfrak{M}_0, \mathfrak{M}_1)$ by using it, regardless of player I's choices.

We define inductively a sequence of equivalence relations on similar relational systems with certain elements selected. Assume $\mathfrak{M}_0$ and $\mathfrak{M}_1$ are similar relational systems and that $\{x_j | j < k\} \subseteq \mathfrak{M}_0$ and $\{y_j | j < k\} \subseteq \mathfrak{M}_1$, then $\langle \mathfrak{M}_0, x_j \rangle_{j<k} \equiv_0 \langle \mathfrak{M}_1, y_j \rangle_{j<k}$ if and only if the map $x_j \longleftrightarrow y_j$ $(j < k)$ is an isomorphism between $\{x_j | j < k\}$ and $\{y_j | j < k\}$ with respect to the relations of $\mathfrak{M}_0$ and $\mathfrak{M}_1$ restricted to these sets. And $\langle \mathfrak{M}_0, x_j \rangle_{j<k} \equiv_{n+1} \langle \mathfrak{M}_1, y_j \rangle_{j<k}$ if and only if for each $x_k \in \mathfrak{M}_0$ there is a $y_k \in \mathfrak{M}_1$ and for each $y_k \in \mathfrak{M}_1$ there is an $x_k \in \mathfrak{M}_0$ such that $\langle \mathfrak{M}_0, x_j \rangle_{j<k+1} \equiv_n \langle \mathfrak{M}_1, y_j \rangle_{j<k+1}$. For the case $k = 0$ we write $\mathfrak{M}_0 \equiv_n \mathfrak{M}_1$. A moments reflection should convince the reader that player II has a winning strategy for $G_n(\mathfrak{M}_0, \mathfrak{M}_1)$ after the k moves $x_0 \longleftrightarrow y_0, \ldots, x_{k-1} \longleftrightarrow y_{k-1}$ $(k \leq n)$ if and only if $\langle \mathfrak{M}_0, x_j \rangle_{j<k} \equiv_{n-k} \langle \mathfrak{M}_1, y_j \rangle_{j<k}$.

Theorem 1 (Ehrenfeucht and Fraissé). Let $\psi(v_0,\ldots,v_{k-1}) = Q_k v_k \ldots Q_{n-1} v_{n-1} \varphi(v_0,\ldots,v_{n-1})$, where Q_i is a quantifier and φ is quantifier free. If $\langle \mathfrak{M}_0, x_j\rangle_{j<k} \equiv_{n-k} \langle \mathfrak{M}_1, y_j\rangle_{j<k}$, then $\mathfrak{M}_0 \models \psi[x_0,\ldots,x_{k-1}]$ if and only if $\mathfrak{M}_1 \models \psi[y_0,\ldots,y_{k-1}]$.

Proof: Due to Ehrenfeucht [4].

We proceed by induction on $n-k$. For $n = k$, the result follows simply from definitions. Assume the result holds for some $k+1 \leq n$. Without loss of generality, assume that Q_k is $\exists$. Let $\psi'(v_0,\ldots,v_k) = Q_{k+1}v_{k+1}\ldots Q_{n-1}v_{n-1}\varphi(v_0,\ldots,v_{n-1})$. Thus $\mathfrak{M}_0 \models \psi[x_0,\ldots,x_{k-1}]$ if and only if $\mathfrak{M} \models \psi'[x_0,\ldots,x_k]$ for some $x_k \in \mathfrak{M}_0$. By the definition of $\equiv_{n-k}$, there is now a $y_k \in \mathfrak{M}_1$ such that $\langle \mathfrak{M}_0, x_j\rangle_{j<k+1} \equiv_{n-k-1} \langle \mathfrak{M}_1, y_j\rangle_{j<k+1}$ and by the inductive hypothesis, this gives $\mathfrak{M}_1 \models \psi'[y_0,\ldots,y_{k-1}]$, hence $\mathfrak{M}_1 \models \psi[y_0,\ldots,y_{k-1}]$. The other direction of the biconditional is similar. Hence the result holds for k.

Theorem 2 (Ehrenfeucht). For every finitary relational system $\mathfrak{M}_0$ and every set of elements $\{x_j | j < k\} \subseteq \mathfrak{M}_0$, there is a formula $\chi(v_0,\ldots,v_{k-1})$ such that for any finitary relational system $\mathfrak{M}_1$ and set of elements $\{y_j | j < k\} \subseteq \mathfrak{M}_1$, $\langle \mathfrak{M}_0, x_j\rangle_{j<k} \equiv_{n-k} \langle \mathfrak{M}_1, y_j\rangle_{j<k}$ if and only if $\mathfrak{M}_1 \models \chi[y_0,\ldots,y_{k-1}]$.

Proof: From Ehrenfeucht [4]. We form a sequence of sets of formulas which we shall refer to as Ehrenfeucht formulas, $\{\chi_{i,j}(v_0,\ldots,v_{n-i-1}) | j < m_j\}$, for $i < n$, such that

a) $\bigvee_{j<m_i} \chi_{i,j}$ and b) $\bigwedge_{j<k<m_i} \neg(\chi_{i,j} \,\&\, \chi_{i,k})$ are tautologies of the predicate calculus. Begin by forming $\{\chi_{0,j}(v_0,\ldots,v_{n-1}) \mid j < m_0\}$ to satisfy a) and b), and such that for any quantifier free formula $\varphi(v_0,\ldots,v_{n-1})$, either $\chi_{0,j} \to \varphi$ or $\chi_{0,j} \to \neg\varphi$ is a tautology of the predicate calculus. That this is possible with finite m_0 is a consequence of the assumption that $\mathcal{M}_0$ and $\mathcal{M}_1$ are finitary relational systems, and hence the language associated with them has only a finite number of relation symbols. Assume that $\{\chi_{i,j} \mid j < m_i\}$ has been formed; let $\chi_{i+1,j}(v_0,\ldots,v_{n-i-2}) = \bigwedge_{k<m_i} (\exists v_{n-i-1}\chi_{i,k}(v_0,\ldots,v_{n-i-1}))^{j(k)}$, where $j \in 2^{m_i}$ (and hence may be viewed both as an integer and as a map $j{:}m_i \to 2$). We claim that one of the $\chi_{i,j}$ satisfies the theorem. This may be shown by induction on n-k. For $n = k$, the result follows by the definitions of $\equiv_0$ and of $\{\chi_{0,j} \mid j < m_0\}$. Now assume the result for some $k+1 \leq n$. Assume that $x_0 \longleftrightarrow y_0,\ldots,x_{k-1} \longleftrightarrow y_{k-1}$ have been chosen. Assume that player I selects x_k at his turn (if he selects y_k, the argument is similar), and that

$$\mathcal{M}_0 \models \chi_{n-k,j_0}[x_0,\ldots,x_{k-1}].$$

There are two cases.

<u>Case 1</u>. $\mathcal{M}_1 \models \chi_{n-k,j_0}[y_0,\ldots,y_{k-1}]$.
Pick j_1 so that $\mathcal{M}_0 \models \chi_{n-k-1,j_1}[x_0,\ldots,x_k]$. By the definition of the Ehrenfeucht formulas, we must have $\mathcal{M}_1 \models \exists v_k\chi_{n-k-1,j_1}[y_0,\ldots,y_{k-1}]$. Player II may then select

any y_k such that $\mathfrak{M}_1 \models \chi_{n-k-1,j_1}[y_0,\ldots,y_k]$. We are then done by the inductive hypothesis.

Case 2. $\mathfrak{M}_1 \models \neg \chi_{n-k,j_0}[y_0,\ldots,y_{k-1}]$.
Then there is a j_1 such that $\mathfrak{M}_0 \models \exists v_k \chi_{n-k-1,j_1}[x_0,\ldots,x_{k-1}]$ and $\mathfrak{M}_1 \models \neg \exists v_k \chi_{n-k-1,j_1}[y_0,\ldots,y_{k-1}]$. Hence player I can win by selecting $x_k \in \mathfrak{M}_0$ such that $\mathfrak{M}_0 \models \chi_{n-k-1,j_1}[x_0,\ldots,x_k]$.

§2. Pairing Functions

Definition. A pairing function J on a set Ω is a one-to-one map $J:\Omega^2 \to \Omega$, not necessarily onto.

We have found it convenient to work with the projections of a pairing function, K and L, which satisfy $K(J(x,y)) = x$, and $L(J(x,y)) = y$. Specifically, we require that the projections satisfy the following two axioms.

(A1) $\forall x\, \forall y\, \exists!z[K(z) = x \,\&\, L(z) = y]$

(A2) $\forall z[\exists x(K(z) = x \vee L(z) = x) \to$
$\exists!x\exists!y(K(z) = x \,\&\, L(z) = y)]$

It has been shown by Hanf (see [13]) and, independently, by Morley [9] that the theory of these two axioms is not decidable. We shall presently give an infinite set of additional axioms from which we will be able to show that the usual pairing functions have decidable theories.

As motivation, consider the case $J(x,y) = 2^x3^y$, and consider $z = 2^5 3^2$. $K(z) = 5$, and $L(z) = 2$; $LK(z)$ is undefined, and so is KLL. The situation is best captured by considering the tree in Figure 1. The edges of the tree are labelled with the $\sigma \in \{K,L\}^*$ in such a way that $\sigma(2^5 3^2)$ is the node below the edge labelled σ.

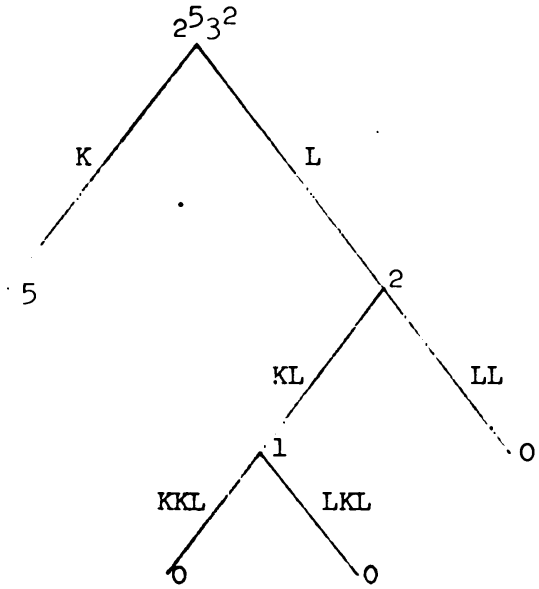

Figure 1

First we partition Ω into two disjoint parts Γ, Δ:

(A3) $\forall z[z \in \Gamma \longleftrightarrow z \notin \Delta]$.

Next, for those z in Γ we demand that the pairing function be what we shall call "acyclic". For each $\sigma \in \{K,L\}^+$ we have the following axiom schema:

(A4) $\forall z[(z \in \Gamma \mathbin{\&} \exists x(\sigma(z) = x)) \rightarrow \sigma(z) \neq z]$.

Furthermore, for z in Γ we demand that z have projections:

(A5) $\forall z[z \in \Gamma \rightarrow \exists x(K(z) = x)]$.

Note that by (A2), axiom (A5) implies the existence of both projections for z in Γ. Finally, once an element lies in Δ, we wish to insure that all iterations of projections from that element also lie in Δ (as long as they are defined):

(A6) $\forall z[(z \in \Delta \;\&\; \exists x(K(z) = x)) \rightarrow (K(z) \in \Delta \;\&\; L(z) \in \Delta)]$.

We shall call a pairing function satisfying A1 to A6 _acyclic except for_ Δ.

Note that the standard pairing functions provide models of these axioms.

Examples:

$J(x,y) = 2^x3^y,\ \Delta = \{z \mid \neg\exists x \exists y(z = 2^x3^y)\}$;

$J(x,y) = \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y),\ \Delta = \{0,1\}$;

$J(x,y) = \max\{x^2 + y, y^2 + 2y - x\},\ \Delta = \{0,1\}$;

$J(x,y) = 2^x(2y + 1) - 1,\ \Delta = \{0,1\}$.

§3. Decidability and Related Results

Following Bell and Slomson [1], if $\mathfrak{M}$ is a relational system and $A \subseteq \mathfrak{M}$, we write $\mathfrak{M} \restriction A$ for the restriction of $\mathfrak{M}$ to A. For the remainder of this section we shall assume that $\mathfrak{M}$, $\mathfrak{M}_0$, and $\mathfrak{M}_1$ are models of the pairing function axioms A1 to A6 (unless otherwise stated), where $\mathfrak{M}_i = \langle \Omega_i, K_i, L_i, \Gamma_i, \Delta_i \rangle$, $i = 0,1$, and $\mathfrak{M} = \langle \Omega, K, L, \Gamma, \Delta \rangle$. By a simple abuse of notation we write $\mathfrak{M} \restriction \Delta$ to mean the restriction of $\mathfrak{M}$ to the set corresponding to the unary relation Δ.

Theorem 3. Let $g(k) = 2^{4^k}$. If $\mathfrak{M}_0 \restriction \Delta_0 \equiv_{g(n)} \mathfrak{M}_1 \restriction \Delta_1$, then one can effectively show that $\mathfrak{M}_0 \equiv_n \mathfrak{M}_1$.

Proof: By the remarks above, it is enough to show that player II has a winning strategy for the Ehrenfeucht game $G_n(\mathfrak{M}_0, \mathfrak{M}_1)$, given a strategy for $G_{g(n)}(\mathfrak{M}_0 \restriction \Delta_0, \mathfrak{M}_1 \restriction \Delta_1)$. We shall show this by induction on the moves of the game.

Enough of the proof will be given here to state the key lemma from which this theorem and others follow. The proof of this lemma however will be deferred to the next section.

There are two cases: either $\Gamma_i = \emptyset$ or $\Gamma_i \neq \emptyset$, for $i < 2$. If $\Gamma_i = \emptyset$, then $\mathfrak{M}_i \restriction \Delta_i = \mathfrak{M}_i$, and $\mathfrak{M}_0 \equiv_n \mathfrak{M}_1$ follows immediately. Hence, in what follows, assume $\Gamma_i \neq \emptyset$.

The proof is inspired by the proof of Lemma 2.3 in Hanf [7]. Our r-neighborhood of an element x stems from his sphere of radius r. We shall want notation with which to define and describe the r-neighborhood of an element. If

$x \in \mathcal{M}$, and if $A \subseteq \mathcal{M}$, then we can define a function, $f_x : \Sigma^* \to \Sigma^*$, and subsets B_x, $C_{x,A}$, and E_x of Σ^* satisfying

a) $f_x(\sigma) = \begin{cases} \text{least } \tau \text{ (w.r.t. } \prec\text{) such that } \sigma(x) = \tau(x), & \text{if } \sigma(x) \text{ is defined,} \\ \sigma & \text{otherwise;} \end{cases}$

b) $\sigma \in B_x$ if and only if $\sigma(x) = \tau(x)$, for some $\tau \prec \sigma$; i.e. $\sigma \in B_x$ if and only if $f_x(\sigma) \neq \sigma$;

c) $\sigma \in C_{x,A}$ if and only if $\sigma(x) \in A$ or $\sigma > \tau$ for some $\tau \in C_{x,A}$;

d) $\sigma \in E_x$ if and only if $\sigma(x) \in \Delta$.

For any analysis we perform, only a finite, recursively bounded portion of f_x, B_x, $C_{x,A}$, or E_x need ever be calculated; but the specific bound will often be suppressed. When it is necessary to indicate a bound, we will use this notation:

a) if $S \subseteq \Sigma^*$, then $S^{(r)} = \{\sigma \mid \sigma \in S \text{ and } \|\sigma\| \leq r\}$;

b) if $g : \Sigma^* \to \Sigma^*$, then $g^{(r)} = g \upharpoonright (\Sigma^*)^{(r)}$.

With the aid of these definitions, we can define the <u>r-frontier of x</u>,

$$F(r,x) = \{\sigma \mid \sigma \in E_x^{(r)} \text{ or } [\|\sigma\| = r \text{ and } \forall\tau(\tau < \sigma \to \tau \notin E_x)]\};$$

the <u>r-interior of x</u>,

$$I(r,x) = \{\sigma \mid \exists\tau(\tau \in F(r,x) \ \&\ \sigma < \tau)\}; \text{ and finally,}$$

the <u>r-neighborhood of x</u>,

$$N(r,x) = \{\sigma(x) \mid \sigma \in \bigcup_{t \leq r} F(t,x)\} = \{\sigma(x) \mid \sigma \in F(r,x) \cup I(r,x)\}.$$

The reader is advised to consider the following simple example. Assume that $J(x,y) = 2^x 3^y$ is the pairing function under consideration, and that $A = \{1\}$. Then the description of $x = \langle\langle\langle 54,288\rangle,10\rangle,\langle 54,288\rangle\rangle$ to three levels in the presence of A can be represented by Figure 2, where a node σ in B_x is indicated by a box (with a dotted arrow pointing to $f_x(\sigma)$), a node in $C_{x,A}$ is indicated by a diamond, and a node in E_x is indicated by a circle. The 3-frontier $F(3,x) = \{KKK, LKK, LK, KKL, LKL, KLL, LLL\}$.

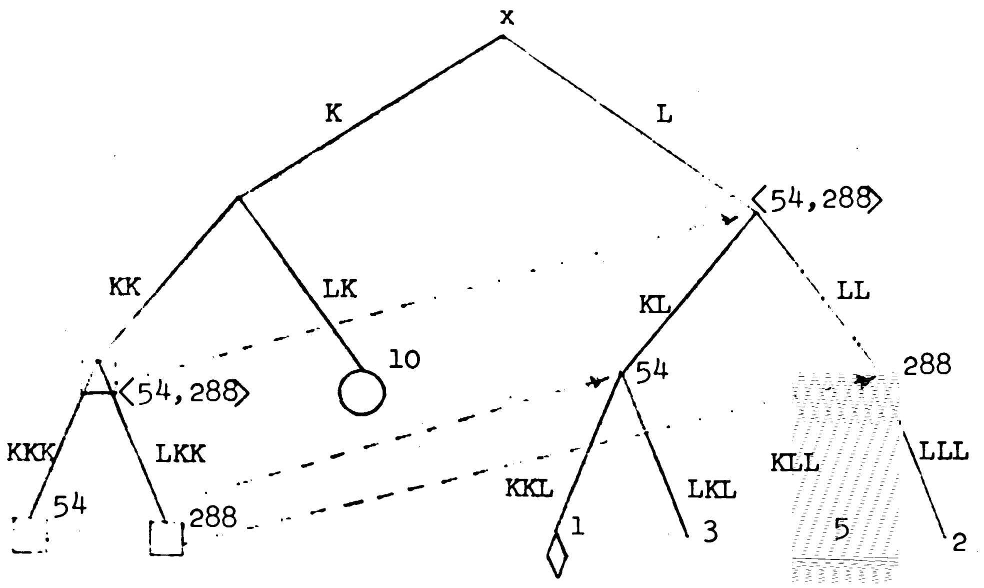

Figure 2

<u>Definition</u>. Assume $\{x_j | j < k\} \subseteq \mathfrak{M}_0$ and $\{y_j | j < k\} \subseteq \mathfrak{M}_1$. Let $N_0 = \bigcup_{j<k} N(r,x_j)$, and $N_1 = \bigcup_{j<k} N(r,y_j)$. N_0 and N_1 are <u>strongly isomorphic</u> (<u>via</u> θ) (in symbols $\theta:N_0 \approx N_1$), if and only if there is an isomorphism $\theta:N_0 \to N_1$ such that

a) $\theta(x_j) = y_j$, for $j < k$, and

b) $\langle \mathfrak{M}_0 \upharpoonright \Delta_0, z_j \rangle_{j<m} \equiv_{g(n-k)} \langle \mathfrak{M}_1 \upharpoonright \Delta_1, \theta(z_j) \rangle_{j<m}$,

where $\{z_j | j < m\} = N_0 \cap \Delta_0$ and $g(i) = 2^{4^i}$. We emphasize that θ is an isomorphism with respect to the relations which are the graphs of the projections K and L (of the pairing function J) as well as with respect to Γ and Δ. Clearly $\approx$ depends on n and k, but their values will be clear from context. It is easily checked that $\approx$ is an equivalence relation.

We can now state the key lemma from which the theorem easily follows.

<u>Lemma 1</u>. Let $g(k) = 2^{4^k}$, and assume that $\{x_j | j < k\} \subseteq \mathfrak{M}_0$ and $\{y_j | j < k\} \subseteq \mathfrak{M}_1$ for some $k < n$. If there is a strong isomorphism $\theta_k: \bigcup_{j<k} N(4^{n-k},x_j) \approx \bigcup_{j<k} N(4^{n-k},y_j)$, then for each $x_k \in \mathfrak{M}_0$ one can effectively find a $y_k \in \mathfrak{M}_1$ and for each $y_k \in \mathfrak{M}_1$ one can effectively find an $x_k \in \mathfrak{M}_0$ such that there is a strong isomorphism

$$\theta_{k+1}: \bigcup_{j<k+1} N(4^{n-k-1},x_j) \approx \bigcup_{j<k+1} N(4^{n-k-1},y_j).$$

We remark that this suffices to prove the theorem, since a winning strategy for player II is to choose $y_k \in \mathfrak{M}_1$ (or

$x_k \in \mathfrak{M}_0$, whichever is appropriate) at move k. At the end of the last move $(k = n-1)$, the definitions of strong isomorphism and of winning show that this strategy works. The proof of this lemma will be given in the next section; the remainder of this section will be devoted to some of its consequences.

<u>Corollary 1</u>. Let $\psi(v_0,\ldots,v_{k-1}) = Q_k v_k \ldots Q_{n-1} v_{n-1} \varphi(v_0,\ldots,v_{n-1})$ be a formula in which φ is quantifier free and Q_i is a quantifier. Assume $\{x_j | j < k\} \subseteq \mathfrak{M}_0$, $\{y_j | j < k\} \subseteq \mathfrak{M}_1$, and $\bigcup_{j<k} N(4^{n-k},x_j) \approx \bigcup_{j<k} N(4^{n-k},y_j)$. Then $\mathfrak{M}_0 \models \psi[x_0,\ldots,x_{k-1}]$ if and only if $\mathfrak{M}_1 \models \psi[x_0,\ldots,y_{k-1}]$.

<u>Proof</u>: Immediate from Lemma 1 and Theorem 1.

Intuitively, in the case $k = 1$, this corollary shows that a formula (in prenex form) with one free variable and $n-1$ quantifiers can describe at most the 4^{n-1} neighborhood of the elements that satisfy it.

The remaining theorems in this section are more semantic in nature, and thus we give a semantic characterization of neighborhoods.

<u>Definition</u>. A formula $\eta^{(r)}(v_0,\ldots,v_{k-1})$ is an <u>r-neighborhood description</u> (<u>of</u> $v_0,\ldots,v_{k-1}$) if and only if $\mathfrak{M} \models \eta^{(r)}[x_0,\ldots,x_{k-1}]$ and $\mathfrak{M} \models \eta^{(r)}[y_0,\ldots,y_{k-1}]$ imply that $\bigcup_{j<k} N(r,x_j) \approx \bigcup_{j<k} N(r,y_j)$.

In the following technical lemma we give a syntactic characterization of a neighborhood description. It is exactly what one would expect: an exhaustive listing of all the relationships that obtain in the neighborhood. For convenience, we will write variables in the form $u_{i,\sigma}$ which will correspond to $\sigma(v_i)$.

<u>Lemma 2</u>. Any r-neighborhood description of $v_0,\ldots,v_{k-1}$ is equivalent to a formula of the form

$$\exists u_{i,K}\,\exists u_{i,L}\ldots\exists u_{j,\sigma}(R(u_{i,K},u_{i,L},\ldots,u_{j,\sigma})$$

$$\& \bigwedge_{(j,\tau)\in I}\Gamma(u_{j,\tau})\ \&\ \bigwedge_{\substack{(i,\sigma)\in I\\(j,\tau)\in I}}(u_{i,\sigma}=u_{j,\tau})^{h(i,\sigma,j,\tau)}$$

$$\& \bigwedge_{\substack{(i,\sigma)\in I\\(j,\tau)\in J}}\neg(u_{i,\sigma}=u_{j,\tau})\ \&\ \bigwedge_{(j,\tau)\in J}\Delta(u_{j,\tau})$$

$$\& \ \chi(u_{j,\tau})_{(j,\tau)\in J})\ ,$$

where $u_{i,\Lambda}$ is understood to be v_i; and if we let $U = \{(i,\sigma)|u_{i,\sigma}$ occurs in the prefix$\}$ and $V = U \cup \{(i,\Lambda)|i< k\}$, then $(i,\sigma)\in U$ if and only if $i < k$, $0 < \|\sigma\| \leq r$, and $\Gamma(u_{i,\tau})$ occurs later in the formula for $\tau = Pd(\sigma)$; R is a conjunction of atomic formulas of the form $K(u,v)\ \&\ L(u,w)$, where $K(u_{i,\sigma},u_{j,\tau})\ \&\ L(u_{i,\sigma},u_{m,\rho})$ occurs in R if and only if $i = j = m$, $K\sigma = \tau$, $L\sigma = \rho$, and each of $(i,\sigma),(j,\tau)$, and $(m,\rho)\in V$; if $(j,\tau)\in I$ then $(j,\sigma)\in I$, for all $\sigma \leq \tau$; $h(i,\sigma,i,\sigma) = 0$; $h(i,\sigma,j,\tau) = h(j,\tau,i,\sigma)$; if $h(i,\sigma,j,\tau) = 0$ and $h(j,\tau,m,\rho) = 0$, then $h(i,\sigma,m,\rho) = 0$; if $\sigma \leq \tau$ then $h(i,\sigma,i,\tau) \neq 0$; $J \cup I = V$; and $\chi(u_{j,\tau})_{(j,\tau)\in J}$ is an Ehrenfeucht formula (described in the proof of Theorem 2).

Proof: Let $\eta(v_0,\ldots,v_{n-1})$ be the formula of the lemma.

In one direction, we show that η is an r-neighborhood description. Assume that $\mathfrak{M} \models \eta[x_0,\ldots,x_{k-1}]$ and that $\mathfrak{M} \models \eta[y_0,\ldots,y_{k-1}]$. η is of the form $\exists u_{i,K}\, \exists u_{i,L} \ldots \exists u_{j,\sigma}\psi$. Pick $z_{i,\tau} \in \mathfrak{M}$ to be a witness for $\exists u_{i,\tau}$, subject to the constraint that $z_{i,\Lambda} = x_i$; pick $z'_{i,\tau}$ similarly with $z'_{i,\Lambda} = y_i$. Then $\theta: N_0 \to N_1$ defined by $\theta(z_{i,\tau}) = z'_{i,\tau}$ is easily seen to be a strong isomorphism.

For the other direction, it is clear that any r-neighborhood can be so described, for the restrictions on η are those forced by the axioms. We examine some of these in detail; the interested reader may complete the examination. The first three restrictions on $h(i,\sigma,j,\tau)$ derive from the fact that equality is an equivalence relation; the last one derives from axiom schema A4. Other restrictions are most easily comprehended semantically. Assume that $\mathfrak{M} \models \eta[x_0,\ldots,x_{n-1}]$. One easily checks that $(i,\sigma) \in U$ if and only if $\sigma \in (F(r,x_i) \cup I(r,x_i)) - \Lambda$; $(i,\sigma) \in I$ if and only if $\sigma(x_i) \in \Gamma$; and $(i,\sigma) \in J$ if and only if $\sigma(x_i) \in \Delta$.

One easily sees from this lemma that there are only a finite number of different r-neighborhood descriptions of $v_0,\ldots,v_{k-1}$, and henceforth we assume that $\{\eta_j^{(r)}(v_0,\ldots,v_{k-1}) \mid j < q(r,k)\}$ is an effective enumeration of them.

Corollary 2. Let $\psi(v_0,\dots,v_{k-1})\ Q_k v_k \dots Q_{n-1} v_{n-1} \varphi(v_0,\dots,v_{n-1})$ be a formula in which Q_i is a quantifier and φ is quantifier free. Let $\eta^{(r)}(v_0,\dots,v_{k-1})$ be an r-neighborhood description, where $r = 4^{n-k}$. Then either $\eta^{(r)}(v_0,\dots,v_{k-1}) \to \psi(v_0,\dots,v_{k-1})$ or $\eta^{(r)}(v_0,\dots,v_{k-1}) \to \neg\psi(v_0,\dots,v_{k-1})$ is a tautology of the predicate calculus.

Proof: Immediate from Lemma 2 and Corollary 1.

For any relational structure $\mathfrak{M}$ we may define its theory, $\mathrm{Th}(\mathfrak{M}) = \{\varphi | \mathfrak{M} \models \varphi\}$. Furthermore, in a manner akin to Feferman [5], we fix a standard Gödel numbering which assigns to each formula φ its Gödel number $\ulcorner\varphi\urcorner$. To a theory T we assign a set of numbers $\ulcorner T\urcorner = \{\ulcorner\varphi\urcorner \mid T \models \varphi\}$, where $T \models \varphi$ means that any model of T is a model of φ. The degree of T, deg(T), is defined to be the degree of unsolvability of $\ulcorner T\urcorner$. Having established this notation, we may now state the following theorem.

Theorem 4. $\deg \mathrm{Th}(\mathfrak{M}) = \deg \mathrm{Th}(\mathfrak{M}\restriction\Delta)$.

Proof: It is clear that $\deg \mathrm{Th}(\mathfrak{M}\restriction\Delta) \leq \deg \mathrm{Th}(\mathfrak{M})$. To show that $\deg \mathrm{Th}(\mathfrak{M}) \leq \deg(\mathfrak{M}\restriction\Delta)$ involves two cases.

Case 1. $\Gamma = \emptyset$. The result is immediate, since $\mathfrak{M} = \mathfrak{M}\restriction\Delta$.

Case 2. $\Gamma \neq \emptyset$. Consider a statement $\psi = Q_0 v_0 \dots Q_{n-1} v_{n-1} \varphi(v_0,\dots,v_{n-1})$, where Q_i is a quantifier and φ is quantifier free. We successively transform ψ by stages, where each stage modifies the result of the previous stage. We then show that the result of these transformations is equivalent to ψ by showing that the output of any stage is equivalent to its input.

Let $\psi_k(v_0,\ldots,v_{k-1}) = Q_k v_k \cdots Q_{n-1} v_{n-1} \varphi(v_0,\ldots,v_{n-1})$; thus $\psi_k = Q_k v_k \psi_{k+1}$.

Stage k: $(k < n)$. Let $r = 4^{n-k-1}$, and let $\{\eta_j(v_0,\ldots,v_k) \mid j < q_{k+1}\}$ be the enumeration of r-neighborhood descriptions mentioned above.

Case 1. Q_k is $\exists$. Replace each occurrence of $\exists v_k \psi_{k+1}$ by $\bigvee_{j<q_{k+1}} \exists v_k(\eta_j(v_0,\ldots,v_k) \;\&\; \psi_{k+1}(v_0,\ldots,v_k))$.

Case 2. Q_k is $\forall$. Replace each occurrence of $\forall v_k \psi_{k+1}$ by $\bigwedge_{j<q_{k+1}} \forall v_k(\eta_j(v_0,\ldots,v_k) \rightarrow \psi_{k+1}(v_0,\ldots,v_k))$. Then replace each occurrence of $\forall v_k(\eta_j \rightarrow \psi_{k+1})$ by $\exists v_k(\eta_j \;\&\; \psi_{k+1}) \vee \neg \exists v_k \eta_j$.

We now verify that the transformation at stage k produces an output equivalent to its input. There are two cases:

Case 1. Q_k is $\exists$. We must show that $\exists v_k \psi_{k+1} \longleftrightarrow \bigvee_{j<q_{k+1}} \exists v_k(\eta_j \;\&\; \psi_{k+1})$. In one direction, $\bigvee_{j<q_{k+1}} \exists v_k(\eta_j \;\&\; \psi_{k+1}) \rightarrow \exists v_k \psi_{k+1}$ follows from the predicate calculus. In the other direction, assume $\exists v_k \psi_{k+1}(v_0,\ldots,v_k)$; then for some u, we have $\psi_{k+1}(v_0,\ldots,v_{k-1},u)$. By the definition of the r-neighborhood description, there must be a $j_0 < q_{k+1}$ such that $\eta_{j_0}(v_0,\ldots,v_{k-1},u)$. Hence

$\exists v_k \eta_{j_0}(v_0,\ldots,v_k)$ & $\psi_{k+1}(v_0,\ldots,v_k)$. Thus

$\bigvee_{j<q_{k+1}} \exists v_k \eta_j(v_0,\ldots,v_k)$ & $\psi_{k+1}(v_0,\ldots,v_k)$.

Case 2. Q_k is $\forall$. That $\forall v_k \psi_{k+1} \longleftrightarrow \bigwedge_{j<q_{k+1}} \forall v_k(\eta_j \to \psi_{k+1})$ follows from Case 1 and the predicate calculus. The predicate calculus also suffices to show that

$\forall v_k(\eta_j \to \psi_{k+1}) \to (\exists v_k(\eta_j \,\&\, \psi_{k+1}) \vee \neg\exists v_k \eta_j)$, as well as that

$\neg\exists v_k \eta_j \to \forall v_k(\eta_j \to \psi_{k+1})$. We now need only show that

$\exists v_k(\eta_j(v_0,\ldots,v_k) \,\&\, \psi_{k+1}(v_0,\ldots,v_k)) \to \forall v_k(\eta_j(v_0,\ldots,v_k) \to \psi_{k+1}(v_0,\ldots,v_k))$.

By Corollary 2, we have either a) $\forall v_k(\eta \to \psi_{k+1})$ or

b) $\forall v_k(\eta \to \neg\psi_{k+1})$, but we assume that $\exists v_k(\eta_j \,\&\, \psi_{k+1})$, hence b) is excluded, and we must have a).

Finally we must show that the sentence that results from this transformation can be decided by a procedure recursive in $Th(\mathfrak{M}\restriction\Delta)$. First we perform another transformation, one which is recursive in $Th(\mathfrak{M}\restriction\Delta)$. Its input is the output of stage $n-1$ of the previous procedure. Replace

$\exists v_{n-1}(\eta_j(v_0,\ldots,v_{n-1}) \,\&\, \varphi(v_0,\ldots,v_{n-1}))$

by
$$\begin{cases} \exists v_{n-1}\eta_j(v_0,\ldots,v_{n-1}) & \text{if } \eta_j(v_0,\ldots,v_{n-1}) \to \varphi(v_0,\ldots,v_{n-1}) \\ \perp \text{ (false)} & \text{if } \eta_j(v_0,\ldots,v_{n-1}) \to \neg\varphi(v_0,\ldots,v_{n-1}). \end{cases}$$

Note that the result of this transformation is again equivalent to its input. The only interesting case is when $\perp$ is substituted, but either $\exists v_{n-1}\eta_j$ (in which case we have $\eta_j \,\&\, \neg\varphi$, whence $\eta_j \,\&\, \varphi$ is false), or $\neg\exists v_{n-1}\eta_j$ (in which case $\eta_j \,\&\, \varphi$ is again false).

Next we note that which of $\eta_j \to \varphi$ or $\eta_j \to \neg\varphi$ holds can be effectively decided on the basis of information about $\mathrm{Th}(\mathfrak{M} \upharpoonright \Delta)$. Consider the form of φ; it must be a Boolean combination of $K(v_i,v_m)$, $L(v_i,v_m)$, and $v_i = v_m$, where $i,m < n$. Clearly it is sufficient to decide each of $\eta_j \to v_i = v_m$, $\eta_j \to K(v_i,v_m)$, and $\eta_j \to L(v_i,v_m)$, since $\to$ distributes over & and $\vee$, and by Corollary 2 either $\eta_j \to \xi(v_i,v_m)$ or $\eta_j \to \neg\xi(v_i,v_m)$, where $\xi(v_i,v_m)$ is one of the atomic formulas above.

Case 1. $\Gamma(v_i)$ and $\Gamma(v_m)$ both occur in η_j. Then $\eta_j \to (v_i = v_n)$ if and only if $(v_i = v_m)$ occurs in η_j; $\eta_j \to K(v_i,v_m)$ if and only if, for some u, $K(v_i,u)$ and $(u = v_j)$ occur in η_j; $\eta_j \to L(v_i,v_m)$ if and only if, for some u, $L(v_i,u)$ and $(u = v_j)$ occur in η_j).

Case 2. $\Gamma(v_i)$ and $\Delta(v_m)$ occur in η_j. Then, from axiom A3 and the definition of η_j, $v_i \neq v_m$ will occur in η_j, thus $\eta_j \to v_i = v_m$ is false; similarly it can be shown that $\eta_j \to K(v_m,v_i)$ and $\eta_j \to L(v_m,v_i)$ are false (using axiom schema A4); $\eta_j \to K(v_i,v_m)$ if and only if, for some u, $K(v_i,u)$ and $\Delta(u)$ occur in η_j and $\chi \to u = v_j \in \mathrm{Th}(\mathfrak{M} \upharpoonright \Delta)$, where χ is the Ehrenfeucht formula occurring in η_j.

Case 3. $\Delta(v_i)$ and $\Delta(v_m)$ occur in η_j. Then if $\xi(v_i,v_m)$ is any of the atomic formulas under consideration, $\eta_j \to \xi(v_i,v_m)$ if and only if $\chi \to \xi \in \mathrm{Th}(\mathfrak{M} \upharpoonright \Delta)$, where χ is the Ehrenfeucht formula occurring in η_j.

The result of all these transformations can be described as a nesting of assertions of the existence or non-existence of elements satisfying certain r-neighborhood descriptions. To show that this sentence can be decided by a procedure recursive in $\mathrm{Th}(\mathfrak{M} \restriction \Delta)$ it suffices to prove the following lemma, which thus completes the proof of Theorem 4.

Lemma 3. Assume that $\eta_{j_i}(v_0, \ldots, v_i)$ is an r-neighborhood description of $r = 4^{n-i-1}$, $i \leq k < n$. Then

$$(\eta_{j_0}(v_0) \,\&\, \eta_{j_1}(v_0, v_1) \,\&\, \ldots \,\&\, \eta_{j_{k-1}}(v_0, \ldots, v_{k-1})) \rightarrow$$

$$(\exists v_k \eta_{j_k}(v_0, \ldots, v_k) \longleftrightarrow \exists u_0 \ldots \exists u_{p-1}(\bigwedge_{\ell < q} \Delta(u_\ell) \,\&\, \chi(u_0, \ldots, u_{q-1}))),$$

where $\chi(u_0, \ldots, u_{q-1})$ is the Ehrenfeucht formula that occurs in η_{j_k}, $\{u_\ell \mid \ell < q\} = J$, and $\{u_\ell \mid \ell < p\} = J \cap U$, where J and U are described in Lemma 2.

The proof of this lemma will be given in the next section, as it parallels the proof of Lemma 1 given there.

We return to the consideration of some of the standard pairing functions, the examples listed earlier. In each case, $\mathrm{Th}(\mathfrak{M} \restriction \Delta)$ is easily seen to be decidable, and hence $\mathrm{Th}(\mathfrak{M})$ is decidable, too.

In an unpublished paper, Rabin and Scott [13]* state that Tarski and Fuhrken independently proved that locally free pairing functions are decidable. The only published references

* The part of this paper that was published by Rabin [12] does not mention the work on pairing functions.

to this appear to be Tarski [15] and Doner [2], from which references we have taken the term "locally free". One easily verifies that a pairing function is locally free if and only if it is acyclic except for Δ, where Δ is empty. As $\mathrm{Th}(\mathfrak{M} \restriction \emptyset)$ is clearly decidable, our results extend those already in the literature.

Having seen that the first-order theories of certain pairing functions are decidable, one might well ask about the weak monadic second-order and the monadic second-order theories of these pairing functions. In the next theorem we exhibit an example that shows these theories to be undecidable.

Theorem. The (weak or full) monadic second-order theory of axioms A1 to A6 is undecidable.

Proof: It is possible to encode the integers into any model of axioms A1 to A6 and to define addition and multiplication on these integers. These definitions may be read as either weak or full monadic second-order definitions. We will do this in terms of a single parameter, a, which must satisfy $a \in \Gamma \,\&\, K(a) \neq L(a)$. Our encoded integers will be denoted by E, where $x \in E$ iff $[x = a \vee \exists y(y \in E \,\&\, J(y,y) = x)]$. To define addition and multiplication, we first define successor and sequence.

$$\mathrm{Succ}(x,y) \equiv [x \in E \;\&\; y \in E \;\&\; J(x,x) = y].$$

Notice that by axiom A2, $(\mathrm{Succ}(x,y) \;\&\; \mathrm{Succ}(z,y) \rightarrow z = x)$.

$$\begin{aligned}\mathrm{Seq}(A) \equiv [&\forall z(z \in A \rightarrow \exists u\, \exists x(u \in E \;\&\; x \in E \;\&\; J(u,x) = z)) \\ &\& \;\forall u\; \forall x\; \forall y((J(u,x) \in A \;\&\; J(u,y) \in A) \rightarrow x = y) \\ &\& \;\exists z(J(a,z) \in A) \;\&\; \forall u\; \forall x((J(u,x) \in A \;\&\; u \neq a) \rightarrow \\ &\qquad \exists v\, \exists y(J(v,y) \in A \;\&\; \mathrm{Succ}(v,u)))].\end{aligned}$$

The meaning of this formula is that anything in A is a pair of integers (thought of as an index and a value), where each index has a unique value, and the index a (which is thought of as zero) has a value, and anything in A not indexed by a is indexed by an index which is the successor of an index of something in A.

It is now easy to define addition.

$$\begin{aligned}b+c = d \equiv \exists A(&\mathrm{Seq}(A) \;\&\; J(a,b) \in A \;\&\; J(c,d) \in A \\ &\& \;\forall u\; \forall v\; \forall x\; \forall y((J(u,x) \in A \;\&\; J(v,y) \in A \;\&\; \mathrm{Succ}(u,v)) \rightarrow \\ &\qquad \mathrm{Succ}(x,y))).\end{aligned}$$

This means that $b+c = d$ if and only if there is a sequence A whose zeroth value is b, and whose c^{th} value is d, and in which successively indexed values are themselves successors.

Similarly, we define multiplication using addition.

$$\begin{aligned}b\cdot c = d \equiv \exists A(&\mathrm{Seq}(A) \;\&\; J(a,a) \in A \;\&\; J(c,d) \in A \\ &\& \;\forall u\; \forall v\; \forall x\; \forall y((J(u,x) \in A \;\&\; J(v,y) \in A \;\&\; \mathrm{Succ}(u,v)) \rightarrow \\ &\qquad x+b = y)).\end{aligned}$$

§4. Proofs of Lemmas 1 and 3.

Lemma 1. Let $g(k) = 2^{4^k}$, and assume that $\{x_j | j < k\} \subseteq \mathfrak{M}_0$ and $\{y_j | j < k\} \subseteq \mathfrak{M}_1$ for some $k < n$. If there is a strong isomorphism $\theta_k: \bigcup_{j<k} N(4^{n-1}, x_j) \approx \bigcup_{j<k} N(4^{n-k}, y_j)$, then for each $x_k \in \mathfrak{M}_0$ one can effectively find a $y_k \in \mathfrak{M}_1$ and for each $y_k \in \mathfrak{M}_1$ one can effectively find an $x_k \in \mathfrak{M}_0$ such that there is a strong isomorphism

$\theta_{k+1}: \bigcup_{j<k+1} N(4^{n-k-1}, x_j) \approx \bigcup_{j<k+1} N(4^{n-k-1}, y_j)$.

Proof: The proof of this lemma is by induction on k, for $k < n$. We make use of the notation developed following the statement of Theorem 1. We first note the following easily proved facts.

Lemma 4.

a) If $f_x(\sigma) = \tau$, then $\tau \preceq \sigma$;

b) $\| f_x(\sigma) \| \leq \| f_x(Pd(\sigma)) \| + 1$;

c) $f_x(\sigma\tau) = f_x(\sigma f_x(\tau))$; and

d) $\mathrm{Range}(f_x) \cap B_x = \emptyset$.

Proof: Left to the reader.

Assume that $\{x_j | j < k\} \subseteq \mathfrak{M}_0$ and $\{y_j | j < k\} \subseteq \mathfrak{M}_1$ satisfy the conditions in the lemma, and assume that $x_k \in \mathfrak{M}_0$ has been selected (the case when $y_k \in \mathfrak{M}_1$ is selected is similar). Let $r = 4^{n-k-1}$ and $s = 2 \cdot 4^{n-k-1}$ and $A = \bigcup_{j<k} N(r, x_j)$. For ease of notation, let B, C, E, F, and I represent B_{x_k}, $C_{x_k, A}$, E_{x_k}, $F(s, x_k)$, and $I(s, x_k)$

respectively. We will show how to construct a $y_k \in \mathcal{M}_1$ satisfying the lemma. This is done by first assigning values to nodes in F-B and then assigning values to the remaining nodes in $F \cup I$ in a bottom-to-top fashion. These two parts may be described as an initialization followed by a procedure.

Initialization (to assign values to $\sigma(y_k)$, for $\sigma \in F-B$): For any $\sigma \in F$, at least one of the following conditions must hold:

0) $\sigma \in B$

1) $\sigma \in C$

2) $\sigma \in E$

3) None of the above.

Let $T_j = \{\sigma \in F \mid \text{condition } j \text{ is the first to hold for } \sigma\}$. We will assign values to $\sigma(y_k)$ for $\sigma \in T_1 \cup T_2 \cup T_3$.

i) For $\sigma \in T_1$, set $\sigma(y_k) = \theta_k(\sigma(x_k))$. Since $A = \bigcup_{j<k} N(r,x_j)$ and $\mathrm{dom}(\theta_k) = \bigcup_{j<k} N(4r,x_j)$, $\theta_k(\sigma(x_k))$ is defined for $\sigma \in C$.

ii) Let $\{\tau_j \mid j < p\}$ be an enumeration of T_2; $t_j = \tau_j(x_k)$, for $j < p$; and $Z = \{z_j \mid j < m\} = \Delta_0 \cap \bigcup_{j<k} N(4^{n-k},x_j)$.

Note: One easily verifies that $p < g(n-k) - g(n-k-1)$; and as $\langle \mathcal{M}_0 \restriction \Delta_0, z_j \rangle_{j<m} \equiv_{g(n-k)} \langle \mathcal{M}_1 \restriction \Delta_1, \theta_k(z_j) \rangle_{j<m}$, there must exist $W = \{w_j \mid j < p\} \subseteq \Delta_1$ such that

$$\langle \mathcal{M}_0 \restriction \Delta_0, z_j, t_i \rangle_{j<m, i<p} \equiv_{g(n-k-1)} \langle \mathcal{M}_1 \restriction \Delta_1, \theta_k(z_j), w_i \rangle_{j<m, i<p}.$$

Set $\tau_i(y_k) = w_i$ for $i < p$.

iii) Now let $\{\tau_j | j < p\}$ be an enumeration of T_3, and let $\{z_j | j < p\} \subseteq \Gamma_1$, where for $i < p$,

$$N(s,z_i) \cap [\bigcup_{j<k} N(s,y_j) \cup \bigcup_{j\neq i} N(s,z_j) \cup W] = \emptyset,$$

where W is the set constructed in part (ii) of this initialization. Clearly there are such z, since $\Gamma_1 \neq \emptyset$ implies $\#(\Gamma_1) \geq \aleph_0$. Set $\tau_i(y_k) = z_i$ for $i < p$.

This completes the initialization.

Procedure $\underset{\sim}{P}$: (to assign values to $\sigma(y_k)$ for some $\sigma \in F \cup I$, working toward the root, Λ).

Stage i $(i < 2^{s+1})$:

(a) Let $t = 2^{s+1}-i-1$. If $\sigma_t(y_k)$ has been defined, or if $\sigma_t \notin F \cup I$, go to stage $i+1$.

(b) If $\sigma_t \in B$, and if $f(\sigma_t)(y_k)$ has been defined, set $\sigma_t(y_k) = f(\sigma_t)(y_k)$, and go to stage $i+1$. If $\sigma_t \in B$, and if $f(\sigma_t)(y_k)$ has not been defined, go to stage $i+1$.

(c) If $\sigma_t \in C$, set $\sigma_t(y_k) = \theta_k\,(\sigma_t(x_k))$, and go to stage $i+1$.

(d) If $(K\sigma_t)(y_k)$ and $(L\sigma_t)(y_k)$ are both defined, then by axiom A1 there is a unique z such that $K(z) = (K\sigma_t)(y_k)$ and $L(z) = (L\sigma_t)(y_k)$; set $\sigma_t(y_k) = z$, and go to stage $i+1$.

(e) If none of the above, go to stage $i+1$.

Stage 2^{s+1}: Stop.

Iterate procedure $\underset{\sim}{P}$ until no stage defines a $\sigma(y_k)$. Clearly 2^{s+1} iterations will more than suffice.

This completes the construction of y_K, but several facts remain to be verified. These verifications occupy the rest of the proof. In order for iterations of the procedure $\underset{\sim}{P}$ to assign any values, at least one $\sigma \in F \vee I$ must have received a value during the initialization; this will certainly happen if $F - B \neq \emptyset$. We now prove that this is the case.

Lemma 5. With the notation established following the statement of Theorem 1: for no q is $F(q,x) \subseteq B_x$.

Proof: Assume the lemma is false, and let q be the least integer such that $F(q,x) \subseteq B_x$. Let $\tau_0 = \sigma_j$, where $j = \text{least } m[\sigma_m \in F(q,x)]$, and let $\tau_{i+1} = K\tau_i$. Let $T = \{\tau_i | i < 2^q\}$. Note that $f_x(\tau_0) \in I(q,x)$ by the choice of τ_0.

Case 1. $T \cap E_x = \emptyset$. Then we claim that $\text{Range}(f_x \restriction T) \subseteq I(q,x)$. The proof of the claim is by induction. By the note above, $f_x(\tau_0) \in I(q,x)$. Assume $f_x(\tau_i) \in I(q,x)$; we show that $f_x(\tau_{i+1}) \in I(q,x)$. Assume not; then by Lemma 4 (b) and the definition of $I(q,x)$, we must have $f_x(\tau_{i+1}) \in F(q,x) \subseteq B_x$. This contradicts Lemma 4 (d). Intuitively, $F(q,x)$ presents a barrier to f_x. Since $I(q,x) \subseteq (\Sigma^*)^{(q-1)}$, $\#(I(q,x)) \leq 2^q-1$. But $\#(T) = 2^q$, and hence there are τ_i, $\tau_j \in T$ such that $i < j$ and $f_x(\tau_i) = f_x(\tau_j)$. This implies that $K^{i-1}\tau_0(x) = K^{j-1}\tau_0(x)$, contradicting axiom schema A4.

Case 2. $T \cap E_x \neq \emptyset$. Let τ_j be such that $K\tau_j \in T \cap E_x$. We claim that $f_x(\tau_j) \in I(q,x)$. The proof is similar to that of the claim in case 1, so we omit it. Since $f_x(\tau_j) \in I(q,x)$ and $K\tau_j \in E_x$, $Kf_x(\tau_j) \in F(q,x) \cap E_x$. (Note the use here of axiom A2.) But then, by Lemma 4 (c), $f_x(Kf_x(\tau_j)) \in F(q,x) \cap E_x \subseteq B_x$, which contradicts Lemma 4 (d).

This completes the proof of the lemma.

The next lemma guarantees that each $\sigma \in F \cup I$ receives a value either during initialization or during the iterations of P. Let $Q = \{\sigma \in F \cup I | \sigma(y_k) \text{ is not defined}\}$.

Lemma 6. $Q = \emptyset$.

Proof: Assume $Q \neq \emptyset$. It is easy to show that if $\tau \in Q - B$, then $\tau(x_k) \in \Gamma_0$. Using this fact, we construct two sequences, $\{\tau\}_{i\in\omega}$ and $\{\tau'\}_{i\in\omega}$ such that for each i, $\tau_i = Pd(\tau_{i+1})$, $\tau_i' \in Q - B$, and $\tau_i(x_k) = \tau_i'(x_k) \in \Gamma_0$. Let $\tau_0 \in Q$, and let $\tau_0' = f(\tau_0)$. Assume $\{\tau_j\}_{j<i}$ and $\{\tau_j'\}_{j<i}$ have been constructed. Since $\tau_i' \in Q - B$ one can easily show that $\{K\tau_i', L\tau_i'\} \cap Q \neq \emptyset$. If $K\tau_i' \in Q$, set $\tau_{i+1}' = f(K\tau_i')$, and set $\tau_{i+1} = K\tau_i$. If $K\tau_i' \notin Q$ (then $L\tau_i' \in Q$), set $\tau_{i+1}' = f(L\tau_i')$, and set $\tau_{i+1} = L\tau_i$. Let $T' = \{\tau_j' | j < 2^s\} \subseteq F \cup I$. By cardinality considerations, there must be distinct $i,j < 2^s$ such that $\tau_i'(x_k) = \tau_j'(x_k)$; hence $\tau_i(x_k) = \tau_j(x_k)$, contradicting axiom schema A4. This completes the proof of the lemma.

It is easy to see that by the very construction of y_k each of the following inclusions holds:

$$B^{(s)}_{x_k} \subseteq B^{(s)}_{y_k}, \quad C^{(s)}_{x_k,A} \subseteq C^{(s)}_{y_k,\theta_k(A)}, \quad \text{and} \quad E^{(s)}_{x_k} \subseteq E^{(s)}_{y_k} .$$

That each of these inclusions can be reversed is a result of the conditions imposed during part (iii) of the initialization, as may be verified by the interested reader; we merely remark here that those conditions were designed to prevent too much structure from occurring in y_k.

All that now remains to check is the existence of a strong isomorphism

$$\theta_{k+1}\colon \bigcup_{j \leq k+1} N(r,x_j) \approx \bigcup_{j \leq k+1} N(r,y_j).$$

Define θ_{k+1} by

$$\theta_{k+1}(z) = \begin{cases} \theta_k(z) & \text{if } z \in \bigcup_{j \leq k} N(r,x_j) \\ \sigma(y_k) & \text{if } z = \sigma(x_k) \text{ and } \|\sigma\| \leq r. \end{cases}$$

We must show that θ_{k+1} is a strong isomorphism (with respect to Γ and Δ and to the graphs of K and L). We carry out only the first step by showing that θ_{k+1} is well defined. After that the reader should have seen enough of the techniques to provide the routine calculations that finish the proof. Assume that $i \leq j \leq k$; $\tau_0, \tau_1 \in (\Sigma^*)^{(r)}$; and $\tau_0(x_i) = \tau_1(x_j)$. We must show that $\theta_{k+1}(\tau_0(x_i)) = \theta_{k+1}(\tau_1(x_j))$.

Case 1. $i \leq j < k$. The result is immediate, since θ_k is well defined.

Case 2. $i = j = k$. Then $\theta_{k+1}(\tau_0(x_i)) = \tau_0(y_k)$, and $\theta_{k+1}(\tau_1(x_j)) = \tau_1(y_k)$. Assume (without loss of generality) that $\tau_0 \prec \tau_1$. Then clearly $\tau_1 \in B$, and $f(\tau_1) = f(\tau_0)$ ($= \tau_0$, if $\tau_0 \notin B$). Thus $\tau_1(y_k)$ will be assigned a value by part (b) of the procedure $\underset{\sim}{P}$ which is equal to $\tau_0(y_k)$.

Case 3. $i < j = k$. Then $\theta_{k+1}(\tau_0(x_i)) = \theta_k(\tau_0(x_i))$, and $\theta_{k+1}(\tau_1(x_j)) = \tau_1(y_k)$. Notice that $\tau_1 \in C$, since $\tau_0(x_i) = \tau_1(x_k)$. Hence $\tau_1(y_k)$ will be assigned a value either by part (i) of the initialization or by part (c) of the procedure $\underset{\sim}{P}$. But in either case, $\tau_1(y_k) = \theta_k(\tau_0(x_i))$, from which the result easily follows.

The remaining calculations are left to the reader.

Lemma 3. Assume that $\eta_{j_i}(v_0,\ldots,v_i)$ is an r-neighborhood description for $r = 4^{n-i-1}$, $i \leq k < n$. Then

$$(\eta_{j_0}(v_0) \,\&\, \ldots \,\&\, \eta_{j_{k-1}}(v_0,\ldots,v_{k-1})) \to$$
$$(\exists v_k \eta_{j_k}(v_0,\ldots,v_k) \longleftrightarrow \exists u_0 \ldots \exists u_{p-1}(\bigwedge_{\ell<q} \Delta(u_\ell) \,\&\, \chi(u_\ell)_{\ell<q})),$$

where $\chi(u_0,\ldots,u_{q-1})$ is the Ehrenfeucht formula that occurs in η_{j_k}, $\{u_\ell \mid \ell < q\} = J$, and $\{u_\ell \mid \ell < p\} = J \cap U$, where J and U are described in Lemma 2.

Proof: It suffices to show that from $\eta_{j_{k-1}}(v_0,\ldots,v_{k-1})$ one can deduce $\exists v_k \eta_{j_k}(v_0,\ldots,v_{k-1})$ $\Longleftrightarrow \exists u_0 \ldots \exists u_{p-1}(\bigwedge_{\ell<q}\Delta(u_\ell) \; \& \; \chi(u_0,\ldots,u_{q-1}))$, where these formulas are described in the statement of the lemma.

In one direction, $\exists v_k \eta_{j_k} \rightarrow \exists u_0 \ldots \exists u_{p-1}(\bigwedge_{\ell<q}\Delta(u_\ell) \; \& \; \chi(u_\ell)_{\ell<q})$ follows immediately from the form of η_{j_k}.

In the other direction, we show that from the 4^{n-k} neighborhood description $\eta_{j_{k-1}}(v_0,\ldots,v_{k-1})$ we may deduce $\exists u_0 \ldots \exists u_{p-1}(\bigwedge_{\ell<q}\Delta(u_\ell) \; \& \; \chi(u_\ell)_{\ell<q}) \rightarrow \exists v_k \eta_{j_k}(v_0,\ldots,v_k)$, where η_{j_k} is a 4^{n-k-1} neighborhood description. We revert to the subscripting convention of Lemma 2, and we assume that $d_{i,\sigma}$ (for $i < k$ and $\|\sigma\| \leq 4^{n-k}$) are variables that witness the existential quantifiers of $\eta_{j_{k-1}}$ ($d_{i,\Lambda}$ is to be understood to be v_i). Furthermore, we let $e_{j,\tau}$ be the witnesses to $\exists u_0 \ldots \exists u_{p-1}(\bigwedge_{\ell<q}\Delta(u_\ell) \; \& \; \chi(u_\ell)_{\ell<q})$, where the subscript on $e_{j,\tau}$ refers back to the subscript originally carried by $u_{j,\tau}$ in η_{j_k} (as described in Lemma 2). By the definitions about Ehrenfeucht games and the choice of the function g, $e_{i,\sigma}$ may be assumed to be $d_{i,\sigma}$ for $i < k$ and $\|\sigma\| \leq 4^{n-k-1}$. (Cf. the note, initialization, part (ii), in the proof of Lemma 1.)

The idea is to mimic the construction in the proof we gave for Lemma 1. To this end, we define B, C, E, F, I in a manner similar to their previous definitions. Let

$r = 4^{n-k-1}$ and $s = 2\cdot 4^{n-k-1}$. Let U be the set associated with η_{j_k}, as described in Lemma 2.

Define

$$f(\sigma) = \begin{cases} \text{least } \tau \text{ (with respect to } \prec\text{) such that } u_{k,\sigma} = u_{k,\tau} \text{ occurs in } \eta_{j_k}, & \text{if } u_{k,\sigma} \in U, \\ \sigma, & \text{otherwise;} \end{cases}$$

$B = \{\sigma | u_{k,\sigma} = u_{k,\tau}$ occurs in η_{j_k} for some $\tau \prec \sigma\}$;

$C = \{\sigma | u_{k,\sigma} = u_{i,\tau}$ occurs in η_{j_k} for some $i < k$ and some τ with $||\tau|| \leq r$, or $\sigma > \sigma'$ for some $\sigma' \in C\}$;

$E = \{\sigma | \Delta(u_{k,\sigma})$ occurs in $\eta_{j_k}\}$;

$F = \{\sigma | \sigma \in E^{(r)}$, or $||\sigma|| = r$ and $\forall\tau(\tau < \sigma \rightarrow \tau \notin E)\}$; and

$I = \{\sigma | \exists\tau(\tau \in F^{(r)}$ and $\sigma < \tau)\}$.

All that is required is to show how to produce witnesses $d_{k,\sigma}$ to the existential quantifiers binding $u_{k,\sigma}$ in η_{j_k}. We again divide the work pieces, corresponding to the initialization and the iterations of $\underset{\sim}{P}$ in the proof of Lemma 1.

First, for $\sigma \in F$, at least one of the following holds:

0) $\sigma \in B$

1) $\sigma \in C$

2) $\sigma \in E$

3) none of the above

Let $T_j = \{\sigma \in F |$ condition j is the first to hold for $\sigma\}$.

i) For $\sigma \in T_1$, $d_{k,\sigma}$ exists since $d_{i,\tau}$ exists, and $u_{k,\sigma} = u_{i,\tau}$ occurs in η_{j_k}.

ii) For $\sigma \in T_2$, $d_{k,\sigma}$ is actually $e_{k,\sigma}$, which exists by assumption.

iii) For $\sigma \in T_3$: let $\{\tau_j | j < p\}$ be an enumeration of T_3. Let $D = \{d_{i,\sigma} | i < k$, and $\|\sigma\| \leq s$, and $\Gamma(u_{i,\sigma})$ occurs in $\eta_{j_{k-1}}\}$, and let $\{d_j | j < q\}$ enumerate D. Represent by $\langle w_0, w_1\rangle$ a variable such that $K(\langle w_0, w_1\rangle) = w_0$ and $L(\langle w_0, w_1\rangle) = w_1$. By repeated use of axiom A1, one can exhibit $\{c_j | j < q\}$ such that $c_0 = d_0$ and $c_{j+1} = \langle c_j, d_{j+1}\rangle$. By axiom A6, $c_j \in \Gamma$ for $j < q$. Let b witness $\exists v \Gamma(v)$, which holds by assumption, and form $b_0 = \langle b, c_{q-1}\rangle$ and $b_{j+1} = \langle b_j, b_j\rangle$. Finally, for $j < p$ and $h(j) = (j+1)4^{n-k}$, let $a_j = b_{h(j)}$. Then one can easily prove that the $a_j (j < p)$ witness a formula corresponding to the restriction placed on $\{z_j | j < p\}$ in part (iii) of the initialization in the proof of Lemma 1. Thus a_j can serve as d_{k,τ_j} for $j < p$.

To prove that $d_{k,\sigma}$ exists for the remaining nodes in $I \cup (B \cap F)$, we mimic the iterations of $\underset{\sim}{P}$.

Stage i $(i < 2^{s+1})$:

a) Let $t = 2^{s+1}-i-1$. If d_{k,σ_t} has already been shown to exist or if $\sigma_t \notin F \cup I$, go to stage i+1.

b) If $\sigma_t \in B$, and if $d_{k,\tau}$ where $\tau = f(\sigma_t)$ has been shown to exist, then d_{k,σ_t} must also exist.

c) If $\sigma_t \in C$, then d_{k,σ_t} exists since $d_{i,\tau}$ does, where $u_{k,\sigma_t} = u_{i,\tau}$ occurs in η_{j_k}.

d) If $d_{k,K\sigma_t}$ and $d_{k,L\sigma_t}$ have both been shown to exist, then by axiom A1 there is a unique $d_{k,\sigma_t} = \langle d_{k,K\sigma_t}, d_{k,L\sigma_t} \rangle$.

Stage 2^{s+1}: Stop.

Iterate $\underset{\sim}{P}$ until no more $d_{i,\sigma}$ are shown to exist in a given iteration.

We then proceed as in the proof of Lemma 1, and for precisely the same reasons as given there, the $d_{i,\sigma}$ witness the existential quantifiers of $\eta_{j_k}(v_0,\ldots,v_k)$, and hence the lemma is proved.

CHAPTER II

DECIDABILITY OF THE FIRST- AND SECOND-ORDER THEORIES OF AN EQUIVALENCE RELATION

§1. Second-Order Ehrenfeucht Games

It is possible to extend the Ehrenfeucht games of the previous chapter to second-order Ehrenfeucht games $G_{m,n}(\mathfrak{M}_0,\mathfrak{M}_1)$ in which there are $m+n$ moves. For the first m moves, players I and II select subsets of $\mathfrak{M}_0$ and $\mathfrak{M}_1$, and for the remaining n moves, they select elements of $\mathfrak{M}_0$ and $\mathfrak{M}_1$. As before, player I has a choice which model he chooses from, and player II is constrained to choose from the other model. For $j < m$, denote by η_j the subset of $\mathfrak{M}_0$ chosen at move j, regardless of which player chose it; similarly $\theta_j \subseteq \mathfrak{M}_1$. For $m \leq j < m+n$, denote by x_{j-m} the element chosen at move j from $\mathfrak{M}_0$; similarly $y_{j-m} \in \mathfrak{M}_1$. Player II wins $G_{m,n}(\mathfrak{M}_0,\mathfrak{M}_1)$ if and only if at the end of the game, the map $x_0 \longleftrightarrow y_0,\ldots,x_{n-1} \longleftrightarrow y_{n-1}$ is an isomorphism between $\{x_i \mid i < n\}$ and $\{y_i \mid i < n\}$ with respect both to the relations of the models and to the sets chosen. In other words, player II wins if and only if $i < n$ and $j < m$, $x_i \in \eta_j \Longleftrightarrow y_i \in \theta_j$ and $\langle \mathfrak{M}_0, x_i\rangle_{i<n} \equiv_0 \langle \mathfrak{M}_1, y_i\rangle_{i<n}$.

If $\mathfrak{M} = \langle \Omega, R_i\rangle_{i\in I}$, then by $\langle \mathfrak{M}, \eta_j\rangle_{j\in J}$ we mean $\langle \Omega, R_i, \eta_j\rangle_{i\in I, j\in J}$. With this notation we introduce an equivalence relation on models with certain subsets selected, $\equiv_{m,n}$, which corresponds to the second-order game. We define

$$\langle \mathcal{M}_0, \eta_j \rangle_{j<k} \equiv_{0,n} \langle \mathcal{M}_1, \theta_j \rangle_{j<k}$$

if and only if

$$\langle \mathcal{M}_0, \eta_j \rangle_{j<k} \equiv_n \langle \mathcal{M}_1, \theta_j \rangle_{j<k},$$

where $\equiv_n$ is the equivalence relation corresponding to the first-order game introduced in the first chapter. And,

$$\langle \mathcal{M}_0, \eta_j \rangle_{j<k} \equiv_{m+1,n} \langle \mathcal{M}_1, \theta_j \rangle_{j<k}$$

if and only if for each $\eta_k \subseteq \mathcal{M}_0$ there is a $\theta_k \subseteq \mathcal{M}_1$ and for each $\theta_k \subseteq \mathcal{M}_1$ there is an $\eta_k \subseteq \mathcal{M}_0$ such that

$$\langle \mathcal{M}_0, \eta_j \rangle_{j<k+1} \equiv_{m,n} \langle \mathcal{M}_1, \theta_j \rangle_{j<k+1}.$$

As the reader may easily verify, this is an equivalence relation, and player II can win $G_{m,n}(\mathcal{M}_0,\mathcal{M}_1)$ after the k moves $\eta_0 \longleftrightarrow \theta_0, \ldots, \eta_{k-1} \longleftrightarrow \theta_{k-1}$ $(k \leq m)$ if and only if $\langle \mathcal{M}_0, \eta_j \rangle_{j<k} \equiv_{m-k,n} \langle \mathcal{M}_1, \theta_j \rangle_{j<k}$.

Corresponding to the first two theorems of the previous chapter we have the following theorems. We omit the proofs, as they are virtually identical with the proofs given in

Chapter I. The reader who wants more details may refer to Le Tourneau [8]*.

<u>Theorem 1</u>. Let $\psi(\xi_0,\ldots,\xi_{k-1})$ $= Q_k\xi_k\cdots Q_{m-1}\xi_{m-1}Q_0v_0\cdots Q_{n-1}v_{n-1}\varphi(\xi_0,\ldots,\xi_{m-1},v_0,\ldots,v_{n-1})$, where φ is quantifier free, and here, as well as below, Q_j is a monadic second-order quantifier and ξ_j is a monadic second-order variable. If

$$\langle \mathfrak{M}_0,\eta_j\rangle_{j<k} \equiv_{m-k,n} \langle \mathfrak{M}_1,\theta_j\rangle_{j<k},$$

then $\mathfrak{M}_0 \models \psi[\eta_0,\ldots,\eta_{k-1}]$ if and only if $\mathfrak{M}_1 \models \psi[\theta_0,\ldots,\theta_{k-1}]$.

<u>Theorem 2</u>. For every finitary relational system $\mathfrak{M}_0$ and every set of subsets $\{\eta_j \subseteq \mathfrak{M}_0 | j < k\}$, there is a formula $\chi(\xi_0,\ldots,\xi_{k-1})$ such that for any finitary relational system $\mathfrak{M}_1$ and any set of subsets $\{\theta_j \subseteq \mathfrak{M}_1 | j < k\}$,

$$\langle \mathfrak{M}_0,\eta_j\rangle_{j<k} \equiv_{m-k,n} \langle \mathfrak{M}_1,\theta_j\rangle_{j<k}$$

if and only if $\mathfrak{M}_1 \models \chi[\theta_0,\ldots,\theta_{k-1}]$.

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* Le Tourneau's development of second-order Ehrenfeucht games in his unpublished thesis [8] was prior to (but independent of) ours which was begun along lines suggested by A. Nerode. We were informed of Le Tourneau's work before we had gone much beyond the basic concepts and a few simple examples. Unlike him, we did not demand that the moves in which sets are chosen precede the moves in which elements are chosen, but we adopt his convention here as it yields slightly simpler definitions and proofs.

§2. The Second-Order Theory of an Equivalence Relation

We use this machinery to show that the monadic second-order theory of an equivalence relation, ~, is decidable by a primitive recursive procedure. We do this by finding a primitive recursive bound $g(m,n)$ on the cardinality of the models that must be examined to determine the validity of any statement with at most m second-order quantifiers followed by at most n first-order quantifiers in prenex form. That this theory is decidable is a known result: it can be encoded in Rabin's S_nS (see [11]). Our proof, however, is quite direct.

In what follows, all models are models of the theory of an equivalence relation, ~, with axioms:

(A1) $\forall x(x \sim x)$

(A2) $\forall x\ \forall y(x \sim y \rightarrow y \sim x)$ and

(A3) $\forall x\ \forall y\ \forall z[(x \sim y\ \&\ y \sim z) \rightarrow x \sim z]$.

Definition 1. For each integer n we define an equivalence relation E_n on the integers as follows:

$$k_0 E_n k_1 \text{ iff } k_0 = k_1 \text{ or } [k_0 \geq n \text{ and } k_1 \geq n].$$

Definition 2. For each integer n we define an equivalence relation $\mathcal{E}_n$ on models as follows:

$$\mathfrak{M}_0\ \mathcal{E}_n\ \mathfrak{M}_1$$

iff

$$\forall k < n[c_0(k) E_n c_1(k)]\ \&\ (\sum_{k \geq n} c_0(k)) E_n (\sum_{k \geq n} c_1(k))$$

where $c_i(k)$ = number of equivalence classes of cardinality k in $\mathcal{M}_i$ $(i = 0,1)$.

Notation. $\#(A)$ = cardinality A.

$[z]$ is the equivalence class of z under the relation $\sim$. Enumerate $\bigcup_{k\in\omega} F_k$, where $F_k = \{f^{(k)} | f^{(k)}: k \to 2\}$ as described below.

$f_i^{(0)}(n)$ is undefined for all n.

$$f_i^{(k+1)}(n) = \begin{cases} f_j^{(k)}(n) & \text{if } n < k-1 \ \& \ j = \text{greatest integer} \leq \frac{i}{2} \\ 0 & \text{if } n = k-1 \ \& \ i \text{ is even} \\ 1 & \text{if } n = k-1 \ \& \ i \text{ is odd} \\ \text{undefined, otherwise.} \end{cases}$$

Remarks.

(1) $f_i^{(k)}(n)$ is the $(n+1)^{st}$ digit in the k digit binary representation of n. E.g., $f_q^{(4)}(n) = 1 \Longleftrightarrow n = 0$ (since $8_{10} = 1000_2$) and $f_q^{(5)}(n) = 1 \Longleftrightarrow n = 1$ (since $8_{10} = 01000_2$).

(2) $f_{2i}^{(k+1)}$ and $f_{2i+1}^{(k+1)}$ both extend $f_i^{(k)}$.

(3) We usually suppress (k) and write f_i for $f_i^{(k)}$.

(4) We refer to this enumeration as the standard enumeration.

More Notation.

$$\begin{aligned} Seq(k,n) &= \text{sequences of length } k \text{ with elements between } 0 \text{ and } n \\ &= \{\vec{i} = \langle \vec{i}(0), \ldots, \vec{i}(k-1)\rangle \,|\, \forall \ell (\vec{i}(\ell) \leq n)\} \\ &= \{\vec{i} \,|\, \vec{i}: k \to n+1\}. \end{aligned}$$

For any set α, $\alpha^0 = \alpha$, and

$\alpha^1 = \bar{\alpha} =$ complement of α.

Definition 3. Given $\langle \mathcal{M}_0, \eta_j \rangle_{j<m}$ and $\langle \mathcal{M}_1, \theta_j \rangle_{j<m}$ where $\eta_j \subseteq \mathcal{M}_0$ and $\theta_j \subseteq \mathcal{M}_1$ (for $j < m$), for each $\vec{i} \in \mathrm{Seq}(2^m, n)$ define

$$\gamma_0(\vec{i}) = \{[x] \subseteq \mathcal{M}_0 \mid \forall \ell < 2^m [\min(\#([x] \cap \bigcap_{k<m} \eta_k^{f_\ell(k)}), n) = \vec{i}(\ell)]\}$$

and

$$\gamma_1(\vec{i}) = \{[y] \subseteq \mathcal{M}_1 \mid \forall \ell < 2^m [\min(\#([y] \cap \bigcap_{k<m} \theta_k^{f_\ell(k)}), n) = \vec{i}(\ell)]\}.$$

Remark. For $\vec{i}, \vec{j} \in \mathrm{Seq}(2^m, n)$, if $\vec{i} \neq \vec{j}$ then

$$\gamma_0(\vec{i}) \cap \gamma_0(\vec{j}) = \emptyset$$

and

$$\gamma_1(\vec{i}) \cap \gamma_1(\vec{j}) = \emptyset.$$

Example. $m = 2$, $n = 3$

$$\begin{aligned} \gamma_0(\langle 1,3,1,2 \rangle) = \{x \mid & \#([x] \cap \eta_0 \cap \eta_1) = 1 \\ & \& \ \#([x] \cap \eta_0 \cap \bar{\eta}_1) \geq 3 \\ & \& \ \#([x] \cap \bar{\eta}_0 \cap \eta_1) = 1 \\ & \& \ \#([x] \cap \bar{\eta}_0 \cap \bar{\eta}_1) = 2\} \end{aligned}$$

To aid the reader's understanding we remark that $\vec{i} \in Seq(2^m, n)$ is used as a sequence of cardinalities modulo the equivalence relation E_n, where each element of the sequence refers to the cardinality of the intersection of an equivalence class in $\mathfrak{M}_0$ with one of the 2^m possible Boolean combinations of the sets η_j, $j < m$. $\gamma_0(\vec{i})$ is just the set of $\sim$ equivalence classes which are characterized by $\vec{i}$. In other words,

$$\gamma_0(\vec{i}) = \{[x] \mid \forall \ell < 2^m [\#([x] \cap \bigcap_{k<m} \eta_j^{f_\ell(k)}) E_n \vec{i}(\ell)]\}.$$

Definition 4. $\langle \mathfrak{M}_0, \eta_j \rangle_{j<m} \, \mathcal{E}_n' \langle \mathfrak{M}_1, \theta_j \rangle_{j<m}$ if and only if for each $\vec{i} \in Seq(2^m, n)$, $\#(\gamma_0(\vec{i})) E_n \#(\gamma_1(\vec{i}))$.

Remark. It is easily seen that

$$\mathfrak{M}_0 \, \mathcal{E}_n \, \mathfrak{M}_1 \iff \langle \mathfrak{M}_0, \eta_j \rangle_{j<0} \, \mathcal{E}_n' \langle \mathfrak{M}_1, \theta_j \rangle_{j<0},$$

thus $\mathcal{E}_n'$ is a generalization of $\mathcal{E}_n$.

Lemma 1. If $\langle \mathfrak{M}_0, \eta_j \rangle_{j<m} \, \mathcal{E}_n' \langle \mathfrak{M}_1, \theta_j \rangle_{j<m}$, then

$$\langle \mathfrak{M}_0, \eta_j \rangle_{j<m} \equiv_n \langle \mathfrak{M}_1, \theta_j \rangle_{j<m}.$$

Before giving the proof of this, we note that by the remark above, this lemma has the following corollary, which is also proved in [10].

Corollary 1. If $\mathfrak{M}_0 \, \mathcal{E}_n \, \mathfrak{M}_1$, then $\mathfrak{M}_0 \equiv_n \mathfrak{M}_1$.

Proof (of Lemma 1): Assume $k < n$ and $x_0 \longleftrightarrow y_0, \ldots, x_{k-1} \longleftrightarrow y_{k-1}$ have been chosen to satisfy the following inductive hypothesis: a) There is a list $\{\vec{i}_0, \vec{i}_1, \ldots, \vec{i}_{k-1} | \vec{i} \in \mathrm{Seq}(2^m, n)\}$ such that for $j < k$, $[x_j] \in \gamma_0(\vec{i}_j)$ and $[y_j] \in \gamma_1(\vec{i}_j)$, and b) for $i < k$, $\ell < 2^m$, $x_i \in \bigcap_{j \in m} \eta_j^{f_\ell(j)} \iff y_i \in \bigcap_{j < m} \theta_j^{f_\ell(j)}$. Assume player I selects $x_k \in \mathcal{M}_0$ (the case when player I selects $y_k \in \mathcal{M}_1$ is similar). We show how player II selects $y_k \in \mathcal{M}_1$ in such a way that the inductive hypothesis again holds. This is clearly a winning strategy.

Let $\ell < 2^m$ be such that $x_k \in \bigcap_{j<m} \eta_j^{f_\ell(j)} = P_0(\ell)$.

Let $P_1(\ell) = \bigcap_{j<m} \theta_j^{f_\ell(j)}$. (These are the ℓ^{th} partition sets determined by the η's and θ's (respectively).)

Case 1. $x_k = x_j$ for some $j < k$. Set $y_k = y_j$.

Case 2. $x_k \in [x_j]$ for some $j < k$ (but $x_k \notin \{x_0, \ldots, x_{k-1}\}$). By assumption, $\#([x_j] \cap P_0(\ell)) E_n \#([y_j] \cap P_1(\ell))$, so there must be a $y \in ([y_j] \cap P_1(\ell)) - \{y_0, \ldots, y_{k-1}\}$. Let y_k be that y. Since $x_k \in [x_j]$, $\vec{i}_k = \vec{i}_j$. Similarly, $y_k \in [y_j]$.

Case 3. $x_k \notin [x_j]$ for each $j < k$, hence $[x_k] \cap [x_j] = \emptyset$ for $j < k$. Let $\vec{i}_k$ be such that $[x_k] \in \gamma_0(\vec{i}_k)$. By assumption, $\#(\gamma_0(\vec{i}_k)) E_n \#(\gamma_1(\vec{i}_k))$, so there must be a $[y]$ such that $[y] \in \gamma_1(\vec{i}_k)$ and $[y] \cap [y_j] = \emptyset$ for $j < k$. Let $y_k \in [y] \cap P_1(\ell)$. There must be such a y_k, since $x_k \in [x_k] \cap P_0(\ell)$ and $\#([y] \cap P_1(\ell)) E_n \#([x_k] \cap P_0(\ell))$.

Lemma 2. There is a primitive recursive function $g(m,n)$ such that if

$$\langle \mathcal{M}_0, \eta_j \rangle_{j<m} \; \mathcal{E}'_{g(m,n)} \langle \mathcal{M}_1, \theta_j \rangle_{j<m},$$

then for each $\eta_m \subseteq \mathcal{M}_0$ there is a $\theta_m \subseteq \mathcal{M}_1$, and for each $\theta_m \subseteq \mathcal{M}_1$ there is an $\eta_m \subseteq \mathcal{M}_0$ such that

$$\langle \mathcal{M}_0, \eta_j \rangle_{j<m+1} \; \mathcal{E}_n \langle \mathcal{M}_1, \theta_j \rangle_{j<m+1}.$$

Proof: We show how to satisfy the requirement of the lemma by picking θ_m once η_m has been chosen (the reverse is similar) under the assumption that

$$g(m,n) = n(n+1)^{2^{m+1}}.$$

This is obviously primitive recursive.

θ_m will be constructed in two phases. First a $\delta_{\vec{j}}$ will be constructed for each $\vec{j} \in Seq(2^{m+1},n)$. Each of these will then be modified to a $\delta'_{\vec{j}}$ in such a way that at the end of the construction, if $[y] \in \gamma_1(\vec{j})$, then $[y] \cap \theta_n \subseteq \delta'_{\vec{j}}$. We will have $\theta_m = \cup\{\delta'_{\vec{j}} \mid \vec{j} \in Seq(2^{m+1},n)\}$.

Phase I. Assume $\vec{j} \in Seq(2^{m+1},n)$.

Let $N = N(\vec{j}) = \min(n, \#(\gamma_0(\vec{j})))$.

Select N unused equivalence classes from $\gamma_0(\vec{j})$ and call them $[x_0],\ldots,[x_{N-1}]$. These $[x_k]$, $k < N$, are now used. Note that $\#(\gamma_0(\vec{j})) < n$ implies $\{[x_0],\ldots,[x_{N-1}]\} = \gamma_0(\vec{j})$. These $[x_k]$, $k < N$, clearly must exist, since $N \leq \#(\gamma_0(\vec{j}))$, and $\vec{i} \neq \vec{j}$ implies $\gamma_0(\vec{i}) \cap \gamma_1(\vec{j}) = \emptyset$; thus $\gamma_0(\vec{j})$ is undisturbed for $\vec{i} \neq \vec{j}$. For $k < N$, let $\vec{i}_k \in Seq(2^m, g(m,n))$ satisfy $[x_k] \in \gamma_0(\vec{i}_k)$. Now pick an unused $[y_k] \in \gamma_1(\vec{i}_k)$, for each $k \in N$. These $[y_k]$ are now used. This can be done, since

$$\#(\gamma_0(\vec{i}_k))E_{g(m,n)}\#(\gamma_1(\vec{i}_k))$$

by assumption, and there are at most $(n+1)^{2^{m+1}}$ $\vec{j}$'s, each of which can use at most n equivalence classes. The idea is now to "carve up" the $[y_k]$'s in the same way the $[x_k]$'s were carved up, putting some of the pieces into θ_m and the rest into $\overline{\theta}_m$. What we want is a procedure which, given an equivalence class $[y_k] \subseteq \mathcal{M}_1$, will produce 2^m disjoint subsets $\alpha_{k,\ell}$ (for $\ell < 2^m$) such that $\alpha_{k,\ell} \subseteq [y_k]$. Of course there are to be produced 2^m disjoint subsets $\beta_{k,\ell}$ ($\ell < 2^m$) such that $[y_k] = \bigcup_{\ell<2^m}(\alpha_{k,\ell} \cup \beta_{k,\ell})$. The actual procedure is described below.

<u>Procedure</u>. This procedure accepts as input an equivalence class $[y_k]$ and a sequence $\vec{j} \in Seq(2^{m+1}, n)$. Its output is two collections of sets $\{\alpha_{k,\ell} \mid \ell < 2^m\}$ and $\{\beta_{k,\ell} \mid \ell < 2^m\}$.

<u>Stage ℓ</u> ($\ell < 2^m$): Let $P_1(\ell) = \bigcap_{t<m} \theta_t^{f_\ell(t)}$, and consider this sketch of $P_1(\ell)$.

$$\begin{array}{c|c|c|}
 & [y_k] & \overline{[y_k]} \\
\hline
\theta_m & \alpha_{k,\ell} & \\
\hline
\overline{\theta}_m & \beta_{k,\ell} & \\
\hline
\end{array}$$

Here $\alpha_{k,\ell} \cup \beta_{k,\ell} = [y_k] \cap P_1(\ell)$.

Let $a = \#(\alpha_{k,\ell})$ and $b = \#(\beta_{k,\ell})$.

Set

(i) $a = \vec{J}(2\ell)$ & $b = \vec{J}(2\ell+1)$, if $\vec{J}(2\ell) < n$ & $\vec{J}(2\ell+1) < n$;

(ii) $a = \vec{J}(2\ell)$ & $b = \#([y_k] \cap P_1(\ell)) - a$, if $\vec{J}(2\ell) < n$ & $\vec{J}(2\ell+1) = n$;

(iii) $a = \#([y_k] \cap P_1(\ell)) - b$ & $b = \vec{J}(2\ell+1)$, if $\vec{J}(2\ell) = n$ & $\vec{J}(2\ell+1) < n$;

(iv) $a = \vec{J}(2\ell)$ & $b = \#([y_k] \cap P_1(\ell)) - a$, if $\vec{J}(2\ell) = n$ & $\vec{J}(2\ell+1) = n$.

Note that in each case, $a+b = \#([y_k] \cap P_1(\ell))$ and that

$$a \; E_n \vec{J}(2\ell) \;\&\; b \; E_n \vec{J}(2\ell+1).$$

<u>Stage 2^m</u>: Stop.

Now set $\delta_{\vec{J}} = \bigcup_{k<N} \left(\bigcup_{\ell<2^m} \alpha_{k,\ell} \right)$. This ends Phase I.

Loosely speaking, if we let $\theta_m = \bigcup\{\delta_{\vec{J}} \mid \vec{J} \in \mathrm{Seq}(2^{m+1},n)\}$ we will have put enough into θ_m to be sure that it works, but not enough to be sure that $\overline{\theta_m}$ works. More specifically, what could happen would be that

$$\#(\gamma_0(\vec{J}_0)) < n \;\&\; \#(\gamma_0(\vec{J}_0)) < \#(\gamma_1(\vec{J}_0))$$

for some $\vec{J}_0 \in \mathrm{Seq}(2^{m+1},n)$ satisfying $\forall \ell \in 2^m[\vec{J}_0(2\ell) = 0]$. Hence we have to find a "dumping place" for the extra equivalence classes in $\gamma_1(\vec{J}_0)$. There must be such a dumping place in $\mathcal{M}_1$ since there was one in $\mathcal{M}_0$. The proof which follows will formalize the following ideas:

Look at all the $\vec{i} \in Seq(2^m, g(m,n))$ which still have unused $[y]$ at the end of Phase I which would lie in $\gamma_1(\vec{j}_0)$ if θ_m were $\cup\delta_{\vec{j}}$. Fix one such $\vec{i}$. From this $\vec{i}$ must have come some $\vec{j}_1$ such that in $\mathcal{M}_0$, $\#(\gamma_0(\vec{j}_1)) \geq n$. By the construction of $\delta_{\vec{j}_1}$ we would have $\#(\gamma_1(\vec{j}_1)) \geq n$, were $\theta_m = \bigcup_{\vec{j}} \delta_{\vec{j}}$. Since we need only have $\#(\gamma_0(\vec{j}_1))E_n\#(\gamma_1(\vec{j}))$, we can put all the unused $[y]$ from $\gamma_1(\vec{i})$ into $\gamma_1(\vec{j}_1)$, which we do by carving up these $[y]$'s into pieces, some of which will be put into θ_m, as we did before.

Phase II. Let $J = \{\vec{j} \in Seq(2^{m+1},n) \mid \forall \ell < 2^m(\vec{j}(2\ell) = 0)\}$. Note that $\vec{j}_0 \in J$ implies that if $[x] \in \gamma_0(\vec{j}_0)$, then $[x] \subseteq \overline{\eta}_m$.

For $\vec{j} \in Seq(2^{m+1},n)$, initialize $\delta'_{\vec{j}} = \delta_{\vec{j}}$. For $\vec{j} \in Seq(2^{m+1},n)$, let $A(\vec{j})$ = ancestors of $\vec{j}$ =
$\{\vec{i} \in Seq(2^m, g(m,n)) \mid \forall \ell < 2^m[(\vec{j}(2\ell) \neq n \,\&\, \vec{j}(2\ell+1) \neq n . \rightarrow \vec{j}(2\ell) + \vec{j}(2\ell+1) = \vec{i}(\ell))$
$\quad \& \ (\vec{j}(2\ell) = n \vee \vec{j}(2\ell+1) = n . \rightarrow \vec{j}(2\ell) + \vec{j}(2\ell+1) \leq \vec{i}(\ell))]\}$.

For $\vec{i} \in Seq(2^m, g(m,n))$, let $P(\vec{i})$ = progeny of $\vec{i}$ $= \{\vec{j} \in Seq(2^{m+1},n) \mid \vec{i} \in A(\vec{j})\}$. Note that $\#(P(\vec{i})) \leq \#(Seq(2^{m+1},n))$. For each $j_0 \in J$, change δ'_{j_0} as indicated below.

Case 1. $\#(\gamma_0(\vec{j}_0)) \geq n$.
Set $\delta'_{\vec{j}_0} = \delta_{\vec{j}_0} \cup \{[y] \mid \exists \vec{i}_1 \in A(\vec{j}_0)([y] \in \gamma_1(\vec{i}_1)$
$\quad \& \ [y] \text{ unused in Phase I})\}$.

<u>Case 2</u>. $\#(\gamma_0(\vec{j}_0)) < n$.

By construction $\#(\delta'_{\vec{j}_0}) = \#(\delta_{\vec{j}_0}) = \#(\gamma_0(\vec{j}_0))$.

<u>Case 2.1</u>. There is no $\vec{i} \in A(\vec{j}_0)$ such that there is a $[y] \in \gamma_1(\vec{i})$ which was unused in Phase I.

In this case, there is nothing that needs to be done for this $\vec{j}_0$.

<u>Case 2.2</u>. There is an $\vec{i} \in A(\vec{j}_0)$ for which there is a $[y] \in \gamma_1(\vec{i})$ which was unused in Phase I.

We claim that for this $\vec{i}$ there is a $\vec{j}_1 \in P(\vec{i})$ such that $\#(\gamma_0(\vec{j}_1)) \geq n$. Clearly $\cup\{\gamma_0(\vec{j}) \mid \vec{j} \in P(\vec{i})\} \supseteq \gamma_0(\vec{i})$. If, contrary to the claim, $\#(\gamma_0(\vec{j})) < n$ for $\vec{j} \in P(\vec{i})$, we would have $g(m,n) > \sum_{\vec{j}\in P(\vec{i})} \#(\gamma_0(\vec{j})) \geq \#(\gamma_0(\vec{i}))$. Then, by the assumption that $\langle \mathcal{M}_0, \eta_{\vec{j}}\rangle_{\vec{j}\in K^m} \ \mathcal{E}'_{g(m,n)} \langle \mathcal{M}_1, \theta_{\vec{j}}\rangle_{\vec{j}\in K^m}$, we would have $\#(\gamma_0(\vec{i})) = \#(\gamma_1(\vec{i}))$. By the construction in Phase I, since $\#(\gamma_0(\vec{j})) < n$ for each $\vec{j} \in P(\vec{i})$, all the $[x] \in \gamma_0(\vec{i})$ are used, and furthermore exactly as many $[x]$'s from $\gamma_0(\vec{i})$ must have been used as $[y]$'s from $\gamma_1(\vec{i})$. Hence all the $[y] \in \gamma_1(\vec{i})$ must have been used, contradicting the assumption that there was an unused $[y] \in \gamma_1(\vec{i})$, and thus establishing the claim.

Let

$$Y = \{[y_0],[y_1],\ldots\} = \{[y] \mid \exists \vec{i}_1 \in A(\vec{j}_0)([y] \in \gamma_1(\vec{i}_1)) \ \& \ [y] \text{ unused in Phase I}\}.$$

Set $\delta'_{\vec{j}_1} = \delta_{\vec{j}_1} \cup \bigcup_{[y_k]\in Y} \left(\bigcup_{\ell<2^m} \alpha_{k,\ell}\right)$, where $\{\alpha_{k,\ell} \mid \ell < 2^m\}$ is the result of applying the procedure of Phase I to $[y_k] \in Y$ and $\vec{j}_1$.

After $\delta'_{\vec{j}_0}$ have been modified for all $j_0 \in J$, set

$$\theta_m = \cup\{\delta'_{\vec{j}} \mid \vec{j} \in \mathrm{Seq}(2^{m+1},n)\}.$$

As the reader may verify, θ_m has been constructed to satisfy the lemma, and we are done.

<u>Corollary 2</u>. Let $g(m,n) = n(n+1)^{2^{m+1}}$, and define $h(m,n)$ by

$$h(0,n) = n$$
$$h(m+1,n) = g(m,h(m,n)).$$

Then $\langle \mathcal{M}_0, \eta_j \rangle_{j<m} \ \mathcal{E}'_{h(k,n)} \langle \mathcal{M}_1, \theta_j \rangle_{j<m}$ implies $\langle \mathcal{M}_0, \eta_j \rangle_{j<m} \equiv_{k,n} \langle \mathcal{M}_1, \theta_j \rangle_{j<m}$.

<u>Proof</u>: By induction on k.

For $k = 0$, the corollary follows from Lemma 1.

Assume the result holds for some k, and assume $\langle \mathcal{M}_0, \eta_j \rangle_{j<m} \ \mathcal{E}'_{h(k+1,n)} \langle \mathcal{M}_1, \theta_j \rangle_{j<m}$. Thus $\langle \mathcal{M}_0, \eta_j \rangle_{j<m} \ \mathcal{E}'_{g(m,h(k,n))} \langle \mathcal{M}_1, \theta_j \rangle_{j<m}$, and by Lemma 2, the inductive hypothesis, and the definition of k, n equivalence,

$$\langle \mathcal{M}_0, \eta_j \rangle_{j<m} \equiv_{k+1,n} \langle \mathcal{M}_1, \theta_j \rangle_{j<m} .$$

<u>Theorem 3</u>. The monadic second-order theory of an equivalence relation is decidable by a primitive recursive procedure.

Proof: By Corollary 2 and Theorem 1, we see that to check the validity of a sentence

$$\psi = Q_0\xi_0 \cdots Q_{m-1}\xi_{m-1}Q_0 v_0 \cdots Q_{n-1}v_{n-1}\varphi(\xi_0, \ldots, \xi_{m-1}, v_0, \ldots, v_{m-1})$$

in which φ is quantifier-free, it is sufficient to verify that $\mathfrak{M} \models \psi$ for those $\mathfrak{M}$ having at most $h(m,n)$ equivalence classes of each cardinality $\leq h(m,n)$. There are $(h(m,n)+1)^{h(m,n)}-1$ of these models, each with at most $\frac{1}{2}h(m,n)^2(h(m,n)+1)$ elements. But all this can be done by a primitive-recursive procedure. (Cf. Nerode [10].)

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