DIMENSIONS OF ORDINALS: SET THEORY, HOMOLOGY THEORY, AND THE FIRST OMEGA ALEPHS

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DIMENSIONS OF ORDINALS:

SET THEORY, HOMOLOGY THEORY,

AND THE FIRST OMEGA ALEPHS

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We describe an organizing framework for the study of infinitary combinatorics. This framework is Čech cohomology. It describes ZFC combinatorial principles distinguishing among higher ω_n. More precisely, it correlates each ω_n with an $(n+1)$-dimensional generalization of Todorcevic's walks technique, and begins to explain that technique's "unreasonable effectiveness" on ω_1. We show in contrast that on higher cardinals κ, the existence of these principles is frequently independent of the ZFC axioms. Finally, we detail implications of these phenomena for the computation of strong homology groups and higher derived limits, deriving independence results in algebraic topology and homological algebra, respectively, in the process.

BIOGRAPHICAL SKETCH

Jeffrey Bergfalk grew up in the American plains region in the advancing neoliberal era. His first teacher in math was Mr. McCalla. His first friend in math was Davesh. He likes a good Kansas rain with thunder and lightning, and likes almost every cat. Grapefruit tastes good to him. His favorite thinker may be Thelonious Monk.

for Sena

Tú repites
la multiplicación del universo.

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Many kindnesses inhere in this work. They cannot be faulted for its shortcomings.

The standing kindnesses of parents come in many ways first, and condition all the rest.

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Students, teachers, and friends, particularly in The People's School, in MATH 2230/40, and in Auburn Correctional Facility, vivified for me the possibilities of living and learning. Friends from home, Hank, Kyle, Jackie, Cyrus, Mike, Howie, Adam, and Shelley, are always in my mind. Fundamental to all my thinking, finally, are three teachers each of whom is of a sort that one finds, with luck, maybe once in a lifetime. They are Ken Irby and Judy Roitman and Justin Moore.

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CHAPTER 1

INTRODUCTION

Why is it that we can say so much and so little, respectively,

about the ZFC combinatorics of ω_1 and of higher ω_n?

This state of affairs would seem to call, if not for remedy, at least for explanation. In what follows, we propose a bit of both. We argue more particularly that homological approaches begin to both technically and conceptually address the above question in ways that more strictly set theoretic approaches hitherto have not. Our argument has the following structure:

Chapter 2: We describe an organizing framework for the study of infinitary combinatorics. This framework is Čech cohomology. Computed with respect to a constant sheaf, for example, $\check{\mathrm{H}}^1$ of any ordinal ξ collects classes of nontrivial coherent sequences of functions on ξ. Such sequences feature centrally in the ZFC combinatorics of ω_1, largely by way of Todorcevic's method of walks on pairs of countable ordinals ([39], [44]). Higher $\check{\mathrm{H}}^n$ describe dramatically underexplored higher-order principles of nontrivial coherence, each of which displays a clear ZFC affinity for ω_n.

Chapter 3: A main sign of this affinity is Barry Mitchell's 1972 result that the homological dimension of ω_n is $n+1$ ([32]). In Chapter 3, we describe and strengthen this result, deriving several ZFC theorems on ω_n in the process. For example, we show that ω_n is the least ordinal admitting no n-dimensional simplicial complexes of a certain type, and we compute in ZFC the nonvanishing of $\check{\mathrm{H}}^n(\omega_n)$. The driving recognition in this chapter is of *algebraicized walks* as lying at the heart of the $n = 1$ case of Mitchell's theorem. We describe witnesses to the higher-dimensional instances of the theorem that have, accordingly, every title to the name *higher-order algebraicized walks.* They're as unpleasant as they sound. From them, though, one may recover higher-order walks techniques more classical in form. We close with their description, exhibiting in particular a higher Tr^n and ρ_2^n which is n-coherent and conjecturally non-n-trivial on ω_n.

Chapter 4: Some readers may find this chapter more earthbound, in that it describes manifestations of the above phenomena in homology computations for closed subsets of $\mathbb{R}^n$. The same readers may find it less earthbound, though, in the sense that these computations are shown to be independent of the axioms of ZFC. It was the author's growing awareness of the centrality of the cofinality of $\mathbb{N}^{\mathbb{N}}$ to these investigations that led to the work of Chapters 2, 3, and 5. More precisely, all these chapters share in a set theoretic analysis of how higher derived limits $\lim^n(\,\cdot\,)$ articulate the combinatorics of indexing orders. Open questions concluding Chapter 4 underscore how much more in this direction remains to be understood.

Chapter 5: Here we consider the independence of the vanishing of $\check{\mathrm{H}}^n(\kappa)$ for various n and κ. We show how to force non-n-trivial n-coherence. We show that in Gödel's constructible universe, $\check{\mathrm{H}}^n(\kappa) \neq 0$ in every instance not ruled out by Goblot's Theorem or by large cardinal properties. In contrast, if κ is ω_1-

strongly compact then $\check{\mathrm{H}}^n(\lambda) = 0$ for every positive n and regular cardinal $\lambda \geq \kappa$. We show finally that the P-Ideal Dichotomy implies that $\check{\mathrm{H}}^1(\varepsilon) \neq 0$ if and only if the cofinality of ε is $\aleph_1$.

After Chapter 2, these chapters are largely self-contained. A set theorist, for example, may find Chapter 5 a more accessible or illuminating approach to this material than Chapter 3, and should feel very free to read these chapters in that order. Assuredly, Chapter 3 is the most taxing. It requires a hands-on mix of set-theoretic and homological techniques; here above all, the author's conscious of working *between* fields.

It should be clear from the above outline that much of this thesis has the character of groundwork. In other words, this work raises at least as many questions as it answers; it calls for further development in several directions and on a variety of scales. Its guiding perception, though, is quite simple, and probably best captured by Stevo Todorcevic's statement:

They each have their own lives.

He was speaking of the ordinals ω_n.

CHAPTER 2

BACKGROUND: WALKS AND COHOMOLOGY

I would formulate *the* basic problem of set-theoretic topology as follows:

To determine which set-theoretic structures

have a connection with the intuitively given material of polyhedral topology

and hence deserve to be considered as geometric figures

- even if very general ones.

Paul Alexandroff, 1932

2.1 Introduction

There's something fantastically remote about the above quote, to contemporary frames of mind. From the beginning, though, its improbability must have been much of its enigma and its appeal.

By 1932, foundational theorems in set theory and "polyhedral" or algebraic topology, respectively, had already long signaled utterly opposite approaches to the concept of dimension:

Theorem 2.1.1 (Cantor, 1877)**.** $|\mathbb{R}^m| = |\mathbb{R}^n|$.

Theorem 2.1.2 (Brouwer, 1912, [6])**.** $\mathbb{R}^m \not\cong \mathbb{R}^n$.

It was clear by then also how much the core *material* of *set theory* and *polyhedral topology*, respectively — uncountable cardinals and Euclidean space — tend to defy comparison:

Theorem 2.1.3. *Let f be*

- *an order-preserving map from ω_1 to $\mathbb{R}$, or*
- *an order-preserving map from $\mathbb{R}$ to ω_1, or*
- *a continuous map from ω_1 to $\mathbb{R}$, or*
- *a continuous map from $\mathbb{R}$ to ω_1.*

Then f is eventually constant.

Alexandroff understood all this. So when he centers *set-theoretic topology* in the question of *which set-theoretic structures [...] deserve to be considered as geometric figures*, he's deliberately thinking a relation from its most tenuous point, from its point of apparent breakdown.

This describes in many ways the present study — which is, most insistently, of the *connection* or applicability of Čech cohomology (and related *material of polyhedral topology*) to that most basic of *set-theoretic structures*, the ordinals themselves. We find that this connection is strong. More particularly, we'll see that classical topological invariants are highly sensitive to the distinctive combinatorics of the various ω_n, and begin even to clarify and describe those combinatorics in ways that strictly set-theoretic approaches hitherto had not.

In this chapter, we briefly review *the method of walks on ordinals*, i.e., that array of techniques announced in [39] and accumulating in [44], that in such large measure consolidate the combinatorics of ω_1. A hallmark of these techniques and, relatedly, of ω_1, is the phenomenon of nontrivial coherence. We turn then to sheaf theory and to Čech cohomology computations; it's our broad argument that this is the "right" framing of nontrivial coherence. For example: let $\mathcal{A}_d$ be the constant sheaf, with values in the abelian group A, and let δ by any ordinal, endowed with the ordinal topology. Then $\check{\mathrm{H}}^1(\delta, \mathcal{A}_d)$ *is* the group of nontrivial coherent A-valued families of functions on δ, in a precise and natural sense. This is the $n = 1$ instance of Theorems 2.6.2 and 2.6.3. The higher-n cases, accordingly, describe for us higher-order combinatorial phenomena which we term *non-n-trivial n-coherence*; these each, for $n > 1$, are profoundly unexplored. This is partly explained by a deduction from Goblot's Vanishing Theorem, which we describe in Section 2.7 — namely, that non-n-trivial n-coherence, for $n > 1$, is a phenomenon entirely alien to any ordinal below ω_2. The fact, on the other hand, that non-n-trivial n-coherence *is* in force at ω_n is this work's guiding recognition. The $n = 1$ case, in this view, is simply the best-understood; we close with (1) a summary discussion of the cohomology groups of ω_1, and (2) a discussion of related approaches, particularly Talayco's [36] and [37].

2.2 Working definitions and conventions

We write Ord for the class of ordinals, and Lim for the class of limit ordinals. We write $\mathrm{otp}(A)$ for the order-type of a set A of ordinals. Topological considerations on any ordinal β will always be with reference to the order topology on β. We write $\tau(X)$ for the topology of any space X. We write $\mathrm{cf}(\beta)$ for the *cofinality of* β, i.e., for the cardinality of the smallest cofinal subset of β. An ordinal β is *regular* (and consequently a cardinal) if $\mathrm{cf}(\beta) = \beta$. We write $\mathrm{Cof}(\beta)$ for the class of ordinals of cofinality β. We reserve the midsection of the Greek alphabet — κ, λ, μ — for cardinals. $\alpha, \beta, \gamma, \delta, \varepsilon$ will typically denote less distinguished ordinals. κ^+ denotes the smallest cardinal larger than κ. The letter n will always denote a finite ordinal. We write $[A]^n$ for the family of n-element subsets of any collection of ordinals A, and frequently identify such subsets with their natural orderings $\vec{\alpha} = (\alpha_0, \dots, \alpha_{n-1})$. We view $\vec{\alpha}^j$, for example, as that element of $[A]^{n-1}$ obtained via deletion of the j-indexed coordinate of $\vec{\alpha}$. In place of tuples, we sometimes concatenate, writing $\varphi_{\alpha\beta}$ or $\beta\gamma\delta \in [\kappa]^3$, for example; we aim in general for alphabetical orderings of symbols consistent with their intended values. We write $A < B$ if $\alpha < \beta$ for every $\alpha \in A$ and $\beta \in B$. And we write $f\restriction_A$ for the restriction of the function f to the domain A, and $f''A$ for $\{f(\alpha) \,|\, \alpha \in A\}$.

G will in general denote a nontrivial group (we table this convention, though, in the more set theoretic Chapter 5). $\mathbb{Z}_k$ means $\mathbb{Z}/k\mathbb{Z}$. Zero is neither a limit ordinal nor a natural number. Ab is the category of abelian groups.

$C_\beta = \{\eta_i^\beta \mid i \in \mathrm{cf}(\beta)\}$ will always denote a closed cofinal subset of β, in increasing enumeration. We term such a C_β a *ladder*, and a family $\{C_\beta \mid \beta \in B\}$ a *ladder system on* B.[1] We'll sometimes tacitly assume that $\mathrm{cf}(\eta_i^\beta) = \mathrm{cf}(i)$ for all $i < \mathrm{cf}(\beta)$. And we write $C^\beta(\alpha)$ for $\min(C_\beta \backslash (\alpha + 1))$.

The closed unbounded subsets of an ordinal are termed *clubs*. The club subsets of an ordinal β of uncountable cofinality form a filterbase on β. Subsets of β intersecting every club subset of β are termed *stationary*. (In the measure analogy, β's club filter collects the "measure one" sets; stationary sets are then the sets of positive outer measure.) A partial function $f : \beta \to \beta$ is *regressive* if $f(\alpha) < \alpha$ for all $\alpha \in \mathrm{dom}(f)$. We'll apply the following lemma repeatedly, below:

Lemma 2.2.1 (The Pressing Down Lemma (Alexandroff, 1929))**.** *Let κ be a regular uncountable cardinal. Let $S \subseteq \kappa$ be stationary, and let $f : S \to \kappa$ be regressive. Then f is constant on some stationary $T \subseteq S$.*

As a typical application, we argue the theorem quoted in this chapter's introduction:

Corollary 2.2.2. *Let κ be a regular uncountable cardinal. A continuous function f from a stationary $S \subseteq \kappa$ to a metric space X is eventually constant. In particular, any continuous $f : \kappa \to \mathbb{R}^n$ is eventually constant.*

[1]There are drawbacks to this terminology: the term "ladder" was meant for other uses, and is, indeed, pictorially misleading, applied to anything with limit points. Still, like many, we use it — in part to signal requirements of C_β not ending at closed unboundedness. Observe in particular our requirement that $\mathrm{otp}(C_\beta) = \mathrm{cf}(\beta)$.

A stronger reverse assertion follows, of course, from the connectedness of $\mathbb{R}^n$: any continuous $f : \mathbb{R}^n \to \kappa$ is promptly and entirely constant. This *incommensurability* of uncountable well-orders and Euclidean space underscores the oddness of much of what follows: techniques developed expressly for the study of polyhedra and manifolds — i.e., for the study of the locally Euclidean — will prove productive, and even well-adapted, for the study of uncountable κ.

Proof of Corollary 2.2.2. Let $f : S \to X$ be continuous. For $n \in \mathbb{N}$ and $\xi \in S \cap \mathrm{Lim}$, there exists some $\xi_n < \xi$ with $f''[\xi_n, \xi] \subseteq B_{1/n}(f(\xi))$. By the Pressing Down Lemma, for each n there exists some $\underline{\xi}_n \in \kappa$ and stationary $T \subseteq S \cap \mathrm{Lim}$ such that $\xi_n = \underline{\xi}_n$ for all $\xi \in T$. This implies that $f''[\underline{\xi}_n, \kappa) \subseteq B_{2/n}(f(\underline{\xi}_n))$. Let $\underline{\xi} = \sup_{n \in \mathbb{N}} \underline{\xi}_n$. Then $f''[\underline{\xi}, \kappa) \subseteq B_\epsilon(f(\underline{\xi}))$ for arbitrary $\epsilon > 0$, hence f is constant above $\underline{\xi}$. □

Similarly,

Fact 2.2.3. Any basic open cover $\mathcal{V} = \{(\alpha_\xi, \beta_\xi) \,|\, \xi \in S\}$ of a stationary $S \subseteq \kappa$ contains some $\{(\alpha, \beta_\xi) \,|\, \xi \in T\}$ as a subset, where $T \subseteq S$ is stationary in κ as well.

In particular, no stationary $S \subseteq \kappa$ is paracompact in the subspace topology. Hence no stationary subspace $S \subseteq \kappa$ is metrizable. The metrizable subspaces of ω_1, in fact, are precisely the nonstationary $N \subseteq \omega_1$; in the measure analogy, they are the "vanishingly small" subsets of ω_1. The technique of walks on the countable ordinals is, against this background, sometimes read as *just what of a metric theory on ω_1 <u>is</u> possible*; see [44] for more.

A sequence of morphisms $\cdot \xrightarrow{f} X \xrightarrow{g} \cdot$ is *exact at* X if $\mathrm{im}(f) = \ker(g)$. A sequence is *exact* if it is exact at every term which is neither initial nor terminal. Beyond this, we'll define most of what we need from homological algebra as we go. A main exception is the derivation of a long exact sequence from a short one. A reader skeptical of this conversion-process should see [16] or [46], for example, for more.

2.3 A review of walks

I have met with but one or two persons in the course of my life
who understood the art of Walking, that is, of taking walks –

We should go forth on *the shortest walk*, perchance,
in the spirit of undying adventure, never to return...
If you are ready to leave father and mother, and brother and sister,
and wife and child and friends, and never see them again
– if you have paid your debts, and made your will,
and settled all your affairs, and are a free man –
then you are ready for a walk.

- Henry David Thoreau, *Walking*, 1851

We begin this section with one further definition:

Definition 2.3.1. A *tree* is a partial order $\triangleleft$ on a set T with the property that

$\{s \in T \mid s \lhd t\}$ is well-ordered, for all $t \in T$. We write $\mathrm{ht}_T(t)$ for the order-type of $\{s \in T \mid s \lhd t\}$. The *height of* T, written $\mathrm{ht}(T)$, is $\sup_{t\in T} \mathrm{ht}_T(t)$. (Here we partake of the standard abuse of conflating a tree with its underlying subset T.) These notions organize T into *levels* $\mathrm{lev}_\xi(T) := \{t \in T \mid \mathrm{ht}_T(t) = \xi\}$ $(\xi \leq \mathrm{ht}(T))$. A *cofinal branch in* T is a totally $\unlhd$-ordered $x \subseteq T$ for which $\sup_{t\in x} \mathrm{ht}_T(t) = \mathrm{ht}(T)$.

Let κ be a regular cardinal. A κ*-tree* is a tree of height κ whose levels are all of size less than κ. If every κ-tree has a cofinal branch then κ *has the tree property.*

Lemma 2.3.2 (König's Infinity Lemma)**.** ω *has the tree property.*

Theorem 2.3.3 (Aronsjazn; reported in [25])**.** ω_1 *does not have the tree property.*

We'll sketch in this section a more modern form of Aronszajn's construction of an ω_1-tree with no cofinal branch. Such a tree is termed an *Aronsjazn tree*, or a κ*-Aronsjazn tree* when κ is greater than ω_1. A κ-Aronsjazn tree T is a textbook instance of what set theorists call *incompactness*, that is, of a phenomenon at κ abruptly unlike phenomena at any $\xi < \kappa$ (see [7]). The phenomenon in this case is the non-existence of a branch in T of height ξ. In contrast, *reflection* or *compactness principles* describe relations of agreement between κ and the ordinals $\xi < \kappa$. These are a characteristic feature of *large cardinals*, that is, of cardinals κ whose existence is a strictly stronger assumption than that of the ZFC axioms alone. A highly relevant example is the following:

Definition 2.3.4. A regular cardinal μ is *weakly compact* if μ has the tree property and $2^\lambda < \mu$ for all $\lambda < \mu$.

A major area of set-theoretic research concerns questions of how far compactness principles like the tree property may be realized on smaller cardinals. William Mitchell's 1972 realization of the tree property at ω_2 was seminal. More generally, Mitchell showed the following:

Theorem 2.3.5 ([33]). *It is consistent with the existence of a weakly compact cardinal above $\lambda = \kappa^{++}$ that λ has the tree property.*

In Chapter 5, we'll describe a forcing $\mathbb{P}(1, \kappa, \mathbb{Z})$ for adding a κ-Aronszajn tree on any κ of uncountable cofinality. Taken together with Theorem 2.3.5, this shows that ZFC alone fails to decide whether ω_n has the tree property, for any $n > 1$. This informs one view of the question heading our introduction. In this view, the combinatorics of $\omega_2, \omega_3, \omega_4$ and so on tend to be independent of ZFC, unlike those of ω_1. The present work complicates that outlook. This is because it begins to describe extensions to ω_n of a ZFC technique which, "despite [its] simplicity, can be used to derive virtually all known other structures that have been defined so far on ω_1" ([44], p. 19). As the quote below suggests, this family of ZFC structures on ω_1 is extensive.

The technique in question is Todorcevic's method of *walks on the ordinals*. This method applies quite generally, but seems somehow tailor-made for ω_1.

> *An interesting phenomenon that one realizes while analyzing walks on ordinals is the special role of the first uncountable ordinal ω_1 in this theory. Any natural coherency requirement on the sets C_ξ $(\xi < \theta)$ that one finds in this theory is satisfiable in the case $\theta = \omega_1$. How natural the*

notion of walk in this context is can be seen from the fact that basically all of its characteristics lead us in one way or the other to some 'critical' structure that shows up in various rough classifications of mathematical structures. For example, any of the characteristics ρ, ρ_0, ρ_1, ρ_2, ρ_3 of the walk that we study here lead us to the canonical linear ordering appearing on the list of five linear orderings that forms a basis for the class of all uncountable linear orderings. The first uncountable cardinal is the only cardinal on which the theory can be carried out without relying on additional axioms of set theory. The first uncountable cardinal is also the place where the theory has its deepest applications as well as its most important open problems. This special role can perhaps be best explained by the fact that many set-theoretical problems, especially those coming from other fields of mathematics, are usually concerned only about the duality between the countable and the uncountable rather than some intricate relationship between two or more uncountable cardinalities. ([44], p. 14)

This is a deeply suggestive counterpoint to the material of this thesis. We now describe the walks technique in the setting of ω_1.

The technique begins with a nonconstructive input: a *ladder system* $\mathcal{C} = \{C_\alpha \subseteq \alpha \mid \alpha < \omega_1\}$. This system determines a sequence $\mathrm{Tr}(\alpha, \beta)$ of "steps" from any $\beta < \omega_1$ down to any α below β, as pictured below. (The diagram's from [44], p. 21.)

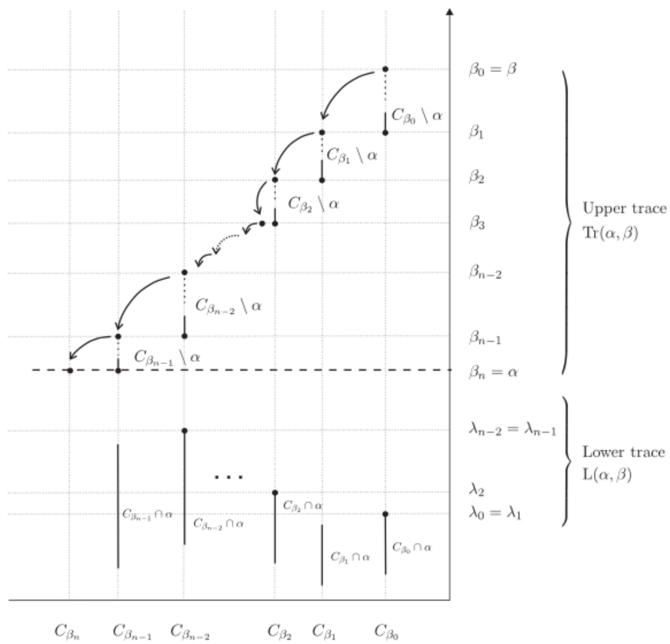


Figure 2.1: The walk and its traces.

These steps are recursively defined; they are those of $\mathrm{Tr}(\,\cdot\,,\,\cdot\,)$, below. We record as well the more prominent *characteristics* of the walk from α to β:

- $\mathrm{Tr}(\alpha, \beta) = \{\beta\} \cup \mathrm{Tr}(\alpha, \min(C_\beta \backslash \alpha))$ (the *upper trace*)
- $\rho_0(\alpha, \beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \min(C_\beta \backslash \alpha))$ (the *full code*)
- $\rho_1(\alpha, \beta) = \max(\rho_0(\alpha, \beta))$ (the *maximal weight*)
- $\rho_2(\alpha, \beta) = |\rho_0(\alpha, \beta)|$ (the *number of steps*)
- $\rho_3(\alpha, \beta) =$ *truth-value of the statement "in the walk from β down to α, the last step weighed the most"* (the *last step function*)
- $L(\alpha, \beta) = (L(\alpha, \min(C_\beta \backslash \alpha)) \backslash \max(C_\beta \cap \alpha)) \cup \{\max(C_\beta \cap \alpha)\}$ (the *lower trace*)

- $F(\alpha, \beta) = F(\alpha, \min(C_\beta \backslash \alpha)) \cup \bigcup_{\xi \in C_\beta \cap \alpha} F(\xi, \alpha)$ (the *full lower trace*)

Grounding each of these recursions is some natural convention at (α, α); for example, $\mathrm{Tr}(\alpha, \alpha) = F(\alpha, \alpha) = \{\alpha\}$, while $L(\alpha, \alpha) = \varnothing$. A reader new to this material is encouraged to play a little with these functions, taking a few "trial walks," for example.

Observe that *as information*, $\rho_0(\cdot, \cdot)$ is equivalent to $\mathrm{Tr}(\cdot, \cdot)$, and maximal among the rho functions, a fact we might in part recast as follows: for all i,

$$T(\rho_i) := \{\rho_i(\cdot, \beta)\restriction_\alpha \mid \alpha \leq \beta < \omega_1\}$$

is an Aronszajn tree (under the extension ordering), with the $i = 1, 2$, and 3 cases quotients of the case $i = 0$. In each case, the narrowness of the tree $T(\rho_i)$ is an effect of the *coherence* of the family $\{\rho_i(\,\cdot\,, \beta) \mid \beta < \omega_1\}$. We'll assign this word concrete mathematical meaning in the next section; here we use it in a more loose and conceptual sense to simultaneously describe phenomena of the following sorts:

Theorem 2.3.6 ([39]). *For all $\alpha < \beta < \omega_1$,*

1. *the function $\rho_1(\,\cdot\,, \beta)\restriction_\alpha - \rho_1(\,\cdot\,, \alpha)$ is a finitely supported function $\alpha \to \mathbb{Z}$;*
2. *the function $\rho_2(\,\cdot\,, \beta)\restriction_\alpha - \rho_2(\,\cdot\,, \alpha)$ is a bounded function $\alpha \to \mathbb{Z}$.*

Most crudely: *functions in a coherent family aren't all that different.* We'll term such a family *trivial* if there's a single-coordinate function $\varphi_i(\,\cdot\,) : \omega_1 \to \mathbb{Z}$ approximating all the functions $\rho_i(\,\cdot\,, \alpha)$ $(\alpha < \omega_1)$ in the sense appropriate to i. The *nontriviality* of the families $\{\rho_i(\,\cdot\,,\beta)\,|\,\beta<\omega_1\}$ then corresponds to the absence of any cofinal branch in $T(\rho_i)$.

In summary, *nontrivial coherence* names *local relations of agreement that can't be globalized.* In some mathematical circles, such phenomena are patently *homological* in nature. In set theoretic circles, such relations are plainly *incompactness* phenomena. It seems odd to this author that these perspectives aren't in better conversation.

2.4 Coherence and cohomology

For generality's sake, we leave the codomain S, below, unspecified.

Definition 2.4.1. A family $\Phi=\{\varphi_\alpha:\alpha\to S\,|\,\alpha\in\delta\}$ is *coherent* if

$$\varphi_\beta\restriction_\alpha=\varphi_\alpha \quad \text{mod finite} \tag{2.1}$$

for all $\alpha\leq\beta$ in δ.

Φ is *trivial* if, for some $\varphi:\delta\to S$,

$$\varphi\restriction_\alpha=\varphi_\alpha \quad \text{mod finite} \tag{2.2}$$

for all α in δ.

We'll term any family satisfying (2.1) but not (2.2) *classically* nontrivial coherent: $\varphi_\beta=\rho_1(\,\cdot\,,\beta)$ or $\rho_3(\,\cdot\,,\beta)$ are canonical examples ([39], [44]).

Elements of δ, of course, are open subsets of δ as well. And structure on the set S is irrelevant to Definition 2.4.1. Hence one natural variant of the above is the following:

Definition 2.4.2. Let $\mathcal{U}$ be an open cover of δ. Let G be a group. A family $\Phi = \{\varphi_U : U \to G \,|\, U \in \mathcal{U}\}$ is *coherent* if

$$\varphi_U\restriction_{U\cap V} - \varphi_V\restriction_{U\cap V} = 0 \quad \text{mod finite}$$

for all U and V in $\mathcal{U}$.

Φ is *trivial* if, for some $\varphi : \delta \to G$,

$$\varphi\restriction_V - \varphi_V = 0 \quad \text{mod finite}$$

for all V in $\mathcal{U}$.

Other standards of comparison also arise: any $\rho_2(\,\cdot\,,\alpha)$ and $\rho_2(\,\cdot\,,\beta)\restriction_\alpha$ ($\alpha < \beta < \omega_1$), for example, may differ infinitely often — but only by a bounded, and in fact locally constant, function $\psi : \alpha \to \mathbb{Z}$. A view of these factors —

1. the index-set or cover,
2. the codomain, and
3. the modulus

— as *parameters* is, broadly, in impulse, sheaf-theoretic.

Definition 2.4.3. A *presheaf* $\mathcal{P}$ *on a topological space* X is a contravariant functor from $\tau(X)$ to the category of abelian groups. It is, in other words, an assignment of a group $\mathcal{P}(U)$ to each $U \in \tau(X)$, together with homomorphisms $p_{UV} : \mathcal{P}(U) \to \mathcal{P}(V)$ for each $U \supseteq V$ in $\tau(X)$, such that $p_{UU} = \text{id}$ and $p_{VW} \circ p_{UV} = p_{UW}$ for all $U \supseteq V \supseteq W$ in $\tau(X)$. It's convenient to require $\mathcal{P}(\varnothing) = 0$ as well.

Elements of $\mathcal{P}(U)$ are termed *sections of* $\mathcal{P}$ *over* U. And in what is, in practice, only rarely an abuse, the maps p_{UV} are termed, and sometimes written as, *restriction maps*, as the maps we consider all will be — just as in the following example.

Example 2.4.4. For any space X and group G, the functor $\mathcal{D}_G : U \mapsto \bigoplus_U G$ is a presheaf.

Definition 2.4.5. A *sheaf on* X is a presheaf $\mathcal{S}$ on X with the following additional property: for any $\mathcal{V} \subseteq \tau(X)$, if $\{s_V \in \mathcal{S}(V) \,|\, V \in \mathcal{V}\}$ are such that $s_V\restriction_{V\cap W} = s_W\restriction_{V\cap W}$ for all $V, W \in \mathcal{V}$, then there exists a unique $s \in \mathcal{S}(\cup\mathcal{V})$ such that $s\restriction_V = s_V$ for all $V \in \mathcal{V}$.

Example 2.4.6. $\mathcal{D}_G$ is not a sheaf, for any infinite Hausdorff space X. Neither is the *constant presheaf* $\mathcal{C}_G : U \mapsto G$, if $\tau(X)$ contains disjoint open sets.

Fact 2.4.7. Write $Sh(\,\cdot\,)$ for the *sheafification* functor: $Sh(\mathcal{P})$ is that universal object, $\mathcal{S}$, in the category of sheaves, through which any morphism from $\mathcal{P}$ to any sheaf uniquely factors. See any standard reference ([5] or [27], for example) for a more precise definition, and construction; for our purposes, it suffices to know that some "optimal" modification, $Sh(\mathcal{P})$, of a presheaf $\mathcal{P}$, exists – often by way of simply relaxing conditions on sections to local ones.

Example 2.4.8. Write $\mathcal{G}_d$ for $Sh(\mathcal{C}_G)$, the sheaf

$$U \mapsto \{f : U \to G \,|\, f \text{ is } \textit{locally} \text{ constant}\}$$

More generally: for G a topological group, write $\mathcal{G}$ for the sheaf

$$U \mapsto \{f : U \to G \,|\, f \text{ is continuous}\}$$

$\mathcal{G}_d$, of course, is a special case of the latter: the subscript "d" signals a discrete topology on G.

All our examples, in fact, will be *presheaves of functions to* G, i.e., $\mathcal{P}(U)$ always will be some subcollection of the functions from U to G, added pointwise.

Any presheaf $\mathcal{P}$ on a space X induces a series of cohomology groups; prominent (and, typically, most concrete) among these are the Čech cohomology groups $\check{\mathrm{H}}^n(X, \mathcal{P})$.

Definition 2.4.9. Fix $\mathcal{V} = \{V_\alpha \,|\, \alpha \in \delta\}$, an open cover of X. Write $\mathrm{H}^n(\mathcal{V}, \mathcal{P})$ for the n^{th} cohomology group of the cochain complex

$$\mathcal{L}(\mathcal{V}, \mathcal{P}): \qquad 0 \to L^0(\mathcal{V}, \mathcal{P}) \to \cdots \to L^j(\mathcal{V}, \mathcal{P}) \xrightarrow{d^j} L^{j+1}(\mathcal{V}, \mathcal{P}) \to \dots \tag{2.3}$$

Here

$$L^j(\mathcal{V}, \mathcal{P}) = \prod_{\vec{\alpha} \in [\delta]^{j+1}} \mathcal{P}(V_{\vec{\alpha}}),$$

where $V_{\vec{\alpha}} = V_{\alpha_0} \cap \cdots \cap V_{\alpha_{j-1}}$. Write then $p_{\vec{\alpha}\vec{\beta}}$ for $p_{V_{\vec{\alpha}}V_{\vec{\beta}}}$, and define $d^j : L^j(\mathcal{V}, \mathcal{P}) \to L^{j+1}(\mathcal{V}, \mathcal{P})$ by

$$d^j f \;:\; \vec{\alpha} \;\mapsto\; \sum_{i=0}^{j+1} (-1)^i p_{\vec{\alpha}^i \vec{\alpha}}(f(\vec{\alpha}^i))$$

Write $\mathcal{V} \leq \mathcal{W}$ if the open cover $\mathcal{W}$ refines $\mathcal{V}$, i.e., if there exists some $r_{\mathcal{W}\mathcal{V}} : \mathcal{W} \to \mathcal{V}$ such that $W \subseteq r_{\mathcal{W}\mathcal{V}}(W)$ for each $W \in \mathcal{W}$. The induced $r^*_{\mathcal{W}\mathcal{V}} : \mathrm{H}^n(\mathcal{V}, \mathcal{P}) \to \mathrm{H}^n(\mathcal{W}, \mathcal{P})$ is independent of the choice of refining map $r_{\mathcal{W}\mathcal{V}}$. Hence these $r^*_{\mathcal{W}\mathcal{V}}$ ($\mathcal{V} \leq \mathcal{W}$) define, in turn, a direct limit

$$\check{\mathrm{H}}^n(X, \mathcal{P}) := \varinjlim_{\mathcal{V} \in \mathrm{Cov}(X)} \mathrm{H}^n(\mathcal{V}, \mathcal{P}) \tag{2.4}$$

This limit is the n^{th} *Čech cohomology group of* X*, with respect to the presheaf* $\mathcal{P}$.

Example 2.4.10. $\check{\mathrm{H}}^0(\omega, \mathcal{D}_G) = \prod_\omega G$. This is easy to see: ordered by refinement, the open covers of ω have a maximum, namely $\mathcal{V}_\omega = \{\{i\} \mid i \in \omega\}$. Hence $\check{\mathrm{H}}^0(\omega, \mathcal{D}_G) = \mathrm{H}^0(\mathcal{V}_\omega, \mathcal{D}_G) = \ker(d^0) = L^0(\mathcal{V}_\omega, \mathcal{D}_G)$ (recall that $\mathcal{D}_G(\varnothing) = 0$), which is, clearly, $\prod_\omega G$. In contrast, $\check{\mathrm{H}}^0(\delta, \mathcal{D}_G) = \bigoplus_\delta G$, for a δ of any cofinality other than ω. This is essentially because for any open cover $\mathcal{V}$ of δ and disjoint $\{V_i \in \mathcal{V} \mid i \in \omega\}$, there's some $k \in \omega$ and $V \in \mathcal{V}$ such that $j > k \Rightarrow V_j \subseteq V$.

A sense of the above example is: $\check{\mathrm{H}}^0(\,\cdot\,, \mathcal{D}_G)$ "picks out" the cofinality $\aleph_0$.

For the remainder of this chapter, let $\mathcal{U}_\delta$ denote $\{\alpha \mid \alpha < \delta\}$, the initial segments open cover of δ. Write $\mathcal{U}_C$, more generally, for any cofinal $C \subseteq \delta$ viewed as a cover.

Lemma 2.4.11. *Let* δ *be an ordinal of uncountable cofinality. A nontrivial coherent* $\Phi = \{\varphi_\alpha : \alpha \to G \mid \alpha \in \delta\}$ *witnesses that* $\mathrm{H}^1(\mathcal{U}_\delta, \mathcal{D}_G) \neq 0$*, in the following sense:*

For $\alpha < \beta < \delta$*, let* $f^\Phi : \alpha\beta \mapsto \varphi_\beta\restriction_\alpha - \varphi_\alpha$*. Then* $0 \neq [f^\Phi] \in \mathrm{H}^1(\mathcal{U}_\delta, \mathcal{D}_G)$*.*

Proof. Coherence ensures that $f^\Phi \in L^1(\mathcal{U}_\delta, \mathcal{D}_G)$. And, being a coboundary, f^Φ is in

$\ker d^1$: for all $\alpha < \beta < \gamma < \delta$,

$$\begin{aligned} d^1 f^\Phi(\alpha\beta\gamma) &= f^\Phi(\beta\gamma) - f^\Phi(\alpha\gamma) + f^\Phi(\alpha\beta) \\ &= (\varphi_\gamma - \varphi_\beta - \varphi_\gamma + \varphi_\alpha + \varphi_\beta - \varphi_\alpha)\restriction_\alpha = 0. \end{aligned} \tag{2.5}$$

If, however, $f^\Phi = d^0(g)$ for some $g \in L^0(\mathcal{U}_\delta, \mathcal{D}_G)$, so that

$$\varphi_\beta - \varphi_\alpha = g(\beta)\restriction_\alpha - g(\alpha) \text{ for all } \alpha < \beta < \delta, \tag{2.6}$$

then

$$\varphi := \varinjlim_{\beta<\delta} (\varphi_\beta - g(\beta)) \tag{2.7}$$

would trivialize the nontrivial family Φ, a contradiction. Hence

$$0 \neq [f^\Phi] \in \mathrm{H}^1(\mathcal{U}_\delta, \mathcal{D}_G).$$

□

Corollary 2.4.12. $\mathrm{H}^1(\mathcal{U}_\delta, \mathcal{D}_G) \neq 0$ *for both* $G = \mathbb{Z}$ *and for* $G = \mathbb{Z}_2$*, for any* δ *of cofinality* $\aleph_1$*.*

Proof. Fix $\delta \in \mathrm{Cof}\,(\aleph_1)$ and a closed and cofinal $C_\delta = \{\eta_i^\delta \mid i < \omega_1\} \subseteq \delta$. For $i = 1$ or 3 and all $\alpha < \beta < \delta$, let

$$\varphi_\beta^{i,\delta}(\alpha) = \begin{cases} \rho_i(\mathrm{otp}(C_\delta \cap \alpha), \mathrm{otp}(C_\delta \cap \beta)) & \text{if } \alpha \in C_\delta \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

Then $\Phi_i^\delta := \{\varphi_\beta^{i,\delta} \mid \beta \in \delta\}$ is a nontrivial coherent family of functions on δ with values in G_i, for $i = 1$ or 3, where $G_1 = \mathbb{Z}$ and $G_3 = \mathbb{Z}_2$.[2] Applying Lemma 2.4.11 then completes the proof. $\square$

A perfectly analogous argument applies to $\mathcal{G}_d$ and ρ_2, for $G = \mathbb{Z}$; that is, ρ_2 witnesses, via an $f : \alpha\beta \mapsto \rho_2(\,\cdot\,,\beta)\restriction_\alpha - \rho_2(\,\cdot\,,\alpha)$, that $\mathrm{H}^1(\mathcal{U}_{\omega_1}, G_d) \neq 0$. And for any $\delta \in \mathrm{Cof}\,(\omega_1)$, a "$C_\delta$-stretched" ρ_2, i.e., the function

$$\rho_2^\delta(\alpha,\beta) := \rho_2(\mathrm{otp}(C_\delta \cap \alpha), \mathrm{otp}(C_\delta \cap \beta)) \quad \text{for } \alpha < \beta < \delta \tag{2.9}$$

will witness that $\mathrm{H}^1(\mathcal{U}_\delta, \mathcal{G}_d) \neq 0$ as well, just as above. For $G = \mathbb{Z}_2$, "raise ρ_3":

$$r\rho_3(\beta,\gamma) = \begin{cases} \rho_3(\alpha,\gamma) & \text{if } \beta = \alpha + 1 \\ 0 & \text{if } \beta \text{ is a limit} \end{cases} \tag{2.10}$$

Then $f : \beta\gamma \mapsto r\rho_3(\,\cdot\,,\gamma)\restriction_\beta - r\rho_3(\,\cdot\,,\beta)$ is locally constant, and witnesses that $\mathrm{H}^1(\mathcal{U}_\delta, \mathcal{G}_d) \neq 0$ for $G = \mathbb{Z}_2$. Such constructions, though, evoke a more general principle — namely, the essential equivalence, in our contexts, of $\mathcal{D}_G$ (encoding the set-theoretic theme of *mod finite* relations) and $\mathcal{G}_d$ (standard in more geometric settings):

Lemma 2.4.13. *For cofinal $C \subseteq \delta \cap \mathit{Lim}$, the cochain complexes $\mathcal{L}(\mathcal{U}_C, \mathcal{D}_G)$ and $\mathcal{L}(\mathcal{U}_C, \mathcal{G}_d)$ are chain-isomorphic. In consequence, $\mathrm{H}^n(\mathcal{U}_C, \mathcal{D}_G) \cong \mathrm{H}^n(\mathcal{U}_C, \mathcal{G}_d)$ for all $n \geq 0$.*

Proof. Fix $\gamma \in C$ and define the following maps: for $\varphi \in \mathcal{G}_d(\gamma)$, let $e\varphi(\beta)$ denote the eventual value of φ below β (in particular, $e\varphi(\alpha+1) = \varphi(\alpha)$). As φ is locally

[2] We can, of course, interpret ρ_3's $\{0,1\}$-codomain in any nontrivial group G, so that in fact $\mathrm{H}^1(\delta, \mathcal{D}_G) \neq 0$ for *any* such G and $\delta \in \mathrm{Cof}\,(\aleph_1)$.

constant, this is well-defined — except at 0: let $e\varphi(0) = 0$. Define then $\partial\varphi \in \mathcal{D}_G(\gamma)$ by

$$\partial\varphi(\beta) = \varphi(\beta) - e\varphi(\beta)$$

for $\beta \in \gamma$. This operation is reversible: for $\psi \in \mathcal{D}_G(\gamma)$, define $\partial^{-1}\psi$ by

$$\partial^{-1}\psi(\beta) = \sum_{\alpha \leq \beta} \psi(\alpha)$$

for $\beta \in \gamma$. Clearly $\partial^{-1} \circ \partial = \mathrm{id} : \mathcal{G}_d(\gamma) \to \mathcal{G}_d(\gamma)$. ∂^{-1} isn't quite an isomorphism from $\mathcal{D}_G$ to $\mathcal{G}_d$, though: $\partial^{-1}\psi \notin \mathcal{G}_d(\gamma)$ if $\mathrm{supp}(\psi) \cap \mathrm{Lim} \neq \varnothing$. But this is easily addressed: for $\psi \in \mathcal{D}_G(\gamma)$, define

$$r\psi(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \in \omega \\ 0 & \text{if } \beta \text{ is a limit} \\ \psi(\alpha) & \text{if } \beta = \alpha + 1 \end{cases} \tag{2.11}$$

for $\beta \in \gamma$. For $\varphi \in \mathcal{G}_d(\gamma)$, define

$$r^{-1}\varphi(\beta) = \begin{cases} \varphi(\beta) & \text{if } \beta \in \omega \\ \varphi(\beta + 1) & \text{otherwise} \end{cases} \tag{2.12}$$

for $\beta \in \gamma$. Then $\partial^{-1} \circ r : \mathcal{D}_G(\gamma) \to \mathcal{G}_d(\gamma)$ and $r^{-1} \circ \partial : \mathcal{G}_d(\gamma) \to \mathcal{D}_G(\gamma)$ are inverse, as desired. And they extend to cochain isomorphisms, as the reader may verify: both

$$\mathcal{L}(\mathcal{U}_C, \mathcal{G}_d) \xrightarrow{r^{-1}\partial} \mathcal{L}(\mathcal{U}_C, \mathcal{D}_d) \xrightarrow{\partial^{-1}r} \mathcal{L}(\mathcal{U}_C, \mathcal{G}_d)$$

and

$$\mathcal{L}(\mathcal{U}_C, \mathcal{D}_d) \xrightarrow{\partial^{-1}r} \mathcal{L}(\mathcal{U}_C, \mathcal{G}_d) \xrightarrow{r^{-1}\partial} \mathcal{L}(\mathcal{U}_C, \mathcal{D}_d)$$

are the identity. □

Operative in the above arguments was the following:

Observation 2.4.14. Nontrivial coherence exhibits degrees of transformation invariance befitting a homological property. In particular, we'll increasingly view the following principles as routine:

1. *Shifts* (*as in* (2.11), (2.12))) *and re-scalings* (*as in* (2.8), (2.9)) *of a nontrivial coherent family are again nontrivial coherent. Ladders* $C_\delta = \{\eta_i^\delta \mid i < \kappa\}$*, in particular, "stretch" or propagate nontrivial coherence, from* κ *to* δ. This holds for higher-order coherence as well; for this reason, we'll tend in later chapters to consider, firstly, regular cardinals κ, leaving arguments' extensions to the class of ordinals $\mathrm{Cof}(\kappa)$, frequently, to the reader.
2. Fix $\eta \in \delta$ and cofinal $C \subseteq (\eta, \delta)$. A rough converse to (1) is that *the restriction,* $\Phi[\eta, C] := \{\varphi_\beta \restriction_{[\eta,\beta)} \mid \beta \in C\}$*, of a nontrivial coherent* $\Phi = \{\varphi_\beta \mid \beta \in \delta\}$ *to cofinal indices and/or to cobounded domain is again nontrivial coherent.*

Both properties hold much more generally. They hold, in particular, for all non-n-trivial n-coherence relations; these we now define.

2.5 Higher coherence: Overview

Henceforth, we write $=^*$ for *equality mod finite.* For readability, we'll also sometimes leave the more obvious restrictions unexpressed, writing $\varphi_{\vec{\alpha}^0}$ for $\varphi_{\vec{\alpha}^0}\restriction_{\alpha_0}$, for example, as in the sum (2.13) below. Observe then that *1-coherence*, below, accords with our working definition(s) of coherence.

Definition 2.5.1. For $n \in \mathbb{N}$, a family $\Phi_n = \{\varphi_{\vec{\alpha}} : \alpha_0 \to G \,|\, \vec{\alpha} \in [\varepsilon]^n\}$ is *n-coherent* if

$$\sum_{i=0}^{n} (\text{-}1)^i \varphi_{\vec{\alpha}^i} =^* 0 \tag{2.13}$$

for all $\vec{\alpha} \in [\varepsilon]^{n+1}$.

Φ_1 is 1-*trivial* if it is trivial.

For $n > 1$, Φ_n is *n-trivial* if there exists a $\Psi_{n-1} = \{\psi_{\vec{\alpha}} : \alpha_0 \to G \,|\, \vec{\alpha} \in [\varepsilon]^{n-1}\}$ such that

$$\sum_{i=0}^{n-1} (\text{-}1)^i \psi_{\vec{\alpha}^i} =^* \varphi_{\vec{\alpha}} \tag{2.14}$$

for all $\vec{\alpha} \in [\varepsilon]^n$.

Remark 2.5.2. Observe that non-n-trivial n-coherence so defined exhibits the signature features of nontrivial coherence. Namely:

1. Rigidity of dimension. Again, non-n-trivial n-coherence names a (mod finite) redundancy of n-dimensional data (2.13) that defies approximation (2.14), nevertheless, on any smaller number of dimensions.
2. Incompactness. Just as for the case $n = 1$, n-coherence and n-triviality are two faces of the same thing: a family is n-coherent if and only if its every proper initial segment is n-trivial. Equivalently, a family is n-trivial if and

only if some n-coherent family properly extends it. (See Example 2.5.3 below for more careful formulations; see 3(a), just following, or Sections 5.4 or 5.5, for framings via more classical incompactness phenomena). In either view, non-n-trivial n-coherence on ε marks an abrupt *change in behavior at* ε — one indeed, as we'll see, that not all ordinals ε can accommodate.

Example 2.5.3. $\Phi_2 = \{\varphi_{\alpha\beta} : \alpha \to G \mid \alpha\beta \in [\kappa]^2\}$ is 2-coherent if

$$\varphi_{\beta\gamma}\restriction_\alpha - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta} =^* 0 \quad \text{for all } \alpha\beta\gamma \in [\kappa]^3 \tag{2.15}$$

(The reader might compare the forms of (2.15) and (2.5), above; their relation reflects the sequence (2.22), below.) Φ_2 is 2-trivial if there exists a $\Psi_1 = \{\psi_\alpha \mid \alpha \in \kappa\}$ such that

$$\psi_\beta\restriction_\alpha - \psi_\alpha =^* \varphi_{\alpha\beta} \quad \text{for all } \alpha\beta \in [\kappa]^2$$

or equivalently if

$$-\psi_\beta\restriction_\alpha - - \psi_\alpha + \varphi_{\alpha\beta} =^* 0 \tag{2.16}$$

for all $\alpha\beta \in [\kappa]^2$. By (2.15), the family $\{\psi_\beta := -\varphi_{\beta\gamma} \mid \beta \in \gamma\}$ fulfills (2.16), for all $\alpha\beta \in [\gamma]^2$; in other words, this family 2-trivializes the initial segment $\{\varphi_{\alpha\beta} \mid \alpha\beta \in [\gamma]^2\}$ of Φ_2.

This holds very generally: fix $n > 1$ and let $\Phi_n = \{\varphi_{\vec{\alpha}} : \alpha_0 \to G \mid \vec{\alpha} \in [\kappa]^n\}$ be n-coherent, and let $\Phi_n^\gamma := \{\varphi_{\vec{\alpha}\gamma} \mid \vec{\alpha} \in [\gamma]^{n-1}\}$ and $\Phi_n\restriction_\gamma := \{\varphi_{\vec{\alpha}} \mid \vec{\alpha} \in [\gamma]^n\}$, for any $\gamma < \kappa$. That

$$(-1)^{n+1}\Phi_n^\gamma \ \ n\text{-trivializes } \ \Phi_n\restriction_\gamma \tag{2.17}$$

is then immediate from Definition 2.5.1. Relatedly, and almost as immediately,

$$\Phi_n^\delta\restriction_\gamma - \Phi_n^\gamma := \{\varphi_{\vec{\alpha}\delta} - \varphi_{\vec{\alpha}\gamma} \,|\, \vec{\alpha} \in [\gamma]^{n-1}\} \text{ is } (n-1)\text{-trivial.} \tag{2.18}$$

Of course,

$$\Phi_n = \bigcup_{\gamma \in \kappa} \Phi_n^\gamma \tag{2.19}$$

and this fact, together with (2.18), shapes the following view:

3. <u>An n-coherent Φ_n is a family of families of functions, agreeing modulo $(n-1)$-triviality.</u>

   This extends the case $n = 1$ in several senses:

   (a) A 1-coherent

   $$\Phi_1 = \{\varphi_\gamma \,|\, \gamma \in \kappa\} = \bigcup_{\gamma \in \kappa} \{\varphi_\gamma\}$$

   determines, as we've seen, a height-κ tree of functions $T(\Phi_1)$ possessing a cofinal branch if and ony if Φ_1 is 1-trivial. Decomposition (2.19) determines a tree of *families of* functions $T(\Phi_n)$, again of height κ, and again possessing a cofinal branch only if Φ_n is n-trivial.

   (b) More immediately, the above suggests an equation of *0-trivial* with *finitely supported*. The level zero nontrivial coherent objects are then, on this suggestion, the very "material" of non-1-trivial 1-coherence: they are the infinite countable ladders themselves.

Definition 2.5.4. A function $\varphi : \beta \to G$ is *0-trivial* if

$$\varphi =^* 0$$

i.e., if φ has finite support. Just as for higher n, then, *0-coherent* means *0-trivial on all initial segments*: φ is *0-coherent* if

$$\varphi\restriction_\alpha =^* 0 \quad \text{for all } \alpha < \beta.$$

Hence a non-0-trivial 0-coherent function is simply a $\varphi : \beta \to G$ supported on some ladder $C_\beta \subseteq \beta \in \mathrm{Cof}(\aleph_0)$.

Under this definition,

- non-0-trivial 0-coherence displays as plain an affinity for ω as non-1-trivial 1-coherence does for ω_1: for both $n = 0$ and 1, the least ordinal and, consistently, the only cofinality,[3] admitting non-n-trivial n-coherent functions, is ω_n.
- a view of (infinite) countable ladders (identified with their characteristic functions) as the canonical (non-0-trivial) 0-coherent functions is natural — much as we might view the rho functions, for example, as canonical non-1-trivial 1-coherent functions on ω_1.

These points, of course, are related: ω_1 is the least ordinal admitting no canonical 0-coherent function, and it's via *walks on* such functions that canonical non-1-trivial 1-coherent families on ω_1 arise — with non-0-trivial 0-coherent functions (the infinite

[3]For the $n = 1$ case, see Theorems 2.7.6 and 5.5.11, below.

ladders) playing the decisive role. In this framing, it's perhaps not, in turn, on *ladders* but on *coherent families* that we should aim to "walk" at ω_2 (with the nontrivial coherent families, presumably, again decisive). And so on. We make some sense of these suggestions in Chapter 2.

Lastly, as noted, higher non-n-trivial n-coherence has all the "resilience" of classical nontrivial coherence:

4. <u>Transformation invariance</u>. The restriction $\Phi[\eta, C]$ (as in Observation 2.4.14) of a non-n-trivial n-coherent family Φ is again non-n-trivial n-coherent. This is largely because, as noted, n-coherent families are n-trivial on initial segments, so that global n-triviality is a question only really of the tail.

2.6 Higher coherence and cohomology

Higher coherence and triviality, of course, are above all cohomological phenomena: for non-n-trivial n-coherent $\Phi = \{\varphi_{\vec{\alpha}} : \alpha_0 \to G \mid \vec{\alpha} \in [\varepsilon]^n\}$ define, as before, elements f^Φ of $L^n(\mathcal{U}_\delta, \mathcal{D}_G)$ by

$$f^\Phi : \vec{\alpha} \mapsto \sum_{i=0}^{n} (\text{-}1)^i \varphi_{\vec{\alpha}^i}$$

Again, then,

$$0 \neq [f^\Phi] \in \mathrm{H}^n(\mathcal{U}_\delta, \mathcal{D}_G)$$

as the reader may verify. In fact something much stronger is true: that *all* the elements of $\mathrm{H}^n(\mathcal{U}_\delta, \mathcal{D}_G)$ are of this form, and that the map $[\Phi] \mapsto [f^\Phi]$ defines an isomorphism between "mod finite" cohomology groups and "finite support" cohomology groups of the next degree (see (2.23) and (2.25), below). This, in turn, is an instance of the following.

Definition 2.6.1. Let $\mathcal{P}_G$ denote a presheaf of functions to G (recall that $\mathcal{D}_G$, $\mathcal{G}_d$, and $\mathcal{G}$ are examples). Write $\mathcal{E}_G$ for the presheaf $U \mapsto \prod_U G$, and $\mathcal{F}_G$ for $\mathcal{E}_G/\mathcal{P}_G$: $U \mapsto (\mathcal{E}_G(U)/\mathcal{P}_G(U))$.

Throughout this section, view the presheaves in question as all defined on some fixed δ of uncountable cofinality.

The natural inclusion and quotient maps, $\mathcal{P}_G \to \mathcal{E}_G$ and $\mathcal{E}_G \to \mathcal{F}_G$, respectively, induce a short exact sequence of cochain complexes —

$$\mathcal{L}(\mathcal{V}, \mathcal{P}_G) \to \mathcal{L}(\mathcal{V}, \mathcal{E}_G) \to \mathcal{L}(\mathcal{V}, \mathcal{F}_G) \tag{2.20}$$

— and, in consequence, the long exact sequence

$$\mathrm{H}^0(\mathcal{V}, \mathcal{P}_G) \hookrightarrow \mathrm{H}^0(\mathcal{V}, \mathcal{E}_G) \to \mathrm{H}^0(\mathcal{V}, \mathcal{F}_G) \xrightarrow{d} \mathrm{H}^1(\mathcal{V}, \mathcal{P}_G) \to \mathrm{H}^1(\mathcal{V}, \mathcal{E}_G) \to \dots \tag{2.21}$$

$$\dots \to \mathrm{H}^n(\mathcal{V}, \mathcal{E}_G) \to \mathrm{H}^n(\mathcal{V}, \mathcal{F}_G) \xrightarrow{d} \mathrm{H}^{n+1}(\mathcal{V}, \mathcal{P}_G) \to \mathrm{H}^{n+1}(\mathcal{V}, \mathcal{E}_G) \to \dots \tag{2.22}$$

We'll see below that $\mathrm{H}^n(\,\cdot\,, \mathcal{E}_G) = 0$, for $n > 0$. By (2.21), then,

$$\frac{\mathrm{H}^0(\mathcal{V}, \mathcal{F}_G)}{\mathrm{im}(\mathrm{H}^0(\mathcal{V}, \mathcal{E}_G))} \overset{d}{\cong} \mathrm{H}^1(\mathcal{V}, \mathcal{D}_G) \tag{2.23}$$

Specialization to the case of $\mathcal{V} = \mathcal{U}_\delta$ and $\mathcal{P}_G = \mathcal{D}_G$ reads

$\mathrm{H}^1(\mathcal{U}_\delta, \mathcal{D}_G)$ *is the group of coherent families of* G*-valued functions indexed by* δ*, quotiented by the group of trivial families of* G*-valued functions indexed by* δ*.* (2.24)

The group operation here is the natural one: for coherent $\Phi = \{\varphi_\alpha \,|\, \alpha < \delta\}$ and $\Psi = \{\psi_\alpha \,|\, \alpha \in \delta\}$, the sum $\Phi + \Psi = \{\varphi_\alpha + \psi_\alpha \,|\, \alpha \in \delta\}$ is again coherent. And the map d is the familiar $[\Phi] \mapsto [f^\Phi]$.

Similarly, (2.22) specializes to

$$\mathrm{H}^n(\mathcal{U}_\delta, \mathcal{F}_G) \cong \mathrm{H}^{n+1}(\mathcal{U}_\delta, \mathcal{D}_G) \qquad \text{for } n > 1, \tag{2.25}$$

again via the mapping $[\Phi] \mapsto [f^\Phi]$. This relation, of course, determined Definition 2.5.1, through which (2.24) generalizes:

Theorem 2.6.2. *For $n > 0$, $\mathrm{H}^n(\mathcal{U}_\delta, \mathcal{D}_G)$ is the group of n-coherent families of G-valued functions indexed by δ, quotiented by the group of n-trivial families of G-valued functions indexed by δ.*

Addition in these groups is "pointwise," as in the case $n = 1$.

Moreover,

Theorem 2.6.3. $\check{\mathrm{H}}^n(\delta, \mathcal{P}_G) \cong \mathrm{H}^n(\mathcal{V}, \mathcal{P}_G)$*, for any $n > 0$ and $\mathcal{V} \geq \mathcal{U}_\delta$ and presheaf $\mathcal{P}_G$ of functions to G. In particular,* $\check{\mathrm{H}}^n(\delta, \mathcal{G}_d) \cong \check{\mathrm{H}}^n(\delta, \mathcal{D}_G) \cong \mathrm{H}^n(\mathcal{U}_\delta, \mathcal{D}_G)$ *for all $n > 0:$ each is naturally isomorphic to the group described in Theorem 2.6.2.*

Proof. The second assertion follows from the first, together with Lemma 2.4.13: $\mathrm{H}^n(\mathcal{U}_\delta, \mathcal{D}_G) \cong \check{\mathrm{H}}^n(\delta, \mathcal{D}_G) \cong \mathrm{H}^n(\mathcal{U}_C, \mathcal{D}_G) \cong \mathrm{H}^n(\mathcal{U}_C, \mathcal{G}_d) \cong \check{\mathrm{H}}^n(\delta, \mathcal{G}_d)$, where $C = \delta \cap \mathrm{Lim}$. We turn then to the first assertion. The heart of the argument is the following square:

$$
\begin{array}{ccc}
\mathrm{H}^n(\mathcal{U}_\delta, \mathcal{F}_G) & \xrightarrow{d} & \mathrm{H}^{n+1}(\mathcal{U}_\delta, \mathcal{P}_G) \\
{\scriptstyle r^*_{\mathcal{V}\mathcal{U}_\delta}}\downarrow & & \downarrow{\scriptstyle r^*_{\mathcal{V}\mathcal{U}_\delta}} \\
\mathrm{H}^n(\mathcal{V}, \mathcal{F}_G) & \xrightarrow{d} & \mathrm{H}^{n+1}(\mathcal{V}, \mathcal{P}_G)
\end{array}
\tag{2.26}
$$

In the case of $n = 0$, replace the left-hand side with the quotients of (2.23): the point in all cases is that

1. The lateral maps, d, are isomorphisms, and
2. The diagram commutes.

Therefore it suffices to show $r^*_{\mathcal{V}\mathcal{U}_\delta} : \mathrm{H}^n(\mathcal{U}_\delta, \mathcal{F}_G) \to \mathrm{H}^n(\mathcal{V}, \mathcal{F}_G)$ to be an isomorphism; $\check{\mathrm{H}}^{k+1}(\delta, \mathcal{P}_G)$ then will be a direct limit of isomorphisms, and the theorem will follow. This in turn follows from two observations:

1. By Fact 2.2.3, $\mathcal{V}$ contains a collection of the form $\{ [\eta, \xi) \mid \xi \in C\}$, for some C cofinal in δ, in the sense that there exists some $\{V_\xi \mid \xi \in C\} \subseteq \mathcal{V}$ with $[\eta, \xi) \subseteq V_\xi$ for all $\xi \in C$.
2. Fix representative $\Phi \in L^k(\mathcal{U}_\delta, \mathcal{E}_G)$ for a class $[\Phi] \in \mathrm{H}^k(\mathcal{U}_\delta, \mathcal{F}_G)$. In this generalized context, the logic of "if Φ is non-n-trivial n-coherent, then so too is $\Phi[\eta, C]$" continues to apply; here it takes the form

$$0 \neq [\Phi] \in \mathrm{H}^k(\mathcal{U}_\delta, \mathcal{F}_G) \text{ implies that } 0 \neq [r_{\mathcal{V}\mathcal{U}_\delta}(\Phi)] \in \mathrm{H}^k(\mathcal{V}, \mathcal{F}_G)$$

This shows that $r^*_{\mathcal{V}\mathcal{U}_\delta} : \mathrm{H}^k(\mathcal{U}_\delta, \mathcal{F}_G) \to \mathrm{H}^k(\mathcal{V}, \mathcal{F}_G)$ is injective. And the reverse observation — that any non-n-trivial n-coherent $\Phi[\eta, C]$ extends to a classically non-n-trivial n-coherent Φ — shows that $r^*_{\mathcal{V}\mathcal{U}_\delta}$ is surjective.

Hence $r^*_{\mathcal{V}\mathcal{U}_\delta} : \mathrm{H}^k(\mathcal{U}_\delta, \mathcal{F}_G) \to \mathrm{H}^k(\mathcal{V}, \mathcal{F}_G)$ is a isomorphism. In consequence, $r^*_{\mathcal{V}\mathcal{U}_\delta} : \mathrm{H}^{k+1}(\mathcal{U}_\delta, \mathcal{P}_G) \to \mathrm{H}^{k+1}(\mathcal{V}, \mathcal{P}_G)$ is as well, as desired. □

Remark 2.6.4. Essentially, $\mathcal{U}_\delta$ functions above as a *good cover*; it marks, in other words, a stage at which the limit $\check{\mathrm{H}}^1(\delta, \mathcal{P})$ has already arrived. This is the ambiguous effect of the Pressing Down Lemma: at once it (1) rules out paracompactness (Fact 2.2.3), which is the usual environment of good covers, and (2) it enforces the effects of good covers.

Theorems 2.6.2 and 2.6.3 together describe a combinatorial translation of mixed effect: in the case of $n = 1$, they cast that most basic of topological invariants — cohomology over the constant sheaf — as a strict measure of that centerpiece of infinitary combinatorics, *nontrivial coherence*. It frames the higher cohomology groups, on the other hand, as phenomena largely without antecedent or meaning, in set-theoretic experience: *non-n-trivial n-coherence*. Why? One reason, as noted, is that non-2-trivial 2-coherence, for example, is simply imperceptible below ω_2. More generally, $\check{\mathrm{H}}^n(\delta, \mathcal{G}_d) = 0$ for any $\delta < \omega_n$, a phenomenon we describe in the following section.

2.7 Derived limits and a vanishing theorem

Our object in this section is threefold:

1. We recall the basics of inverse systems and their derived functors $\lim^n$.
2. We relate such higher derived limits to the Čech cohomology of the associated index-set.
3. We report a vanishing theorem for $\lim^n$ with implications for our main line of argument.

Good and relevant treatments of $\lim^n$ are [20] and [46] and, particularly, Chapter 3 of [29].

Definition 2.7.1. An *inverse system over* I is a family of abelian groups A_i, indexed by a partial order I, together with *bonding homomorphisms* $p^A_{ij} : A_j \to A_i$ satisfying $p^A_{ij} \circ p^A_{jk} = p^A_{ik}$ for all $i \leq j \leq k$.[4] We'll often record this data as a triple (A_i, p^A_{ij}, I); we'll write inverse systems and the maps between them in boldface. These maps are simply the natural transformations between the contravariant functors $I \to \mathrm{Ab}$ that inverse systems over I are; they are the morphisms, in other words, in the category $\mathrm{Ab}^{I^{op}}$. More concretely, a map $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a collection of maps $f_i : a_i \to b_i$ $(i \in I)$ commuting with the bonding morphisms p^A_{ij} and p^B_{ij}.

[4] As [20]observes: for our primary concern, which is the vanishing (or not) of $\lim^n$, there's no loss of generality in restricting our attention to abelian groups.

Wherever our framings are in terms of inverse systems, the associated index-sets will be ordinals (Chapter 3) or products of ordinals, ordered pointwise (the ω^ω or ω^κ of Chapter 4).

Definition 2.7.2. The *inverse limit* $\lim \mathbf{B}$ of a system $\mathbf{B} = (B_i, p^B_{ij}, I)$ is an abelian group B together with homomorphisms $g_B = \{g_i : B \to B_i \,|\, i \in I\}$ satisfying the following universal property: *For any abelian group A and collection of homomorphisms $\{f_i : A \to B_i \,|\, i \in I\}$ satisfying $f_i = p_{ij} f_j$ for all $i < j$ in I, there exists a unique homomorphism $f : A \to B$ with $f_i = g_i f$ for all $i \in I$.* Such a group B is unique up to isomorphism. Like inverse systems, inverse limits admit more concrete description:

$$\lim \mathbf{B} = \{(b_i) \in \prod_{i\in I} B_i \mid p^B_{ij}(b_j) = b_i \text{ for all } i < j \text{ in } I\} \tag{2.27}$$

As is common, we'll tend to leave the maps g_B unmentioned; in the above framing, they're the natural projections.

For $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, one may similarly define $\lim \mathbf{f}$. In this way, one defines an additive functor $\lim : \mathrm{Ab}^{I^{op}} \to \mathrm{Ab}$. Much of the trouble and/or interest of this functor is that it's not exact; more precisely, it's *left exact* — meaning that lim applied to a short exact sequence of systems

$$\mathbf{0} \to \mathbf{A} \xrightarrow{\mathbf{i}} \mathbf{B} \xrightarrow{\mathbf{j}} \mathbf{C} \to \mathbf{0} \tag{2.28}$$

may fail to preserve exactness on the right-hand side: $\lim \mathbf{j}$ may fail to be surjective in the image

$$\mathbf{0} \to \lim \mathbf{A} \xrightarrow{\lim \mathbf{i}} \lim \mathbf{B} \xrightarrow{\lim \mathbf{j}} \lim \mathbf{C} \to \mathbf{0} \tag{2.29}$$

It's this failure that the *higher derived limits* $\lim^n$ of lim are typically viewed as measuring. Relatedly, they emend the potentially non-exact sequence (2.29) to the following long exact sequence:

$$0 \to \lim \mathbf{A} \xrightarrow{\lim \mathbf{i}} \lim \mathbf{B} \xrightarrow{\lim \mathbf{j}} \lim \mathbf{C} \xrightarrow{\theta_0} \lim^1 \mathbf{A} \xrightarrow{\lim^1 \mathbf{i}} \lim^1 \mathbf{B} \to \dots \tag{2.30}$$

In light of this sequence, it's natural to index lim itself by zero. This permits an abstract characterization of the family of derived limits:

Theorem 2.7.3 (See [29])**.** *The functors* $\lim^n$*, together with the connecting morphisms* Θ_n*, form a universal connected sequence of functors.*

We refer the reader to ([29] §11) for definitions; what counts for us here is that Theorem 2.7.3 affords us an identification of $(\lim^n, \Theta_n)$ with any other universal connected sequence of functors $\mathrm{Ab}^{I^{op}} \to \mathrm{Ab}$ taking the same initial value. This, in turn, affords us concrete presentations of the functors $\lim^n$. which we now describe.

The key such identification is the following: for any abelian group G, let $\mathbf{\Delta}_I(G) = (G, \mathrm{id}, I)$. This is the *diagonal functor* embedding Ab into $\mathrm{Ab}^{I^{op}}$. Observe that for any inverse system $\mathbf{B} = (B_i, p^B_{ij}, I)$ and morphism $\mathbf{f} : \mathbf{\Delta}_I(\mathbb{Z}) \to \mathbf{B}$,

$$\{(f_i(1)) \in \prod_{i \in I} B_i \mid p^B_{ij}(f_j(1)) = f_i(1) \text{ for all } i < j \text{ in } I\} \tag{2.31}$$

is an element of lim $\mathbf{B}$; moreover, all elements of lim $\mathbf{B}$ are of this form. The induced identification of connected sequences of functors then takes the following form:

$$\mathrm{H}^n(\mathrm{Hom}(\mathsf{P}, \mathbf{B})) \cong \mathrm{Ext}^n(\mathbf{\Delta}_I(\mathbb{Z}), \mathbf{B}) \cong \lim^n \mathbf{B} \tag{2.32}$$

where P is any projective resolution of $\mathbf{\Delta}_I(\mathbb{Z})$. We'll pursue (and define) these themes further in Chapter 3. In particular, we'll work extensively there with a standard projective resolution $\mathsf{P}(\varepsilon)$ of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$, for any ordinal ε. For this resolution, $\mathrm{Hom}(\mathsf{P}(\varepsilon), \mathbf{B})$ is isomorphic to the following cochain complex:

$$K^j(\mathbf{B}) = \prod_{\vec{\alpha} \in [\varepsilon]^{j+1}} B_{\alpha_0} \tag{2.33}$$

with $d^j : K^j(\mathbf{B}) \to K^{j+1}(\mathbf{B})$ defined by

$$d^j(f)(\vec{\alpha}) = p^B_{\alpha_0\alpha_1}(f(\vec{\alpha}^0)) + \sum_{i=1}^{j}(-1)^i f(\vec{\alpha}^i)$$

We denote this cochain complex $\mathcal{K}(\mathbf{B})$. We've arrived to the concrete characterization of $\lim^n$ alluded to above:

Theorem 2.7.4 (Nöbeling, Roos; see [29])**.** $\lim^n \mathbf{B} \cong \mathrm{H}^n(\mathcal{K}(\mathbf{B}))$.

The concrete description (2.27) of $\lim = \lim^0$ is a special case of the theorem. Consider now a presheaf on the ordinal ε. For all $\alpha \in \varepsilon$ let $B_\alpha = \mathcal{P}(\alpha)$ and let $p^B_{\alpha\beta}$ be the $\mathcal{P}$-restriction map from $\mathcal{P}(\beta)$ to $\mathcal{P}(\alpha)$. This defines an inverse system $\mathbf{B}(\mathcal{P})$. Moreover, the cochain complexes $\mathcal{K}(\mathbf{B}(\mathcal{P}))$ and $\mathcal{L}(\mathcal{U}_\varepsilon, \mathcal{P})$ (see Definition 2.4.9) are plainly identical. In consequence, by Theorems 2.7.4 and 2.6.3,

Lemma 2.7.5. *For any presheaf $\mathcal{P}$ of functions on ε,*

$$\lim^n \mathbf{B}(\mathcal{P}) \cong \mathrm{H}^n(\mathcal{U}_\varepsilon, \mathcal{P}) \cong \check{\mathrm{H}}^n(\varepsilon, \mathcal{P})$$

This connects Goblot's 1967 vanishing theorem to our main orbit of concerns:

Theorem 2.7.6 (Goblot, [15]). *Let* $\mathbf{B}$ *be an inverse system with index-set* I *of cofinality* $\aleph_k$. *Then* $\lim^n \mathbf{B} = 0$ *for all* $n > k+1$.

By Lemma 2.7.5 and Theorem 2.7.6, if $\mathcal{P}$ is a presheaf of functions over an ordinal ε of cofinality $\aleph_k$ then $\check{\mathrm{H}}^n(\varepsilon, \mathcal{P}) = 0$ for all $n > k+1$. When $\mathcal{P} = \mathcal{G}_d$ or $\mathcal{D}_G$, though, we can do a little better. For a presheaf $\mathcal{P}$ of either sort, let ε be an uncountable limit ordinal and let $C \subseteq \varepsilon$ be a cofinal collection of successor ordinals. Let $\mathbf{A}(\mathcal{P}) = (\mathcal{P}(\alpha), \mathrm{id}, C)$ and let $\mathbf{U}(\mathcal{P}) = (\mathcal{P}([\alpha, \varepsilon)), p^{\mathbf{U}}_{\alpha\beta}, C)$, where $p^{\mathbf{U}}_{\alpha\beta} : \mathcal{P}([\beta, \varepsilon)) \to \mathcal{P}([\alpha, \varepsilon))$ is the inclusion map. Define $\mathbf{B}(\mathcal{P})$ as above. Then

$$\mathbf{0} \to \mathbf{U}(\mathcal{P}) \to \mathbf{A}(\mathcal{P}) \to \mathbf{B}(\mathcal{P}) \to \mathbf{0}$$

is an exact sequence. This and the fact that $\lim^n \mathbf{A}(\mathcal{P}) = 0$ for $n \geq 1$ the reader may verify (for the latter fact, use Theorem 2.7.4). Hence for any $n \in \omega$, the induced long exact sequence consists of exact sequences

$$\mathbf{0} \to \lim{}^{n+1}\mathbf{B}(\mathcal{P}) \to \lim{}^{n+2}\mathbf{U}(\mathcal{P}) \to \mathbf{0}$$

By Theorem 2.7.6, when $\mathrm{cf}(\varepsilon) = \mathrm{cf}(C) = \aleph_k$ and $n \geq k$, the term $\lim^{n+2}\mathbf{U}(\mathcal{P})$ vanishes; hence $\lim^{n+1}\mathbf{B}(\mathcal{P})$ must as well. As in Lemma 2.7.5 (by the reasoning behind Theorem 2.6.3), $\lim^{n+1}\mathbf{B}(\mathcal{P}) \cong \check{\mathrm{H}}^{n+1}(\varepsilon, \mathcal{P})$. Hence we've shown the following:

Corollary 2.7.7. *Let* G *be an abelian group and let* ε *be an ordinal of cofinality* $\aleph_k$ *and let either* $\mathcal{P} = \mathcal{D}_G$ *or* $\mathcal{G}_d$. *Then* $\check{\mathrm{H}}^n(\varepsilon, \mathcal{P}) = 0$ *for all* $n > k$.

In later chapters, this corollary takes the routine shape of recognitions like *any* 2*-coherent family indexed by* ε *of cofinality* $\aleph_1$ *is* 2*-trivial.* Technically, we didn't consider the countable cofinality case when arguing the corollary above but it, too, follows from the recognition that *any coherent family indexed by an ε of countable cofinality is trivial.* We close with a word on Goblot's Theorem: [15], [20], and [29] all give standard proofs of the theorem. We've not included those proofs here, for the following reason: by (2.32), Goblot's Theorem is equivalent to the statement that *In any projective resolution of the system $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$, the $(n+2)^{nd}$ term is projective.* We'll define these terms, and show something a little stronger, in Chapter 3.

2.8 Conclusion

To summarize:

Theorem 2.8.1. *For any ordinal δ and abelian group G,*

$$\check{\mathrm{H}}^n(\delta, \mathcal{G}_d) = \begin{cases} \text{the group of} & \\ \text{0-coherent functions } \delta \to G & \text{if } n = 0 \\ & \\ \text{the group of} & \\ n\text{-coherent } \{\varphi_{\vec{\alpha}} \to G \mid \vec{\alpha} \in [\delta]^n\} & \\ \text{modulo the group of} & \\ n\text{-trivial } \{\varphi_{\vec{\alpha}} \to G \mid \vec{\alpha} \in [\delta]^n\} & \text{if } n > 0 \end{cases} \tag{2.34}$$

Moreover, if δ is of cofinality $\aleph_k$, then $\check{\mathrm{H}}^n(\delta, \mathcal{G}_d) = 0$ for all $n > k$.

For δ of countable cofinality, the above reduces to

$$\check{\mathrm{H}}^n(\delta, \mathcal{G}_d) = \begin{cases} \text{the group of} & \\ \text{0-coherent functions } \delta \to G & \text{if } n = 0 \\ & \\ 0 & \text{if } n > 0 \end{cases} \tag{2.35}$$

And for $\delta \in \mathrm{Cof}(\omega_1)$,

$$\check{\mathrm{H}}^n(\delta, \mathcal{G}_d) = \begin{cases} \text{the group of} & \\ \text{0-coherent functions } \delta \to G & \text{if } n = 0 \\ & \\ \text{the group of} & \\ \text{coherent } \{\varphi_\alpha \to G \,|\, \alpha \in \delta\} & \\ \text{modulo the group of} & \\ \text{trivial } \{\varphi_\alpha \to G \,|\, \alpha \in \delta\} & \text{if } n = 1 \\ & \\ 0 & \text{if } n > 1 \end{cases} \tag{2.36}$$

In particular, for any group G, ω_1 is the least ordinal δ for which $\check{\mathrm{H}}^1(\delta, \mathcal{G}_d)$ is is nonzero. It is also, as we'll see in the Chapter 5, the greatest regular cardinal κ for which the vanishing (or not) of $\check{\mathrm{H}}^n(\kappa, \mathcal{G}_d)$ is, for each n, entirely decided in ZFC. Before closing, we should remark as well the centrality of the above homology groups; for example:

Lemma 2.8.2. *If $\Phi = \{\varphi_\alpha \to G \,|\, \alpha \in \omega_1\}$ is a nontrivial coherent family of functions satisfying*

$$(\varphi_\beta - \varphi_\alpha)\restriction_{\alpha \cap \mathrm{Lim}} = 0 \text{ for all } \alpha < \beta < \omega_1 \tag{2.37}$$

then $0 \neq [f^\Phi] \in \mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{K})$ *where* K *is any metrizable abelian group with* G *as a discrete topological subgroup.*

Recall that $r\rho_3$ in (2.10), for example, defined such a Φ. Lemma 2.8.2 has the following bizarre consequence.[5]

Corollary 2.8.3. $\check{\mathrm{H}}^1(\omega_1, \mathcal{K}) \neq 0$ *for any metrizable abelian group* K*: for* $K = \mathbb{Z}_2$*,* $\mathbb{Z}$*,* $\mathbb{Q}$*,* $\mathbb{R}$*,* $\mathbb{C}$*, or the circle group* $\mathbb{T}$*, for example.*

Proof. Apply Lemma 2.8.2 and Theorem 2.6.3 to the function $r\rho_3$. □

Proof of Lemma 2.8.2. Fix Φ as in the statement of the lemma. If $f^\Phi = d^0(g)$ for some $g \in L^0(\mathcal{U}_\delta, \mathcal{K})$, then the restrictions to β of

$$\varphi := \varinjlim_{\beta<\omega_1} (\varphi_\beta - g(\beta))$$

would equal φ_β modulo the continuous function $g(\beta) : \beta \to K$, for each $\beta < \omega_1$. Condition 2.37 implies that $(g(\beta) - g(\alpha))\restriction_{\alpha \cap \mathrm{Lim}} = 0$ for all $\alpha \leq \beta < \omega_1$, so

$$\varinjlim_{\beta<\omega_1} g(\beta)\restriction_{\omega_1 \cap \mathrm{Lim}}$$

defines a continuous $\underline{g} : \omega_1 \cap \mathrm{Lim} \to K$. By Corollary 2.2.2, the function $i \mapsto \underline{g}(i)$ on $\omega_1 \cap \mathrm{Lim}$ is eventually constant. Modifying the original Φ and g accordingly if necessary, we may assume $\underline{g}$ to be eventually zero. Take then an open $U \subseteq K$ about

[5]When $K = \mathbb{R}$, for example, the sheaf $\mathcal{K}$ is *soft* — a technical condition implying that $\check{\mathrm{H}}^n(X, \mathcal{K}) = 0$ for all $n \geq 1$, for any paracompact X (see [5]). Lemma 2.8.2 and Corollary 2.8.3 are odd for showing that such customarily acyclic presheaves determine nonzero cohomology groups on ω_1.

0 disjoint from $G \setminus \{0\}$, and an open symmetric V about 0 with $V + V \subseteq U$. For all $i \in \omega_1 \cap \mathrm{Lim}$ there exists some $k(i) < i$ with $g(i+1)\restriction_{[k(i),i]} \subseteq V$ and, hence, some stationary $S \subseteq \omega_1 \cap \mathrm{Lim}$ and ℓ such that $i \in S$ implies that $k(i) = \ell$. By equation 2.6, then, for all $i \leq j$ in S and $\xi \in [\ell, i]$,

$$(g(j+1) - g(i+1))(\xi) \in G \cap (V + V) = \{0\}$$

Hence φ_{i+1} $(i \in S)$ all agree above ℓ. This implies Φ trivial: a contradiction. $\square$

Remark 2.8.4. Our characterizations of $\check{\mathrm{H}}^n(\delta, \mathcal{G}_d)$ are fairly coarse: each gauges non-n-trivial n-coherence on δ, and each vanishes, or not. More structural characterizations seem to us, in general, hard. The question of whether $\check{\mathrm{H}}^n(\delta, \mathcal{G}_d)$ is free (posed by Viale), even in so basic a case as $n = 1$ and $\delta = \omega_1$ and $G = \mathbb{Z}$, for example, is already interesting. Similarly, Talayco's computation of the cardinality of $\mathrm{H}^n(\mathcal{U}_{\omega_1}, D_{\mathbb{Z}})$ is nontrivial ([37]). Which brings us to our next point:

Remark 2.8.5. As noted, a recognition of combinatorial phenomena on ω_1 as cohomological in form is not original: Talayco's *Applications of cohomology to set theory I* and *II* are the major precedent ([36], [37]). $\mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{D}_{\mathbb{Z}})$ and $\mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{D}_{\mathbb{Z}_2})$, in particular, figure centrally in the latter; Talayco identifies their elements with classes of tree-function pairings, and computes the cardinality of each group to be $2^{\aleph_1}$. These are technically and conceptually valuable works. We distinguish our own approach in three main respects:

1. In principle, the distinction between $\mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{P})$ and $\check{\mathrm{H}}^1(\omega_1, \mathcal{P})$ is something like

that between the *order* and *topological* structures of ω_1.[6] (Indeed, as we saw in Section 2.7, the δ of $\mathrm{H}^k(\mathcal{U}_\delta, \mathcal{P})$ is naturally viewed as an *index*, of an inverse system.) By Theorem 2.6.3, though, this is a distinction more *in principle* than *in fact*.[7] This recognition, however, fosters in turn a view of the ordinals as *interesting topological spaces in their own right*, i.e., of infinitary combinatorics and some very classical topological invariants as deeply intertwined.

2. In prior approaches, cohomology had played an essentially *descriptive* role, refining our perception of known objects. At the level of ω_1, this is the case here as well, but with some difference of emphasis, connecting point (1) above and (3) below: here gaps and Aronszajn trees are viewed as somewhat secondary phenomena, as tending to express the core *topological* phenomenon of nontrivial coherence.[8]

3. Cohomology on higher ω_n, on the other hand, plays a largely *prospective* role: higher $\check{\mathrm{H}}^n$ evoke no objects of prior or more immediate interest; they point, rather, to what the objects of interest on ω_n may be. It's this role and these objects that we pursue in Chapter 2.

[6] In the literature, $\mathrm{H}^n(\mathcal{U}_\kappa, \mathcal{P})$, also, is sometimes cast as a Čech group ([20]); in such framings, though, $\tau(\kappa)$ is only $\mathcal{U}_\kappa$.

[7] But one needs, of course, Theorem 2.6.3 in order to see this.

[8] Recall again the observation that "despite [their] simplicity, [walks] can be used to derive virtually all known other structures that have been defined so far on ω_1." ([44], p. 19).

CHAPTER 3

THE FIRST OMEGA ALEPHS: FROM SIMPLICES TO HIGHER WALKS

3.1 Introduction

Operative in Čech cohomology are functor or "presheaf" categories $\mathrm{Ab}^{\tau(X)^{op}}$; recall that $\tau(X)^{op}$ denotes the collection of open subsets of X, reverse-ordered by inclusion. More generally, let T denote any partial order; important for our purposes is the resemblance of Ab^T to R-module categories ${}_R\mathrm{Mod} \cong \mathrm{Ab}^R$, where R is a ring. Under this view, just as module theory is the representation theory of rings R, the study of the category Ab^T might be thought of as "the representation theory of orders." Central to the field is Barry Mitchell's 1972 result [32] that the homological dimension of ω_n is $n+1$; we'll discuss this and related works further in Chapter 3. Our broad argument in this chapter is that Mitchell's theorem and the walks technique can only really be understood in terms of one another. The essential implication of this argument is the existence on higher ω_n of rich (ZFC) generalizations of the ω_1 walks techniques.

3.2 Working definitions and conventions 2

3.2.1 Simplicial complexes and the systems $\mathbf{P}_n(\varepsilon)$

By *simplicial complex B on β* we mean a simplicial complex whose vertices are the elements of β. We may more generally identify any n-dimensional face of B with the size-$(n+1)$ set of its vertices. Best suited for our purposes, in other words, are *abstract simplicial complexes on β*: $\subseteq$-downward-closed collections of finite subsets of β. Writing B^n for the set of n-dimensional faces of B, we then have

1. $B^n \subseteq [\beta]^{n+1}$, and
2. $\bigcup_{k \leq n} B^k$ is the n-skeleton of B.

For any such simplicial complex B, let

$$C_n(B) = \bigoplus_{B^n} \mathbb{Z} \tag{3.1}$$

and for all $\vec{\alpha} \in B^n$, write $\langle \vec{\alpha} \rangle$ for the associated generator of $C_n(B)$. The maps

$$\langle \vec{\alpha} \rangle \mapsto \sum_{i \leq n} (-1)^i \langle \vec{\alpha}^i \rangle$$

then induce boundary maps $\partial_n : C_n(B) \to C_{n-1}(B)$, and homology groups

$$H_n^{\Delta}(B) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

for $n \geq 0$. (We let $C_{-1}(B)$ and, hence, ∂_0 equal zero.)

When β is of cofinality $\aleph_k$, its order-structure manifests as a k-dimensional topological condition on the family of simplicial complexes B on β. This is the content of Theorem 3.3.5, below. The mechanism of this unlikely rapport is a *grading* of simplicial complexes, for which inverse systems are a convenient framework. To that end, for $n \geq 0$ and A a collection of ordinals, define

$$P_n(A) = \bigoplus_{[A]^{n+1}} \mathbb{Z} \quad \text{and} \quad R_n(A) = \prod_{[A]^{n+1}} \mathbb{Z}$$

In the framework of (3.1) above, $P_n(A)$ is $C_n(B)$, where B is the complete n-dimensional simplicial complex on A. For both $P_n(A)$ and $R_n(A)$, again write $\langle\vec{\alpha}\rangle$ for the generator associated to $\vec{\alpha} \in [A]^{n+1}$. Again boundary maps on these $\langle\vec{\alpha}\rangle$ determine maps

$$d_n : P_n(A) \to P_{n-1}(A)$$

for $n \geq 1$. For any ordinal ε and $n \geq 0$ define then the inverse system

$$\mathbf{P}_n(\varepsilon) = (P_n([\alpha, \varepsilon)), p_{\alpha\beta}, \varepsilon)$$

with $p_{\alpha\beta} : P_n([\beta, \varepsilon)) \to P_n([\alpha, \varepsilon))$ the natural inclusion map, for $\alpha \leq \beta < \varepsilon$. (Similarly for $\mathbf{R}_n$.) Observe that $p_{\alpha\beta}$ and d_n commute. These d_n therefore determine, in turn, a boundary map

$$\mathbf{d}_n : \mathbf{P}_n(\varepsilon) \to \mathbf{P}_{n-1}(\varepsilon)$$

Lastly, write $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ for the inverse system $(\mathbb{Z}, \mathrm{id}, \varepsilon)$; it is the *diagonal functor* $\mathbf{\Delta}_\varepsilon(\,\cdot\,)$ embedding Ab into $\mathrm{Ab}^{\varepsilon^{op}}$, evaluated at $\mathbb{Z}$. These objects then assemble in the following exact sequence:

$$\cdots \xrightarrow{\mathbf{d}_{n+1}} \mathbf{P}_n(\varepsilon) \xrightarrow{\mathbf{d}_n} \mathbf{P}_{n-1}(\varepsilon) \xrightarrow{\mathbf{d}_{n-1}} \cdots \xrightarrow{\mathbf{d}_1} \mathbf{P}_0(\varepsilon) \xrightarrow{\mathbf{e}} \mathbf{\Delta}_\varepsilon(\mathbb{Z}) \longrightarrow \mathbf{0} \qquad (\mathsf{P}(\varepsilon))$$

with $\mathbf{e} = \{e_\alpha \,|\, \alpha \in \varepsilon\}$ defined by $e_\alpha : \langle \beta \rangle \mapsto 1$, for all $\alpha \leq \beta < \varepsilon$.

Simple as it might appear, the sequence $\mathsf{P}(\varepsilon)$ will be a main object of study in this chapter. A main part of our argument, in other words, will frequently be the manipulation of algebraic relations between n-tuples of ordinals. For this work, a clear but flexible notation is critical; we therefore pause to collect and augment its more scattered description above:

1. For A a collection of ordinals, we write $\vec{\beta} \in [A]^n$ to mean that $\vec{\beta}$ is an increasing n-tuple $(\beta_0, \ldots, \beta_{n-1})$ of ordinals in A. We'll typically write a 1-tuple (β) as β. For $\vec{\beta} \in [A]^n$ and $0 \leq i < n$, we write $\vec{\beta}^i$ for $\vec{\beta}$ with the i^{th} coordinate removed. If $\vec{\beta}$ is a 1-tuple, then $\vec{\beta}^0 = \varnothing$. As we did when defining simplicial complexes above, we'll sometimes simply view $\vec{\beta}$ as an n-element subset of A.

2. As for $\mathbf{e}$ and $\mathbf{d}_n$, above, we'll define maps among inverse systems largely by way of their action on generators $\langle \vec{\alpha} \rangle$; at times, we'll conflate maps between terms (like d_n) and maps between inverse systems (like $\mathbf{d}_n$), as well. Relatedly, we'll tend not to formally distinguish between a generator $\langle \vec{\gamma} \rangle \in P_n([\beta, \varepsilon))$ and its images $p_{\alpha\beta}(\langle \vec{\gamma} \rangle)$. When we do, it will be to regard $\langle \vec{\gamma} \rangle$ as an element of the "highest possible" term of $\mathbf{P}_n(\varepsilon)$ — namely, $P_n([\gamma_0, \varepsilon))$.

3. We'll at times write sums of generators inside the angled brackets, preferring expressions like $\langle d_k \vec{\alpha}, \vec{\beta} \rangle$ to $\sum_{i=0}^{k} (-1)^i \langle \vec{\alpha}^i, \vec{\beta} \rangle$. As they do here, commas can render such expressions more readable. In subscripts, however, such commas typically have more of an effect of clutter. In these cases we omit them, denoting concatenations of coordinates, as in $(\beta_0, \ldots, \beta_{m-1}, \gamma_0, \ldots, \gamma_{n-1})$, as concatena-

tions of tuples, as in $\vec{\beta}\vec{\gamma}$. Putting all this together: the tuple $(\beta_0, \beta_2, \delta)$ would typically appear in a subscript as $C_{\vec{\beta}^1\delta}$, for example; it would appear in a generator probably as $\langle \vec{\beta}^1, \delta \rangle$. Lastly, an expression like $d_k\mathcal{B}$ means $\{d_k\langle \vec{\alpha} \rangle \mid \langle \vec{\alpha} \rangle \in \mathcal{B}\}$.

3.2.2 Free and projective inverse systems

Definition 3.2.1. For any object $\mathbf{P}$ in $\mathrm{Ab}^{\gamma^{op}}$, let $\mathbf{id}$ denote the identity morphism. $\mathbf{P}$ is *projective* if for any epimorphism $\mathbf{e} : \mathbf{R} \to \mathbf{P}$ there exists a morphism $\mathbf{s} : \mathbf{P} \to \mathbf{R}$ such that $\mathbf{e}\,\mathbf{s} = \mathbf{id}$. We'll sometimes term such a right-inverse to an epimorphism a *section*. Dually, we'll often term a left-inverse $\mathbf{r}$ to a monomorphism $\mathbf{m}$ a *retract*.

An object $\mathbf{X}$ in $\mathrm{Ab}^{\gamma^{op}}$ is *free* if there exists some $\mathcal{B} \subseteq \cup_{\alpha<\gamma} X_\alpha$ such that any x in any X_α has a unique $\mathcal{B}$-decomposition

$$x = \sum_{i<k} a_i q_{\alpha\beta_i}(b_i)$$

with $b_i \in X_{\beta_i}$ for all $i < k$.

Example 3.2.2. The system $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ is free if and only if ε is a successor, i.e., if $\mathrm{cf}(\varepsilon) = 1$. The system $\mathbf{P}_n(\varepsilon)$ is free, on the other hand, for any ordinal ε and $n \in \omega$. By an argument exactly as in more standard settings, it follows that every $\mathbf{P}_n(\varepsilon)$ is projective as well.

The reverse question of whether a projective system is free (or, conversely, of whether a nonfree system is nonprojective) is in general much subtler. Even the simplest instance is less than obvious: let ε be a limit ordinal.

$$\text{Is } \mathbf{\Delta}_\varepsilon(\mathbb{Z}) \text{ projective?} \tag{3.2}$$

The question involves a different order of quantification from that of freeness: it quantifies over the collection of morphisms in $\mathrm{Ab}^{\varepsilon^{op}}$. Arguably the obscurity — or, in another view, the power — of the notion of projective consists, simply, in this quantification. To see that a question like (3.2) is as much about the ambient category as it is about the object itself, consider the following:

Definition 3.2.3. Let κ-*Ab* denote the category of abelian groups with generating sets of cardinality less than κ.

Theorem 3.2.4. *For κ an infinite cardinal, $\mathbf{\Delta}_\kappa(\mathbb{Z})$ is projective in $(\kappa\text{-}Ab)^{\kappa^{op}}$ if and only if κ has the tree property.*

ω-Ab, in particular, is the category of finitely generated abelian groups. Hence by König's Infinity Lemma, $\mathbf{\Delta}_\omega(\mathbb{Z})$ is projective in the category of *height-ω inverse systems of finitely generated abelian groups.* $\mathbf{\Delta}_\omega(\mathbb{Z})$ is *not* projective, however, in the wider category of *height-ω inverse systems of abelian groups.* We'll argue this latter fact as the base case in the inductive proof of Theorem 3.5.1.

Proof of Theorem 3.2.4. Suppose that κ has the tree property, and consider an epimorphism $\mathbf{e} = \{e_\xi : Q_\xi \to \mathbb{Z} \,|\, \xi < \kappa\}$ from some $\mathbf{Q} = (Q_\xi, q_{\eta\xi}, \kappa)$ in $(\kappa\text{-Ab})^{\kappa^{\mathrm{op}}}$ to $\mathbf{\Delta}_\kappa(\mathbb{Z})$. We'll show that $\mathbf{e}$ has a right-inverse $\mathbf{s}$. As $\mathbf{e}$ is arbitrary, this will show that

$\mathbf{\Delta}_\kappa(\mathbb{Z})$ is projective in $(\kappa\text{-Ab})^{\kappa^{\mathrm{op}}}$. Observe that

$$\Big(\bigcup_{\xi<\kappa} e_\xi^{-1}(1), \trianglelefteq\Big)$$

defines a κ-tree T, where $x \trianglelefteq y$ iff $q_{\eta\xi}(y) = x$ for some $\eta \leq \xi < \kappa$. By the tree property, T contains a cofinal branch $\{x_\xi \,|\, \xi < \kappa\}$. Setting $s_\xi(1) = x_\xi$ for $\xi < \kappa$ then defines an $\mathbf{s} : \mathbf{\Delta}_\kappa(\mathbb{Z}) \to \mathbf{Q}$ right-inverse to $\mathbf{e}$.

Given, on the other hand, a κ-Aronszajn tree T, let

$$Q_\xi = \bigoplus_{\mathrm{lev}_\xi(T)} \mathbb{Z}$$

be the free group generated by the ξ^{th} level of T and define $q_{\eta\xi} : Q_\xi \to Q_\eta$ by $q_{\eta\xi}(\langle x\rangle) =$ *the Q_η-generator corresponding to the η^{th}-level predecessor of x.* Mappings $\langle x\rangle \mapsto 1$ for $x \in \mathrm{lev}_\xi(T)$ then induce $e_\xi : Q_\xi \to \mathbb{Z}$, and, hence, an $\mathbf{e} : \mathbf{Q} \to \mathbf{\Delta}_\kappa(\mathbb{Z})$ with no right-inverse $\mathbf{s} = \{s_\xi : \mathbb{Z} \to Q_\xi \,|\, \xi < \kappa\}$, since for any such inverse, $\{s_\xi(1) \,|\, \xi < \kappa\}$ would define a cofinal branch in T. □

The above remarks and theorem were something of a digression, meant to help frame the recognition in this chapter that a number of projective inverse systems are free.[1] To apply this recognition, we'll want the following standard lemma:

Lemma 3.2.5. *An inverse system* $\mathbf{X}$ *is projective if and only if* $\mathbf{X}$ *is a direct summand of a free system* $\mathbf{Y}$*.*

[1] The systems we consider are indeed "big," so we're recording a fact somewhat described by Hyman Bass's 1963 title *Big projective modules are free* ([2], discovered by this author fairly recently). Note that by Theorem 3.2.4, though, that title's far from describing the situation for inverse systems in any unqualified generality: assume the tree property of some infinite cardinal κ (readers wary of large cardinals may take κ to be ω). Then $\Delta_\kappa(\mathbb{Z})$ is a projective system in $(\kappa\text{-Ab})^{\kappa^{\mathrm{op}}}$ which, by Theorem 3.5.1 below, is not projective in $\mathrm{Ab}^{\kappa^{\mathrm{op}}}$, and therefore cannot be free.

Proof. For the *only if* direction, fix an epimorphism $\mathbf{e}$ from a free system $\mathbf{Y}$ to $\mathbf{X}$. As $\mathbf{X}$ is projective, $\mathbf{e}$ admits a right-inverse $\mathbf{s}$, so that $\mathbf{Y} \cong \mathbf{s}(\mathbf{X}) \oplus \ker(\mathbf{e}) \cong \mathbf{X} \oplus \ker(\mathbf{e})$. For the *if* direction, observe that if $\mathbf{Y} = \mathbf{X} \oplus \mathbf{Z}$ then any epimorphism $\mathbf{e} : \mathbf{R} \to \mathbf{X}$ naturally extends to an epimorphism $\mathbf{e}' : \mathbf{R} \oplus \mathbf{Z} \to \mathbf{X} \oplus \mathbf{Z}$. As $\mathbf{Y}$ is free, $\mathbf{e}'$ has a right-inverse $\mathbf{s}'$, which restricts to an $\mathbf{s} : \mathbf{X} \to \mathbf{R}$ right-inverse to $\mathbf{e}$. □

As the lemma suggests, it's not in general true that subsystems of free inverse systems of abelian groups are free, or even projective. A central concern of this chapter, in fact, is the question of whether the subsystem $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ of the free system $\mathbf{P}_{n-1}(\varepsilon)$ is projective. This question, we'll see, is fundamentally a question of the cofinality of ε. Observe in this connection that we can truncate the exact sequence $\mathsf{P}(\varepsilon)$ at any $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ to form a shorter exact sequence as follows:

$$\cdots \to \mathbf{0} \longrightarrow \mathbf{d}_n\mathbf{P}_n(\varepsilon) \xrightarrow{\mathbf{i}} \mathbf{P}_{n-1}(\varepsilon) \xrightarrow{\mathbf{d}_{n-1}} \cdots \xrightarrow{\mathbf{d}_1} \mathbf{P}_0(\varepsilon) \xrightarrow{\mathbf{e}} \mathbf{\Delta}_\varepsilon(\mathbb{Z}) \longrightarrow \mathbf{0} \qquad (3.3)$$

If $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ is projective then (3.3) shares with $\mathsf{P}(\varepsilon)$ the feature that all terms except the "target" $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ are projective.

Definition 3.2.6. A *projective resolution* of an inverse system $\mathbf{X}$ is an exact sequence ending in $\mathbf{X}$ as in $\mathsf{P}(\varepsilon)$ or (3.3), above, in which all nonzero terms except possibly the rightmost are projective. Such resolutions are sometimes written $\mathsf{P} \to \mathbf{X} \to \mathbf{0}$. The *length* of P is the supremum of the indices of its nonzero terms — where P's terms are indexed, as above, from right to left, beginning with zero. Possibly all of P's terms are nonzero; its length in this case is ∞. The *homological dimension* of $\mathbf{X}$, written $\mathrm{hd}(\mathbf{X})$, is the minimal length of a projective resolution of $\mathbf{X}$.

Example 3.2.7. $\mathbf{X}$ is projective if and only if $\cdots \to \mathbf{0} \to \mathbf{X} \xrightarrow{\mathbf{id}} \mathbf{X} \to \mathbf{0}$ is a projective resolution, if and only if $\mathrm{hd}(\mathbf{X}) = 0$. More generally, homological dimension measures "how far" a system is from being projective.

We conclude this section with several summary remarks. Our interest is in projective resolutions of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$, for two related reasons:

1. They translate order-theoretic information into algebraic information.
2. They're of computational value.

In point (2), we have in mind the following: the functor $\mathbf{\Delta}_\varepsilon(\,\cdot\,) : \mathrm{Ab} \to \mathrm{Ab}^{\varepsilon^{\mathrm{op}}}$ is left-adjoint to the functor $\lim(\,\cdot\,) : \mathrm{Ab}^{\varepsilon^{\mathrm{op}}} \to \mathrm{Ab}$. As discussed in Section 2.7, this has as consequence the following formula:

$$\mathrm{H}^n(\mathrm{Hom}(\mathsf{P}, \mathbf{X})) \cong \mathrm{Ext}^n(\mathbf{\Delta}_\varepsilon(\mathbb{Z}), \mathbf{X}) \cong \lim{}^n \mathbf{X} \tag{3.4}$$

Here $\mathbf{X}$ is any system in $\mathrm{Ab}^{\varepsilon^{\mathrm{op}}}$ and P is any projective resolution of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$, such as $\mathsf{P}(\varepsilon)$ or (3.3) above. Via equation 3.4, the "standard" projective resolution $\mathsf{P}(\varepsilon)$ uniformizes the computation of higher derived limits, and this indeed was the source of the explicit formula for $\lim^n \mathbf{X}$ described in Theorem 2.7.4. On the other hand, the eventual zeros of a resolution like (3.3) translate on the left-hand side of equation 3.4 to vanishing cohomology groups above some n. These manifest on the right-hand side of the equation as the following implication:

the functor $\lim^n(\,\cdot\,) : \mathrm{Ab}^{\varepsilon^{\mathrm{op}}} \to \mathrm{Ab}$ *is trivial for any* $n > \mathrm{hd}(\mathbf{\Delta}_\varepsilon(\mathbb{Z}))$

Sensitivities of $\mathrm{hd}(\mathbf{\Delta}_\varepsilon(\mathbb{Z}))$ to the cofinality of ε transmit in this manner to functors of broad application and importance — namely, to the higher derived limits $\lim^n(\,\cdot\,)$.

One may then wonder in what the sensitivities of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ or $\mathsf{P}(\varepsilon)$ to order-theoretic considerations consist. This returns us to point (1) above, in the form of a question:

What is it in the ordinals — the ordinals ω_n, in particular — that these algebraic structures are capturing?

This is the guiding question of the remaining sections of this chapter.

3.3 Good simplicial complexes

It's natural to consider, for a given $\gamma \in \omega_1$, the family of all walks $\mathrm{Tr}(\alpha, \gamma)$ from γ down to some $\alpha < \gamma$. Such a family is most concisely conceived, perhaps, as the graph

$$\bigcup_{\alpha<\gamma} \mathrm{Tr}^2(\alpha, \gamma) \tag{3.5}$$

on $\gamma + 1$, where $\mathrm{Tr}^2(\alpha, \gamma)$ records the steps of $\mathrm{Tr}(\alpha, \gamma)$ as edges $\{\{\gamma_i, \gamma_{i+1}\} \mid i < \rho_2(\alpha, \gamma) - 1\}$. It's an effect of the extension properties and directedness of walks that any such graph is well-behaved or *good* in the following senses:

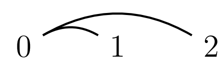
Definition 3.3.1. A graph G on an ordinal γ is *good* if

(i) G is acyclic, and

(ii) $G\restriction_{[\alpha,\gamma)}$ is connected, for all $\alpha < \gamma$.

Example 3.3.2. Consider the following graphs on the ordinal 4:

G_0



G_1

G_0 is a good graph. On the other hand, $G_1\restriction_{[1,3)}$ is disconnected, so G_1 is not good.

In fact, G_1 is the *forbidden configuration*: *a connected graph is good if and only if it contains no copy of* G_1 *(i.e., contains no* $\{\{\alpha,\beta\},\{\alpha,\gamma\}\}$*, for* $\alpha < \beta < \gamma$*).* A consequence is the following theorem, one measure of the difficulty of extending the technique of minimal walks beyond the countable ordinals.

Theorem 3.3.3. ω_1 *is the least ordinal admitting no good graph.*

Proof. Suppose for contradiction that ω_1 admitted a good graph G. As G is connected, there exists for each $\gamma \in \lim(\omega_1)$ some least $\gamma_1 \geq \gamma$ such that $(\xi, \gamma_1) \in G$ for some $\xi < \gamma$. Let γ_0 denote the least such ξ. The function $\gamma \mapsto \gamma_0$ is then a regressive function and, hence, constantly α on some stationary $S \subseteq \lim(\omega_1)$. But for any $\beta < \gamma$ in S above α, then, $\{\{\alpha,\beta\},\{\alpha,\gamma\}\}$ is a copy of G_1 in G — a contradiction.

On the other hand, (3.5) defines a good graph on any $\gamma < \omega_1$. In fact, the more abbreviated $\{\{\alpha, C^\gamma(\alpha)\} \mid \alpha < \gamma\}$ defines a good graph on *any* γ of countable cofinality (with G_0, above, a simple instance). $\square$

These phenomena generalize.

Definition 3.3.4. An n-dimensional simplicial complex G on an ordinal γ is *good* if $G^{n-1} = [\gamma]^n$ and for all $\alpha < \gamma$,

$$\mathrm{H}^{\Delta}_n(G\restriction_{[\alpha,\gamma)}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Good n-dimensional G on γ, in other words, have a complete $(n-1)$-skeleton, and are acyclic and connected on any tail of γ, just as in Definition 3.3.1.[2]

Theorem 3.3.5. *ω_n is the least ordinal which does not admit a good n-dimensional simplicial complex, for each $n \geq 1$.*

In particular, there is some least number of dimensions — namely, $n+1$ — in which ω_n *can* support a good simplicial complex.

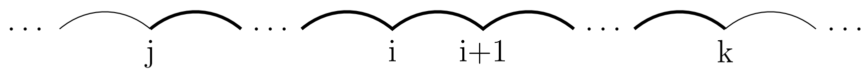
We'll argue Theorem 3.3.5 by way of an algebraic translation, which we motivate as follows:

Example 3.3.6. Recall that $P_1(\omega) = \bigoplus_{[\omega]^2} \mathbb{Z}$, and let $I = \{\langle i, i+1\rangle \,|\, i \in \omega\}$. For every $j < k$ in ω, the difference $\langle k\rangle - \langle j\rangle$ has a unique $d_1 I$-decomposition

$$\sum_{j \leq i < k} (\langle i+1\rangle - \langle i\rangle).$$

In other words, $d_1 I$ is a basis for $d_1 P_1(\omega)$. Pictorially, edges $\{i, i+1\}$ connect the points j and k as below:

[2]The requirement of a complete $(n-1)$-skeleton in Definition 3.3.1 simplifies the argument of Theorem 3.3.7. It's not clear whether it's needed there.



These edges evidently define a unique path between any two points in ω. Put differently, $d_1 I$ defines a good graph G_I on ω. More precisely, the spanning and linear independence properties of $d_1 I$ manifest as the connectedness of, and lack of cycles in, G_I, respectively. And "goodness" captures the fact that these properties persist on any restriction of $d_1 I$ and $d_1 P_1(\omega)$ to a tail $[n, \omega)$ of ω — the fact, in other words, that $d_1 I$ defines a basis for the *inverse system* $\mathbf{d}_1\mathbf{P}_1(\omega)$.

These seemingly rudimentary considerations are surprisingly sensitive to the cofinality of the index-set of $\mathbf{d}_n\mathbf{P}_n(\gamma)$. For example: by Theorems 3.3.3 and 3.3.7 below, the least ordinal γ for which $\mathbf{d}_1\mathbf{P}_1(\gamma)$ is not free is ω_1; in fact, $\mathbf{d}_1\mathbf{P}_1(\omega_1)$ is not even projective.

Theorem 3.3.7. *For $n \geq 1$, the system $\mathbf{d}_n\mathbf{P}_n(\gamma)$ is free if and only if γ admits a good n-dimensional simplicial complex.*

Proof. For the forward direction of the proof, suppose that $\mathbf{d}_n\mathbf{P}_n(\gamma)$ is free. We'll want the following fact:

Fact 3.3.8. If $\mathbf{d}_n\mathbf{P}_n(\gamma)$ is projective and $\mathrm{cf}(\gamma) = \omega_\xi$, then $\xi < n$.

This fact is immediate from Theorem 3.5.1 below. In the following section, we construct for any $\mathbf{d}_n\mathbf{P}_n(\gamma)$ as in Fact 3.3.8 a basis of the form $d_n\mathcal{B} = \{d_n\langle\vec{\alpha}\rangle \mid \vec{\alpha} \in B\}$, with $B \subseteq [\gamma]^{n+1}$. Let $\mathbf{d}_n\mathbf{P}_n(\gamma)$ be free and fix such a basis, and write $\underline{B}$ for the $\subseteq$-downward closure of B. In other words, $\underline{B}$ is the natural interpretation of B as a simplicial complex. We show that $\underline{B}$ is good:

As $d_n\mathcal{B}$ is linearly independent,

$$\ker(\partial_n : C_n(\underline{B}) \to C_{n-1}(\underline{B})) = 0$$

hence $H_n^{\Delta}(\underline{B}) = 0$.

Claim 3.3.9. *$\underline{B}^{n-1} = [\gamma]^n$.*

Proof. Towards contradiction, suppose instead that $\vec{\beta}^i \in [\gamma]^n \backslash \underline{B}^{n-1}$ for some $\vec{\beta} \in [\gamma]^{n+1}$. Then no linear combination of elements of $d_n\mathcal{B}$ can supply the summand $\langle \vec{\beta}^i \rangle$ of $d_n\langle \vec{\beta} \rangle$, hence $d_n\mathcal{B}$ does not span $\mathbf{d}_n\mathbf{P}_n(\gamma)$. $\square$

By the claim, $\underline{B}^k = [\gamma]^{k+1}$ for all $k < n$. Therefore

(i) $\underline{B}$ is connected: $H_0^{\Delta}(\underline{B}) = \mathbb{Z}$, and

(ii) $H_k^{\Delta}(\underline{B})$ for $0 < k < n$ is nothing other than the homology of the chain complex

$$P_n(\gamma) \xrightarrow{d_n} P_{n-1}(\gamma) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1(\gamma) \xrightarrow{d_1} P_0(\gamma)$$

As noted in Section 3.2.1, this sequence is exact, so $H_k^{\Delta}(\underline{B}) = 0$. By definition these arguments hold on any tail of γ; goodness follows.

For the reverse direction of the proof, simply observe that the above argument is reversible. In other words, any good n-dimensional simplicial complex is determined by its collection, B, of n-faces, which in turn define a basis $d_n\mathcal{B} = \{d_n\langle\vec{\alpha}\rangle \mid \vec{\alpha} \in B\}$ for $\mathbf{d}_n\mathbf{P}_n(\gamma)$, just as above. $\square$

Theorem 3.3.5 then takes the following form:

Theorem 3.3.10. *For $n \geq 1$, ω_n is the least ordinal ε such that $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ is not free.*

The theorem, like 3.3.3 and 3.3.5 above, conjoins both positive and negative statements, namely that

1. $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ *does* admit a basis, for $\varepsilon < \omega_n$, while
2. $\mathbf{d}_n\mathbf{P}_n(\omega_n)$ does not.

We argue (1) and (2), respectively, in Sections 3.4 and 3.5 below. To show (1), we define an explicit basis for each eligible $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$; this we do by an expanded, or compound, use of ladders. We argue (2) from Mitchell's computation of the homological dimension of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ for all ordinals ε. Outwardly, in fact, Theorem 3.3.10 is only a minor strengthening of Mitchell's computation, upgrading the term "projective" to "free." Our real object in all of this, though, is the function $\mathtt{f}_n \in \prod_{\vec{\alpha}\in[\omega_n]^{n+1}} R_{n+1}([\alpha_0, \omega_n))$ defined *in the course of* these arguments. This witness to hitherto abstract facts like $\mathrm{hd}(\mathbf{\Delta}_{\omega_n}(\mathbb{Z})) = n+1$ generalizes the walks apparatus. In particular, $\mathtt{f}_1$ fluently arrays all the essential data of the technique of walks on the countable ordinals (see Section 3.7). And the higher $\mathtt{f}_n$ array higher-order extensions of that technique (Sections 3.8 and 3.9).

3.4 From ladders to bases

In Rousseau's Ninth Walk, he reminisces about earlier walks,
which seem to slide out from each other like sections of a telescope ...

- Rebecca Solnit, Wanderlust

To streamline statements, define the cofinality of successor ordinals to be $\aleph_{-1}$. For all ε of cofinality $\aleph_k$ and positive $n > k$, we'll define a $\mathcal{B} \subseteq \mathbf{P}_n(\varepsilon)$ such that $d_n\mathcal{B}$ is a basis for $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$. Ladders on $\beta \in \varepsilon \cap \{\gamma \,|\, \mathrm{cf}(\gamma) < \mathrm{cf}(\varepsilon)\}$ will structure the construction of $\mathcal{B}$. For this purpose, fix for each relevant β a closed unbounded $C_\beta \subseteq \beta$ such that $\mathrm{otp}(C_\beta) = \mathrm{cf}(\beta)$, and such that for $i < \mathrm{cf}(\beta)$ the cofinality of the i^{th} element of C_β is $\mathrm{cf}(i)$.

Definition 3.4.1. For $\beta < \gamma$, define $C_{\beta\gamma}$ to be $\pi^{-1}(C_\alpha)$, where $\alpha = \mathrm{otp}\,(C_\gamma \cap \beta)$ and π is the order-isomorphism $C_\gamma \cap \beta \to \alpha$.

We may continue in this fashion, defining $C_{\vec{\gamma}}$ by induction on the length of $\vec{\gamma}$ much as above: Suppose $C_{\vec{\gamma}}$ is defined and $\beta < \gamma_0$. Let $\alpha = \mathrm{otp}(C_{\vec{\gamma}} \cap \beta)$ and let $\pi : C_{\vec{\gamma}} \cap \beta \to \alpha$ be the order-isomorphism. Define $C_{\beta\vec{\gamma}}$ to be $\pi^{-1}(C_\alpha)$.

Extend the notation $C^\gamma(\alpha) = \min(C_\gamma \backslash (\alpha+1))$ to $C^{\vec{\gamma}}(\alpha) := \min(C_{\vec{\gamma}} \backslash (\alpha+1))$.

Finally, define $\vec{\alpha} = (\alpha_0, \ldots, \alpha_n)$ to be *internal to* C_β iff

1. $\alpha_n \in C_\beta$,
2. $\alpha_i \in C_{\alpha_{i+1}\ldots\alpha_n\beta}$ for all $i < n$, and

3. $\mathrm{cf}(\alpha_0) \leq \mathrm{cf}(\alpha_i) < \mathrm{cf}(\alpha_j) < \mathrm{cf}(\beta)$ for $0 < i < j \leq n$.

Observation 3.4.2. The following observations are straightforward:

- If α is a limit of $C_{\vec{\beta}}$ then $C_{\alpha\vec{\beta}}$ is a ladder on α.
- If ε is a successor ordinal, then $C_{\alpha\varepsilon} = \varnothing$ for all $\alpha < \varepsilon$.
- If $\varepsilon = \omega_k = C_\varepsilon$ for some $k \geq 0$ then $C_{\vec{\beta}\varepsilon} = C_{\vec{\beta}}$ for any $\vec{\beta} \in [\varepsilon]^{<\omega}$.
- If β_0 is a limit ordinal above α, then $C^{\vec{\beta}}(\alpha)$ is a successor ordinal.
- If $\alpha > \sup(C_{\vec{\beta}})$ then $C^{\vec{\beta}}(\alpha)$ is undefined and $C_{\alpha\vec{\beta}} = C_{\vec{\beta}}$.
- If $\vec{\gamma}$ is a tail of $\vec{\beta}$ then $C_{\vec{\beta}} \subseteq C_{\vec{\gamma}}$.
- If $\vec{\gamma}$ is a tail of $\vec{\beta}$ and $C^{\vec{\gamma}}(\alpha)$ is defined then $C^{\vec{\beta}}(\alpha)$ also is, and $C^{\vec{\gamma}}(\alpha) \geq C^{\vec{\beta}}(\alpha)$.
- For ε of cofinality ω_k, the inductions of Definition 3.4.1 are only meaningful for $k+2$ many steps. $\vec{\beta}$ *is internal to* C_ε, in particular, implies that $|\vec{\beta}| \leq k+2$.

The next definition is crucial.

Definition 3.4.3. For ε of cofinality $\aleph_k$ and positive $n > k$, let $\mathcal{B}_n(\varepsilon)$ denote the collection of $\langle \vec{\alpha}, \vec{\beta} \rangle$ satisfying the following:

1. $\vec{\alpha} \in [\varepsilon]^{i+1}$ for some $i < n$.
2. $\vec{\beta} \in [\varepsilon]^{n-i}$ is internal to C_ε.
3. $C^{\vec{\beta}^0\varepsilon}(\alpha_i) = \beta_0$.

Where we wish to emphasize the choice of the parameter C_ε in the above definition, we write $\mathcal{B}_n(\varepsilon)[C_\varepsilon]$.

Example 3.4.4. For ε a successor, e.g., $\varepsilon = \delta + 1$,

$$\mathcal{B}_n(\varepsilon) = \{\langle \vec{\alpha}, \delta \rangle \mid \vec{\alpha} \in [\delta]^n\}$$

Here there's only one possibility for C_ε, and the only $\vec{\beta}$ which is internal to C_ε is the 1-tuple δ. Hence the $C^{\vec{\beta}^0 \varepsilon}(\alpha_i)$ of Definition 3.4.3 is constantly equal to $C^\varepsilon(\alpha_i) = \delta$ for all $\vec{\alpha} < \vec{\beta}$.

Lemma 3.4.5. *If ε is a successor ordinal then $d_n\mathcal{B}_n(\varepsilon)$ is a basis for $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$.*

Proof. Here and below, it will suffice to check that $d_n\mathcal{B}_n(\varepsilon)$ uniquely decomposes each generator $d_n\langle\vec{\beta}\rangle$ of $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$. Again let $\varepsilon = \delta + 1$. Uniqueness follows from the fact that whenever z_j ($j \in J$) are nonzero coefficients,

$$\sum_{j \in J} z_j d_n \langle \vec{\alpha}_j, \delta \rangle = 0 \quad \text{implies} \quad \sum_{j \in J} z_j \langle \vec{\alpha}_j \rangle = 0\,. \tag{3.6}$$

Hence the only $d_n\mathcal{B}_n(\varepsilon)$ decomposition of 0 is the trivial one. Therefore, since $d_n d_{n+1}\langle\vec{\beta}, \delta\rangle = 0$ for any $\vec{\beta} \in [\delta]^{n+1}$,

$$d_n\langle\vec{\beta}\rangle = \sum_{i=0}^{n-1} (-1)^{n+i} d_n \langle \vec{\beta}^i, \delta \rangle$$

is the unique $d_n\mathcal{B}_n(\varepsilon)$ decomposition of $d_n\langle\vec{\beta}\rangle$. □

Theorem 3.4.6. *For any ladder C_ε on $\varepsilon \in \mathrm{Cof}(\aleph_k)$ and positive $n > k$, the collection $d_n(\mathcal{B}_n(\varepsilon)[C_\varepsilon])$ is a basis for $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$. In particular, $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ is free.*

Proof. The proof is by induction on ε. Denote the following inductive hypothesis IH(ε):

If $\delta < \varepsilon$ *and* $\mathrm{cf}(\delta) = \aleph_k$ *and* $n > max(0, k)$*, then for any ladder* C_δ *on* δ*,*

$$d_n(\mathcal{B}_n(\delta)[C_\delta]) \textit{ is a basis for } \mathbf{d}_n\mathbf{P}_n(\delta).$$

Notice that if ε is a limit ordinal and IH(ξ) holds for all $\xi < \varepsilon$, then IH(ε) holds. Hence we only need to show that IH(ε) implies IH($\varepsilon + 1$). If ε is a successor ordinal, then IH($\varepsilon + 1$) follows from IH(ε) by the preceding example and lemma. If ε is a limit ordinal of cofinality greater than $\aleph_\omega$, then IH($\varepsilon + 1$) and IH(ε) are equivalent assertions. This leaves just one case of interest: limit ε of cofinality less than $\aleph_\omega$.

First, a lemma:

Lemma 3.4.7. *For* $\delta \in \mathrm{Lim} \cap C_\varepsilon$*,*

$$\mathcal{B}_{n-1}(\delta)[C_{\delta\varepsilon}] = \{\langle\vec{\alpha}\rangle \mid \langle\vec{\alpha}, \delta\rangle \in \mathcal{B}_n(\varepsilon)[C_\varepsilon]\}$$

Proof of Lemma. Term the longest proper tail-segment of $\vec{\alpha}$ which is internal to C_ε the C_ε*-tail of* $\vec{\alpha}$. Write $(\vec{\beta}, \delta)$ for the C_ε-tail of $\langle\vec{\alpha}, \delta\rangle \in \mathcal{B}_n(\varepsilon)[C_\varepsilon]$. As δ is a limit, $\vec{\beta} \neq \varnothing$. Moreover, $\vec{\beta}$ is the $C_{\delta\varepsilon}$-tail of $\langle\vec{\alpha}\rangle$ if and only if $(\vec{\beta}, \delta)$ is the C_ε-tail of $\langle\vec{\alpha}, \delta\rangle$. The lemma follows. □

We return to the proof of the theorem. Assume IH(ε), with ε a limit ordinal of cofinality less than $\aleph_\omega$. Enumerate the elements of C_ε as $\{\eta_i^\varepsilon \mid i \in \mathrm{cf}(\varepsilon)\}$.

Claim 1. $d_n\mathcal{B}_n(\varepsilon)$ is linearly independent.

Proof of Claim 1. Towards contradiction, suppose instead that

$$\sum_{j<\ell} z_j d_n\langle\vec{\alpha}_j, \beta_j\rangle = 0 \tag{3.7}$$

for nonzero coefficients z_j ($j < \ell$) and $\{\langle\vec{\alpha}_j, \beta_j\rangle \,|\, j < \ell\} \subset \mathcal{B}_n(\varepsilon)$. Let $\delta = \max\{\beta_j \,|\, j < \ell\}$, and let $J = \{j \,|\, \beta_j = \delta\}$. Note that all β_j are elements of C_ε. In particular, δ is an element of C_ε. Note also that the $\vec{\alpha}_j$ indexed by J are all distinct. By (3.7),

$$\sum_{j\in\ell\backslash J} z_j d_n\langle\vec{\alpha}_j, \beta_j\rangle + \sum_{j\in J} z_j\langle d_{n-1}\vec{\alpha}_j, \delta\rangle + (-1)^n \sum_{j\in J} z_j\langle\vec{\alpha}_j\rangle = 0 \tag{3.8}$$

Case 1: δ is a limit ordinal. Then together with the induction hypothesis, $\sum_J z_j\langle d_{n-1}\vec{\alpha}_j, \delta\rangle = 0$, which follows from (3.8), contradicts Lemma 3.4.7.

Case 2: δ is a successor ordinal: $\delta = \eta^\varepsilon_{i+1}$, for some $i < \mathrm{cf}(\varepsilon)$. Hence $(\vec{\alpha}_j)_{n-1} \geq \eta^\varepsilon_i$ for $j \in J$. By (3.8),

$$\sum_{j\in\ell\backslash J} z_j d_n\langle\vec{\alpha}_j, \beta_j\rangle + (-1)^n \sum_{j\in J} z_j\langle\vec{\alpha}_j\rangle = 0 \tag{3.9}$$

By definition, $\beta_j \leq \eta^\varepsilon_i$ for $j \in \ell\backslash J$. This implies that $(\vec{\alpha}_j)_{n-1} \leq \eta^\varepsilon_i$ for $j \in J$. Hence $(\vec{\alpha}_j)_{n-1} = \eta^\varepsilon_i$ for $j \in J$. Therefore the $\vec{\alpha}^{n-1}_j$ indexed by J are all distinct. By (3.9), though,

$$0 = \sum_{j\in J} z_j d_{n-1}\langle\vec{\alpha}_j\rangle$$

and we may conclude as in equation 3.6 that $\sum_{j\in J} z_j\langle\vec{\alpha}^{n-1}_j\rangle = 0$, a contradiction. □

Claim 2. $d_n\mathcal{B}_n(\varepsilon)$ generates $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$.

Proof of Claim 2. We argue by induction on $\delta \in C_\varepsilon$. Let

$$\mathcal{B}_n(\varepsilon)\restriction_\delta = \{\langle\vec{\alpha}\rangle \in \mathcal{B}_n(\varepsilon) \mid \alpha_n < \delta\}$$

We show that if $d_n\mathcal{B}_n(\varepsilon)\restriction_{\gamma+1}$ generates $\mathbf{d}_n\mathbf{P}_n(\gamma+1)$ for all $\gamma \in \delta\cap C_\varepsilon$, then $d_n\mathcal{B}_n(\varepsilon)\restriction_{\delta+1}$ generates $\mathbf{d}_n\mathbf{P}_n(\delta+1)$.

The base case, $\delta = \eta_0^\varepsilon$, is exactly as in Example 3.4.4.

<u>Case 1</u>: δ is a limit ordinal. Consider then $d_n\langle\vec{\alpha},\delta\rangle \in \mathbf{d}_n\mathbf{P}_n(\delta+1)$. Let

$$\sum_{j<\ell} z_j d_{n-1}\langle\vec{\beta}_j\rangle$$

be the $\mathcal{B}_{n-1}(\delta)[C_{\delta\varepsilon}]$ decomposition of $d_{n-1}\langle\vec{\alpha}\rangle$. Then

$$\begin{aligned} d_n\langle\vec{\alpha},\delta\rangle &= \langle d_{n-1}\langle\vec{\alpha}\rangle,\delta\rangle + (-1)^n\langle\vec{\alpha}\rangle \\ &= \sum_{j<\ell} z_j d_n\langle\vec{\beta}_j,\delta\rangle + (-1)^n(\sum_{j<\ell} z_j\langle\vec{\beta}_j\rangle + \langle\vec{\alpha}\rangle) \end{aligned}$$

By Lemma 3.4.7, the left-hand summands of the last line are all from $d_n\mathcal{B}_n(\varepsilon)\restriction_{\delta+1}$, while the rightmost sum is in $\mathbf{d}_n\mathbf{P}_n(\eta+1)$ for some $\eta \in \delta\cap C_\varepsilon$. By our induction hypothesis, this concludes Case 1.

<u>Case 2</u>: δ is a successor ordinal. Let $\delta = \eta_{i+1}^\varepsilon$. Consider $\vec{\beta} \in [\delta]^n$. If $\beta_{n-1} \geq \eta_i^\varepsilon$ then $\langle\vec{\beta},\delta\rangle \in \mathcal{B}_n(\varepsilon)$. If $\beta_{n-1} < \eta_i^\varepsilon$, then $d_n d_{n+1}\langle\vec{\beta},\eta_i^\varepsilon,\delta\rangle = 0$, hence

$$d_n\langle\vec{\beta},\delta\rangle = (-1)^{n-1} d_n\langle d_{n-1}\vec{\beta},\eta_i^\varepsilon,\delta\rangle + d_n\langle\vec{\beta},\eta_i^\varepsilon\rangle \tag{3.10}$$

Again the right-hand summand decomposes by hypothesis, while those on the left are from $\mathcal{B}_n(\varepsilon)\restriction_{\delta+1}$.

We will be done if we show that, for any $\vec{\alpha} \in [\delta]^{n+1}$ with $\alpha_n > \eta_i^\varepsilon$, $d_n\langle\vec{\alpha}\rangle$ has a $\mathcal{B}_n(\varepsilon)\restriction_{\delta+1}$ decomposition. Again, though, since $d_n d_{n+1}\langle\vec{\alpha}, \delta\rangle = 0$,

$$d_n\langle\vec{\alpha}\rangle = \sum_{j=0}^{n}(-1)^{n+j+1} d_n\langle\vec{\alpha}^j, \delta\rangle$$

and all summands on the right are as discussed above: either of type (3.10), or from $\mathcal{B}_n(\varepsilon)\restriction_{\delta+1}$, directly. This concludes the proof of Claim 2. $\square$

Together with the induction hypothesis IH(ε), Claims 1 and 2 establish IH($\varepsilon+1$). This concludes the proof of Theorem 3.4.6. $\square$

Lemma 3.4.7 in the above argument bears comparison with square principles (see Section 5.2.3): structuring both is a certain uniformity at the limit points δ of a club $C_\varepsilon \subseteq \varepsilon$. In $\square(\kappa)$, this condition takes the form

$$C_\varepsilon \cap \delta = C_\delta \text{ for all } \delta \in C'_\varepsilon$$

In our basis construction, it comes at the cost of an additional coordinate:

$$\mathcal{B}_{n-1}(\delta)[C_{\delta\varepsilon}] = \{\langle\vec{\alpha}\rangle \mid \langle\vec{\alpha}, \delta\rangle \in \mathcal{B}_n(\varepsilon)[C_\varepsilon]\} \text{ for all } \delta \in C'_\varepsilon$$

Moreover, whenever δ is in C_ε and $\gamma \in \mathrm{Lim} \cap C_{\delta\varepsilon}$ then

$$\mathcal{B}_{n-2}(\gamma)[C_{\gamma\delta\varepsilon}] = \{\langle\vec{\alpha}\rangle \mid \langle\vec{\alpha}, \gamma\rangle \in \mathcal{B}_{n-1}(\delta)[C_{\delta\varepsilon}]\} = \{\langle\vec{\alpha}\rangle \mid \langle\vec{\alpha}, \gamma, \delta\rangle \in \mathcal{B}_n(\varepsilon)[C_\varepsilon]\}$$

Hence these additional coordinates accrue. In other words, more room is needed to carry out the construction on higher cofinality ε; this is one heuristic for the associated rise in homological dimension. Internal tails record these accruing coordinates and are the key to our further constructions. More particularly, internal tails organize the $d_n\mathcal{B}_n(\varepsilon)$-decomposition of $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ to such a degree that the associated map $\mathbf{d}_n\mathbf{P}_n(\varepsilon) \to \mathbf{P}_n(\varepsilon)$ extends to all of $\mathbf{P}_{n-1}(\varepsilon)$, as we describe in the following section. The basic principle is the following: a fact used in the proof of Lemma 3.4.7 is that any $\langle\vec{\gamma}\rangle$ has a maximal proper internal tail $\vec{\beta}$. Hence $\langle\vec{\gamma}\rangle = \langle\vec{\alpha}, \vec{\beta}\rangle$ for some i and $\vec{\alpha} \in [\varepsilon]^{i+1}$, and $\langle\vec{\gamma}\rangle$ has some "nearest" basis element $b(\vec{\gamma})$, if $C^{\vec{\beta}\varepsilon}(\alpha_i)$ is defined:

Definition 3.4.8. Given an ε and C_ε as in the above construction, the *maximal proper internal tail* of $\langle\vec{\gamma}\rangle \in \mathbf{P}_n(\varepsilon)$ is the longest tail $\vec{\beta}$ of $\vec{\gamma}$ which is internal to C_ε and not all of $\vec{\gamma}$. In this case, if $C^{\vec{\beta}\varepsilon}(\alpha_i)$ is defined, let

$$\mathtt{b}(\vec{\gamma}) = \langle\vec{\alpha}, C^{\vec{\beta}\varepsilon}(\alpha_i), \vec{\beta}\rangle$$

If $C^{\vec{\beta}\varepsilon}(\alpha_i)$ is not defined, let $\mathtt{b}(\vec{\gamma}) = 0$.

The function $\mathtt{b}$ will feature centrally in the following sections.

3.5 Mitchell's Theorem

For the results of Section 3.3, it remains to be shown that $\mathbf{d}_n\mathbf{P}_n(\omega_n)$ is *not* free, for any $n > 0$. This is immediate, though, from the following theorem and proposition:

Theorem 3.5.1 ([32]). *If $\varepsilon > 0$ is an ordinal of cofinality $\aleph_\xi$ and ξ is finite, then the homological dimension of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ is $\xi + 1$. If ξ is infinite, then the homological dimension of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ is ∞.*

We know both from Goblot's Theorem and Theorem 3.4.6 above that when ε is of cofinality $\aleph_n$ and n is finite then the homological dimension of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ is at most $n + 1$. Hence to see Theorem 3.5.1, we only need to see the following:

Proposition 3.5.2. *Let ε be of cofinality $\aleph_\xi$. Then $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ is not projective, for any finite ordinal $n \leq \xi$.*

Together with the work of the previous section, the proposition establishes Theorems 3.3.10 and, hence, 3.3.5 as well. We sketch its proof, referring the reader to [32] or [29] for details (we should caution, though, that our notation diverges from both sources).

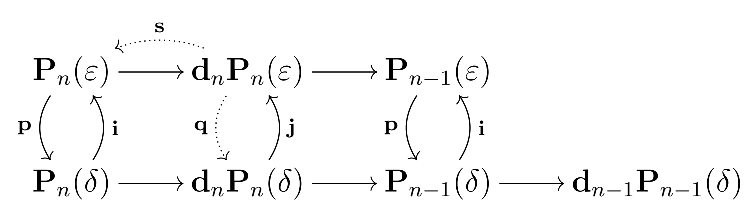
Sketch of proof of Proposition 3.5.2. The argument is by induction on ξ. In the base case $\xi = 0$ we verify that $\mathbf{e}\,\mathbf{P}_0(\varepsilon) \cong \mathbf{\Delta}_\varepsilon(\mathbb{Z})$ is not projective if ε is a limit ordinal. For the induction step we show that if $\mathbf{d}_{n-1}\mathbf{P}_{n-1}(\delta)$ isn't projective for some $\delta < \varepsilon$ with $\mathrm{cf}(\delta) < \mathrm{cf}(\varepsilon)$ then $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ isn't projective either. (In the $n = 1$ case, $\mathbf{d}_{n-1}$ should be read as $\mathbf{e}$.)

<u>The base case</u>: By the following claim, if ε is a limit ordinal then the epimorphism $\mathbf{e} : \mathbf{P}_0(\varepsilon) \to \mathbf{\Delta}_\varepsilon(\mathbb{Z})$ has no right-inverse. Hence $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ is not projective.

Claim 3.5.3. *Let ε be a limit ordinal. Then the only morphism $\mathbf{f} : \mathbf{\Delta}_\varepsilon(\mathbb{Z}) \to \mathbf{P}_0(\varepsilon)$ is the zero morphism.*

Proof. Such an $\mathbf{f}$ is a collection of morphisms $\{f_\alpha : \mathbb{Z} \to \bigoplus_{[\alpha,\varepsilon)} \mathbb{Z} \,|\, \alpha < \varepsilon\}$ commuting with the bonding maps in $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ and $\mathbf{P}_0(\varepsilon)$. This requirement means in this case that $f_\alpha(1)$ must equal $f_\beta(1)$ for all $\alpha < \beta < \varepsilon$. This is not possible if $\beta > \min(\mathrm{supp}(f_\alpha(1)))$. Hence $\min(\mathrm{supp}(f_\alpha(1)))$ must not be defined for any $\alpha < \varepsilon$. Thus $\mathbf{f}$ is the zero morphism. $\square$

<u>The induction step</u>: This consists in showing that if $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ is projective, then for any regular $\kappa < \mathrm{cf}(\varepsilon)$, there exists a $\delta \in \mathrm{Cof}(\kappa) \cap \varepsilon$ with $\mathbf{d}_{n-1}\mathbf{P}_{n-1}(\delta)$ projective as well. This is argued via the following diagram:



(3.11)

Rows are "telescopings" of the projective resolutions of $\mathbf{\Delta}_\varepsilon(\mathbb{Z})$ and $\mathbf{\Delta}_\delta(\mathbb{Z})$, respectively; they are, in other words, the natural decompositions of the differentials $\mathbf{d}_n : \mathbf{P}_n \to \mathbf{P}_{n-1}$ into $\mathbf{P}_n \twoheadrightarrow \mathbf{d}_n\mathbf{P}_n$ followed by $\mathbf{d}_n\mathbf{P}_n \hookrightarrow \mathbf{P}_{n-1}$. Between each pair $\mathbf{P}_n(\varepsilon)$ and $\mathbf{P}_n(\delta)$, there are natural projections $\mathbf{p} : \mathbf{P}_n(\varepsilon) \to \mathbf{P}_n(\delta)$ and inclusions $\mathbf{i} : \mathbf{P}_n(\delta) \to \mathbf{P}_n(\varepsilon)$. Similarly, $\mathbf{d}_n\mathbf{P}_n(\delta)$ naturally includes into $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$; what is perhaps surprising is that this inclusion $\mathbf{j}$ may have no left-inverse.[3] This is the first

[3] $\mathbf{j} : \mathbf{d}_1\mathbf{P}_1(5) \to \mathbf{d}_1\mathbf{P}_1(\omega)$, for example, does have a left-inverse, while $\mathbf{j} : \mathbf{d}_1\mathbf{P}_1(\omega) \to \mathbf{d}_1\mathbf{P}_1(\omega_1)$ does not, as the reader is encouraged to verify.

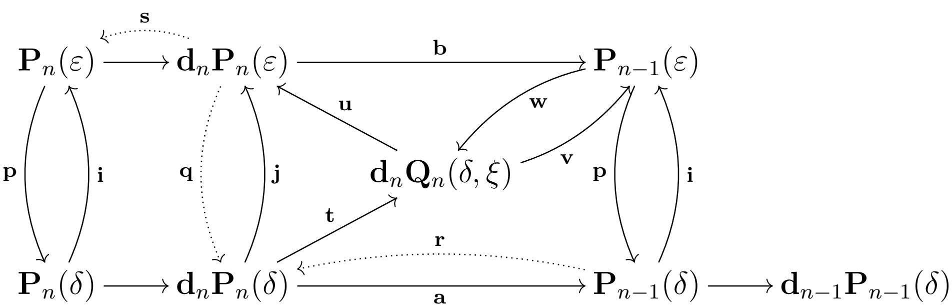
key observation in the induction: if $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ is projective and, hence, admits some section $\mathbf{s}$ of the map $\mathbf{d}_n$, then at closure points δ of $\mathbf{s}$, a left-inverse to $\mathbf{j}$ does exist — namely, $\mathbf{q} = \mathbf{d}_n\,\mathbf{p}\,\mathbf{s}$.

The second key observation is that $\mathbf{q}$, together with the space in ε above δ, may be used to define a retract $\mathbf{r}$ of the inclusion $\mathbf{d}_n\mathbf{P}_n(\delta) \hookrightarrow \mathbf{P}_{n-1}(\delta)$. The existence of such an $\mathbf{r}$ implies that $\mathbf{P}_{n-1}(\delta) \cong \mathbf{d}_n\mathbf{P}_n(\delta) \oplus \mathbf{d}_{n-1}\mathbf{P}_{n-1}(\delta)$, by the exactness of the sequence

$$\mathbf{0} \longrightarrow \mathbf{d}_n\mathbf{P}_n(\delta) \underset{\overset{\mathbf{r}}{\curvearrowleft}}{\longrightarrow} \mathbf{P}_{n-1}(\delta) \longrightarrow \mathbf{d}_{n-1}\mathbf{P}_{n-1}(\delta) \longrightarrow \mathbf{0}$$

Hence $\mathbf{d}_{n-1}\mathbf{P}_{n-1}(\delta)$ is a summand of the free system $\mathbf{P}_{n-1}(\delta)$. By Lemma 3.2.5, $\mathbf{d}_{n-1}\mathbf{P}_{n-1}(\delta)$ is therefore projective. If we've shown that $\mathrm{hd}(\mathbf{\Delta}_\delta(\mathbb{Z})) > n$, then this is a contradiction; hence our assumption that $\mathbf{d}_n\mathbf{P}_n(\varepsilon)$ is projective was false. In consequence, $\mathrm{hd}(\mathbf{\Delta}_\varepsilon(\mathbb{Z})) > n + 1$.

In the preceding paragraph, we referenced "the space in ε above δ": fix $\xi \in \varepsilon\backslash\delta$. The key device in this second part of the argument — i.e., in the derivation of a retract $\mathbf{r}$ from $\mathbf{q}$ — is the formation of a cone over $\mathbf{P}_{n-1}(\delta)$ in $\mathbf{P}_n(\varepsilon)$. By this we mean the following: let $\mathbf{Q}_n(\delta, \xi)$ be the subsystem of $\mathbf{P}_n(\varepsilon)$ generated by $\{\langle\vec{\alpha}, \xi\rangle \mid \vec{\alpha} \in [\delta]^n\}$. As the reader may verify, there are natural inclusion-relations between $\mathbf{d}_n\mathbf{Q}_n(\delta, \xi)$ and many of the main terms in the diagram 3.11. These we denote $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$ in the diagram below.



(3.12)

What the cone construction critically affords us is a retract, $\mathbf{w}$, of $\mathbf{v}$. This is defined as follows: for $\vec{\beta} \in [\delta]^n$, let

$$\mathbf{w}(\langle\vec{\beta}\rangle) = \begin{cases} (-1)^n d_n\langle\vec{\beta},\xi\rangle & \text{if } \vec{\beta} \in [\delta]^n \\ 0 & \text{otherwise} \end{cases}$$

For a generator $d_n\langle\vec{\alpha},\xi\rangle$ of $\mathbf{d}_n\mathbf{Q}_n(\delta,\xi)$,

$$\mathbf{wv}(\mathbf{d}_n\langle\vec{\alpha},\xi\rangle) = \mathbf{w}\left(\sum_{i=0}^{n-1}(-1)^i\langle\vec{\alpha}^i,\xi\rangle + (-1)^n\langle\vec{\alpha}\rangle\right) = d_n\langle\vec{\alpha},\xi\rangle$$

Hence $\mathbf{w}$ is a retract of $\mathbf{v}$, as desired. The point is the following:

Write $\mathbf{a}$ for $\mathbf{d}_n\mathbf{P}_n(\delta) \hookrightarrow \mathbf{P}_{n-1}(\delta)$, as in diagram 3.12 above. Then given a $\mathbf{q}$ left-inverse to $\mathbf{j}$, the map $\mathbf{r} = \mathbf{quwi}$ is left-inverse to $\mathbf{a}$:

$$\mathbf{ra} = \mathbf{quwia} = \mathbf{quwbj} = \mathbf{quwbut} = \mathbf{quwvt} = \mathbf{qut} = \mathbf{qj}$$

The equation records a diagram-chase on (3.12) above, together with the fact that $\mathbf{wv} = \mathbf{id}$. It shows that $\mathbf{r}$ is indeed a retract of $\mathbf{a}$, and thereby concludes the induction step. □

We included the above argument for completeness. Reading it, though, one can't escape the feeling of having missed something. The next section might be thought of as *a closer look*; here the present material begins to connect back to that of Chapter Two.

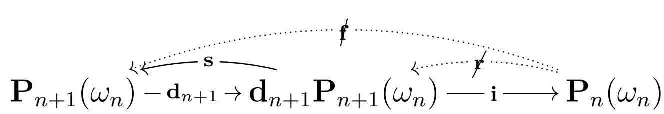
3.6 Witnesses to higher nontrivial coherence

Let ε be ω_n. For simplicity later, also let C_ε equal ω_n. Theorem 3.5.1 then admits the following reading:

In the projective resolution

$$\cdots \longrightarrow \mathbf{P}_{n+1}(\omega_n) \xrightarrow{\mathbf{d}_{n+1}} \mathbf{P}_n(\omega_n) \xrightarrow{\mathbf{d}_n} \cdots \xrightarrow{\mathbf{d}_1} \mathbf{P}_0(\omega_n) \xrightarrow{\mathbf{e}} \mathbf{\Delta}_{\omega_n}(\mathbb{Z}) \longrightarrow \mathbf{0},$$

the interval $\mathbf{P}_{n+1}(\omega_n) \longrightarrow \mathbf{P}_n(\omega_n)$ *is distinctive; it unpacks as follows:*



Here maps $\mathbf{d}_{n+1}$ and $\mathbf{i}$ in and out of $\mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n)$, respectively, *do* and *do not* admit right and left inverses $\mathbf{s}$ and $\mathbf{r}$. And it's from just this discord of factorizations, ultimately, that nonvanishing $\lim^{n+1}$ and non-n-trivial n-coherence will more generally derive.

More particularly, observe that

1. $\mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n)$ is projective; hence there exists a section $\mathbf{s} : \mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n) \to \mathbf{P}_{n+1}(\omega_n)$ right-inverse to $\mathbf{d}_{n+1}$. We've defined, in fact, a basis for $\mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n)$ affording explicit definition of such an $\mathbf{s}$: let

$$s_\alpha \ : \ x \mapsto \sum_{j=0}^{k} z_j \langle \vec{\alpha}_j \rangle \tag{3.13}$$

for any $x \in d_{n+1}P_{n+1}([0, \omega_n))$, where

$$x = \sum_{j=0}^{k} z_j d_{n+1} \langle \vec{\alpha}_j \rangle$$

is the $d_{n+1}\mathcal{B}_{n+1}(\omega_n)$-decomposition of x.

2. $\mathbf{d}_n\mathbf{P}_n(\omega_n)$ is *not* projective; hence there exists *no* retract $\mathbf{r} : \mathbf{P}_n(\omega_n) \to \mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n)$ left-inverse to the inclusion map $\mathbf{i} : \mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n) \to \mathbf{P}_n(\omega_n)$. (If such an $\mathbf{r}$ existed, then $\mathbf{d}_n\mathbf{P}_n(\omega_n)$ would be a summand of $\mathbf{P}_n(\omega_n) \cong \mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n) \oplus \mathbf{d}_n\mathbf{P}_n(\omega_n)$ and, hence, would be projective: a contradiction.) Therefore there exists no $\mathbf{f} : \mathbf{P}_n(\omega_n) \to \mathbf{P}_{n+1}(\omega_n)$ extending $\mathbf{s}$: for any such $\mathbf{f}$, the map $\mathbf{d}_{n+1}\mathbf{f}\,\mathbf{i} = \mathbf{d}_{n+1}\mathbf{s}$ would be the identity, so that $\mathbf{d}_{n+1}\mathbf{f}$ would define just such a left-inverse $\mathbf{r}$ of $\mathbf{i}$ as we've seen cannot exist.

In consequence,

Theorem 3.6.1. *The function* $\mathbf{s}\,\mathbf{d}_{n+1}$ *witnesses the nonvanishing of* $\lim^{n+1}\mathbf{P}_{n+1}(\omega_n)$.

Proof. Recall again from [29] the following: for any object $\mathbf{X}$ in $Ab^{\varepsilon^{op}}$,

$$\lim{}^n \mathbf{X} \cong \mathrm{Ext}^n(\mathbf{\Delta}_\varepsilon(\mathbb{Z}), \mathbf{X}) \cong \mathrm{H}^n(\mathrm{Hom}(\mathsf{P}(\varepsilon), \mathbf{X})) \tag{3.14}$$

with $\mathsf{P}(\omega_n)$ our standard projective resolution of $\boldsymbol{\Delta}_{\omega_n}(\mathbb{Z})$. Apply the functor $\mathrm{Hom}(\,\cdot\,,\mathbf{P}_{n+1}(\omega_n))$ to $\mathsf{P}(\omega_n)$. Write ∂^n for $\mathrm{Hom}(\mathbf{d}_n,\mathbf{P}_{n+1}(\omega_n))$. Then $\partial^{n+2}\mathbf{g} = \mathbf{g}\,\mathbf{d}_{n+2}$, hence

$$\ker\Big(\,\partial^{n+2}\;:\;\mathrm{Hom}(\mathbf{P}_{n+1}(\omega_n),\mathbf{P}_{n+1}(\omega_n))\rightarrow\mathrm{Hom}(\mathbf{P}_{n+2}(\omega_n),\mathbf{P}_{n+1}(\omega_n))\Big)$$

is just those maps $\mathbf{g}\;:\;\mathbf{P}_{n+1}(\omega_n)\rightarrow\mathbf{P}_{n+1}(\omega_n)$ whose restriction to $\mathbf{d}_{n+2}\mathbf{P}_{n+2}(\omega_n)$ is zero. Clearly $\mathbf{s}\,\mathbf{d}_{n+1}$ is such a map. But $\mathbf{s}\,\mathbf{d}_{n+1}$ is in

$$\mathrm{im}\Big(\,\partial^{n+1}\;:\;\mathrm{Hom}(\mathbf{P}_{n}(\omega_n),\mathbf{P}_{n+1}(\omega_n))\rightarrow\mathrm{Hom}(\mathbf{P}_{n+1}(\omega_n),\mathbf{P}_{n+1}(\omega_n))\Big)$$

if and only if $\mathbf{s}$ extends to some $\mathbf{f}\;:\;\mathbf{P}_n(\omega_n)\rightarrow\mathbf{P}_{n+1}(\omega_n)$. Thus, by observation (2) above, $0\neq[\mathbf{s}\,\mathbf{d}_{n+1}]\in\mathrm{H}^{n+1}(\mathrm{Hom}(\mathcal{P}(\omega_n),\mathbf{P}_{n+1}(\omega_n)))\cong\lim^{n+1}\mathbf{P}_{n+1}(\omega_n)$. □

The essential condition above — that no map $\mathbf{f}\;:\;\mathbf{P}_n(\omega_n)\rightarrow\mathbf{P}_{n+1}(\omega_n)$ extends $\mathbf{s}$ — should be qualified: $\mathbf{s}$ *does* extend to a map $\mathbf{f}$ on $\mathbf{P}_n(\omega_n)$, but that map will require terms in the codomain of infinite support. In other words, $\mathbf{f}$ will map $\mathbf{P}_n(\omega_n)$ to $\mathbf{R}_{n+1}(\omega_n)$.

Example 3.6.2. The case of ω: Here ladder system $\langle\, C_{i+1}=\{i\}\,|\,i<\omega\,\rangle$ determines basis $\mathcal{B}_1(\omega)=\{\,\langle i,i+1\rangle\,|\,i<\omega\,\}$; hence

$$\mathbf{s}\,\mathbf{d}_1(\langle j,k\rangle)=\sum_{i=j}^{k-1}\langle i,i+1\rangle$$

for any $j<k<\omega$. An $\mathbf{f}:\mathbf{P}_0(\omega)\rightarrow\mathbf{R}_1(\omega)$ for which $\mathbf{f}\restriction_{\mathbf{d}_1\mathbf{P}_1(\omega)}=\mathbf{s}$ would satisfy

$$\mathbf{f}(\langle j+1\rangle-\langle j\rangle)=\mathbf{s}(\langle j+1\rangle-\langle j\rangle)=\langle j,j+1\rangle$$

for any $j < \omega$. But this amounts to a definition: the implicit formula

$$\mathbf{f}(\langle j \rangle) = -\langle j, j+1 \rangle + \mathbf{f}(\langle j+1 \rangle)$$

in fact fully determines $\mathbf{f}$. This is because $\mathbf{f} = \{f_j : P_0([j,\omega)) \to R_1([j,\omega)) \mid j < \omega\}$, and $\mathbf{f}(\langle j \rangle)$ by definition is $f_j(\langle j \rangle)$. This value, falling in $R_1([j,\omega) = \prod_{[[j,\omega)]^2} \mathbb{Z}$, can involve no coordinates less than j. Hence the formula

$$\mathbf{f}(\langle 0 \rangle) = -\langle 0, 1 \rangle + \mathbf{f}(\langle 1 \rangle) \tag{3.15}$$

entirely determines the "0-column" of $\mathbf{f}(\langle 0 \rangle)$. Similarly, the formula

$$\mathbf{f}(\langle 1 \rangle) = -\langle 1, 2 \rangle + \mathbf{f}(\langle 2 \rangle)$$

entirely determines the "1-column" of $\mathbf{f}(\langle 1 \rangle)$ and hence, by (3.15), that of $\mathbf{f}(\langle 0 \rangle)$ as well — and so on. This defines $\mathbf{f}$ on $\{\langle j \rangle \mid j \in \omega\}$ and therefore on all of $\mathbf{P}_0(\omega)$.

This technique very generally applies. Recall from Definition 3.4.8 the function $\mathtt{b}$, which inserts a single coordinate into $\vec{\alpha}$, if possible, to take $\vec{\alpha}$ to some "nearest" $\mathtt{b}(\vec{\alpha}) \in \mathcal{B}$.[4] Recall also that $\mathbf{s}$ was defined so that $\mathbf{s}\,\mathbf{d}_n\restriction_{\mathcal{B}_n} = \mathbf{id}\restriction_{\mathcal{B}_n}$ (see equation 3.13). Therefore if $\mathbf{f}_n : \mathbf{P}_n(\omega_n) \to \mathbf{R}_{n+1}(\omega_n)$ extends $\mathbf{s}$, then

$$\sum_{i=0}^{n+1} (\text{-}1)^i \mathbf{f}_n(\mathtt{b}(\vec{\alpha})^i) = \mathbf{f}_n(\mathbf{d}_{n+1}(\mathtt{b}(\vec{\alpha}))) = \mathbf{s}(\mathbf{d}_{n+1}(\mathtt{b}(\vec{\alpha}))) = \mathtt{b}(\vec{\alpha}) \tag{3.16}$$

Just as above, equation 3.16 may be read in another direction as *defining* such an $\mathbf{f}_n$: let $\vec{\alpha} = (\vec{\beta}, \vec{\gamma}) \in [\omega_n]^{n+1}$, with $|\vec{\beta}| = j+1$ and $\vec{\gamma}$ the maximal proper internal tail

[4]Since $\mathtt{b}(\vec{\alpha})$ is a generator, not an n-tuple, the notation $\mathtt{b}(\vec{\alpha})^i$ hereabouts is a minor abuse; still, the meaning should be clear.

of $\vec{\alpha}$, so that $\mathtt{b}(\vec{\alpha})$ is either 0 or $\langle\vec{\beta}, C^{\vec{\gamma}}(\beta_j), \vec{\gamma}\rangle$. In the former case, let

$$\mathbf{f}_n(\langle\vec{\alpha}\rangle) = 0 \tag{3.17}$$

In the latter case, $\langle\vec{\alpha}\rangle = \mathtt{b}(\vec{\alpha})^{j+1}$; hence equation 3.16 entails that

$$\mathbf{f}_n(\langle\vec{\alpha}\rangle) = (\text{-}1)^{j+1}\Big[\mathtt{b}(\vec{\alpha}) - \sum_{i=0}^{j}(\text{-}1)^i\mathbf{f}_n(\mathtt{b}(\vec{\alpha})^i) - \sum_{i=j+2}^{n+1}(\text{-}1)^i\mathbf{f}_n(\mathtt{b}(\vec{\alpha})^i)\Big] \tag{3.18}$$

Unlike in Example 3.6.2, equations 3.17 and 3.18 alone don't fully determine $\mathbf{f}_n$.[5] However, these equations *do* share with that of Example 3.6.2 a canonical solution, namely the function associating to $\langle\vec{\alpha}\rangle$ just those generators $\mathtt{b}(\,\cdot\,) \in \mathcal{B}_{n+1}(\omega_n)$ appearing in the expansion of equation 3.18. It's this algebraic extension of the identification of each $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$ with its formal expansion via (3.18) and (3.17) that we'll denote hereafter as $\mathbf{f}_n$. The legitimacy of this operation, of course, requires argument.

Lemma 3.6.3. *Equations 3.17 and 3.18 determine a canonical* $\mathbf{f}_n : \mathbf{P}_n(\omega_n) \to \mathbf{R}_{n+1}(\omega_n)$ *extending* $\mathbf{s} : \mathbf{d}_{n+1}\mathbf{P}_{n+1}(\omega_n) \to \mathbf{P}_{n+1}(\omega_n)$*.*

Proof. We've described an $\mathbf{f}_n$ extending $\mathbf{s}$ by definition. What remains to be checked is that this description makes sense. What "making sense" in this context entails will be clearest via an example. We'll take the occasion to establish some vocabulary as well.

The typical instance of the case $n = 1$ of equation 3.18 is

$$\mathbf{f}_1(\langle\alpha,\gamma\rangle) = -\langle\alpha, C^\gamma(\alpha), \gamma\rangle + \mathbf{f}_1(\langle C^\gamma(\alpha), \gamma\rangle) + \mathbf{f}_1(\langle\alpha, C^\gamma(\alpha)\rangle)$$

[5] In particular, in crucial distinction to Example 3.6.2, they fail to fully determine "columns."

This engenders two further "steps" or computations $\mathbf{f}_1(\langle C^\gamma(\alpha), \gamma\rangle)$ and $\mathbf{f}_1(\langle \alpha, C^\gamma(\alpha)\rangle)$. We term the totality of such steps the *expansion* of $\mathbf{f}_1(\langle\vec{\alpha}\rangle)$. Similarly for $\mathbf{f}_n$. The collection of all $\mathcal{B}_{n+1}$-elements output in the expansion of $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$ is the *support* of $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$, written $supp(\mathbf{f}_n(\langle\vec{\alpha}\rangle))$. Elements of the support of $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$ are called *values* of $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$. Those $\mathbf{f}_n(\,\cdot\,)$ which appear in the course of the expansion of $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$ and direct its further computation are called its *terms*. In our example, $\langle \alpha, C^\gamma(\alpha), \gamma\rangle$ is a value, and $\mathbf{f}_1(\langle C^\gamma(\alpha), \gamma\rangle)$ and $\mathbf{f}_1(\langle \alpha, C^\gamma(\alpha)\rangle)$ are terms, of $\mathbf{f}_1(\langle \alpha, \gamma\rangle)$. Equations 3.17 and 3.18 comprise a sensible definition if

1. No $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$ appears among the terms of its expansion, and
2. No value appears infinitely often in the expansion of any $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$.

Since the nontrivial portion of the function $\vec{\alpha} \mapsto \mathtt{b}(\vec{\alpha})$ is injective and the values of $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$ are output by its terms, to see both items above it suffices to see item (1) only. Item (1) is quite clear in the case of our example: any step — converting a term $\mathbf{f}_1(\langle \alpha, \gamma\rangle)$ to $\mathbf{f}_1(\langle \alpha, C^\gamma(\alpha)\rangle)$ and $\mathbf{f}_1(\langle C^\gamma(\alpha), \gamma\rangle)$ — may be thought of as subdividing the interval $[\alpha, \gamma]$. The term $\mathbf{f}_1(\langle \alpha, \gamma\rangle)$ cannot recur in the course of such steps or subdivisions any more than the interval $[\alpha, \gamma]$ can. The logic is the same for higher n, only more tedious to argue: once one's "shed" the coordinate γ, as in the passage from $\mathbf{f}_1(\langle \alpha, \gamma\rangle)$ to $\mathbf{f}_1(\langle \alpha, C^\gamma(\alpha)\rangle)$, there's no getting it back.[6] More precisely, the question of whether an $\mathbf{f}_n(\langle\vec{\alpha}\rangle)$ recurs in its expansion is a question entirely of the

[6] We note in passing that subdivision remains an intriguingly good heuristic for the sequence $\vec{\alpha} \mapsto \mathtt{b}(\vec{\alpha}) \mapsto d_{n+1}\mathtt{b}(\vec{\alpha})$ for higher n as well.

argument $\vec{\alpha}$: it is a question of whether we can recover $\vec{\alpha}$ in a sequence of the form

$$\vec{\alpha} \to \mathtt{b}(\vec{\alpha}) \to \mathtt{b}(\vec{\alpha})^{k_1} \to \mathtt{b}(\mathtt{b}(\vec{\alpha})^{k_1}) \to \mathtt{b}(\mathtt{b}(\vec{\alpha})^{k_1})^{k_2} \to \dots \tag{3.19}$$

where k_i is other than the index of the coordinate added by the i^{th} application of $\mathtt{b}$. As usual, let $\vec{\alpha} = (\vec{\beta}, \vec{\gamma})$ with $\vec{\gamma}$ the maximum proper internal tail of $\vec{\alpha}$, so that $\mathtt{b}(\vec{\alpha}) = (\vec{\beta}, C^{\vec{\gamma}}(\beta_i), \vec{\gamma})$ for some i.

Case 1: $\mathtt{b}(\vec{\alpha})^{k_1} = (\vec{\beta}, C^{\vec{\gamma}}(\beta_i), \vec{\gamma}^j)$ for some j.

In this case, γ_j will reappear in no later term in the sequence 3.19, for the following reason: as $(C^{\vec{\gamma}}(\beta_i), \vec{\gamma}^j)$ is internal, $\mathtt{b}(\vec{\beta}, C^{\vec{\gamma}}(\beta_i), \vec{\gamma}^j)$ can only introduce a coordinate below $C^{\vec{\gamma}}(\beta_i)$. It's of course also possible that $\mathtt{b}(\vec{\beta}, C^{\vec{\gamma}}(\beta_i), \vec{\gamma}^j)$ is zero. These remain the only possibilities at every subsequent step of (3.19), even if one of those steps removes $C^{\vec{\gamma}}(\beta_i)$; this is because of the general principle that $C^{\vec{\delta}}(\alpha) \leq C^{\gamma\vec{\delta}}(\alpha)$ whenever $C^{\vec{\delta}}(\alpha)$ is defined (cf. Observation 3.4.2). Since γ_j cannot reappear in any later term of (3.19), $\vec{\alpha}$ cannot either.

Case 2: $\mathtt{b}(\vec{\alpha})^{k_1} = (\vec{\beta}^j, C^{\vec{\gamma}}(\beta_i), \vec{\gamma})$ for some $j \leq i$.

In this case, if $j < i$, then $\mathtt{b}(\mathtt{b}(\vec{\alpha})^{k_1}) = 0$. This is because $C^{\vec{\gamma}}(\beta_i)\vec{\gamma}$ is the maximal proper internal tail of $\mathtt{b}(\vec{\alpha})^{k_1}$, hence computing $\mathtt{b}(\mathtt{b}(\vec{\alpha})^{k_1}) = 0$ entails computing $C^{C^{\vec{\gamma}}(\beta_i)\vec{\gamma}}(\beta_i)$, which is undefined. If $j = i$, then either $\mathtt{b}(\mathtt{b}(\vec{\alpha})^{k_1})$ again equals 0 or $C^{C^{\vec{\gamma}}(\beta_i)\vec{\gamma}}(\beta_{i-1}) < \beta_i$. In the latter case, we may argue as in Case 1 that the coordinate β_i can never reappear. In all these cases, clearly, the argument $\vec{\alpha}$ never reappears.

Cases 1 and 2 are all the relevant possibilities, so we've shown item (1) above.

This concludes the proof. ∎

From the above analysis, we may also conclude the following:

Lemma 3.6.4. *The values* $\langle\vec{\alpha}\rangle$ *in* $\mathbf{f}_n(\langle\vec{\beta}\rangle)$ *all have coefficients plus or minus one. In other words,* $\mathbf{f}_n(\langle\vec{\beta}\rangle)$ *may be viewed as an element of* $\prod_{\mathcal{B}_{n+1}(\omega_n)}\mathbb{Z}$ *in which each coordinate is* -1*,* 0*, or* 1*.*

Also by the above analysis, we might equally have framed $\mathbf{f}_n$ in terms of pointwise convergence to an element of $\prod_{\mathcal{B}_{n+1}(\omega_n)}\mathbb{Z}$. Some readers may find this approach more formally satisfactory.

It's the next lemma, though, that connects all this to Chapter Two.

Lemma 3.6.5. $\mathbf{f}_n\,\mathbf{d}_{n+1}$ *maps* $\mathbf{P}_{n+1}(\omega_n)$ *to* $\mathbf{P}_{n+1}(\omega_n)$*. But there exists no* $\mathbf{g}:\mathbf{P}_{n-1}(\omega_n)\to\mathbf{R}_{n+1}(\omega_n)$ *with* $\mathbf{f}_n-\mathbf{g}\,\mathbf{d}_n$ *a mapping from* $\mathbf{P}_n(\omega_n)$ *to* $\mathbf{P}_{n+1}(\omega_n)$*.*

Proof. The first part follows from Claim 3.6.3. For the second, observe that for such a $\mathbf{g}$,

$$(\mathbf{f}_n-\mathbf{g}\,\mathbf{d}_n)\,\mathbf{d}_{n+1}=\mathbf{f}_n\,\mathbf{d}_{n+1}=\mathbf{s}\,\mathbf{d}_{n+1}$$

Hence $\mathbf{f}_n-\mathbf{g}\,\mathbf{d}_n$ would be just such an $\mathbf{f}:\mathbf{P}_n(\omega_n)\to\mathbf{P}_{n+1}(\omega_n)$ extending $\mathbf{s}$ as we've seen cannot exist. ∎

Here the outlines of what we're getting at begin to emerge: in the above, the codomain $\mathbf{P}_{n+1}(\omega_n)$ imposes a finiteness condition on various combinations of outputs of $\mathbf{f}_n$ or of $\mathbf{f}_n$ and $\mathbf{g}$. Lemma 3.6.5 should be read as *the function* $\mathbf{f}_n$ *agrees, or "coheres," mod finite, on boundaries; but no function* $\mathbf{g}$ *approximates or "trivializes"* $\mathbf{f}_n$*, mod finite, on lower-order boundaries.* It's this reading we'll pursue in the next sections. It's useful first to slightly simplify our functions. Define

$$\mathtt{f}_n \in K^n(\mathbf{R}_{n+1}(\omega_n)) := \prod_{\vec{\alpha}\in[\omega_n]^{n+1}} R_{n+1}([\alpha_0, \omega_n))$$

by $\phi_n(\vec{\alpha}) = \mathbf{f}_n(\langle\vec{\alpha}\rangle)$. As $\mathbf{f}_n$ is determined by its behavior on generators, the procedure defines in fact a chain complex isomorphism, $\mathrm{Hom}(\mathcal{P}(\omega_n), \mathbf{X}) \cong \mathcal{K}(\mathbf{X}) := (K^\ell(\mathbf{X}), \partial^\ell)$, in an obvious fashion (cf. [29], p. 262; this is the $\mathcal{K}(\mathbf{X})$ of equation 2.33 above). By Lemma 3.6.5 together with Lemma 2.7.5,

$$0 \neq [\mathtt{f}_n] \in \mathrm{H}^n(\mathcal{K}(\mathbf{R}_{n+1}(\omega_n)/\mathbf{P}_{n+1}(\omega_n))) \cong \lim{}^n(\mathbf{R}_{n+1}(\omega_n)/\mathbf{P}_{n+1}(\omega_n)).$$

Hence we've argued by concrete means what might more typically and neatly be argued by means of the long exact $\lim^k$ sequence associated to

$$\mathbf{P}_{n+1}(\omega_n) \hookrightarrow \mathbf{R}_{n+1}(\omega_n) \twoheadrightarrow (\mathbf{R}_{n+1}(\omega_n)/\mathbf{P}_{n+1}(\omega_n))$$

Namely, we've extended the equivalence

$$\mathrm{hd}(\mathbf{\Delta}_{\omega_n}(\mathbb{Z})) = n+1 \;\Leftrightarrow\; \lim{}^{n+1}\mathbf{P}_{n+1}(\omega_n) \neq 0$$

to the equivalence

$$\mathrm{hd}(\mathbf{\Delta}_{\omega_n}(\mathbb{Z})) = n+1 \;\Leftrightarrow\; \lim{}^{n}(\mathbf{R}_{n+1}(\omega_n)/\mathbf{P}_{n+1}(\omega_n)) \neq 0$$

What we have connected in the process are *ladder systems* and *higher nontrivial coherence*. These connect in the explicit formulae for the functions $\mathtt{f}_n$. The investigation of these functions will occupy the remainder of this chapter.

3.7 A return to ω_1

We turn now to the case $n = 1$. The function $\mathtt{f}_n$ of the previous section specializes here to a function $\mathtt{f}_1 \in K^1(\mathbf{R}_2(\omega_1))$ with the property that

$$\mathtt{f}_1(\beta,\gamma) - \mathtt{f}_1(\alpha,\gamma) + \mathtt{f}_1(\alpha,\beta) =^* 0 \text{ for all } \alpha < \beta < \gamma < \omega_1 \tag{3.20}$$

Claim 3.6.5 translates in this context to the condition that no $\mathtt{f}_0 \in K^0(\mathbf{R}_2(\omega_1))$ satisfies the following property:

$$\mathtt{f}_0(\beta) - \mathtt{f}_0(\alpha) =^* \mathtt{f}_1(\alpha,\beta) \text{ for all } \alpha < \beta < \omega_1 \tag{3.21}$$

Statements like "$\mathtt{f}_1 \in K^1(\mathbf{R}_2(\omega_1))$" mainly describe the support of $\mathtt{f}_1$ (i.e., $\mathtt{f}_1(\alpha,\beta) \in R_2^\alpha(\omega_1)$ for all $\alpha < \beta < \omega_1$), but we can be much more precise: in the case $n = 1$, the definition of $\mathtt{f}_n$ via equations 3.17 and 3.18 takes a particularly straightforward form:

$$\mathtt{f}_1(\alpha,\beta) = \begin{cases} 0 & \text{if } \beta = \alpha + 1 \\ -\langle \alpha, C^\beta(\alpha), \beta\rangle + \mathtt{f}_1(\alpha, C^\beta(\alpha)) + \mathtt{f}_1(C^\beta(\alpha),\beta) & \text{otherwise} \end{cases} \tag{3.22}$$

It follows immediately that

$$\operatorname{supp}(\mathtt{f}_1(\alpha,\beta)) \subseteq [\,[\alpha,\beta]\,]^3 \tag{3.23}$$

This facilitates sufficiently "spatial" readings that we introduce the following notation: for $A, B \subseteq [\xi]^{<\omega}$, let $A \otimes B$ denote the collection of tuples $(\vec{\alpha},\vec{\beta}) \in A \times B$ for which $\vec{\alpha} < \vec{\beta}$. Extensions of this notation should be self-explanatory. For example, it follows from equations 3.20 and 3.23 that

$$\mathtt{f}_1(\alpha,\beta) - \mathtt{f}_1(\alpha,\gamma)\restriction_{[\alpha,\beta)\otimes[\omega_1]^2} =^* 0 \text{ for all } \alpha < \beta < \gamma < \omega_1 \tag{3.24}$$

and, hence, that

$$\mathtt{f}_1(\alpha,\gamma)\restriction_{[\alpha,\beta)\otimes[(\beta,\omega_1)]^2} =^* 0 \text{ for all } \alpha<\beta<\gamma<\omega_1 \tag{3.25}$$

It follows also from equation 3.23 that for any "trivializing" $\mathtt{f}_0$ as in (3.21),

$$\mathtt{f}_0(\beta) =^* \mathtt{f}_0(\alpha)\restriction_{[(\beta,\omega_1)]^3} \text{ for all } \alpha<\beta<\omega_1 \tag{3.26}$$

hence the data of such an $\mathtt{f}_0$ is entirely present (mod finite) in $\mathtt{f}_0(0)$. In other words, there exists an $\mathtt{f}_0$ is as in 3.21 if and only if for some $A \in \prod_{[\omega_1]^3}\mathbb{Z}$

$$A\restriction_{\beta\otimes[\omega_1]^2} =^* \mathtt{f}_1(0,\beta) \text{ for all } \beta<\omega_1 \tag{3.27}$$

(For the "if" direction, let $\mathtt{f}_0(\beta) = A\restriction_{[\beta,\omega_1)\otimes[\omega_1]^2}$.) Now let

$$\varphi^x_\beta(\alpha) := \mathtt{f}_1(0,\beta)\restriction_{\{\alpha\}\otimes[\omega_1]^2} \tag{3.28}$$

for $\alpha<\beta<\omega_1$. By equations 3.24 and 3.27,

Lemma 3.7.1. $\{\varphi^x_\beta \mid \beta\in\omega_1\}$ *is a nontrivial coherent family in the classical sense.*

(See again Definition 2.4.1 and just after for definitions.) The superscript "x" indicates, of course, that $\varphi^x_\beta(\alpha)$ outputs the $x=\alpha$ "slice" of $\mathtt{f}_1(0,\beta)$. Claim 3.7.1 holds equally for families $\{\varphi^y_\beta \mid \beta\in\omega_1\}$ and $\{\varphi^z_\beta \mid \beta\in\omega_1\}$, defined by

$$\varphi^y_\beta(\alpha) := \mathtt{f}_1(0,\beta)\restriction_{\omega_1\otimes\{\alpha\}\otimes\omega_1}, \text{ and} \tag{3.29}$$

$$\varphi^z_\beta(\alpha) := \mathtt{f}_1(0,\beta)\restriction_{[\omega_1]^2\otimes\{\alpha\}}, \text{ respectively.} \tag{3.30}$$

In the x and y cases, the codomain (the S, in other words, of Definition 2.4.1) may be uniformly viewed as $\bigoplus_{[\omega_1]^2}\mathbb{Z}$, or, hence, as $\bigoplus_{\omega_1}\mathbb{Z}$, simply. The $z=\alpha$ slices may be construed as smaller, being bounded by α — but for small codomain we can do much better: fix a bijection $\theta : \omega_1 \to [\omega_1]^3$. Then $E_\theta = \{\beta \mid \theta''\beta = [\beta]^3\}$ is a club subset of ω_1; for $\beta \in E_\theta$, let

$$\varphi^\theta_\beta(\alpha) := \begin{cases} 1 & \text{if } \theta(\alpha) \in \mathrm{supp}(\mathtt{f}_1(0,\beta)) \\ 0 & \text{otherwise} \end{cases} \tag{3.31}$$

Then

Lemma 3.7.2. $\{\varphi^\theta_\beta : \beta \to \mathbb{Z}_2 \mid \beta \in E_\theta\}$ *is a nontrivial coherent family of functions.*

One may, in short, derive nontrivial coherence in a variety of directions from that of $\mathtt{f}_1$ itself. The connection of $\mathtt{f}_1$ to signature walks phenomena only begins there. Consider the "outermost" slices of $\mathtt{f}_1(\alpha,\beta)$, for example: by equation 3.23,

$$\mathtt{f}_1(\alpha,\beta)\restriction_{\{\alpha\}\otimes[\omega_1]^2} = -\langle\alpha, C^\beta(\alpha), \beta\rangle - \langle\alpha, C^{C^\beta(\alpha)}(\alpha), C^\beta(\alpha)\rangle - \dots \tag{3.32}$$

$$\mathtt{f}_1(\alpha,\beta)\restriction_{[\omega_1]^2\otimes\{\beta\}} = -\langle\alpha, C^\beta(\alpha), \beta\rangle - \langle C^\beta(\alpha), C^\beta(C^\beta(\alpha)), \beta\rangle - \dots \tag{3.33}$$

The first, restricted to either the 2^{nd} or 3^{rd} coordinate, bears copies (minus the first or last element, respectively) of the walk from β down to $\alpha+1$. The second, similarly, is an image of the ladder on β above α. For limit β, of course, these ladders C_β are infinite; by equation 3.24, $\mathtt{f}_1(\alpha,\gamma)$ must contain all but finitely much of each of these C_β-images (where β ranges through $(\alpha,\gamma)\cap\mathrm{Lim}$). This is a requirement in some tension with equation 3.25 — a tension nontrivial coherence phenomena may be viewed as expressing (see particularly the functions $m(\cdot\,,\gamma)$, below).

It's clear that once β appears as 2^{nd} coordinate in a value $\langle\xi,\beta,\eta\rangle$ in the expansion of $\mathtt{f}_1(\alpha,\gamma)$, some tail of the sequence (3.33) will follow, by way of the term $\mathtt{f}_1(\xi,\beta)$. In other words, $\mathtt{f}_1(\alpha,\gamma)$ at some point "joins" the level $[\omega_1]^2 \otimes \{\beta\}$; it will, by definition, from above, i.e. via the expansion of some term in $\mathtt{f}_1(\alpha,\gamma)\restriction_{[\alpha,\beta)\otimes[(\beta,\omega_1)]^2}$. Note that

$$-\langle \xi_i^L, \xi_i^T, \eta_i \rangle$$

is in the latter only if

$$\xi_i^L < \beta < \xi_i^T = C^{\eta_i}(\xi_i^L) < \eta_i$$

and that in this case the indexing makes sense; there is, in other words, a *next* value in $\mathtt{f}_1(\alpha,\gamma)\restriction_{[\alpha,\beta)\otimes[[\beta,\omega_1)]^2}$, namely

$$-\langle \max(\xi_i^T, \max(C_{\xi_i^T} \cap \beta)), C^{\xi_i^T}(\max(\xi_i^L, \max(C_{\xi_i^T} \cap \beta))), \xi_i^T \rangle$$

which we denote $-\langle \xi_{i+1}^L, \xi_{i+1}^T, \eta_{i+1} \rangle$. Beginning with $\mathtt{f}_1(\alpha,\gamma)$, particularly when $\alpha = 0$, the right and left sides of this sequence (ending when the middle coordinate is β) are again familiar:

$$\begin{aligned} \eta_0 > \eta_1 > \cdots > \beta &= \\ \gamma > \xi_0^T > \cdots > \xi_k^T &= \mathrm{Tr}(\beta,\gamma), \text{ and} \qquad (3.34) \\ \xi_0^L \le \xi_1^L \le \cdots \le \xi_{k-1}^L &= \mathrm{L}(\beta,\gamma), \qquad (3.35) \end{aligned}$$

where

$$k = |\,\mathrm{supp}(\mathtt{f}_1(0,\gamma)\restriction_{[0,\beta)\otimes[\,[\beta,\omega_1)]^2})\,| = \rho_2(\beta,\gamma) \qquad (3.36)$$

(See again Section 2.3 for definitions). Varying $\beta \in (\alpha,\gamma)$, we see that all the entries of $\mathtt{f}_1(\alpha,\gamma)$ fall in some such sequence, so that $\mathtt{f}_1$ might fairly be viewed as a knitting together, simply, of the upper and lower traces, in a strikingly comprehensive fashion.

$\rho_0(\beta,\gamma)$, for example, and, hence $\rho_1(\beta,\gamma)$, are legible (by (3.34)) in the 2^{nd} coordinate of $\widetilde{\mathtt{f}}_1(0,\gamma)\restriction_{[0,\beta)\otimes[\,[\beta,\omega_1)]^2})$, where

$$\widetilde{\mathtt{f}}_1(\alpha,\beta)=\begin{cases} 0 & \text{if } \beta=\alpha+1 \\ -\langle\alpha,|\alpha\cap C_\beta|,\beta\rangle+\widetilde{\mathtt{f}}_1(\alpha,C^\beta(\alpha))+\widetilde{\mathtt{f}}_1(C^\beta(\alpha),\beta) & \text{otherwise}\end{cases}$$

The "materialization" of so much of the classical walks apparatus — ρ_0, ρ_1, ρ_2, Tr, L, for example, and the ladders themselves — as elementary spatial features of the $\mathtt{f}_1$ system may not itself be altogether surprising, given Definition 3.22. The *interesting* point is that $\mathtt{f}_1$ is only the first in an infinite series of such systems $\mathtt{f}_n$, of broadly similar spatial organization.

A spatial view, moreover, at times illuminates the classical: by (3.36), for example,

$$|\rho_2(\alpha,\gamma)-\rho_2(\alpha,\beta)|\leq|\,\mathrm{supp}(\mathtt{f}_1(0,\gamma)\restriction_{\beta\otimes[\omega_1]^2}-\mathtt{f}_1(0,\beta))\,| \tag{3.37}$$

for all $\alpha<\beta<\gamma<\omega_1$. In other words, for any such α,β,γ the difference between $\rho_2(\alpha,\gamma)$ and $\rho_2(\alpha,\beta)$ manifests as difference between $\mathtt{f}_1(0,\gamma)\restriction_{\beta\otimes[\omega_1]^2}$ and $\mathtt{f}_1(0,\beta)$ — which is finitely supported, by equation 3.24; this imposes a uniform bound. In strong senses, too, that difference in turn — or, even more plainly, the $\mathtt{f}_1(0,\gamma)-\mathtt{f}_1(0,\beta)+\mathtt{f}_1(\beta,\gamma)$ of equation 3.20 — is an aggregate of recognizably classical forms: of the upper and lower traces from γ down to β, and the ladders C_β and C_γ. We view this as some confirmation — (as if any more were needed) — that the classical approach was organized around the "right" questions.

We close this section by foregrounding a value somewhat more prominent in these framings than classically:

Consider again the function $\varphi^y_\gamma : \beta \mapsto \mathtt{f}_1(0,\gamma)\restriction_{\omega_1 \otimes \{\beta\} \otimes \omega_1}$. By (3.22), each β between 0 and γ appears exactly once, as y-coordinate, in the expansion of $\mathtt{f}_1(0,\gamma)$; hence $\varphi^y_\gamma(\beta) = \langle \xi, \beta, \zeta \rangle$ for some unique ξ and ζ. Denote that unique ξ by $m(\beta,\gamma)$. Note that wherever it's defined, $\max(\mathrm{L}(\beta,\gamma))$ equals $m(\beta,\gamma)$; in the framing of (3.35), $m(\beta,\gamma)$ is ξ^-_{k-1}, the x-coordinate of $\mathtt{f}_1(0,\gamma)$'s "arrival" to the limit level $[\omega_1]^2 \otimes \{\beta\}$. Hence for all $\beta < \gamma < \omega_1$,

1. $m(\cdot\,,\gamma) : \mathrm{Lim} \cap \gamma \to \omega_1$ is a regressive function,
2. $m(\cdot\,,\gamma) : \mathrm{Lim} \cap \gamma \to \omega_1$ is finite-to-one (by (3.25)), and
3. $m(\cdot\,,\gamma)\restriction_{\mathrm{Lim}\cap\beta} =^* m(\cdot\,,\beta)$ (by (3.20) and (3.23)).

By the Pressing Down Lemma, no $m : \mathrm{Lim} \cap \omega_1 \to \omega_1$ can simultaneously satisfy (1) and (2), above. Consequently, no $m : \omega_1 \to \omega_1$ trivializes $\{m(\cdot\,,\gamma) \mid \gamma \in \omega_1\}$:

Lemma 3.7.3. $\{m(\cdot\,,\gamma) \mid \gamma \in \omega_1\}$ *is a nontrivial coherent family. So, too, is the induced* $\{n(\cdot\,,\gamma) : \gamma \to \mathbb{N} \mid \gamma \in \omega_1\}$ *given by* $n(\,\cdot\,,\gamma) : \beta \mapsto |C_\beta \cap m(\beta,\gamma)|$.

A motif in classical arguments is that, for $\alpha < \beta < \gamma < \omega_1$,

$$\max(L(\beta,\gamma)) < \alpha \Rightarrow \beta \in \mathrm{Tr}(\alpha,\gamma).$$

The strengthening, here, to the statement

$$m(\beta,\gamma) < \alpha \text{ if and only if } \beta \in \mathrm{Tr}(\alpha,\gamma)$$

is equivalent to the identification, above, of $\mathrm{Tr}(\alpha,\gamma)$ and $\mathtt{f}_1(0,\gamma)\restriction_{[0,\alpha)\otimes[\,[\alpha,\omega_1)]^2}$.

3.8 A turn to ω_2, and ω_n

Consider now $\mathtt{f}_2 \in K^2(\mathbf{R}_3(\omega_2))$. This is a function $[\omega_2]^3 \to \prod_{[\omega_2]^4} \mathbb{Z}$ for which

$$\mathtt{f}_2(\beta,\gamma,\delta) - \mathtt{f}_2(\alpha,\gamma,\delta) + \mathtt{f}_2(\alpha,\beta,\delta) - \mathtt{f}_2(\alpha,\beta,\gamma) =^* 0 \text{ for all } \alpha < \beta < \gamma < \delta < \omega_2 \tag{3.38}$$

but for which no $g_1 \in K^1(\mathbf{R}_3(\omega_2))$ satisfies

$$g_1(\beta,\gamma) - g_1(\alpha,\gamma) + g_1(\alpha,\beta) =^* \mathtt{f}_2(\alpha,\beta,\gamma) \text{ for all } \alpha < \beta < \gamma < \omega_2 \tag{3.39}$$

Again the statement "$g_1 \in K^1(\mathbf{R}_3(\omega_2))$" abbreviates

$$g_1 : [\omega_2]^2 \to \prod_{[\omega_2]^4} \mathbb{Z} \text{ and } \operatorname{supp}(g_1(\beta,\gamma)) \subseteq [\,[\beta,\omega_2)\,]^4 \text{ for all } \beta < \gamma < \omega_2 \tag{3.40}$$

And again $\mathtt{f}_2$ admits a straightforward definition, that of (3.17) and (3.18):

$$\mathtt{f}_2(\alpha,\beta,\gamma) = \begin{cases} 0 & \text{if } \beta \in C_\gamma \text{ but} \\ & C^{\beta\gamma}(\alpha) \text{ is undefined} \\ -\langle \alpha, C^{\beta\gamma}(\alpha), \beta, \gamma\rangle + \mathtt{f}_2(C^{\beta\gamma}(\alpha),\beta,\gamma) & \\ +\mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha),\gamma) - \mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha),\beta) & \text{if } \beta \in C_\gamma \text{ and} \\ & C^{\beta\gamma}(\alpha) \text{ is defined} \\ \langle \alpha, \beta, C^{\gamma}(\beta), \gamma\rangle - \mathtt{f}_2(\beta, C^{\gamma}(\beta),\gamma) & \\ +\mathtt{f}_2(\alpha, C^{\gamma}(\beta),\gamma) + \mathtt{f}_2(\alpha,\beta, C^{\gamma}(\beta)) & \text{if } \beta \notin C_\gamma \end{cases} \tag{3.41}$$

(If $\beta \in C_\gamma$, then $C^{\beta\gamma}(\alpha)$ is undefined when $C_{\beta\gamma} \subseteq \alpha+1$. The case $\gamma = \beta+1$, wherein $C_{\beta\gamma} = \varnothing$, is an instance.) And it's immediate again, in turn, from this definition, that

$$\operatorname{supp}(\mathtt{f}_2(\alpha,\beta,\gamma)) \subseteq [\,[\alpha,\gamma]\,]^4 \tag{3.42}$$

for all $\alpha < \beta < \gamma < \omega_2$. Support considerations again afford us a reduction, as follows, of equation 3.38:

$$(\mathtt{f}_2(0,\beta,\gamma) - \mathtt{f}_2(0,\alpha,\gamma) + \mathtt{f}_2(0,\alpha,\beta))\restriction_{\alpha\otimes[\omega_2]^3} =^* 0 \text{ for all } \alpha < \beta < \gamma < \omega_2 \tag{3.43}$$

Now suppose that $g(0,\,\cdot\,) : \omega_2 \to \prod_{[\omega_2]^4} \mathbb{Z}$ satisfied

$$(g(0,\beta) - g(0,\alpha))\restriction_{\alpha\otimes[\omega_2]^3} =^* \mathtt{f}_2(0,\alpha,\beta)\restriction_{\alpha\otimes[\omega_2]^3} \quad \text{for all } \alpha < \beta < \omega_2 \tag{3.44}$$

Then

$$g_1(\alpha,\beta) = \begin{cases} g(0,\beta) & \text{if } \alpha = \beta \\ (g(0,\beta) - g(0,\alpha) + \mathtt{f}_2(0,\alpha,\beta))\restriction_{[\alpha,\omega_2)\otimes[\omega_2]^3} & \text{otherwise} \end{cases} \tag{3.45}$$

would define a g_1 which, by (3.40), cannot exist. Hence $g(0,\,\cdot\,)$ cannot exist, either. Let w, x, y, z denote the four coordinates of $[\omega_2]^4$ (viewed, as usual, as increasing 4-tuples of ordinals). Much as in the case of ω_1, we've just described a relation of non-2-trivial 2-coherence. In particular, let

$$\varphi^w_{\beta\gamma}(\alpha) := \mathtt{f}_2(0,\beta,\gamma)\restriction_{\{\alpha\}\otimes[\omega_2]^3} \tag{3.46}$$

for $\alpha < \beta < \gamma < \omega_2$. Then

Proposition 3.8.1. $\{\varphi^w_{\beta\gamma} : \beta \to \bigoplus_{\omega_2} \mathbb{Z} \mid \beta\gamma \in [\omega_2]^2\}$ *is a non-2-trivial 2-coherent family of functions.*

Proof. The slices $\varphi_{\beta\gamma}(\alpha)$ along $w = \alpha$, for $\alpha < \beta$, record $\mathtt{f}_2(0,\beta,\gamma)\restriction_{\beta\otimes[\omega_2]^3}$ entirely. Non-2-trivial 2-coherence therefore follows directly from the analysis above. In the lemma's statement, the codomain might more scrupulously have been indexed by

$[\omega_2]^3$: we've simply re-indexed by ω_2. The deeper matter is the direct sum, i.e., the assertion that the support of $\mathtt{f}_2(0,\beta,\gamma)\restriction_{\{\alpha\}\otimes[\omega_2]^3}$ is finite. This may be seen via the following:

Lemma 3.8.2. *Let $\trianglelefteq$ denote the product ordering of n-tuples of ordinals; that is, $\vec{\alpha} \trianglelefteq \vec{\beta}$ if and only if $\alpha_i \leq \beta_i$ for all $i < n$. Each new nontrivial step in the expansion of $\mathtt{f}_2(\alpha,\beta,\gamma)\restriction_{\{\alpha\}\otimes[\omega_2]^3}$ outputs a value $\triangleleft$-below the last one.*

Proof. This is a simple analysis of cases.

<u>Case 1</u>: $\mathtt{f}_2(\alpha,\beta,\gamma)$ is of the following form:

$$-\langle\alpha, C^{\beta\gamma}(\alpha), \beta, \gamma\rangle + \mathtt{f}_2(C^{\beta\gamma}(\alpha), \beta, \gamma) + \mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha), \gamma) - \mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha), \beta)$$

Hence $\mathtt{f}_2(\alpha,\beta,\gamma)\restriction_{\{\alpha\}\otimes[\omega_2]^3}$ is of the following form:

$$-\langle\alpha, C^{\beta\gamma}(\alpha), \beta, \gamma\rangle + \mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha), \gamma) - \mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha), \beta)$$

The term $\mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha), \gamma)$ can output the values $\langle\alpha, C^{C^{\beta\gamma}(\alpha)\gamma}(\alpha), C^{\beta\gamma}(\alpha), \gamma\rangle$ or 0. The nontrivial output is less than or equal to $\langle\alpha, C^{\beta\gamma}(\alpha), \beta, \gamma\rangle$ on every coordinate.

The term $\mathtt{f}_2(\alpha, C^{\beta\gamma}(\alpha), \beta)$ can output values of the form $\langle\alpha, C^{C^{\beta\gamma}(\alpha)\beta}(\alpha), C^{\beta\gamma}(\alpha), \beta\rangle$ or $\langle\alpha, C^{\beta\gamma}(\alpha), C^{\beta}(C^{\beta\gamma}(\alpha)), \beta\rangle$ or 0. The nontrivial outputs are each less than or equal to $\langle\alpha, C^{\beta\gamma}(\alpha), \beta, \gamma\rangle$ on every coordinate.

<u>Case 2</u>: $\mathtt{f}_2(\alpha,\beta,\gamma)$ is of the following form:

$$\langle\alpha, \beta, C^{\gamma}(\beta), \gamma\rangle - \mathtt{f}_2(\beta, C^{\gamma}(\beta), \gamma) + \mathtt{f}_2(\alpha, C^{\gamma}(\beta), \gamma) + \mathtt{f}_2(\alpha, \beta, C^{\gamma}(\beta))$$

Hence $\mathtt{f}_2(\alpha,\beta,\gamma)\restriction_{\{\alpha\}\otimes[\omega_2]^3}$ is of the following form:

$$\langle\alpha,\beta,C^\gamma(\beta),\gamma\rangle+\mathtt{f}_2(\alpha,C^\gamma(\beta),\gamma)+\mathtt{f}_2(\alpha,\beta,C^\gamma(\beta))$$

The term $\mathtt{f}_2(\alpha,C^\gamma(\beta),\gamma)$ can output values of the form $\langle\alpha,C^{C^\gamma(\beta)\gamma}(\alpha),C^\gamma(\beta),\gamma\rangle$ or 0. The nontrivial output is less than or equal to $\langle\alpha,\beta,C^\gamma(\beta),\gamma\rangle$ on every coordinate.

The term $\mathtt{f}_2(\alpha,\beta,C^\gamma(\beta))$ can output values of the form $\langle\alpha,\beta,C^{C^\gamma(\beta)}(\beta),C^\gamma(\beta)\rangle$ or $\langle\alpha,C^{\beta C^\gamma(\beta)}(\alpha),\beta,C^\gamma(\beta)\rangle$ or 0. The nontrivial outputs are each less than or equal to $\langle\alpha,C^{\beta\gamma}(\alpha),\beta,\gamma\rangle$ on every coordinate.

Cases 1 and 2 above are all the nontrivial possibilities. □

By the lemma, $\mathtt{f}_2(\alpha,\beta,\gamma)\restriction_{\{\alpha\}\otimes[\omega_2]^3}$ is finite, for each $\alpha<\beta<\gamma$ in ω_2. By a compactness argument, if there are infinitely many terms $\mathtt{f}_2(\alpha,\beta_i,\gamma_i)$ beginning with α in the expansion of $\mathtt{f}_2(0,\beta,\gamma)$, then infinitely many fall in the expansion of one of them. The lemma rules this out. Hence $\mathtt{f}_2(0,\beta,\gamma)\restriction_{\{\alpha\}\otimes[\omega_2]^3}$ is finite; the proposition follows. □

Corollary 3.8.3. $\check{\mathrm{H}}^2(\omega_2,\mathcal{D}_A)\neq 0$*, for* $A=\bigoplus_{\omega_2}\mathbb{Z}$.

The witness is the family $\{\varphi^w_{\beta\gamma}:\beta\to\bigoplus_{\omega_2}\mathbb{Z}\,|\,\beta\gamma\in[\omega_2]^2\}$. Witnesses to $\check{\mathrm{H}}^2(\omega_2,\mathcal{D}_A)\neq 0$ for smaller A, of course, are desirable. An ω_2 analogue of the family $\{\varphi^\theta_\beta:\beta\to\mathbb{Z}_2\,|\,\beta\in C_\theta\}$ as in Lemma 3.7.2 is a natural place to look. The author's given considerable energy to the question of whether such a family must be non-2-trivial. Certainly its non-2-triviality is consistent with ZFC; emergent in that search, moreover, are combinatorial principles on ω_2 of some independent interest. The ZFC non-2-triviality of such a family, though, remains obscure. In general, it appears that if these sorts of "cheaper" derivations of non-2-coherence from $\mathtt{f}_2$ are non-2-trivial, there's nothing cheap about the reason — with $\{\varphi^w_{\beta\gamma} : \beta \to \bigoplus_{\omega_2} \mathbb{Z} \,|\, \beta\gamma \in [\omega_2]^2\}$ the one clear exception. This appears to be the case for higher n as well. In short:

1. First-coordinate slices of the $\mathtt{f}_n$-systems provide witnesses to higher-n analogues of Corollary 3.8.3.
2. The description smaller-codomain ZFC non-n-trivial n-coherent families awaits our deeper understanding of the functions $\mathtt{f}_n$, and of *their* principles of non-n-triviality. A *systemic* understanding of this sort would likely be a theory of higher-order walks.

In the remainder of this section, we'll argue (1). We'll close out this chapter in the following section with a discussion of (2).

An organizing question for $\mathtt{f}_2$, no less than $\mathtt{f}_1$, is: when does the expansion of a function $\mathtt{f}_2(0, \gamma, \delta)$ arrive to the level $z = \beta$? One reason for that question's importance is that, for both $n = 1$ and $n = 2$, it's in the z-hyperplanes that non-$(n-1)$-triviality appears. For δ of uncountable cofinality, for example, $\mathtt{f}_2(0, \cdot\,, \delta)\restriction_{z=\delta}$ is a nontrivial coherent family, in precisely the sense that $\{\mathtt{f}_1(0, \beta) \,|\, \beta \in \omega_1\}$ is. $\mathtt{f}_2(0, \cdot\,, \delta)\restriction_{z=\delta}$ is, more precisely, $\{\mathtt{f}_1(0, \beta) \,|\, \beta \in \omega_1\}$ "relativized" to the ladder C_δ, which suggests the notation $\mathtt{f}^\delta_1(\cdot\,, \cdot)$, defined so that

1. $\mathtt{f}_2(\alpha, \beta, \delta) = \langle \mathtt{f}^\delta_1(\alpha, \beta), \delta \rangle - \mathtt{f}''_2 \mathtt{f}^\delta_1(\alpha, \beta)$, for all $\alpha < \beta < \delta < \omega_2$, and

2. $\mathtt{f}_1^{\omega_1}(\,\cdot\,,\,\cdot\,) = \mathtt{f}_1(\,\cdot\,,\,\cdot\,)$.

More generally, define $\mathtt{f}_n^\delta$ so that

1. $\mathtt{f}_n(\vec{\alpha},\delta) = (-1)^n\langle \mathtt{f}_{n-1}^\delta(\vec{\alpha}),\delta\rangle - \mathtt{f}_n''\mathtt{f}_{n-1}^\delta(\vec{\alpha})$, and
2. $\mathtt{f}_n^{\omega_n}(\,\cdot\,) = \mathtt{f}_n(\,\cdot\,)$.

This facilitates inductive arguments, such as that for the following:

Theorem 3.8.4. *$\check{\mathrm{H}}^n(\omega_n,\mathcal{D}_A)\neq 0$, for $A=\bigoplus_{\omega_n}\mathbb{Z}$, for all $n\geq 0$.*

Proof. Again our witness is $\Phi_n = \{\varphi_{\vec{\beta}} : \beta_0 \to \bigoplus_{\omega_n}\mathbb{Z} \,|\, \vec{\beta}\in[\omega_n]^n\}$, where

$$\varphi_{\vec{\beta}} : \alpha \mapsto \mathtt{f}_n(0,\vec{\beta})\restriction_{\{\alpha\}\otimes[\omega_n]^{n+1}}$$

As in the cases $n=1$ and $n=2$, both the n-coherence and non-n-triviality of Φ derive in a straightforward fashion from the $(n+1)$-coherence and non-$(n+1)$-triviality of the function $\mathtt{f}_n$. What requires argument is the assertion that $\mathtt{f}_n(0,\vec{\beta})\restriction_{\{\alpha\}\otimes[\omega_n]^{n+1}}$, or more generally $\mathtt{f}_n(\vec{\gamma})\restriction_{\{\alpha\}\otimes[\omega_n]^{n+1}}$ with $\alpha\in\omega_n$ and $\vec{\gamma}\in[\omega_n]^{n+1}$, has finite support. This we argue by induction on n. The base case $n=1$ follows from the well-foundedness of the ordinals, and in fact we verified the case $n=2$ in the course of the proof of Proposition 3.8.1. If the assertion holds at n it holds of all relativized $\mathtt{f}_n^\delta$ as well. Then for any $\alpha\in\omega_{n+1}$ and $(\vec{\gamma},\delta)\in[\omega_{n+1}]^{n+2}$, by the above definition $\mathtt{f}_{n+1}(\vec{\gamma},\delta)\restriction_{\{\alpha\}\otimes[\omega_{n+1}]^{n+2}}$ equals

$$(-1)^{n+1}\langle \mathtt{f}_n^\delta(\vec{\gamma})\restriction_{\{\alpha\}\otimes[\omega_{n+1}]^{n+1}},\delta\rangle - (\mathtt{f}_{n+1}''\mathtt{f}_n^\delta(\vec{\gamma}))\restriction_{\{\alpha\}\otimes[\omega_{n+1}]^{n+1}} \tag{3.47}$$

The point is this: by the inductive hypothesis, $\mathtt{f}_n^\delta(\vec{\gamma})\restriction_{\{\alpha\}\otimes[\omega_{n+1}]^{n+1}}$ contributes only finitely many values to the left- or right-hand sides of (3.47). From these inputs, terms on the right-hand side will again output only finitely many values with first coordinate α, but on lower z-levels than δ. This association of z-levels to steps defines a finitely branching downward growing tree in the ordinals, which by well-foundedness must be finite. This concludes the induction step. □

3.9 Higher rho functions

For higher n the $\mathtt{f}_n$ functions grow ugly, unwieldy, cumbrous with redundant information, in increasing contrast to the elegance and concision of classical walks. It's no less clear, at the same time, that they encode rich and intricate combinatorial phenomena. We therefore close this chapter with a reduction of these functions to what seems to be their essential information. It's this we propose as the higher-degree walk.

We work from the view — confirmed, in the case of $n = 1$ — that the walk from $\vec{\beta} \in [\omega_n]^n$ down to $\alpha < \beta_0$ should record those terms of $\mathtt{f}_n(0, \vec{\beta})$ with a formal possibility of bearing on the level $[\omega_n]^{n+1} \otimes \{\alpha\}$. The guiding form, in other words, is $\mathtt{f}_n(0, \vec{\beta})\restriction_{\alpha\otimes[[\alpha,\omega_n)]^{n+1}}$. As in the case of $n = 1$, these "walks," with α ranging in the interval $(0, \beta_n]$, largely constitute $\mathtt{f}_n(0, \vec{\beta})$. For higher n, however, they involve two additional considerations:

1. Sign. Recall that the values of $\mathtt{f}_1$ are all negative. For higher $\mathtt{f}_n$, sign plays a far more variable and meaningful role. This again evokes geometric contexts, in which orientation only assumes its full complexity or significance in dimensions greater than two.

2. The branching. When $\alpha < \beta_0$, the term $\mathtt{f}_n(0, \vec{\beta}) \restriction_{\alpha \otimes [[\alpha, \omega_n)]^{n+1}}$ outputs $n+1$ terms; only n of these have any real possibility of being on the level $z = \alpha$. If the walk from β down to α is, intuitively, a *path* tracking the terms of $\mathtt{f}_1(0, \beta) \restriction_{\alpha \otimes [[\alpha, \omega_n)]^2}$, higher "walks" should be analogously viewed as *n-branching trees.*

The naive approach, therefore, is to record the data of (1) and (2) data alongside the associated ordinals, guided by the defining formulae (3.17) and (3.18) for $\mathtt{f}_n$. In the case of $n = 2$, a recursive formula paralleling that of $\mathrm{Tr}(\,\cdot\,,\,\cdot\,)$ then takes the following form:

Definition 3.9.1. The function Tr^2 takes as input a tuple of the form $(sign, \sigma, \alpha, \beta, \gamma)$ and outputs a set of triples of the form $(sign, \sigma, \xi)$. Here (α, β, γ) is an increasing sequence of ordinals and σ is an element of $2^{<\omega}$ and *sign* is a plus or a minus.

$$\mathrm{Tr}^2(\pm, \sigma, \alpha, \beta, \gamma) =$$

Case: $\beta \in C_\gamma$:

$$\begin{aligned} &\{\,(\mp, \sigma, \min(C_{\beta\gamma} \backslash \alpha)\,)\} \\ &\cup \operatorname{Tr}^2(\pm, \sigma^\frown 0, \alpha, \min(C_{\beta\gamma} \backslash \alpha), \gamma) \\ &\cup \operatorname{Tr}^2(\mp, \sigma^\frown 1, \alpha, \min(C_{\beta\gamma} \backslash \alpha), \beta) \end{aligned}$$

Case: $\beta \notin C_\gamma$:

$$\begin{aligned} &\{\,(\pm, \sigma, \min(C_\gamma \backslash \beta))\,\} \\ &\cup \operatorname{Tr}^2(\pm, \sigma^\frown 0, \alpha, \min(C_\gamma \backslash \beta), \gamma) \\ &\cup \operatorname{Tr}^2(\pm, \sigma^\frown 1, \alpha, \beta, \min(C_\gamma \backslash \beta)) \end{aligned}$$

If $\beta = \alpha$ or $\beta = \gamma$, then let $\operatorname{Tr}^2(\pm, \sigma, \alpha, \beta, \gamma) = \varnothing$.

On ω_2, then, Tr^2 fully encodes the non-2-trivial 2-coherent functions $\mathtt{f}_2$. Higher Tr^n would be similarly defined, with the modification that $\sigma \in n^{<\omega}$ and $\alpha, \beta, \gamma, \ldots$ should be increasing $(n+1)$-tuples of ordinals in the input. Formulae (3.17) and (3.18) would continue to shape the output. This framework, unlike that of the $\mathtt{f}_n$ functions, recovers some of the clarity and ease of classical walks computations; again the process is a rhythm of two natural questions:

1. Is β in C_γ?
2. What is $\min(C_\gamma \backslash \beta)$ (or $\min(C_{\beta\gamma} \backslash \alpha)$, as the case may be)?

Finally, consider the case of $n = 1$. In this case, σ is an element of $1^{<\omega}$:

$$\operatorname{Tr}^1(\pm, \sigma, \alpha, \beta) = \{\,(\mp, \sigma, \min(C_\beta \backslash \alpha))\,\} \cup \operatorname{Tr}^1(\pm, \sigma^\frown 0, \alpha, \min(C_\beta \backslash \alpha))$$

This is visibly equivalent to the classical upper trace; there, the data of orientation ($\pm$), being constantly negative, had gone without saying. The branching data (σ) had appeared as the indices of the elements of the upper trace.

In the case $\beta \in C_\gamma$ of Definition 3.9.1, with $\alpha, \beta, \gamma \in \omega_2$, the cardinality of $\alpha \cap C_{\beta\gamma}$ is a natural number; such considerations suggest conversions of Tr^n to functions ρ_0^n in which the ordinal-values of Tr^n are replaced by natural numbers. In these cases, unlike in the classical case, some information may be lost; still, these forms are intriguing and strongly merit further investigation. While the classical ρ_0 output finite strings of natural numbers, for example, the output of ρ_0^n may be thought of finite $\leq n$-branching trees, whose nodes are labeled by integers.

Our more immediate interest, though, is ρ_2^n. The naive definition would be

$$\rho_2^n(\alpha, \beta, \gamma) = |\mathrm{Tr}^n(+, \varnothing, \alpha, \beta, \gamma)| \tag{3.48}$$

The correct definition, though, appears to be the following:

Definition 3.9.2. Let $\mathrm{neg}(\mathrm{Tr}^n(+, \varnothing, \alpha, \beta, \gamma))$ denote the number of minuses appearing in $\mathrm{Tr}^n(+, \varnothing, \alpha, \beta, \gamma)$. Let $\mathrm{pos}(\mathrm{Tr}^n(+, \varnothing, \alpha, \beta, \gamma))$ denote the number of pluses appearing in $\mathrm{Tr}^n(+, \varnothing, \alpha, \beta, \gamma)$. Let

$$\rho_2^n(\alpha, \beta, \gamma) := \mathrm{neg}(\mathrm{Tr}^n(+, \varnothing, \alpha, \beta, \gamma)) - \mathrm{pos}(\mathrm{Tr}^n(+, \varnothing, \alpha, \beta, \gamma)) \tag{3.49}$$

Unlike in the naive attempt, in this definition

$$|\rho_2^2(\alpha, \gamma, \delta) - \rho_2^2(\alpha, \beta, \delta) + \rho_2^2(\alpha, \beta, \gamma)| \leq k(\beta, \gamma, \delta) \quad \text{for all } \alpha < \beta \tag{3.50}$$

where

$$k(\beta,\gamma,\delta) = |\text{supp}(f_2(0,\gamma,\delta) - f_2(0,\beta,\delta) + f_2(0,\beta,\gamma))\restriction_{\beta\otimes[\omega_2]^3}|$$

This ensures that this difference is, in fact, a locally constant function of α, so that ρ_2^2 defines a 2-cocycle in $\mathcal{L}(\mathcal{U}_{\omega_2}, \mathbb{Z}_d)$. Similarly for higher ρ_2^n. The question of whether Definition 3.9.2 *really is* the right generalization of ρ_2 hangs then on the question of whether ρ_2^n defines an n-coboundary, i.e., of whether it's n-trivial (modulo locally constant functions). We close this chapter with the conjecture that it is non-n-trivial and, hence, is the generalization that we're after.

Conjecture 3.9.3. *ρ_2^n defines a non-n-trivial function, modulo locally constant functions, on ω_n. In consequence, it witnesses the fact that $\check{\mathrm{H}}^n(\omega_n, \mathbb{Z}_d) \neq 0$.*

CHAPTER 4

STRONG HOMOLOGY: HISTORY, INDEPENDENCE PHENOMENA, OPEN QUESTIONS.

4.1 Introduction

The strong homology theory $\overline{H}_*$, defined for all topological spaces, has the following desirable properties:

1. It satisfies all the Eilenberg-Steenrod axioms on paracompact pairs (X, A).
2. It is strong shape invariant.
3. It is a Steenrod-type homology theory (and therefore Alexander dual to $\check{H}^*$); it satisfies, in other words, two of the three axioms Milnor proposed to supplement Eilenberg and Steenrod's ([30], [31]; see [29]).

It remains an open question on what class of spaces it may satisfy the third of those axioms, *additivity*, the condition that every mapping

$$i : \bigoplus_A \overline{H}_p(X_\alpha) \to \overline{H}_p(\coprod_A X_\alpha)$$

induced by inclusion maps $i_\alpha : X_\alpha \hookrightarrow \coprod_A X_\alpha$ be an isomorphism.

It was in investigation of this question that Mardešič and Prasolov computed the strong homology of $Y^{(k)}$, the topological sum of countably many k-dimensional

Hawaiian earrings. They showed, in particular, that $\overline{H}_p(Y^{(k)}) = \lim^{k-p}\mathbf{A}$ for $0 < p < k$, where $\mathbf{A}$ is an abelian pro-group indexed by $\mathbb{N}^{\mathbb{N}}$ (see below). For a single k-dimensional Hawaiian earring $X^{(k)}$, $\overline{H}_p(X^{(k)}) = 0$ for $0 < p < k$; thus additivity requires at least that $\lim^n \mathbf{A} = 0$ for $n > 0$. Mardešič and Prasolov then showed that the continuum hypothesis implies that $\lim^1 \mathbf{A} \neq 0$ [28]. Shortly thereafter, Dow, Simon, and Vaughan showed that the Proper Forcing Axiom (PFA) implies that $\lim^1 \mathbf{A} = 0$ and, hence, that the vanishing of $\lim^1 \mathbf{A}$ is independent of the axioms of ZFC [12]. This vanishing, in fact, is a question of broad interest in its own right; see [41], for instance, and the discussion therein.

In this chapter we extend those investigations. The chapter is largely self-contained, even to the point of recalling definitions from Chapter 2. We'll show the vanishing of $\lim^2 \mathbf{A}$ also to be independent of the axioms of ZFC and characterize, more generally, the higher $\lim^n \mathbf{A}$. In particular, we show that, under PFA, strong homology is not additive, not even on the category of, e.g., closed subspaces of $\mathbb{R}^4$ (our witness in this case is $\overline{H}_1(Y^{(3)})$). In Section 4.3, for κ infinite, we let $\mathbf{A}_\kappa$ denote a pro-group analogous to $\mathbf{A}$ but indexed by $\mathbb{N}^\kappa$; we show $\lim^1 \mathbf{A}_\kappa = 0$ if and only if $\lim^1 \mathbf{A} = 0$. Extending, as it does, the topological significance of the system $\mathbf{A}$, this is the main theorem of the paper. In Section 4.4, we list some open problems.

In this section, we define our notation, the system $\mathbf{A}$, and briefly review the derived functors $\lim^n$ of lim. This chapter aims to interest readers in both homological algebra and set theory, and therefore — with a few mild exceptions — assumes no more than a basic knowledge of either. In particular, no knowledge of forcing is presumed; the reader need only understand that the Proper Forcing Axiom, $\diamondsuit(S_1^2)$, $\mathrm{MA}_{\aleph_1}$, wKH, and $\mathfrak{d} = \aleph_1$ (or $\aleph_2$) are prominent set-theoretic hypotheses independent of the axioms of ZFC. For more on the Proper Forcing Axiom, see in particular [34]. For more on set theory generally, see [19] or [24]. For further on lim and its derived functors, see ([29] §11) and [20].

Our inverse systems all will consist of abelian groups X_f and "bonding" homomorphisms $p_{fg} : X_g \to X_f$ for every $f \leq g$. Our index-set will typically be $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$, ordered coordinatewise: $f \leq g$ if and only if $f(i) \leq g(i)$ for all $i \in \mathbb{N}$. Our particular focus is $\mathbf{A} = (A_f, p_{fg}, \mathcal{N})$, where

$$A_f = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}^{f(i)}$$

with projection mappings p_{fg}. Relatedly, $\mathbf{B} = (B_f, p_{fg}, \mathcal{N})$, where

$$B_f = \prod_{i \in \mathbb{N}} \mathbb{Z}^{f(i)}$$

We consider only level morphisms $\mathbf{F} : \mathbf{X} \to \mathbf{Y}$ among such systems, i.e., collections of functions $F_f : X_f \to Y_f$ which commute with all the bonding maps. Likewise, terms of the quotient $\mathbf{Y}/\mathbf{X}$ are of the form Y_f/X_f, so that

$$0 \to \mathbf{A} \xrightarrow{\mathbf{I}} \mathbf{B} \xrightarrow{\mathbf{Q}} \mathbf{B}/\mathbf{A} \to 0 \tag{4.1}$$

for example, is exact. In the language of category theory, we study the abelian category $\mathcal{A}b^{\mathcal{N}}$.

An abelian group X together with $\mathbf{p} = \{p_f : X \to X_f \mid f \in \mathcal{N}\}$ is an inverse limit of $\mathbf{X}$ if

$$p_f = p_{fg} p_g \text{ for all } f \leq g \in \mathcal{N} \tag{4.2}$$

and for any $(Y, \mathbf{q})$ satisfying (4.2) there exists a unique $q : Y \to X$ such that $\mathbf{p}q = \mathbf{q}$. Such an X and $\mathbf{p}$ are unique up to isomorphism; we henceforth write $X = \lim \mathbf{X}$ for the group alone. X admits the following description:

$$\lim \mathbf{X} = \{\langle x_f \rangle \in \prod_{f \in \mathcal{N}} X_f \,|\, p_{fg}(x_g) = x_f \text{ for all } f \leq g \in \mathcal{N}\} \tag{4.3}$$

Note that

$$\lim \mathbf{B} \cong \prod_{i \in \mathbb{N}} \prod_{j \in \mathbb{N}} \mathbb{Z}$$

$$\lim \mathbf{A} \cong \bigoplus_{i \in \mathbb{N}} \prod_{j \in \mathbb{N}} \mathbb{Z}$$

Returning to (4.3), for $\mathbf{F} : \mathbf{X} \to \mathbf{Y}$, define $\lim \mathbf{F} : \lim \mathbf{X} \to \lim \mathbf{Y}$ as the induced mapping of products. We define thereby a functor $\lim : \mathcal{A}b^{\mathcal{N}} \to \mathcal{A}b$. We are interested in the following phenomenon: lim applied to sequence (4.1), for example, may fail to be exact. More precisely, lim is left exact: $\lim \mathbf{I}$ will be injective, but $\lim \mathbf{Q}$ may fail to be surjective, to a degree the long exact sequence

$$0 \to \lim \mathbf{A} \xrightarrow{\lim \mathbf{I}} \lim \mathbf{B} \xrightarrow{\lim \mathbf{Q}} \lim \mathbf{B}/\mathbf{A} \xrightarrow{\theta_0} \lim{}^1 \mathbf{A} \xrightarrow{\lim^1 \mathbf{I}} \lim{}^1 \mathbf{B} \dots \tag{4.4}$$

in some sense measures. The non-exactness of lim induces, in other words, a sequence of derived functors $\lim^n$ connected, for any short exact sequence in $\mathcal{A}b^{\mathcal{N}}$, by a long exact sequence of abelian groups of the above form, with connecting transformations θ_n. These functors $\lim^n$, like lim, admit explicit description; see the proof of Theorem 4.1.6, below. From this description, the reader may verify the following:

(i) For any constant system $\mathbf{X} = (X_f, p_{fg}, \mathcal{N})$, i.e., any system with $X_f = X$ and $p_{fg} = id$ for all $f \leq g \in \mathcal{N}$, $\lim^n \mathbf{X} = 0$ for $n \geq 1$.

(ii) $\lim^n \mathbf{B} = 0$ for $n \geq 1$.

Returning to (4.4): by (ii), $\lim^1 \mathbf{A} = 0$ if and only if $\lim \mathbf{Q}$ is surjective. To better articulate that equivalence, we introduce the following conventions, basic to all that follows:

For $f \in \mathcal{N}$, let $I_f = \{(i,j) \,|\, j \leq f(i)\}$. For $f, g \in \mathcal{N}$ write $f <^* g$ if $\{i \,|\, f(i) \not< g(i)\}$ is finite. Write $f \leq^* g$ if $\{i \,|\, f(i) \not\leq g(i)\}$ is finite or, equivalently, if $I_f \subseteq^* I_g$. ϕ_f will denote a function of the form $I_f \to \mathbb{Z}$. Write $\phi =^\star \psi$ if $\{x \in \mathrm{dom}(\phi) \cap \mathrm{dom}(\psi) \,|\, \phi(x) \neq \psi(x)\}$ is finite; note that this is not, in general, an equivalence relation. A collection $\Phi = \{\phi_f \,|\, f \in \mathcal{N}\}$ is *coherent* if $\phi_f =^\star \phi_g$ for all $f, g \in \mathcal{N}$. Φ is *trivial* if there exists $\phi : \mathbb{N}^2 \to \mathbb{Z}$ such that $\phi =^\star \phi_f$ for all $f \in \mathcal{N}$. We may view any ϕ_f as an element of B_f; write $[\phi_f]$ for its image in B_f/A_f. We may view Φ, likewise, as an element of $\prod_{\mathcal{N}} B_f$; writing $[\Phi]$ for $\{[\phi_f] \,|\, f \in \mathcal{N}\}$, then,

$$\lim \mathbf{B}/\mathbf{A} \cong \{[\Phi] \,|\, \Phi \text{ is coherent}\}$$

Hence $\lim \mathbf{Q}$ is surjective if and only if every coherent $[\Phi]$ equals $\{[\phi\restriction_{I_f}] \,|\, f \in \mathcal{N}\}$ for some $\phi : \mathbb{N}^2 \to \mathbb{Z}$ in $\lim \mathbf{B}$. In other words,

Theorem 4.1.1. *[28] $\lim^1 \mathbf{A} = 0$ if and only if every coherent family of functions $\Phi = \{\phi_f \,|\, f \in \mathcal{N}\}$ is trivial.*

It's this observation we generalize in section 3.

We recall, finally, the following notions from set theory. The cofinality of a partial order P is

$$\mathrm{cf}(P) = \min\{|Q| \mid \text{for all } p \in P \text{ there exists a } q \in Q \text{ with } q \geq p\}$$

The cofinality of an inverse system is the cofinality of its index-set. Observe that $\mathrm{cf}(\mathcal{N}, <) = \mathrm{cf}(\mathcal{N}, <^*)$. We write $\mathfrak{d}$ for either.

$$\mathfrak{b} = \min\{|\mathcal{F}| \mid \text{for all } g \in \mathcal{N} \text{ there exists an } f \in \mathcal{F} \text{ with } f \not<^* g\}$$

Symbols α, β, ξ denote ordinals; κ denotes a cardinal. $[\kappa]^{<\kappa} = \{y \subset \kappa \mid \kappa > |y|\}$. For $A \subseteq \mathrm{dom}(f)$, $f''A = \{f(a) \mid a \in A\}$. A cofinal subset C of β is *club* if it is closed in β under the topology induced by the membership relation. $S \subseteq \beta$ is *stationary* if it intersects all club subsets of β.

Definition 4.1.2. $\diamondsuit(S^2_1)$ is the assertion that there exists a family $\mathcal{S} = \{S_\beta \mid \beta < \omega_2 \text{ and } \mathrm{cof}(\beta) = \aleph_1\}$ such that, for any $A \subset \omega_2$, the set $\{\beta \mid A \cap \beta = S_\beta\}$ is stationary.

The reader may verify that the intersection of two club subsets of β is a club and, hence, that the intersection of a club and a stationary subset of β is stationary; these facts and the straightforward implication $\diamondsuit(S^2_1) \Rightarrow 2^{\aleph_0} \leq \aleph_2$ play a role in the proof of Theorem 4.2.1.

Let $\mathcal{N}^{[n]} = \{(f_0, \ldots, f_{n-1}) \in \mathcal{N}^n \mid f_i \leq f_j \text{ for all } i < j\}$ for $n > 0$, and let $\mathcal{N}^{[0]} = \{\varnothing\}$. Let $\vec{f}^{\,i}$ denote the $(n-1)$-tuple obtained by deleting f_i from $\vec{f} \in \mathcal{N}^{[n]}$; $\phi_{\vec{f}}$ will denote a function of the form $I_{f_0} \to \mathbb{Z}$ unless $\vec{f} = \varnothing$, in which case $\phi_{\vec{f}} : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$.

Definition 4.1.3. A collection $\Phi = \{\phi_{\vec{f}} \,|\, \vec{f} \in \mathcal{N}^{[n]}\}$ is *n-coherent* if, for all $\vec{f} \in \mathcal{N}^{[n+1]}$,

$$\phi_{\vec{f}^0} \restriction_{I_{f_0}} + \sum_{i=1}^{n} (-1)^i \phi_{\vec{f}^i} =^{\star} 0.$$

For readability, we henceforth write such sums, simply, as $\sum_{i=0}^{n} (-1)^i \phi_{\vec{f}^i}$.

Definition 4.1.4. Φ is *n-trivial* if there exists $\{\psi_{\vec{f}} \,|\, \vec{f} \in \mathcal{N}^{[n-1]}\}$ such that, for all $\vec{f} \in \mathcal{N}^{[n]}$,

$$\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{f}^i} =^{\star} \phi_{\vec{f}}.$$

Observe that every n-trivial Φ is n-coherent. The following instance will play a role in the proof of Theorem 4.1.

Example 4.1.5. If $\Phi_1 = \{\phi_f \,|\, f \in \mathcal{N}\}$ 2-trivializes $\Phi_2 = \{\phi_{fg} \,|\, fg \in \mathcal{N}^{[2]}\}$, then for all $(f, g, h) \in \mathcal{N}^{[3]}$

$$\phi_{fg} - \phi_{fh} + \phi_{gh} =^{\star} \phi_g - \phi_f - (\phi_h - \phi_f) + \phi_h - \phi_g =^{\star} 0$$

i.e., Φ_2 is 2-coherent. (Here again for readability we've suppressed the obvious restrictions.)

The question of whether every n-coherent Φ is n-trivial is subtler.

Theorem 4.1.6. *For $n > 0$, $\lim^n \mathbf{A} = 0$ if and only if every n-coherent $\Phi = \{\phi_{\vec{f}} \,|\, \vec{f} \in \mathcal{N}^{[n]}\}$ is n-trivial.*

Proof. Define cochain complex $\mathcal{K}(\mathbf{B}) : 0 \to K^0(\mathbf{B}) \to K^1(\mathbf{B}) \to \dots$ by

$$K^n(\mathbf{B}) = \prod_{\vec{f} \in \mathcal{N}^{[n+1]}} B_{f_0}$$

with differential $d^n : K^{n-1}(\mathbf{B}) \to K^n(\mathbf{B})$ defined, for $n > 0$, by

$$(d^n\Psi)(\vec{f}) = \sum_{i=0}^{n} (-1)^i \Psi(\vec{f^i})$$

Analogously define $\mathcal{K}(\mathbf{A})$, a subcomplex of $\mathcal{K}(\mathbf{B})$. View any Φ as in the statement of the theorem as an element of $K^{n-1}(\mathbf{B})$; observe that Φ is n-coherent if and only if $d^n\Phi \in K^n(\mathbf{A})$, and is n-trivial if and only if $\Phi - d^{n-1}\Psi \in K^{n-1}(\mathbf{A})$ for some $\Psi \in K^{n-2}(\mathbf{B})$.

Nöbeling and Roos independently established that, in general, $\lim^n \mathbf{X} \cong H^n\mathcal{K}(\mathbf{X})$ (see [29] for proof; the reader may more immediately verify that $H^0(\mathcal{K}(\mathbf{X})) = \lim \mathbf{X}$). In particular, $\lim^n \mathbf{A} = 0$ if and only if, in $\mathcal{K}(\mathbf{A})$, every n-cocycle is an n-coboundary. Assume the latter, and take $n \geq 2$ (the case $n = 1$ was Theorem 4.1.1): if Φ is n-coherent, then $d^n\Phi \in K^n(\mathbf{A})$ is an n-cocycle and hence, by assumption, equals $d^n\Upsilon$ for some $\Upsilon \in K^{n-1}(\mathbf{A})$. Since $\lim^{n-1} \mathbf{B} = 0$, cocycle $(\Phi - \Upsilon)$ equals $d^{n-1}\Psi$ for some $\Psi \in K^{n-2}(\mathbf{B})$; in other words, $\Phi - d^{n-1}\Psi \in K^{n-1}(\mathbf{A})$, i.e., Φ is n-trivial.

On the other hand, if every n-coherent Φ is n-trivial, take n-cocycle $\Upsilon \in K^n(\mathbf{A})$. Since $\lim^{n-1} \mathbf{B} = 0$, $\Upsilon = d^n\Phi$ for some $\Phi \in K^{n-1}(\mathbf{B})$. Φ is n-coherent, so by assumption, $\Phi - d^{n-1}\Psi \in K^{n-1}(\mathbf{A})$ for some $\Psi \in K^{n-2}(\mathbf{B})$; hence $\Upsilon = d^n(\Phi - d^{n-1}\Psi)$ is an n-coboundary in $\mathcal{K}(\mathbf{A})$. □

We will sometimes consider systems indexed by orders extending or contained in

$\mathcal{N}$; the appropriate modification of definitions should in such cases be obvious.

Early indications of the relevance of set-theoretic considerations to higher derived limits were the following:

Theorem 4.1.7. *[15] For any inverse system* $\mathbf{X}$ *of cofinality* $\aleph_k$*, lim*$^n\,\mathbf{X} = 0$ *for all* $n \geq k+2$.

Theorem 4.1.8. *[32] For every* $k \geq 0$ *there exists an inverse system* $\mathbf{X}$ *of cofinality* $\aleph_k$ *with lim*$^{k+1}\,\mathbf{X} \neq 0$.

Corollary 4.1.9. *If* $\mathfrak{d} = \aleph_k$*, then lim*$^n\,\mathbf{A} = 0$ *for all* $n \geq k+1$.

Proof. Immediate, by Theorem 4.1.7, for $n > k+1$. Let $\mathbf{D} = (D_f, p^d_{fg}, \mathcal{N})$, with $D_f = \{\phi : \mathbb{N}^2 \backslash I_f \to \mathbb{Z} \,|\, \mathrm{supp}(\phi) \text{ is finite}\}$ and p^d_{fg} be the inclusion map; let $\mathbf{E} = (E_f, p^e_{fg}, \mathcal{N})$, with $E_f = \{\phi : \mathbb{N}^2 \to \mathbb{Z} \,|\, \mathrm{supp}(\phi) \text{ is finite}\}$ and p^e_{fg} the identity. Form short exact sequence

$$0 \to \mathbf{D} \to \mathbf{E} \to \mathbf{A} \to 0$$

inducing long exact sequence

$$\cdots \to \lim{}^{k+1}\,\mathbf{E} \to \lim{}^{k+1}\,\mathbf{A} \to \lim{}^{k+2}\,\mathbf{D} \to \ldots$$

As noted, for $k \geq 0$, $\lim^{k+1}\mathbf{E} = 0$, so by Theorem 4.1.7, $0 = \lim^{k+2}\mathbf{D} = \lim^{k+1}\mathbf{A}$. □

By the corollary, together with the following theorem, $\mathfrak{d} = \aleph_1$ fully determines when $\lim^n\mathbf{A} = 0$.

Theorem 4.1.10. *[12] If* $\mathfrak{d} = \aleph_1$ *then lim*$^1\,\mathbf{A} \neq 0$.

4.2 PFA and $\lim^2 \mathbf{A}$

By the following theorem, PFA fully determines when $\lim^n \mathbf{A} = 0$, as well - but in a different direction.

Theorem 4.2.1. *If* $\mathfrak{b} = \mathfrak{d} = \aleph_2$ *and* $\diamondsuit(S^2_1)$ *then* $\lim^2 \mathbf{A} \neq 0$.

Corollary 4.2.2. *Assuming the Proper Forcing Axiom,* $\lim^n \mathbf{A} \neq 0$ *if and only if* $n = 2$.

Proof of Corollary 4.2.2. Among the consequences of PFA:

1. $\mathfrak{d} = \aleph_2$ ([45], [4]). So by Corollary 4.1.9, $\lim^n \mathbf{A} = 0$ for $n > 2$.
2. $\lim^1 \mathbf{A} = 0$ ([12]). This and $\mathfrak{b} = \aleph_2$ follow in fact from a strictly weaker assumption, the Open Coloring Axiom, a consequence of PFA ([40]).
3. $\diamondsuit(S^2_1)$ ([3], [45]).

Theorem 4.2.1 then completes the proof. □

The condition $\mathfrak{b} = \mathfrak{d} = \aleph_2$ is equivalent to the existence of an ω_2-scale.

Definition 4.2.3. A γ-chain in $\mathcal{N}$ is a collection $\{f_\alpha \,|\, \alpha < \gamma\} \subset \mathcal{N}$ such that $\alpha < \beta$ implies $f_\alpha <^* f_\beta$. A γ-scale is a γ-chain which is $<^*$-cofinal in $\mathcal{N}$.

Theorem 4.1.10 is perhaps better understood as a ZFC phenomenon:

Theorem 4.2.4. *For any ω_1-chain $\mathcal{F}$ in $\mathcal{N}$, there exists a nontrivial coherent $\Phi^{\mathcal{F}} = \{\phi_f \mid f \in \mathcal{F}\}$.*

In other words, $\lim^1 \mathbf{A}^{\mathcal{F}} \neq 0$, where $\mathbf{A}^{\mathcal{F}} = (A_f, p_{fg}, \mathcal{F})$. Theorem 4.2.4 is simply a recasting of [4] pages 96-98, which inscribes a gap in any $\subset^*$-increasing ω_1-chain of subsets of $\mathbb{N}$. Let $\mathcal{F}^* = \{g \in \mathcal{N} \mid g \leq^* f \text{ for some } f \in \mathcal{F}\}$; write $\Phi^{\mathcal{F}}$ for $\{\phi_f \mid f \in \mathcal{F}\}$, as above. Any coherent $\Phi^{\mathcal{F}}$ extends to a coherent $\Phi^{\mathcal{F}^*}$, so the theorem gives a nontrivial coherent $\Phi^{\mathcal{G}}$ for any $\mathcal{G} \subseteq \mathcal{N}$ of cofinality $\aleph_1$ in the $<^*$-ordering. Such $\Phi^{\mathcal{G}}$ admit no "upwards" extensions:

Observation 4.2.5. For any h with $g \leq^* h$ for all $g \in \mathcal{G}$, no nontrivial coherent $\Phi^{\mathcal{G}}$ extends to a coherent $\Phi^{\mathcal{G} \cup \{h\}}$.

For if it did, then any $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ extending ϕ_h would trivialize $\Phi^{\mathcal{G}}$, a contradiction. This is one key to the proof below. The other is the following:

Observation 4.2.6. If $\Phi_1 = \{\phi_f \mid f \in \mathcal{N}\}$ and $\Upsilon_1 = \{\upsilon_f \mid f \in \mathcal{N}\}$ 2-trivialize the same $\Phi_2 = \{\phi_{fg} \mid (f,g) \in \mathcal{N}^{[2]}\}$ then they differ by a 1-coherent $\Psi_1 = \{\psi_f \mid f \in \mathcal{N}\}$.

For, letting $\psi_f = \phi_f - \upsilon_f$,

$$\psi_g - \psi_f = (\phi_g - \upsilon_g) - (\phi_f - \upsilon_f) = (\phi_g - \phi_f) - (\upsilon_g - \upsilon_f) =^\star \phi_{fg} - \phi_{fg} = 0$$

for all $f \leq g$. Hence Ψ_1 is 1-coherent.

Proof of Theorem 4.2.1. Fix an ω_2-scale $\mathcal{F} = \{f_\alpha \mid \alpha < \omega_2\}$ and an $\mathcal{S}$ witnessing $\diamondsuit(S^2_1)$. Let $\mathcal{F}_\beta = \{f \in \mathcal{N} \mid f \leq^* f_\alpha \text{ for some } \alpha < \beta\}$. Let $Y_\beta = \bigcup_{f \in \mathcal{F}_\beta} \mathbb{Z}^{I_f}$ and

$Y = \bigcup_{\beta<\omega_2} Y_\beta$, and fix a bijection $\rho : \omega_2 \to Y$. For any $\Phi_1 = \{\phi_f \mid f \in \mathcal{N}\}$ those β for which $\Phi_1 \cap Y_\beta \subseteq \rho''\beta$ form a club subset of ω_2, so the set $S(\Phi_1) = \{\beta \in S^2_1 \mid \rho'' S_\beta = \Phi_1 \cap Y_\beta\}$ is stationary.

We show $\lim^2 \mathbf{A} \neq 0$ by constructing, in stages Φ_2^β ($\beta < \omega_2$), a non-2-trivial 2-coherent Φ_2: each Φ_2^β will be of the form

$$\{\phi_{fg} \mid f \leq g \leq^* f_\alpha \text{ for some } \alpha < \beta\}$$

and Φ_2 will be their union. By the argument of Corollary 4.1.9, $\lim^2 \mathbf{A}^{\mathcal{F}_\beta} = 0$ for every $\beta < \omega_2$, so every 2-coherent Φ_2^β is 2-trivial, and therefore extends to some 2-trivial (hence 2-coherent) $\Phi_2^{\beta+1}$. For limit β, let $\Phi_2^\beta = \bigcup_{\gamma<\beta} \Phi_2^\gamma$. At stage β, if $\rho'' S_\beta$ is of the form $\{\phi_f \mid f \in \mathcal{F}_\beta\}$ and 2-trivializes Φ_2^β, we extend with greater care. Since $\mathrm{cf}(\beta) = \aleph_1$, there exists by Theorem 4.2.4 a nontrivial coherent family $\Psi_1^{\mathcal{F}_\beta}$. Take any extension $\Upsilon_1^{\beta+1} = \{\upsilon_f : f \in \mathcal{F}_{\beta+1}\}$ of $\Upsilon_\beta = \rho'' S_\beta + \Psi_1^{\mathcal{F}_\beta}$. Letting $\phi_{fg} = \upsilon_g \restriction_{I_f} - \upsilon_f$ for any $g \in \mathcal{F}_{\beta+1} \backslash \mathcal{F}_\beta$ defines a 2-coherent extension $\Phi_2^{\beta+1}$ of Φ_2^β which is 2-trivialized by $\Upsilon_1^{\beta+1}$.

Clearly Φ_2 is 2-coherent. Suppose for contradiction that Φ_1 2-trivializes Φ_2. Then for $\beta \in S(\Phi_1)$, $\Phi_1 \cap Y_\beta$ and $\Phi_1 \cap Y_{\beta+1}$ 2-trivialize Φ_2^β and $\Phi_2^{\beta+1}$, respectively. By construction, $\Upsilon_1^{\beta+1}$ also 2-trivializes $\Phi_2^{\beta+1}$. By Observation 2, then, $\Upsilon_1^{\beta+1} - (\Phi_1 \cap Y_{\beta+1})$ is a coherent family extending nontrivial coherent family $\Upsilon_1^\beta - (\Phi_1 \cap Y_\beta) = \Psi_1^{\mathcal{F}_\beta}$, contradicting Observation 1. □

Beginning from a model of PFA, Todorcevic forced $\lim^1 \mathbf{A} \neq 0$ while preserving $\mathrm{MA}_{\aleph_1}$. The forcing in question is ω_2-distributive; in consequence, ¬wKH and, hence,

$\diamondsuit(S^2_1)$ are preserved (see [41], [3]). Therefore,

Theorem 4.2.7. *Under the assumption of the Proper Forcing Axiom, $MA_{\aleph_1}$ is consistent with "$\lim^n \mathbf{A} \neq 0$ if and only if $n \leq 2$".*

Remark. The large cardinal consistency strength of PFA isn't really needed here. More precisely, it's unneeded in Todorcevic's construction. One may then follow his forcing (over a model of $\mathfrak{b} = \mathfrak{d} = \aleph_2$) with the usual forcing for $\diamondsuit(S^2_1)$: the latter, being ω_2-closed, adds no reals, hence preserves $\lim^1 \mathbf{A} \neq 0$, and, clearly, $\mathrm{MA}_{\aleph_1}$ as well.

4.3 Wider systems

Let $\mathbf{A}_\kappa = (A_f, p_{fg}, \omega^\kappa)$, where $A_f = \bigoplus_{\alpha<\kappa} \mathbb{Z}^{f(\alpha)} = \{\phi_f : I_f \to \mathbb{Z} \,|\, \mathrm{supp}(f) \text{ is finite}\}$ and $p_{fg} : \phi_f \mapsto \phi_f \restriction_{I_g}$, as before. $\mathbf{A}_\kappa$ generalizes $\mathbf{A}$ both in form ($\mathbf{A}_\omega = \mathbf{A}$) and in content: $\bar{H}_p(Y^{(k)}) = \lim^{k-p} \mathbf{A}_\kappa$ for $Y^{(k)}$ the disjoint union of κ many k-dimensional Hawaiian earrings when $0 < p < k$. We show the following relation:

Theorem 4.3.1. *$\lim^1 \mathbf{A} = 0$ if and only if $\lim^1 \mathbf{A}_\kappa = 0$ for all infinite κ.*

Replacing $\mathcal{N}$ with ω^κ in Definitions 2.13 and 2.14, the arguments of Theorem 4.1.6 apply equally to $\lim^n \mathbf{A}_\kappa$, so one direction of the theorem is clear: if every n-coherent family $\{\phi_{\vec{f}} \,|\, \vec{f} \in (\omega^\kappa)^{[n]}\}$ is n-trivial, so too must be every n-coherent family $\{\phi_{\vec{f}} \,|\, \vec{f} \in (\omega^\omega)^{[n]}\}$. In other words:

Observation 4.3.2. For $n > 0$, $\lim^n \mathbf{A}_\kappa = 0$ implies $\lim^n \mathbf{A} = 0$.

For the other direction of Theorem 4.3.1, we assume $\lim^1 \mathbf{A} = 0$, fix a coherent family $\Phi = \{\phi_f \mid f \in \omega^\kappa\}$ and show it trivial. This we'll argue by transfinite induction on κ. The argument separates into the two cases $\mathrm{cof}(\kappa) = \omega$ and $\mathrm{cof}(\kappa) > \omega$. The hypothesis in all cases is that $\lim^1 \mathbf{A}_\lambda = 0$ for $\lambda < \kappa$; hence $\Phi\restriction_x = \{\phi_f \restriction_{I_f \cap (x \times \omega)} \mid f \in \omega^\kappa\}$ is trivial for any $x \in [\kappa]^{<\kappa}$. We'll want to measure the failure of various ϕ to trivialize Φ; for this the notation $e(\phi, \psi) = \{\alpha \mid \phi(\alpha, i) \neq \psi(\alpha, i) \text{ for some } i\}$ will be useful.

Proof. Case 1: $\mathrm{cof}(\kappa) = \omega$. Fix $\{\beta_j \mid j < \omega\}$ cofinal in κ, with $\beta_0 = 0$. Let $L_j = [\beta_j, \beta_{j+1})$ and fix, for all $j < \omega$, some $\phi_j : L_j \times \omega \to \mathbb{Z}$ trivializing $\Phi\restriction_{L_j}$. For all $\alpha < \kappa$, there's a unique $j(\alpha)$ such that $\alpha \in L_{j(\alpha)}$. Define $\phi : \kappa \times \omega \to \mathbb{Z}$ by $\phi(\alpha, i) = \phi_{j(\alpha)}(\alpha, i)$. Let $\mathrm{err}(\phi) = \{f \in \omega^\kappa \mid \phi_f \neq^\star \phi\}$, i.e., $\mathrm{err}(\phi)$ collects those f such that $e(\phi_f, \phi)$ is infinite. Note that $e(\phi_f, \phi)$ is countable for every f, and that $\mathrm{err}(\phi) = \varnothing$ if and only if ϕ trivializes Φ.

Say x *bounds* a collection $\mathcal{C} \subset P(\kappa)$ if $c \subset^* x$ for all $c \in \mathcal{C}$. For any $x \in [\kappa]^{<\kappa}$ bounding $\{e(\phi_f, \phi) : f \in \mathrm{err}(\phi)\}$, it is our induction hypothesis that some $\psi : x \times \omega \to \mathbb{Z}$ trivializes $\Phi\restriction_x$. Define $\phi' : \kappa \times \omega \to \mathbb{Z}$:

$$\phi'(\alpha, i) = \begin{cases} \psi(\alpha, i) & \alpha \in x \\ \phi(\alpha, i) & \text{otherwise} \end{cases}$$

Observe that ϕ' trivializes Φ.

We show that such an x must always exist. If not, then there exist $f_\xi \in \mathrm{err}(\phi)$ $(\xi < \omega_1)$ such that $u(\xi) = e(\phi_{f_\xi}, \phi) \backslash \bigcup_{\eta < \xi} e(\phi_{f_\eta}, \phi)$ is infinite for every ξ. Define $g : \kappa \to \omega$ by $g(\alpha) = f_\xi(\alpha)$ if $\alpha \in u(\xi)$, $g(\alpha) = 0$ otherwise. For some $j < \omega$, $A = \{\xi < \omega_1 \,|\, e(\phi_{f_\xi}, \phi_g) < \beta_j\}$ is uncountable. For some $k \geq j$, $\{\xi \in A \,|\, u(\xi) \cap L_k \neq \varnothing\}$ is uncountable as well. But this gives uncountably many $\alpha_\xi \in L_k$ such that, for some i, $\phi_g(\alpha_\xi, i) = \phi_{f_\xi}(\alpha_\xi, i) \neq \phi(\alpha_\xi, i) = \phi_k(\alpha_\xi, i)$. Hence ϕ_k does not trivialize $\Phi\restriction_{L_k}$, a contradiction.

Case 2: $\mathrm{cof}(\kappa) > \omega$. *Stacked functions* are natural attempts to trivialize Φ:

Definition 4.3.3. A collection of functions $f_j \in \omega^\kappa$ such that $\bigcup_{j\in\omega} I_{f_j} = \kappa \times \omega$ is a *stack*. $\phi : \kappa \times \omega \to \mathbb{Z}$ is *stacked* if $\phi : (\alpha, i) \mapsto \phi_{f_k}(\alpha, i)$ for some stack $\mathcal{F} = \langle f_j \rangle$, where $k = \min\{j : (\alpha, i) \in I_{f_j}\}$.

If $\mathcal{F}$ so determines ϕ, write $\phi = \phi^{\mathcal{F}}$.

Lemma 4.3.4. *For any stacked functions ϕ, ψ, there exists $\delta < \kappa$ such that $\phi(\alpha, i) = \psi(\alpha, i)$ whenever $\alpha > \delta$.*

Proof. Let $\mathcal{F} = \langle f_j \rangle$, $\mathcal{G} = \langle g_k \rangle$ determine ϕ and ψ, respectively. Were $e(\phi, \psi) = \{\alpha : \phi(\alpha, i) \neq \psi(\alpha, i) \text{ for some } i\}$ uncountable, so too would be $e(\phi_{f_j}, \phi_{g_k})$ for some $j, k \in \omega$. This cannot be; hence $e(\phi, \psi)$ is bounded below κ. □

Applying the induction hypothesis, for $\beta < \kappa$ fix $\phi_\beta : \beta \times \omega \to \mathbb{Z}$ trivializing $\Phi\restriction_\beta$. Note that these ϕ_β "cohere": $e(\phi_\beta, \phi_\gamma)$ is finite, for every $\beta < \gamma < \kappa$. Now fix a stack $\mathcal{F} = \langle f_j \,|\, 0 < j < \omega \rangle$. Note the index-shift: though $\phi = \phi^{\mathcal{F}}$ is defined, we've left room at index 0 for one more function f_0 (room, in other words, to revise $\phi^{\mathcal{F}}\restriction_{I_{f_0}}$ to ϕ_{f_0}). Now assume, towards contradiction, that Φ is nontrivial.

For all $\alpha < \kappa$ there exists a least $\alpha^+ < \kappa$ such that $e(\phi_{\alpha^+}, \phi) \cap [\alpha, \alpha^+)$ is infinite; if for some $\beta < \kappa$ this were not so, then

$$\phi'(\alpha, i) = \begin{cases} \phi_\beta(\alpha, i) & \alpha < \beta \\ \phi(\alpha, i) & \text{otherwise} \end{cases}$$

would trivialize Φ. Let $A = \{\alpha < \kappa \,|\, \alpha \in e(\phi_{\alpha^+}, \phi)\}$. If $\alpha \in A$ let $f_0(\alpha) = \min\{i \,|\, \phi_{\alpha^+}(\alpha, i) \neq \phi(\alpha, i)\}$. For $\alpha \in \kappa \backslash A$ let $f_0(\alpha) = 0$.

Let $\psi = \psi^{\mathcal{F} \cup \{f_0\}}$; by Lemma 4.3.4, take $\delta < \kappa$ such that $\psi(\alpha, i) = \phi(\alpha, i)$ for all $\alpha > \delta$. By the coherence of $\{\phi_\beta \,|\, \beta < \kappa\}$, $\alpha^+ = \delta^+$ for $\alpha \in A \cap [\delta, \delta^+)$. So $\phi_{\delta^+}(\alpha, i) \neq \phi(\alpha, i)$ for infinitely many $(\alpha, i) \in I_{f_0} \cap ([\delta, \delta^+) \times \omega)$, by the definition of f_0. But $\psi(\alpha, i) = \phi_{f_0}(\alpha, i)$ for such (α, i), and $\phi_{f_0}(\alpha, i) = \phi_{\delta^+}(\alpha, i)$ for all but finitely many (α, i), hence $\psi(\alpha, i) \neq \phi(\alpha, i)$ for some $\alpha > \delta$ - a contradiction. □

4.4 Open questions

The foregoing suggests a number of further questions:

1. *For* $n > 1$ *does* $\lim^n \mathbf{A} = 0$ *imply* $\lim^n \mathbf{A}_\kappa = 0$*?*
2. *Does* $\lim^n \mathbf{A}_\kappa = 0$ *for all* $n > 0$, $\kappa \geq \omega$ *imply strong homology additive on, e.g., locally compact metric spaces?*

Here there are two questions, really, in play. Andrei Prasolov has exhibited a paracompact, non-metrizable ZFC counterexample to the additivity of strong homology (see [35]). So a first question is *On what class of spaces can strong homology be additive?* Prasolov's example is a kind of upper bound. Secondly: *On that class of spaces, are nonzero* $\lim^n \mathbf{A}_\kappa$ *the only obstructions to additivity?*

3. *Is it consistent that* $\lim^n \mathbf{A}_\kappa = 0$ *for all* $n > 0$, $\kappa \geq \omega$*?*

   This extends a question of Moore's (see [34]): *Is it consistent that* $\lim^1 \mathbf{A} = \lim^2 \mathbf{A} = 0$*?*

4. *Is it consistent that* $\lim^3 \mathbf{A} \neq 0$*?*

   Arguments like ours for Theorem 4.2.1 would require higher analogues of Theorem 4.2.4. An affirmative answer to 4, in other words, would follow from an affirmative answer to 5, in the case $n = 2$.

5. *Given an* ω_n*-chain* $\mathcal{F} \subset \mathcal{N}$, *does* $\lim^n \mathbf{A}^{\mathcal{F}} \neq 0$*?*

6. *Can a witness to* $\lim^n \mathbf{A} \neq 0$ *be analytic?*

   Todorcevic has given a negative answer in the case $n = 1$ [41].

CHAPTER 5

"PURER" SET THEORY

5.1 Introduction

Shaping Chapters 2 and 3 was a programmatic avoidance of independence techniques, a restriction to the resources of ZFC. Nontrivial (or trivial) coherence, though, is very plainly a generic phenomenon, as in Sections 5.2.1 and 5.2.2, below. So, too, is non-n-trivial n-coherence, as we'll see in Section 5.3. The arguments of that section, which transfer in Section 5.4 to L, are patterned on the case of $n = 1$, and we therefore begin this chapter with the following orienting observations:

1. Section 5.2.1: The collection of coherent families of successor height less than κ, ordered by extension, defines a forcing notion which adds a nontrivial coherent family Φ of height κ. Moreover, Φ is a κ-Suslin coherent sequence.
2. Section 5.2.2: The collection of functions in the uniformization of a coherent family Φ, ordered by extension, defines a forcing notion which adds a trivialization of Φ.
3. Definition 5.2.9: A finite conditions forcing adds a nontrivial coherent family on ω_1 which no further forcing can trivialize without collapsing ω_1.
4. Proposition 5.2.12: An $(\omega + 1)$-strategically closed forcing cannot trivialize a nontrivial coherent family.

5. Examples 5.2.15 and 5.2.14: One may derive a nontrivial coherent family on κ in any of several ways from a $\square(\kappa)$-sequence.

The first two of the above examples readily generalize to higher-n nontrivial coherence; the fifth does as well, though in a more conditional sense. The questions of whether the third or fourth generalize seem to us the most interesting open questions in this area, for reasons we'll describe in greater detail below. Their main interest, though, is their potential value for proving results of the type $\check{\mathrm{H}}^n(\kappa, \mathcal{A}_d) = 0$, which are in general harder to come by than the positive results of Sections 5.2 through 5.4. Large cardinals calibrate this difficulty, a phenomenon we explore in Section 5.5.

We'll understand non-n-trivial n-coherence everywhere in this chapter to mean *mod finite*, with the partial exception of Example 5.2.14. (The structures of Sections 4.2 and 4.3, in fact, may in large degree be viewed as generalizing Cohen forcing; see Remark 5.3.7.) Recall also from Chapter 1 the observation that $\check{\mathrm{H}}^n(\varepsilon, \mathcal{A}_d) = 0$ if and only if $\check{\mathrm{H}}^n(\mathrm{cf}(\varepsilon), \mathcal{A}_d) = 0$; a focus on regular κ below, therefore, is much less restrictive than it may at first appear to be.

This chapter draws broadly on arguments and techniques developed in the study of gaps and of square principles. It draws more directly on three main conversations:

1. Discussions with Chris Lambie-Hanson, dating to the Fruška Gora SETTOP conference of July 2016.
2. Discussions since 2016 with Stevo Todorcevic, as well as his unpublished 1992 note *Some Reflections on Souslin Trees* [38].

3. Discussions with Assaf Rinot, principally in Budapest, Hungary, in July 2017.

Central to both [38] and to discussions with Rinot is a motif dating at least to Kunen's *Saturated Ideals* [23]: an analysis of the interrelated final and intermediate models of a two-step forcing to add, and then destroy, a Suslin tree. Our debt to all these sources is quite general; still, we've aimed to record contributions more particularly below, wherever possible.

5.2 Forcing nontrivial coherence

In this chapter, we assume of the reader some basic background in set theory, such as is contained in [24]. We'll recall the definitions of a few further principles as they arise. The following is the most immediate; we follow its statement in [9]:

Definition 5.2.1. For $\mathbb{P}$ and β an ordinal, define the game $G_\beta(\mathbb{P})$ as follows: players Even and Odd take turns choosing successive entries, to form a decreasing sequence $p_0 > \cdots > p_\alpha > \ldots$ $(\alpha < \beta)$. Odd plays at odd stages, while Even plays at even stages – and we view all limit ordinals as even. If at some stage of $G_\beta(\mathbb{P})$, Even cannot play, then Odd wins. Even wins otherwise. $\mathbb{P}$ is *β-strategically closed* if Even has a winning strategy in the game $G_\beta(\mathbb{P})$, and $\mathbb{P}$ is *$<\beta$-strategically closed* if $\mathbb{P}$ is α-strategically closed for all $\alpha < \beta$.

Fact 5.2.2. If $\mathbb{P}$ is $(\lambda + 1)$-strategically closed, then forcing with $\mathbb{P}$ adds no λ-sequences of ordinals. In particular, $\mathbb{P}$ then preserves all cardinals less than or equal to λ^+. Similarly, a $<\lambda$-strategically closed forcing adds no ordinal sequences of length less than λ, and, hence, preserves all cardinals less than or equal to λ.

5.2.1 The forcing $\mathbb{P}(1, \lambda, A)$

Definition 5.2.3. For A an abelian group and $\lambda \geq \omega_1$ a regular cardinal, let $\mathbb{P}(1, \lambda, A)$ denote the following partial order:

Conditions: coherent families $p = \{\varphi_\beta : \beta \to A \,|\, \beta \in \delta_p\}$, with $\delta_p \in \lambda \setminus \mathrm{Lim}$.

Order: $q \leq p$ if and only if $q \supseteq p$.

Lemma 5.2.4. *Let G be $\mathbb{P}(1, \lambda, A)$-generic. Then $\cup G$ is a nontrivial coherent family of functions.*

Proof. Coherence is clear. Write Φ for $\cup G$, and suppose for contradiction that $p \Vdash \mathring{\varphi}$ trivializes $\mathring{\Phi}$. Define then $p > p_0 > \cdots > p_i > \dots$ such that

$$p_i \Vdash \mathring{\varphi}(\alpha_i) = a_i$$

and $\delta_{p_i} < \alpha_j \leq \delta_{p_j}$, for all $i < j < \omega$. Let $\beta = \sup_{i \in \omega} \delta_{p_i} = \sup_{i \in \omega} \alpha_i$, and let $\varphi_\beta : \beta \to A$ trivialize $\bigcup_{i \in \omega} p_i$. We may assume (by modifying φ_β, if necessary, on the coordinates α_i) that $\varphi_\beta(\alpha_i) \neq a_i$ for all i. Hence $q := \varphi_\beta \cup \bigcup_{i \in \omega} p_i$ forces that $\mathring{\varphi}$ does not trivialize $\mathring{\Phi}$. But $q < p$: a contradiction. □

If λ is greater than ω_1, then $\mathbb{P}(1, \lambda, A)$ is not $<\lambda$-closed: consider a cofinal sequence $\vec{p} = \{p_\alpha \,|\, \alpha < \omega_1\}$ of initial segments of a nontrivial coherent family. Any lower bound for $\vec{p}$ would include among its elements a trivialization of that family; hence no such lower bound exists. In all games $G_\alpha(\mathbb{P}(1,\lambda,A))$ ($\alpha < \lambda$), though, Even can easily play to avoid sequences like $\vec{p}$:

Lemma 5.2.5. *The forcing $\mathbb{P}(1,\lambda,A)$ is $<\lambda$-strategically closed. In consequence, $\mathbb{P}(1,\lambda,A)$ preserves all cardinals less than or equal to λ.*

Proof. Fix $\gamma < \lambda$. Write φ^p for the function in p which has the largest domain. In the game $G_\gamma(\mathbb{P}(1,\lambda,A))$, Even's winning strategy is simple: at any even successor stage $\alpha+2$, she plays so that $\varphi^{p_{\alpha+2}}$ extends φ^{p_α}, and at any limit stage β, she plays

$$p_\beta := \{\bigcup_{\alpha\in E(\beta)} \varphi^{p_\alpha}\} \cup \bigcup_{\alpha\in\beta} p_\alpha \tag{5.1}$$

where $E(\beta)$ denotes the even ordinals below β. Even ensures by this strategy that any sequence of plays will be trivial and, hence, will have a lower bound. □

In summary: *forcing with the family of coherent families of functions of size less than λ adds a nontrivial coherent family Φ on λ, without collapsing λ.*

5.2.2 Forcing triviality

Forcing with the uniformization of Φ, in turn, trivializes Φ. By "uniformization" we mean simply the underlying set of $\mathbb{T}(\Phi)$, in the following example.

Definition 5.2.6. Let $\Phi = \{\varphi_\beta : \beta \to A \mid \beta \in \lambda\}$ be a coherent family of functions. Then $\mathbb{T}(\Phi)$ denotes the following partial order:

Conditions: $t : \alpha_t \to A$ such that $\alpha_t < \lambda$ and $t =^* \varphi_{\alpha_t}$.

Order: $s \leq t$ if and only if $s \supseteq t$.

Clearly $\cup G$, for a $\mathbb{T}(\Phi)$-generic G, trivializes Φ; the question in general is at what cost. When $\Phi_{\mathbb{P}}$ is the $\mathbb{P}(1, \lambda, A)$-generic nontrivial coherent family of Example 5.2.1 above, $\mathbb{T}(\Phi_{\mathbb{P}})$ is mild.

Lemma 5.2.7. *The two-step forcing $\mathbb{P}(1, \lambda, A) * \mathbb{T}(\Phi_{\mathbb{P}})$ has a λ-closed dense subset. In particular, $\mathbb{P}(1, \lambda, A) * \mathbb{T}(\Phi_{\mathbb{P}})$ adds a nontrivial coherent family, then trivializes it, without collapsing λ.*

Proof. The set

$$D = \{(p, \mathring{t}) \in \mathbb{P}(1, \lambda, A) * \mathbb{T}(\Phi_{\mathbb{P}}) \,|\, p \Vdash \mathring{t} = \check{f}_t, \text{ and } \delta_p = \text{dom}(f_t) + 1\}$$

is dense in $\mathbb{P} * \mathbb{T}$. It is evidently λ-closed. □

Lemma 5.2.7 suggests in turn that $\Phi_{\mathbb{P}}$ is in some sense nice. Indeed it is:

Lemma 5.2.8. *For $\lambda \geq \omega_1$ a regular cardinal and A an abelian group of size less than λ, the generic family $\Phi_{\mathbb{P}}$ added by $\mathbb{P}(1, \lambda, A)$ is λ-Suslin, in the sense that any $\sqsubseteq$-antichain $\{\varphi_\alpha \in \Phi_{\mathbb{P}} \,|\, \alpha \in X\}$ is of size less than λ. Hence the trees $T(\Phi_{\mathbb{P}})$ and $\mathbb{T}(\Phi_{\mathbb{P}})$ are λ-Suslin as well.*

Proof. Suppose

$$p \Vdash \{\varphi_\alpha \in \mathring{\Phi} \,|\, \alpha \in \mathring{X}\} \text{ is an } \sqsubseteq\text{-antichain}$$

Define a decreasing chain $p > p_0 > \cdots > p_i > \dots$ $(i < \lambda)$ so that for any finitely supported $f : \lambda \to A$ there's a j with $\delta_{p_j} \supseteq \mathrm{supp}(f)$ such that either

$$p_j \Vdash (\mathring{\varphi}_{\delta_{p_j}} + f{\restriction}_{\delta_{p_j}}) \not\sqsubseteq \mathring{\varphi}_\varepsilon \text{ for all } \varepsilon \in (\delta_{p_j}, \lambda) \cap \mathring{X} \tag{5.2}$$

or for some $\alpha < \delta_{p_j}$

$$p_j \Vdash (\mathring{\varphi}_{\delta_{p_j}} + f){\restriction}_\alpha = \mathring{\varphi}_\alpha \text{ and } \alpha \in \mathring{X} \tag{5.3}$$

More concretely, enumerate by odd ordinals $j(f)$ the finitely supported functions $f : \lambda \to A$. At stage $j(f)$, Odd plays to decide between possibilities (5.2) and (5.3). Even plays according to the strategy outlined in the proof of Lemma 5.2.5. In this case, a limit closure point of the function

$$\beta \mapsto \sup\{\delta_{p_{j(f)}} \,|\, \mathrm{supp}(f) \subseteq \beta\}$$

is also the supremum of $\{\delta_{p_i} \,|\, i < k\}$ for some limit $k < \lambda$. Again let $E(k)$ denote the even ordinals less than k. Then the condition

$$p_k := \{\bigcup_{i \in E(k)} \varphi_{\delta_{p_i}}\} \cup \bigcup_{i<k} p_i$$

forces that every φ_ε with $\varepsilon \in X \backslash \delta$ extends some φ_α with $\alpha \in \delta \cap X$. Hence the antichain X is contained in δ.

Now observe that a size-λ antichain in $T(\Phi_{\mathbb{P}})$ is without loss of generality of the form $\{t_\alpha \,|\, \alpha \in X\}$, with $\mathrm{ht}(t_\alpha) = \alpha$ and X cofinal in λ. Any map $\xi : X \to \lambda$ satisfying $\varphi_{\xi(\alpha)}{\restriction}_\alpha = t_\alpha$ for all α then determines an $\sqsubseteq$-antichain $\{\varphi_{\xi(\alpha)} \in \Phi_{\mathbb{P}} \,|\, \alpha \in X\}$ of cofinal index $\xi''X$ in λ, such as we've just shown cannot exist. Hence $T(\Phi_{\mathbb{P}})$ is λ-Suslin as well. We see no comparably easy deduction for $\mathbb{T}(\Phi_{\mathbb{P}})$; however, a forcing argument just as for $\Phi_{\mathbb{P}}$ shows that $\mathbb{T}(\Phi_{\mathbb{P}})$, also, is λ-Suslin. $\square$

In general, $\mathbb{T}(\Phi)$ is not so well-behaved: it trivializes $\Phi(\rho_1) = \{\rho_1(\,\cdot\,,\alpha) \mid \alpha < \omega_1\}$, for example, only by collapsing ω_1. The issue is that $\Phi(\rho_1)$ is nontrivial *for a reason*: its constituent functions are all finite-to-one mappings to $\mathbb{N}$, as no function from ω_1 to $\mathbb{N}$ can be.[1] By *reason*, here, we mean some such local principle of nontriviality; we'll generally term such a principle a *Hausdorff condition*, referencing Hausdorff's original use of such a principle to build an (ω_1, ω_1^*)-gap ([17]). One can force strong forms of nontriviality by forcing such conditions:

Definition 5.2.9. For A an abelian group, let $\mathbb{Q}_1(A)$ denote the following partial order:[2]

Conditions: $p : [\vec{\xi}_p]^2 \to A$ with $\vec{\xi}_p \in [\omega_1]^{<\omega}$.

Ordering: $q \leq p$ iff

(a) $q \supseteq p$

(b) $q(\alpha,\beta) = q(\alpha,\gamma)$ for all $\beta < \gamma$ in $\vec{\xi}_p$ and $\alpha \in \beta \cap (\vec{\xi}_q \backslash \vec{\xi}_p)$

(c) $q(\beta,\gamma) \neq q(\alpha,\gamma)$ for all $\beta < \gamma$ in $\vec{\xi}_p$ and $\alpha \in \gamma \cap (\vec{\xi}_q \backslash \vec{\xi}_p)$

Requirements (b) and (c) enforce coherence and finite-to-one-ness, respectively.

[1] The general principle at work here may be more precisely described as the conjunction of a sentence which is Σ_1 in the parameters ρ_1 and ω_1 with the assertion that ω_1 is uncountable.

[2] This forcing is adapted from that of Lemma 1.7 in [43].

Lemma 5.2.10. *If A is countable, then $\mathbb{Q}_1$ is ccc and, in consequence, preserves all cardinals.*

Proof. Fix $\{q_i \,|\, i \in \omega_1\} \subseteq \mathbb{Q}_1$. Thinning the collection if necessary, we may assume that $\{\vec{\xi}_{q_i} \,|\, i \in \omega_1\}$ forms a Δ-system with root $r \in [\omega_1]^{<\omega}$. We may assume as well that all q_i agree on $[r]^2$. By the pigeonhole principle, r is an initial segment of both $\vec{\xi}_{q_j}$ and $\vec{\xi}_{q_k}$ for some $j < k$ in ω_1. Define

$$f : (\vec{\xi}_{q_j} \backslash r \otimes \vec{\xi}_{q_k} \backslash r) \cup (\vec{\xi}_{q_k} \backslash r \otimes \vec{\xi}_{q_j} \backslash r) \to A$$

so that each $f(\alpha, \cdot)$ is, where meaningful, a constant function, while each $f(\cdot, \beta)$ is, where meaningful, injective. Then q_j and q_k are comparable: $q_j \cup q_k \cup f$ is a condition in $\mathbb{Q}_1$ refining them both. $\square$

In summary: $\mathbb{Q}_1$ *adds an "indestructible" nontrivial coherent family* Φ *on* ω_1, *i.e., a* Φ *that no forcing can trivialize without collapsing* ω_1.

Of course one might have taken some more general parameter ε in ω_1's place, when defining the conditions of $\mathbb{Q}_1$; write $\mathbb{Q}_1(\varepsilon)$ for such a variant. But this gains us little: when $\varepsilon < \omega_1$, a $\mathbb{Q}_1(\varepsilon)$-generic is trivial – and when $\varepsilon > \omega_1$, $\mathbb{Q}_1(\varepsilon)$ is no longer ccc. The latter is perhaps most easily seen via the observation that $\mathbb{Q}_1(\varepsilon)$ adds a finite-to-one function from ω_1 to ω, thereby rendering ω_1 countable. The $\mathbb{Q}_1$ ordering, in other words, is "made just for ω_1." This motivates the following question:

Question 5.2.11. *Can a forcing by finite conditions add a non-2-trivial 2-coherent family of functions on some cardinal* κ*?*

The cardinal κ would presumably be ω_2 in such a case. An affirmative answer would likely point to Hausdorff conditions for higher nontrivial coherence.

By the above considerations, the forcing $\mathbb{T}(\Phi)$ of Definition 5.2.2 has, in general, very weak closure properties. This is necessarily so, by the following proposition:

Proposition 5.2.12 (B., Lambie-Hanson)**.** *An* $(\omega+1)$*-strategically closed forcing* $\mathbb{P}$ *cannot trivialize a nontrivial* Φ*.*

Proof. Suppose instead, for contradiction, that some $p \Vdash \mathring{\varphi}$ trivializes Φ.

Fact. Let ε denote the height of Φ. For all $\alpha < \varepsilon$ there exist $\beta \in (\alpha, \varepsilon)$ and $q_0, q_1 < p$ such that

$$q_0 \Vdash \mathring{\varphi}(\beta) = z_0$$
$$q_1 \Vdash \mathring{\varphi}(\beta) = z_1$$

and $z_0 \neq z_1$.

(For if at some α this Fact were false, then

$$\varphi_\alpha \cup \{(\xi, z) \,|\, \xi \in [\alpha, \varepsilon) \text{ and } q \Vdash \mathring{\varphi}(\xi) = z, \text{ for some } q < p\}$$

would be a ground-model trivialization of Φ — a contradiction.) Using the Fact and strategic closure strategy, construct descending q_0^i and q_1^i with $q_j^i \Vdash \mathring{\varphi}(\beta_i) = z_j^i$ for $j = 0, 1$, and $z_0^i \neq z_1^i$, for all i; lower bounds $\underline{q}_0$ and $\underline{q}_1$ then define ground-model partial trivializations of $\Phi(\varepsilon)$ in infinite disagreement — a contradiction. $\square$

There's some symmetry between the arguments of Proposition 5.2.12 and those of Lemma 5.2.4, above: in both cases, we diagonalize against potential trivializations of a nontrivial Φ. In the proposition, Φ itself is known, while the forcing conditions are not; in the example, $\mathring{\Phi}$ isn't yet decided, though the conditions are.

For an application of Proposition 5.2.12 and the question of higher-order versions, see Section 5.5.2 below.

5.2.3 Square constructions

As noted, for some readers, much of the above will strongly recall the study of square principles. This is no coincidence: square principles record *local agreements that cannot be globalized* close in spirit to nontrivial coherence. We recall one main such principle, first isolated in [39]:

Definition 5.2.13. For regular uncountable λ, the principle $\square(\lambda)$ is the assertion that there exists a sequence $\mathcal{C} = \{C_\alpha \mid \alpha \in \lambda\}$ satisfying

1. C_α is a closed unbounded subset of α, for each α.
2. $C_\beta \cap \alpha = C_\alpha$, for every limit point α of C_β.
3. No club $C \subseteq \lambda$ satisfies $C \cap \alpha = C_\alpha$ for every limit point α of C.

We note two applications.

Example 5.2.14. By Theorem 6.32 and Lemma 7.1.10 in [44], walks along a $\square(\lambda)$ sequence determine a nontrivial coherent ρ_2 (in the locally constant sense; of course, the function readily converts to a mod finite family, via the "differential" d of Lemma 2.4.13). Here the nontriviality of a $\square(\lambda)$ sequence secures the classical "unboundedness" condition of the ρ_2 function — and, hence, the nontriviality of ρ_2 — on cardinals $\lambda > \omega_1$.

Example 5.2.15. In [22], König uses $\square(\lambda)$ to construct a nontrivial coherent family $\{\varphi_\beta : \beta \to \beta \mid \beta \in \lambda\}$. Essentially, the functions φ_β record the sequence $\mathcal{C}$; as is usual, some care is required at points of cofinality ω. The construction is easily modified to take codomain 2.

As Todorcevic notes in [39], for example, if $\square(\lambda)$ fails, then λ is weakly compact in L. Hence, by the above examples, the consistency strength of the statement $\check{\mathrm{H}}^1(\lambda, \mathcal{A}_d) = 0$ is at least one weakly compact cardinal, for any abelian group A. For considerations in the other direction, see Section 5.5.

Given the above examples, it's natural to ask, lastly, whether nontrivial coherence on λ in turn implies $\square(\lambda)$. In general it does not. To see this, consider again the forcing $\mathbb{P} * \mathbb{T} = \mathbb{P}(1, \omega_2, \mathbb{Z}_2) * \mathbb{T}(\Phi_{\mathbb{P}})$, over a model M of Martin's Maximum. Write $\mathbb{P} * \mathbb{T}$, for short. By the argument of Lemma 5.2.7, $\mathbb{P} * \mathbb{T}$ is equivalent to an $<\omega_2$-directed closed forcing and hence, by [26], preserves Martin's Maximum. By [13] and [45], $\square(\omega_2)$ fails in any model of Martin's Maximum. As forcing with $\mathbb{T}$ would preserve any $\square(\omega_2)$-sequence, there must be no such sequence in $M^{\mathbb{P}}$, though $M^{\mathbb{P}}$ does have a nontrivial coherent family of functions of height ω_2 — namely, the family

$\Phi_{\mathbb{P}}$. This argument is due to Rinot; its idea, in his term, is "implicit" in [10].

Similar arguments organize Todorcevic's [38]. It, too, treats a variant of $\mathbb{P}(1, \omega_2, \mathbb{Z}_2)$:

Conditions: $p : \delta_p \to 2$ with $\delta_p \in \omega_2$.

Ordering: $q \leq p$ if and only if $p \subseteq^* q$.

Denote this forcing $\mathbb{P}$. Observe that, by the above arguments, $\mathbb{P}$ adds an ω_2-Suslin nontrivial coherent family of functions Φ of height ω_2, and that $\mathbb{P} * \mathbb{T}(\Phi)$ is equivalent to the Cohen forcing $\mathrm{Fn}(\omega_2, 2, \omega_2)$. In consequence, forcing with $\mathbb{P}$ over models of strong forcing axioms like PFA^+ preserves principles like OCA, MA, $2^{\aleph_0} = \aleph_2$, stationary reflection, etc., while adding an ω_2-Suslin tree.

In the following section we generalize Examples 5.2.1 and 5.2.2. The rhythm, broadly, is the same: strategic closure preserves cardinals; genericity ensures non-n-triviality. More precisely, one may use a non-$(n-1)$-trivial $(n-1)$-coherent family to diagonalize against initial segments of any n-trivialization, as in the case of $n = 1$.

This is a diagonalization we implement in L, in Section 5.4.

5.3 Forcing higher nontrivial coherence

5.3.1 The forcings $\mathbb{P}(n, \lambda, A)$

In this section, we generalize to the cases $n \geq 1$ the arguments of Section 5.2. To build intuition, we detail that generalization first for the case of $n = 2$. This entails redundancy; in consequence, having understood either the general or $n = 2$ arguments, the reader might safely skim those for the other case.

Definition 5.3.1. For A an abelian group and $\lambda \geq \omega_2$ a regular cardinal, let $\mathbb{P}(2, \lambda, A)$ denote the following forcing:

Conditions: 2-coherent families $p = \{\varphi^p_{\beta\gamma} : \beta \to A \mid \beta\gamma \in [\delta_p]^2\}$, with $\delta_p \in \lambda \setminus \mathrm{Lim}$.

Order: $q \leq p$ if and only if $q \supseteq p$.

Lemma 5.3.2. *The forcing $\mathbb{P}(2, \lambda, A)$ is $<\lambda$-strategically closed. In consequence, $\mathbb{P}(2, \lambda, A)$ preserves all cardinals less than or equal to λ.*

Proof. Let ε be an ordinal less than λ. We describe a winning strategy for Even in the game $G_\varepsilon(\mathbb{P}(2, \lambda, A))$. For any sequence of plays $p_0 > \cdots > p_i > \ldots$ $(i < \eta)$, let

$$\mathrm{Ev}(\eta) = \{\delta_{p_i} - 1 \mid i < \eta \text{ and } i \text{ is even}\}$$

Note that for any such sequence of plays and $\beta\gamma \in [\cup_{i<\eta} \delta_{p_i}]^2$, the notation $\varphi_{\beta\gamma}$ is unambiguous. Even's strategy is to play to maintain the following condition at all stages $\eta < \varepsilon$:

$$\text{For all } \beta < \gamma < \delta \text{ in } \mathrm{Ev}(\eta), \qquad \varphi_{\gamma\delta}\restriction_\beta - \varphi_{\beta\delta} + \varphi_{\beta\gamma} = 0 \tag{$*(\eta)$}$$

Even does so as follows: suppose $(*(\eta))$ holds at some even stage η, and let $\delta = \sup\{\delta_{p_i} \,|\, i < \eta\}$. For $\beta \in \mathrm{Ev}(\eta)$ and $\xi < \beta$ let $\alpha_\xi = \min(\mathrm{Ev}(\eta)\backslash\xi + 1)$. Adopt the convention that any $\varphi_{\beta\beta}$ is constantly zero, and for all $\xi < \gamma$ let $\varphi_{\beta\delta}(\xi) = -\varphi_{\alpha_\xi\beta}(\xi)$. Then for $\beta < \gamma$ in $\mathrm{Ev}(\eta)$ and $\xi < \beta$,

$$\varphi_{\gamma\delta}(\xi) - \varphi_{\beta\delta}(\xi) + \varphi_{\beta\gamma}(\xi) = -\varphi_{\alpha_\xi\gamma}(\xi) + \varphi_{\alpha_\xi\beta}(\xi) + \varphi_{\beta\gamma}(\xi) \tag{5.4}$$

The right-hand side equals zero when $\alpha_\xi < \beta$ by our inductive hypothesis; it equals zero when $\alpha_\xi = \beta$ by arrangement. In short,

$$\text{For all } \beta < \gamma \text{ in } \mathrm{Ev}(\eta), \qquad \varphi_{\gamma\delta}\restriction_\beta - \varphi_{\beta\delta} + \varphi_{\beta\gamma} = 0$$

Hence, by playing any 2-coherent extension

$$p_\eta := \{\varphi_{\gamma\delta} \,|\, \gamma \in \delta\} \cup \bigcup_{\alpha<\eta} p_\alpha$$

of the 2-coherent family $\{\varphi_{\gamma\delta} \,|\, \gamma \in \mathrm{Ev}(\eta)\} \cup \bigcup_{\alpha<\eta} p_\alpha$, Even ensures that $(*(\eta + 2))$ will hold as well. (Such an extension exists by Observation 2.4.14). In this way, Even maintains the condition $(*(\eta))$ up through the stage $\eta = \varepsilon$, ensuring thereby that the sequence $p_0 > \cdots > p_i > \ldots$ $(i < \varepsilon)$ has a lower bound. □

Lemma 5.3.3. *Let G be a $\mathbb{P}(2, \lambda, A)$-generic filter. Then $\cup G$ is a non-2-trivial 2-coherent family of functions $\Phi = \{\varphi_{\beta\gamma} : \beta \to A \,|\, \beta\gamma \in [\lambda]^2\}$.*

Proof. For $\beta < \lambda$, the set $D_\beta = \{p \,|\, \delta_p > \beta\}$ is dense in $\mathbb{P}(2, \lambda, A)$. Hence $\cup G := \Phi$ is a 2-coherent family of functions indexed by $[\lambda]^2$. We argue by contradiction that Φ is non-2-trivial: if not, then

$$p \Vdash \mathring{\Psi} = \{\mathring{\psi}_\beta \,|\, \beta < \lambda\} \text{ 2-trivializes } \mathring{\Phi} \tag{5.5}$$

for some $\mathbb{P}(2, \lambda, A)$-name $\mathring{\Psi}$ and $p \in \mathbb{P}(2, \lambda, A)$. By Lemma 5.3.2, $\mathbb{P}(2, \lambda, A)$ adds no functions $\xi \to A$, for any $\xi < \lambda$. There therefore exists a descending chain of conditions p_i for which

$$p_i \Vdash \{\mathring{\psi}_\beta \,|\, \beta < \xi_i\} = \{\check{\psi}_\beta \,|\, \beta < \xi_i\}$$

with $\delta_{p_i} \leq \xi_i < \delta_{p_j}$ for all $i < j < \omega_1$. Let $\varepsilon = \sup_{i<\omega_1} \xi_i$. Fix a nontrivial coherent family $\Upsilon = \{\upsilon_\beta \to A \,|\, \beta \in \varepsilon\}$. For $\beta < \varepsilon$, let $\varphi_{\beta\varepsilon} = \upsilon_\beta - \psi_\beta$, and let

$$q = \{\varphi_{\beta\varepsilon} : \beta \to A \,|\, \beta < \varepsilon\} \cup \bigcup_{i \in \omega_1} p_i$$

Claim 5.3.4. *q is 2-coherent and, hence, is a condition in* $\mathbb{P}(2, \lambda, A)$.

Proof. Any $\delta < \varepsilon$ is less than some δ_{p_i} with $i < \omega_1$; in this case the relation

$$\varphi_{\gamma\delta}\restriction_\beta - \varphi_{\beta\delta} + \varphi_{\beta\gamma} =^* 0 \tag{5.6}$$

holds for all $\beta < \gamma < \delta$ by the definition of p_i. When $\delta = \varepsilon$, the left side of (5.6) equals

$$\upsilon_\gamma\restriction_\beta - \psi_\gamma\restriction_\beta - \upsilon_\beta + \psi_\beta + \varphi_{\beta\gamma} = (\upsilon_\gamma\restriction_\beta - \upsilon_\beta) + (\varphi_{\beta\gamma} - (\psi_\gamma\restriction_\beta - \psi_\beta)) =^* 0$$

□

Claim 5.3.5. *q forces that $\mathring{\Psi}$ does not 2-trivialize $\mathring{\Phi}$.*

Proof. Any $r \leq q$ forcing that $\mathring{\Psi}$ 2-trivializes $\mathring{\Phi}$ forces that

$$\varphi_{\beta\varepsilon} =^* \mathring{\psi}_\varepsilon\restriction_\beta - \psi_\beta \text{ for all } \beta < \varepsilon$$

In other words,

$$r \Vdash \mathring{\psi}_\varepsilon\restriction_\beta =^* \varphi_{\beta\varepsilon} + \psi_\beta = \upsilon_\beta \text{ for all } \beta < \varepsilon$$

hence

$$r \Vdash \text{the family } \Upsilon \text{ is trivial}$$

— a contradiction. This establishes Claim 5.3.5. □

Claim 5.3.5, in turn, contradicts equation 5.5. In consequence,

$$\Vdash_{\mathbb{P}(2,\lambda,A)} \mathring{\Phi} \text{ is a non-2-trivial family of functions.}$$

□

The above arguments generalize:

Theorem 5.3.6 (B., Lambie-Hanson)**.** *For $n \geq 1$ and A an abelian group and $\lambda \geq \omega_n$ a regular cardinal, let $\mathbb{P}(n, \lambda, A)$ denote the following partial order:*

Conditions: n-coherent families $p = \{\varphi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\delta_p]^n\}$, with $\delta_p \in \lambda \setminus Lim$.

Order: $q \leq p$ if and only if $q \supseteq p$.

Then $\mathbb{P}(n, \lambda, A)$ preserves cardinals less than or equal to λ, and adds an n-coherent A-valued family of functions, Φ_n, indexed by λ. If any $\kappa < \lambda$ indexes a non-$(n-1)$-trivial $(n-1)$-coherent A-valued family of functions, then Φ_n is non-n-trivial as well.

We'll argue the theorem in a series of lemmas paralleling the $n = 2$ case, above.

Remark 5.3.7. The $n = 0$ case of the above family of forcings is

Conditions: 0-coherent functions $p : \delta_p \to A$, with $\delta_p \in \lambda \setminus \mathrm{Lim}$.

Order: $q \leq p$ if and only if $q \supseteq p$.

Recall that, on a successor ordinal, 0-*coherent* means *finitely supported* (Definition 2.5.4). Hence $\mathbb{P}(0, \omega, \mathbb{Z}_2)$ is Cohen forcing. $\mathbb{P}(0, \lambda, A)$ more generally adds a sequence cofinal in λ which is of ordertype ω, thereby collapsing any regular $\lambda > \omega$. The $n = 0$ case is in this sense unlike the cases $n > 0$:

Lemma 5.3.8. *For $n \geq 1$, the forcing $\mathbb{P}(n, \lambda, A)$ is $<\lambda$-strategically closed. In consequence, $\mathbb{P}(n, \lambda, A)$ preserves all cardinals less than or equal to λ.*

Proof. As in Lemma 5.3.2, we describe for any $\varepsilon < \lambda$ a winning strategy for Even in the game $G_\varepsilon(\mathbb{P}(2, \lambda, A))$. Again for any sequence of plays $\{p_i \,|\, i < \eta\}$ let $\mathrm{Ev}(\eta) = \{\delta_{p_i} - 1 \,|\, i < \eta \text{ and } i \text{ is even}\}$. Again for any such sequence of plays and $\vec{\beta} \in [\cup_{i<\eta} \delta_{p_i}]^n$, the notation $\varphi_{\vec{\beta}}$ is unambiguous. Even's strategy is to play to maintain the following condition at all stages $\eta < \varepsilon$:

$$\text{For all } \vec{\beta} \in [\mathrm{Ev}(\eta)]^{n+1}, \qquad \sum_{j=0}^{n} (-1)^j \varphi_{\vec{\beta}^j} = 0 \tag{$\dagger(\eta)$}$$

(Here again for readability we've suppressed the obvious restriction of $\varphi_{\vec{\beta}^0}$ to β_0.) Even does so as follows: suppose ($\dagger(\eta)$) holds at some even stage η, and let $\delta = \sup\{\delta_{p_i} \mid i < \eta\}$. For $\beta_0 \in \mathrm{Ev}(\eta)$ and $\xi < \beta_0$ let $\alpha_\xi = \min(\mathrm{Ev}(\eta) \backslash \xi + 1)$. Adopt again the convention that any $\varphi_{\vec{\beta}}$ indexed by a "degenerate" $\vec{\beta}$, in other words by a tuple $\vec{\beta}$ with repeated terms, is constantly zero. For $\vec{\beta} \in [\mathrm{Ev}(\eta)]^{n-1}]$ and $\xi < \beta_0$ let $\varphi_{\vec{\beta}\delta}(\xi) = (-1)^{n-1} \varphi_{\alpha_\xi \vec{\beta}}(\xi)$. Then by the same principle as in (5.4), the following holds:

$$\text{For all } \vec{\beta} \in [\mathrm{Ev}(\eta)]^{n}, \qquad \sum_{j=0}^{n-1} (-1)^j \varphi_{\vec{\beta}^j \delta} + (-1)^n \varphi_{\vec{\beta}} = 0$$

Hence, by playing any n-coherent extension

$$p_\eta := \{\varphi_{\vec{\beta}\delta} \mid \vec{\beta} \in [\delta]^{n-1}\} \cup \bigcup_{\alpha < \eta} p_\alpha$$

of the n-coherent family $\{\varphi_{\vec{\beta}\delta} \mid \vec{\beta} \in [\mathrm{Ev}(\eta)]^{n-1}\} \cup \bigcup_{\alpha<\eta} p_\alpha$, Even ensures that ($\dagger(\eta+2)$) will hold as well. Even in this way maintains the condition ($\dagger(\eta)$) up through the stage $\eta = \varepsilon$, ensuring thereby that the sequence $p_0 > \cdots > p_i > \ldots$ $(i < \varepsilon)$ has a lower bound. □

Lemma 5.3.9. *Suppose for $n \geq 1$ some $\kappa < \lambda$ indexes a non-$(n-1)$-trivial $(n-1)$-coherent A-valued family of functions. Let G be a $\mathbb{P}(n, \lambda, A)$-generic filter. Then $\cup G$ is a non-n-trivial n-coherent family of functions $\Phi = \{\varphi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\lambda]^n\}$.*

Proof. For $\beta < \lambda$, the set $D_\beta = \{p \,|\, \delta_p > \beta\}$ is dense in $\mathbb{P}(n, \lambda, A)$. Hence $\cup G := \Phi$ is an n-coherent family of functions indexed by $[\lambda]^n$. We argue by contradiction that Φ is non-n-trivial: if not, then

$$p \Vdash \mathring{\Psi} = \{\mathring{\psi}_{\vec{\beta}} \,|\, \vec{\beta} \in [\lambda]^{n-1}\} \text{ n-trivializes } \mathring{\Phi} \tag{5.7}$$

for some $\mathbb{P}(n, \lambda, A)$-name $\mathring{\Psi}$ and $p \in \mathbb{P}(n, \lambda, A)$. Define via Even's strategic closure strategy a descending chain of conditions p_i for which

$$p_i \Vdash \{\mathring{\psi}_{\vec{\beta}} \,|\, \vec{\beta} \in [\xi_i]^{n-1}\} = \{\check{\psi}_{\vec{\beta}} \,|\, \vec{\beta} \in [\xi_i]^{n-1}\}$$

with $\delta_{p_i} \leq \xi_i < \delta_{p_j}$ for all odd $i < j < \kappa$. Let $\varepsilon = \sup_{i<\kappa} \xi_i$. Our assumption on κ, together with Observation 2.4.14, ensures us a non-$(n-1)$-trivial $(n-1)$-coherent family $\Upsilon = \{\upsilon_{\vec{\beta}} : \beta_0 \to A \,|\, \vec{\beta} \in [\varepsilon]^{n-1}\}$. For $\vec{\beta} \in [\varepsilon]^{n-1}$, let $\varphi_{\vec{\beta}\varepsilon} = \upsilon_{\vec{\beta}} + (-1)^{n-1}\psi_{\vec{\beta}}$, and let

$$q = \{\varphi_{\vec{\beta}\varepsilon} : \beta_0 \to A \,|\, \vec{\beta} \in [\varepsilon]^{n-1}\} \cup \bigcup_{i\in\kappa} p_i$$

Claim 5.3.10. *q is n-coherent and, hence, is a condition in $\mathbb{P}(n, \lambda, A)$.*

Proof. Any $\beta_n < \varepsilon$ is less than some δ_{p_i} with $i < \kappa$; hence the relation

$$\sum_{j=0}^{n} (-1)^j \varphi_{\vec{\beta}^j} =^* 0 \tag{5.8}$$

holds for all $\vec{\beta} \in [\varepsilon]^{n+1}$ by the definition of p_i. For $\vec{\beta}$ of the form $\vec{\alpha}\varepsilon$, with $\vec{\alpha} \in [\varepsilon]^n$, the left side of (5.8) equals

$$(-1)^n \varphi_{\vec{\alpha}} + \sum_{j=0}^{n-1} (-1)^j \varphi_{\vec{\alpha}^j\varepsilon} = (-1)^n \Big(\varphi_{\vec{\alpha}} - \sum_{j=0}^{n-1} (-1)^j \psi_{\vec{\alpha}^j}\Big) + \sum_{j=0}^{n-1} (-1)^j \upsilon_{\vec{\alpha}^j} =^* 0$$

This is because Ψ n-trivializes Φ, while Υ is $(n-1)$-coherent. □

Claim 5.3.11. *q forces that $\mathring{\Psi}$ does not n-trivialize $\mathring{\Phi}$.*

Proof. We've proven the claim already for the case $n = 2$. So fix $n > 2$. Any $r \leq q$ forcing that $\mathring{\Psi}$ n-trivializes $\mathring{\Phi}$ forces that

$$\varphi_{\vec{\beta}\varepsilon} =^* \sum_{j=0}^{n-1}(-1)^j \mathring{\psi}_{(\vec{\beta}\varepsilon)^j} = (-1)^{n-1}\psi_{\vec{\beta}} + \sum_{j=0}^{n-2}(-1)^j \mathring{\psi}_{(\vec{\beta}^j\varepsilon)} \text{ for all } \vec{\beta} \in [\varepsilon]^{n-1}$$

In other words,

$$r \Vdash \sum_{j=0}^{n-2}(-1)^j \mathring{\psi}_{(\vec{\beta}^j\varepsilon)} =^* \varphi_{\vec{\beta}\varepsilon} + (-1)^n \psi_{\vec{\beta}} = \upsilon_{\vec{\beta}} \text{ for all } \vec{\beta} \in [\varepsilon]^{n-1}$$

hence

$$r \Vdash \text{the family } \Upsilon \text{ is } n\text{-trivial}$$

– a contradiction. This establishes Claim 5.3.11. □

Claim 5.3.11, in turn, contradicts equation 5.7. In consequence,

$$\Vdash_{\mathbb{P}(n,\lambda,A)} \mathring{\Phi} \text{ is a non-}n\text{-trivial family of functions.}$$

□

Together the above lemmas establish Theorem 5.3.6.

Among the conditions of the theorem is the existence of a non-$(n-1)$-trivial $(n-1)$-coherent family of A-valued functions on some $\kappa \geq \omega_{n-1}$. For many A, as we've seen, the existence of such a family in ZFC remains conjectural. These families are easily added, on the other hand, by sequences of $\mathbb{P}(i, \kappa_i, A)$ $(i < n)$. In consequence:

Corollary 5.3.12. *For any $n \geq 1$ and abelian group A and regular $\lambda \geq \omega_n$, it is consistent with the ZFC axioms that $\check{\mathrm{H}}^n(\lambda, \mathcal{A}_d) \neq 0$.*

Proof. Fix A and argue by induction. The case $n = 1$ is Lemmas 5.2.4 and 5.2.5, above. Therefore let M witness the $n-1$ and $\lambda = \omega_{n-1}$ instance of the corollary. For the induction step, observe that, for any $\kappa \geq \omega_n$, forcing over M with $\mathbb{P}(n, \kappa, A)$ produces a ZFC model of $\check{\mathrm{H}}^n(\kappa, \mathcal{A}_d) \neq 0$, by Theorem 5.3.6. □

In this sense, Goblot's Theorem (Section 2.7) is sharp: it rules out precisely those instances of nontrivial cohomology that can be ruled out. The above arguments provide some further heuristic, as well, for the affinity of $\check{\mathrm{H}}^n$ and ω_n pervading these pages: a non-n-trivial n-coherent family Ψ can be used to diagonalize against initial segments of $(n+1)$-trivializations of an $(n+1)$-coherent family Φ. Necessarily, though, in these diagonalizations, Φ must be of greater height than Ψ. In other words, the "jumps" in cardinality corresponding to higher cohomology groups reflect simply the "space needed" to build up higher nontriviality relations.

5.3.2 Forcing n-triviality

As in the case $n = 1$, it's natural to turn from $\mathbb{P}(n, \lambda, A)$ to n-trivializing forcings:

Definition 5.3.13. Let n be greater than one, and let $\Phi = \{\varphi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\lambda]^n\}$ be an n-coherent family of functions. For $\delta < \lambda$, let $\Phi\restriction_\delta := \{\varphi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\xi]^n\}$. Let $\mathbb{T}_n(\Phi)$ denote the following forcing:

Conditions: $t = \{\psi^t_{\vec{\beta}} : \beta_0 \to A \,|\, \vec{\beta} \in [\delta_t]^{n-1}\}$ with $\delta_t < \lambda$, such that t is an n-trivialization of $\Phi\restriction_{\delta_t}$.

Order: $t \leq s$ if and only if $t \supseteq s$.

Again the two-step forcing $\mathbb{P}(n, \lambda, A) * \mathbb{T}(\Phi_{\mathbb{P}})$ is mild.

Theorem 5.3.14. *The forcing $\mathbb{P}(n, \lambda, A) * \mathbb{T}(\Phi_{\mathbb{P}})$ is equivalent to a $<\lambda$-closed poset. In particular, $\mathbb{P}(n, \lambda, A) * \mathbb{T}(\Phi_{\mathbb{P}})$ adds a non-n-trivial n-coherent family, then n-trivializes it, without collapsing λ.*

Proof. Again write the two-step forcing as $\mathbb{P} * \mathbb{T}$. The set

$$D = \{(p, \mathring{t}) \in \mathbb{P} * \mathbb{T} \,|\, p \Vdash \mathring{t} = \check{\Psi}_t, \text{ for some } \Psi_t = \{\psi_{\vec{\beta}} \,|\, \vec{\beta} \in [\delta_p]^{n-1}\}\,\}$$

is dense in $\mathbb{P} * \mathbb{T}$. Again D is evidently $<\lambda$-closed. □

5.4 Non-n-trivial n-coherence in L

As is often the case, square and diamond principles allow us to carry out the forcing arguments of Section 5.3 *within* a model M. These principles are plentiful enough in Gödel's constructible universe L, for example, that $\check{\mathrm{H}}^n(\varepsilon, \mathcal{A}_d) \neq 0$ holds there everywhere it possibly can — that is, everywhere not ruled out by either Corollary 2.7.7 of Goblot's Theorem or by large cardinal properties.

Theorem 5.4.1 (B., Lambie-Hanson)**.** *Suppose $V = L$, and $n \geq 1$, and $\lambda \geq \aleph_n$ is a regular cardinal that is not weakly compact. Then $\check{\mathrm{H}}^n(\lambda, \mathcal{A}_d) \neq 0$, for any nontrivial abelian group A. In particular, there exists an A-valued non-n-trivial n-coherent family of functions on λ.*

The theorem follows from a "stepping-up" theorem of more general application. Recall the following definitions:

Definition 5.4.2. Let $\kappa < \lambda$ be a regular infinite cardinals. Write S^λ_κ for $\lambda \cap \mathrm{Cof}(\kappa)$. For any stationary subset S of λ, the principle $\diamondsuit_\lambda(S)$ is the assertion that there exists a sequence $\{S_\beta \,|\, \beta \in S\}$ with $S_\beta \subseteq \beta$ for each $\beta \in S$, such that

$$\text{for any } X \subseteq \lambda \text{ the collection } \{\beta \in S \,|\, S_\beta = X \cap \beta\} \text{ is stationary in } \lambda.$$

For any countable group A and $n \in \mathbb{N}$, this principle is equivalent to the assertion that there exists a sequence $\{F^n_\beta \,|\, \beta \in S\}$ with $F^n_\beta = \{f^\beta_{\vec{\alpha}} : \alpha_0 \to A \,|\, \vec{\alpha} \in [\beta]^n\}$ for each $\beta \in S$, such that

$$\text{for any } \Phi = \{\varphi_{\vec{\alpha}} : \alpha_0 \to A \,|\, \vec{\alpha} \in [\lambda]^n\} \text{ the set } \{\beta \in S \,|\, F^n_\beta = \Phi\restriction_\beta\} \text{ is stationary in } \lambda.$$

(Recall that $\Phi\restriction_\beta = \{\varphi_{\vec{\alpha}} : \alpha_0 \to A \,|\, \vec{\alpha} \in [\beta]^n\}$.)

Definition 5.4.3. Let λ be a regular uncountable cardinal, and let S be a stationary subset of λ. For any $C \subseteq \lambda$, write C' for $\{\alpha \in \lambda \,|\, \sup(C \cap \alpha) = \alpha\}$. $\square(\lambda, S)$ is the assertion that there exists a sequence $\{C_\alpha \,|\, \alpha \in \lambda\}$ such that:

1. For all $\alpha \in \lambda$, C_α is club in α.

2. For all $\alpha < \beta$ in λ, if $\alpha \in C'$ then $C'_\beta \cap \alpha = C_\alpha$.

3. For all $\alpha \in \lambda$, $C'_\alpha \cap S = \varnothing$.

Theorem 5.4.4 (B., Lambie-Hanson)**.** *Fix a countable abelian group A and $n \geq 1$ and let $\kappa < \lambda$ be regular uncountable cardinals satisfying the following:*

1. κ indexes a non-n-trivial n-coherent family of A-valued functions.

2. There exists a stationary $S \subseteq S^\lambda_\kappa$ such that both $\diamondsuit_\lambda(S)$ and $\square(\lambda, S)$ hold.

Then λ indexes a non-$(n+1)$-trivial $(n+1)$-coherent family of A-valued functions.

Theorem 5.4.1 is then immediate from Theorem 5.4.4, coupled with the following fact:

Fact 5.4.5 (Jensen; see [11])**.** If $V = L$, then for any regular uncountable $\kappa < \lambda$ which are not weakly compact there is a stationary $S \subseteq S^\lambda_\kappa$ for which the principles $\diamondsuit_\lambda(S)$ and $\square(\lambda, S)$ both hold.

Proof of Theorem 5.4.4. Fix a $\diamondsuit_\lambda(S)$-sequence $\mathcal{F} = \{F^n_\beta \mid \beta \in S\}$ and a $\square(\lambda, S)$-sequence $\mathcal{C} = \{C_\beta \mid \beta \in \lambda\}$ witnessing condition (2) of the theorem. We recursively construct, at stages $\delta < \lambda$, the initial segments $\Phi\restriction_{\delta+1}$ of a non-$(n+1)$-trivial $(n+1)$-coherent family $\Phi = \{\varphi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\lambda]^{n+1}\}$. By maintaining the following inductive hypothesis, we avoid the danger of achieving $(n+1)$-coherence too early:

$$\varphi_{\vec{\alpha}\beta} = \varphi_{\vec{\alpha}\gamma} \text{ for all limit } \gamma \in (\delta+1)\backslash S \text{ and } \beta \in C'_\gamma \text{ and } \vec{\alpha} \in [\beta]^n$$

In other words, at stage δ of the construction, $\Phi\restriction_\delta$ is given. The task is to define functions $\varphi_{\vec{\beta}\delta} : \beta_0 \to A$ for $\vec{\beta} \in [\delta]^n$ respecting our requirements. There are several possibilities:

Case 1: δ is a successor. Any $(n+1)$-coherent extension of $\Phi\restriction_\delta$ to $\Phi\restriction_{\delta+1}$ will do.

Case 2: $\delta \in \mathrm{Lim}\backslash S$, with $\delta > \sup(C'_\delta)$. In this case $\mathrm{cf}(\delta) = \omega$, so there exists an $(n+1)$-trivialization Υ of $\Phi\restriction_\delta$. Observe that $\psi_{\vec{\beta}} := (-1)^n \varphi_{\vec{\beta}\gamma}$ for $\vec{\beta} \in [\gamma]^n$ defines an $(n+1)$-trivialization, Ψ, of $\Phi\restriction_\gamma$. Hence $(-1)^n(\Psi - \Upsilon\restriction_\gamma)$ $(n+1)$-trivializes a family of constantly zero functions indexed by $[\gamma]^n$, and therefore extends to some Ξ which $(n+1)$-trivializes a family of constantly zero functions indexed by $[\delta]^n$. Let then $\Theta = (-1)^n \Upsilon + \Xi$ and let $\varphi_{\vec{\beta}\delta} = \theta_{\vec{\beta}}$ for all $\vec{\beta} \in [\delta]^n$. As desired, $\varphi_{\vec{\beta}\delta} = \varphi_{\vec{\beta}\gamma}$ for all $\vec{\beta} \in [\gamma]^n$, and $\Phi\restriction_{\delta+1}$ is $(n+1)$-coherent.

Case 3: $\delta \in \mathrm{Lim}\backslash S$ with $\sup(C'_\delta) = \delta$. For $\vec{\beta} \in [\delta]^n$ let $\varphi_{\vec{\beta}\delta} = \varphi_{\vec{\beta}\gamma}$ where $\gamma = \min C'_\delta \backslash (\beta_{n-1} + 1)$. Then $\Phi\restriction_{\delta+1}$ is $(n+1)$-coherent, and the induction hypothesis is maintained.

Case 4: $\delta \in S$ and F^n_δ does not $(n+1)$-trivialize Φ_δ. Proceed as in Case 3.

Case 5: $\delta \in S$ and F^n_δ does $(n+1)$-trivialize Φ_δ. By hypothesis (1) of our theorem, if δ is in S then there exists a non-n-trivial n-coherent $\Psi^\delta_n = \{\psi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\delta]^n\}$. Let $\varphi_{\vec{\beta}\delta} = (-1)^n f^\delta_{\vec{\beta}} + \psi_{\vec{\beta}}$ for all $\vec{\beta} \in [\delta]^n$. The verification that $\Phi\restriction_{\delta+1}$ is $(n+1)$-coherent is as in Claim 5.3.10, and is left to the reader.

Proceeding in this way through stages $\delta < \lambda$, we construct an $(n+1)$-coherent

Φ. Suppose Υ $(n+1)$-trivializes Φ. Then $\Upsilon\restriction_\delta = F^n_\delta$ for some δ. If we let $\theta_{\vec\beta} = \upsilon_{\vec\beta\delta}$ for $\vec\beta \in [\delta]^{n-1}$ then the family $\{\theta_{\vec\beta} \mid \vec\beta \in [\delta]^{n-1}\}$ n-trivializes the family Ψ^δ_n: for all $\vec\beta \in [\delta]^n$

$$\varphi_{\vec\beta\delta} = \sum_{i=0}^{n}(-1)^i \upsilon_{(\vec\beta\delta)^i} = (-1)^n f^\delta_{\vec\beta} + \sum_{i=0}^{n-1}(-1)^i\theta_{\vec\beta^i} = \varphi_{\vec\beta\delta} - \psi_{\vec\beta} + \sum_{i=0}^{n-1}(-1)^i\theta_{\vec\beta^i}$$

hence for all $\vec\beta \in [\delta]^n$

$$\psi_{\vec\beta} = \sum_{i=0}^{n-1}(-1)^i\theta_{\vec\beta^i}$$

But this contradicts the non-n-triviality of Ψ^δ_n. Hence Φ is non-$(n+1)$-trivial. $\square$

5.5 Large cardinals and independence

One might summarize the above sections, simply, by the statement that *it's easy to add non-n-trivial n-coherent families of functions indexed by regular cardinals* $\kappa \geq \omega_n$. It's accordingly hard, but not impossible, to get rid of them, at least on large cardinals κ. The question of whether non-n-trivial n-coherent families of functions must exist on smaller cardinals remains mostly open, with the strong and provocative exception of the case $n = 1$; we record here some relevant recognitions and partial results. In particular, we show in Section 5.5.1 that, for any abelian group A and $n \geq 1$,

1. If κ is a weakly compact cardinal, then $\check{\mathrm{H}}^n(\kappa, \mathcal{A}_d) = 0$.
2. If κ is an ω_1-strongly compact cardinal, then $\check{\mathrm{H}}^n(\lambda, \mathcal{A}_d) = 0$ for all regular $\lambda > \kappa$.

Somewhat similarly, in Section 5.5.2, we show that for any abelian group A,

1. $\check{\mathrm{H}}^1(\kappa^+, \mathcal{A}_d) = 0$ in the Lévy collapse $\mathrm{Col}(\kappa, {<}\lambda)$ of a measurable cardinal λ, for any $\kappa \geq \omega_1$.
2. The P-Ideal Dichotomy implies that for all ordinals ε, $\check{\mathrm{H}}^1(\varepsilon, \mathcal{A}_d) \neq 0$ if and only if $\mathrm{cf}(\varepsilon) = \omega_1$.

5.5.1 Large cardinals

Recall the following characterization of a weakly compact cardinal:

Definition 5.5.1. A cardinal κ is Π^n_m*-indescribable* if whenever $U \subset V_\kappa$ and σ is a Π^n_m sentence such that $(V_\kappa, \in, U) \vDash \sigma$, then for some $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \vDash \sigma$.

Theorem 5.5.2 (Hanf-Scott; see [21])**.** *A cardinal κ is Π^1_1-indescribable if and only if it is weakly compact.*

Proposition 5.5.3. *Let κ be a weakly compact cardinal. Then for any $n \geq 1$, any n-coherent family $\Phi_n = \{\varphi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\kappa]^n\}$ is n-trivial.*

Proof. If Φ_n is non-n-trivial then $(V_\kappa, \in, \Phi_n)$ satisfies the following Π^1_1-sentence:

$$\forall \{\psi_{\vec{\beta}} : \beta_0 \to A \mid \vec{\beta} \in [\mathrm{Ord}]^{n-1}\} \, \exists \vec{\beta} \in [\mathrm{Ord}]^n \left[\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{\beta}^i} \neq^* \varphi_{\vec{\beta}} \right] \qquad (\sigma)$$

(We've preferred a somewhat more readable statement of σ to a more formally correct one.) For some $\alpha < \kappa$, then, $(V_\alpha, \in, \Phi_n \cap V_\alpha)$ satisfies σ. Hence $\Phi_n \cap V_\alpha$ is non-n-trivial and therefore extends to no larger n-coherent family: a contradiction. □

Definition 5.5.4. A cardinal κ is ω_1-strongly compact if and only if for every set X, every κ-complete filter on X extends to a countably complete ultrafilter on X.

The following observation is due entirely to Chris Lambie-Hanson:

Theorem 5.5.5 (Lambie-Hanson)**.** *Let A be an abelian group and let $\kappa > |A|$ be an ω_1-strongly compact cardinal. Then $\check{\mathrm{H}}^n(\lambda, \mathcal{A}_d) = 0$ for any $n \geq 1$ and regular $\lambda \geq \kappa$.*

Proof. Fix $n > 1$ and A and λ as above, and extend the filter $\{(\alpha, \lambda) \,|\, \alpha \in \lambda\}$ to a countably complete ultrafilter $\mathcal{U}$ on λ. Let $\Phi = \{\varphi_{\vec{\gamma}} : \gamma_0 \to A \,|\, \vec{\gamma} \in [\lambda]^n\}$ be n-coherent. For $\vec{\beta} \in [\lambda]^{n-1}$ and $\xi < \beta_0$ let $\psi_{\vec{\beta}}(\xi)$ be that $a \in A$ such that

$$U_{\vec{\beta},\xi} := \{\gamma \in (\beta_{n-2}, \lambda) \,|\, \varphi_{\vec{\beta}\gamma}(\xi) = a\} \in \mathcal{U}$$

Then $\Psi = \{(-1)^n \psi_{\vec{\beta}} \,|\, \vec{\beta} \in [\lambda]^{n-1}\}$ n-trivializes Φ. For if it did not, then for some $\vec{\gamma} \in [\lambda]^n$ and $x \in [\gamma_0]^{\aleph_0}$,

$$\sum_{i=0}^{n-1} (-1)^i \psi_{\vec{\gamma}^i}(\xi) \neq (-1)^{n-1} \varphi_{\vec{\gamma}}(\xi) \text{ for all } \xi \in x \tag{5.9}$$

Take then some

$$\delta \in \bigcap_{(i,\xi) \in I} U_{\vec{\gamma}^i,\xi} \backslash \gamma_{n-1}$$

where (for readability) I is the index-set $(n-1) \times x$. By (5.9),

$$\sum_{i=0}^{n-1} (-1)^i \varphi_{\vec{\gamma}^i\delta}(\xi) \neq (-1)^{n-1} \varphi_{\vec{\gamma}}(\xi) \text{ for all } \xi \in x$$

hence Φ is non-n-coherent: a contradiction. Hence Φ is n-trivial. The argument for the case $n = 1$ requires only cosmetic adjustments. □

5.5.2 Independence

We describe in this section two techniques for establishing the consistency of

$$\check{\mathrm{H}}^1(\kappa, \mathcal{A}_d) = 0 \text{ for all abelian groups } A \tag{5.10}$$

Only when $\mathrm{cf}(\kappa) \neq \omega_1$, of course, is this statement even plausibly consistent with ZFC. We show, modulo the existence of large cardinals, that this is the *only* constraint on the consistency of statement (5.10). We begin by recalling two facts.

Definition 5.5.6. Let κ be a regular cardinal, and let λ be an ordinal above κ. The *Lévy collapse* $\mathrm{Col}(\kappa, {<}\lambda)$ is the following partial order:

Conditions: $p : \kappa \times \lambda \to \lambda$ such that $|p| < \kappa$ and $p(\alpha, \beta) < \beta$ for all $(\alpha, \beta) \in \mathrm{dom}(p)$.

Ordering: $q \leq p$ if and only if $q \supseteq p$.

The following is Theorem 10.5 in [8]:

Theorem 5.5.7. *Let λ be measurable, let U be a normal measure on λ and let $j : V \to M = Ult(V, U)$ be the ultrapower map. Let κ be an uncountable regular cardinal less than λ. Let $\mathbb{P} = Col(\kappa, {<}\lambda)$ and let G be $\mathbb{P}$-generic. There is a κ-closed forcing poset $\mathbb{Q} \in M$ such that for any H a $\mathbb{Q}$-generic filter, j can be lifted to an elementary embedding $j_G : V[G] \to M[G * H]$.*

By our "branch lemma" Proposition 5.2.12, then, the following is straightforward:

Theorem 5.5.8. *Let A be an abelian group, and let λ be a measurable cardinal, and let κ be a regular uncountable cardinal below λ. Then the forcing $\mathbb{P} = Col(\kappa, {<}\lambda)$ forces that $\check{\mathrm{H}}^1(\kappa^+, \mathcal{A}_d) = 0$.*

Proof. Apply Theorem 5.5.7. In other words, fix an ultrapower map $j : V \to M$ with $\mathrm{crit}(j) = \lambda$ and a $\mathrm{Col}(\kappa, {<}\lambda)$-generic filter G and a $\mathbb{Q}$-generic filter H and an elementary embedding $j_G : V[G] \to M[G{*}H]$ extending j. Observe that $\mathrm{crit}(j_G) = \lambda$, which in $V[G]$ has the value κ^+. Let $\Phi = \{\varphi_\beta : \beta \to A \,|\, \beta \in \lambda\}$ be an element of $V[G]$. Then $\Phi = j_G'' \Phi \subsetneq j_G(\Phi)$, so Φ is trivial in $M[G * H]$ (the element φ_λ of $j_g(\Phi)$, for example, trivializes Φ). As $M \subseteq V$, Φ is trivial in $V[G * H]$ as well. By Proposition 5.2.12, the ${<}\kappa$-closed poset $\mathbb{Q}^{V[G]}$ trivializes no nontrivial coherent families of functions. Hence Φ is trivial in $V[G]$. □

For the above result, a weakly compact cardinal is almost certainly sufficient, as Theorem 5.4.1 and 5.5.3 suggest. A value of the above approach, though, is transparency, particularly with regard to what independence arguments for higher $\check{\mathrm{H}}^n$ might entail. Clearly, for example, applying the following conjecture in place of Proposition 5.2.12 in the above argument would secure the relative consistency of the statement $\check{\mathrm{H}}^n(\kappa^+, \mathcal{A}_d) = 0$, for any $1 \leq n < \omega_n \leq \kappa$ and abelian group A.

Conjecture 5.5.9. *For all $n \geq 1$ the following is true: an $(\omega_{n-1} + 1)$-strategically closed forcing $\mathbb{P}$ cannot n-trivialize a non-n-trivial Φ.*

Consider now a rather different, and much more total, approach to making $\check{\mathrm{H}}^1(\kappa)$ vanish.

Definition 5.5.10. Let X be a set. A *P-ideal on X* is an ideal $\mathcal{I} \subseteq [X]^{\leq \aleph_0}$ containing every finite subset of X such that

1. $\mathcal{I}$ contains all finite subsets of X, and
2. For every $\{x_n \mid n \in \omega\} \subseteq \mathcal{I}$ there exists an $x \in \mathcal{I}$ satisfying $x_n \subseteq^* x$ for all n.

The *P-Ideal Dichotomy* is the following assertion ([42]):

If $\mathcal{I}$ is a P-ideal on X, then exactly one of the following alternatives holds:

1. *There exists an uncountable $B \subseteq X$ such that $[B]^{\leq \aleph_0} \subseteq \mathcal{I}$.*
2. *$X = \bigcup_{i \in \omega} B_i$ with $[B_i]^{\aleph_0} \cap \mathcal{I} = \varnothing$ for each i.*

Stevo Todorcevic described to the author the P-ideal $\mathcal{I}$ of the argument below in the spring of 2016.

Theorem 5.5.11 (Todorcevic)**.** *Assume the P-Ideal Dichotomy, and let A be an abelian group. Then $\check{\mathrm{H}}^n(\varepsilon, \mathcal{A}_d) \neq 0$ if and only if cf$(\varepsilon) = \omega_1$.*

Proof. The "if" implication is a main (ZFC) argument of Chapter 2. For the implication in the other direction, fix a coherent family of functions $\Phi = \{\varphi_\alpha : \alpha \to A \mid \alpha < \varepsilon\}$, where $\mathrm{cf}(\varepsilon) > \omega_1$. For any $\alpha < \beta$ in ε let

$$e(\alpha, \beta) = \{\xi < \alpha \mid \varphi_\alpha(\xi) \neq \varphi_\beta(\xi)\}$$

and let

$$\mathcal{I} = \{b \in [\varepsilon]^{\leq \aleph_0} \mid \exists\, \beta \geq \sup(b)\, \forall\, n \in \omega\, \{\alpha \in b \mid |e(\alpha, \beta)| \leq n\} \text{ is finite}\}$$

Claim 5.5.12. *$\mathcal{I}$ is a P-ideal of countable subsets of ε.*

Proof. For $b_i \in \mathcal{I}$ $(i \in \omega)$, take $\gamma \geq \sup(\cup_{i\in\omega} b_i)$, and for $k \in \omega$ let

$$b_i(k,\gamma) = \{\beta \in b_i \mid |e(\beta,\gamma)| = k\}$$

By assumption, each $b_i(k,\gamma)$ is finite; in consequence, γ witnesses that

$$b := \cup\{b_i(k,\gamma) \mid k > i\} \text{ is in } \mathcal{I}$$

Clearly $b_i \subseteq^* b$ for all $i \in \omega$. □

Assuming the P-Ideal Dichotomy, there are now two possibilities.

Case 1: In the first possibility, there exists a $B \in [\varepsilon]^{\aleph_1}$ with $[B]^{\leq\aleph_0} \subseteq \mathcal{I}$. Write B as an increasing union of $b_i \in [\varepsilon]^{\leq\aleph_0}$ $(i \in \omega_1)$. For $\gamma \geq \sup(B)$ there exists some $n \in \omega$ and increasing $\{\alpha_j \mid j \in \omega_1\} \subseteq B$ such that $|e(\alpha_j,\gamma)| = n$ for all $j \in \omega_1$. Fix a b_i containing $\{\alpha_j \mid j \in \omega\}$. Then there exists a β witnessing that $b_i \in \mathcal{I}$. Therefore:

For all $k \in \omega$ there exists an $\ell(k)$ such that $j > \ell(k) \Rightarrow |e(\alpha_j,\beta)| > n + k$

As $|e(\alpha_j,\gamma)| = n$, though, this implies that $|e(\beta,\gamma)| > k$ for all $k \in \omega$. This contradicts the premise that Φ is coherent.

Case 2: The second possibility, which by the above argument must hold, is that $\varepsilon = \bigcup_{i\in\omega} B_i$, with no infinite subset of any B_i an element of $\mathcal{I}$. In consequence, there exists some stationary $B = B_i \subseteq \lambda$ such that for all $\gamma \in B$,

$$n^B_\gamma = \max\{|e(\beta,\gamma)| \mid \beta \in B \cap \gamma\}$$

is well-defined.

Claim 5.5.13. *For some stationary subset $S \subseteq B$, there exists an $N \in \omega$ such that*

1. $n_\gamma^S = \max\{|e(\beta,\gamma)| \mid \beta \in S \cap \gamma\} \leq N$ *for all* $\gamma \in S$, *and*
2. $T = \{\gamma \in S \mid \sup\{\beta \in S \cap \gamma \mid |e(\beta,\gamma)| = N\} = \gamma\}$ *is stationary in* λ.

Proof. Thin B first to a stationary S satisfying (a). If (b) fails, then

$$\gamma \mapsto \sup\{\beta \in S \cap \gamma \mid |e(\beta,\gamma)| = N\}$$

is a regressive function on some stationary $E \subseteq S$, and is therefore constantly β on some stationary $S_1 \subseteq E \backslash \beta$. Now there exists an $N_1 < N$ such that $n_\gamma^{S_1} \leq N_1$ for all $\gamma \in S_1$. If

$$T = \{\gamma \in S_1 \mid \sup\{\beta \in S_1 \cap \gamma \mid |e(\beta,\gamma)| = N\} = \gamma\}$$

is again nonstationary, we may repeat this process, defining in turn an $N_2 < N_1$. In other words, this process, defining a strictly decreasing sequence of natural numbers, must at some finite stage i terminate; at this point, $S = S_i$ is as claimed. □

We now argue by induction on $N = n$ that an S as in Claim 5.5.13 witnesses the triviality of Φ. The case $N = 0$ is clear: $\bigcup_{\gamma \in S} \varphi_\gamma$ trivializes Φ. For the induction step, assume that any S as in Claim 5.5.13 with $N = m < n$ witnesses the triviality of Φ, and consider an S as in Claim 5.5.13 with $N = n$. For all $\gamma \in T$, let $a_\gamma = \{\alpha \in S \mid |e(\alpha,\gamma)| = n\}$. Consider then any $\gamma \in T$ and $\delta > \gamma$ in S. If $e(\gamma,\delta) \neq \varnothing$, then $e(\alpha,\gamma) \cap e(\gamma,\delta) \neq \varnothing$ for all $\alpha \in a_\gamma \cap (\max(e(\gamma,\delta)),\gamma)$, since by assumption $|e(\alpha,\gamma)| = n$ and $n \geq |e(\alpha,\delta)| \geq |e(\alpha,\gamma) \triangle e(\gamma,\delta)|$. Therefore one of the two following possibilities holds:

1. For all γ in some stationary $U \subseteq T$ there exists some $\beta_\gamma < \gamma$ such that $e(\alpha,\gamma) \cap \beta_\gamma \neq \varnothing$ for all $\alpha \in a_\gamma \cap (\beta_\gamma, \gamma)$. By the Pressing Down Lemma, $\beta_\gamma = \beta$ for all γ in some stationary $V \subseteq U$. Claim 5.5.13 then holds for some $N < n$ and stationary set $S \subseteq V$ for the coherent family

$$\Phi\restriction_{(\beta,\gamma)} := \{\varphi_\gamma\restriction_{\beta,\gamma)} \mid \gamma \in (\beta,\varepsilon)\}$$

   By the induction hypothesis, $\Phi\restriction_{(\beta,\gamma)}$ is trivial; in consequence, Φ is, as well.

2. For all γ in some stationary $U \subseteq T$,

$$e(\gamma,\delta) = \varnothing \text{ for all } \delta \in U \backslash \gamma$$

   (This alternative follows from our argument above that if $e(\gamma,\delta) = \varnothing$ for *any* $\delta \in S$ then there exists some $\beta_\gamma < \gamma$ as in item 1.) In this case, $\bigcup_{\gamma \in U} \varphi_\gamma$ trivializes Φ.

$\square$

CHAPTER 6

CONCLUSION

We close with a summary diagram of the above computations. For uniformity, regard the cohomology pictured as with respect to $\mathcal{A}_d$, where $A = \bigoplus_{\aleph_\omega} \mathbb{Z}$.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\check{\mathrm{H}}^3$	**0**	**0**	**0**	**nonzero**	consistently nonzero	$\dots$
$\check{\mathrm{H}}^2$	**0**	**0**	**nonzero**	consistently nonzero	consistently nonzero	$\dots$
$\check{\mathrm{H}}^1$	**0**	**nonzero**	*independent*	*independent*	*independent*	$\dots$
$\check{\mathrm{H}}^0$	**nonzero**	**nonzero**	**nonzero**	**nonzero**	**nonzero**	$\dots$
	ω	ω_1	ω_2	ω_3	ω_4	$\dots$

Pictured, plainly, are phenomena of dimension; the evocation of spheres in the above diagram is strong. zin this research, a tentative perspective emerges in which the combinatorics we, as set theorists, have tended to look for have been first-cohomological, two-dimensional, or modeled perhaps too insistently on the combinatorics of ω_1. Recall again Theorem 3.3.5, in which ω_n is the least ordinal admitting no good n-dimensional simplicial complex, for example: this points, if nothing else, to *something about* ω_n *that* n *dimensions don't accommodate, but that* $n + 1$ *dimensions do.* We've aimed here to show that "something" deserving of our attention for some time to come.

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