TOWARD A POLYNOMIAL BOUND ON DP:
A SPECIAL CASE

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In [3], Davis and Putnam exhibited an algorithm for determining whether or not a formula in the propositional calculus, in conjunctive normal form (CNF) is unsatisfiable, by eliminating one variable after another from certain derived formulas. The dual of this algorithm determines whether a proposition in DNF (disjunctive normal form) is a tautology. The number of iterations required is no larger than the number of variables, but there is no obvious sub-exponential bound on the time and space taken for each iteration, as a function of the initial space used in a standard propositional representation of the formula.

In [1], Cook showed that a polynomial bound on this would yield polynomial bounds on some other interesting problems, including the testing of an arbitrary propositional formula for truth, and in general any problem solvable in polynomial time by a nondeterministic Turing machine. In response to an inquiry last summer, Mr. Cook sent me two examples bearing on this question [2]. One shows that if variables are eliminated in an unfavorable order, the space required by the Davis and Putnam algorithm will be exponential, regardless of any evident modifications that could be made to the procedure. The other is an example in which, regardless of the order of elimination of variables, the algorithm, if followed literally (as by a computer) would require an exponential space. The former example can be set aside for the moment by agreeing to eliminate the variables in what appears,
as one proceeds, to be a favorable order. The latter example is the one I am concerned with here. What I have shown is that following Cook's suggestion in his communication [2], namely, at each iteration, to eliminate redundant clauses, yields a polynomial bound on a class of formulas that includes the one he exhibited. (Mr. Simon [5] independently discovered the same class of formulas, but did not suggest a way to simplify their analysis.)

For the sake of simplicity in some of my proofs, I have adopted a notation which represents formulas in DNF (and by a slight change, in CNF). The variables of the propositional calculus are indexed by the natural numbers. I am concerned then with formulas $A \lor B \lor \ldots \lor C$, where $A$ might be for example $a_1 \land \overline{a}_{13} \land a_{25}$, $B$ might be $a_2 \land a_3 \land a_4 \land \overline{a}_{4}$, etc. More explicitly,

Def$^R$: Pick $N > 0$. Let "literal" mean $a_i$ or $\overline{a}_i$ for $1 \leq i \leq N$. Let "clause" mean "conjunct of literals" (i.e. a string of literals separated by "and", written "\land"). Then a "formula in DNF" is any (finite) disjunct of clauses (i.e. string of clauses separate by $\lor$).

I will replace a formula in DNF by "a DNF", defined as follows:

Def$^R$: An "element" means a member of the Boolean lattice $\{0, 1, +, \}$, where $+v = 1$, $+a = 0$. All the usual relations are defined as usual e.g. $0 < +, 1 > -, +2 = +$, etc. In particular $x \equiv y$ iff $xy = y$. A "row" means a vector or string of $N$ elements, e.g. $(+1, +, -, -)$ or $11++++1$ ($N$ being 5 and 7 respectively). All the operations of the lattice are extended elementwise, e.g.

$$(0, +, 1, +, -) \prec (+, +, 1, 1, -); (0++)v(1-0) = 11.$$
Parentheses will be used only for readability. If \( R \) is a row, \( R[i] \) is the \( i^{th} \) element of \( R \). A "DNF" is a set of rows, all of the same length. If \( A \) is a DNF, then \( \text{VARS}(A) \) means the length of any row of \( A \).

**Def**: If \( S \) is a clause in a formula in \( N \) variables (i.e. a formula containing \( a_N \) but containing no variable with subscript greater than \( N \)), then the "DNF for \( S \)" is the row \( R \) such that

1. if \( a_1 \) does not occur in \( S \), \( R[i]=1; \)
2. if \( a_1 \) and \( \bar{a}_1 \) are literals in \( S \), \( R[i]=0; \)
3. if \( a_1 \) is a literal in \( S \), but not \( \bar{a}_1 \), then \( R[i]=+; \)
4. if \( \bar{a}_1 \) is a literal in \( S \), but not \( a_1 \), then \( R[i]=-.. \)

If \( B \) is a formula in \( N \) variables in DNF, then \( A \) is the DNF for \( B \) iff \( A=[R|S \text{ is a clause of } B \text{ and } R \text{ is the DNF for } S] \) (thus \( \text{VARS}(A)=N \) and \( A \) has no more rows than \( B \) has clauses; since \( A \) is a set, it has no repetitions of rows).

**Remark 1**: To represent a formula in CNF, interchange 0 and 1, \( A \) and \( V \), "unsatisfiable" and "tautology", "true" and "false", throughout this paper.

**Examples:**

1. \( E=\neg v(a\land a)v(a\land \neg a)v(a\land \neg \neg a) \)
   \[ A=\{+,0\} \]
2. \( B=(a_1\land a_2)v(a_1\land \neg a_3)v(a_1\land a_2\land a_4) \)
   \[ A=\{+1+1,+1-1,+1++\} \]

For convenience one can write \( A \) as

\[
A = \begin{pmatrix}
+1+1 \\
+1-1 \\
+1++
\end{pmatrix} = \begin{pmatrix}
+11- \\
+1+1 \\
+1++
\end{pmatrix}
\]

(changing order of rows does not change the set \( A \)). The latter scheme is handy for computing, and also allows us to refer to
Columns of $A$. This term will not be defined, as it is not needed in proofs.

Remark 2: Since some DNF's have only one row, all statements and definitions concerning DNF's apply to rows likewise; this fact will be used tacitly hereinafter.

I have chosen the notation so that the logical functions in formulas are equivalent to the Boolean functions in DNF's. Thus if 0 represents False, and 1, True, then

$$(a_1 \land a_2) \land (\overline{a_2} \land \overline{a_3}) = (a_1 \land a_2 \land \overline{0}) = 0;$$

in row notation,

$$+1 \land +1 = - + + 0;$$

I shall describe later how to reflect the fact that this represents an unsatisfiable clause.

From here on I shall phrase all definitions and proofs in the "row-DNF" terminology; the reader is referred to [3] for an alternate description and proof of the Davis-Putnam procedure for propositional formulas.

Convention: Throughout, $A$ and $B$ are arbitrary DNF's.

Def$: $B$ is a valuation of $A$ if $\text{VARS}(B) = \text{VARS}(A)$ and there is a function $\nu$ (called the valuation function), satisfying

$$\nu: \{1, \ldots, \text{VARS}(A)\} \rightarrow \{0, 1\};$$

and $R$ is a row of $B$ iff there is a row $S$ of $A$ with

$$R[i] = \begin{cases} 
1 & \text{if } S[i] = 1 \\
1 & \text{or } S[i] = + \text{ and } \nu(i) = 1 \\
0 & \text{or } S[i] = - \text{ and } \nu(i) = 0 \\
0 & \text{otherwise}
\end{cases}$$

Def$: If $\nu$ is a valuation function, then $U$, also called a valuation function, is the function such that $U(R) =$ the valuation of $R$
under u.

Convention: If X is a symbol, "X's" means "occurrences of X".

Def^n: B is a valuation iff it is a DNF containing only 0's and 1's.

Def^n: 1 (an underlined one) is a row of all 1's of the appropriate length (and 0 a row of all 0's).

Def^n: A valuation B is true iff 1 is a row of B. (B is false otherwise.)

Def^n: A is a tautology (or true) iff every valuation of A is true. (A is false otherwise.)

Def^n: A is equivalent to B iff VARS(A) = VARS(B) and for every valuation v, v(A) is true iff v(B) is true.

Remark 3: If the symmetrical set difference of A and B (i.e. (B ∩ A^c) U (A ∩ B^c)) contains only rows that have 0 as some element, then A and B are equivalent.

Pf: Said rows can never contain all 1's. QED. (This is how unsatisfiable clauses may be dealt with.)

The notion of "no redundant clauses" is specified by the term "reduced". Intuitively, a formula in DNF is reduced if you are never wasting your time (for the purpose of finding a tautology) looking at some clause; two ways you know for sure you are wasting your time looking at a clause are (1) The clause is never true, i.e. contains a and 5; (2) If the clause in question is true only when some other clause is true. This motivates the following definition:

Def^n: A DNF A is reduced iff

(1) for every row R and every i, 1 ≤ i ≤ VARS(A), R[i] ≠ 0; and

(2) for every pair of rows R and S, RVS≠R (and ≠S).
\textbf{Def.} We say $R$ is subsumed by $S$ iff $RV_S=S$ (i.e. $R \leq S$). *

\textbf{Def.} The reduced form of a DNF $A$ is a DNF $B$ such that

1. $B$ is reduced;
2. $R$ is a row of $B$ only if $R$ is a row of $A$;
3. if $R$ is a row of $A$, then either $R$ contains a 0 or there is some row $S$ of $B$, $RV_S=S$.

"To reduce" $A$ means to replace $A$ by a reduced form of $A$.

\textbf{Remark 4.} Since $A$ is a set of rows, "$R \in A$" iff "$R$ is a row of $A$".

\textbf{Lemma 1.} The reduced form of $A$ is unique.

\textbf{Pf.} Let $B,C$ be reduced forms of $A$.

Let $R$ be a row of $B$.

To show $R$ is a row of $C$ (hence $B \leq C$, but since $B$ and $C$ are arbitra\begin{array}{c}C \leq B \text{ and } C=\text{C}\end{array}$). Well, $R \in B \Rightarrow R \in A$ by defn of "reduced form".

Hence $\exists S$, $S \in C$, $R \leq S$. But $S \in C \Rightarrow S \in A$, so $\exists T$, $T \in B$, $S \leq T$.

By transitivity of $\leq$, $R \leq T$, so since $R$ and $T$ are both rows of $B$ and $B$ is reduced and $R \leq T$, $R=T$. QED.

\textbf{Notation:} If $S=f(R)$ then $(f(R))[i]$ denotes $S[i]$.

\textbf{Lemma 2.} Reduction preserves truth, i.e. $A$ is a tautology iff the reduced form of $A$ is a tautology.

\textbf{Pf.} We will show in fact that if $B$ is the reduced form of $A$, and $v$ is a valuation function, then the valuation of $A$ under $v$ is true iff the valuation of $B$ under $v$ is true

\begin{itemize}
  \item[(+) Let $R \in A$, $V(R)=1$, then $\exists S \in B$ and $R \leq S$ (since if $V(R)=1$, $R$ contains no zeroes). Now if $R[i] \neq 0$ yet $R[i] \leq S[i]$ then either $R[i]=S[i]$ or $S[i]=1$. Hence $(V(R))[i]=1+(V(S))[i]=1$, so $V(R)=1+V(S)=1$ and the valuation of $B$ under $v$ is indeed true.
  \end{itemize}

*My apologies for having reversed the usual definition.
(+) $B \subseteq A$, so if $R \in B$ with $V(R) = 1$, then $R \in A$ and $V(R) = 1$ and the valuation of $A$ under $v$ is true. Q.E.D.

I will next define the equivalent of one iteration of Davis and Putnam's procedure. "Rule 3" of [3] is called "PRODUCT" here; Rules 1 and 2 of [3] are taken care of by the reduction procedure; they are merely special cases of Rule 3. I will not prove that this method is equivalent to the one they define.

Def: $\otimes$ is an operation between elements defined by

\[ 1 \otimes 1 = 1, \quad 1 \otimes b = 0 \text{ otherwise.} \]

Def: $\oplus$ is an operation between rows of equal length defined by

\[ (R \oplus S)[i] = (R[i] \otimes S[i]) \cup (R[i+1] \cap S[i+1]), \]

$1 \leq i \leq \text{VARS}(R)-1$, and $\text{VARS}(R \oplus S) = \text{VARS}(R)-1$.

Def: PRODUCT is a function on DNF's defined by

\[ \text{PRODUCT}(E) = \{ R \oplus S \mid R, S \text{ are rows of } E \} \text{ (including } R R \text{ for each } R, \text{ of course).} \]

Although I will not prove that this is equivalent to Rule 3 of [3], I will indicate why it is: namely, if $a_1$ is the variable being eliminated, then (1) if $R$ represents a clause $R'$ not containing $a_1$, then $R \oplus R$ (which still represents $R'$) will occur in PRODUCT($A$); (2) if $R'$ contains $a_1$ and $S'$ contains $\overline{a}_1$, then $R \oplus S$ occurs, which represents $(R' \text{ with } a_1 \text{ deleted}) \cup (S' \text{ with } \overline{a}_1 \text{ deleted})$. There are also some extra clauses inserted (other than unsatisfiable ones), but they all represent $R' \cap S'$ where $R'$ and $S'$ are both clauses not containing $a_1$, and hence $R'$ will also occur, and $R' \cap S'$ will be subsumed by $R'$ in the reduced form of PRODUCT($A$).

Def: PROD($A$) = the reduced form of PRODUCT($A$).

Notation: For any function $F$, $F_A = F(A)$. I now show an important
result (which saves proving the equivalence of PROD with one
iteration of DP using subsumption):

**Lemma 3:** PROD preserves truth, i.e. A is true iff PROD(A) is true

**Pf:** Since I have shown that reduction preserves truth, it remain
to show that PRODUCT does also.

(\(\rightarrow\)) Given that A is true and \(v\) is a valuation of \(B=\text{PRODUCT} A\);
to show that for some \(T \in B\), \(v(T)=\perp\). Note that there are two
possible extensions of \(v\) to a valuation of \(A\), namely \(u_1\) and \(u_2\)
given by:

\[
\begin{align*}
  u_1(i) &= \begin{cases} 
    v(i-1) & \text{if } i>1 \\
    1 & \text{if } i=1 \\
    u_2(i) & \text{if } i>1 \\
    0 & \text{if } i=1 
  \end{cases} \\
\end{align*}
\]

and

\[
\begin{align*}
  u_1(i) &= \begin{cases} 
    v(i) & \text{if } i>1 \\
    1 & \text{if } i=1 \\
    u_2(i) & \text{if } i>1 \\
    0 & \text{if } i=1 
  \end{cases} \\
\end{align*}
\]

There are 2 cases:

1. a row \(R\) of \(A\) exists such that \(u_1(R)=u_2(R)=\perp\);
2. \(u_1(R)=\perp\) only if \(R[1]=+\), and \(u_2(R)=\perp\) only if \(R[1]=-\).

In case (1), \(V(R\oplus R)=\perp\) and we are done.

In case (2), let \(R,S\) be chosen so that \(u_1(R)=u_2(S)=\perp\).

**Claim:** \(V(R\land S)=V(R)\land V(S)\)

**Pf:** WLOG \(\text{VARS}(R)=\text{VARS}(S)=1\)

By cases:

(i) if \(R=1\) then

\[
V(R\land S)=V(1\land S)=V(S)
\]

and

\[
V(R\land S)=V(1)\land V(S)=1\land V(S)=V(S)
\]

(ii) if \(R=0\) then

\[
V(R\land S)=V(0\land S)=V(0)=0
\]

and

\[
V(R)\land V(S)=V(1)\land V(S)=1\land V(S)=V(S)
\]

(iii) if \(R=S\) then

\[
V(R\land S)=V(R\land R)=V(R)
\]

\[
V(R\land S)=V(R)\land V(R)=V(R)
\]

(iv) if \(R=+\) and \(S=-\) then

\[
V(R\land S)=V(+\land -)=V(0)=0
\]

and

\[
V(+)\land V(-)=1\land 0=0, \text{ so } \land 0=0=0
\]

Since \(\land\) is symmetric, this proves the claim.
Now consider $V(R \oplus S)$.

$$(R \oplus S)[i] = (R \wedge S)[i+1], \text{ so}$$

$$(V(R \oplus S))[i] = (U_1(R \wedge S))[i+1]$$

$$= (U_1(R))[i+1] \wedge (U_2(S))[i+1]$$

$$= (U_1(R))[i+1] \wedge (U_2(S))[i+1] \text{ since}$$

$$U_1(i) = U_2(i), \ i > 1$$

$$= 1 \wedge 1 \text{ since } U_1(R) = U_2(S) = 1.$$ 

This is what we wanted to prove.

(+) Suppose $A$ is false, let $B$ be a false valuation of $A$, then I will show that if $v$ is the valuation function for $B$, and if $u(i) = v(i+1), 1 \leq i \leq \text{VARS}(A) - 1$, then the valuation $C$ of $\text{PRODUCT } A$ under $u$ is false, i.e. every row of $C$ contains a 0.


Since $T[i] = R[i+1] \wedge S[i+1]$, and since $V(R \wedge S) = V(R) \wedge V(S)$, we need only show that for some $j > 1$, either $(V(R))[i] = 0$ or $(V(S))[i] = 0$.

If $R[1] = S[1] = 1$ then $(V(R))[1] = 1$, and since $V(R)$ contains a 0, $(V(R))[j] = 0$ for some $j > 0$, as required. Now if $v(1) = 1$, then $(V(R))[1] = 1$, so again $(V(R))[j] = 0$, for some $j > 1$. If $v(1) = 0$, then $(V(S))[1] = 1$, so now $(V(S))[j] = 0$ for some $j > 0$, as required.

Q.E.D.

A few more definitions are needed before defining my version of the Davis-Putnam procedure.

**Def:** If $\sigma$ is a permutation of $\{1 | 1 \leq i \leq \text{VARS} A\}$, then $\sigma A = B$, where $R$ is a row of $B$ iff $r S$, $S \cap A$ and $R[1] = S[\sigma i]$, $1 \leq i \leq \text{VARS}(A)$. (In other words, $\sigma A$ is a permutation of the columns of $A$ [among columns, not within columns]).

**Def:** $|X|$ = cardinality of $X$ (thus $|A|$ = the number of rows of $A$).
Def: $\text{PLUS}(A,i) = \{ R \mid R \text{ is a row of } A \text{ and } R[i] = + \}$

$\text{MINUS}(A,i) = \{ R \mid R \in A, R[i] = - \}$

Def: $B$ is a **locally optimal** form of $A$ if $B$ is a permutation of $A$ and $\text{PLUS}(B,1) \cdot \text{MINUS}(B,1) = \min\{ \text{PLUS}(A,i) \cdot \text{MINUS}(A,i) \mid 1 < i < \text{VARS}(A) \}$

Remark 5: Local optimization is not unique, e.g. $A = \begin{pmatrix} +1+ \\ +1- \\ 1-- \end{pmatrix}$.

Def: The Davis-Putnam procedure for DNF's (denoted DPDNF) is as follows (in bastardized program form):

- **Given the DNF $A$**
- **Begin:** $A \leftarrow$ Locally optimal form of $A$
- $A \leftarrow \text{PROD} A$
- If $A = 1$ or $A = \{+, -\}$ then say "yes" and halt.
- If $A$ is empty then say "no" and halt.
- **GO TO BEGIN**

As I indicated earlier, I wish to investigate DNF's $A$ such that every permutation of $A$ is a locally optimal form of $A$. A simple class satisfying that condition is the class of symmetric DNF's, as defined below (an extension of the class will be discussed later). In terms of propositional formulas, a formula is symmetric iff it is invariant under interchanging names of variables. This is my own definition, but I presume it to be the one Mr. Cook had in mind in his communication [2].

Def: RSDNF stands for "reduced symmetric DNF", i.e. a DNF which is both symmetric and reduced.

Convention: Hereinafter $\sigma$ denotes an arbitrary permutation, and $E$ and $F$ will in general denote RSDNF's (exceptions will be noted or evident from context).

A more useful characterization of symmetry is as follows:
Lemma 4: A DNF $E$ is symmetrical iff $RFE$ and $\sigma$ is a permutation imply $\sigma RE$.

Pf: (+) if $E$ is symmetrical, and $\sigma$ is a permutation, then $\sigma E = E$; if $R$ is a row of $E$, $\sigma R$ is a row of $\sigma E$, and hence by definition of set equality, $\sigma RE$.

(+) if $RFE + \sigma RE$, then $\sigma E = E$; since permutations are bijective, $\sigma E = E$.

Q.E.D.

Defn: For $R$ a row, $R^+$ denotes the number of +'s in $R$, and $R^-$, the number of −'s.

Convention: Unless otherwise stated, "a row" means "a row containing no 0's".

Notice that by Lemma 4, and since permutations are bijective, an RSDNF $E$ is characterized by (1) $\text{VARS}(E)$, and

(2) the number of different numbers of +'s and −'s in the various rows.

(2) is formalized as follows:

Defn: A row-type is an ordered pair of integers. The row-type of $R$, for $R$ a row, is $(R^+, R^-)$. $(a, b)$ is a row-type of a DNF $A$ if $3R, RCA, (R^+, R^-) = (a, b)$.

Thus an RSDNF $E$ is completely specified by $\text{VARS}(E)$ and {row-types of $E$}.

Defn: $(a, b)$ subsumes $(c, d)$ iff $(a, b) \preceq (c, d)$, i.e. $a \leq c$ and $b \leq d$.

Remark 6: If $(R^+, R^-)$ subsumes $(S^+, S^-)$, then $3\sigma, \sigma R$ subsumes $S$.

Pf: Since $R^+ \leq S^+$, and $R^- \leq S^-$, we can find $\sigma$ so that

$(\sigma R)[i] = +$ only if $S[i] = +$
$(\sigma R)[i] = -$ only if $S[i] = -$
Thus (since \( R \) and \( S \) have no 0's, by convention) \((\sigma R)\Sigma=\sigma R\), i.e. \( \sigma R \) subsumes \( S \). Q.E.D.

**Note:** \( R \preceq S \iff (R^+,R^-)\preceq (S^+,S^-) \) (i.e. \( \preceq \) has the opposite meaning for rows as for row-types).

Thus we can characterize an RSDNF as a symmetric DNF \( E \) such that if \((a,b)\) and \((c,d)\) are distinct row-types of \( E \), then \((a,b)\) does not subsume \((c,d)\).

**Lemma 5:** If \( E \) is an RSDNF, then for every pair of distinct row-types \((a,b)\) and \((c,d)\) of \( E \), either \( a\prec c \) and \( b\succ d \), or \( a\succ c \) and \( b\prec d \).

**Pf:** Otherwise \( (a,b)\nsim (c,d) \) or \((a,b)\nsim (c,d)\).

**Def:** \( (a,b)\nsim (c,d) \) iff \( a\succ c \) and \( b\prec d \).

**Def:** \( (a,b)\nsim (c,d) \) iff \((c,d)\nsim (a,b)\).

(Note that \( \nsim \) is transitive)

Thus Lemma 5 may be restated:

**Coro 5:** If \( E \) is a RSDNF and \( p,q \) are distinct row-types of \( E \), either \( p\nsim q \) or \( q\nsim p \).

**Def:** \( (a,b) \) is adjacent (in \( E \)) to \((c,d)\) iff \((a,b)\) and \((c,d)\) are row-types of \( E \), \((a,b)\nsim (c,d)\) and if \((e,f)\) is a row-type of \( E \), then either \((e,f)\nsim (a,b)\) or \((c,d)\nsim (e,f)\). "\((a,b)\) is adjacent to \((c,d)\)" is denoted "\((a,b) \text{ adj} (c,d)\)".

**Remark 7:** Adjacency is unique, i.e. if \((a,b) \text{ adj} (c,d)\) and \((a,b) \text{ adj} (e,f)\) then \((c,d)\equiv (e,f)\).

**Def:** \( R \text{ adj} S \) (in \( E \)) iff \((R^+,R^-) \text{ adj} (S^+,S^-) \) (in \( E \)).

**Def:** \( \text{RT}(E)\!=\!(\text{row-types of } E) \), so that "\((a,b)\) is a row-type of \( E \)" may be written "\((a,b)\in \text{RT}(E)\)".
Remark 8: If $E$ is an RSDNF, and $(a,b), (c,d) \in RT(E)$, then 
$(a,b) \Theta (c,d)$ iff $a > c$; and $(a,b) \text{ adj} (c,d)$ iff $a > c$ and no $S \in E$ is $a > S > c$.

Remark 9: $U \oplus U > U \ominus V$ if $U[1] = 1$.

Pf: If $U \ominus V \neq \emptyset$, then $(U \oplus V)[i] = (U \ominus V)[i+1] \leq U[i+1]$; since $U \neq V$, the inequality is strict for some $i$.

Remark 10: If $A$ is any DNF and $(a,b)$ is a row-type of $A$, then $a+b \in \text{VAR}(A)$.

Lemma 6: If $(a,b) \in RT(E)$ with $a+b \in \text{VAR}(E)$, then for some $(c,d) \in RT(\text{PROD} E)$, $(c,d) \leq (a,b)$.

Pf: For some $R$ of row-type $(a,b)$, $R[1] = 1$. For this $R$, 
$R \oplus R \in \text{PRODUCT} E$ and has no 0's, so either $R \in \text{PROD} E$ or $R$ is 
subsumed by some $S \in \text{PROD} E$. By Remark 6 plus the definitions of 
subsumption,

$(s^+, S^-) \leq (R^+, R^-)$

Q.E.D.

Notation: A, B sets; A-B means the set difference of A and B, 
$A \cap B^c$. The motivation for the next definition is Lemma 7 below.

Some examples may be found at the end of the paper.

Def$: If (a,b) \Theta (c,d)$ then $(a,b) \oplus (c,d) = (a-1, d-1)$.

Lemma 7: If $(a,b) \in RT(\text{PROD} E) - RT(E)$, then $(a,b) = (a+1, x) \oplus (y, b+1)$,
where $(a+1, x) \text{ adj} (y, b+1)$ in $E$.

Pf: Let $W \in \text{PROD} E$ with $(W^+, W^-) = (a,b)$. Then $W = U \ominus V$ with 

If $U[1] = V[1] = 1$, then $U^+ + U^- \in \text{VAR}(E)$, so by Lemma 6, there is some 
row-type $(c,d)$ of $\text{PROD} E$ with $(c,d) \leq (U^+, U^-)$; but then by Remark 9, 
since $W = U \ominus V$ with $U[1] = V[1] = 1$, $W$ is subsumed by $U \oplus U$, which is
impossible because PROD \( E \) is reduced. Hence \( U[1]=+ \), \( V[1]=- \).

Then since \( W \) has no 0's, for \( 2 < i \in \text{VARS}(E) \), \( U[i]=1 \) or \( V[i]=1 \) or \( U[i]=V[i] \). Hence, (1) \( W^- \Rightarrow U^- \), \( W^+ \Rightarrow V^+ \)
(2) \( W^- \Rightarrow V^- \), \( W^+ \Rightarrow U^+ \).

Now either \( U^+ \Rightarrow V^+ \) or \( U^+ \Rightarrow V^- \).

If \( U^+ \Rightarrow V^+ \), then \( U^+ + U^- \leq \text{VARS}(E) \) (because for some \( i \geq 2 \), \( V[i]=+ \) and not \( U[i], \) so \( U[i]=1 \)).

But then, by Lemma 6 \( \exists (c,d), (c,d) \in \text{RT}(\text{PROD} \ E), (c,d) \leq (U^+ , U^-) \)
but \( W^+ \Rightarrow V^+ \), so \( W^+ \Rightarrow U^+ \) by the hypothesis that \( V^+ \Rightarrow U^+ \). But \( W^- \Rightarrow U^- \), so \( (W^+ , W^-) \geq (c,d) \) which is impossible since \( \text{PROD} \ E \) is reduced and both \( (W^+ , W^-) \) and \( (c,d) \) are row-types of \( \text{PROD} \ E \). Hence, \( U^+ \Rightarrow V^+ \), \( U^- \Rightarrow V^- \). Now suppose \( U^+ + U^- \leq \text{VARS}(E) \). Then \( W^+ = U^+ - 1 \). Suppose \( U^+ + U^- \leq \text{VARS}(E) \). Then \( (c,d) \in \text{RT}(\text{PROD} \ E) \) with \( (c,d) \leq (U^+ , U^-) \), so if \( W^+ = U^+ \), inequations (1) above imply that \( (W^+ , W^-) \geq (c,d) \); this is impossible. Hence, \( W^+ \leq U^+ \) and (by (2)) \( W^+ \geq U^+ - 1 \), so \( W^+ = U^+ - 1 \). By a similar argument \( W^- = V^+ - 1 \). We now have \((a,b)=(a+1,x) \oplus (y,b+1)\) with \((a+1,x) \oplus (y,b+1)\); we now need to show that for no \((c,d)\) is \((a+1,x) \oplus (c,d) \oplus (y,b+1)\) in \( E \).

Suppose the contrary.

If \( c+d \leq \text{VARS}(E) \), then by Lemma 6, \((a,b)\) is subsumed. But if \( c+d = \text{VARS}(E) \), then since \( a+1 \geq c+1 \), and \( b+1 \geq d+1 \), \( a+b \geq \text{VARS}(E) \), so \( a+b \geq \text{VARS}(\text{PROD} \ E) = \text{VARS}(E) - 1 \), and \((a,b)\) is not a row-type of \( \text{PROD} \ E \).

Hence, there can be no such \((c,d)\), and \((a+1,x) \oplus (y,b+1)\) as claimed.

Q.E.D.

**Notation:** \( \oplus \) means "which is a contradiction".

**Lemma 8:** If \( E \) is an RSDNF then the row-types of \( \text{PROD} \ E \) include
all the row-types \((a, b)\) satisfying "\((a+1, x)\) and \((y, b+1)\) are adjacent row-types of \(E\), and \(a+b < \text{VARS}(E)\)". A useful form of this Lemma is: if \((a_1, b_1) \text{ adj } (a_2, b_2)\) in \(E\) with \(a_1 + b_2 < \text{VARS}(E) + 2\), then \((a_1 - 1, b_2 - 1) \in \text{RT}(\text{PROD } E)\).

Pf: By the proof of Lemma 7, \((a, b)\) will certainly be a row-type of \(\text{PRODUCT } E\); it remains to show that it is not subsumed by some other row-type of \(\text{PRODUCT } E\). Suppose \((c, d)\) subsumes \((a, b)\).

Then either \((c, d) \in \text{RT}(E)\) or by Lemma 7, \((c, d) = (c+1, m) \oplus (n, d+1), (c+1, m) \text{ adj } (n, d+1)\) in \(E\). But if \((c, d) \in \text{RT}(E)\) and \((a+1, x) \text{ adj } (y, b+1)\), \(c < y\) (because \(c < a < y\)) and \(d < b + 1\) (since \(d < b\)), so \((c, d)\) subsumes \((y, b+1) \obs \) since \(E\) is reduced.

But if \((c, d) = (c+1, m) \oplus (n, d+1)\) then since \(c < a, c+1 \leq a+1; d \leq b, \) so \(d+1 \leq b+1\), and since \((a+1, x) \text{ adj } (y, b+1)\), \(d+1 \leq x\) also; \(m < d+1\), so \(m < x\); therefore \((c+1, m) \not\subset (a+1, x)\) (and \(\not\subset\) since \(E\) is reduced. Hence there is no such \((c, d)\).

Q.E.D.

Corollary 8: If \((a, b) \in \text{RT}(E)\) and either \((a+1, x) \in \text{RT}(E)\) or \((y, b+1) \in \text{RT}(E)\), then \((a, b) \not\in \text{RT}(\text{PROD}(E))\).

Pf: If \((a+1, x) \in \text{RT}(E)\) then \((a+1, x) \text{ adj } (a, b)\) by Remark 8, so \((a+1, x) \oplus (a, b) = (a, b-1) \in \text{RT}(\text{PROD } E)\) by Lemma 8; and \((a, b-1)\) subsumes \((a, b)\). Similarly if \((y, b+1) \in \text{RT}(E)\) then \((a-1, b) \in \text{RT}(\text{PROD } E)\).

Q.E.D.

Lemma 9: If \(E\) is an RSDNF and \((a, x) \text{ adj } (y, b)\) in \(E\) and \(a+b > \text{VARS } E+2\), then \((a, x) \oplus (y, b) \not\in \text{RT}(\text{PROD } E)\).

Pf: \((a, x) \oplus (y, b) = (a-1, b-1)\) but \(a-1+b-1 = a+b-2 > \text{VARS } E+2-2\) by hypothesis.

\[ = \text{VARS } \text{PROD } E+1 > \text{VARS } \text{PROD } E\]
so by Remark 10, \((a-1,b-1)\) cannot be a row-type of \(\text{PROD } E\).

Q.E.D.

The following lemma, along with a simple combinational lemma contains the heart of the bound on DP-symmetrical.

**Lemma 10:** \((a_1,b_1) \, \text{adj} \, (a_2,b_2)\) in \(\text{PROD}(E)\), \(E\) an RSDNF.

1. If \((a_1,b_1) \not\in \text{RT}(E)\), either \(b_2 = b_1 + 1\) or \(a_1 + b_2 > \text{VARS}(E) + 1\);
2. If \((a_2,b_2) \not\in \text{RT}(E)\), either \(a_1 = a_2 + 1\) or \(a_1 + b_2 > \text{VARS}(E) + 1\);
3. If \((a_1,b_1)\) and \((a_2,b_2)\) are both row-types of \(E\), then \(a_1 + b_2 > \text{VARS}(E) + 1\).

**Pf:**

1. Suppose \((a_1,b_1) \not\in \text{RT}(E)\) and \((a_2,b_2) \not\in \text{RT}(E)\). (i.e. (1) and (2) both hold) We are done if \(a_1 = a_2 + 1\) and \(b_2 = b_1 + 1\), so assume \(a_1 > a_2 + 2\) or \(b_2 > b_1 + 2\).

Let \((a_1,b_1) = (a_1 + 1,a) \circ (b,b+1)\)

\((a_2,b_2) = (a_2 + 1,c) \circ (d,d+1)\)

Now either \((a)b = a_2 + 1\) and \(c > b_1 + 1\) or (since \(E\) is reduced)

\((b)b > a_2 + 1\) and \(c > b_1 + 1\). If \((a)\) is true, i.e.

\((b,b_1 + 1) = (a_2 + 1,c) = (a_2 + 1,b_1 + 1)\),

then by Lemma 6, either \(a_2 + 1 + b_1 + 1 > \text{VARS}(E)\), or for some \((e,f) \in \text{RT(\text{PROD} E)\), \((e,f) \leq (a_2 + 1,b_1 + 1)\). If \(a_2 + 1 + b_1 + 1 > \text{VARS} E\), the \(a_1 + b_2 > a_2 + 1 + b_1 + 1 + 1 > \text{VARS}(E) + 1\) as desired.

If \((e,f) \in \text{RT(\text{PROD} E)\) subsumes \((a_2 + 1,b_1 + 1)\), then (i) if \(a_2 + 1 < a_1\), \(e < a_1\) and \(f < b_2\). But if \(e < a_1\) then by Remark 8 \(e \leq a_2\), so \((e,f)\) subsumes \((a_2,b_2)\). Hence \((e,f) = (a_2,b_2)\), since \(\text{PROD}(E)\) is reduced at \((a_2,b_2) \in \text{RT}(\text{PROD} E)\). But \(b_2 = f \leq b_1 + 1 \leq b_2\), so \(b_2 = b_1 + 1\) as desired.
Similarly (ii) if $b_1+1 < b_2$, $f < b_2$, and we find that $e = a_1 - a_2 + 1$ as desired. If (b) is true, then by Remark 8,

$$(b, b_1 + 1) \oplus (a_2 + 1, c) \notin RT(\text{PROD} \ E)$$

(Since either $a_1 > a_2 + 1 > a_2$, and $b - 1 > a_2 + 1$, or $b_2 > b_1 + 1 > b_1$, and $c - 1 > b_1 + 1$).

Hence by Lemma 8,

$$b + C \geq VARS \ E + 2$$

but

$$a_1 \geq b \text{ and } b_2 + 1 > C$$

or

$$a_1 + 1 \geq b \text{ and } b_2 \geq C$$

so

$$a_1 + b_2 + b + C - 1 > VARS(E) + 1 \text{ as desired.}$$

(1) $(a_1, b_1) \notin RT(E)$ and $(a_2, b_2) \in RT(E)$.

Then either

(a) $(a_1, b_1) = (a_1 + 1, b) \oplus (a_2, b_2)$

or (b) $(a_1, b_1) = (a_1 + 1, b) \oplus (c, b_1 + 1)$, and

by Lemma 7, $(C, b_1 + 1) > (a_2, b_2)$.

If (a) holds, then $(a_1, b_1) = (a_1, b_2 - 1)$, so $b_2 = b_1 + 1$ as desired.

If (b) holds, then by Lemma 8,

$$C + b_2 \geq VARS(E) + 2,$$

and since $a_1 > C$, $a_1 + b_2 > VARS(E) + 2$

$$b + VARS(E) + 1 \text{ as claimed.}$$

(2) $(a_1, b_1) \in RT(E)$ and $(a_2, b_2) \notin RT(E)$.

(This is a "dual" result to (1); the proof is identical in form.)

(a) $(a_2, b_2) = (a_1, b_1) \oplus (a, b_2 + 1)$

$$= (a_1 - 1, b_2) \text{ and } a_1 = a_2 + 1$$

(b) $(a_2, b_2) = (a_2 + 1, b) \oplus (C, b_2 + 1)$,

$$(a_1, b_1) \oplus (a_2 + 1, b)$$
Lemma 8: \( a_1 + b \geq \text{VARS}(E)+2 \), so 
\( a_1 + b_2 \geq \text{VARS}(E)+1 \) as desired.

(3) \((a_1, b_1) \in \text{CRT}(E)\) and \((a_2, b_2) \in \text{CRT}(E)\).

By Lemma 8 either 
\((a_1, b_1) \otimes (a_1-1, b_2-1) \otimes (a_2, b_2)\) (and so \((a_1, b_1)\) is not adjacent to \((a_2, b_2)\) or else \(a_1 + b_2 \geq \text{VARS}(E)+2\), as claimed. Since \((a_1, b_1)\) is indeed adjacent to 
\((a_2, b_2)\) by hypothesis, we are done.

**Def**: PROD^0_E = E; for \( n > 1 \), PROD^n_E = PROD(\text{PROD}^{n-1}_E).

**Remark 11**: For RSDNF's, application of \text{PROD}^n to E is the same as \( n \) iterations of DPDNF, until the step at which DPDNF halts. Note that \(|\text{RT}(E)|\) is the number of row-types of E.

**Coro 10**: If E is an RSDNF and \( n > 1 \) and \((a_1, b_1) \text{ adj} (a_2, b_2)\) in \( \text{PROD}^n_E \), then either 
\( a_1 + b_2 \geq \text{VARS}(\text{PROD}^n_E)+2 \) 
or \( a_1 = a_2 + 1 \) and \( b_2 = b_1 + 1 \) 
or \((a_1, b_1) \in \text{CRT}(E)\) with \( a_1 = a_2 + 1 \) 
or \((a_2, b_2) \in \text{CRT}(E)\) with \( b_2 = b_1 + 1 \)

**Pf**: (By induction on n)

**Basis**: \( n = 1 \): If \((a_1, b_1) \in \text{CRT}(E)\) and \((a_2, b_2) \in \text{CRT}(E)\), then by Lemma 10 (2),
either \( a_1 = a_2 + 1 \) 
or \( a_1 + b_2 \geq \text{VARS}(E)+1 = \text{VARS}(\text{PROD} E)+2 \).

If \((a_2, b_2) \in \text{CRT}(E)\) and \((a_1, b_1) \in \text{CRT}(E)\) then by Lemma 10(1) 
either \( b_2 = b_1 + 1 \) or \( a_1 + b_2 \geq \text{VARS}(\text{PROD} E)+2 \).
If neither \((a_1, b_1)\) nor \((a_2, b_2)\) is a row-type of \(E\), then by Lemma 10 (1) and (2), either \(a_1 + b_2 \geq \text{VAR}(\text{PROD} E) + 2\), or \(a_1 = a_2 + 1\) and \(b_2 = b_1 + 1\). If both \((a_1, b_1)\) and \((a_2, b_2)\) are row-types of \(E\), then by Lemma 10(3), \(a_1 + b_2 \geq \text{VAR}(\text{PROD} E) + 2\). This exhausts the cases.

**Inductive step.** Given that Coro 10 holds for \(n-1\), to show it holds for \(n, n \geq 2\)

(1) If \((a_1, b_1) \in \text{RT}(\text{PROD}^{n-1} E)\), and \((a_2, b_2) \in \text{RT}(\text{PROD}^{n-1} E)\), then by Lemma 10(2) either

\[ a_1 + b_2 \geq \text{VAR}(\text{PROD}^{n-1} E) + 1 \geq \text{VAR}(\text{PROD}^n E) + 2, \text{ or } a_1 = a_2 + 1. \]

I now show that if \(a_1 = a_2 + 1\), then \((a_1, b_1) \in \text{RT}(E)\).

Since \((a_2, b_2) \in \text{RT}(\text{PROD}^{n-1} E)\),

\[ (a_2, b_2) = (c, d) \oplus (e, f). \]

By defn of \(\oplus\), \(f = b_2 + 1\) and \(c = a_2 + 1\);

Since \(a_1 = a_2 + 1, a = c, \) so since \(\text{PROD}^{n-1} E\) is reduced, \(d = b_1\).

Therefore

\[ (a_2, b_2) = (a_1, b_1) \oplus (e, b_2 + 1), \text{ and by Lemma 7, } \]

\[ (a_1, b_1) \text{ adj } (e, b_2 + 1) \text{ in } \text{PROD}^{n-1} E. \]

Moreover by Remark 10, \(a_2 + b_2 \leq \text{VAR}(\text{PROD}^n E)\).

But \(a_2 + b_2 = a_1 + 1 + b_2\), so \(a_1 + b_2 + 1 \leq \text{VAR}(\text{PROD}^n E) + 1\)

\[ = \text{VAR}(\text{PROD}^{n-1} E) \]

\[ < \text{VAR}(\text{PROD}^{n-1} E) + 2 \]

So applying the induction hypothesis to \((a_1, b_1) \text{ adj } (e, b_2 + 1)\) in \(\text{PROD}^{n-1} E\), we have

either (a) \(a_1 = e + 1\) and \(b_2 + 1 = b_1 + 1\)

or (b) \((a_1, b_1) \in \text{RT}(E)\) and \(a_1 = e + 1\)
or \((c)\) \((a_2, b_2) \in \text{RT}(E)\) and \(b_2 + 1 = b_1 + 1\)

(a) and (c) are impossible because they imply \(b_1 = b_2\).

Hence (b) holds, as desired.

(2) If \((a_2, b_2) \in \text{RT}(\text{PROD}^{n-1}E)\) and \((a_1, b_1) \in \text{RT}(\text{PROD}^{n-1}E)\) then

a "dual" of the preceding proof yields \((a_2, b_2) \in \text{RT}(E)\) and

\(b_2 = b_1 + 1\).

(3) If both \((a_1, b_1)\) and \((a_2, b_2)\) are row-types of \(\text{PROD}^{n-1}E\),

then by Lemma 10(3)

\[ a_1 + b_2 > \text{VARS}(\text{PROD}^{n-1}E) + 1 \]

\[ = \text{VARS}(\text{PROD}^nE) + 2 \]

(4) If neither \((a_1, b_1)\) nor \((a_2, b_2)\) is a row-type of \(\text{PROD}^{n-1}E\),

then by Lemma 10(1) and (2),

\[ a_1 = a_2 + 1 \text{ and } b_2 = b_1 + 1. \]

Q.E.D.

**Def**: \(R_n(E) = |\text{RT}(\text{PROD}^nE)|\)

**Thm**: \(R_n(E) < R_0(E) + n(R_0(E) - 1)\)

**Pf**: Trivial for \(n=0\).

By Coro 10 and Coro 8, with \(n \geq 1\), if \((a, b) \text{ adj } (c, d) \in \text{PROD}^{n-1}E\),
and if \((a, b) \nabla (c, d) \in \text{RT}(\text{PROD}^nE)\), then \((a, b) \in \text{RT}(\text{PROD}^nE) \rightarrow (a, b) \in \text{RT}(\text{PROD}^{n-1}E) \rightarrow (c, d) \in \text{RT}(E)\).

Thus the number of row-types that are in \(\text{PROD}^{n-1}E\) but are not in \(\text{PROD}^nE\) is at least \(R_{n-1}(E) - R_0(E)\). Moreover adjacency is unique, so the number of row-types in \(\text{PROD}^nE\) but not in \(\text{PROD}^{n-1}E\) is at most \(R_{n-1}(E) - 1\). Therefore \(R_n(E) - R_{n-1}(E)\)

\[ \leq (R_{n-1}(E) - 1) - (R_{n-1}(E) - R_0(E)) \]

\[ = R_0(E) - 1 \]
\[ R_n(E) = R_0(E) + \sum_{i=1}^{n} (R_1(E) - R_{i-1}(E)) \]
\[ \leq R_0(E) + \sum_{i=1}^{n} (R_0(E) - 1) \text{ by the above} \]
\[ = R_0(E) + n(R_0(E) - 1) \]
Q.E.D.

Now that we know how many row-types PROD\(^n\)E has, we need to know how many rows each row-type can have.

**Lemma 11:** If \( E \) is an RSDNF, \( VARS(E) = V, (a,b) \in RT(E) \), then the number of rows in \( E \) of row-type \( (a,b) \) is
\[
\frac{(V)(a+b)}{a!b!}, \text{ where} \]
\[
(m) \equiv \frac{m!}{q!} = m \cdot (m-1) \cdot \ldots \cdot (m-q+1) \]

**Pf:** By Lemma 4, we must count all rows of length \( V \) with a +'s and b -'s. There are \( V \) places to put the first + or -, \( V-1 \) to put the second, \ldots, \( V-a-b+1 \) places to put the last, and these choices are independent, so there are \( V \cdot (V-1) \cdot \ldots \cdot (V-a+b+1) = (V)_{a+b} \) ways to pick spots for the +'s and -'s. However, since +'s are indistinguishable this number must be divided by the \( a! \) ways to interchange +'s, and we must divide again by \( b! \) for the -'s, whence \( \frac{(V)(a+b)}{a!b!} \) (For a more careful proof, see Feller, Introduction to Probability, 3rd Ed., Vol.1, Ch.II.4, Thm.2 (p.37))

Next, the simple combinatorial lemma mentioned earlier.

**Lemma 12:** If \( N \geq b+1 \), then
\[
\frac{(N-1)(a+b)}{a!b!} \leq \frac{(N)(a+b+1)}{a!(b+1)!} \]
Pf: \( A \subseteq B \) iff \( A \cdot \frac{1}{B} \leq 1 \). So, consider

\[
\begin{align*}
\frac{(N-1)(a+b)}{a!b!} & \cdot \frac{a!(b+1)!}{(N)(a+b+1)} \\
\frac{(N-1)(a+b)}{a!b!} & \cdot \frac{a!(b+1)b!}{N \cdot (N-1)(a+b)} \\
\frac{b+1}{N} & \leq 1 \text{ by hypothesis.}
\end{align*}
\]

Q.E.D.

**Def**\(^n\): Let \( \text{RMAX}_n(E) = \max(\text{number of rows of row-type } (a, b)|(a, b)) \) is a row-type of \( \text{PROD}^n(E) \). (i.e. \( \text{RMAX}_n(E) \) is the greatest number of rows that are of any given row-type, in \( \text{PROD}^n(E) \)).

**Lemma 13**: If \( n \geq 1 \) then \( \text{RMAX}_n(E) \leq \text{RMAX}_{n-1}(E) \)

Pf: Note: Throughout this proof, "less than" means "\( \leq \)". For some \((a, b) \in \text{CRT}(\text{PROD}^{n-1}E)\), \( \text{RMAX}_{n-1}(E) \) the number of rows of row-type \((a, b)\) in \( \text{PROD}^{n-1}E \) (since \( |\text{RT}(\text{PROD}^{n-1}E)| \) is finite)

\[
\frac{(VARS \cdot E-n-1)(a+b)}{a!b!} \text{ by Lemma 11.}
\]

Since this is the maximum, if \((c, d) \in \text{CRT}(\text{PROD}^{n-1}E)\), then

\[
\frac{(VARS \cdot E-n+1)(c+d)}{c!d!} \leq \frac{(VARS \cdot E-n+1)(a+b)}{a!b!}.
\]

If \((a', b') \in \text{CRT}(\text{PROD}^nE)\), then by Lemmas 9 and 10, either

(a) \((a', b') = (c, d)\) for some \((c, d)\) a row-type of \( \text{PROD}^{n-1}E \);

or (b) \((a', b') = (a'+1, d_1) \oplus (a', b'+1)\)

or (c) \((a', b') = (a'+1, b') \oplus (c, b'+1)\)

In case (a), \((c, d)\) has

\[
\frac{(VARS(E)-n)(c+d)}{c!d!} \text{ rows, which is clearly less than}
\]

\[
\frac{(VARS(E)-n)(a+b)}{a!b!}.
\]
\((\text{VARS}(E)-n+1)_{c+d}\), which we know is less than \(\text{RMAX}_{n-1}(E)\).

In case (b), \((a',b')\) has

\((\text{VARS}(E)-n)_{a'+b'}\) rows, which by Lemma 12 is less than \(a!b!\)

\((\text{VARS}(E)-n+1)_{a'+b'+1}\), which is less than \(\text{RMAX}_{n-1}(E)\).

\(a'!(b'+1)!\)

In case (c), we have the same number or rows as in (b), which is less than

\((\text{VARS}(E)-n+1)_{a+b+1}\) by Lemma 12.

\((a+1)!(b+1)!\)

Thus every row-type in \(\text{PROD}^nE\) has fewer rows than some row-type of \(\text{PROD}^{n-1}E\), so \(\text{RMAX}_nE \leq \text{RMAX}_{n-1}E\).

Q.E.D.

Thm 2: \(|\text{PROD}^nE| \leq n|\text{PROD} E|\), n\geq 1. (remember \(|A|\) = number of rows of A)

Pf: \(|\text{PROD}^nE| = \sum_{(a,b) \in \text{RT}(\text{PROD}^nE)} (\text{number of rows of row-type } (a,b))

\leq \sum_{(a,b) \in \text{RT}(\text{PROD}^nE)} \text{RMAX}_nE

\text{ (by defn of max)}

R_n(E) \cdot \text{RMAX}_n(E)

\text{ (by defn of } R_n)

R_{n-1}(\text{PROD } E) \cdot \text{RMAX}_n(E)

\text{ (by defn of } R_n \text{ and } \text{PROD }^n)

\leq [R_0(\text{PROD } E)+(n-1)(R_0(\text{PROD } E)-1)] \cdot \text{RMAX}_n(E),

\text{ by Thm 1}

\leq [R_0(\text{PROD } E)+(n-1)(R_0(\text{PROD } E)-1)] \cdot \text{RMAX}_1(E) \text{ by Lemma 13}

= [R_1(E)+(n-1)(R_1(E)-1)] \cdot \text{RMAX}_1(E) \text{ (by defn of } R_1)

= [nR_1(E)-(n-1)] \cdot \text{RMAX}_1(E) \text{ (rearranging terms)}

\leq nR_1(E) \cdot \text{RMAX}_1(E) \text{ (adding } (n-1) \cdot \text{RMAX}_1(E))

\leq n|\text{PROD } E| \text{ Q.E.D.}
**Lemma 14**: For $E$ an RSDNF, $|\text{PROD } E| \leq 2|E|$.

**Pf:** $RT(\text{PROD } E) \leq 2 RT(E) - 1$, because the only possible row-type of \text{PROD } $E$ are the row-types of $E$ and $p \oplus q$ for $p \text{ adj } q$ in $E$.

But by uniqueness of $\text{adj}$ there are at most $RT(E) - 1$ such pairs $p, q$. Also $\text{RMAX}_1(E) \leq \text{RMAX}_0(E)$ by Lemma 13. Hence $|\text{PROD } E| \leq (2 RT(E) - 1) \cdot \text{RMAX}_0(E) \leq 2|E|$. Q.E.

**Remark 12:** From example (4) below it is seen that the 2 is necessary.

**Thm 3:** The time required by DPDNF for an RSDNF $E$ is no more than $36|E|^2(\text{VARS}(E))^4$, assuming element operations to take unit time.

**Pf:** Let $V = \text{VARS}(E)$.

The procedure terminates in no more than $V$ steps, so $n \leq V$ below.

The $n$-th application of \text{PRODUCT} takes $(|\text{PROD } E|^n)^2$ applications of $\oplus$; in this iteration, each application of $\oplus$ takes $2(V-n+\wedge$ operations and one $\odot$ operation.

Take upper bounds: By Thm 2 and Lemma 14, the $n$-th application of \text{PRODUCT} takes no more than $(2n|E|)^2 \leq (2V|E|)^2$ applications $\oplus$, and an application of $\oplus$ takes no more than $2V+3$ operations so each application of \text{PRODUCT} takes no more than $(2V+3)\cdot (2V|E|) \leq 3\cdot V\cdot (2V|E|)^2$ operations, for $V \geq 3$.

Each application of the reduction procedure takes no more than twice as long as each application of \text{PRODUCT} (you do as many element operations, and then possibly that many transfers as well), so each application of \text{PROD} takes no more than $9V(2V|E|)^2$. There are at most $V$ iterations, so the whole
procedure terminates in

\[ v \cdot 9 \cdot v \cdot (2v|z|)^2 = 36v^4|z|^2 \text{ operations.} \]

Q.E.D.

The constant 36 could be reduced by using tighter bounds, but not by very much; and it is the degree of the polynomial that is of interest; using the best bounds I have derived here will not improve this degree; neither will the obvious simplifications of PROD (e.g. omit \( R \oplus S \) unless: \( R = S \) and \( R(1) = 1 \); or \( R(1) = +, S(1) = + \)).

I mentioned that these results apply to an extension of the symmetrical class of DNF's. I call this class "quasi-symmetrical".*

**Def**:

\( f \) is the opposite (in parity) of an element \( e \) iff

\[
    f = \begin{cases} 
        0 & \text{if } e = 0 \\
        1 & \text{if } e = 1 \\
        - & \text{if } e = + \\
        + & \text{if } e = -
    \end{cases}
\]

The opposite of \( e \) is denoted \( \oplus e \)

**Def**: \( B \) is a parity change (over \( I \)) of \( A \) iff \( I \subseteq \{1, \ldots, \text{VARS}(A)\} \), and \( R \) is a row of \( B \) iff \( \exists S \in A \),

\[
    R[i] = \begin{cases} 
        S[i] & \text{if } i \in I \\
        \oplus S[i] & \text{if } i \notin I
    \end{cases}
\]

*The functions represented by quasi-symmetrical DNF's are called "symmetrizable".
Remark 13: If B is a parity change of A, then \( \text{VARS}(B) = \text{VARS}(A) \) and \(|B| = |A|\).

Pf: Clearly \( \text{VARS}(B) = \text{VARS}(A) \). Also \(|B| \leq |A|\). If \( S \neq S' \) and \( R, R' \) are parity changes over \( I \) of \( S \) and \( S' \) respectively, then \( R \neq R' \), since \( S[i] \neq S'[i] \rightarrow R[i] \neq R'[i] \). Hence \(|B| \geq |A|\) and \(|B| = |A|\). Q.E.D.

Lemma 15: Change of parity commutes with PROD, i.e. If B is a parity change over \( I \) of A, then PROD B is the parity change of PROD A over \( \{j | j+1 \in I \} \).

Pf: It is clear that change of parity commutes with reduction. To show it commutes with PRODUCT. Suppose for the moment that \( i \in I \). Then if \( i+1 \in I \), and \( R, S \in B \) with \( R[1] = S[1] = 1 \) or \( R[1] = + \) and \( S[1] = - \), then \((R \oplus S)[i] = R[i+1] \oplus S[i+1] \) by defn

\[ = R'[i+1] \oplus S'[i+1], \]

where \( T' \) means the row in A corresponding to \( T \in B \)

\[ = (R'[i+1] \oplus S'[i+1]) \text{ by inspection} \]

\[ = (((R' \oplus S') [i]) \text{ by defn} \]

and this is what we want.

If \( i \notin I \), then \( R \oplus S = S' \oplus R' \), and we still get the same rows since \( a \oplus b = b \oplus a \).

Q.E.D.

Coro 15-1: If B is a parity change of A, then DPDNF takes exactly as long on B as on A.

Def3: B is quasi-symmetrical denoted QSDNF iff B is a parity change of an RSDNF A.

Coro 15-2: If A is a QSDNF then the time required for DPDNF to halt on A is no more than \( 0.36 |A|^2 (\text{VARS}(A))^4 \).
Pf: Thm 3 and Coro 15-1.

It is my conjecture that DPDNF always halts in polynomial time, but I have not been able to find even a plausibility argument for this.

**Examples**

1. An RSDNF

\[ A = \begin{pmatrix} +1 \\ +1+ \\ 1+ \end{pmatrix} \]

2. A QSDNF derived from A over \{2\}

\[ B = \begin{pmatrix} +1 \\ +1+ \\ 1- \end{pmatrix} \]

3. Products of row types

\[
\begin{align*}
(2,2) \oplus (2,2) &: (+1+-) \oplus (-1++) = (1000) & 1 \\
(2,2) \oplus (2,2) &: (+1+-) \oplus (-++1-) = (++-) & 2 \\
(2,2) \oplus (2,2) &: (+1+-) \oplus (-1+--) = (10-) & 3 \\
(2,2) \oplus (1,3) &: (+1--+) \oplus (+1--) = (+1-) & 4 \\
(2,2) \oplus (1,3) &: (+1--+) \oplus (-1--+) = (++--) & 5 \\
(1,3) \oplus (2,2) &: (+1---) \oplus (-++1+) = (+0+) & 6 \\
(1,2) \oplus (2,1) &: (++--11) \oplus (-11++) = (---+) & 7 \\
(2,1) \oplus (1,2) &: (++1111) \oplus (---11) = (+-11) & 8 \\
(2,1) \oplus (1,2) &: (++1111) \oplus (---11) = (+-11) & 9 \\
(2,2) \oplus (2,2) &: (++++--1) \oplus (+++--1) = (+++++-) & 10 \\
(2,2) \oplus (2,2) &: (++++--1) \oplus (+++--1) = (+++++-) & 11 \\
(4,1) \oplus (1,4) &: (++++---1) \oplus (+1---) = (++++0--) & 12
\end{align*}
\]
Reference table for above examples:

Lemmas and Coros \(\rightarrow\) (L=Lemmas, C=Coro)

<table>
<thead>
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</table>

(In most cases the examples are to be compared with adjacent ones.)

(4) A sample of DPDNF, using row-types and VARS.

\(n\) VARS(\(\text{PROD}^n\)E) Row-types of \(\text{PROD}^n\)E

\(0\) 100 \((50,0)\) \((40,10)\) \((10,40)\) \((0,50)\)

\(1\) 99 \((50,0)\) \((49,9)\) \((40,10)\) \((39,39)\) \((10,40)\) \((9,49)\) \((0,50)\)

\(2\) 98 \((50,0)\) \((49,8)\) \((48,9)\) \((40,10)\) \((39,38)\)

\((38,39)\) \((10,40)\) \((9,48)\) \((8,49)\) \((0,50)\)

\(3\) 97 \((50,0)\) \((49,7)\) \((48,8)\) \((47,9)\) \((40,10)\)

\((39,37)\) \((38,38)\) \((37,39)\) \((10,40)\)

\((9,47)\) \((8,48)\) \((7,49)\) \((0,50)\)

etc.

I have not investigated a general test to circumvent DPDNF on RSDNF's to see when and how they are decided.
References:


[2] Cook, Stephen A., "Examples for the Davis-Putnam Procedure", (mimeographed communication from University of Toronto), June 1971 (I have one copy of this).


