DISCRETE HEAT KERNEL ESTIMATES IN INNER UNIFORM DOMAINS

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DISCRETE HEAT KERNEL ESTIMATES IN INNER UNIFORM DOMAINS

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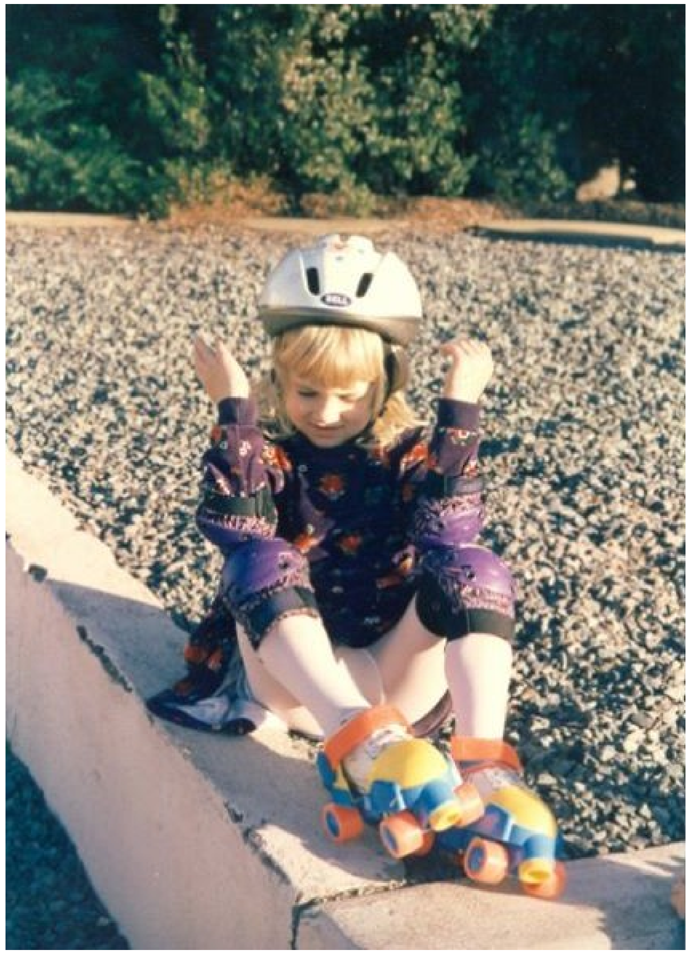
We prove a variety of estimates for the heat kernel on domains with discrete space and discrete time. First, we give a novel proof of the fact that, for a fixed graph G with heat semigroup generator P and measure m, the following are equivalent: (1) (G, P, m) satisfies volume doubling and the Poincaré inequality; (2) the heat kernel on (G, P, m) satisfies two-sided Gaussian bounds; and (3) solutions to the heat equation on (G, P, m) satisfy the Harnack inequality. This useful equivalence—which connects two geometric conditions to stochastic bounds—was first shown in the continuous case by L. Saloff-Coste and A. Grigor'yan and later, in the discrete case, by T. Delmotte. Our proof avoids complications of previous discrete proof.

Given a graph (G, P, m) that satisfies the three equivalent conditions above, which subgraphs $U \subseteq G$ also satisfy these conditions? We prove that (U, P_N, m) does, where U is an inner uniform domain and P_N is the Neumann heat semigroup. Then, we prove that (U, P_h, m_{h^2}) does, where U is an inner uniform domain and P_h is the heat semigroup obtained through Doob's h-transformation. The kernel for P_h is directly related to the kernel for P_D, the heat semigroup on U with Dirichlet boundary conditions, and therefore, we are able to derive bounds for the Dirichlet heat kernel in inner uniform domains. All work in this thesis is joint with Laurent Saloff-Coste.

BIOGRAPHICAL SKETCH



Roughly nineteen years after this photo was taken, Kelsey graduated from Reed College.

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CHAPTER 1

INTRODUCTION

The general setting for this thesis will be a graph G, with vertices V_G and edges E_G, with Markov operator P and vertex measure m, which we denote using the triple (G, P, m). We consider an edge e to be an ordered pair of vertices, i.e., $e = \{x, y\}$ or $x, y \in V_G$, and will always assume that $\{x, y\} \in E_G$ implies that $\{y, x\} \in E_G$. We will assume, without comment, that G is a countably infinite, connected graph. (To simplify notation, we write $x \in G$ to indicate $x \in V_G$.)

Let $\mu_{xy} = \mu_{yx} \geq 0$ denote symmetric weights on the edges of the graph, where x and y are neighbors (denoted $x \sim y$) if $\mu_{xy} \neq 0$. For convenience, we define $\mu_{xy} = 0$ for any two vertices x and y which are not connected by an edge. The vertices are given weights $m(x) = \sum_y \mu_{xy}$.

We will consider a nearest neighbor random walk, where the probability transition function is given by $p(x, y) = \mu_{xy}/m(x)$. Note that p is reversible with respect to the measure m. Let P denote the Markov operator

$$Pf(x) = \sum_y p(x, y)f(y)$$

and Δ denote the associated discrete Laplace operator

$$\Delta u(x) = (P - Id)u(x) = \sum_y (u(y) - u(x))p(x, y).$$

Throughout this thesis, we use $p(x, y)$ to indicate the probability of moving from $x \in G$ to $y \in G$, a probabilist's notation. Note that p is not symmetric, i.e., $p(x, y) \neq p(y, x)$ in general, but that

$$q(x, y) = \frac{p(x, y)}{m(y)} = \frac{p(y, x)}{m(x)}$$

is symmetric and a better analogy to the kernel of an operator in continuous space.

1.1 Discrete calculus

1.1.1 Topology of the graph

Using the natural graph metric, where $d(x, y)$ is the minimal path length between x and y, let

$$B(x, r) = \{y : d(x, y) \leq r\}$$

denote a ball of radius r centered at x and let

$$V(x, r) = m(B(x, r)) = \sum_{y \in B(x,r)} m(y)$$

be the volume of the ball.

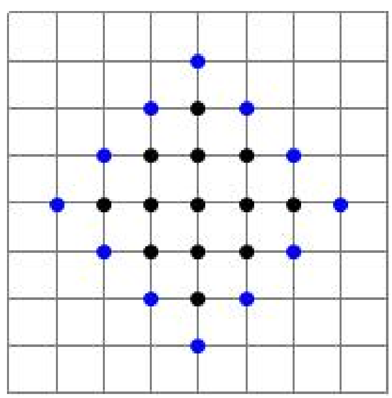


Figure 1.1: The discrete ball $B(0, 2)$ is indicated in black, and ∂B is indicated in blue.

For a fixed subset $U \subseteq G$, we need a well-defined notion of the *boundary* of U. Throughout this thesis, we use the convention,

$$\partial U = \{x \in G : x \notin U,\ d(x, U) = 1\}.$$

Under this definition, all subsets U are analogous to open sets in the continuous sense. (Alternatively, we could have defined the boundary to be $x \in U$ such that $d(x, G \setminus U) = 1$, which would make U analogous to a closed set.) We define $\overline{U} = U \cup \partial U$.

1.1.2 Analogs of continuous operators

In this thesis, we will primarily be concerned with the discrete Laplace operator,

$$\Delta u(x) = \sum_y (u(y) - u(x))p(x, y)$$

and discrete heat equation,

$$u_{k+1}(x) - u_k(x) = \sum_y (u_k(y) - u_k(x))p(x, y),$$

which simplifies to

$$u_{k+1}(x) = \sum_y u_k(y)p(x, y).$$

Of course, these are intended to be the discrete analogs of the standards Laplacian,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

and heat equation,

$$\frac{\partial}{\partial t} u_t(x) = \Delta u_t(x)$$

in $\mathbb{R}^n$. For more information about the heat equation in $\mathbb{R}^n$ we refer the reader to books by Evans [9] and Grigor'yan [13].

We will additionally use several other discrete analogs of concepts from standard calculus in continuous space. They are summarized in the chart below.

$(\mathbb{R}^n, P, \lambda)$	(G, P, m)
$\frac{\partial u}{\partial x_i}(x)$	$\nabla_{xy} u = u(y) - u(x)$
$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x)\right)$	-
$\lvert\nabla u(x)\rvert^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}(x)\right)^2$	$\lvert\nabla u(x)\rvert^2 = \frac{1}{2}\sum_y (u(x) - u(y))^2 p(x, y)$
$\lVert\nabla u\rVert_2^2 = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}(x)\right)^2 d\lambda(x)$	$\lVert\nabla u\rVert_2^2 = \frac{1}{2}\sum_{x,y} (u(x) - u(y))^2 \mu_{xy}$
$\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x)$	$\Delta u(x) = \sum_y (u(y) - u(x)) p(x, y)$

In general, the outgoing edges at a vertex serve as the discrete analog of the basis vectors in $\mathbb{R}^n$—they represent all possible directions of motion from a point. However, this creates two big differences between the discrete and continuous cases. (1) The structure of the edges connected to a vertex—their number, length and direction—is likely to be different for different vertices. In other words, the possible directions of motion might vary from vertex to vertex. To accommodate this, we use the notation $\nabla_{xy} u = u(y) - u(x)$ to indicate the discrete directional derivative at x in the direction of y, a neighboring vertex. (2) Unlike with basis vectors, a linear combination of adjacent edges is not necessary an adjacent edge. Consequently, there is no obvious discrete analog of the gradient. However, we can still define the magnitude of the gradient at a point.

Now, we collect two propositions to use later. Both propositions highlight the analogy between the model continuous space $(\mathbb{R}^n, P, \lambda)$ and the definitions we have constructed on (G, P, m).

Proposition 1.1 (Discrete integration by parts)**.** *For function $u, v : G \to \mathbb{R}$,*

$$\sum_x \Delta u(x)v(x)m(x) = -\frac{1}{2}\sum_{x,y}(\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy}.$$

Proof. Using the fact that $\mu_{xy} = \mu_{yx}$,

$$\begin{aligned}
2\sum_x \Delta u(x)v(x)m(x) &= 2\sum_{x,y}(u(y) - u(x))v(x)p(x,y)m(x) \\
&= \sum_{x,y}(u(y) - u(x))v(x)\mu_{xy} + \sum_{y,x}((u(x) - u(y))v(y)\mu_{yx} \\
&= \sum_{x,y}(u(y) - u(x))(v(x) - v(y))\mu_{xy} \\
&= -\sum_{x,y}(\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy}.
\end{aligned}$$

□

Proposition 1.2 (Discrete maximum principle)**.** *For some subset $U \subseteq G$, assume $u : \overline{U} \to \mathbb{R}$ is defined on $\overline{U}$ and harmonic, i.e., $\Delta u \equiv 0$, on $U \subseteq G$. Then, either u is constant, or*

$$\sup_{\overline{U}} u(x) > \sup_U u(x) \qquad \textit{and} \qquad \inf_{\overline{U}} u(x) < \inf_U u(x).$$

1.1.3 Discrete heat equation

The standard heat equation

$$\frac{\partial}{\partial t} u_t(x) = \Delta u_t(x) \qquad t \in \mathbb{R}^+, x \in \mathbb{R}^n$$

is defined in continuous space and continuous time. In this thesis, we will study an analog of the heat equation defined in discrete space and discrete time. For a general reference, see M. Barlow [2].

The discrete calculus established in Section 1.1 is sufficient to understand the Laplacian on a graph. To work in discrete time—indexed by $k \in \mathbb{N}$, with $k = 0$ indicating the initial condition—we need to define the *discrete time derivative*,

$$\partial_k u_k(x) = u_{k+1}(x) - u_k(x).$$

With these definitions in place, a clear definition of the discrete heat equation emerges.

Definition 1.3 (Subsolution of the heat equation)**.** *A function $u : \mathbb{N} \times G \to \mathbb{R}$ is called a* subsolution of the heat equation *if*

$$u_{k+1}(x) - u_k(x) \leq \sum_y (u_k(y) - u_k(x))p(x, y)$$

for all $x \in G$ and $k \in \mathbb{N}$, or equivalently,

$$u_{k+1}(x) \leq \sum_y u_k(y)p(x, y).$$

A function $u : \mathbb{N} \times G \to \mathbb{R}$ is said to be a *solution of the heat equation* if both u and $-u$ are subsolutions, or, equivalently, $u_{k+1}(x) = \sum_{y \sim x} p(x, y)u_k(y)$ for all $x \in G$ and $k \in \mathbb{N}$. Note that if u is a solution to the heat equation, then $u_k : G \to \mathbb{R}$ is harmonic for any fixed k.

1.1.4 Gaussian bounds

Definition 1.4 (Gaussian bounds). *A graph (G, P, m) is said to satisfy Gaussian bounds if there exists positive constants c_l, C_l, c_u, C_u such that, for any $k \in \mathbb{N}^+$ and $x, y \in G$ with $d(x, y) \leq k$,*

$$\frac{C_l}{V(x, \sqrt{k})} \exp(-c_l d(x,y)^2/k) \leq q^k(x,y) \leq \frac{C_u}{V(x, \sqrt{k})} \exp(-c_u d(x,y)^2/k). \quad (1.1)$$

Note that $d(x, y) > k$ implies that $p^k(x, y) = 0$.

1.1.5 Harnack inequality

Definition 1.5 (Parabolic Harnack inequality). *A graph (G, P, m) is said to satisfy the discrete time parabolic Harnack inequality if the following holds: Given any $\eta \in (0, 1)$ and constants $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$, there exists constant $C > 0$ such that, for any $x \in G$, $k, r \in \mathbb{N}$, and any nonnegative solution $u_k(x)$ of the heat equation on the cylinder $Q = [k, k + \theta_4 r^2] \times B(x, r)$, we have*

$$u_{k_\ominus}(x_\ominus) \leq C u_{k_\oplus}(x_\oplus) \quad (1.2)$$

for all $(k_\ominus, x_\ominus) \in Q_\ominus$ and $(k_\oplus, x_\oplus) \in Q_\oplus$ such that

$$d(x_\ominus, x_\oplus) \leq k_\oplus - k_\ominus,$$

where $Q_\ominus = [k + \theta_1 r^2, k + \theta_2 r^2] \times B(x, \eta r)$ and $Q_\oplus = [k + \theta_3 r^2, k + \theta_4 r^2] \times B(x, \eta r)$.

1.2 Graph geometry

Throughout this thesis, we will generally assume we are working on graphs with three geometric properties: ellipticity, volume doubling, and the Poincaré inequality. We will refer to these three properties as (**E**), (**V**), and (**P**), The constants ζ, δ, and ρ will always refer to the constants from the assumptions (**E**), (**V**), and (**P**), respectively.

The first property, ellipticity, is necessary to ensure that the heat kernel diffuses is all directions.

Definition 1.6 (Ellipticity)**.** *There exists some $\zeta > 0$ such that, for all $x, y \in G$ with $x \sim y$,*

$$p(x, y) \geq \zeta \tag{\textbf{E}}$$

Additionally, we assume $x \sim x$ for all $x \in G$, so as a consequence, $p(x, x) \geq \zeta$.

1.2.1 Volume doubling

Definition 1.7 (Volume doubling)**.** *A graph G has the* volume doubling *property if there exists some $\delta > 0$ such that*

$$V(x, 2r) \leq \delta V(x, r) \tag{\textbf{V}}$$

for all $x \in G$ and $r > 0$.

Here we collect some consequences of volume doubling:

- *Locally uniformly finite:* Letting $r = 1/2$, the hypothesis (**V**) implies that $V(x, 1) \leq \delta V(x, 1/2)$. Therefore,

  $$m(y) \leq \sum_{z \sim x} m(z) \leq \delta m(x)$$

  holds for any $x, y \in G$ with $x \sim y$. This implies G is locally uniformly finite, with the number of neighbors bounded by δ^2. Furthermore,

  $$\delta^{-1} p(y, x) \leq p(x, y) \leq \delta p(y, x)$$

for any $x, y \in G$ with $x \sim y$.

- *Volume regularity:* For $x, y \in G$ and $r, s \in \mathbb{N}^+$

$$\frac{V(x,r)}{V(y,s)} \leq C\left(\frac{r}{s}\right)^{\theta}$$

where $C, \theta > 0$ depend only on the volume doubling constant δ. To prove this, we first note that if $x = y$, then the result follows by iterating the volume doubling inequality $\lceil \log_2(r/s) \rceil$ times. If $y \neq x$, then

$$\frac{V(x,r)}{V(y,s)} \leq \frac{V(y,r+s)}{V(y,s)} \leq C\left(\frac{r+s}{s}\right)^{\theta} \leq C2^{\theta}\left(\frac{r}{s}\right)^{\theta}.$$

- *Quasi-converse:* There exists a sufficiently large A such that, for any $x \in G$ and $r \in \mathbb{N}$,

$$2V(x,r) \leq V(x,Ar).$$

To prove this, fix $y \in G$ such that $d(x,y) = 3r$ (such a y is guaranteed to exist since our graph is infinite and connected). By volume regularity, $V(y,r) \geq \epsilon V(x,3r)$ for some positive ϵ. Then

$$V(x,4r) \geq V(x,r) + V(y,r) \geq (1+\epsilon)V(x,r)$$

which iterates to give the desired result.

1.2.2 Poincaré inequality

Definition 1.8 (Poincaré inequality)**.** *A graph G satisfies the Poincaré inequality if there exists some $\rho > 0$ such that $\forall n \in \mathbb{N}$ and balls $B(x,r) \subseteq G$ of fixed radius r,*

$$\sum_{x \in B} (f(x) - f_B)^2 m(x) \leq \rho r^2 \sum_{x \in B} |\nabla f(x)|^2 m(x) \qquad \textbf{(P)}$$

or equivalently,

$$\sum_{x\in B}(f(x)-f_B)^2 m(x) \leq \rho r^2 \sum_{x,y\in B}(f(x)-f(y))^2\mu_{xy}$$

for any $f: G \to \mathbb{R}$*, where* $f_B = V(B)^{-1}\sum_{x\in B} f(x)m(x)$ *denotes the average of* f *in* B*.*

Note that the weak Poincaré inequality,

$$\sum_{x\in B}(f(x)-f_B)^2 m(x) \leq Cr^2 \sum_{x,y\in kB}(f(x)-f(y))^2\mu_{xy},$$

i.e., the Poincaré inequality where the right-hand side is summed over a scaled ball kB, is actually sufficient to prove the standard Poincaré inequality above. We discuss this in Section 4.4.4.

1.3 Outline of main results

1.3.1 Discrete Harnack inequality

The first major result, in Chapter 3, shows the equivalence of the Harnack inequality, two sided Gaussian bounds, and three geometric conditions: the Poincaré inequality, volume doubling and ellipticity.

Theorem 1.9. *For a fixed graph* (G, P, m)*, the following are equivalent:*

(1) (G, P, m) *satisfies* $(\mathbf{V})$, $(\mathbf{P})$ *and* $(\mathbf{E})$.

(2) The discrete Gaussian bounds in Definition 1.4 hold.

(3) The discrete parabolic Harnack inequality in Definition 1.5 hold.

The continuous analog of Theorem 1.9 was proved independently by L. Saloff-Coste [24] and A. Grigor'yan [12]. Those articles and several others were motivated by a 1964 paper by J.Moser's [19] in which he proves that the parabolic Harnack inequality (1.2) holds for uniformly elliptic operators in divergence form in $\mathbb{R}^n$. Moser's method crucially uses the Poincaré inequality on $\mathbb{R}^n$ and is thus adaptable to other settings. See M. Murugan and L. Saloff Coste's monograph [20, Ch. 1] for a history of the Harnack inequality—and its geometric characterizations—in a variety of settings.

Theorem 1.9, in the discrete form presented here, was first proved in 1999 by T. Delmotte [7]. To prove that a graph (G, P, m) satisfying the geometric conditions (**V**) and (**P**) must also satisfy the discrete Harnack inequality, T. Delmotte's argument uses an intermediate step: a continuous time Harnack inequality. He proves this discrete space, continuous time Harnack inequality using a modified version of Moser's method. These continuous time estimates yield discrete time estimates, from which he proves the full discrete space, discrete time Harnack inequality. The proof of Theorem 1.9 presented in this thesis, which combines techniques from [7] and [25], is more direct and does not use continuous time as an intermediary step. As with the continuous case, it is not difficult to derive the geometric conditions from the Harnack inequality; the majority of the argument is devoted to using the geometric conditions to prove the Gaussian inequality, which fairly directly implies the Harnack inequality.

We also note two papers which prove versions of Theorem 1.9 in other contexts. M. Murugan and L. Saloff-Coste [20] proved an analog of Theorem 1.9 with discrete time and any metric measure space as the domain, which encompasses both manifolds and graphs. Andres, Deuschel, and Slowik [1] proved a continuous time, discrete space variant of Theorem 1.9 that replaces the uniform bounds on edge weights μ_{xy} with L^p bounds with the goal of studying the random conductance model.

1.3.2 Harnack inequality on inner uniform domains

Given a graph (G, P, m) satisfying the Harnack inequality, it is natural to wonder what subgraphs $U \subseteq G$ also satisfy the Harnack inequality. In light of Theorem 1.9, we can rephrase the question: Given a graph (G, P, m) satisfying volume doubling and the Poincaré inequality, what subgraphs $U \subseteq G$ with Neumann operator $P_{U,N}$, i.e., reflecting boundary conditions, also satisfy volume doubling and the Poincaré inequality? We give a partial answer to this question in Chapter 4.

Theorem 1.10. *Fix a graph (G, P, m) satisfying the Harnack inequality in Definition 1.5. Let $U \subseteq G$ be an inner uniform domain. Then $(U, P_{N,U}, m)$ also satisfies the Harnack inequality.*

P. Gyrya and L. Saloff-Coste [14] prove a continuous analog of Theorem 1.10 using a Whitney covering of the inner uniform domain. The argument given in this thesis closely mirrors their technique.

1.3.3 Harnack inequality on domain with h-transform process

Chapter 5 discusses the discrete analog of Doob's h-transform. Given an infinite connected subgraph $U \subseteq G$, the h-function associated to U is a function satisfying:

(1) $\Delta h(x) = 0$ in U; (2) $h(x) > 0$ in U; and (3) $h(x) = 0$ on ∂U. On U, we consider the modified stochastic process with Markov operator $P_{h,U} = H^{-1} \circ P_{D,U} \circ H$ where $H(f)(x) = h(x)f(x)$ and $P_{D,U}$ is the operator with Dirichlet boundary conditions. This modified process is pushed away from the boundary. The reversible measure for the process is $m_{h^2}(x) = m(x)h(x)^2$.

The main result of Chapter 5 shows that $(U, P_{h,U}, m_{h^2})$, an inner uniform subdomain of a Harnack space G with the h-transformed process, satisfies the Harnack inequality.

Theorem 1.11. *Let (G, P, m) be a graph satisfying the Harnack inequality and assume that $U \subseteq G$ is an inner uniform domain. Then $(U, P_{h,U}, m_{h^2})$ satisfies the Harnack inequality.*

For example,

$$\mathbb{Z}_+^2 = \{(x, y) \in \mathbb{Z}^2 : x, y > 0\}$$

with

$$\partial\mathbb{Z}_+^2 = \{(x, y) : x = 0, y > 0 \text{ or } x > 0, y = 0\}$$

is an inner uniform domain and thus, by Theorem 1.10, $(\mathbb{Z}_+^2, P_{N,\mathbb{Z}_+^2}, m)$ satisfies the Harnack inequality with Neumann boundary conditions. The h-function for $\mathbb{Z}_+^2$ is given by $h(x, y) = xy$. Theorem 1.11 tells us that $(\mathbb{Z}_+^2, P_{h,\mathbb{Z}_+^2}, m_{h^2})$ also satisfies the Harnack inequality.

In Chapter 5 we describe how this characterization of $(\mathbb{Z}_+^2, P_{h,\mathbb{Z}_+^2}, m_{h^2})$ has many interesting consequences. In particular, the h-function for a domain is nicely related to $q_{D,U}^k(x, y)$, the heat kernel for the process killed at the boundary, i.e., with Dirichlet boundary conditions. In Corollary 5.8 we derive,

$$\frac{C_l h(x)h(y)}{V_{m_{h^2}}(x, \sqrt{k})} \exp(-c_l d(x, y)^2/k) \le q_{D,U}^k(x, y) \le \frac{C_u h(x)h(y)}{V_{m_{h^2}}(x, \sqrt{k})} \exp(-c_u d(x, y)^2/k).$$

Using several approximations given by the h-function and specific properties of $\mathbb{Z}_+^2$, we have that $q_{D,U}^k((x_0, y_0), (x_1, y_1))$ is bounded above and below by expressions of the form

$$\frac{Cx_0x_1y_0y_1}{(x_0+\sqrt{k})(x_1+\sqrt{k})(y_0+\sqrt{k})(y_1+\sqrt{k})k}\exp(-cd(x,y)^2/k). \tag{1.3}$$

and a similar upper bound. Using this explicit bound, we can make several observations about the heat kernel. Consider two points near the y-axis boundary, but far from the x-axis: $(1, y_0)$ and $(1, y_1)$ where $\sqrt{k} << y_0, y_1$ and $|y_1 - y_0| = \sqrt{k}$. Then, (1.3) simplifies to show that $q_{D,U}^k((1, y_0), (1, y_1))$ is similar to $\frac{1}{k^2}$. Or, consider two points far from the origin: (x_0, y_0) and (x_1, y_1) where $\sqrt{k} << x_0, x_1, y_0, y_1$ and $d((x_0, y_0), (x_1, y_1)) = \sqrt{k}$. Then, (1.3) simplifies to show that $q_{D,U}^k((x_0, y_0), (x_1, y_1))$ is similar to $\frac{1}{k}$. The difference between the two examples shows the influence of Dirichlet boundary.

The Gaussian-type estimates for the Dirichlet heat kernel highlighted above have several interesting consequences. For example, if $\mathbb{P}_x(\tau_U > k)$ denotes the probability that a process started at x survives past time k, i.e., has not been killed the Dirichlet boundary, then

$$c\frac{h(x)}{h(x_{\sqrt{k}})} \leq \mathbb{P}_x(\tau_U > k) \leq C\frac{h(x)}{h(x_{\sqrt{k}})},$$

where $x_{\sqrt{k}}$ is a point in U such that $d_U(x, x_{\sqrt{k}}) \leq \frac{\sqrt{k}}{4}$ and $d(x_{\sqrt{k}}) \geq \frac{\tilde{\kappa}\sqrt{k}}{8}$. In the case of $\mathbb{Z}_+^2$, this simplifies to saying that $\mathbb{P}_{(x,y)}(\tau_U > k)$ behaves like $\frac{xy}{(x+\sqrt{k})(y+\sqrt{k})}$.

Furthermore, the h-function gives us an easy way to understand the process conditioned to not hit the boundary in the following sense. Define a probability transition function as

$$p_{c,U}^k(x, y) = \lim_{n\to\infty} \mathbb{P}_x(X_k = y \mid \tau_U > n).$$

This turns out to be equal to $p^k_{h,U}(x,y)$, the probability transition function for the h process.

We conclude Chapter 5 with a discussion of the effect that a finite perturbation of the boundary of an inner uniform domain has on its h-function.

CHAPTER 2

DIRICHLET FORMS

For general information about Dirichlet forms, we refer the reader to [11]. In this thesis, we will use the standard Dirichlet form in continuous space and then give the discrete analog. After, we will develop the cable process and its Dirichlet form—a combination of the discrete and continuous forms.

2.1 Dirichlet forms

2.1.1 Euclidean space

Our model continuous space will be $\mathbb{R}^n$, equipped with the Lebesgue measure λ. There is a long literature (see, e.g., [13]) of studying the heat kernel on $\mathbb{R}^n$ (i.e., $p_t(x_1, \ldots, x_n)$) via the Dirichlet form $\mathcal{E}$, defined on $\mathcal{D}(\mathcal{E})$, some dense subset of $L^2(\mathbb{R}^n, \lambda)$. Ultimately, we would like to imitate this in discrete space, but first, we will outline the continuous version.

The *Dirichlet form on* $(\mathbb{R}^n, \lambda)$ is

$$\mathcal{E}(f, g) = \int \nabla f \cdot \nabla g \; d\lambda, \;\; f, g \in \mathcal{D}(\mathcal{E}),$$

where $\mathcal{D}(\mathcal{E}) = H^1(\mathbb{R}^n, \lambda) \subseteq L^2(\mathbb{R}^n, \lambda)$ is the space all functions whose gradient is in $L^(\mathbb{R}^n, \lambda)$,

$$H^1(\mathbb{R}^n, \lambda) = \{f \in L^2(\mathbb{R}^n, \lambda) : \nabla f \in L^(\mathbb{R}^n, \lambda)\}.$$

Analogously, for any open subset $U \subseteq \mathbb{R}^n$, define the Dirichlet form on U as

$$\mathcal{E}^U(f, g) = \int_U \nabla f \cdot \nabla g \; d\lambda, \;\; f, g \in \mathcal{D}(\mathcal{E}^U),$$

where $\mathcal{D}(\mathcal{E}^U) = H^1(U) \subseteq L^2(U, \lambda)$.

2.1.2 Connection to heat kernel

Note that the Dirichlet form has a special relationship to the Laplacian,

$$\mathcal{E}(f,g) = \int \nabla f \cdot \nabla g = \int -\Delta f \cdot g = \langle -\Delta f, g \rangle.$$

One can prove this fact using an integration by parts argument.

From here, we can define the *heat semigroup associated with the Laplacian* as $P^t = e^{-t\Delta}$. More generally, the *heat semigroup associated with an operator* $\mathcal{L}$ is $P^t_{\mathcal{L}} = e^{-t\mathcal{L}}$.

We can write the heat semigroup operator as an integral,

$$P^t f(x) = \int_{\mathbb{R}^n} q^t(x) f(y) \, dy$$

where

$$q^t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$$

is the *heat kernel*.

In this sense, there are five interrelated objects we wish to study:

- the Dirichlet form, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$
- the Laplace operator, Δ
- the heat semigroup, $P^t f(x)$
- the heat kernel $q^t(x)$
- the stochastic process, Brownian motion.

2.1.3 Discrete space

We want to define a Dirichlet form on a graph domain which is analogous to the Dirichlet form on $\mathbb{R}^n$ defined in the previous section. Fix a graph G with symmetric edges weights μ_{xy}, and assign a measure on the vertices $m(x) = \sum_y \mu_{xy}$.

The *discrete Dirichlet form on* (G, P, m) is

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{(x,y)\in G\times G} (f(y) - f(x))(g(y) - g(x))\mu_{xy},\ \ f, g \in \mathcal{D}(\mathcal{E})$$

where $\mathcal{D}(\mathcal{E}) \subseteq L^2(G, P, m)$ is the space of functions whose discrete gradient is in $L^2(G, P, m)$,

$$\mathcal{D}(\mathcal{E}) = \{f \in L^2(G, P, m) : \sum_{x\in G}(f(y) - f(x))^2 m(x) < \infty\}.$$

Analogously, for any subset $U \subseteq G$, define the Dirichlet form on U as

$$\mathcal{E}^U(f,g) = \frac{1}{2} \sum_{(x,y)\in U\times U} (f(y) - f(x))(g(y) - g(x))\mu_{xy},\ \ f, g \in \mathcal{D}(\mathcal{E}^U)$$

where $\mathcal{D}(\mathcal{E}^U) \subseteq L^2(U, m)$ is defined as $\mathcal{D}(\mathcal{E})$, but restricted to U.

Similar to the continuous case, we are interested in this Dirichlet form because it is associated with the Laplacian. That is, $\mathcal{E}(f, g) = \langle \Delta f, g\rangle_{(G,P,m)}$, where the inner product is taken in the space, (G, P, m).

$$\begin{aligned}
2\mathcal{E}(f,g) &= \sum_{(x,y)}(f(y) - f(x))(g(y) - g(x))\mu_{xy} \\
&= \sum_{(x,y)}(f(x)g(x) - f(y)g(x))\mu_{xy} + \sum_{(x,y)}(f(y)g(y) - f(x)g(y))\mu_{yx} \\
&= \sum_x \left(f(x) - \sum_y p(x,y)f(y) \right) g(x)m(x) \\
&\quad + \sum_y \left(f(y) - \sum_x p(y,x)f(x) \right) g(y)m(y) \\
&= 2\langle -\Delta f, g\rangle_{(G,P,m)}
\end{aligned}$$

2.2 The cable process

The cable process is a continuous stochastic process whose domain is a continuous analog of a graph. More precisely, given a graph G (i.e., a collection of vertices V_G and edges E_G), let I_e be a unit length interval associated with e, and let $\sim$ be an equivalence relationship on $\bigcup_{e\in E} I_e$ where $I_e(x) \sim I_{e'}(x')$ if e and e' intersect at a common vertex v and $I_e(x) = I_{e'}(x') = v$. The domain for the cable process is $\bigcup\limits_{e\in E} I_e/\sim$.

The cable process can either be constructed from Brownian motion, which we outline in Sections 2.2.1 and 2.2.2, or defined using a Dirichlet form, which we give in Section 2.2.3. For more information, we recommend D. Revuz and M. Yor's book [22, Ch. 9] or M. Folz's article [10].

2.2.1 Brownian excursions

Let $B(t)$ for $t \in [0, \infty)$ be a standard one-dimensional Brownian motion, and let $\mathcal{Z}$ be the zero set of Brownian motion,

$$\mathcal{Z} = \{t : B(t) = 0\}.$$

Note that, by the a.s. continuity of Brownian motion, $\mathcal{Z}$ is a closed set.

Define $\tau_0 = \sup\{t : t \in \mathcal{Z} \text{ and } t \leq 1\}$ to be the last zero before $t = 1$, and $\tau_1 = \inf\{t : t \in \mathcal{Z} \text{ and } t \geq 1\}$ to be the first zero after $t = 1$. We claim that, a.s., $\tau_0 < 1 < \tau_1$: Because $\mathcal{Z}$ is closed, the sup and inf in the definition of τ_0 and τ_1 actually obtain, i.e., they are a max and min. But, a.s., $B(1) \neq 0$.

A *Brownian excursion* a process $\{e(t) : 0 \leq t \leq 1\}$ defined by

$$e(t) = \frac{|B(\tau_0(1-t) + \tau_1(t)|}{\sqrt{(\tau_1 - \tau_0)}}.$$

In other words, a Brownian excursion is the Brownian path between the last zero before $t = 1$ and the first zero after $t = 1$, shifted and scaled to occur on the interval $[0, 1]$.

2.2.2 The cable process from Brownian excursions

Using scaled Brownian excursions and the law of the zeros of Brownian motions, we can recover Brownian motion. Conversely, using a standard Brownian motion $\{B_t\}_{t\geq 0}$, we can extract the a.s. countable excursion set $\{e_i(t) : 0 \leq t \leq T_i\}_{i=1}^{\infty}$, where T_i indicates the duration of the e_i excursion, i.e., the time scaling of the standard excursion.

Now we can describe $\{C_{x,U}(t) : t \geq 0\}$, the *cable process* on $U \subseteq G$ starting at x. By fiat, the process starts at $x \in U$, i.e. $C_{x,U}(0) = x$. If x is inside an interval, perform a standard one-dimensional Brownian motion until the process reaches a vertex. Otherwise, randomly select an edge adjacent to x, uniformly from the neighbors in U, and perform a scaled excursion, according to $e_1(t)$, along that edge. If the Brownian excursion is not long enough to reach another vertex, i.e. $\max\{e_1(t) : 0 \leq t \leq T_i\} < 1$, uniformly randomly select a neighboring vertex in U and perform a new scaled Brownian excursion according to $\{e_2(t)\}$. If the scaled excursion is long enough to reach another vertex, then stop at the neighboring vertex as soon as it is reached; using the strong Markov property, uniformly select an edge adjacent to the current vertex position and perform a new excursion. Repeat this process.

2.2.3 The Dirichlet form of the cable process

Equip M_U with λ, the one-dimensional Lebesgue measure. For $f \in L^2(M_U)$, define $f_e = f_{I_e}$ to be the restriction of f to the interval $I_e \subseteq M_U$. Then, define the Dirichlet form

$$\tilde{\mathcal{E}}_U(f,g) = \sum_{e \in E_U} \int_{I_e} \frac{\partial f_e}{\partial x} \cdot \frac{\partial g_e}{\partial x} \mu_e dx \tag{2.1}$$

for $f, g \in \mathcal{D}_{\mathcal{E}} \subseteq \mathcal{C}_c(M_U)$.

This Dirichlet form gives the cable process, whose construction from Brownian excursions is described in the preceding sections. To see this intuitively, we can take a local point of view: inside an interval I_e the Dirichlet form $\tilde{\mathcal{E}}$ looks like the Dirichlet form for the Laplacian, and hence, it generates Brownian motion; at each vertex v, the Dirichlet form $\tilde{\mathcal{E}}$ makes a uniform choice among the edges. These descriptions align with the description of the cable process from Brownian excursions.

2.3 Strictly local, regular Dirichlet forms

To study the general properties of Dirichlet forms, consider a Dirichlet form $\mathcal{E}$ defined on some $\mathcal{D}(\mathcal{E}) \subseteq L^2(X,\mu)$, where X is any space equipped with a measure μ.

A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *local* if $\mathcal{E}(f,g) = 0$ for any $f, g \in \mathcal{D}(\mathcal{E})$ where $\text{supp}(f)$ and $\text{supp}(g)$ are disjoint compact sets. To weaken this condition, we say a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *strictly local* if $\mathcal{E}(f,g) = 0$ for any $f, g \in \mathcal{D}(\mathcal{E})$ where $\text{supp}(f)$ and $\text{supp}(g)$ are compact and g is constant on a neighborhood of $\text{supp}(g)$. The Dirichlet form on the graph domain is not local: $\mathcal{E}(\delta_x, \delta_y) = -\mu_{xy}$ for $x \sim y$. However, both the Dirichlet forms on continuous settings—in $\mathbb{R}^n$ and for the cable process—are local.

A *core* for $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a subset $\mathbf{C} \subseteq \mathcal{C}_C(X) \cap \mathcal{D}(\mathcal{E})$ such that (1) $\mathbf{C}$ is dense in $\mathcal{C}_C(X)$ using the uniform norm, and (2) $\mathbf{C}$ is dense in $\mathcal{D}(\mathcal{E})$ using the $\mathcal{E}_1$ norm,

$$\|f\|_{\mathcal{E}_1}^2 = \|f\|_2^2 + \mathcal{E}(f, f).$$

A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called *regular* if it admits a core. Notice that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular if and only if $\mathcal{C}_C(X) \cap \mathcal{D}(\mathcal{E})$ is a core.

CHAPTER 3

HARNACK INEQUALITY

The primary goal of this chapter is to prove Theorem 1.9, which states that, for a fixed graph (G, P, m), the following are equivalent:

(1) (G, P, m) satisfies (**V**), (**P**) and (**E**).

(2) The discrete Gaussian bounds in Definition 1.4 hold.

(3) The discrete parabolic Harnack inequality in Definition 1.5 hold.

Throughout the first five sections of this chapter we assume, without note, hypotheses (**V**), (**P**) and (**E**). Sections 3.1 and 3.2 state the elliptic Harnack inequality and prove the mean value inequality, respectively. We will use these results later in sections 3.3 and 3.4, which prove that (1) $\Rightarrow$ (2). Section 3.5 shows that (2) $\Rightarrow$ (3). And finally, Section 3.6 shows that (3) $\Rightarrow$ (1).

The arguments in this chapter use elements from T. Delmotte's proof [6] of the elliptic Harnack inequality, two papers [15, 16] by W. Hebisch and L. Saloff-Coste, a paper [5] by T. Coulhon and A. Grigor'yan, and a monograph [20] by M. Murugan and L. Saloff-Coste.

3.1 Elliptic Harnack inequality

In 1999, T. Delmotte proved the following elliptic Harnack inequality on a graph.

Theorem 3.1 (Elliptic Harnack inequality)**.** *Assume G satisfies* (**V**)*,* (**P**) *and* (**E**)*. If $u : G \to \mathbb{R}$ is harmonic and positive on $2B$ for some ball B, then*

$$\sup_{\overline{B}} u \leq C \inf_{\overline{B}} u$$

where C is a constant which depends only on δ and ρ, the constants from (**V**) *and* (**P**). *Notably, the constant is independent of u and B.*

T. Delmotte's proof relies on the quantity

$$\phi(u,p,B) = \left(V(B)^{-1} \sum_{x \in B} u(x)^p m(x) \right)^{1/p}.$$

The central idea of the proof is that $\phi(u,p,B) \to \sup_B u$ as $p \to +\infty$ and $\phi(u,p,B) \to \inf_B u$ as $p \to -\infty$. Using the technique of Moser iteration from [18] and [19], one can compare $\phi(u,p,B)$ for different values of p to obtain the elliptic Harnack inequality. For the detailed proof, see [6].

Next, we state and prove two important corollaries of the elliptic Harnack inequality.

3.1.1 Hölder-type continuity for harmonic equations

Corollary 3.2 (Hölder-type continuity for harmonic equations)**.** *Assume G satisfies* (**V**), (**P**) *and* (**E**). *Let $u : G \to \mathbb{R}$ be positive and harmonic on B, a ball of radius r. Then there exists constants $A, \alpha > 0$ such that*

$$|u(x) - u(y)| \leq A \left(\frac{d(x,y)}{r} \right)^{\alpha} \sup_{\overline{B}}(u)$$

for any $x, y \in \frac{1}{2}B$.

Proof. If u is constant, the result clearly holds. If not, note that both $(\sup_{\overline{B}} u - u)$ and $(u - \inf_{\overline{B}} u)$ are positive and harmonic on B by Proposition 1.2, the discrete maximum principle. Applying Theorem 3.1 to $(\sup_{\overline{B}} u - u)$, we get

$$\sup_{1/2\overline{B}} \left(\sup_{\overline{B}} u - u \right) \leq C \inf_{1/2\overline{B}} \left(\sup_{\overline{B}} u - u \right)$$

which simplifies to

$$\sup_{\overline{B}} u - \inf_{1/2\overline{B}} u \leq C \left(\sup_{\overline{B}} u - \sup_{1/2\overline{B}} u \right). \tag{3.1}$$

Similarly, applying Theorem 3.1 to $(u - \inf_{\overline{B}} u)$, we get

$$\sup_{1/2\overline{B}} \left(u - \inf_{\overline{B}} u \right) \leq C \left(\inf_{1/2\overline{B}} (u - \inf_{\overline{B}} u) \right)$$

which simplifies to

$$\sup_{1/2\overline{B}} u - \inf_{\overline{B}} u \leq C \left(\inf_{1/2\overline{B}} u - \inf_{\overline{B}} u \right) \tag{3.2}$$

Adding together (3.1) and (3.2) produces

$$\sup_{1/2\overline{B}} u - \inf_{1/2\overline{B}} u \leq \frac{C-1}{C+1} \left(\sup_{\overline{B}} u - \inf_{\overline{B}} u \right). \tag{3.3}$$

Since $0 < d(x,y) < r$, we can select n such that $r2^{-(n+1)} < d(x,y) < r2^{-n}$. Then, iterate (3.3) to get

$$\sup_{d(x,y)\overline{B}} u - \inf_{d(x,y)\overline{B}} u \leq \sup_{2^{-n}\overline{B}} u - \inf_{2^{-n}\overline{B}} u \leq \left(\frac{C-1}{C+1} \right)^n \left(\sup_{\overline{B}} u - \inf_{\overline{B}} u \right). \tag{3.4}$$

Fix α such that $\frac{C-1}{C+1} = 2^{-\alpha}$. Then,

$$\left(\frac{C-1}{C+1} \right)^n \leq 2^{-\alpha n} = 2^{\alpha} \left(2^{-(n+1)} \right)^{\alpha} \leq 2^{\alpha} \left(\frac{d(x,y)}{r} \right)^{\alpha} \tag{3.5}$$

Taking the constant $A = 2^{\alpha}$, which only depends on the constant C from Theorem 3.1 (and hence, on δ and ρ from (**V**) and (**P**)), the result follows from (3.4) and (3.5). □

3.1.2 Liouville theorem

The classical Liouville theorem states that any harmonic function on $\mathbb{R}^n$ which is bounded above or below is a constant function. Here we state and prove the analogous version for graphs.

Corollary 3.3 (Liouville's theorem)**.** *Assume G satisfies* $(\mathbf{V})$, $(\mathbf{P})$ *and* $(\mathbf{E})$*. Assume $u : G \to \mathbb{R}$ is harmonic and bounded above or below. Then u is constant.*

Proof. We can assume u is bounded below. (If it is bounded above, consider $-u$.) By Proposition 1.2, the discrete maximum principle, we know that $u - \inf_G u$ is positive on any ball B. Applying Theorem 3.1, the elliptic Harnack inequality,

$$\sup_B \left(u - \inf_G u\right) \leq C \inf_B \left(u - \inf_G u\right)$$

for any ball B. Letting the radius of B tend to infinity,

$$\inf_B \left(u - \inf_G u\right) \to 0$$

and so, $\sup_B u - \inf_G u$, which is non-negative and increasing with the radius of B, must be zero. Therefore, u is constant. □

3.2 Mean value inequalities

The primary goal of this section is to prove Theorem 3.5, the L^1 parabolic mean value inequality. We will first establish Theorem 3.4, the L^2 parabolic mean value inequality, and then show that this implies the L^1 version.

Let $\mathfrak{m}(x) = \max\{x, x^{-1/\nu}\}$, where ν is a constant which will be fixed in Theorem 3.10 and depend only on δ and ρ from $(\mathbf{V})$ and $(\mathbf{P})$.

Theorem 3.4 (L^2 parabolic mean value inequality, [5])**.** *For any nonnegative subsolution $u_k(x)$ with $k \in \mathbb{N}$ and $x \in G$, there exists a constant C (depending only on δ, ρ, and ζ from our assumptions* $(\mathbf{V})$, $(\mathbf{P})$, *and* $(\mathbf{E})$*) such that for any $z \in G$ and any $R, T \in \mathbb{N}^+$*

$$u_T(z)^2 \leq \frac{C\mathfrak{m}(T/R^2)}{m(\Psi)} \sum_{k,x\in\Psi} u_k(x)^2 m(x) \qquad (\mathbf{MV2})$$

where $\Psi = [0, 2T] \times B(z, R)$ *is a cylinder and* $m(\Psi) = 2TV(z, R)$ *is its volume.*

Theorem 3.5 (L^1 parabolic mean value inequality, [5])**.** *For any nonnegative subsolution* $u_k(x)$ *with* $k \in \mathbb{N}$ *and* $x \in G$*, there exists a constant* C *(depending only on* δ, ρ*, and* ζ *from our assumptions* (**V**)*,* (**P**)*, and* (**E**)*) such that for any* $z \in G$ *and any* $R, T \in \mathbb{N}^+$

$$u_T(z) \leq \frac{C\mathfrak{m}(T/R^2)}{m(\Psi)} \sum_{k,x\in\Psi} u_k(x)m(x) \qquad (\mathbf{MV1})$$

where $\Psi = [0, 2T] \times B(z, R)$ *is a cylinder and* $m(\Psi) = 2TV(z, R)$ *is its volume.*

The arguments in this section follow T. Coulhon and A. Grigor'yan in [5]. To establish the L^2 mean value inequality on a cylinder Ψ, we form a sequence of nested subcylinders Ψ_n and compare the L^2 norm of a function u in these nested subcylinders. In Section 3.2.1 we prove the Cacciopoli inequality, which we use in Section 3.2.2 to compare the L^2 norms of two neighboring subcylinders in terms of a function Λ. Section 3.2.3 establishes the Faber-Krahn inequality, which provides a specific function Λ and gives an explicit comparison between the L^2 norm of u on neighboring subcylinders. In Section 3.2.4, we iterate this comparison to establish (**MV2**). Finally, in Section 3.2.4, we show that (**MV2**) implies (**MV1**).

3.2.1 Cacciopoli inequality

By the definition of a subsolution,

$$u_{k+1}^2 \leq [Pu_k]^2 = (u_k + \Delta u_k)^2 = u_k^2 + 2(\Delta u_k)u_k + (\Delta u_k)^2. \qquad (3.6)$$

Rearranging equation (3.6) proves the following lemma.

Lemma 3.6. *For every non-negative subsolution* u*,*

$$\partial_k u_k^2 \leq 2(\Delta u_k)u_k + (\Delta u_k)^2.$$

Proposition 3.7. *(Cacciopoli inequality) There exists $c, A > 0$ depending on ζ (the constant from assumption (**E**)) such that, for any non-negative subsolution u and any function ϕ with finite support*

$$\sum_x \partial_k u_k^2 \phi^2 m + c \sum_{x,y} (\nabla_{xy}(u_k\phi))^2 \mu_{xy} \leq A \sum_{x,y} (\nabla_{xy}\phi)^2 u_k^2 \mu_{xy}.$$

Proof. Throughout the proof, we drop the subscript from u_k. Applying Lemma 3.6 to the non-negative function u, and multiplying both sides by ϕm, where m is the measure of the vertex,

$$\partial_k u^2 \phi^2 m \leq 2(\Delta u)u\phi^2 m + (\Delta u)^2 \phi^2 m. \tag{3.7}$$

Because ϕ is finitely supported, we can sum (3.7) over all $x \in G$ to obtain

$$\sum_x \partial_k u^2 \phi^2 m \leq 2 \sum_x (\Delta u)u\phi^2 m + \sum_x (\Delta u)^2 \phi^2 m. \tag{3.8}$$

Now we want to bound the right side of (3.8). Apply the integration by parts formula to the first summand,

$$2 \sum_x (\Delta u)u\phi^2 m = -\sum_{x,y} \nabla_{xy} u \nabla_{xy}(u\phi^2)\mu_{xy}.$$

To bound the second summand, we use a technique from [7] that relies of the assumption (**E**):

$$\begin{aligned}
\sum_x (\Delta u)^2 \phi^2 m &= \sum_x \left(\left(\sum_y u(y)p(x,y) \right) - u(x) \right)^2 \phi(x)^2 m(x) \\
&= \sum_x \left(\sum_{y \neq x} (\nabla_{xy} u) p(x,y) \right)^2 \phi(x)^2 m(x) \\
&\leq \sum_x \left(\sum_{y \neq x} p(x,y) \right) \left(\sum_y (\nabla_{xy} u)^2 p(x,y) \right) \phi(x)^2 m(x) \\
&= (1-\zeta) \sum_{x,y} (\nabla_{xy} u)^2 \phi(x)^2 \mu_{xy}.
\end{aligned}$$

Applying these bounds to (3.8), we get

$$\sum_x \partial_k u^2 \phi^2 m \leq (1-\zeta)\sum_{x,y}(\nabla_{xy}u)^2\phi(x)^2\mu_{xy} - \sum_{x,y}\nabla_{xy}u\nabla_{xy}(u\phi^2)\mu_{xy}. \tag{3.9}$$

Additionally, using an analog of the product rule, we note that,

$$\begin{aligned}\sum_{x,y}(\nabla_{xy}(u\phi))^2\mu_{xy} &= \sum_{xy}((\nabla_{xy}u)\phi(x) + (\nabla_{xy}\phi)u(x))^2\mu_{xy} \\ &\leq \sum_{x,y} 2((\nabla_{xy}u)^2\phi(x)^2 + (\nabla_{xy}\phi)^2u(x)^2)\mu_{xy}\end{aligned} \tag{3.10}$$

Adding together (3.9) and (3.10), we have that, for any $c > 0$,

$$\sum_x \partial_k u^2\phi^2 m + c\sum_{x,y}(\nabla_{xy}(u_k\phi))^2\mu_{xy} \tag{3.11}$$

$$\begin{aligned}&\leq (1-\zeta+2c)\sum_{x,y}(\nabla_{xy}u)^2\phi(x)^2\mu_{xy} \\ &+ 2c\sum_{x,y}(\nabla_{xy}\phi)^2u(x)^2\mu_{xy} - \sum_{x,y}\nabla_{xy}u\nabla_{xy}(u\phi^2)\mu_{xy}.\end{aligned} \tag{3.12}$$

It suffices to show that

$$\begin{aligned}&(1-\zeta+2c)\sum_{x,y}(\nabla_{xy}u)^2\phi(x)^2\mu_{xy} \\ &\qquad \leq \sum_{x,y}\nabla_{xy}u\nabla_{xy}(u\phi^2)\mu_{xy} + (A-2c)\sum_{x,y}(\nabla_{xy}\phi)^2u^2(x)\mu_{xy}\end{aligned} \tag{3.13}$$

for some $c, A > 0$. Using the product and chain rule, we have

$$\nabla_{xy}(u\phi^2) = (\nabla_{xy}u)\phi(x)^2 + 2(\nabla_{xy}\phi)u(x)\phi(x) + (\nabla_{xy}\phi)^2u(x)$$

which reduces the proof of (3.13) to showing that

$$\begin{aligned}0 \leq (2c-\zeta)&\sum_{x,y}(\nabla_{xy}u)^2\phi(x)^2\mu_{xy} + 2\sum_{x,y}(\nabla_{xy}u)(\nabla_{xy}\phi)u(x)\phi(x)\mu_{xy} \\ &+ \sum_{x,y}\nabla_{xy}u(\nabla_{xy}\phi)^2u(x)\mu_{xy} + (A-2c)\sum_{x,y}(\nabla_{xy}\phi)^2u(x)^2\mu_{xy}.\end{aligned}$$

Fixing $c < \zeta/2$ and A large enough so that

$$0 \leq (2c-\zeta)(\nabla_{xy}u)^2\phi(x)^2 + 2(\nabla_{xy}u)(\nabla_{xy}\phi)u(x)\phi(x) + (A-2c-2)(\nabla_{xy}\phi)^2u(x)^2$$

by a quadratic identity, we can reduce the above problem to showing that

$$0 \leq \sum_{x,y} (\nabla_{xy} u)(\nabla_{xy} \phi)^2 u(x) \mu_{xy} + 2 \sum_{x,y} (\nabla_{xy} \phi)^2 u(x)^2 \mu_{xy}.$$

Swapping x and y we obtain the identity

$$\begin{aligned} -\sum_{x,y} \nabla_{xy} u (\nabla_{xy} \phi)^2 u(x) \mu_{xy} &= -\frac{1}{2} \sum_{x,y} ((\nabla_{xy} u) u(x) + (\nabla_{yx} u) u(y))(\nabla_{xy} \phi)^2 \mu_{xy} \\ &= \frac{1}{2} \sum_{x,y} (f(x) - f(y))^2 (\nabla_{xy} \phi)^2 \mu_{xy} \\ &\leq \sum_{x,y} (u(x)^2 + u(y)^2)(\nabla_{xy} \phi)^2 \mu_{xy} \\ &= 2 \sum_{x,y} (\nabla_{xy} \phi)^2 u(x)^2 \mu_{xy} \end{aligned}$$

which proves the desired statement. □

Proposition 3.7, the Cacciopoli inequality, is an inequality about u_k, the subsolution at a fixed time k. However, we want to use this result to compare subsolutions u within a space-time cylinder. This requires use to prove the following corollary, which extends the Cacciopoli inequality to an entire space-time cylinder $[0, k_0] \times G$.

Corollary 3.8. *Let $\eta_k(x)$ be a function with $k \in \mathbb{N}$ and $x \in G$, such that (1) η is supported on a finite set $U \subseteq G$; (2) $\eta_0(x) \equiv 0$; and (3) there exists a constant M such that $(\nabla_{xy} \eta_k)^2 \leq M$ and $\partial_k(\eta_k^2) \leq M$ for all $k \in \mathbb{N}$. Then, for any non-negative subsolution $u_k(x)$ and time $k_0 \in \mathbb{N}$,*

$$\sum_{x \in U} u_{k_0}^2 \eta_{k_0}^2 m + c \sum_{k=0}^{k_0-1} \sum_{x,y \in G} (\nabla_{xy}(u_k \eta_k))^2 \mu_{xy} \leq 2AM \sum_{k=0}^{k_0} \sum_{x \in \tilde{U}} u_k^2 m$$

where $\tilde{U}$ is the 1-neighborhood of U, i.e., $\tilde{U} = \{x \in G : d(x, U) \leq 1\}$, and $c, A > 0$ are the constants from Proposition 3.7, the Cacciopoli inequality, which depend only on ζ, the constant from $(\mathbf{E})$.

Proof. Applying Proposition 3.7 to $\phi(x) = \eta_k(x)$ and summing over all $k \in \{0, ..., k_0 - 1\}$, we obtain

$$\sum_{k=0}^{k_0-1} \sum_{x \in G} \partial_k(u_k^2)\eta_k^2 m + c \sum_{k=0}^{k_0-1} \sum_{x,y \in G} |\nabla_{xy}(u_k\eta_k)|^2 \mu_{xy} \leq A \sum_{k=0}^{k_0-1} \sum_{x,y} |\nabla_{xy}\eta_k|^2 u_k(x)^2 \mu_{xy}. \tag{3.14}$$

Focusing on the left-most summand, observe that

$$\begin{aligned}\sum_{k=1}^{k_0-1} \sum_{x \in G} \partial_k(u_k^2)\eta_k^2 m &= \sum_{k=1}^{k_0-1} \sum_{x \in G} (\partial_k(u_k^2\eta_k^2) - \partial_k(\eta_k^2)u_{k+1}^2)m \\ &= \sum_{x \in G} u_k^2 \eta_{k_0}^2 m - \sum_{k=0}^{k_0-1} \sum_{x \in G} \partial_k(\eta^2)u_{k+1}^2 m.\end{aligned}$$

Therefore, all that remains to be shown is

$$\sum_{k=0}^{k_0-1} \sum_{x \in G} \partial_k(\eta_k^2)u_{k+1}^2 m + A \sum_{k=0}^{k_0-1} \sum_{x,y \in G} (\nabla_{xy}\eta_k)^2 u_k(x)^2 \mu_{xy} \leq 2AM \sum_{k=0}^{k_0} \sum_{x \in \tilde{U}} u_k^2 m. \tag{3.15}$$

We assumed that η is supported on U, so we can reduce the domain on the left side of (3.15) to $\tilde{U}$. Furthermore, we assumed that both $\partial_k(\eta^2)$ and $(\nabla_{xy}\eta)^2$ are bounded above by M, so (3.15) simplifies to

$$\sum_{k=0}^{k_0-1} \sum_{x \in \tilde{U}} u_{k+1}^2 m + A \sum_{k=0}^{k_0-1} \sum_{x,y \in \tilde{U}} u_k^2 \mu_{xy} \leq 2A \sum_{k=0}^{k_0} \sum_{x \in \tilde{U}} u_k^2 m. \tag{3.16}$$

To see that (3.16) holds, note that in the second summand on the left side, $\sum_y \mu_{xy} = m(x)$. $\square$

3.2.2 Comparing subcylinders

In this section, we will use Corollary 3.8 to compare the L^2 norm of u on neighboring cylinders in a nested sequence of subcylinders. The comparison will be given in terms of a function Λ, which we explicitly give in the next section.

For integers $1 \le t' < t$ and $1 \le r' < r-1$, and a point $z \in G$ we introduce cylinders in $\mathbb{N} \times G$:

$$\Psi = [0,t] \times B(z,r),\ \ \Psi' = [t', t-1] \times B(z,r').$$

Let u be a non-negative subsolution and define:

$$I = \sum_{k,x \in \Psi} u_k^2(x) m(x),\ \ I' = \sum_{k,x \in \Psi'} (u_k(x) - \theta)_+^2 m(x)$$

where $\theta > 0$ is a constant. Then I and I' are roughly, the square of the L^2 norm of u in Ψ and Ψ' respectively. Clearly, $\Psi' \subseteq \Psi$ and hence $I' \le I$, but we would like to quantify this difference. To do this, we vary θ and $D = \min\{(r-r')^2, t'\}$.

Lemma 3.9. *Assume $\Lambda : \mathbb{R} \to \mathbb{R}$ is a fixed, decreasing, positive function such that $\Lambda(m(U)) \le \lambda_1(U)$, the first Dirichlet eigenvalue of the Laplacian in U, for any $U \subseteq G$. For cylinders Ψ and Ψ' defined above, the inequality*

$$I' \le \frac{C}{D\Lambda(\frac{CI}{D\theta^2})} I$$

holds for a constant C depending only on ζ, the constant from (**E**).

Proof. Let $r_a = (r+r')/2$ and define

$$\eta_1(x) = \begin{cases} 1 & \text{if } d(x,z) \le r_a \\ 0 & \text{if } d(x,z) \ge r \\ \frac{r - d(x,z)}{r - r_a} & \text{otherwise,} \end{cases}$$

and

$$\eta_2(k) = \begin{cases} 1 & \text{if } k \ge t' \\ \frac{k}{t'} & \text{if } k < t'. \end{cases}$$

Define $\eta_k(x) = \eta_1(x)\eta_2(k)$. Note that η satisfies the hypotheses of Corollary 3.8: (1) η is supported on $B(z, r-1)$; (2) $\eta_0(x) \equiv 0$; and (3) both $(\nabla_{xy}\eta_k)^2 \le M$ and

$\partial_k(\eta_k^2) \leq M$, where $M = 4D^{-1}$. Recall that D was defined as $D = \min\{(r-r')^2, t'\}$. To prove hypothesis (3), note that:

$$(\nabla_{xy}\eta)^2 \leq \left(\frac{1}{r - r_a}\right)^2 = \left(\frac{2}{r - r'}\right)^2 = 4D^{-1}$$

and

$$\partial_k(\eta^2) = 2(\partial_k\eta)\eta + (\partial_k\eta)^2 \leq \frac{2}{t'} + \frac{1}{t'^2} \leq \frac{3}{t'} \leq 4D^{-1}.$$

We only need a weaker form of Corollary 3.8, using the left summand, and taking $U = B(x, r-1)$,

$$\sum_{x \in B(x,r-1)} u_{k_0}(x)^2 \eta_{k_0}(x)^2 m(x) \leq 8AD^{-1} \sum_{k=0}^{l} \sum_{x \in B(x,r)} u_k(x)^2 m(x) \tag{3.17}$$

for any $k_0 \in [0, t]$. On the right side of (3.17) appears I, the quantity we are trying to approximate. Since $\eta \equiv 1$ in $B(z, r_a)$, and $B(z, r_a) \subseteq B(x, r-1)$, this inequality becomes

$$\sum_{x \in B(z,r_a)} u_{k_0}^2(x) m(x) \leq 8AD^{-1} I. \tag{3.18}$$

Now we repeat the above procedure, applying Corollary 3.8, but to slightly different functions. We will use the subsolution $\tilde{u} = (u - \theta)_+$, where $\theta > 0$ is a constant that will vary, and the finitely supported function $\tilde{\eta}_k(x) = \tilde{\eta}_1(x)\eta_2(k)$, where

$$\tilde{\eta}_1(x) = \begin{cases} 1 & \text{if } d(x,z) \leq r' \\ 0 & \text{if } d(x,z) \geq r_a \\ \frac{r' - d(x,z)}{r' - r_a} & \text{otherwise,} \end{cases}$$

and $\eta_2(k)$ is the same as above. We again apply a weaker form of Corollary 3.8, but this time we use only the right summand,

$$\sum_{k=0}^{t-1} \sum_{x,y \in G} (\nabla_{xy}(\tilde{u}_k \tilde{\eta}_k))^2 \mu_{xy} \leq c^{-1} 8AD^{-1} \sum_{k=0}^{t} \sum_{x \in B(z,r_a)} \tilde{u}_k^2(x) m(x). \tag{3.19}$$

Using that $\tilde{u} \leq u$ and $B(x, r_a) \subseteq B(x, r)$, we have that

$$\sum_{k=0}^{t-1} \sum_{x,y \in G} (\nabla_{xy}(\tilde{u}_k \tilde{\eta}_k))^2 \mu_{xy} \leq 8c^{-1} A D^{-1} I. \tag{3.20}$$

Recall that the variational definition of λ_1 is given by $\lambda_1(U) = \inf\{\frac{\|\nabla f\|_2^2}{\|f\|_2^2}\}$ where the infimum is taken over all functions supported on U.

Let $U_k = \{x \in B(z, r_a) \mid \tilde{u}_k(x) > 0\}$ be the support set for $\tilde{u}_k$. Applying the hypothesis that, $\Lambda(m(U_k)) \leq \lambda_1(U_k)$ and the variational definition of λ_1,

$$\Lambda(m(U_k)) \sum_{x \in U_k} (\tilde{u}_k \tilde{\eta}_k)^2 m \leq \sum_{x,y \in U_k} (\nabla_{xy}(\tilde{u}_k \tilde{\eta}_k))^2 \mu_{xy}. \tag{3.21}$$

We can also estimate $m(U_k)$ using the fact that $\theta < u_k$ on U_k and (3.18),

$$m(U_k) = \sum_{x \in U_k} m \leq \theta^{-2} \sum_{x \in B(z, r_a)} u_k^2 m \leq 8 A D^{-1} I. \tag{3.22}$$

Combining (3.21) and (3.22), we have

$$\Lambda\left(\frac{8AI}{\theta^2 D}\right) \sum_{x \in U_k} (\tilde{u}_k \tilde{\eta}_k)^2 m \leq \sum_{x,y \in U_k} (\nabla_{xy}(\tilde{u}_k \tilde{\eta}_k))^2 \mu_{xy}. \tag{3.23}$$

But, $\tilde{\eta} \equiv 1$ for $k \in [t', t-1]$ and $x \in B(z, r')$, and so (3.23) simplifies to an inequality on subcylinders,

$$\Lambda\left(\frac{8AI}{\theta^2 D}\right) \sum_{k=t'}^{t-1} \sum_{x \in B(z, r')} \tilde{u}_k^2 m \leq \sum_{k=t'}^{t-1} \sum_{x,y \in B(x, r')} (\nabla_{xy}(\tilde{u}_k \tilde{\eta}_k))^2 \mu_{xy}. \tag{3.24}$$

Plugging in the definition of I' and the inequality (3.20) to (3.24) gives,

$$\Lambda\left(\frac{8AI}{\theta^2 D}\right) I' \leq 8c^{-1} A D^{-1} I$$

Finally, we set the constant $C = 8c^{-1}A$, and the proof is complete. □

3.2.3 Faber-Krahn inequality

The Faber-Krahn inequality gives us a specific example of a function Λ which satisfies the hypothesis of Lemma 3.9, i.e., $\Lambda(m(U)) \leq \lambda_1(U)$ for all $U \subseteq G$.

Theorem 3.10 (Faber-Krahn)**.** *There exist constants $\alpha, \nu > 0$ such that, for $x \in G$ and $r \geq 1/2$, and any nonempty subset $U \subseteq B(x, r)$,*

$$\frac{a}{r^2}\left(\frac{V(x,r)}{m(U)}\right)^{\nu} \leq \lambda_1(U)$$

where $\lambda_1(U)$ denotes the first Dirichlet eigenvalue of the Laplace operator in U. Notably, the constants a, ν depend only on δ and ρ from ($\mathbf{V}$) and ($\mathbf{P}$).

In fact, Faber-Krahn is strictly weaker than the conjunction of ($\mathbf{V}$) and ($\mathbf{P}$): in [5], Coulhon and Grigoryan showed that Faber-Krahn is equivalent to the ($\mathbf{V}$) plus the Gaussian upper bound. Following their technique, we will use the Faber-Krahn inequality to obtain the Gaussian upper bound in Section 3.3, but we will need a stronger result to prove the Gaussian lower bound in Section 3.4.

Proof. Fix $x \in G$ and $r \geq 1/2$. The following Nash inequality [Theorem 4.3][6] holds for any f supported in $B(x, r)$:

$$\|f\|_2^{\nu+1} \leq C\frac{r}{(V(x,r))^{\nu/2}}\|f\|_1^{\nu}(\|\nabla f\|_2 + r^{-1}\|f\|_2) \tag{3.25}$$

for some positive constant C. Fix $k > 1$, which we will select later. Since any f supported on $B(x, r)$ is also supported on $B(x, kr)$, we can rewrite (3.25) as

$$\|f\|_2^{\nu+1} \leq C\frac{ar}{(V(x,kr))^{\nu/2}}\|f\|_1^{\nu}(\|\nabla f\|_2 + a^{-1}r^{-1}\|f\|_2). \tag{3.26}$$

Since supp$f \subseteq B(x, r)$ the Hölder inequality gives us that $\|f\|_1 \leq (V(x,r))^{1/2}\|f\|_2$. Applying this to (3.26), we get

$$\|f\|_2^{\nu+1} \leq C\frac{kr}{(V(x,kr))^{\nu/2}}\|f\|_1^{\nu}\|\nabla f\|_2 + \frac{C}{(V(x,kr))^{\nu/2}}(V(x,r))^{\nu/2}\|f\|_2^{\nu+1}. \tag{3.27}$$

As noted in Section 1.2.1, one of the consequences of volume doubling is a quasi-converse: we can select k large enough so that $C(V(x,kr))^{-\nu/2}(V(x,r))^{\nu/2} \leq 1/2$. This reduces (3.27) to

$$\|f\|_2^{\nu+1} \leq C\frac{ar}{(V(x,kr))^{\nu/2}}\|f\|_1^{\nu}\|\nabla f\|_2 + \frac{1}{2}\|f\|_2^{\nu+1},$$

which, after rearranging and altering the constant, gives

$$\|f\|_2^{\nu+1} \leq \tilde{C}\frac{r}{(V(x,r))^{-\nu/2}}\|f\|_1^{\nu}\|\nabla f\|_2.$$

Squaring both sides and rearranging, we get

$$\frac{\tilde{C}^{-2}}{r^2}(V(x,r))^{\nu}\left(\frac{\|f\|_2^2}{\|f\|_1^2}\right)^{\nu} \leq \frac{\|\nabla f\|_2^2}{\|f\|_2^2}.$$

The Hölder inequality implies that $\frac{1}{m(U)} \leq \frac{\|f\|_2^2}{\|f\|_1^2}$. Taking the infimum over both sides and using the variational definition of $\lambda_1(U)$ we arrive at the Faber-Krahn inequality. □

The Faber-Krahn inequality shows that $\Lambda(\xi) = \frac{\alpha}{r^2}V(z,r)^{\nu}\xi^{-\nu}$ satisfies the hypothesis of Lemma 3.9. We note the consequence here.

Corollary 3.11. *For cylinders Ψ and Ψ' and L^2-type norms I and I', as described in Lemma 3.9,*

$$I' \leq \frac{C}{D\frac{\alpha}{r^2}V(z,r)^{\nu}(\frac{CI}{D\theta^2})^{-\nu}}I$$

where $C, \alpha,$ and ν are constants depending only on $\delta, \rho,$ and ζ from our assumptions (**V**), (**P**), *and* (**E**). *The terms D and θ are positive and will vary in applications.*

The Faber-Krahn inequality gives a specific formulation of volume regularity that will be useful later. Recall that, as a consequence of volume doubling, $\frac{V(x,r)}{V(x,s)} \leq C\left(\frac{r}{s}\right)^{\theta}$, where C and θ depend only only δ, the volume doubling constant. The following corollary restates volume regularity using the Faber-Krahn constant.

Corollary 3.12 (Volume regularity with Faber-Krahn constant)**.** *For any $x, y \in G$ and $r, s \in \mathbb{N}^+$,*

$$\frac{V(x,r)}{V(x,s)} \leq C\left(\frac{r}{s}\right)^{2/\nu}$$

where ν is the constant from the Faber-Krahn inequality.

Proof. Let $f(y) = (s - d(x,y))_+$. Then,

$$\|\nabla f\| \begin{cases} = 0 & \text{for } y \notin B(x, s+1) \\ \leq 1 & \text{for } y \in B(x, s+1) \end{cases}$$

and

$$f(y) \geq s/2 \text{ for } y \in B(x, s/2).$$

Applying the Faber-Krahn inequality, Theorem 3.10, with $U = B(x,s)$,

$$\begin{aligned} \frac{a}{r^2}\left(\frac{V(x,r)}{V(x,s)}\right)^{\nu} &\leq \lambda_1(B(x,s)) \\ &\leq \frac{\|\nabla f\|_2^2}{\|f\|_2^2} \\ &\leq \frac{4V(x,s+1)}{s^2 V(x,s/2)} \\ &\leq \frac{4CV(x,s)}{s^2 V(x,s/2)} \end{aligned}$$

where the last line follows by volume doubling. This rearranges to

$$\left(\frac{s^2 a}{4Cr^2} V(x,r)^{\nu} V(x,s/2)\right)^{\frac{1}{1+\nu}} \leq V(x,s). \tag{3.28}$$

Now, apply 3.28 to $B(x, \frac{s}{2^i})$ for $i = 1, \ldots, j$ and iterate to get

$$\left(\frac{s^2 a}{4Cr^2} V(x,r)^{\nu}\right)^{\sum_{i=1}^{j} \frac{1}{(1+\nu)^i}} \left(V(x, \frac{s}{2^j})\right)^{\frac{1}{1+\nu}^j} \leq V(x,s). \tag{3.29}$$

Letting $j \to \infty$, note that $\sum_{i=1}^{\infty} \frac{1}{(1+\nu)^i} = \frac{1}{\nu}$. Therefore, (3.29) becomes

$$\left(\frac{s^2 a}{4Cr^2}\right)^{1/\nu} V(x,r) \leq V(x,s)$$

which rearranges to give volume regularity with the constant $\frac{4C}{a}$. $\square$

3.2.4 Iterate to get (MV2)

We are now ready to prove Theorem 3.4, the L^2 mean value inequality. To do so, we will repeatedly apply Corollary 3.11 to a collection of nested cylinders.

From the statement of the L^2 mean value inequality, fix $z \in G$, and any $R, T \in \mathbb{N}^+$. Let $\Psi = [0, 2T] \times B(z, R)$ be a cylinder and $m(\Psi) = 2TV(z, R)$ be its volume.

Define two sequences $\{R_n\}_{n=0}^N$ and $\{T_n\}_{n=0}^N$ as follows: Let $R_0 = R$, $T_0 = T$ and

$$R_n = \lceil R_{n-1}/2 \rceil, \quad T_n = \lceil T_{n-1}/2 \rceil$$

where N is the maximum integer such that $R_N > 2, T_n > 2$, and $N < T$. We have selected these values for $\{R_n\}$ and $\{T_n\}$ because they are strictly decreasing finite sequences of positive integers satisfying

$$R_n < R_{n-1} - 1, \quad T_n > n - T.$$

Note that the second inequality can be written $2T - n > T - T_n$.

We define a sequence of cylinders

$$\Psi_n = [T - T_n, 2T - n] \times B(z, R_n)$$

and wish to apply Corollary 3.11 iteratively to these cylinders. To that end, define

$$D_n = \min\{T_{n-1} - T_n, (R_{n-1} - R_n)^2\}$$

and $D = \min\{T, R^2\}$. For some fixed $\theta > 0$ let

$$\theta_n = \theta(2 - 2^{-n}).$$

Finally, for a fixed non-negative subsolution u let

$$I_n = \sum_{k,x \in \Psi_n} (u(k,x) - \theta_n)_+^2.$$

Applying Corollary 3.11 we get

$$I_n \leq \frac{C}{\beta D_n (\frac{C I_{n-1}}{D_n(\theta_n - \theta_{n-1})^2})^{-\nu}} I_{n-1} \tag{3.30}$$

where $\beta = \frac{\alpha}{R^2} V(z,R)^\nu$. Rearranging (3.30) and applying the definition of θ_n gives

$$I_n \leq \frac{C}{\beta D_n^{1+\nu} 4^{-\nu n} \theta^{2\nu}} I_{n-1}^{1+\nu}$$

where C is a new constant. Throughout the remainder of this proof, the value of C will change, but the constant will always depend only on α and ν from the Faber-Krahn inequality (and hence, on δ, ρ and ζ).

Note that $T_{n-1} - T_n \geq \frac{1}{4} T_{n-1}$ and similarly, $R_{n-1} - R_n \geq \frac{1}{4} R_n$. Iterating this, $D_n \geq 4^{-n-1} D$. Applying this we get

$$I_n \leq \frac{C 4^{2n\nu + n}}{\beta D^{1+\nu} \theta^{2\nu}} I_{n-1}^{1+\nu}$$

where the constant C has absorbed a factor of $4^{\nu+1}$. Rewrite this as

$$I_n \leq \frac{C \exp(n \log 4^{2\nu+1})}{\beta D^{1+\nu} \theta^{2\nu}} I_{n-1}^{1+\nu}$$

which iterates to give

$$I_N \leq \frac{\exp(\log 4^{2\nu+1}) \sum_{i=1}^{\infty} i(1+\nu)^{N-i}}{(C^{-1} \beta D^{1+\nu} \theta^{2\nu})^{1+(1+\nu)+(1+\nu)^2+\cdots+(1+\nu)^{N-1}}} I^{(1+\nu)^N}.$$

Then

$$I_N^{(1+\nu)^{-N}} \leq \frac{\exp(\log(4^{2\nu+1}) \sum_{i=1}^{\infty} i(1+\nu)^{-i})}{(C^{-1}\beta D^{1+\nu}\theta^{2\nu})^{\nu^{-1}(1-(1+\nu)^{-N})}} I \tag{3.31}$$

$$\leq C \left(\frac{1}{\beta^{\nu^{-1}} D^{1+\nu^{-1}} \theta^{2(1-(1+\nu)^{-N})}} \right) \left(\beta^{\nu^{-1}(1+\nu)^{-N}} D^{(1+\nu^{-1})(1+\nu)^{-N}} \right) I. \tag{3.32}$$

Volume regularity tells us that

$$\beta = \frac{a}{R^2} V(x,R)^\nu \leq \frac{C}{R^2}(R^{2/\nu} m(z))^\nu = Cm(z)^\nu$$

so $\beta^{\nu^{-1}(1+\nu)^{-N}} \leq Cm(z)^{(1+\nu)^{-N}}$. Also note that

$$(1+\nu)^N = \exp(N \log(1+\nu)) \geq C \exp(\frac{1}{2} \log(1+\nu) \log_2 D) = CD^\epsilon$$

where $\epsilon = \frac{1}{2}\log_2(1+\nu) > 0$. This gives

$$D^{(1+\nu^{-1})(1+\nu)^{-N}} \leq \exp\left(\frac{(1+\nu^{-1})\log D}{CD^\epsilon}\right) \leq C.$$

Plugging in these two estimates into (3.31), we get

$$I_N^{(1+\nu)^{-N}} \leq \frac{C}{\beta^{\nu^{-1}} D^{(1+\nu^{-1})} \theta^{2(1-(1+\nu)^{-N})}} m(z)^{(1+\nu)^{-N}}. \tag{3.33}$$

Since $T \in [T - T_N, 2T - N]$ and $\theta_N \leq 2\theta$, we have

$$I_N \geq (u(T,z) - 2\theta)_+^2 m(z)$$

which, combined with (3.33), gives

$$(u(T,z) - 2\theta)_+^{2(1+\nu)^{-N}} \theta^{2(1-(1+\nu)^{-N})} \leq \frac{C}{\beta^{\nu^{-1}} D^(1+\nu^{-1})} I.$$

Finally, we let $\theta = \frac{1}{3}u(T,z)$. Substituting in $\beta = \frac{a}{R^2}V(z,R)^\nu$ and $D = \min(T, R^2)$, this gives our desired result:

$$\begin{aligned} u(T,z)^2 &\leq \frac{CR^{2/\nu}}{V(z,R)\min(T,R^2)^{1+\nu^{-1}}} \sum_{k,x\in\Psi} u(k,x)^2 \\ &= \frac{C}{m(\Psi)T^{-1}\min(T^{1+\nu^{-1}}R^{-2/\nu}, R^2)} \sum_{k,x\in\Psi} u(k,x)^2 \end{aligned}$$

which completes the proof of Theorem 3.4.

3.2.5 (**MV2**) implies (**MV1**)

Now we want to show that (**MV2**) implies (**MV1**). Throughout this section, assume $u_k(x)$ is a fixed nonnegative subsolution on $\mathbb{N} \times G$.

For $k, R \in \mathbb{N}^+$ and $z \in G$, let

$$\Phi = [k - R^2, k + R^2] \times B(z, R) \tag{3.34}$$

be a cylinder with volume $m(\Phi) = 2kV(z, R)$. The overall strategy for this section will be to prove a version of the L^1 mean value inequality

$$u_k(z) \leq \frac{C}{m(\Phi)} \sum_{(l,y)\in\Phi} u_l(y)m(y) \tag{3.35}$$

on cylinders of this special form (3.34) and then extend the result to the full (**MV1**) for arbitrary cylinders of the form $\Psi = [0, 2T] \times B(z, R)$. Notice that, on these special cylinders, the $\mathfrak{m}$ term disappears from (**MV2**) and (**MV1**).

Define two sequences of cylinder of this special form in (3.34). These sequences of nested cylinders will all sit inside $\Phi = [k - R^2, k + R^2] \times B(z, R)$. First, let $R_0 = \lfloor R/2 \rfloor$ and $R_n = \lfloor R_{n-1}/2 \rfloor$ for $n \geq 1$. Let N be such that $r_N \geq 1$ but $r_{N+1} < 1$. Then, for $1 \neq n \neq N$, define cylinders

$$\Phi_n = [k - (R - R_n)^2, k + (R - R_n)^2] \times B(z, R - R_n).$$

Second, for $(\kappa_{n-1}, \xi_{n-1}) \in \Phi_{n-1}$, a specific point which we will select later, define

$$\tilde{\Phi}_n = [\kappa_{n-1} - r_n^2, \kappa_{n-1} + r_n^2] \times B(\xi_{n-1}, r_n)$$

for $1 \leq n \leq N$. Note that $\tilde{\Phi}_n \subseteq \Phi_n$.

Applying (**MV2**) to the cylinders $\tilde{\Phi}_n$,

$$\begin{aligned} u_{\kappa_{n-1}}(\xi_{n-1})^2 &\leq \frac{C}{m(\tilde{\Phi}_n)} \sum_{(l,y)\in\tilde{\Phi}_n} u_l(y)^2 m(y) \\ &\leq \max_{\tilde{\Phi}_n}(u) \frac{C}{m(\tilde{\Phi}_n)} \sum_{(l,y)\in\tilde{\Phi}_n} u_l(y) m(y) \\ &\leq \max_{\tilde{\Phi}_n}(u) \frac{C}{m(\Phi)} \cdot \frac{m(\Phi)}{m(\tilde{\Phi}_n)} \sum_{(l,y)\in\Phi} u_l(y) m(y) \end{aligned} \tag{3.36}$$

Recall from Corollary 3.12 that the Faber-Krahn inequality gave a volume regularity inequality of the form

$$\frac{V(x,r)}{V(x,s)} \leq C \left(\frac{r}{s}\right)^{2/\nu}.$$

We can expand this volume regularity estimate on cylinders of the special form Φ by noting that

$$\frac{m(\Phi_1)}{m(\Phi_2)} = \frac{m([k-r_1^2, k+r_1^2] \times B(z,r_1))}{m([k-r_2^2, k+r_2^2] \times B(z,r_2))} = \frac{2r_1^2 V(z,r_1)}{2r_2^2 V(z,r_2)} \leq C \left(\frac{r_1}{r_2}\right)^{2+2/\nu} \tag{3.37}$$

where C, ν are still only dependent on the volume doubling constant δ. Applying (3.37) to Φ and $\tilde{\Phi}_n$, we get

$$\frac{m(\Phi)}{m(\tilde{\Phi}_n)} \leq C \left(\frac{R}{R_n}\right)^{2+2/\nu} \leq C(4^{n+1})^{2+2/\nu}, \tag{3.38}$$

where the second inequality follows because $R_N \leq R_{n-1}/4$ implies that $R_n \geq R/4^{n+1}$.

Applying (3.38) to (3.36), we get

$$u_{\kappa_{n-1}}(\xi_{n-1})^2 \leq \max_{\tilde{\Phi}_n}(u) C \frac{4^{n(2+2/\nu)}}{m(\Phi)} \sum_{(l,y)\in\Phi} u_l(y) m(y). \tag{3.39}$$

Recalling that $(\kappa_{n-1}, \xi_{n-1})$ is some point in Φ_{n-1}, we can rewrite (3.39) as

$$\left(\max_{\tilde{\Phi}_{n-1}}(u)\right)^2 \leq \exp(n \log(4^{2+2/\nu})) \frac{C}{m(\Phi)} \sum_{(l,y)\in\Phi} u_l(y) m(y) \cdot \max_{\tilde{\Phi}_n}(u). \tag{3.40}$$

Iterating equation (3.40) for $0 \le n \le N$ gives

$$\left(\max_{\tilde{\Phi}_0}(u)\right)^{2^N} \le \exp(\sum_{i=1}^{N} i2^{N-i} \cdot \log(4^{2+2/\nu})) \left(\frac{C}{m(\Phi)} \sum_{(l,y)\in\Phi} u_l(y)m(y)\right)^{1+2+2^2+\cdots+2^{N-1}} \cdot \max_{\tilde{\Phi}_n}(u)$$

which implies that

$$\max_{\tilde{\Phi}_0}(u) \le C' \left(\frac{C}{m(\Phi)} \sum_{(l,y)\in\Phi} u_l(y)m(y)\right)^{1-2^N} \cdot (\max_{\tilde{\Phi}_N}(u))^{2^{-N}} \tag{3.41}$$

In the limit, we now have the desired result (3.35) for specific cylinders of the form $[k-R^2, k+R^2]\times B(z,R)$. To expand it to the general case, let $\Psi = [0,2T]\times B(z,R)$. Let $\tilde{\Psi} = [T-\tilde{R}^2, T+\tilde{R}^2]\times B(z,\tilde{R})$ so that $\tilde{\Psi} \subseteq \Psi$. By (3.35) for special cylinders,

$$u_T(z) \le \frac{C}{\tilde{R}^2 V(z,r)} \sum_{k=0}^{2T} \sum_{x\in B(x,R)} u_k(x)m(x).$$

To obtain the full result note that

$$\frac{TV(z,R)}{\tilde{R}^2 V(z,\tilde{R})} \le C \max\left\{\frac{T}{R^2}, \left(\frac{R^2}{T}\right)^{1/\nu}\right\}.$$

by (3.37), volume regularity for cylinders.

3.3 Gaussian upper bound

In this section, we will use (**MV1**) to show that (**E**), (**V**), and (**P**) imply the Gaussian upper bound in Definition 1.4. The argument follows [5]. Together with the next section, Section 3.4, this proves one implication of Theorem 1.9.

Lemma 3.13. *There exists constants $C, c > 0$ such that for any finite subsets U_1 and U_2 of G, and any $k \in \mathbb{N}^+$,*

$$\sum_{x\in U_1}\sum_{y\in U_2} p^k(x,y)m(x) \le C\exp\left(-c\frac{d^2(U_1,U_2)}{k}\right) m(U_1)^{1/2} m(U_2)^{1/2}$$

where $d(U_1,U_2) = \inf\{d(x,y) : x \in U_1, y \in U_2\}$.

Proof. Let $s \in \mathbb{R}^+$. (We will select a specific value for s later.) Consider the operator P_s given by kernel

$$p_s(x,y) = \exp(sd(x,U_2))p(x,y)\exp(-sd(y,U_2)).$$

The iterated kernel is

$$p_s^k(x,y) = \exp(sd(x,U_2))p^k(x,y)\exp(-sd(y,U_2)).$$

In [15] it is shown that $\|P_s\|_{2\to 2} \le C\exp(Cs^2k)$ for some fixed constant C and any $s \in \mathbb{R}^+, k \in \mathbb{N}^+$. Then:

$$\begin{aligned}
\sum_{x\in U_1}\sum_{y\in U_2} p^k(x,y)m(x) &= \sum_{x,y\in G} \mathbb{1}_{U_1}(x)p^k(x,y)\mathbb{1}_{U_2}(y)m(x) \\
&= \sum_{x,y\in G} \mathbb{1}_{U_1}(x)p^k(x,y)\exp(-sd(y,U_2))\mathbb{1}_{U_2}(y)m(x) \\
&= \sum_{x,y\in G} \mathbb{1}_{U_1}(x)\exp(-sd(x,U_2))p_s^k(x,y)\mathbb{1}_{U_2}(y)m(x) \\
&\le \exp(-sd(U_1,U_2))\sum_{x,y\in G} \mathbb{1}_{U_1}(x)p_s^k(x,y)\mathbb{1}_{U_2}(y)m(x) \\
&= \exp(-sd(U_1,U_2))\langle \mathbb{1}_{U_1}, P_s^k\mathbb{1}_{U_2}\rangle \\
&\le C\exp(Cs^2k - sd(U_1,U_2))\|\mathbb{1}_{U_1}\|_2\|\mathbb{1}_{U_2}\|_2
\end{aligned}$$

by letting $s = d(U_1,U_2)/2Ck$ the lemma is proved. □

Theorem 3.14. *Assume* (**E**), (**V**), *and* (**P**). *Then, there exists* $C, c > 0$ *such that, for any* $k \in \mathbb{N}^+$ *and any* $x, y \in G$,

$$q^k(x,y) \le \frac{C}{V(x,\sqrt{k})}\exp\left(-c\frac{d^2(x,y)}{k}\right).$$

Proof. We will show that

$$q^k(x_0,y_0) \le \frac{C}{\sqrt{V(x_0,\sqrt{k})V(y_0,\sqrt{k})}}\exp(-c\frac{d^2(x_0,y_0)}{k}) \tag{3.42}$$

any fixed $x_0, y_0 \in G$ and $k \in \mathbb{N}^+$, with independent constants $c, C > 0$. This formulation suffices to prove that theorem because, by volume regularity, $V(y_0, \sqrt{k})$ differs from $V(x_0, \sqrt{k})$ only by an exponential function of their distance, and so we can simply adjust the c value.

First, we will prove (3.42) when $k < 3$. Using Lemma 3.13 with $U_1 = \{x_0\}$ and $U_2 = \{y_0\}$ we have

$$p^k(x_0, y_0)m(x_0) \leq C \exp(-c\frac{d^2(x_0, y_0)}{k})m(x_0)^{1/2}m(y_0)^{1/2}$$

or, equivalently,

$$q^k(x_0, y_0) \leq C \exp(-c\frac{d^2(x_0, y_0)}{k})m(x_0)^{-1/2}m(y_0)^{-1/2}.$$

Using the volume doubling constant δ we can bound

$$m(x_0)^{-1/2}m(y_0)^{-1/2} \leq C\delta^2 V(x_0, \sqrt{k})^{-1/2}V(y_0, \sqrt{k})^{-1/2}.$$

Now let $k \geq 3$. Using (**MV1**), Theorem 3.5, for a fixed $x_0, y_0 \in G$,

$$q^k(x_0, y_0) \leq \frac{C\mathfrak{m}(l/r^2)}{lV(x_0, r)} \sum_{i=k-l}^{k+l} \sum_{x \in B(x_0, r)} q^i(x, y_0)m(x)$$

and

$$q^i(x, y_0) \leq \frac{C\mathfrak{m}(l/r^2)}{lV(y_0, r)} \sum_{j=i-l}^{i+l} \sum_{y \in B(y_0, r)} q^j(x, y)m(y)$$

where $l, r \in \mathbb{N}^+$ such that $3l \leq k$ (which required $k \geq 2$).

By combining the previous two inequalities and extending the summation from $j \in [i-l, i+l]$ to $j \in [k-2l, k+2l]$ we have

$$q^k(x_0, y_0) \leq \frac{C\mathfrak{m}(l/r^2)^2}{l^2 V(x_0, r)V(y_0, r)} \sum_{i=k-l}^{k+l} \sum_{j=k-2l}^{k+2l} \sum_{x \in B(x_0, r)} \sum_{y \in B(y_0, r)} q^j(x, y)m(x)m(y). \tag{3.43}$$

Lemma 3.13 gives that

$$\sum_{x\in B(x_0,r)}\sum_{y\in B(y_0,r)} q^j(x,y)m(x)m(y) = \sum_{x\in B(x_0,r)}\sum_{y\in B(y_0,r)} p^j(x,y)m(x)$$
$$\leq C\exp\left(-c\frac{(d(x_0,y_0)-2r)_+^2}{j}\right)V(x_0,r)^{1/2}V(y_0,r)^{1/2}.$$

Applying this to (3.43) and manipulating the constants (which uses the fact that i,j,l are all of order k)

$$q^k(x_0,y_0) \leq \frac{C\mathfrak{m}(l/r^2)}{l^2V(x_0,r)V(y_0,r)}l^2\exp\left(-c\frac{(d(x_0,y_0)-2r)_+^2}{k}\right)V(x_0,r)^{1/2}V(y_0,r)^{1/2}$$
$$\leq \frac{C\mathfrak{m}(k/r^2)}{\sqrt{V(x_0,r)V(y_0,r)}}\exp\left(-c\frac{(d(x_0,y_0)-2r)_+^2}{k}\right)$$

Select $r = \lceil\sqrt{k}\rceil$. If $d(x_0,y_0) \geq 3r$ then $d(x_0,y_0)-2r \geq 1/3d(x_0,y_0)$ and the desired result holds by changing c. Similarly, if $d(x_0,y_0)\leq 3r$ then we still get out desired result because $d(x_0,y_0)^2$ is bounded by a constant multiple of k. □

3.4 Gaussian lower bound

In this section, we show that (**E**), (**V**), and (**P**) imply the Gaussian lower bound in Definition 1.4. The proof is less direct than the one given in Section 3.3 for the Gaussian upper bound. First, we establish an on-diagonal lower bound,

$$q^k(x,x) \geq \frac{C}{V(x,\sqrt{k})},$$

and then extend it to a near-diagonal lower bound,

$$q^k(x,y) \geq \frac{C}{V(x,\sqrt{k})},$$

for $x,y\in G$ sufficiently close. Finally, we extend the lower bound to all x,y such that $d(x,y)\leq k$. (If $d(x,y)>k$, then $p^k(x,y)=0$.)

3.4.1 On-diagonal lower bound

Lemma 3.15. *For any $x \in G, k \in \mathbb{N}^+$ and $r > 0$*

$$\sum_{y \notin B(x,r)} p^k(x,y) \le C \exp\left(-c\frac{r^2}{k}\right).$$

Proof. Using the Gaussian upper bound and volume doubling,

$$\begin{aligned}
\sum_{y \notin B(x,r)} p^k(x,y) &\le C \sum_{y \notin B(x,y)} \frac{m(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \\
&= C \sum_{i=1}^{\infty} \sum_{y \in B(x,2^{i+1}r)/B(x,2^i r)} \frac{m(y)}{V(x,\sqrt{k})} \exp\left(-c\frac{d^2(x,y)}{k}\right) \\
&\le C \sum_{i=1}^{\infty} \frac{V(x,2^{i+1}r)}{V(x,\sqrt{k})} \exp\left(-c\frac{2^{2i}r^2}{k}\right) \\
&\le C \sum_{i=1}^{\infty} \left(\frac{2^{i+1}r}{\sqrt{k}}\right)^{\theta} \exp\left(-c\frac{2^{2i}r^2}{k}\right) \\
&\le C \exp\left(-c\frac{r^2}{k}\right)
\end{aligned}$$

where the constants c, C shift throughout the proof. □

Theorem 3.16. *There exists $C > 0$ such that for all $k \in \mathbb{N}^+$, $x \in G$,*

$$q^k(x,x) \ge \frac{C}{V(x,\sqrt{k})}.$$

Proof. As a consequence of Lemma 3.15, we can select A large enough and independent of x and k so that

$$\sum_{y \notin B(x,A\sqrt{k})} p^k(x,y) \le C\exp(-cA^2) \le 1/2.$$

Using this value of A,

$$p^{2k}(x,x) = \sum_{y\in G} p^k(x,y)p^k(y,x) = m(x)\sum_{y\in G}\frac{p^2(x,y)}{m(y)} \ge m(x) \sum_{y \notin B(x,A\sqrt{k})} \frac{p^2(x,y)}{m(y)}.$$

By Hölder's inequality,

$$\sum_{y\notin B(x,A\sqrt{k})} \frac{p^2(x,y)}{m(y)} \geq \frac{1}{V(x,A\sqrt{k})}\left(\sum_{y\notin B(x,A\sqrt{k})} p(x,y)\right)^2 \geq \frac{1}{4V(x,A\sqrt{k})}.$$

And (**V**) gives us that $V(x,A\sqrt{k}) = CV(x,\sqrt{k})$ for some constant C independent of x and k. We now have the desired result for even k. To extend to all k, use (**E**),

$$p^{2k+1}(x,x) \geq p^{2k}(x,x)p(x,x) \geq \frac{Cm(x)}{V(x,\sqrt{k})}$$

where the constant C has been multiplied by ζ. □

Let $q_B^k(x,y)$ be the heat kernel inside a ball B with Dirichlet boundary (and let $p_B^k(x,y)$ be the corresponding probability transition function). For the next section, extension to near diagonal, we will use the following corollary to Theorem 3.16.

Corollary 3.17. *There exists $C>0$ such that for a sufficiently large A and for all $k\in\mathbb{N}^+, x\in G$,*

$$q_B^k(x,x) \geq \frac{C}{V(x,\sqrt{k})}$$

where $B=B(x,r)$ with $r\geq A\sqrt{k}$.

Proof. We approximate $q_B(x,x)$ using $q(x,x)$ and the Dynkin-Hunt formula. By the on-diagonal lower bound, Theorem 3.16, and the Gaussian upper bound, Theorem 3.14:

$$\begin{aligned}
h_B^k(x,x) &= q^k(x,x) - \mathbb{E}_x[q_{k-l}(X_l,x)\mathbb{1}_{\{l\leq k\}}]\\
&\geq \frac{\tilde{C}}{V(x,\sqrt{k})} - \sup_{0<l<k}\frac{C}{V(x,\sqrt{l})}\exp(-c\frac{d(x,y)^2}{l})\\
&\geq \frac{1}{V(x,\sqrt{k})}\left[\tilde{C} - C\sup_{0<l<k}\frac{V(x,\sqrt{k})}{V(x,\sqrt{l})}\exp(-\frac{A^2k}{l})\right]\\
&\geq \frac{1}{V(x,\sqrt{k})}\left[\tilde{C} - C\delta\sup_{0<l<k}(k/l)^{\log_2(\delta)/2}\exp(-\frac{A^2k}{s})\right]\\
&\geq \frac{1}{V(x,\sqrt{k})}\left[\tilde{C} - C\delta(\log_2(\delta)/2A^2)^{\log_2(\delta)/2}\right]
\end{aligned}$$

where δ is the volume doubling constant. For A large enough so that

$$\tilde{C} - C\delta(\log_2(\delta)/2A^2)^{\log_2(\delta)/2} = \tilde{C}/2 > 0$$

the proof is complete. □

3.4.2 Extension to near-diagonal

The goal of this section is to extend this on-diagonal lower bound to a near-diagonal lower bound. In Lemma 3.19 we will bound the quantity $|q_B^k(x,x) - q_B(x,y)|$ for y sufficiently close to x, which, combined with Corollary 3.17 gives the desired near-diagonal lower bound. Before proving the theorem, we will need to prove a proposition and two lemmas.

The argument presented in this Section is adapted from W. Hebisch and L. Saloff-Coste [16].

Proposition 3.18. *Fix $k \in \mathbb{N}^+$ and $x \in G$. Let A be the large enough constant in Corollary 3.17 and set $B = B(x, A\sqrt{k})$. Then:*

(1) There exists C_1, c_1 such that

$$\forall n \in \mathbb{N}^+, y, z \in B \quad q_B^n(y,z) \leq \frac{C_1}{V(y,\sqrt{n})} \exp\left(-c_1 \frac{d^2(y,z)}{n}\right).$$

(2) There exists C_2 such that

$$\forall y, z \in B \quad |\partial_k q_B^k(z,y)| \leq \frac{C_2 A^{(\log_2(\delta))}}{kV(x,\sqrt{k})}$$

where δ is the volume doubling constant.

(3) There exists $c > 0$ and, for any $0 < \theta < 1$, there exists C_θ such that for any $n \in \mathbb{N}^+$ with $n > \theta k$,

$$\forall z, y \in B, \quad q_B^n(y,z) \leq \frac{C_\theta A^{(\log_2(\delta))}}{V(x,\sqrt{k})} \exp\left(-c\frac{n}{A^2 k}\right).$$

Proof. (1) This follows from Theorem 3.14 because $p_B^n(y,z) \leq p^n(y,z)$ for all $n \in \mathbb{N}^+$, $y, z \in G$.

(2) Using the semigroup property, the spectral theorem and part (1):

$$\begin{aligned}
|\partial_k q_B^k(z,y)| &= |\sum_{\xi \in B} \partial_k q_B^{3k/4}(z,\xi) q_B^{k/4}(\xi,y) m(\xi)| \\
&\leq \|\Delta^B q_B^{3k/4}(z,\cdot)\|_2 \|q_B^{k/4}(\cdot,y)\|_2 \\
&\leq \frac{1}{k} \|q_B^{k/4}(z,\cdot)\|_2 \|q_B^{k/4}(\cdot,y)\|_2 \\
&\leq \frac{1}{k} \sqrt{q_B^{t/2}(z,z) q_B^{t/2}(y,y)} \\
&\leq \frac{C A^{\log_2(\delta)}}{kV(x,\sqrt{k})}
\end{aligned}$$

(3) Using a theorem in [16] we have

$$\begin{aligned}
\sup_{y,z \in B} \{q_B^n(z,y)\} &= \sup_{z \in B} \{\|q_B^{n/2}(z,\cdot)\|_2^2\} \\
&= \|P_B^{n/2}\|_{2\to\infty}^2 \\
&\leq \|P_B^{n/2-\theta k/4}\|_{2\to 2}^2 \|P_B^{\theta k/4}\|_{2\to\infty}^2 \\
&\leq \frac{C_\theta A^{\log_2(\delta)}}{V(x,\sqrt{k})} \exp(-c\frac{n}{A^2 k})
\end{aligned}$$

□

Lemma 3.19. *For any $\sigma > 0$ and any $A \geq 1$, there exist two positive reals $C_{\sigma,A}$ and ϵ_A such that for all $x \in G$, $y \in B$ and $k \in \mathbb{N}^+$,*

$$|q_B^k(x,y) - q_B^k(x,x)| \leq \left[\sigma + C_{\sigma,A}\left(\frac{d(x,y)}{\sqrt{k}}\right)^\alpha\right] \frac{1}{V(x,\sqrt{k})}$$

where $B = B(x, A\sqrt{k})$ and α is the Hölder exponent in Corollary 3.2.

Proof. Let $g(x,y) = \sum_{k=1}^{\infty} q^k(x,y)$ (and similarly, define $g_B(x,y) = \sum_{k=1}^{\infty} q_B^k(x,y)$) be the Green's function.

For a fixed $y, z \in B$:

$$\begin{aligned} q_B^k(x,y) - q_B^k(x,z) &= (\Delta_y^B)^{-1}(\Delta_y^B) q_B^k(x,y) - (\Delta_z^B)^{-1}(\Delta_z^B) q_B^k(x,z) \\ &= (\Delta_y^B)^{-1}(\partial_k q_B^k(x,y)) - (\Delta_z^B)^{-1}(\partial_k q_B^k(x,z)) \\ &= \sum_{\xi \in B} g_B(y,\xi)\partial_k q_B^k(x,\xi) m(\xi) - \sum_{\xi \in B} g_B(z,\xi)\partial_k q_B^k(x,\xi) m(\xi) \\ &= \sum_{\xi \in B} (g_B(y,\xi) - g_B(z,\xi))\partial_k q_B^k(x,\xi) m(\xi) \end{aligned}$$

Letting $z = x$, this gives

$$|q_B^k(x,y) - q_B^k(x,x)| \leq \sum_{\xi \in B} |g_B(y,\xi) - g_B(x,\xi)||\partial_k q_B^k(x,\xi)| m(\xi).$$

For $\eta \in (0,1)$ let

$$W_1 = \{\xi \in B : d(x,\xi) \leq \eta\sqrt{t}\},\ W_2 = \{\xi \in B : d(y,\xi) \leq \eta\sqrt{t}\}$$

and

$$W = \{\xi \in B : d(x,\xi) \geq \eta\sqrt{t} \text{ and } d(y,\xi) \geq \eta\sqrt{t}\}.$$

Because $B = W \sqcup W_1 \sqcup W_2$, we break sum on the right-hand side into the three separate summations over the three sets. We estimate them in the following two lemmas. □

Lemma 3.20. *Given the setup of Lemma 3.19, define*

$$I_i = \sum_{\xi \in W_i} |g_B(y,\xi) - g_B(x,\xi)||\partial_k q_B^k(x,\xi)| m(\xi)$$

for $i = 1,2$. For a fixed τ, A, there exists $\eta_{\tau,A} > 0$ small enough so that $I_i \leq \frac{\tau}{V(x,\sqrt{k})}$.

Proof. We will prove the case for I_1, but the same argument will also provide an estimate for I_2. First, bound I_1 by

$$I_1 \le \sup_{\xi \in B}\{|\partial_k q_B^k(x,\xi)|\} \sum_{\xi \in W_1} (g_B(x,\xi) + g_B(y,\xi))m(\xi)$$

noting that Proposition 3.18 (2) gives us that

$$\sup_{\xi \in B}\{|\partial_k q_B(x,\xi)|\} \le \frac{C_2 A^{\log_2(\delta)}}{kV(x,\sqrt{k})}.$$

We now simultaneously estimate $\sum_{W_1} g_B(x,\xi)m(\xi)$ and $\sum_{W_1} g_B(y,\xi)m(\xi)$. For any $z \in B$, Proposition 3.18 (3) gives the estimate

$$\begin{aligned}
\sum_{\xi \in W_1} g_B(z,\xi)m(\xi) &= \sum_{s=0}^{\theta k-1} \sum_{\xi \in W_1} q_B^s(z,\xi)m(\xi) + \sum_{s=\theta k}^{\infty} \sum_{\xi \in W_1} h_s^B(z,\xi)m(\xi) \\
&\le \sum_{s=0}^{\theta k-1} \sum_{\xi \in W_1} q_B^s(z,\xi)m(\xi) + \sum_{s=\theta k}^{\infty} \sum_{\xi \in W_1} \frac{C_\theta A^{(\log_2(\delta))}}{V} \exp\left(-\frac{a_1 s}{A^2 t}\right) \\
&\le (\theta k - 1) + \frac{C_\theta A^{(\log_2(\delta))} V(W_1)}{V} \sum_{s=\theta k}^{\infty} \exp\left(-\frac{a_1 s}{A^2 k}\right) \\
&\le \theta k + \frac{C_\theta A^{(\log_2(\delta))+2} k V(W_1)}{a_1 V} \exp\left(\frac{-a_1 \theta}{A^2}\right) \\
&\le \left(\theta + \frac{C_\theta A^{(\log_2(\delta))+2} \eta^\gamma}{a_1 \beta}\right) k
\end{aligned}$$

where η will be selected later and $\gamma, \beta > 0$ exist by volume regularity.

Combining the last two results gives

$$I_1 \le \frac{2C_2 A^{(\log_2(\delta))}}{V(x,\sqrt{k})} \left(\theta + \frac{C_\theta A^{\log_2(\delta)+2} \eta^\gamma}{a_1 \beta}\right).$$

For any $\tau > 0$, pick θ so that

$$\theta = \frac{\tau}{4C_2 A^{(\log_2(\delta))}}$$

and η so that

$$\frac{C_\theta A^{2+(\log_2(\delta))} \eta^\gamma}{a_1 \beta} = \frac{\tau}{4C_2 A^{(\log_2(\delta))}}$$

which gives the desired inequality. $\square$

Lemma 3.21. *Given the setup of Lemma 3.19, for any $A, \tau > 0$, with $\eta = \eta_{\tau,A}$ from Lemma 3.20, let*

$$W = \{\xi \in B : d(x,\xi) \geq \eta\sqrt{t} \text{ and } d(y,\xi) \geq \eta\sqrt{t}\}.$$

There exists $C_{\tau,A}$ such that

$$J = \sum_{\xi \in W} |g_B(x,\xi) - g_B(y,\xi)| q_B^t(x,\xi) m(\xi) \leq C_{\tau,A} \left(\frac{d(x,y)}{\sqrt{t}}\right)^\alpha V(x,\sqrt{t})^{-1}$$

where α is the Hölder exponent from Corollary 3.2.

Proof. For now, let $\xi, z \in B$ with $d(z,\xi) \geq \frac{\eta\sqrt{t}}{2}$. Then

$$\begin{aligned} g_B(z,\xi) &= \sum_{k=0}^{t-1} q^k(z,\xi) + \sum_{k=t}^{\infty} q_B^k(z,\xi) \\ &\leq \frac{C}{V(x,\sqrt{t})} \sum_{k=0}^{t-1} \frac{V(x,\sqrt{t})}{V(z,\sqrt{k})} \exp(-c\frac{\eta^2 t}{k}) + \frac{C}{V(x,\sqrt{t})} \sum_{k=t}^{\infty} \exp(-\frac{ak}{A^2 t}) \\ &\leq \left(\frac{CA^{\log_2(\delta)}}{\eta^{\log_2(\delta)}} + CA^2\right) \frac{t}{V(x,\sqrt{t})} \end{aligned} \tag{3.44}$$

using Proposition 3.18 and volume regularity.

We wish to show that

$$|g_B(x,\xi) - g_B(y,\xi)| \leq C_{\tau,A} \left(\frac{d(x,y)}{\sqrt{t}}\right)^\alpha \frac{t}{V(x,\sqrt{t})} \tag{3.45}$$

for any $y \in B(x, A\sqrt{t})$ and $\xi \in W$, where α is the Hölder exponent in Corollary 3.2. We show this in two cases. First, let $y \notin B(x, \eta\sqrt{t}/2)$. Then, by (3.44) we have

$$|g_B(x,\xi) - g_B(y,\xi)| \leq g_B(x,\xi) + g_B(y,\xi) \leq C_{\tau,A} \frac{t}{V(x,\sqrt{t})}.$$

Next, let $y \in B(x, \eta\sqrt{t}/2)$. Since $g_B(\cdot,\xi)$ is harmonic, we apply the Hölder estimate and Proposition 3.18:

$$|g_B(x,\xi) - g_B(y,\xi)| \leq C \left(\frac{d(x,y)}{\eta\sqrt{t}}\right)^\alpha \sup_{B(x,\eta\sqrt{t}/2)} \{g^B(\cdot,\xi)\} \leq C_{\tau,A} \left(\frac{d(x,y)}{\sqrt{t}}\right)^\alpha \frac{t}{V(x,\sqrt{t})}.$$

In either case we have that

$$J = \sum_{\xi \in W} |g_B(x,\xi) - g_B(y,\xi)| h_t^B(x,\xi) m(\xi) \leq \sum_{\xi \in W} C_{\tau,A} \left(\frac{d(x,y)}{\sqrt{t}}\right)^\alpha \frac{t}{V(x,\sqrt{t})} h_B^t(x,\xi) m(\xi)$$
$$\leq C_{\tau,A} \left(\frac{d(x,y)}{\sqrt{t}}\right)^\alpha V(x,\sqrt{t})^{-1}$$

where the second line is justified by Proposition 3.18 and volume regularity. □

Theorem 3.22. *There exists $\epsilon, C > 0$ such that*

$$q^k(x,y) \geq \frac{C}{V(x,\sqrt{k})}$$

for any $k \in \mathbb{N}^+$ and $x, y \in G$ with $d(x,y)^2 \leq \epsilon k$.

Proof. Let A be large enough to apply Corollary 3.17 and let C be the constant given by the result. Fix $x \in G, k \in \mathbb{N}^+$ and $y \in B(x, A\sqrt{k})$. We will apply Lemma 3.19 with $\sigma = C/2$.

$$\begin{aligned} q^k(x,y) &\geq q_B^k(x,y) \\ &\geq q_B^k(x,x) - \left[\frac{C}{2} + c\left(\frac{d(x,y)}{\sqrt{k}}\right)^\alpha\right] \frac{1}{V(x,\sqrt{k})} \\ &\geq \frac{C}{V(x,\sqrt{k})} - \left[\frac{C}{2} + c\left(\frac{d(x,y)}{\sqrt{k}}\right)^\alpha\right] \frac{1}{V(x,\sqrt{k})} \\ &\geq \frac{1}{V(x,\sqrt{k})} \left[\frac{C}{2} - c\left(\frac{d(x,y)}{\sqrt{k}}\right)^\alpha\right] \end{aligned}$$

where α is the Hölder exponent. Fix ϵ such that $c\epsilon \leq \frac{C}{4}$. Then, for any $y \in G$ with $d(x,y) \leq \epsilon k$, the result holds. □

The theorem applies to x, y such that $d(x,y)^2 \leq \epsilon k$. Here, we prove a corollary which applies to any $d(x,y)^2 \leq k$.

Corollary 3.23. *There exists $C > 0$ such that*

$$q^k(x,y) \geq \frac{C}{V(x,\sqrt{k})}$$

for any $k \in \mathbb{N}^+$ and $x, y \in G$ with $d(x,y)^2 \leq k$.

Proof. To remove the dependence on the constant ϵ from Theorem 3.22 we consider two cases. Fix $x, y \in G$ such that $d(x,y)^2 \leq k$.

First, if $\sqrt{k} \leq \epsilon^{-2}$ then $d(x,y) \leq \epsilon^{-1}$. By the assumption of ellipticity (**E**) we know $p_k(x,y) \geq \zeta^{1/\epsilon}$. Because $y \in B(x, \sqrt{k})$, we have $\frac{m(y)}{V(x,\sqrt{k})} \leq 1$, we have the desired result with the constant $C = \zeta^{1/\epsilon}$.

If $\sqrt{k} \geq \epsilon^{-2}$, then $1 \leq \sqrt{k}\epsilon^2$. We construct a chain of balls $B(x_1, r_1), B(x_2, r_2), ..., B(x_j, r_j)$. Let $j = 1/\epsilon^2$. Let $k_i = \lfloor \sqrt{k}\epsilon^2 \rfloor$ or $\lfloor \sqrt{k}\epsilon^2 \rfloor$ so that $\sum_{i=1}^{j} k_i = \sqrt{k}$. Similarly, let $r_i = \lfloor d(x,y)\epsilon^2 \rfloor$ or $\lfloor d(x,y)\epsilon^2 \rfloor + 1$ so that $\sum_{i=1}^{j} r_i \geq d(x,y)$. Select $\{x_i\}_{i=1}^{j}$ so that $x_0 = x, x_j = y$ and $d(x_i, x_{i+1}) \leq r_i$. Then, repeatedly applying Theorem 3.22,

$$\begin{aligned}
p^k(x,y) &\geq \sum_{(z_1,\dots,z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} p^{k_1}(x, z_1) p^{k_2}(x_1, x_2) \cdots p^{k_j}(z_{j-1}, y) \\
&\geq \sum_{(z_1,\dots,z_{j-1}) \in B_1 \times \cdots B_{j-1}} \frac{Cm(z_1)}{V(x, \sqrt{k_1})} \\
&\geq C^j \sum_{(z_1,\dots,z_{j-1}) \in B_1 \times \cdots B_{j-1}} \frac{m(z_1)}{V(x, \sqrt{k_1})} \frac{m(z_2)}{V(B_1)} \cdots \frac{m(y)}{V(B_j)} \\
&= \frac{Cm(y)}{V(x, \sqrt{k_1})} \kappa^{j-1} \\
&\geq \frac{Cm(y)}{V(x, \sqrt{k})} \kappa^{1/\epsilon^2}
\end{aligned}$$

where the κ constant comes from volume doubling: $V(z, \sqrt{k_{i+1}}) \leq \kappa V(x_i, r_i)$ for $z \in B(x_i, r_i)$. Multiplying the constant C by κ^{1/ϵ^2} we get the decided Corollary. □

3.4.3 Full Gaussian lower bound

We will extend the results for Corollary 3.23 to show a Gaussian lower bound. The argument follows from T. Delmotte [7].

Theorem 3.24. *There exists c, C such that*

$$q^k(x,y) \geq \frac{C}{V(x,\sqrt{k})} \exp\left(-c\frac{d(x,y)^2}{k}\right)$$

for any $k \in \mathbb{N}^+$ and $x, y \in G$ with $d(x,y) \leq k$.

Proof. We want to produce $k = k_1 + k_2 + \cdots + k_j$, $x = x_0, ..., x_j = y$ and $B_0 = \{x\}, B_i = B(x_i, r_i), B_j = \{y\}$ such that

- $j - 1 \leq a_1 \frac{d(x,y)^2}{k}$,
- $r_i \geq a_2\sqrt{k_{i+1}}$, which ensures that $V(z, \sqrt{k_{i+1}}) \leq a_3 V(B_i)$ for $z \in B_i$ by $[\mathcal{VD}]$
- $\sup_{\substack{z \in B_{i-1} \\ z' \in B_i}} d(z,z')^2 \leq k_i$, which ensures that $p_{k_i}(z,z') \geq \frac{Cm(z')}{V(z,\sqrt{k_i})}$ by Theorem 3.22.

It we have such a chain of balls between x and y, this proves the Gaussian lower bound because

$$\begin{aligned}
p^k(x,y) &\geq \sum_{(z_1,\dots,z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} p^{k_1}(x,z_1) p^{k_2}(x_1,x_2) \cdots p^{k_j}(z_{j-1},y) \\
&\geq \sum_{(z_1,\dots,z_{j-1}) \in B_1 \times \cdots B_{j-1}} \frac{Cm(z_1)}{V(x,\sqrt{k_1})} \\
&\geq C^j a_3^{1-j} \sum_{(z_1,\dots,z_{j-1}) \in B_1 \times \cdots B_{j-1}} \frac{m(z_1)}{V(x,\sqrt{k_1})} \frac{m(z_2)}{V(B_1)} \cdots \frac{m(y)}{V(B_j)} \\
&= \frac{Cm(y)}{V(x,\sqrt{k_1})} \left(\frac{C}{a_3}\right)^{j-1} \\
&\geq \frac{Cm(y)}{V(x,\sqrt{k})} \left(\frac{C}{a_3}\right)^{-a_1 \frac{d(x,y)^2}{k}} \\
&= \frac{Cm(y)}{V(x,\sqrt{k})} \exp\left(-a_1 \frac{d(x,y)^2}{k} \log\left(\frac{C}{a_3}\right)\right)
\end{aligned}$$

which gives the Gaussian lower bound if we choose $c \leq a_1 \log\left(\frac{C}{a_3}\right)$.

Now we show that it is possible to create a chain of balls which satisfy the above criteria. In the first case, if $d(x,y)^2 \le k$, then the theorem follows immediately by the lemma and we are done.

In the second case, if $d(x,y) \ge k/100$, then let $j = k, k_i \equiv 1$ and set a path $x = x_0, x_1, ..., x_{j-1} = y$ with $d(x_i, x_{i+1}) \le 1$. Let $B_i = \{x_i\}$. Then one can easily check that the above criteria are satisfied.

In the final case, let

$$j = \left\lfloor 10\frac{d(x,y)^2}{k} \right\rfloor \ge 10$$

which ensures that

$$9\left(\frac{d(x,y)}{j}\right)^2 \le \frac{k}{j}.$$

Then set $k_i = \lfloor k/j \rfloor$ or $k_i = \lfloor k/j \rfloor + 1$ so that $\sum k_i = k$. Similarly, let $r_i = \lfloor \frac{d(x,y)}{j} \rfloor$ or $\lfloor \frac{d(x,y)}{j} \rfloor + 1$. Select x_i such that $d(x_i, x_{i+1}) \le r_i$. Again, it is straightforward to check that this construction satisfies the desired properties. □

3.5 Parabolic Harnack inequality

The goal of this section is to prove that any graph (V, μ) which satisfies *Gaussian bounds*, i.e. there exists positive constants c_L, C_L, c_U, C_U such that, for any $k \in \mathbb{N}^+$ and $x, y \in G$ with $d(x,y) \le k$,

$$\frac{C_L}{V(x,\sqrt{k})} \exp(-c_L d(x,y)^2/k) \le q^k(x,y) \le \frac{C_U}{V(x,\sqrt{k})} \exp(-c_U d(x,y)^2/k),$$

also satisfies the *discrete-time parabolic Harnack inequality*, i.e. given any $\eta \in (0,1)$ and constants $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$, there exists positive constants C, R such that, for any $x \in G$, $k, r \in \mathbb{N}$ with $r \ge R$, and any nonnegative solution $u(k,x)$ of the heat equation on the cylinder $Q = [k, k+\theta_4 r^2] \times B(x,r)$, we have

$$u_{k_\ominus}(x_\ominus) \leq C u_{k_\oplus}(x_\oplus)$$

for all $(k_\ominus, x_\ominus) \in Q_\ominus$ and $(k_\oplus, x_\oplus) \in Q_\oplus$ such that

$$d(x_\ominus, x_\oplus) \leq k_\oplus - k_\ominus,$$

where $Q_\ominus = [k+\theta_1 r^2, k+\theta_2 r^2] \times B(x, \eta r)$ and $Q_\oplus = [k+\theta_3 r^2, k+\theta_4 r^2] \times B(x, \eta r)$. This will complete the implication (2) ⇒ (3) of Theorem 1.9. This section largely follows the arguments of T. Delmotte in [7].

Lemma 3.25. *For (G, P, m) which satisfy the Gaussian bounds in Definition 1.4, there exists $\epsilon, c > 0$ such that*

$$q_B^k(y,z) \geq \frac{c}{V(x, 2\epsilon r)}$$

for any ball $B = B(x,r)$ with $(\epsilon r)^2 \leq k \leq (2\epsilon r)^2$, $y \in B(x, \epsilon r)$, $z \in B(x, 2\epsilon r)$ and $d(y,z)^2 \leq k$.

Proof. Using the Gaussian lower bound, Theorem 3.24,

$$p^k(y,z) \geq \frac{2cm(z)}{V(x, 2\epsilon r)}$$

where the constant $c = C_L \exp(-9c_L)/2$ absorbs the exponential term, which uses the hypothesis that $d(y,z)^2 \leq k$. Let

$$r(k,y) = p^k(y,z) - p_B^k(y,z) = \sum_{\substack{\xi \in \partial B \\ \kappa \leq k}} a(\kappa, \xi) p^{k-\kappa}(x, \xi)$$

for some $a(\kappa, \xi) \geq 0$. We will show that $r(k,y)$ is small (i.e. $p^k(y,z) \sim p_B^k(y,z)$) for y sufficiently far from the boundary of the cylinder. Note that $\frac{m(\xi)}{m(z)} a(\kappa, \xi)$ is the probability of reaching first reaching the boundary of B at $\xi \in \partial B$ after κ steps. Then:

$$\begin{aligned}
1 &= \sum_{y \in G} \frac{m(y)}{m(z)} p^k(y,z) \\
&\geq \sum_{y \in G} \frac{m(y)}{m(z)} r(k,y) \\
&= \sum_{y,\kappa,\xi} \frac{m(y)}{m(z)} a(\kappa,\xi) p^{k-\kappa}(y,\xi) \\
&= \sum_{\kappa,\xi} a(\kappa,\xi) \left(\sum_{y \in G} \frac{m(y)}{m(z)} p^{k-\kappa}(y,\xi) \right) \\
&= \sum_{\kappa,\xi} a(\kappa,\xi) \frac{m(\xi)}{m(z)}
\end{aligned}$$

Using the Gaussian upper bound, Theorem 3.14,

$$\begin{aligned}
\frac{m(z)}{m(\xi)} p^{k-\kappa}(y,\xi) &\leq \frac{C_U m(z)}{V(y, \sqrt{k-\kappa})} \exp\left(-c_U d^2(y,z)/(k-\kappa)\right) \\
&\leq \left(C_U \frac{V(y, 2\epsilon r)}{V(y, \sqrt{k-\kappa})} \exp(-c_U((1-\epsilon)r)^2/(k-\kappa)) \right) \frac{m(z)}{V(y, 2\epsilon r)} \\
&\leq \frac{c m(z)}{V(x, 2\epsilon r)}
\end{aligned}$$

for the correct choice of ϵ. This allows us to bound $r(k,y)$:

$$\begin{aligned}
r(k,y) &= \sum_{\substack{\xi \in \partial B \\ \kappa \leq k}} a(\kappa,\xi) p^{k-\kappa}(y,\xi) \\
&= \sum_{\substack{\xi \in \partial B \\ \kappa \leq k}} \left(a(\kappa,\xi) \frac{m(\xi)}{m(z)} \right) \left(\frac{m(z)}{m(\xi)} p^{k-\kappa}(y,\xi) \right) \\
&\leq \frac{c m(z)}{V(x, 2\epsilon r)} \sum_{\substack{\xi \in \partial B \\ \kappa \leq k}} a(\kappa,\xi) \frac{m(\xi)}{m(z)} \\
&\leq \frac{c m(z)}{V(x, 2\epsilon r)}.
\end{aligned}$$

which proves the lemma. □

Theorem 3.26. *If (G, P, m) satisfy the Gaussian bounds in Definition 1.4, then (G, P, m) satisfy the Harnack inequality in Definition 1.5.*

Proof. We will prove the Harnack inequality with specific constants $\eta = \epsilon$, $\theta_1 = \epsilon^2/2$, $\theta_2 = \epsilon^2$, $\theta_3 = 2\epsilon^2$, and $\theta_4 = 4\epsilon^2$ from Lemma 3.25 and use the same radius r. Even though this proof is for specific values of $\theta_1, \theta_2, \theta_3, \theta_4, \eta$, the results extend to any $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3, \tilde{\theta}_4, \tilde{\eta} \in (0,1)$. Given any $(k_\ominus, x_\ominus) \in \tilde{Q}_\ominus$ and $(k_\oplus, x_\oplus) \in \tilde{Q}_\oplus$ we can create a finite chain $x_\ominus = x_0, ..., x_n = x_\oplus$ and $k_\ominus = k_0 < \cdots < k_n = k_\oplus$ such that $u(k_i, x_i) \leq Cu(k_{i+1}, x_{i+1})$, which gives a new constant $\tilde{C} = C^n$. The length of the chain, n, depends only on the $\eta, \tilde{\eta}, \theta_i$ and $\tilde{\theta}_i$.

Let $u(k,x)$ be a nonnegative solution of the heat equation on the cylinder

$$Q = [0, 4\epsilon^2 r^2] \times B(x,r).$$

Let $Q_\ominus = [\epsilon^2/2r^2, \epsilon^2 r^2] \times B(x, \epsilon r)$ and $Q_\oplus = [2\epsilon^2 r^2, 4\epsilon^2 r^2] \times B(x, \epsilon r)$. In Lemma 3.25 we defined a restriction of the transition function to the cylinder. Similarly, we now define a restriction of our solution. Decompose u as

$$v(k,y) = \sum_{\substack{\kappa<k \\ \xi \in \partial B(x,2\epsilon r) \\ \text{or} \\ \kappa=0 \\ \xi \in B(x,2\epsilon r)}} a(\kappa, \xi) p^{k-\kappa}_{B(x,r)}(y, \xi),$$

with nonnegative $a(\kappa, \xi)$ such that $u(k,y) = v(k,y)$ if $y \in B(x, 2\epsilon r)$. Note that $v \leq u$ everywhere. Since they are equal in the interior, we will prove that Harnack inequality for v.

Therefore, it suffices to show that, for $B = B(x,r)$,

$$p_B^{k_\ominus - \kappa}(x_\ominus, \xi) \leq C p_B^{k_\oplus - \kappa}(x_\oplus, \xi)$$

for $(k_\ominus, x_\ominus) \in Q_\ominus$, $(k_\oplus, x_\oplus) \in Q_\oplus$, $\kappa < \epsilon^2 r^2$ and $d(x_\ominus, x_\oplus) \leq k_\oplus - k_\ominus$.

If $d(x_\oplus, \xi) > k_\oplus - \kappa$, then

$$d(x_\ominus, \xi) \geq d(x_\oplus, \xi) - d(x_\oplus, x_\ominus) > (k_\oplus - \kappa) - (k_\oplus - k_\ominus) = k_\ominus - \kappa$$

in which case $p_B^{k_\ominus-\kappa}(x_\ominus,\xi)=0$ and we are done.

If $d(x_\oplus,\xi)\le k_\oplus-\kappa$, then by Lemma 3.25 we have that

$$p_B^{k_\oplus-\kappa}(x_\oplus,\xi)\ge \frac{cm(\xi)}{V(x,2\epsilon r)}.$$

In this case we now compute a similar upper bound on $p_B^{k_\ominus-\kappa}(x_\ominus,\xi)$. First, the Gaussian estimate gives:

$$p_B^{k_\ominus-\kappa}(x_\ominus,\xi)\le p^{k_\ominus-\kappa}(x_\ominus,\xi)\le \frac{C_U m(\xi)}{V(x_\ominus,\sqrt{k_\ominus-\kappa})}\exp(-c_U d(x^2_{\ominus,\xi}/(k_\ominus-\kappa)).$$

Note that $\epsilon^2/r^2\le k_\ominus\le \epsilon^2 r^2$. We bound this quantity by looking at two separate cases. First, if $\kappa=0$ and $\xi\in B(x,2\epsilon r)$, then we can simply ignore the exponential term and use volume doubling to obtain the upper bound of $\frac{Cm(\xi)}{V(x,2\epsilon r)}$. In the second case, when $\kappa>0$ and $\xi\in\partial B(x,2\epsilon r)$, we have that $d(x_\ominus,\xi)\ge\epsilon r$. Then, making use of the Gaussian term and volume doubling gives the same bound. □

3.6 Harnack inequality implies (V), (P) and (E)

In this section, we will show that (3) ⇒ (1) in Theorem 1.9. In other words, the Harnack inequality implies (**V**), (**P**), and (**E**).

The proof of ellipticity is fairly direct. Fix $x\in U$. Since $u_k(y)=p^k(x,y)$ is a solution to the heat equation, the Harnack inequality implies that

$$p^0(x,x)\le Cp^1(x,y)$$

for all $y\in U$ such that $d(x,y)\le 1$. Since $p^0(x,x)=1$, we have the ellipticity with $\zeta=1/C$.

3.6.1 Harnack implies (V)

Theorem 3.27. *A graph (G, P, m) satisfying the parabolic Harnack inequality of Definition 1.5 also satisfies volume doubling* (**V**).

Proof. The constant $C > 0$ shifts throughout the proof, but remains dependent only on the Harnack inequality constant. Fix $x \in G$ and $k \in \mathbb{N}^+$. Define

$$u(l, y) = \begin{cases} 1 & l \leq k \\ \sum_{y \in B(x,\sqrt{k})} p^{l-k}_{B(x,\sqrt{k})}(y, z)m(z) & l > k \end{cases}.$$

Then by the Harnack inequality,

$$1 = u(k, x) \leq Cu(2k, x) = C \sum_{y \in B(x,\sqrt{k})} p^{k}_{B(x,\sqrt{k})}(x, z)m(z).$$

Applying Hölder's inequality to this, we have

$$\begin{aligned} 1 &\leq CV(x, \sqrt{k}) \sum_{y \in B(x,\sqrt{k})} \left(p^{k}_{B(x,\sqrt{k})}(x, z)\right)^2 m(z) \\ &= CV(x, \sqrt{k}) p^{2k}_{B(x,\sqrt{k})}(x, x) \end{aligned}$$

by reversibility. This gives

$$V(x, \sqrt{k})^{-1} \leq Cp^{2k}_{B(x,\sqrt{k})}(x, x) \leq Cp^{2k}(x, x). \tag{3.46}$$

Next we will apply the Harnack inequality again to estimate p^{2k}. Define $q(l, y) = p^{l+2}(x, y)$. Then $q(l, y)$ is a solution to the heat equation, and by the Harnack inequality

$$p^n(x, x) \leq Cp^{2n}(x, y). \tag{3.47}$$

This gives,

$$\sum_{y \in B(x,\sqrt{n})} p^n(x, x) \leq C \sum_{y \in B(x,\sqrt{n})} p^{2n}(x, y)$$

or

$$p^n(x,x) \le CV(x,\sqrt{n})^{-1}.$$

Iterating (3.46) and using (3.47) with $n = 4k$ we get that

$$p^{2k}(x,x) \le Cp^{4k}(x,x) \le CV(x,2\sqrt{k})^{-1}. \tag{3.48}$$

By (3.46) and (3.48)

$$V(x,2\sqrt{k}) \le CV(x,\sqrt{k}).$$

□

3.6.2 Harnack implies (P)

Lemma 3.28. *Given a graph (G,P,m) satisfying the parabolic Harnack inequality of Definition 1.5, for a fixed $x \in G$ and $k \in \mathbb{N}^+$,*

$$p^{4k}_{B(x,\sqrt{k})}(x,y) \ge \frac{C}{V(x,\sqrt{k})}$$

for any $y \in B(x,\sqrt{k})$.

Proof. Fix $x \in G$ and $k \in \mathbb{N}^+$. Let $u^l(y) = p^{k+l}_{B(x,\sqrt{k})}(x,y)$. Since $u^l(y)$ is a solution to the heat equation, we apply the Harnack inequality to get

$$p^{2k}_{B(x,\sqrt{k})}(x,x) \le C \inf_{y \in B(x,\sqrt{k}/2)} p^{3k}_{B(x,\sqrt{k})}(x,y).$$

Similarly, $v^l(y) = p^{2k+l}_{B(x,\sqrt{k})}(x,y)$ is a solution to the heat equation and, by the Harnack inequality, gives us

$$p^{3k}_{B(x,\sqrt{k})}(x,y) \le C \inf_{z \in B(x,\sqrt{k})} p^{4k}_{B(x,\sqrt{k})}(z,y).$$

Combining these with (3.46) we get

$$CV(x,\sqrt{k})^{-1} \leq \inf_{y \in B(x,\sqrt{k})} p^{4k}_{B(x,\sqrt{k})}(x,y)$$

which proves the lemma. $\square$

Theorem 3.29. *A graph (G,P,m) satisfying the parabolic Harnack inequality of Definition 1.5 also satisfies the Poincaré inequality* $(\mathbf{P})$*.*

Proof. Let $f : G \to \mathbb{R}$ and fix $x \in G$, $k \in \mathbb{N}^+$. Let Q be the Markov operator associated with the iterated Dirichlet kernel $p^{4k^2}_{B(x,k)}$. Then $Q[(f - Qf(x))^2]$ is a solution to the heat equation. Using Lemma 3.28

$$\begin{aligned} Q[(f - Qf(y))^2](y) &\geq \sum_{z \in B(x,k)} (f(z) - Qf(y))^2 p^{4k^2}_{B(x,k)}(y,z) \\ &\geq \sum_{z \in B(x,k)} (f(z) - Qf(y))^2 \frac{Cm(z)}{V(y,2k)} \\ &\geq \frac{C}{V(x,3k)} \sum_{z \in B(x,k)} m(z)(f(z) - F_{B(x,k)})^2 \end{aligned}$$

where the last line is justified because the quadratic form is minimized by the mean. Summing over $B(x,2k)$ we have

$$\begin{aligned} \sum_{y \in B(x,2k)} Q[(f - Qf(y))^2](y) &\geq \frac{C}{V(x,3k)} \sum_{y \in B(x,2k)} \sum_{z \in B(x,k)} m(z)(f(z) - f_{B(x,k)})^2 \\ &\geq C \sum_{z \in B(x,k)} m(z)(f(z) - f_{B(x,k)})^2 \end{aligned}$$

by volume doubling, Theorem 3.27,

$$\sum_{y \in B(x,2k)} Q[(f - Qf(y))^2](y) = \|f\|^2_{L^2(B)} - \|Qf\|^2_{L^2(B)}.$$

From the above two lines we have

$$\sum_{z \in B(x,k)} (f(z) - f_{B(x,k)})^2 m(z) \leq C(\|f\|^2_{L^2(B)} - \|Qf\|^2_{L^2(B)}). \tag{3.49}$$

To prove the Poincaré inequality, it suffices to show that

$$\|f\|^2_{L^2(B)} - \|Qf\|^2_{L^2(B)} \leq Ck^2 \|\nabla f\|^2_{L^2(B)}. \tag{3.50}$$

Let P_B be the Markov operator associated with the Dirichlet kernel $p^B(x, y)$, so that $Q = P_B^{4k^2}$. Then

$$\|f\|^2_{L^2(B)} - \|Qf\|^2_{L^2(B)} \leq 4k^2 (\|f\|^2_{L^2(B)} - \|P_B f\|^2_{L^2(B)})$$

making iterative use of the fact that $\|Pf\|^2_{L^2(B)} \leq \|f\|^2_{L^2(B)}$. We can bound the left-hand side using inequality $a^2 - b^2 \leq 2a(a - b)$:

$$\begin{aligned}
\|f\|^2_{L^2(B)} - \|P_B f\|^2_{L^2(B)} &= \sum_{y \in B(x,k)} m(y) \left(f(x)^2 - \left(\sum_{z \in B(x,k)} p^B(y, z) f(z) \right)^2 \right) \\
&\leq \sum_{y \in B(x,k)} 2m(y) f(y) \left(f(y) - \left(\sum_{z \in B(x,k)} p^B(y, z) f(z) \right) \right) \\
&= 2 \sum_{y,z \in B(x,k)} m(y) p^B(y, z) f(y)(f(y) - f(z)) \\
&= \sum_{y,z \in B(x,k)} \mu_{yz} (f(y) - f(z))(f(y) - f(z)) \\
&= \|\nabla f\|^2_{L^2(B)}.
\end{aligned}$$

This, combined with (3.49) prove the Poincaré inequality. □

3.7 Hölder-type continuity for solutions to the heat equation

In Section 3.1 we proved a Hölder-type estimate for harmonic equations using the elliptic Harnack inequality. Analogously, the parabolic Harnack inequality can be used to prove a Hölder-type estimate for solutions to the heat equation. The proof below follows [7].

Theorem 3.30. *Assume (G, P, m) is a graph satisfying the Poincaré inequality and volume doubling (or, by Theorem 1.9, equivalently, the parabolic Harnack inequality). Then, there exist $A, \alpha > 0$ such that, for all $z \in G$ and $K, N \in \mathbb{N}^+$, and solutions to the heat equation $u_k(x)$ in the cylinder $Q = [K - 2N^2, K] \times B(z, 2N)$,*

$$|u_k(x) - u_{\tilde{k}}(y)| \leq A \left(\frac{\sup\{\sqrt{|k - \tilde{k}|}, d(x,y)\}}{N} \right)^{\alpha} \sup_Q |u|,$$

where $k, \tilde{k} \in [K - N^2, K]$ and $x, y \in B(z, N)$.

Proof. We can assume $k > \tilde{k}$. For each $i \geq 0$ (until $k - 2^{2i} < 0$), define:

$$Q(i) = [k - 2^{2i}, K] \times B(x, 2^i),$$

$$M(i) = \sup_{Q(i)} u,$$
$$m(i) = \inf_{Q(i)} u,$$
$$\omega(i) = M(i) - m(i).$$

To apply the Harnack inequality, Theorem 1.9, to each cylinder $Q(i)$, define

$$Q_{\ominus}(i) = [k - 2^{2i}, k - 2^{2(i-1)}] \times B(x, 2^{i-1}),$$

$$Q_{\oplus}(i) = [k - 2^{2(i-1)}, K] \times B(x, 2^{i-1}).$$

Note that $Q_{\oplus}(i) = Q(i-1)$. Using this separation of $Q(i)$, apply the Harnack inequality to $(u - m(i))$,

$$\sup_{Q_{\ominus}(i)} (u_n(z) - m(i)) \leq C \inf_{Q_{\oplus}(i)} (u_n(z) - m(i)),$$

which simplifies to,

$$\sup_{Q_{\ominus}(i)} u_n(z) - m(i) \leq C(m(i-1) - m(i)). \tag{3.51}$$

Using the fact that $(k-2^{2i-1},x) \in Q_{\ominus}(i)$, (3.51) becomes,

$$u_{k-2^{2i-1}}(x) - m(i) \leq C(m(i-1) - m(i)). \tag{3.52}$$

Similarly, applying the Harnack inequality to $(M(i) - u)$ in $Q(i)$ yields

$$M(i) - u_{k-2^{2i-1}}(x) \leq C(M(i) - M(i-1)). \tag{3.53}$$

Combining (3.52) and (3.53) shows that

$$\omega(i-1) \leq \frac{C-1}{C}\omega(i). \tag{3.54}$$

Fix I such that

$$2^{I-1} \leq \sup\{\sqrt{|k-\tilde{k}|}, d(x,y)\} \leq 2^{I}$$

fix $\tilde{I}$ such that

$$2^{\tilde{I}} \leq N \leq 2^{\tilde{I}+1}.$$

This implies that

$$\omega(I) \geq |u_k(x) - u_{\tilde{k}}(y)| \tag{3.55}$$

$$\omega(\tilde{I}) \leq 2\sup_{Q}|u| \tag{3.56}$$

Iteratively applying (3.54) between I and $\tilde{I}$ yields

$$\omega(I) \leq \left(\frac{C-1}{C}\right)^{\tilde{I}-I}\omega(\tilde{I}), \tag{3.57}$$

which, combined with (3.55) and (3.56) and the definitions of I and $\tilde{I}$, proves the proposition. □

CHAPTER 4

INNER UNIFORM DOMAINS

We say that U is an *inner uniform* domain if there exists constants $\kappa, \tilde{\kappa} > 0$ such that any $x, y \in U$ can be connected by a path γ_{xy} with the following two properties: (1) the length of γ_{xy} is at most $\kappa d_U(x, y)$; and (2) for any $z \in \gamma_{xy}$,

$$d(z, \partial U) \geq \tilde{\kappa} \frac{d_U(z, x) d_U(z, y)}{d_U(x, y)}, \tag{4.1}$$

where $d_U(x, y)$ is the length of the shortest path between x and y that is contained within U.

The primary goal of this chapter is to prove Theorem 1.10, which states that if (G, P, m) satisfies the Harnack inequality and $U \subseteq G$ is an inner uniform domain, then $(U, P_{N,U}, m)$ also satisfies the Harnack inequality. The operator $P_{N,U}$ is the restriction of P to U with Neumann, or reflecting, boundary conditions. To be more precise, for each $x \in U$, define $\mu^N_{xx} = \mu_{xx} + \sum_{y: y \in \partial U} \mu_{xy}$ and $\mu^N_{xy} = \mu_{xy}$ for $x, y \in U$. Then the Neumann probability transition function is $p_{N,U}(x, y) = \frac{\mu^N(xy)}{m(x)}$ and the kernel for the operator $P_{N,U}$ is $\frac{p_{N,U}(x,y)}{m(y)}$.

To prove that U satisfies the Harnack inequality, we show that U is volume doubling (in Theorem 4.1) and satisfies the Poincaré inequality (in Theorem 4.3). Then we invoke Theorem 1.9, which states that the Harnack inequality is equivalent to the conjunction of volume doubling and the Poincaré inequality.

4.1 Examples of inner uniform domains

Intuitively, we think of an inner uniform domain as a domain with "banana" region around paths between any two points. The path must curve away from the boundary by condition (2) in the definition of inner uniform, but cannot veer too far from the most direct path by condition (1).

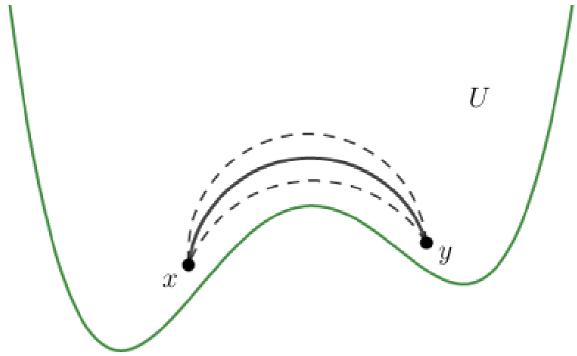


Figure 4.1: Path from x to y contained in banana shaped region

Our first example of an inner uniform domain is the discrete upper half-plane with vertical slits extending up x-axis removed.

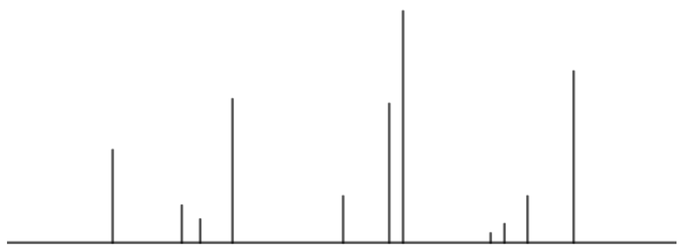


Figure 4.2: Slitted upper half-place

More precisely, let $s = \{(x_i, y_i) : 0 < y_i\}_{i=1}^{\infty}$ be a countable collection of points in $\mathbb{Z}^2$. These are the tops of the slits. The slitted upper half-plane is then given by

$$\mathbf{S} = \{(x, y) : x \in \mathbb{Z},\ y \in \mathbb{Z}^+\} \setminus S$$

where S is the collection of slits,

$$S = \bigcup_{i=1}^{\infty} \{(x_i, y) : 0 < y < y_i\}.$$

Then $\mathbf{S}$ is inner uniform if and only if there exists some constant $\epsilon > 0$ such that $\epsilon \min\{y_i, y_j\} \leq d(x_i, x_j)$ for all $i \neq j$.

This example is interesting because **S** is not uniform. A domain $U \subset G$ is *uniform* if there exists constants $k, \tilde{k}$ such that any $x, y \in U$ can be connected by a path γ_{xy} with the following two properties: (1) the length of γ_{xy} is at most $kd(x, y)$ and (2) for any $z \in \gamma_{xy}$,

$$d(z, \partial U) \geq \tilde{k}\frac{d(z, x)d(z, y)}{d(x, y)}$$

where $d(x, y)$ is the length of the shortest path in G. Notice that the definition of uniform is analogous to the definition of inner uniform, but uses the metric on G instead of the metric on U.

4.1.1 Compliment of convex set

Let $V \subseteq \mathbb{Z}^2$ be a closed and convex set. Then $U = \mathbb{Z}^2 \setminus V$ is an inner uniform domain. See [14, 6.4.1] for a proof of this fact. For our purposes, an interesting instance of this class of inner uniform domains is the region outside a parabola, e.g., $V = \{(x, y) \in \mathbb{Z}^2 : y \leq x^2\}$.

4.1.2 Non-example

Let $\tilde{\mathbf{S}}$ be the slitted upper half plane where the tops of the slits are given by $\tilde{s} = \{(x, x) : x \in 2\mathbb{N}\}$. As we progress along the positive x-axis, the slits, which extend up from the even values of x, become taller and taller.

As noted in the previous example, $\tilde{\mathbf{S}}$ is not an inner uniform domain because there does exist an ϵ such that $\epsilon \min\{y_i, y_j\} \leq d(x_i, x_j)$ for $i \neq j$. To see this more explicitly, note that, for a given $n \in \mathbb{N}$ consider a path extending from $(2n + 1, 1)$

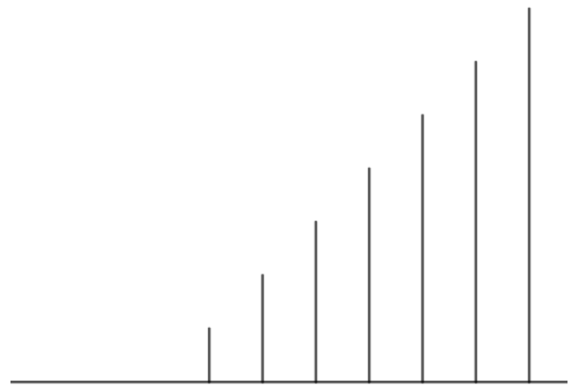


Figure 4.3: Example that is not an inner uniform domain

to $(2n+3,1)$. As $n \to \infty$, it would be impossible to create paths satisfying both conditions of inner uniformity, i.e., either the κ or $\tilde{\kappa}$ would be unbounded.

4.2 Volume doubling in inner uniform domains

Theorem 4.1. *Let (G, P, m) be volume doubling and let $U \subseteq G$ be an inner uniform domain. Then (U, m_U), where m_U is the restriction of m to U, is volume doubling.*

Proof. For fixed $x \in U$ and $r > 0$, we will use $B_G(x,r)$ and $B_U(x,r)$ to denote the set of points at most distance r from x within G and U, respectively. Similarly, $V_G(x,r)$ and $V_U(x,r)$ denote the volumes of $B_G(x,r)$ and $B_U(x,r)$, respectively Note that, for any $x \in U$ and $r > 0$ such that $B_G(x,r) = B_U(x,r)$, i.e., the ball is sufficiently far from the boundary of U, volume doubling holds immediately because

$$V_U(x, 2r) \leq V_G(x, 2r) \leq \delta V_G(x,r) = \delta V_U(x,r).$$

For the general case, we will show that any ball $B_U(x,r)$ contains a ball $\tilde{B}$

whose radius is uniformly comparable to r and such that $\tilde{B}_U = \tilde{B}_G$. The existence of such a ball implies that the volume of B_U and B are uniformly comparable, which gives the desired result.

To construct $\tilde{B}$, begin by fixing $z \in B_U(x,r)$ such that $d_U(x,z) \geq r/2$. Connect x and z by γ_{xz}, the path guaranteed to exist by the inner uniformity condition. Then, (1) $|\gamma_{xz}| \leq \kappa d_U(x,z) = \kappa r$ and (2) $d(y, \partial U) \geq \tilde{\kappa} \frac{d_U(x,y) d_U(y,z)}{d_U(x,z)}$ for any $y \in \gamma_{xz}$. Fix a $y \in \gamma_{xz}$ such that $d_U(x,y) \geq r/4$. Then, applying condition (2),

$$\begin{aligned} d_U(y, \partial U) &\geq d(y, \partial U) \\ &\geq \tilde{\kappa} \frac{d_U(x,y)}{d_U(y,z)} d_U(x,z) \\ &\geq \frac{\tilde{\kappa} r}{4} \frac{d_U(y,z)}{d_U(x,z)} \\ &\geq \frac{\tilde{\kappa} r}{4} \frac{(d_U(x,z) - d_U(x,y)}{d_U(x,z)} \\ &\geq \frac{\tilde{\kappa} r}{4} \left(1 - \frac{r/4}{r/2}\right) \\ &= \frac{\tilde{\kappa} r}{8} \end{aligned}$$

Therefore, $\tilde{B} = B(y, \frac{\kappa r}{8})$ is a ball inside $B_U(x,r)$ such that $\tilde{B}_U = \tilde{B}_G$.

Note that $B_U(x, 2r) \subseteq B_G(y, 4r)$. Then,

$$V_U(x, 2r) \leq V_G(y, 4r) \leq \tilde{\delta} V_G(y, \frac{\tilde{\kappa} r}{8}) = \tilde{\delta} V_U(y, \frac{\tilde{\kappa} r}{8}) \leq \tilde{\delta} V_U(x, r)$$

where we have applied the volume doubling hypothesis on G (perhaps several times) to compare the volumes of $V_G(y, 4r)$ and $V_G(y, \frac{\tilde{\kappa} r}{8})$. Therefore, $\tilde{\delta}$, the new volume doubling constant, depends only on δ and $\tilde{\kappa}$. $\square$

4.3 Poincaré inequality on example domains

For a ball B, we let V_B denote the vertices within B and E_B denote the edges within B. The volume of the ball is $V(B)$. For $e \in E_B$, we denote the origin vertex as e^- and the target vertex as e^+, i.e., $e = (e^-, e^+)$. The following lemma proves a variant of the Poincaré inequality on a ball B, but there is a constant A_B, dependent on B, where we would like Cr^2. In the following section we will give examples where A_B is uniformly bounded by Cr^2 for all balls B of a fixed radius r, and hence, the Poincaré inequality is satisfied.

Lemma 4.2. *Given a ball B and a fixed set of paths $\{\gamma_{xy}\}_{x,y\in B}$,*

$$\sum_{x\in V_B} (f(x) - f_B)^2 m(x) \leq A_B \sum_{x,y\in V_B} (f(x) - f(y))^2 \mu_{xy}$$

where

$$A_B = \max_{e\in E_B} \left\{ \frac{\sum_{\gamma_{xy}\ni e} |\gamma_{xy}| m(x)m(y)}{2V(B)m(e^+)p(e^+, e^-)} \right\}.$$

Proof. Fix a ball B and a set of paths $\{\gamma_{xy}\}_{x,y\in B}$. Let $|\gamma_{xy}|$ denote the length of the path from x to y. Then

$$
\begin{aligned}
\sum_{x\in B}(f(x)-f_B)^2 m(x) &= \frac{1}{2V(B)}\sum_{x,y\in V_B}(f(x)-f(y))^2 m(x)m(y)\\
&= \frac{1}{2V(B)}\sum_{x,y\in V_B}\left[\sum_{e\in\gamma_{xy}}(f(e^+)-f(e^-))\right]^2 m(x)m(y)\\
&\le \frac{1}{2V(B)}\sum_{x,y\in V_B}\left[\sum_{e\in\gamma_{xy}}(f(e^+)-f(e^-))^2\right]|\gamma_{xy}|m(x)m(y)\\
&= \frac{1}{2V(B)}\sum_{e\in E_B}(f(e^+)-f(e^-))^2\left[\sum_{\gamma_{xy}\ni e}|\gamma_{xy}|m(x)m(y)\right]\\
&= \sum_{e\in E_B}(f(e^+)-f(e^-))^2 m(e^+)p(e^+,e^-)\left[\frac{\sum_{\gamma_{xy}\ni e}|\gamma_{xy}|m(x)m(y)}{2V(B)m(e^+)p(e^+,e^-)}\right]\\
&\le A_B\sum_{x,y\in V_B}(f(x)-f(y))^2 m(x)m(y)
\end{aligned}
$$

where the first inequality follows by Cauchy-Schwarz. □

4.3.1 Upper half-plane

Consider a simple random walk in the upper half-plane,

$$\mathbb{Z}^2_{>0} = \{(x,y)\in\mathbb{Z}^2 \ : \ y>0\}.$$

Note that $\partial\mathbb{Z}^2_{>0}$ is the y-axis, i.e., $\partial\mathbb{Z}^2_{>0} = \{(x,y)\in\mathbb{Z}^2 \ : \ y=0\}$. For a simple random walk, the edge weights are $\mu((x_0,y_0),(x_1,y_1)) = \frac{1}{4}$ for any two points (x_0,y_0) and (x_1,y_1) that are distance one apart. Then,

$$p((x_0,y_0),(x_1,y_1)) = \frac{1}{4} \text{ and } \quad m((x,y))\equiv 1$$

for points that are not adjacent to the boundary.

To prove the Poincaré inequality for $(\mathbb{Z}^2_{>0}, m)$ we need to uniformly bound the constants A_B in Lemma 4.2 by Cr^2 for all balls B of radius r, where C is independent of r. To do this, we describe a set of paths connecting any pair of points (x_0, y_0) and (x_1, y_1) within a fixed ball B.

At least one of (x_0, y_1) or (x_1, y_0) must also be in B. Thus, we can define the path from (x_0, y_0) to (x_1, y_1) as the L-shaped path (or, rotated L-shape) where the corner is (x_0, y_1)—or, if (x_0, y_1) is not in B, then the corner is (x_1, y_0).

To prove the Poincaré inequality we compute the constant A_B from Lemma 4.2,

$$A_B = \max_{e \in E_B} \left\{ \frac{\sum_{\gamma_{xy} \ni e} |\gamma_{xy}|}{2V(B)(1/4)} \right\} \leq C \frac{r \cdot r^3}{r^2} = Cr^2$$

since $\max\{|\gamma_{xy}|\} \leq 2r$ and the maximum number of paths through a given edge is bounded above by Cr^3. This is because, if $|\gamma_{xy}|$ goes through a fixed edge e then x must lie on the same line as e (hence, there are r choices) and y can be anywhere (hence, there are r^2 choices).

4.3.2 Cone in $\mathbb{R}^2$

Let U be a cone centered at the origin. Then U is an inner uniform domain. Using the same L-shaped paths from the previous section, we can see that U also satisfies the Poincaré inequality.

4.4 Poincaré inequality on general inner uniform domains

Theorem 4.3 (Poincaré inequality on inner uniform domains). *Let (G, P, m) be a graph satisfying the Harnack inequality. From Chapter 3, we know that (G, P, m) is volume doubling and satisfies the Poincaré inequality and ellipticity. Assume* *that $U \subseteq G$ is an inner uniform domain. Then (U, P, m_U) satisfies the Poincaré inequality.*

We know that the Poincaré inequality holds for balls that are sufficiently far inside U, relative to their radius. Therefore, the rest of this section is devoted to proving Theorem 4.3 for balls whose radius is too large, relative to the assumptions, in the following sense.

Assume the Poincaré inequality holds for balls $B(x,r)$ with $d(x, \partial U) \geq Nr$. Then, fix a ball $B_0 = B(x_0, r_0) \subseteq U$ such that

$$d(x_0, \partial U) \leq N r_0. \tag{4.2}$$

We will prove the Poincaré inequality for B_0 satisfying (4.2) in an inner uniform domain U in four steps. The first two steps describe the basic machinery: Whitney coverings and chaining arguments. The third step uses this machinery to establish the weak Poincaré inequality on a ball $B(x,r)$:

$$\sum_{x \in B} (f(x) - f_B)^2 m(x) \leq Cr^2 \sum_{x,y \in kB} (f(x) - f(y))^2 \mu_{xy} \tag{4.3}$$

where the right-hand side is taken over some dilated ball kB. The final step shows that the weak Poincaré inequality implies the standard Poincaré inequality.

Note that 4.2 is more general than we need. If G satisfies the Harnack inequality and $U \subseteq G$ is an inner uniform domain, then any ball $B(x,r)$ such that $d(x, \partial U) \geq 2r$ inherits the Poincaré inequality from G. Therefore, the proof of Theorem 4.3 only requires us to prove the Poincaré inequality on balls of the form 4.2 with $N = 2$.

4.4.1 Whitney covering

The primary tool used to prove the Poincaré inequality on inner uniform domains is Whitney coverings, a collection of balls whose radius is small relative to its distance to the boundary. We can essentially cover a fixed ball B_0 with balls in our Whitney covering and then apply the Poincaré inequality to those balls. This will prove the Poincaré inequality on the original ball B_0.

Definition 4.4 (Whitney covering)**.** *Fix $\epsilon > 0$. An ϵ_1, ϵ_2-*Whitney covering *of $U \subseteq G$ is a collection of disjoint balls $\mathcal{W} = \{B_i : B_i = B(x_i, r_i) \subseteq U\}$ such that*

1. *The enlarged balls $\{3B_i\}$ cover U, i.e. $\bigcup_{B_i \in \mathcal{W}} B(x_i, 3r_i) = U$*
2. *Relative to their radii, the balls are far from the boundary of U, i.e. for all $B_i = B(x_i, r_i) \in \mathcal{W}$,*

$$\epsilon_1 d(x_i, \partial U) \leq r_i \leq \epsilon_2 d(x_i, \partial U). \tag{4.4}$$

For a fixed ball $B_0 \subseteq U$, define $\mathcal{W}(B_0)$ to be the Whitney balls whose triples intersect B_0, i.e.,

$$\mathcal{W}(B_0) = \{B = B(x, r) : B \in \mathcal{W}, \text{ and } 3B \cap B_0 \neq \emptyset\}.$$

Proposition 4.5 (Bounded intersection property)**.** *Let $\mathcal{W}$ be an ϵ_1, ϵ_2-Whitney covering of a subset $U \subseteq G$ with $\epsilon_1 \leq \epsilon_2 < \frac{1}{4}$. Equip the graph G with a volume doubling measure m. Then there exists a constant A such that, for all $k \leq \epsilon_2^{-1}$,*

$$\sum_{B \in \mathcal{W}} \chi_{kB} \leq A, \tag{4.5}$$

where χ is the characteristic function. In other words, there is a uniform bound on the number of dialed balls kB that any point x is in.

Proof. Fix $x_0 \in U$ and $k \leq \epsilon_2^{-1}$. By the first property of Whitney coverings, there exists some $B(x, r) \in \mathcal{W}$ such that $x \in 3B$.

Assume that $x_0 \in k\tilde{B}$ for some $\tilde{B} = B(\tilde{x}, \tilde{r}) \in \mathcal{W}$. We aim to prove that only a uniformly bounded number of such Whitney balls can exist.

We begin with two observations, both of which follow by the triangle inequality:

$$d(x_0, \partial U) \leq d_U(x_0, \tilde{x}) + d(\tilde{x}, \partial U) \leq k\tilde{r} + \frac{\tilde{r}}{\epsilon_1} \tag{4.6}$$

and

$$d(x_0, \partial U) \leq d_U(x_0, x) + d(x, \partial U) \leq 3r + \frac{r}{\epsilon_1}. \tag{4.7}$$

Using the Whitney covering condition (4.4) and the estimate (4.6),

$$r \leq \epsilon_2 d(x, \partial U) \leq \epsilon_2(d(x, x_0) + d(x_0, \partial U)) \leq \epsilon_2(3r + k\tilde{r} + \frac{\tilde{r}}{\epsilon_1}).$$

This simplifies to

$$r \leq \frac{\epsilon_2}{1 - 3\epsilon_2}(k + \frac{1}{\epsilon_1})\tilde{r} \leq 3\epsilon_2(k + \frac{1}{\epsilon_1})\tilde{r} \leq (3 + 3\frac{\epsilon_2}{\epsilon_1})\tilde{r} \leq c_1\tilde{r}, \tag{4.8}$$

where c_1 depends only on ϵ_1, ϵ_2 and not B_0, B or $\tilde{B}$. A similar analysis, using the Whitney covering condition (4.4) and the estimate (4.7) shows that

$$\tilde{r} \leq \frac{\epsilon_2}{1 - k\epsilon_2}(3 + \frac{1}{\epsilon_1})r \leq k\epsilon_2(3 + \frac{1}{\epsilon_1})r \leq (3 + k\frac{\epsilon_1}{\epsilon_2})r \leq c_2 k r, \tag{4.9}$$

where again c_2 depends only on ϵ_1, ϵ_2.

By the triangle inequality and estimate (4.9)

$$d(\tilde{x}, x) \leq d(\tilde{x}, x_0) + d(x_0, x) \leq k\tilde{r} + 3r \leq (c_2 k^2 + 3)r$$

and so $\tilde{x} \in (c_2 k^2 + 3)B$. Then, $\tilde{B} \subseteq (c_2 k^2 + 3 + \tilde{r})B$, which, using (4.9), simplifies to $\tilde{B} \subseteq (2c_2 k^2 + 3)B$. Since $k^2 \leq \epsilon_2^{-2}$, this implies $\tilde{B} \subseteq (2c_1\epsilon_2^{-2} + 3)B$.

Because the measure m is volume doubling, there can only be a uniformly bounded number of disjoint Whitney balls of radius $\tilde{r} \geq c_2^{-1} r$ contained in $(2c_1\epsilon_2^{-2} + 3)B$. This uniform bound gives us the constant A. □

Proposition 4.6 (Comparability of neighboring balls)**.** *Let $\mathcal{W}$ be an ϵ_1, ϵ_2-Whitney covering of a subset $U \subseteq G$ with $\epsilon_1 \leq \epsilon_2 < \frac{1}{4}$. Equip the graph G with a volume doubling measure m. Then there exist constants d and D such that, for any $B = B(x,r) \in \mathcal{W}$ and $\tilde{B} = B(\tilde{x}, \tilde{r}) \in \mathcal{W}$ with $3B \cap 3\tilde{B} \neq \emptyset$,*

$$d^{-1} r \leq \tilde{r} \leq dr$$

and

$$|f_{4B} - f_{4\tilde{B}}| \leq Dr \left(\frac{1}{V(B)} \sum_{x,y \in 16B} (f(x) - f(y))^2 \mu_{xy} \right)^{1/2}$$

for any function $f : U \to \mathbb{R}$.

Proof. The inequality $d^{-1} r \leq \tilde{r} \leq dr$ follows from (4.8) and (4.9) where $d = \max\{c_1, 3c_2\}$. Recall that c_1 and c_2 depend only on ϵ_1 and ϵ_2. Here, we use that the Poincaré inequality applies to the balls in the Whitney covering.

$$\begin{aligned}
|f_{4B} - f_{4\tilde{B}}|^2 &= m(4B \cap 4\tilde{B})^{-1} \sum_{x \in 4B \cap 4\tilde{B}} |f_{4B} - f_{4\tilde{B}}|^2 m(x) \\
&\le m(4B \cap 4\tilde{B})^{-1} \\
&\quad \cdot 2 \left(\sum_{x \in 4B \cap 4\tilde{B}} |f - f_{4B}|^2 m(x) + \sum_{x \in 4B \cap 4\tilde{B}} |f - f_{4\tilde{B}}|^2 m(x) \right) \\
&\le m(4B \cap 4\tilde{B})^{-1} \\
&\quad \cdot 2 \left(\sum_{x \in 4B} |f - f_{4B}|^2 m(x) + \sum_{x \in 4\tilde{B}} |f - f_{4\tilde{B}}|^2 m(x) \right) \\
&\le m(4B \cap 4\tilde{B})^{-1} \\
&\quad \cdot 2 \left(Cr^2 \sum_{x,y \in 4B} (f(x) - f(y))^2 \mu_{xy} + C\tilde{r}^2 \sum_{x,y \in 4\tilde{B}} (f(x) - f(y))^2 \mu_{xy} \right) \\
&\le m(4B \cap 4\tilde{B})^{-1} \\
&\quad \cdot 2 \left(Cr^2 \sum_{x,y \in 16B} (f(x) - f(y))^2 \mu_{xy} + C\tilde{r}^2 \sum_{x,y \in 16B} (f(x) - f(y))^2 \mu_{xy} \right) \\
&\le m(4B \cap 4\tilde{B})^{-1}\ 2Cd^2 r^2 \sum_{x,y \in 16B} (f(x) - f(y))^2 \mu_{xy}
\end{aligned}$$

where the last two inequalities use the fact that $4\tilde{B} \subseteq 16B$ and $\tilde{r} \le dr$. There exists some constant c depending only on the m such that $m(4B \cap 4\tilde{B}) \ge cV(B)$ by the assumptions that m is volume doubling. Setting $D^2 = c^{-1}Cd^2$ gives the desired result. □

4.4.2 Chaining estimate

Now, we continue the proof of the Poincaré inequality for a fixed ball $B_0 = B(x_0, r_0)$ satisfying (4.2), i.e., $d(x_0, \partial U) \le Nr_0$. The first step in the chaining estimate is to construct a *central ball* $B_{\mathcal{C}} \in \mathcal{W}(B_0)$.

Lemma 4.7. *Let U be an inner uniform domain. For a fixed ball $B_0 = B(x_0, r_0) \subseteq U$ with $d(x_0, \partial U) \leq N r_0$ and $r_0 \geq 4$, there exists a point $y \in B_0$ such that $d(y, \partial U) \geq \frac{\tilde{\kappa} r_0}{8}$ and $d_U(y, x_0) \leq \frac{r_0}{4}$, where $\tilde{\kappa}$ is the inner uniformity constant from* (4.1)*.*

Proof. Select $z \in B(x_0, r_0) \setminus B_(x_0, \frac{r_0}{2})$. Let γ be the path from x_0 to z that is guaranteed by the inner uniformity assumption on U. Select y to be a point on the path such that $d_U(y, x_0) \leq \frac{r_0}{4}$. Then, using the inner uniformity assumption,

$$\begin{aligned} d(y, \partial U) &\geq \tilde{\kappa} \frac{d_U(x_0, y) d_U(y, z)}{d_U(x_0, z)} \\ &\geq \tilde{\kappa} \frac{\frac{r_0}{4}(d_U(x_0, z) - d_U(x_0, y))}{d_U(x_0, z)} \\ &\geq \tilde{\kappa} \frac{\frac{r_0}{4}(\frac{r_0}{2} - \frac{r_0}{4})}{\frac{r_0}{2}} \\ &= \frac{\tilde{\kappa} r_0}{8} \end{aligned}$$

□

Let $B_{\mathcal{C}} = B(x_{\mathcal{C}}, r_{\mathcal{C}})$ be the ball in the Whitney covering $\mathcal{W}$ of U such that $y \in 3B_{\mathcal{C}}$. Note that, by construction

$$d(B_{\mathcal{C}}, \partial U) \geq \frac{\tilde{\kappa} r_0}{16}. \tag{4.10}$$

We want to compare the value of a function on $B_{\mathcal{C}}$ with the value of a function on any other $B \in \mathcal{W}(B_0)$. To do so, we'll create chains of Whitney balls. Specifically, for a fixed $B \in \mathcal{W}(B_0)$, the inner uniformity consider implies that there exists a path γ of length at most $\kappa d_U(B_{\mathcal{C}}, B)$ connecting a point in $\mathcal{C}$ to a point in B. Define a chain of Whitney balls $B_{\mathcal{C}} = W_0^B, W_1^B, ..., W_l^B = B$ (with $W_i^B = B(\omega_i, \rho_i) \in \mathcal{W}$)

whose triples connect $\mathcal{C}$ to B along the path γ, with the properties that

$$3W_i^B \cap 3W_{i+1}^B \neq \emptyset \tag{4.11}$$

and

$$3W_i^B \cap \gamma \neq \emptyset \tag{4.12}$$

for all $0 \leq i \leq l$.

Lemma 4.8. *It is always possible to construct the chain of Whitney balls described above such that, for all balls W_i^B in chain*

$$W_i^B \subseteq 4\kappa B_0 \tag{4.13}$$

and

$$B \subseteq K W_i^B \tag{4.14}$$

where K is a constant which is independent of B_0 and B, i.e., K is uniform for all chains.

Proof. Note that $B \subseteq 2B_0$ for any $B \in \mathcal{W}(B_0)$ (including, $B_{\mathcal{C}}$. This is because B and B_0 are roughly the same distance from the boundary—they are at most $2r$ apart. But r, the radius of B, is assumed to be relatively small by the Whitney covering condition (4.4), while r_0, the radius of B_0, is assumed to be relatively large by (4.2). We can assume ϵ_2 is small enough so that $B \subseteq 2B_0$.

Using the path estimate in the definition of an inner uniform domain and the previous fact,

$$d_U(W_i^B, B_{\mathcal{C}}) \leq \kappa d_U(B_{\mathcal{C}}, B) \leq 2\kappa r_0, \tag{4.15}$$

which implies (4.13).

The proof of (4.14) is split into two cases. First, assume $d_U(W_i^B, B_{\mathcal{C}}) \geq d_U(B_{\mathcal{C}}, \partial U)$. Using (4.4), the estimate of radii of balls in a Whitney covering, and (4.1) in the definition of inner uniform domain,

$$r_{W_i^B} \geq \epsilon_1 d_U(W_i^B, \partial U) \geq \epsilon_1 \tilde{\kappa} \frac{d_U(W_i^B, B_{\mathcal{C}}) d_U(W_i^B, B)}{d_U(B_{\mathcal{C}}, B)} \tag{4.16}$$

where $r_{W_i^B}$ is the radius of W_i^B. Now, we estimate the terms on the left side of (4.16). As noted in the proof of (4.13) above, $d_U(B_{\mathcal{C}}, B) \leq 2r_0$, and from (4.10) we have $d_U(B_{\mathcal{C}}, \partial U) \geq \frac{\tilde{\kappa}}{16}$. Applying these estimates to (4.16) and the assumption that $d_U(W_i^B, B_{\mathcal{C}}) \geq d_U(B_{\mathcal{C}}, \partial U)$, we get

$$r_{W_i^B} \geq \frac{\epsilon_1 \tilde{\kappa}^2}{32} d_U(W_i^B, B). \tag{4.17}$$

For the other case, assume that $d_U(W_i^B, B_{\mathcal{C}}) < d_U(B_{\mathcal{C}}, \partial U)$. Using (4.4), the estimate of radii of balls in a Whitney covering, and the triangle inequality,

$$r_{W_i^B} \geq \epsilon_1 d_U(W_i^B, \partial U) \geq \epsilon_1 (d_U(B_{\mathcal{C}}, \partial U) - d_U(W_i^B, B_{\mathcal{C}})). \tag{4.18}$$

Applying the assumption that $d_U(W_i^B, B_{\mathcal{C}}) < d_U(B_{\mathcal{C}}, \partial U)$ and the bound $d_U(B_{\mathcal{C}}, \partial U) \geq \frac{\tilde{\kappa}}{16}$ from (4.10) to (4.18),

$$r_{W_i^B} \geq \epsilon_1 d_U(B_{\mathcal{C}}, \partial U) \geq \frac{\epsilon_1 \tilde{\kappa} r_0}{16} \geq \frac{\epsilon_1 \tilde{\kappa}}{32\kappa} d_U(W_i^B, B) \tag{4.19}$$

where the last inequality follows by the path length bound in an inner uniform domain,

$$d_U(W_i^B, B) \leq |\gamma_{B_{\mathcal{C}}, B}| \leq \kappa d_U(B_{\mathcal{C}}, B) \leq 2\kappa r_0.$$

Letting $K = \max\{\frac{\epsilon_1 \tilde{\kappa}^2}{32}, \frac{\epsilon_1 \tilde{\kappa}}{32\kappa}\}$, the inequalities (4.17) and (4.19) prove (4.14). □

We collect one final lemma using Whitney chains which will be helpful in the next section.

Lemma 4.9. *For a fixed ball B_0, let $B_{\mathcal{C}}$ be the central ball described above. Let $B \in \mathcal{W}(B_0)$ and $\{W_i^B\}_{i=1}^l$ be the chain of Whitney balls connecting B to $B_{\mathcal{C}}$. Then*

$$|f_{4B} - f_{4B_{\mathcal{C}}}|\chi_B \leq C \sum_{A \in \mathcal{W}(B_0)} r_A \left(\frac{1}{V(A)} \sum_{x,y \in 16A} (f(x) - f(y))^2 \mu_{xy} \right)^{1/2} \chi_B \chi_{KA} \quad (4.20)$$

where C, K depend only on the constants ϵ_1, ϵ_2 from the Whitney covering and $\kappa, \tilde{\kappa}$ from the definition of inner uniform domain.

Proof. Note that, by Lemma 4.8, $\chi_B = \chi_B \chi_{KW_i^B}$ for any W_i^B in the Whitney chain.

$$\begin{aligned} |f_{4B} - f_{4B_{\mathcal{C}}}|\chi_B &\leq \sum_{i=1}^{l} |f_{4W_i^B} - f_{4W_{i-1}^B}|\chi_B \chi_{KW_i^B} \\ &\leq \sum_{i=1}^{l} C r_{W_i^B} \left(\frac{1}{V(W_i^B)} \sum_{x,y \in 16W_i^B} (f(x) - f(y))^2 \mu_{xy} \right)^{1/2} \chi_B \chi_{KW_i^B} \quad (4.21) \\ &\leq C \sum_{A \in \mathcal{W}(B_0)} r_A \left(\frac{1}{V(A)} \sum_{x,y \in 16A} (f(x) - f(y))^2 \mu_{xy} \right)^{1/2} \chi_B \chi_{KA}, \end{aligned}$$

where (4.21) follows by applying Proposition 4.6 to the neighboring balls in the Whitney chain $\{W_i^B\}$. □

4.4.3 Weak Poincaré inequality

In this section, we will prove (4.3), the weak Poincaré inequality, for the fixed ball B_0,

$$\sum_{x \in B_0} (f(x) - f_{B_0})^2 m(x) \leq C r_0^2 \sum_{x,y \in kB_0} (f(x) - f(y))^2 \mu_{xy}.$$

Note that the left side of (4.3) can be written as $\min_{\xi \in \mathbb{R}} \sum_{x \in B_0} (f(x) - \xi)^2 m(x)$. The first step to prove (4.3) is to bound this quantity using the Whitney covering of B_0 and the central ball $B_{\mathcal{C}}$ established in the previous section.

$$\begin{aligned}\min_{\xi\in\mathbb{R}} \sum_{x\in B_0}(f(x)-\xi)^2 m(x) &\le \sum_{x\in B_0}(f(x)-f_{4B_{\mathcal{C}}})^2 m(x)\\ &\le \sum_{B\in\mathcal{W}(B_0)}\sum_{x\in 3B}(f(x)-f_{4B_{\mathcal{C}}})^2 m(x)\\ &\le 2\sum_{B\in\mathcal{W}(B_0)}\left(\sum_{x\in 4B}(f_{4B}-f_{4B_{\mathcal{C}}})^2 m(x)+\sum_{x\in 4B}(f(x)-f_B)^2 m(x)\right)\end{aligned} \tag{4.22}$$

Using the fact that the balls in the Whitney covering are sufficiently far inside U to apply the standard Poincaré inequality, and that their radius differs by a constant from r_0, we can bound the second term on the right side of (4.22),

$$\begin{aligned}\sum_{B\in\mathcal{W}(B_0)}\sum_{x\in 4B}(f(x)-f_{4B})^2 m(x) &\le Cr_0^2\sum_{B\in\mathcal{W}(B_0)}\left(\sum_{x,y\in 4B}(f(x)-f(y))^2\mu_{xy}\right)\\ &\le Cr_0^2\sum_{x,y\in 2B_0}\left(\sum_{B\in\mathcal{W}(B_0)}\chi_{4B}\right)(f(x)-f(y))^2\mu_{xy}\\ &\le CAr_0^2\sum_{x,y\in 2B_0}(f(x)-f(y))^2\mu_{xy}\end{aligned} \tag{4.23}$$

where the last inequality follows by Proposition 4.5.

Now, to prove the weak Poincaré inequality we only need to estimate the first term on the right side of (4.22). To do so, we first apply volume doubling and then

Lemma 4.9 as follows,

$$\begin{aligned}
&\sum_{B\in\mathcal{W}(B_0)}\sum_{x\in 4B}(f_{4B}-f_{4B_{\mathcal{C}}})^2 m(x) \qquad (4.24)\\
&\le \sum_{B\in\mathcal{W}(B_0)}(f_{4B}-f_{4B_{\mathcal{C}}})^2\delta^2\sum_{x\in B} m(x)\\
&\le \delta^2\sum_{x\in U}\sum_{B\in\mathcal{W}(B_0)}(f_{4B}-f_{4B_{\mathcal{C}}})^2\chi_B(x)m(x)\\
&\le \delta^2 C\sum_{x\in U}\sum_{B\in\mathcal{W}(B_0)}\left(\sum_{A\in\mathcal{W}(B_0)} r_A\left(\frac{1}{V(A)}\sum_{y,z\in 16A}(f(y)-f(z))^2\mu_{yz}\right)^{1/2}\chi_B(x)\chi_{KA}(x)\right)^2 m(x)\\
&= \delta^2 C\sum_{x\in U}\left(\sum_{B\in\mathcal{W}(B_0)}\chi_B\right)\left(\sum_{A\in\mathcal{W}(B_0)} r_A\left(\frac{1}{V(A)}\sum_{y,z\in 16A}(f(y)-f(z))^2\mu_{yz}\right)^{1/2}\chi_{KA}(x)\right)^2 m(x)\\
&\le \delta^2 C\sum_{x\in U}\left(\sum_{A\in\mathcal{W}(B_0)} r_A\left(\frac{1}{V(A)}\sum_{y,z\in 16A}(f(y)-f(z))^2\mu_{yz}\right)^{1/2}\chi_{KA}(x)\right)^2 m(x)
\end{aligned} \tag{4.25}$$

where the last line follows because the balls in a Whitney covering are disjoint, and hence $\sum_{B\in\mathcal{W}(B_0)}\chi_B \le 1$.

Lemma 4.10. *Let (G,P,m) satisfy the Harnack inequality and let $U\subseteq G$ satisfy volume doubling. Fix $K\ge 1$. Then, for any sequence of balls $\{B_i\}$ with $B_i\subseteq U$ and real numbers $\{b_i\}$ with $b_i>0$,*

$$\sum_{x\in U}\left(\sum_{i=1} b_i\chi_{KB_i}\right)^2 m(x)\le C\sum_{x\in U}\left(\sum_{i=1} b_i\chi_{B_i}\right)^2 m(x)$$

for some constant C.

For a proof of Lemma 4.10, see [25]. Continuing from (4.25) and immediately applying Lemma 4.10,

$$\begin{aligned}
&\sum_{B\in\mathcal{W}(B_0)}\sum_{x\in 4B}(f_{4B}-f_{4B_C})^2 m(x)\\
&\le \delta^2 C\sum_{x\in U}\left(\sum_{A\in\mathcal{W}(B_0)} r_A\left(\frac{1}{V(A)}\sum_{y,z\in 16A}(f(y)-f(z))^2\mu_{yz}\right)^{1/2}\chi_A(x)\right)^2 m(x)\\
&\le \delta^2 C\sum_{x\in U}\sum_{A\in\mathcal{W}(B_0)}\frac{r_A^2}{V(A)}\left(\sum_{y,z\in 16A}(f(y)-f(z))^2\mu_{yz}\right)\chi_A(x)m(x)\\
&\le \delta^2 C\sum_{A\in\mathcal{W}(B_0)}\frac{r_A^2}{V(A)}\left(\sum_{y,z\in 16A}(f(y)-f(z))^2\mu_{yz}\right)V(A) \qquad (4.26)\\
&\le \delta^2 C r_0^2\sum_{A\in\mathcal{W}(B_0)}\chi_{16A}\left(\sum_{y,z\in 16A}(f(y)-f(z))^2\mu_{yz}\right) \qquad (4.27)\\
&\le \delta^2 C r_0^2\sum_{A\in\mathcal{W}(B_0)}\chi_{16A}\left(\sum_{y,z\in kB_0}(f(y)-f(z))^2\mu_{yz}\right) \qquad (4.28)\\
&\le \delta^2 C r_0^2\sum_{y,z\in kB_0}(f((y)-f(z))^2\mu_{xy} \qquad (4.29)
\end{aligned}$$

where the constant shifts each line. Inequality (4.26) follows because the A are disjoint, and inequality (4.29) follows by Proposition 4.5. In inequality (4.28), the constant k comes from the fact that $B\subseteq 4\kappa B_0$ by (4.13).

4.4.4 Weak Poincaré implies Poincaré

The last step is to show that the weak Poincaré inequality

$$\sum_{x\in B}(f(x)-f_B)^2 m(x)\le Cr^2\sum_{x,y\in kB}(f(x)-f(y))^2\mu_{xy}$$

implies the standard Poincaré inequality

$$\sum_{x\in B_0}(f(x)-f_B)^2 m(x)\le Cr^2\sum_{x,y\in B}(f(x)-f(y))^2\mu_{xy}.$$

The proof of this fact is described in the continuous case in [25]. The argument is very similar to Whitney covering and chaining argument presented in the first three steps of this proof. Instead of repeating the details, we will just sketch the argument, highlighting the analogies with the previous sections.

We know the Poincaré inequality holds for balls whose distance to the boundary of the inner uniform domain U is large relative to their radius, so our proof of the weak Poincaré inequality focused on the potentially problematic balls—those that are relatively near the boundary, (i.e., those balls $B(x,r)$ such that $d(x,\partial U) \leq Nr$). We then defined a Whitney covering: each point $y \in B$ is contained in $3W_i$, where W_i is a ball that is far enough inside U for the Poincaré inequality to hold. We can then chain each W_i to the center of B using a series of Whitney balls. The inner uniformity condition bounds this chain is such a way that B inherits the Poincaré inequality from W_i. However, the Whitney balls might only be contained in a scaled ball kB, so we can only prove the weak Poincaré inequality.

Now, assume we know that the weak Poincaré inequality holds within a ball $B = (x,r)$. Cover B by balls $\tilde{B}$ such that $k\tilde{B} \subseteq B$. Using the definition of a ball, each $\tilde{B}$ is connected to the center of B by a path of length at most r. Along this path, form a chain of balls with the property that a k-multiple of the ball is still in B. Then we can apply a nearly identical argument to this covering to prove the standard Poincaré inequality.

CHAPTER 5

THE HARMONIC PROFILE

The previous chapter considered the heat kernel with Neumann boundary conditions in an inner uniform domain. Now, we would like to study a weighted random walk in a domain $U \subseteq G$ with Dirichlet boundary, i.e., the process governed by the operator

$$P_{D,U}(f)(x) = \mathbb{1}_U \cdot P(f \cdot \mathbb{1}_U) = \sum_{y \sim x} \mathbb{1}_U(x)\mathbb{1}_U(y)p(x,y)f(y)$$

with associated transition function,

$$p_{D,U}(x,y) = \mathbb{1}_U(x)\mathbb{1}_U(y)p(x,y),$$

and reversible kernel,

$$q_{D,U}(x,y) = \frac{p_{D,U}(x,y)}{m(y)}.$$

The transition function $p_{D,U}$ restricts the original p inside the set U, but is sub-Markovian due to the killing at the boundary. We can slightly modify the kernel—by, for example, making each boundary state absorbing—so that it becomes Markovian.

To start, we will focus on another method for modifying the kernel, known as Doob's h-transform. The h-transform has a nice relationship to $q_{D,U}(x,y)$, the heat kernel with Dirichlet boundary, and by studying the h-transform we will end up with estimates for $q_{D,U}(x,y)$.

5.1 Doob's h-transform

Fix $U \subseteq G$ and recall that the boundary is defined as

$$\partial U = \{x \in G : x \notin U, d(x,U) = 1\}.$$

Define the *harmonic profile*, or *h-function*, of U to be a function $h : U \to \mathbb{R}_{\geq 0}$, where h is zero on ∂U, harmonic in the interior of U, and positive inside each infinite component of U. For additional references, see Doob's 1984 book [8], L.C.G. Rogers and D. Williams's book [23], and H. Kesten's article [17].

Proposition 5.1 (Existence of discrete h-function)**.** *Let G be a graph and let $U \subseteq G$ be an infinite subgraph. Then there exists an h-function, or harmonic profile, for U.*

Proof. Notice that, if U is finite, then the maximum principle implies that any harmonic function with zero boundary condition is zero everywhere in U, and hence, an h-function does not exist. We will assume U is infinite and has one connected component. If U has several infinite components, use the following procedure to create an h-function on each component; the h-function for entire space will then be a piecewise combination of these h-functions.

Create a sequence $\{U_i\}_{i=1}^{\infty}$ of finite subgraphs of U such that (1) $U_i \subseteq U_j$ for $i \leq j$ and (2) $\lim_{i\to\infty} U_i \nearrow U$. Within each U_i, define $u_i : U_i \cup \partial U_i \to \mathbb{R}$ by: (1) $u_i = 0$ on ∂U; (2) $u_i = 1$ on $\partial U_i \setminus \partial U$; and (3) $\Delta u_i = 0$ in U_i. Fix $x_0 \in U_1$ (and hence, $x_0 \in U_i$ for all i). Let $v_i(x) = \frac{u_i(x)}{u_i(x_0)}$, i.e., v_i is a scaled version of u_i so that $v_i(x_0) = 1$ for all n. Note that v_i is also a harmonic function in U_i satisfying the conditions: (1) $v_i = 0$ on ∂U; (2) $v_i = \frac{1}{u_i(x_0)}$ on $\partial U_i \setminus \partial U$; and (3) $\Delta v_i = 0$ in U_i.

Enumerate the remaining points of U by $\{x_k\}_{k=1}^{\infty}$ and assume $x_1 \sim x_0$. Then,

$$1 = v_i(x_0) = \sum_{y\sim x_0} p(x_0, y)v_i(y) \geq p(x_0, x_1)v_i(x_1)$$

for all n, and hence, $\frac{1}{p(x_0,x_1)} \geq v_i(x_1)$. Since $\{v_i(x_1)\}_{i=1}^{\infty}$ is a bounded sequence, we can form a convergent subsequence $\{v_{n_1(i)}(x_1)\}_{i=1}^{\infty}$ by the Bolzano-Weierstrass theorem. Note: $n_1(i)$ is a subsequence of i. Select another point $x_2 \sim x_1$. (A similar idea works if $x_2 \sim x_0$.) Reasoning as above, we can see that $v_n(x_1) \geq p(x_1, x_2)v_n(x_2)$ for all n, and hence, $\frac{1}{p(x_0,x_1)p(x_1,x_2)} \geq v_n(x_2)$. Again, $\{v_{n_{k_1(i)}}(x_2)\}$ is a bounded sequence and so we can form a convergent subsequence $\{v_{n_{k_2(i)}}(x_2)\}$, where $k_2(i)$ is a subsequence of $k_1(i)$. Continue this process, connecting each point x_k back to x_0, using that path to bound $v_n(x_i)$, and then forming a convergent subsequence.

Note that the sequence of function $\{v_{n_{K(i)}}\}$ converges pointwise at each point x_k with $k \leq K$. Unfortunately, if we take the limit $J \to \infty$, the sequence may be empty. Instead, define a new sequence index: $\tilde{n}(i) = n_i(i)$. Then $\{v_{\tilde{n}(i)}\}$ converges pointwise for all x_k and the limit function v is a harmonic profile of U. $\square$

Now, we wish to modify our stochastic process $(U, P_{D,U}, m)$ using the harmonic profile. Define the operator $P_{h,U}$,

$$P_{h,U}(f)(x) = H \circ P_{D,U} \circ H^{-1}(f)(x)$$

where $H(f)(x) = h(x) \cdot f(x)$, which simplifies to

$$P_{h,U}(f)(x) = \sum_{y:x\sim y} \frac{1}{h(x)} pD, U(x,y)h(y)f(y), \tag{5.1}$$

with the associated probability transition function

$$p_{h,U}(x,y) = \frac{h(y)}{h(x)} p_{D,U}(x,y) \; x, y \in U$$

with respect to the measure m. Note that $p_{h,U}$ is Markovian (unlike p_U), because

$$p_{h,U}(\mathbb{1}_U)(x) = \frac{1}{h(x)} \sum_{y:y\sim x} \mathbb{1}_U(x)\mathbb{1}_U(y)p_{D,U}(x,y)h(y) = \frac{1}{h(x)} \sum_{y\in U:y\sim x} p_{D,U}(x,y)h(y) = 1.$$

The first equality holds because $h(y) = 0$ for $y \in \partial U$, and the second equality holds because h is harmonic in U. In this way, we have reduced the problem of studying $P_{D,U}$ and $p_{D,U}$, the process which is killed at the boundary of U, to studying the h function in U.

Proposition 5.2. *The transition function $p_{h,U}$ is reversible w.r.t. to measure $m_{h^2}(x) = h^2(x)m(x)$.*

Proof. Given $x, y \in U$,

$$\begin{aligned} p_{m,h,U}(x,y)m_h(x) &= \left(\frac{h(y)}{h(x)}p(x,y)\right) h^2(x)m(x) \\ &= p(x,y)h(x)h(y)m(x) \\ &= p(y,x)h(x)h(y)m(y) \\ &= \left(\frac{h(x)}{h(y)}p(y,x)\right) h^2(y)m(y) \\ &= p_{m,h,U}(y,x)m_h(y) \end{aligned}$$

□

In light of Proposition 5.2, the symmetric kernel for $p_{h,U}$ with respect to the reversible measure is given by,

$$q_{h,U}(x,y) = \frac{p_{h,U}(x,y)}{m(y)h(y)^2} = \frac{p_{D,U}(x,y)}{m(y)h(x)h(y)}.$$

The main result of this chapter, Theorem 1.11, states that, if (G, P, m) is a graph satisfying the Harnack inequality and $U \subseteq G$ is an inner uniform domain, then $(U, P_{h,U}, m_{h^2})$ also satisfies the Harnack inequality.

The proof relies on Theorem 1.9, which states that the Harnack inequality is equivalent to the conjunction of volume doubling and the Poincaré inequality. In the following two sections, we prove Theorem 5.4, which states that $(U, P_{h,U}, m_{h^2})$ is volume doubling, and Theorem 5.7, which states that $(U, P_{h,U}, m_{h^2})$ satisfies the Poincaré inequality.

Before we prove Theorem 1.11, we will comment on the uniqueness of the harmonic profile. It is easy to verify that any constant multiple of an h-function for a domain, i.e., $\tilde{h}(x) = ch(x)$, is also an h-function. Furthermore, it is possible for a domain to have several h-functions which are not scale multiples of each other: given two disconnected domains U and W with h-functions h_U and h_V, we can form an h-function for the domain $W = U \cup V$ using any scalar multiples of h_V and h_U, i.e., $h_W = c_1 h_U \chi_U + c_2 h_V \chi_V$.

However, any inner uniform domain (which is necessarily connected) has a unique h-function, up to scaling. This is a consequence of Theorem 1.11, which we will state and prove here to justify our use of the phrase "the h-function" for U in the following sections. We note that a paper by A. Bouaziz, S. Mustapha, and M. Sifi [4] proves the existence and uniqueness of harmonic functions on orthants of $\mathbb{Z}^d$. Because an orthant of $\mathbb{Z}^d$ is an inner uniform domain, the following corollary generalizes their result. (Also see K. Raschel [21].)

Corollary 5.3. *Let (G, P, m) be a graph satisfying the Harnack inequality and assume $U \subseteq G$ is an inner uniform domain. Then the h-function for U is unique, up to scaling.*

Proof. Let h and $\tilde{h}$ be two h-functions for U. Then $\frac{\tilde{h}}{h}$ (defined in U) is $P_{h,U}$ harmonic:

$$P_{h,U}\left(\frac{\tilde{h}}{h}\right) = H^{-1} \circ P_{D,U} \circ H\left(\frac{\tilde{h}}{h}\right) = H^{-1} \circ P_{D,U}(\tilde{h}) = \frac{\tilde{h}}{h}.$$

By Theorem 1.11 $(U, P_{h,U}, m_{h^2})$ satisfies the Harnack inequality, and so we can apply Corollary 3.3, Liouville's theorem, which states that any harmonic function that is bounded above or below is constant. Therefore $\frac{\tilde{h}}{h}$ is constant. □

5.2 The h-transform is volume doubling

As a first step toward proving that $(U, P_{h,U}, m_{h^2})$ satisfies the Harnack inequality, we will prove that it is volume doubling. We will use $V_m(x, r)$ and $V_{m_{h^2}}(x, r)$ to indicate the volume of a ball of radius r with center x with respect to the measure m and m_{h^2}, respectively.

Theorem 5.4. *Let (G, P, m) be a graph satisfying the Harnack inequality. From Chapter 3, we know that (G, P, m) is volume doubling and satisfies the Poincaré inequality. Assume that $U \subseteq G$ is an inner uniform domain. Then $(U, P_{h,U}, m_{h^2})$ is volume doubling.*

In fact, we can say even more about the measure m_{h^2}: the volume of a ball under m_{h^2} is roughly proportional to h^2 multiplied by the volume of a ball under m. In other words, $V_{m_{h^2}}(x, r) \simeq h(x_r)^2 V_m(x, r)$.

Proposition 5.5. *Let (G, P, m) be a graph satisfying the Harnack inequality, and assume that $U \subseteq G$ is an inner uniform domain. Then*

$$ch(x_r)^2 V_m(x, r) \leq V_{m_{h^2}}(x, r) \leq Ch(x_r)^2 V_m(x, r)$$

where $x_r \in U$ is such that $d_U(x, x_r) \leq r/4$ and $d(x_r, \partial U) \geq \frac{\tilde{\kappa} r}{8}$, and $c, C > 0$ only depend on δ, ρ (the volume doubling and Poincaré constants) and $\kappa, \tilde{\kappa}$ (the inner uniformity constants). Note that Lemma 4.7 guarantees the existence of such an x_r.

Before proving Proposition 5.5, we will show how it is used to prove Theorem 5.4. Note that we can apply the elliptic Harnack inequality on G, Theorem 3.1, to any ball which is sufficiently far inside U. (Specifically, it applies to balls B for which $B_G(x, r) = B_U(x, r)$, where B_G and B_U denote the ball in G and U, respectively.)

Proof of Theorem 5.4. Fix $x \in U$ and $r > 0$. Then,

$$
\begin{aligned}
V_{m_{h^2}}(x, 2r) &\le Ch(x_{2r})^2 V_m(x, 2r) \\
&\le Ch(x_r)^2 V_m(x, 2r) \qquad (5.2)\\
&\le C\delta h(x_r)^2 V_m(x, r) \qquad (5.3)\\
&\le cC\delta V_{m_{h^2}}(x, r)
\end{aligned}
$$

where the first and last lines follows by Proposition 5.5. The third inequality (5.3) follows because we assumed m is volume doubling in U. To prove he second inequality (5.2), form a path $\gamma_{x_r,x_{2r}}$ from x_r to x_{2r} which is given by the inner uniformity condition. Then $|\gamma_{x_r,x_{2r}}| \le \kappa r$. Create a chain of overlapping balls $\{B_i\}_{i=1}^{\mathcal{I}} \subseteq U$ connecting x_r to x_{2r} along $\gamma_{x_r,x_{2r}}$ such that $2B_i \subseteq U$. The number of balls in such a chain is uniformly bounded because each B_i has radius at least cr, where c depends only on $\tilde{\kappa}$. This follows because (1) $d(x_r, \partial U) \ge \frac{\tilde{\kappa} r}{8}$ and $d(x_{2r}, \partial U) \ge \frac{\tilde{\kappa} r}{4}$ and (2) the inner uniformity condition on the path $\gamma_{x_r,x_{2r}}$. Within each B_i we can apply the elliptic Harnack inequality, Theorem 3.1. Then iterate to get that,

$$\sup_{B_1} h \le C\mathcal{I} \inf_{B_{\mathcal{I}}} h,$$

which implies that $h(x_r) \simeq h(x_{2r})$. □

Now, to complete the proof of Theorem 5.4, we need to prove Proposition 5.5, which requires the cable process introduced in the next section.

5.2.1 The h-transform of the cable process

The cable process introduced in Section 2.2 essentially allows us to transfer results from the better-studied continuous domains to our discrete domains. Starting from a continuous domain with a Dirichlet form (e.g., $\mathbb{R}^n$ with $\mathcal{E}(f,f) = \int |\nabla f|^2 \, d\lambda$) and an inner uniform subset, one can take its h-transform using the same process described in Section 5.1. It is proved in [14] that the resulting process and measure satisfy the Harnack inequality. Because the cable process has a continuous domain, we can directly apply the results of [14] to it. But the cable process is a very good approximation to the discrete process, so we can extend these results to the discrete process.

Starting from the graph with discrete process $(U, P_{D,U}, m)$ we form the associated cable process $(\tilde{U}, \tilde{P}_{D,U}, \tilde{m})$. The continuous domain $\tilde{U}$ consists of the both the vertices and edges of U (including the edges connecting $x \in U$ to $y \in \partial U$), where the edges are considered to be continuous unit length intervals inside the domain. (Note: we include at most one edge interval edge between each pair of vertices, and we only include edges with positive weight μ_{xy}.) The process $\tilde{P}_{D,U}$ is the one associated with the Dirichlet form given in (2.1),

$$\tilde{\mathcal{E}}_U(f,g) = \sum_{e \in E_U)} \int_{I_e} \frac{\partial f_e}{\partial x} \cdot \frac{\partial g_e}{\partial x} \mu_e dx.$$

The measure $\tilde{m}$ is simply the one-dimensional Lebesgue measure.

We want to define $\tilde{h}$ to be the h-function associated to $\tilde{U}$, i.e., a function such that (1) $\tilde{h}(x) = 0$ on $\partial\tilde{U}$, (2) $\tilde{h}(x) > 0$ in $\tilde{U}$, and (3) $\Delta\tilde{h}(x) = 0$ for all $x \in \tilde{U}$. But, to do so requires several careful definitions. First, let $\partial\tilde{U}$ be exactly the vertices in ∂U; all vertices in U and edges, considered as intervals, connecting vertices in $U \cup \partial U$ are in $\tilde{U}$. Second, we define $\Delta\tilde{h}(x) = 0$ differently, depending on whether x is in an edge internal or a vertex: (1) if x is in an edge interval, it the standard Laplacian in $\mathbb{R}^1$; and (2) if x is a vertex, then

$$\Delta\tilde{h}(x) = \sum_{y \sim x} \tilde{h}'_{xy}(x)$$

where $\tilde{h}'_{xy}(x)$ is the derivative of $\tilde{h}$ at the vertex x along the interval between x and y, which is oriented toward x.

The definition of Laplacian at a vertex ensures that harmonic functions are in the domain of the Laplacian. It is motivated by analogy with the following: in $\mathbb{R}^1$, we say that f is in the domain of the Laplacian if

$$\int_U \nabla f(x) \nabla g(x)\; d\lambda(x) = -\int_U \Delta f(x) g(x)\; d\lambda(x) \tag{5.4}$$

for any test function g defined on U. In the cable process, for a given vertex x, standard $\mathbb{R}^1$ integration by parts along each adjacent edge gives,

$$\sum_{y \in x} \int_x^y f'_{xy}(z) g'_{xy}(z) \mu_{xy} d\lambda(z) = \sum_{y \sim x} \left([f'_{xy}(z) g_{xy}(z) \mu_{xy}]_x^y - \int_x^y f''_{xy}(z) g_{xy}(z) \mu_{xy} d\lambda(z) \right).$$

To have the form of (5.4), $\sum_{y \sim x} f'_{xy}(x) g_{xy}(x) \mu_{xy} = 0$ for all test functions g supported near x. (Note: one boundary term disappears since $g_{xy}(y) = 0$.) This is only true for every test function g if $\sum_{y \sim x} f'_{xy}(x) \mu_{xy} = 0$. When the edge μ_{xy} is a loop, i.e., $x = y$, then $f'_{xx}(x)$ must be interpreted as $f'_{\overleftarrow{xx}}(x) + f'_{\overrightarrow{xx}}(x)$, where $\overleftarrow{xx}$ and $\overrightarrow{xx}$ are the two directions of the loop at x.

Proposition 5.6. *Let U be an inner uniform graph domain and let $\tilde{U}$ be its cable analog, i.e., the continuous domain consisting of the vertices of U and the edges of $U \cap \partial U$, with boundary $\partial \tilde{U} = \partial U$. Let h and $\tilde{h}$ be the h-functions on U and $\tilde{U}$, respectively. Then, there exists a constant c such that*

$$h(x) = c\tilde{h}(x)$$

for all $x \in U$.

Proof. If $h(x)$ is the h-function defined on U, then we can define

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in U \\ th(y) + (1-t)h(z) & \text{if } x \in (y,z),\ x = ty + (1-t)z \end{cases}.$$

In other words, $\tilde{h}$ is defined as h on the vertices and the linear interpolation on the edges. It is easy to check that $\tilde{h}$ satisfies the properties of an h-function up to scaling.

We can also reverse this construction: given $\tilde{h}$ on $\tilde{U}$, define $h(x) = \tilde{h}(x)$ for each $x \in U$. We claim that $h(x) = \sum_{y \sim x} h(y)$ and thus, h is the h-function for U. To see this claim, first note that $\tilde{h}$ must be linear along each edge—those are the only harmonic functions on $\mathbb{R}^1$. Since each edge is unit length, we can we write $f(y) = f(x) + f'_{xy}(x)$ for each y adjacent to x. Therefore,

$$\sum_{y \sim x} f(y)p(x,y) = \sum_{y \sim x} (f'_{xy}(x)p(x,y) + f(x)p(x,y)) = f(x),$$

because $\sum_{y \sim x} f'_{xy}(x)\mu_{xy} = 0$ implies that $\sum_{y \sim x} f'_{xy}(x)p(x,y) = 0$. □

Note that h-functions are unique in continuous inner uniform domains [14], and so Proposition 5.6 provides another proof that h-functions are unique in discrete inner uniform domains. However, in a domain which is not inner uniform and has several h-functions which are not scaled versions of each other, an analog of Proposition 5.6 still holds: given a graph domain V with any fixed boundary conditions and its cable analog $\tilde{V}$ with the same boundary conditions, there exists a bijection between the distinct h-functions of V and the h-functions of $\tilde{V}$.

Using the tools of the h-transformation, we are finally in a position to the finish the proof of Proposition 5.5.

Proof of Proposition 5.5. First, we give an easy proof of the lower bound,

$$ch(x_r)^2 V_m(x,r) \leq V_{m_{h^2}}(x,r). \tag{5.5}$$

Fix x_r as specified in Proposition 5.5 and let $\tilde{r} = \frac{\tilde{\kappa} r}{16}$. Then,

$$\begin{aligned} V_{m_{h^2}}(x,r) &\geq V_{m_{h^2}}(x_r,\tilde{r}) \\ &\geq \tilde{c}h(x_r)^2 V_m(x_r,\tilde{r}) \qquad (5.6) \\ &\geq ch(x_r)^2 V_m(x,r). \qquad (5.7) \end{aligned}$$

The second inequality (5.6) follows because $B(x_r,\tilde{r})$ is sufficiently far inside U that the elliptic Harnack inequality applies and the final inequality (5.7) follows by Theorem 4.1, which states that (U,P,m) is volume doubling.

To prove an accompanying upper bound,

$$V_{m_{h^2}}(x,r) \leq Ch(x_r)^2 V_m(x,r), \qquad (5.8)$$

we will use the cable process. Let $(\tilde{U},\tilde{P}_U,\tilde{m})$ be the cable process associated with (U,P_U,m), and let $(\tilde{U},\tilde{P}_{h,u},\tilde{m}_{h^2})$ be the h-transform of $(\tilde{U},\tilde{P}_U,\tilde{m})$. In other words, $\tilde{P}_{h,U} = \tilde{H}\circ\tilde{P}_U\circ\tilde{H}^{-1}$ (where $\tilde{H}(f)(x) = \tilde{h}(x)\cdot f(x)$) and $\tilde{m}_{h^2} = \tilde{h}^2\tilde{m}$.

The continuous process $(\tilde{U},\tilde{P}_{D,U},\tilde{m})$ and its h-transform, $(\tilde{U},\tilde{P}_{h,U},\tilde{m}_{h^2})$, are subject to the theorems in [14]. In particular, their fourth chapter demonstrates that, for any two harmonic functions u and v on an inner uniform subset of a domain satisfying the Harnack inequality,

$$\frac{u(x)}{u(x')} \leq C\frac{v(x)}{v(x')}, \qquad (5.9)$$

where x,x' are in a ball in U and C depends only on the uniformity constants and Harnack constant. In other words, the ratio of any two harmonic functions only differs by a constant. In [14], they use this to bound the h-function by the ratio of a Green function, which is uniformly bounded by a constant, to prove

$$\frac{\tilde{h}(y)}{\tilde{h}(x_r)} \leq C \qquad (5.10)$$

where $y \in B(x,r)$ and x_r is as described in Proposition 5.5 and C depends only on the uniformity constants and the Harnack constant.

Because $h = \tilde{h}$ on the vertices, we can directly transfer 5.9 and 5.10 from the cable process to the process on the graph. Therefore,

$$h(y) \leq Ch(x_r)$$

for all $y \in B(x,r) \subseteq U$. This easily gives us the upper bound 5.8,

$$V_{m_{h^2}}(x,r) = \sum_{y \in B(x,r)} h(y)^2 m(y) \leq Ch(x_r)^2 \sum_{y \in B(x,r)} m(y) = Ch(x_r)^2 V(x,r).$$

□

5.3 The h-transform satisfies Poincaré

The final step to proving Theorem 1.11, which states that the h-transform on U is volume doubling, is to show that $(U, P_{h,U}, m_{h^2})$ satisfies the Poincaré inequality.

Theorem 5.7. *Let (G,P,m) be a graph satisfying the Harnack inequality. From Chapter 3, we know that (G,P,m) is volume doubling and satisfies the Poincaré inequality and ellipticity. Assume that $U \subseteq G$ is an inner uniform domain. Then $(U, P_{h,U}, m_{h^2})$ satisfies the Poincaré inequality.*

Because we know $(U, P_{h,U}, m_{h^2})$ is volume doubling, we can use a Whitney covering argument to prove that $(U, P_{h,U}, m_{h^2})$ satisfies the Poincaré inequality. The proof is nearly identical to the one in Section 4.4 and is omitted here.

Corollary 5.8. *Let (G,P,m) satisfy the Harnack inequality and let $U \subseteq G$ be an inner uniform domain. By Theorem 1.11 we know that $(U, P_{h,U}, m_{h^2})$ satisfies the*

Harnack inequality, and thus, Gaussian bounds. Then, for x, y *such that* $d(x, y) \leq k$*,*

$$\frac{C_l h(x)h(y)}{V_{m_{h^2}}(x, \sqrt{k})} \exp(-c_l d(x, y)^2/k) \leq q^k_{D,U}(x, y) \leq \frac{C_u h(x)h(y)}{V_{m_{h^2}}(x, \sqrt{k})} \exp(-c_u d(x, y)^2/k)$$

because $q^k_h(x, y)h(x)h(y) = q^k_U(x, y)$*.*

Using the fact that m_{h^2} is volume doubling and modifying the constants, we can rewrite the conclusion of Corollary 5.8 as

$$\begin{aligned} &\frac{C_l h(x)h(y)}{\sqrt{V_{m_{h^2}}(x, \sqrt{k})V_{m_{h^2}}(y, \sqrt{k})}} \exp(-c_l d(x, y)^2/k) \leq q^n_{D,U}(x, y) \\ &\leq \frac{C_u h(x)h(y)}{\sqrt{V_{m_{h^2}}(x, \sqrt{k})V_{m_{h^2}}(y, \sqrt{k})}} \exp(-c_u d(x, y)^2/k), \end{aligned} \tag{5.11}$$

for x, y such that $d(x, y) \leq k$.

5.3.1 Average time that the process stays in U

Theorem 5.9. *Let* (G, P, m) *be a graph satisfying the Harnack inequality and let* $U \subseteq G$ *be an inner uniform domain. Let* τ_U *be the first time that the Markov process* $(U, P_{D,U}, m)$ *exits* U*, i.e., is killed at the boundary, and* $\mathbb{P}_x(\tau_U > k)$ *denotes the probability that a process started at* x *has survived past time* k*. Then*

$$c\frac{h(x)}{h(x_{\sqrt{k}})} \leq \mathbb{P}_x(\tau_U > k) \leq C\frac{h(x)}{h(x_{\sqrt{k}})},$$

where $c, C > 0$ *are constants and* $x_{\sqrt{k}}$ *is, as defined in Proposition 5.5, an arbitrary point in* U *satisfying* $d_U(x, x_{\sqrt{k}}) \leq \frac{\sqrt{k}}{4}$ *and* $d(x_{\sqrt{k}}, \partial U) \geq \frac{\tilde{\kappa}\sqrt{k}}{8}$*.*

Proof. Following [14], we first prove the upper bound:

$$\begin{aligned}
\mathbb{P}_x(\tau_U > k) &= \sum_{y \in U} p_U^k(x,y) \\
&= \sum_{y \in U} q_U^k(x,y) m(y) \\
&= \sum_{y \in U} q_{h,U}^k(x,y) h(x) h(y) m(y) \\
&\geq h(x) \sum_{y \in B(x,\sqrt{k})} q_{h,u}^k(x,y) h(y) m(y) \\
&\geq \frac{Ch(x)}{V_{m_{h^2}}(x,\sqrt{k})} \sum_{y \in B(x,\sqrt{k})} h(y) m(y) \qquad (5.12) \\
&\geq \frac{C'h(x)}{V_{m_{h^2}}(x,\sqrt{k}) h(x_{\sqrt{k}})} \sum y \in B(x,\sqrt{k}) h(y)^2 m(y) \qquad (5.13) \\
&= C' \frac{h(x)}{h(x_{\sqrt{k}})}
\end{aligned}$$

where (5.12) follows by Corollary 5.8 and (5.13) follows by (5.10).

The argument for the lower bound uses a similar technique:

$$\begin{aligned}
\mathbb{P}_x(\tau_U > k) &= \sum_{y \in U} p_U^k(x,y) \\
&\leq \sum_{y \in U} p^k(x,y) \\
&\leq c \sum_{y \in U} p^k(x,y) \frac{h(x)h(y)}{h(x_{\sqrt{k}}) h(y_{\sqrt{k}})} \qquad (5.14) \\
&\leq c' \frac{h(x)}{h(x_{\sqrt{k}})} \sup_y \left\{ \frac{h(y)}{h(y_{\sqrt{k}})} \right\} \qquad (5.15) \\
&\leq c'' \frac{h(x)}{h(x_{\sqrt{k})}} \qquad (5.16)
\end{aligned}$$

where (5.14) follows by two applications of the elliptic Harnack inequality to h and (5.16) follows by 5.10. □

5.4 Compared to conditioned process

Both the killed process, $p_U(x, y)$ and the h-transform, $p_{h,U}(x, y)$, have the effect of restricting the process $p(x, y)$, defined on all of G, to a subset U. The h-transform process is a way to condition the process to not hit the boundary of U.

Alternatively, we could condition the process to stay inside U as follows. Let X_k be a process governed by P, and let τ_U be the first that X_k exists U. Then the transition probability for a random walk conditioned to stay inside U is given by

$$p^k_{c,U}(x, y) = \mathbb{P}_x(X_k = y \mid \tau_U = \infty),$$

where $\tau_U = \infty$ means that $\tau_U > N$ for all $N \in \mathbb{N}$. Let

$$q^k_{c,U}(x, y) = \frac{p_{c,U}(x, y)}{m(y)}$$

denote the kernel. In fact, this notion of "conditioned to not hit the boundary" coincides with the h-transform process, i.e,

$$p^k_{c,U}(x, y) = p^k_{h,U}(x, y),$$

and

$$q^k_{c,U}(x, y) = q_{h,U}(x, y)h(x)h(y).$$

A version of the following theorem is outlined in [3].

Theorem 5.10. *Let (G, P, m) be a graph satisfying the Harnack inequality and let $U \subseteq G$ be an inner uniform domain. Then,*

$$p^k_{c,U}(x, y) = p^k_{h,U}(x, y)$$

for $x, y \in U$. In other words, the conditioned process and h process have the same probability transition function.

Proof. Recall that $p^k_{h,U}(x,y) = \frac{h(y)}{h(x)} p^k_{D,U}(x,y)$. We will prove that $p^k_{c,U}(x,y) = \frac{h(y)}{h(x)} p^k_{D,U}(x,y)$.

Fix $K \in \mathbb{N}$ and any $k \leq K$. Temporarily fix N, but we're going to let it tend to infinity.

$$\mathbb{P}_x(X_k = y, \tau_U > k \mid \tau_U > N) = \frac{\mathbb{P}_x(\tau_U > N \mid X_k = y, \tau_U > k)\ \mathbb{P}_x(X_k = y, \tau_U > k)}{\mathbb{P}_x(\tau_U > N, \tau_U > k)} \tag{5.17}$$

Since we are going to let N tend to infinity, we can assume $N > K$, so (5.17) becomes

$$\mathbb{P}_x(X_k = y \mid \tau_U > N) = \frac{\mathbb{P}_x(\tau_U > N \mid X_k = y, \tau_U > k)\ \mathbb{P}_x(X_k = y, \tau_U > k)}{\mathbb{P}_x(\tau_U > N)} \tag{5.18}$$

or equivalently,

$$\mathbb{P}_x(X_k = y \mid \tau_U > N) = \frac{\mathbb{P}_x(\tau_U > N \mid X_k = y, \tau_U > k)}{\mathbb{P}_x(\tau_U > N)} p^k_U(x,y) \tag{5.19}$$

Because X_k is a Markov chain,

$$\begin{aligned} \frac{\mathbb{P}_x(\tau_U > N \mid X_k = y,\ \tau_U > k)}{\mathbb{P}_x(\tau_U > N)} &= \frac{\mathbb{P}_y(\tau_U > N - k)}{\mathbb{P}_x(\tau_U > N)} \\ &= \frac{\sum_{z \in U} p^{N-k}_{D,U}(y,z)}{\sum_{z \in U} p^N_{D,U}(x,z)} \\ &= \frac{\sum_{z \in U} p^{N-k}_{h,U}(y,z) \frac{h(y)}{h(z)}}{\sum_{z \in U} p^N_{h,U}(x,z) \frac{h(x)}{h(z)}}. \end{aligned}$$

Plugging this in to (5.19), we have

$$\mathbb{P}_x(X_k = y \mid \tau_U > N) = \left[\frac{\sum_{z \in U} p^{N-k}_{h,U}(y,z) h(z)^{-1}}{\sum_{z \in U} p^N_{h,U}(x,z) h(z)^{-1}} \right] \frac{h(y)}{h(x)} p^k_U(x,y) \tag{5.20}$$

In the limit, as $N \to \infty$, the left-hand side is $p^k_{c,U}(x,y)$. To finish the proof of Theorem 5.10 we need to show that

$$\frac{\sum_{z \in U} p^{N-k}_{h,U}(y,z) h(z)^{-1}}{\sum_{z \in U} p^N_{h,U}(x,z) h(z)^{-1}} \to 1 \text{ as } N \to \infty, \tag{5.21}$$

which is the content of the following, Lemma 5.11. □

Lemma 5.11. *Let (G, P, m) be a graph satisfying the Harnack inequality and let $U \subseteq G$ be an inner uniform domain. Fix some $K \in \mathbb{N}$ and distance $d_0 > 0$. For any $x, y \in U$ such that $d(x, y) \le d_0$,*

$$\frac{\sum_{z\in U} p^{n}_{h,U}(y,z)h(z)^{-1}}{\sum_{z\in U} p^{\tilde{n}}_{h,U}(x,z)h(z)^{-1}} \to 1 \text{ as } n, \tilde{n} \to \infty \tag{5.22}$$

where we assume that $|n - \tilde{n}|$ is always smaller than some fixed $K \in \mathbb{N}$. In other words, $n, \tilde{n}$ increase at the essentially the same rate.

Proof. For a fixed $n, \tilde{n}$ with $|n - \tilde{n}| \le K$,

$$\left| \sum_{z\in U} p^{n}_{h,U}(y,z)h(z)^{-1} - \sum_{z\in U} p^{\tilde{n}}_{h,U}(x,z)h(z)^{-1} \right| \le A \left(\frac{\sqrt{K} + d_0}{\sqrt{n}} \right)^{\alpha} \sum_{z\in U} p^{2n}_{h,U}(x,z)h(z)^{-1}$$

by the parabolic Hölder inequality, Theorem 3.30.

Dividing, we get

$$\left| 1 - \frac{\sum_{z\in U} p^{n}_{h,U}(y,z)h(z)^{-1}}{\sum_{z\in U} p^{\tilde{n}}_{h,U}(x,z)h(z)^{-1}} \right| \le A \left(\frac{\sqrt{K} + d_0}{\sqrt{n}} \right)^{\alpha} \frac{\sum_{z\in U} p^{2n}_{h,U}(x,z)h(z)^{-1}}{\sum_{z\in U} p^{n}_{h,U}(x,z)h(z)^{-1}}$$

and $\frac{\sum_{z\in U} p^{2n}_{h,U}(x,z)h(z)^{-1}}{\sum_{z\in U} p^{n}_{h,U}(x,z)h(z)^{-1}}$ is bounded, and therefore, the right side of the inequality goes to zero as n approaches infinity. □

5.5 Applications

The theorems of the preceding sections demonstrate that we can uncover useful estimates for the discrete heat kernel in a domain U if we know the h-function for U.

5.5.1 Examples of continuous h-functions

Given a particular inner uniform domain, there is no direct way to determine the associated h-function. However, [14, Chapter 6] gives many nice examples of h-functions for domains in $\mathbb{R}^n$, and particularly, in $\mathbb{R}^2$, where conformal maps can be used to derive h-functions. Below, we highlight several of their key examples.

Example 5.12 (Complement of convex set)**.** *The complement of any closed, convex set in $\mathbb{R}^n$ is an inner uniform domain, and therefore, has a unique h-function.*

In particular, let $U \subseteq \mathbb{R}^2$ be the area outside of the standard parabola $y = x^2$,

$$U = \{(x, y) : y < x^2\}.$$

Then, U is an inner uniform domain with unique h-function,

$$h(x, y) = \left(2\left((x^2 + (1/4 - y)^2)^{1/2} + 1/4 - y\right)\right)^{1/2} - 1.$$

Example 5.13 (Lipschitz domain)**.** *The graph above a Lipschitz domain in $\mathbb{R}^n$ is an inner uniform domain, and therefore, has a unique h-function.*

Example 5.14 (Arbitrary cone)**.** *Let $\Omega \subseteq \mathbb{S}^{n-1}$ with smooth boundary, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, and let ϕ be the first Dirichlet eigenfunction of the spherical Laplacian on Ω with eigenvalue λ. Let $U = \mathbb{R}_+ \times \Omega \subseteq \mathbb{R}^n$ be a cone in $\mathbb{R}^n$. Then, the h-function for U is*

$$h(x) = |x|^\alpha \phi\left(\frac{x}{|x|}\right),$$

where

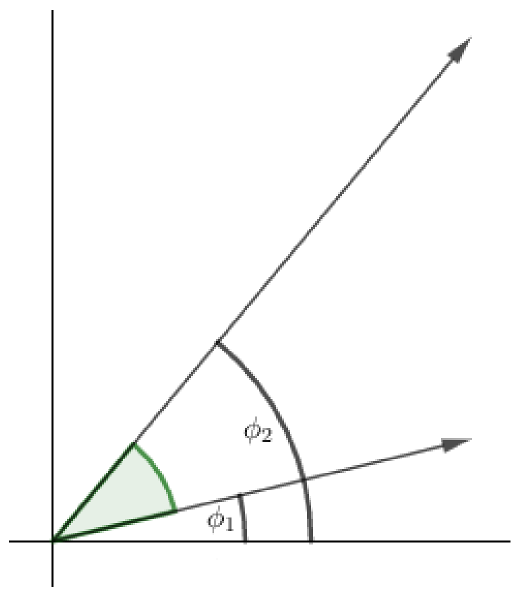
$$\alpha = \frac{((n-2)^2 + 4\lambda)^{1/2} - (n-2)}{2},$$

for $x \in \mathbb{R}^n$.

5.5.2 Approximating the discrete h-function

Plugging $n = 2$ into Example 5.14, we can see that the h-function associated with a cone in $\mathbb{R}^2$ given in polar coordinates by $\{(r, \theta) : 0 < r,\ \phi_1 < \theta < \phi_2\}$ is

$$h(r, \theta) = r^{\pi/(\phi_2 - \phi_1)} \sin((\theta - \phi_1)(\frac{\pi}{\phi_2 - \phi_1})). \tag{5.23}$$



We would like to produce similar examples of h-functions for discrete subsets of $\mathbb{Z}^n$ by transferring the h-function from the analogous continuous subset of $\mathbb{R}^n$. This process is relatively straightforwards in the cases where the boundary of the discrete domain is entirely contained in boundary of the continuous domain. For example, using (5.23) (and shifting to rectangular coordinates and scaling), we know that the h-function of a cone extending from $\theta = 0$ to $\theta = \pi/4$ in $\mathbb{R}^2$,

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : 0 < x,\ 0 < y < x\},$$

is given by $h_{\mathcal{C}}(x, y) = x^3y - xy^3$ on $\mathcal{C} \cup \partial\mathcal{C}$. For the analogous cone in $\mathbb{Z}^2$,

$$\mathcal{C}_{\mathbb{Z}} = \{(x, y) \in \mathbb{Z}^2 : 0 < x,\ 0 < y < x\},$$

the h-function is identical: $h_{\mathcal{C}_\mathbb{Z}}(x,y) = x^3y - xy^3$.

This works for a cone from $\theta = 0$ to $\theta = \pi/4$ because $\partial\mathcal{C}_\mathbb{Z} \subseteq \partial\mathcal{C}$, and therefore, the h-function is already zero at those points. However, this does not work for all cones. Consider another cone of angle $\pi/2$, but extending from $\theta = \pi/8$ to $\theta = 3\pi/8$,

$$\mathcal{C}' = \{(x,y) \in \mathbb{R}^2 : 0 < x,\ (\sqrt{2}-1)x < y < (\sqrt{2}+1)x\}$$

and its discrete version

$$\mathcal{C}'_\mathbb{Z} = \{(x,y) \in \mathbb{Z}^2 : 0 < x,\ (\sqrt{2}-1)x < y < (\sqrt{2}+1)x\}.$$

Notice that $(1,0) \in \partial\mathcal{C}'_\mathbb{Z}$ and hence $h_{\mathcal{C}'_\mathbb{Z}}(1,0) = 0$. But $(1,0) \notin \mathcal{C}' \cup \partial\mathcal{C}'$ and so $h_{\mathcal{C}'}(1,0)$ is not defined. If we tried to extend $h_{\mathcal{C}'}$ beyond $\mathcal{C}' \cup \partial\mathcal{C}'$ it would either be negative or not harmonic. Therefore, we cannot directly transfer the continuous h-function to the discrete domain.

Even though the h-function on a discrete domain will not be exactly the same as the h-function on the analogous continuous domain, one can hope that the two h-functions are roughly comparable, i.e., there exist $c, C > 0$ such that

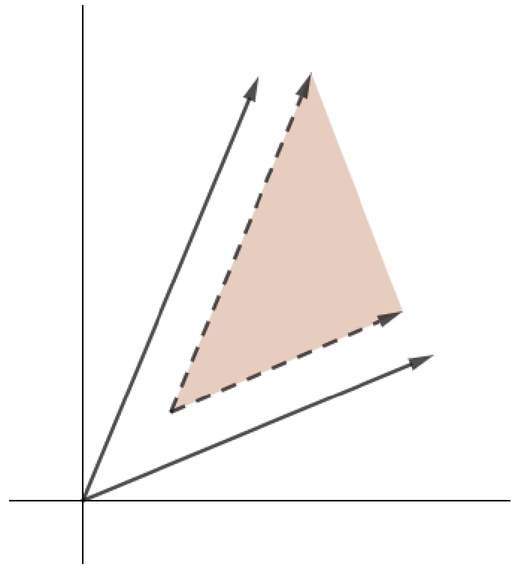
$$c h_\mathcal{C}(x,y) \leq h_{\mathcal{C}_\mathbb{Z}}(x,y) \leq C h_\mathcal{C}(x,y) \tag{5.24}$$

for any points $(x,y) \in \mathcal{C}_\mathbb{Z}$ that are sufficiently far away from the boundary.

Fortunately, a 2009 article by N. Varopoulos [26] tells us that, if U is a Lipschitz domain with discrete h-function h_d and continuous h-function h_c, then

$$|h_c(x) - h_d(x)| \leq C\frac{h_c(x)}{d(x, \partial U)},$$

where x is sufficiently far inside U.



Consider the cone $\mathcal{C}$ of angle $\pi/4$ off the x-axis and the tilted version, $\mathcal{C}'$, extending from $\theta = \pi/8$ to $\theta = 3\pi/8$. In $\mathbb{R}^n$, tilting a cone should not change its h-function, i.e.,

$$h_{\mathcal{C}}(r,\theta) = h_{\mathcal{C}'}(r, \theta + \pi/8).$$

We cannot directly extend this formula for the h-function in $\mathcal{C}'$ to the h-function in $\mathcal{C}'_{\mathbb{Z}}$ because their boundaries do not coincide. However, applying the result of Varopoulos, we can see that $h_{\mathcal{C}'}$ and $h_{\mathcal{C}'_{\mathbb{Z}}}$ are comparable sufficently far inside $\mathcal{C}'$.

5.6 Perturbing the boundary

For a given cone $\mathcal{C}$ with discrete version $\mathcal{C}_{\mathbb{Z}}$, tote that there exists a continuous domain $\tilde{\mathcal{C}}$ whose boundary exactly coincides with $\mathcal{C}_{\mathbb{Z}}$, and that the boundary of $\tilde{\mathcal{C}}$ is some finite perturbation of the boundary of $\mathcal{C}$. The main result of this section, Theorem 5.18, implies that

$$ch_{\mathcal{C}}(x,y) \leq h_{\tilde{\mathcal{C}}}(x,y) \leq Ch_{\mathcal{C}}(x,y).$$

Using the cable process, we can see that $h_{\tilde{\mathcal{C}}}(x,y) = h_{\mathcal{C}_{\mathbb{Z}}}$, which proves that $h_{\mathcal{C}}(x,y)$ is comparable to $h_{\mathcal{C}_{\mathbb{Z}}}$.

5.6.1 Example cones

Example 5.15 (Upper half-plane)**.** *Let $U = \{(x,y) : y \geq 0\}$ be the upper half-plane with associated h-function $h_U(x,y) = y$. Let $V = \{(x,y) : y \geq f(x)\}$ be the domain above some function $f(x)$, where f is some bounded nonnegative function: $0 \leq f(x) \leq \tilde{y}$. We assume that V has an associated h-function $h^V(x,y) : V \to \mathbb{R}_{\geq 0}$. Then, there exist constants c, C depending only on $\tilde{y}$ such that*

$$ch^U(x,y) \leq h^V(x,y) \leq Ch^U(x,y)$$

for all (x,y) with $y > \tilde{y}$.

Proof. Let $U_n = [-n/2, n/2] \times [0,n] \subseteq \mathbb{R}^2$ be an $n \times n$ box whose bottom edge lies on the line $y = 0$. Let $u_n(x,y) : U_n \to \mathbb{R}_{\geq 0}$ be the solution to

$$u_n(x,y) = \begin{cases} 0 & \text{for } y = 0 \\ 1 & \text{for } y = n \\ y/n & \text{for } x = -n/2 \text{ or } x = n/2 \\ \Delta u_n = 0 & \text{else .} \end{cases}$$

That is u_n is zero on the lower boundary, one on the upper boundary, and monotonically increasing on the sides. It is easy to verify that $u_n(x,y) = y/n$ is the unique solution.

We normalize relative to a fixed point (x_0, y_0), independent of n, and set

$$h_n^U(x,y) = \frac{u_n(x,y)}{u_n(x_0,y_0)} = \frac{y}{y_0}.$$

Now, define $\tilde{U}_n = [-(n-\tilde{y})/2, (n-\tilde{y})/2] \times [\tilde{y}, n] \subseteq \mathbb{R}^2$ be an $(n-\tilde{y}) \times (n-\tilde{y})$ box whose bottom edge lies on the line $y = r$. Analogously, we define $\tilde{u}_n(x,y)$: $\tilde{U}_n \to \mathbb{R}_{\geq 0}$ by the solution to

$$\tilde{u}_n(x,y) = \begin{cases} 0 & \text{for } y = \tilde{y} \\ 1 & \text{for } y = n \\ y/n & \text{for } x = -(n-\tilde{y})/2 \text{ or } x = (n-\tilde{y})/2 \\ \Delta u_n = 0 & \text{else .} \end{cases}.$$

Similarly, $\tilde{u}_n(x,y) = \frac{y-\tilde{y}}{n-\tilde{y}}$, and the normalized version is given by

$$h_n^{\tilde{U}}(x,y) = \frac{\tilde{u}_n(x,y)}{\tilde{u}_n(x_0,y_0)} = \frac{y-\tilde{y}}{y_0-\tilde{y}}.$$

Finally, we define $V_n = V \cap U_n$, and we similarly let $v_n(x,y) : V_n \to \mathbb{R}_{\geq 0}$ be zero on the lower boundary, one on the upper boundary and linearly increasing on the sides. The normalized version is $h_n^V(x,y) = \frac{v_n(x,y)}{v_n(x_0,y_0)}$.

By the maximum principle, for any $(x,y) \in \tilde{U}_n$,

$$\tilde{u}_n(x,y) \leq v_n(x,y) \leq u_n(x,y).$$

To obtain a similar inequality in the other direction, note that

$$\frac{u_n(x,y)}{\tilde{u}_n(x,y)} = \frac{y}{n} \cdot \frac{n-\tilde{y}}{y-\tilde{y}} \leq \tilde{y} + 1.$$

Hence, there exists a constant $C = \tilde{y} + 1$ such that $u_n(x,y) \leq C\tilde{u}_n(x,y)$.

Merging these, we have that

$$h_n^V(x,y) = \frac{v_n(x,y)}{v_n(x_0,y_0)} \leq \frac{u_n(x,y)}{\tilde{u}_n(x,y)} \leq \frac{u_n(x,y)}{(1/C)u_n(x,y)} = Ch_n^U(x,y)$$

and

$$h_n^U(x,y) = \frac{u_n(x,y)}{u_n(x_0,y_0)} \leq \frac{C\tilde{u}_n(x,y)}{v_n(x_0,y_0)} \leq \frac{Cv_n(x,y)}{v_n(x_0,y_0)} = Ch_n^V(x,y).$$

The constant C depends only on $\tilde{y}$, so it holds in the limit. □

Example 5.16 (Upper right quadrant)**.** *Let $U = \{(x, y) : x \geq 0, y \geq 0\}$ be the upper right quadrant with associated h-function $h^U(x, y) = xy$. Let $V = \{(x, y) : x \geq f(y), y \geq g(x)\}$ where f, g are bounded nonnegative functions: $0 \leq f(y) \leq \tilde{x}$ and $0 \leq g(x) \leq \tilde{y}$. We assume that V has an associated h-function $h^V(x, y) : V \to \mathbb{R}_{\geq 0}$. Then, there exist constants c, C depending only on $\tilde{y}$ such that*

$$ch^U(x, y) \leq h^V(x, y) \leq Ch^U(x, y)$$

for all (x, y) with $x > \tilde{x}$ and $y > \tilde{y}$.

Proof. Similar to Example 5.15, construct three domains, with associated functions:

$$U_n = [0, n] \times [0, n],\ \ u_n(x, y)$$

$$V_n = V \cap U_n,\ \ v_n(x, y)$$

$$\tilde{U}_n = [\tilde{x}, n] \times [\tilde{y}, n],\ \ \tilde{u}_n(x, y)$$

where the associated functions are zero on the lower and left boundary, linearly increasing on the top and right boundary, and harmonic on the interior.

Then the maximum principle implies that

$$\tilde{u}_n(x, y) \leq v_n(x, y) \leq u_n(x, y)$$

for any $(x, y) \in \tilde{U}_n$, and, for $C = 4\tilde{x}^2\tilde{y}^2$,

$$u_n(x, y) \leq C\tilde{u}_n(x, y).$$

Again, as in Example 5.15,

$$h_n^V(x, y) = \frac{v_n(x, y)}{v_n(x_0, y_0)} \leq \frac{u_n(x, y)}{\tilde{u}_n(x, y)} \leq \frac{u_n(x, y)}{(1/C)u_n(x, y)} = Ch_n^U(x, y)$$

and

$$h_n^U(x,y) = \frac{u_n(x,y)}{u_n(x_0,y_0)} \leq \frac{C\tilde{u}_n(x,y)}{v_n(x_0,y_0)} \leq \frac{Cv_n(x,y)}{v_n(x_0,y_0)} = Ch_n^V(x,y).$$

The constant C depends only on $\tilde{x}$ and $\tilde{y}$, so it holds in the limit. □

5.6.2 Arbitrary cone

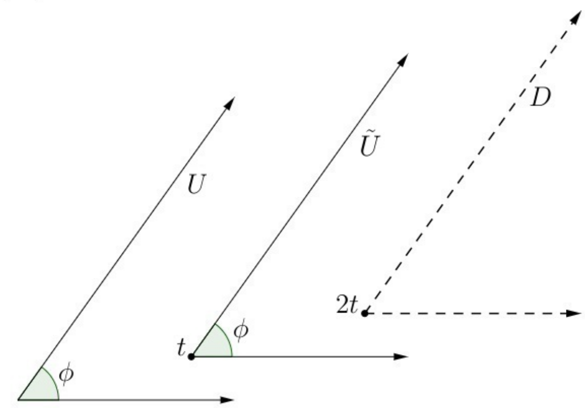
Proposition 5.17. *Let $U = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq r,\ \phi_1 \leq \theta \leq \phi_2\}$ be a cone in $\mathbb{R}^2$ centered at the origin. For a fixed point t inside U, define $\tilde{U} = \{u+t : u \in U\}$ to be copy of the cone U translated by the vector $\vec{t}$, where $\vec{t}$ is defined as extending from the origin to t. Let $h_U(r,\theta) : U \to \mathbb{R}_{\geq 0}$ and $h_{\tilde{U}}(r,\theta) : \tilde{U} \to \mathbb{R}_{\geq 0}$ be the h-functions in U and $\tilde{U}$, respectively. Then, there exist constants c, C such that*

$$ch_U(r,\theta) \leq h_{\tilde{U}}(r,\theta) \leq Ch_U(r,\theta)$$

for $(r,\theta) \in D$, where $D = \{u + 2t : u \in U\}$ is another copy of U translated by t inside of $\tilde{U}$. The constants c, C depend on $|t|$ and the angle between $\vec{t}$ and the boundary of U.

Proof. We begin by noting that the set-up is invariant under rotation, which allows us to assume that the lower boundary of U is on the x-axis, i.e., $U = \{(r,\theta) : 0 \leq r, 0 \leq \theta \leq \phi\}$ for $\phi \in (0, 2\pi)$.

There are two ways to describe the cone $\tilde{U}$. (1) Using an internal coordinate system, we can define $\tilde{U} = \{(r_{\tilde{U}}, \theta_{\tilde{U}}) : 0 \leq r_{\tilde{U}}, 0 \leq \theta_{\tilde{U}} \leq \phi\}$ where $r_{\tilde{U}}$ and $\theta_{\tilde{U}}$ are the radius and angle considered from the point t. (2) In terms of U, we can define $\tilde{U} = \{(r,\theta) + t : (r,\theta) \in U\}$.



From [14] we know that the h-function in U is

$$h_U(r,\theta) = r^{\pi/\phi}\sin(\theta \cdot \frac{\pi}{\phi})$$

for $(r,\theta) \in U$. We have two similar descriptions of $h_{\tilde{U}}$, the h-function in $\tilde{U}$. (1) Using an internal coordinate system, $h_{\tilde{U}}(r_{\tilde{U}}, \theta_{\tilde{U}}) = r_{\tilde{U}}^{\pi/\phi}\sin(\theta_{\tilde{U}} \cdot \frac{\pi}{\phi})$ where $(r_{\tilde{U}}, \theta_{\tilde{U}})$ is measured from t. (2) We can express the h-function in $\tilde{U}$ in terms of h_U as

$$h_{\tilde{U}}(r,\theta) = h_U((r,\theta) - \vec{t})$$

where $(r,\theta) \in \tilde{U}$ and is measured from the origin.

Our goal is to compare the values of h_U and $h_{\tilde{U}}$ within D, a copy of U translated by $\vec{2t}$ so that it sits within $\tilde{U}$, i.e., $D = \{(r,\theta) + 2t : (r,\theta) \in U\}$. We wish to find two constants C_1, C_2 which bound

$$\frac{h_U(r,\theta)}{h_{\tilde{U}}(r,\theta)} = \frac{h_U(r,\theta)}{h_U(\tilde{r},\tilde{\theta})} = \frac{r^{\pi/\phi}\sin(\theta \cdot \frac{\pi}{\phi})}{\tilde{r}^{\pi/\phi}\sin(\tilde{\theta} \cdot \frac{\pi}{\phi})} \tag{5.25}$$

and

$$\frac{h_{\tilde{U}}(r,\theta)}{h_U(r,\theta)} = \frac{h_U(\tilde{r},\tilde{\theta})}{h_U(\tilde{r},\tilde{\theta})} = \frac{\tilde{r}^{\pi/\phi}\sin(\tilde{\theta} \cdot \frac{\pi}{\phi})}{r^{\pi/\phi}\sin(\theta \cdot \frac{\pi}{\phi})} \tag{5.26}$$

respectively, where (r,θ) is any point in D measured from the origin (i.e. the vertex of U) and $(\tilde{r},\tilde{\theta}) = (r,\theta) - t$.

By the triangle inequality,

$$\tilde{r} - |t| \leq r \leq \tilde{r} + |t|$$

and therefore it suffices to find constant bounds for

$$\frac{\sin(\theta \cdot \frac{\pi}{\phi})}{\sin(\tilde{\theta} \cdot \frac{\pi}{\phi})} \tag{5.27}$$

and

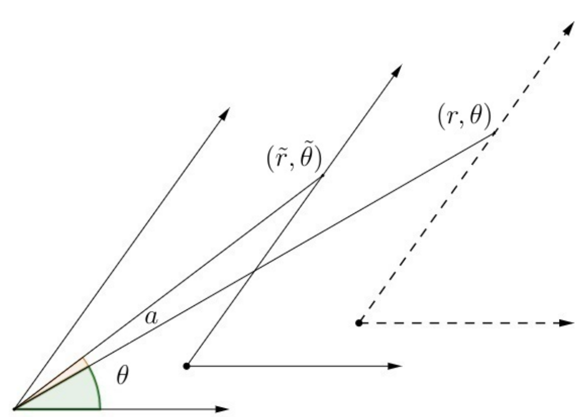
$$\frac{\sin(\tilde{\theta} \cdot \frac{\pi}{\phi})}{\sin(\theta \cdot \frac{\pi}{\phi})} \tag{5.28}$$

that work for any $(r, \theta) \in D$.

Let θ_t denote the angle of $\vec{t}$, measured off the x-axis. For now, we will consider points (r, θ) such that $\theta \geq \theta_t$. Note that $\tilde{\theta} > \theta$ since $(\tilde{r}, \tilde{\theta})$ is shifted from (r, θ) by $\vec{t}$ and $\theta > \theta_t$. This implies that $\tilde{\theta} \cdot \frac{\pi}{\phi} > \theta \cdot \frac{\pi}{\phi}$ and both angles are between $\pi/2$ and π. Therefore $\sin(\tilde{\theta} \cdot \frac{\pi}{\phi}) < \sin(\theta \cdot \frac{\pi}{\phi})$ and (5.28) is bounded.

Bounding (5.27) requires slightly more work. Define $a = \tilde{\theta} - \theta$. We rewrite (5.27) as

$$\frac{\sin(\theta \cdot \frac{\pi}{\phi})}{\sin(\tilde{\theta} \cdot \frac{\pi}{\phi})} = \frac{\sin(\theta \cdot \frac{\pi}{\phi})}{\sin((\theta + a)\frac{\pi}{\phi})} = \frac{\sin((\phi - \theta)\frac{\pi}{\phi})}{\sin((\phi - (\theta + a))\frac{\pi}{\phi})} \tag{5.29}$$



To bound (5.29) we need to consider the points (r, θ) in D where $\theta + a$ approaches ϕ, sending the denominator to zero. This happens at points (r, θ) on the upper boundary of D, as $r \to \infty$. We restrict our attention to the boundary. In this case, both $\theta \to \phi$ and $\theta + a \to \phi$, but we need to compare their relative rate to show (5.29) is bounded.

Since (r, θ) is on the boundary of D, $(\tilde{r}, \tilde{\theta})$ is on the boundary of $\tilde{U}$. Construct a circle whose radius is given by $\tilde{r}$. The arc of the circle between the upper boundaries of U and $\tilde{U}$ represents the angle $\phi - \tilde{\theta}$, or equivalently $\phi - \theta - a$. The arc of the circle between the upper boundary of $\tilde{U}$ and the upper boundary of D (possibly extended down as a line) is at least as big as a. Call these two arcs $A_1(r)$ and $A_2(r)$, respectively. Notice that $A_1(r) \leq A_2(r)$ and asymptotically, $\lim r \to \infty A_1(r) = A_2(r)$. This implies that there must be some boundary point such that $A_1(r) \geq \frac{1}{2} \cdot A_2(r)$. Define (R, Θ) be the point on the upper boundary of U such that (1) $\Theta > \phi - \frac{\pi}{4}$ and (2) $A_1(r) \geq \frac{1}{2} \cdot A_2(r)$ for (r, θ) above (R, Θ).

Consider $(\tilde{r}, \tilde{\theta})$ on the boundary of U above (R, Θ).

$$\phi - \theta - a = \phi - \tilde{\theta} = A_1(\tilde{r}) \geq \frac{1}{2} \cdot A_2(\tilde{r}) \geq \frac{a}{2} \tag{5.30}$$

Rearranging (5.30) gives that

$$\phi - (\theta + a) \geq \frac{1}{3}(\phi - \theta).$$

For x sufficiently small, $x/2 < \sin x < x$. In particular, this holds for $x < \frac{\pi}{4}$. For $(\tilde{r}, \tilde{\theta})$ above (R, Θ) we can bound (5.29) above by

$$2\frac{(\phi - \theta)\frac{\pi}{\phi}}{(\phi - (\theta + a))\frac{\pi}{\phi}} \leq 2\frac{\phi - \theta}{\frac{1}{3}(\phi - \theta)} = 6. \tag{5.31}$$

Consider a point $(\tilde{r}, \tilde{\theta})$ on the boundary of U below (R, Θ) (but necessarily above t). Then $\phi - \tilde{\theta} = \phi - \theta - a$ is bounded away from zero. Therefore, the denominator of (5.29) is also bounded away from zero, and the fraction has an upper bound.

In the case of $(r,\theta) \in D$ with $\theta \geq \theta_t$, we have given upper bounds on (5.27) (through (5.29) and (5.31)) and (5.28). These, in turn, bound (5.25) and (5.26), after we incorporate the radial estimates. For points (r,θ) with $\theta \leq \theta_t$, a nearly identical argument gives the same bounds, but in this case, the extreme points to analyze fall on the lower boundary of D. □

Take an arbitrary cone U in $\mathbb{R}^2$ and perturb its boundary by a finite amount. The following theorem proves that this does not significantly change the h-function.

Theorem 5.18. *Let* $U = \{(r,\theta) \in \mathbb{R}^2 : 0 \leq r,\ \phi_1 \leq \theta \leq \phi_2\}$ *be a cone in* $\mathbb{R}^2$ *centered at the origin, with h-function* $h^U(r,\theta) : U \to \mathbb{R}_{\geq 0}$. *Let V be a domain whose boundary is a finite perturbation of the boundary of U, which has an h-function* $h^V(r,\theta) : V \to \mathbb{R}_{\geq 0}$. *Then there exists constants $c, C > 0$ such that*

$$ch^U(r,\theta) \leq h^V(r,\theta) \leq Ch^U(r,\theta)$$

for $(r,\theta) \in \{u + 2t : u \in U\}$, *where t is a vector such that* $V \subseteq \{u + t : u \in U\}$.

Proof. As in the proof of Proposition 5.17 we can assume that the lower boundary of U is on the x-axis, i.e., $U = \{(r,\theta) : 0 \leq r,\ 0 \leq \phi\}$ where $\phi = \phi_1 = \phi_2$. Also as in the proof of Proposition 5.17 we fix a point t inside U and define $\tilde{U} = \{u+t : u \in U\}$ to be a copy of the cone U translated by the vector $\vec{t}$. But this time we require that t lie on the angle bisector and $|t|$ is large enough that $\tilde{U} \subseteq V$.

For $R > 0$, define $U_R = \{(r,\theta) : 0 \leq r \leq R,\ 0 \leq \theta \leq \phi\}$. Then define $\tilde{U}_R = \tilde{U} \cap U_R$ and $V_R = V \cap U_R$.

Let $u_R(r,\theta) : U_R \to \mathbb{R}_{\geq 0}$ be the solution to

$$u_R(r,\theta) = \begin{cases} 0 & \text{for } \theta = 0 \text{ or } \theta = \phi \\ \sin(\theta \cdot \frac{\pi}{\phi}) & \text{for } r = R \\ \Delta u_R = 0 & \text{else .} \end{cases}$$

That is, u_R is zero on the sides of the cone, positive on the arc and increasing to one in the center, and harmonic on the interior. It is easy to verify that $u_R(r,\theta) = (\frac{r}{R})^{\pi/\phi} \sin(\theta \cdot \frac{\pi}{\phi})$ is the unique solution.

Similarly, we define $\tilde{u}_R(r,\theta) : \tilde{U}_R \to \mathbb{R}_{\geq 0}$ be the solution to

$$\tilde{u}_R(r,\theta) = \begin{cases} 0 & \text{for } \theta = 0 \text{ or } \theta = \phi \\ \sin(\theta \cdot \frac{\pi}{\phi}) & \text{for } r = \tilde{R} \\ \Delta \tilde{u}_R = 0 & \text{else} \end{cases}$$

where (r,θ) is measured from t, the vertex of $\tilde{U}$, and $\tilde{R} = R - |t|\cos(\phi/2)$ is the radius of $\tilde{U}_R$. Then $\tilde{u}_R$ satisfies the same criteria: zero on the sides of the cone, positive on the arc and increasing to one in the center, and harmonic on the interior. It is easy to verify that $\tilde{u}_R(r,\theta) = (\frac{r}{R})^{\pi/\phi} \sin(\theta \cdot \frac{\pi}{\phi})$ is the unique solution, where again (r,θ) is measured from t.

Let $v_R(r,\theta) : V_R \to \mathbb{R}_{\geq 0}$ be the analogous function on V_R, where (r,θ) is measured from the origin, i.e., the vertex of U. More precisely, v_R is zero on the side boundaries of V_R, harmonic on the interior, and uses the sine function to interpolate along the arc $r + R$ between the side boundaries of V and the center point $(R, \phi/2)$. To make the last condition more precise, we define θ_1 to be the lower intersection of V and the arc $r = R$, and θ_2 to be the upper intersection of V and the arc $r = R$. Then, for $\theta \in (\theta_1, \phi/2)$, the boundary condition is

$(R,\theta) = \sin((\theta - \theta_1) \cdot \frac{\pi}{2(\phi/2-\theta_1)})$. And for $\theta \in (\phi/2, \theta_2)$ the boundary condition is $(R,\theta) = \sin((\theta_2 - \theta) \cdot \frac{\pi}{2(\theta_2-\phi/2)})$.

The maximum principle implies that

$$\tilde{u}_R(r,\theta) \leq v_R(r,\theta) \leq u_R(r,\theta) \tag{5.32}$$

for any $(r,\theta) \in \tilde{U}_R$ where (r,θ) is measured from the origin, i.e., the vertex of U.

Note that the ratio $\frac{\tilde{R}}{R}$ is bounded above as $R \to \infty$ since $\tilde{R} = R - |t|\cos(\phi/2)$. Applying this and Proposition 5.17,

$$u_R(r,\theta) = \left(\frac{1}{R}\right)^{\pi/\phi} h^U(r,\theta) \leq c\left(\frac{1}{R}\right)^{\pi/\phi} h^{\tilde{U}}(r,\theta) = c\left(\frac{\tilde{R}}{R}\right)^{\pi/\phi} \tilde{u}_R(r,\theta) \leq C\cdot\tilde{u}_R(r,\theta) \tag{5.33}$$

for any $(r,\theta) \in \tilde{U}_R$ where C depends only on $|t|$.

Fix a point (r_0,θ_0). Define $h^U_R = \frac{u_R(r,\theta)}{u_R(r_0,\theta_0)}$ and $h^V_R = \frac{v_R(r,\theta)}{v_R(r_0,\theta_0)}$. Applying estimates (5.32) and (5.33),

$$h^V_R(r,\theta) = \frac{v_R(r,\theta)}{v_R(r_0,\theta_0)} \leq \frac{u_R(r,\theta)}{\tilde{u}_R(r_0,\theta_0)} \leq C \cdot \frac{u_R(r,\theta)}{u_R(r_0,\theta_0)} = C\dot{h}^U_R(r,\theta) \tag{5.34}$$

and

$$h^U_R(r,\theta) = \frac{u_R(r,\theta)}{u_R(r_0,\theta_0)} \leq C \cdot \frac{\tilde{u}_R(r,\theta)}{v_R(r_0,\theta_0)} \leq C \cdot \frac{v_R(r,\theta)}{v_R(r_0,\theta_0)} = C\dot{h}^V_R(r,\theta). \tag{5.35}$$

The constant C does not depend on R, so the inequalities hold in the limit: $h^V(r,\theta) \leq C \cdot h^U(r,\theta)$ and $h^U(r,\theta) \leq C \cdot h^V(r,\theta)$. □

BIBLIOGRAPHY

[1] S. Andres, J. Deuschel, and M. Slowik. Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probability Theory and Related Fields*, 164(3–4):931–977, 2016.

[2] M. Barlow. *Random Walks and Heat Kernels on Graphs*. Cambridge University Press, 2017.

[3] A. Bloemendal. Doob's h-transform: theory and examples. 2010.

[4] A. Bouaziz, S. Mustapha, and M. Sifi. Discrete harmonic functions on an orthant in $\mathbb{Z}^d$. *Electronic Communications in Probability*, 20(52):1–15, 2015.

[5] T. Coulhon and A. Grigor'yan. Random walks on graphs with regular volume growth. *Geometric & Functional Analysis*, 8(4):656–701, Sept. 1998.

[6] T. Delmotte. Inégalité de Harnack elliptique sur les graphes. *Colloquium Mathematicum*, 72(1):19–37, 1997.

[7] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Revista Matematica Iberoamericana*, 15(1):181–232, 1999.

[8] J.L. Doob. *Classical Potential Theory and Its Probabilistic Counterpart*. Springer-Verlag New York, 1984.

[9] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, 2nd edition, 2010.

[10] M. Folz. Volume growth and stochastic completeness of graphs. *Transactions of the American Mathematical Society*, 366(4):2089–2119, 2018.

[11] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes*. de Gruyter, 1994.

[12] A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. *Matematicheskii Sbornik*, 182(1):55–87, 1991. English translation in Math. USSR-Sb. 72(1):47-77, 1992.

[13] A. Grigor'yan. *Heat Kernel and Analysis on Manifolds*. American Mathematical Society, 2009.

[14] P. Gyrya and L. Saloff-Coste. *Neumann and Dirichlet Heat Kernels in Inner Uniform Domains*. Astérisque; 336. Société mathématique de France, Paris, 2011.

[15] W. Hebisch and L. Saloff-Coste. Gaussian estimates for Markov chains and random walks on groups. *The Annals of Probability*, 21(2):673–709, 1993.

[16] W. Hebisch and L. Saloff-Coste. On the relation between elliptic and parabolic Harnack inequalities. *Annales de l'Institut Fourier*, 51(5):1437–1481, 2001.

[17] H. Kesten. *Relations Between Solutions to a Discrete and Continuous Dirichlet Problem*, pages 309–321. Birkhäuser Boston, Boston, MA, 1991.

[18] J. Moser. On Harnack's theorem for elliptic differential equations. *Communications on pure and applied mathematics*, 14(3):577–591, 1961.

[19] J. Moser. A Harnack inequality for parabolic differential equations. *Communications on pure and applied mathematics*, 17:101–134, 1964.

[20] M. Murugan and L. Saloff-Coste. Harnack inequalities and Gaussian estimates for random walks on metric measure spaces. *ArXiv e-prints, 1506.07539*, 2015.

[21] K. Raschel. Random walks in the quarter plane, discrete harmonic functions and conformal mappings. *Stochastic Processes and their Applications*, 124:3147–3178, 2014.

[22] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Grundlehren der mathematischen Wissenschaften, 293. Springer, Berlin, third edition, 2001.

[23] L.C.G. Rogers and D. Williams. *Diffusions, Markov processes, and Martingales*. Cambridge University Press, Cambridge, 2000.

[24] L. Saloff-Coste. Parabolic Harnack inequality for divergence form second order differential operators. *Potential Analysis*, 4(4):429–467, 1995.

[25] L. Saloff-Coste. *Aspects of Sobolev-type inequalities*. London Mathematical Society lecture note series; 289. Cambridge University Press, Cambridge, 2002.

[26] N. Varopoulos. The discrete and classical Dirichlet problem. *Milan Journal of Mathematics*, 2009.