

# ON THE GORENSTEINIZATION OF SCHUBERT VARIETIES VIA BOUNDARY DIVISORS

A Dissertation

Presented to the Faculty of the Graduate School  
of Cornell University

in Partial Fulfillment of the Requirements for the Degree of  
Doctor of Philosophy

by

Sergio Mathew Luis Da Silva

May 2018

© 2018 Sergio Mathew Luis Da Silva

ALL RIGHTS RESERVED

ON THE GORENSTEINIZATION OF SCHUBERT VARIETIES VIA  
BOUNDARY DIVISORS

Sergio Mathew Luis Da Silva, Ph.D.

Cornell University 2018

We will describe a one-step “Gorensteinization” process for a Schubert variety by blowing-up along its boundary divisor. The local question involves Kazhdan-Lusztig varieties which can be degenerated to affine toric schemes defined using the Stanley-Reisner ideal of a subword complex. The blow-up along the boundary in this toric case is in fact Gorenstein. We show that there exists a degeneration of the blow-up of the Kazhdan-Lusztig variety to this Gorenstein scheme, allowing us to extend this result to Schubert varieties in general. The potential use of this one-step Gorensteinization to describe the non-Gorenstein locus of Schubert varieties is discussed, as well as the relationship between Gorensteinizations and the convergence of the Nash blow-up process in the toric case.

## BIOGRAPHICAL SKETCH

Sergio received his Bachelor of Science degree at the University of Toronto with a specialty in pure and applied mathematics in 2011. After completing a Masters of Science the following year, he moved to Cornell University, completing his PhD in 2018.

It was as early as his undergraduate studies that he began his research in algebraic geometry. In its simplest form, algebraic geometry studies objects which arise as the zero sets of polynomials. It is a broad subject, having been generalized considerably in the 20th century using the language of schemes, which intersects many other fields of mathematics like number theory and analytic geometry.

His earliest research projects dealt with topics related to the minimal model program. His work with Kalle Karu in 2009 at the University of British Columbia concerned the factorization of birational maps between toric varieties into a sequence of blow-up and blow-down maps. This project on Oda's Strong Conjecture resulted in his first publication. His subsequent work with Edward Bierstone and Pierre Milman at the University of Toronto produced a more specialized version of the characteristic 0 desingularization algorithm which detected and avoided simple normal crossings in the resolution process.

While at Cornell, Sergio collaborated on a group project led by Dan Barbasch in representation theory. The orbits of the  $G$ -diagonal action on multiple flag varieties in the exceptional cases were studied. Results for the  $F_4$  case were published and work is ongoing for the  $E_6, E_7, E_8$  cases.

His most recent projects have included a number of proposed problems from his advisor Allen Knutson including the application of characteristic 0 desingularization techniques to Frobenius split varieties. This idea can be realized in

low dimensions, providing a desingularization for certain F-pure singularities.

In addition, Sergio considered problems involving Schubert varieties, an important class of varieties for algebraic combinatorists. It is well known that a Schubert variety can be desingularized using a Bott-Samelson resolution; it does not however do this strictly (it is not a bijection over the smooth locus). On the other hand, Hironaka's result tells us that such varieties can be desingularized strictly. Sergio's research in this area compares these two very different approaches to end up with a hybrid desingularization process which is strict as in Hironaka's method, yet similar to the Bott-Samelson construction (see Appendix A for more information).

His most recent work, the content of this thesis, has dealt with understanding what happens when one blows-up the boundary divisor of a Schubert variety (or locally, the Kazhdan-Lusztig variety). He shows that it takes only one blow-up along the boundary divisor to get a Gorenstein variety, providing a convenient (almost canonical) description for a Gorensteinization of Schubert varieties.

While Sergio has worked on problems involving the minimal model program (resolution of singularities, Frobenius splittings, factoring birational maps etc.) or algebraic combinatorics (Schubert varieties, toric varieties etc.), he is excited to work with experts in other areas of algebraic geometry. In particular, he has accepted a PIMS postdoctoral fellowship at the University of Manitoba, and looks forward to beginning this new position in August 2018.

To Ernestine – your love and support knows no bounds.

Home is behind, the world ahead.

To our family – thank you for your sacrifice and encouragement ... and of course your guilt trip every time we leave. We shall return.

## ACKNOWLEDGEMENTS

I would like to give special thanks to my advisor Allen Knutson for his direction and guidance all of these years. I have learned a great deal from him and am thankful for the many projects we have worked on.

I would also like to thank my committee members Yuri Berest, Michael Stillman and Ed Swartz whose support during this process has been invaluable.

Finally, a special thanks to Dan Barbasch, Michel Brion, Karl Schwede, Jarosław Włodarczyk, Alexander Woo and Alexander Yong for their advice on a variety of topics related to my research.

## TABLE OF CONTENTS

Biographical Sketch . . . . .	iii
Dedication . . . . .	v
Acknowledgements . . . . .	vi
Table of Contents . . . . .	vii
<b>1 Introduction</b>	<b>1</b>
<b>2 Preliminaries</b>	<b>7</b>
2.1 Kazhdan-Lusztig varieties . . . . .	8
2.2 The Gorenstein property . . . . .	10
2.3 The subword complex . . . . .	11
2.4 Frobenius splittings . . . . .	13
2.4.1 The trace map . . . . .	14
2.4.2 Frobenius splittings for schemes . . . . .	15
2.5 Bott-Samelson resolutions . . . . .	16
2.6 Geometric vertex decomposition . . . . .	18
<b>3 Degenerating to the toric case</b>	<b>21</b>
3.1 Combinatorics . . . . .	23
3.2 Boundary divisors and Gorenstein toric varieties . . . . .	29
<b>4 Degenerating blow-up algebras</b>	<b>36</b>
<b>5 The blow-up of <math>X^w</math> along <math>\partial X^w</math></b>	<b>42</b>
5.1 Syzygies of products of ideals . . . . .	48
5.2 Viewing syzygies of intersections in terms of products . . . . .	50
5.3 Specializing to bigrassmannians . . . . .	52
<b>6 Understanding the total transform</b>	<b>54</b>
6.0.1 Universal property of blow-ups . . . . .	54
6.1 Extending the frobenius splitting . . . . .	56
6.2 Weakly normal varieties . . . . .	57
6.3 Reducing to $T$ -fixed points . . . . .	59
<b>A Strict Bott-Samelson varieties</b>	<b>61</b>
A.1 Schubert variety singularities . . . . .	63
A.2 Hironaka’s desingularization . . . . .	64
A.3 Comparing Bott-Samelson resolutions to blow-ups . . . . .	66
A.4 Finding strict Bott-Samelson resolutions . . . . .	71
A.4.1 The $n = 5$ case . . . . .	74
A.4.2 The $n = 6$ case . . . . .	75
<b>Bibliography</b>	<b>79</b>



# CHAPTER 1

## INTRODUCTION

Gorenstein varieties provide useful representatives for birational equivalence classes as their canonical (and anticanonical) bundles are invertible. It is therefore useful to find methods to achieve this marked improvement, a so called *Gorensteinization*. Such a Gorensteinization can be difficult to describe, even for toric varieties in general, except in the simplest cases. The fact that Schubert varieties can be locally degenerated to these simple toric cases allows for the existence of a simple Gorensteinization by blowing-up an explicit Weil divisor.

Schubert varieties are a well-studied class of schemes that have useful descriptions which reduce many otherwise difficult operations to simple combinatorics. Let  $G$  be a simple complex Lie group, and fix a maximal torus  $T$  as well as a Borel subgroup  $B$  containing  $T$ . By the Bruhat decomposition,  $B$  acts on  $G/B$  with finitely many orbits indexed by  $W$ , where  $W = N_G(T)/T$  is the Weyl group of  $G$ . The closures of these orbits  $X^w := \overline{BwB/B}$  are called Schubert varieties.

We can define the boundary of  $X^w$  as

$$\partial(X^w) = \bigcup_{v < w} X^v,$$

using  $v$  which are covered by  $w$  in strong Bruhat order. There is a Frobenius splitting on  $X^w$  for which  $\partial(X^w)$  is compatibly split (see [6]). We can localize our question and reduce to blowing-up a Kazhdan-Lusztig variety  $X^w \cap X_v^o$  along its boundary. Here  $\partial(X^w) \cap X_v^o$  is an anticanonical divisor, and it is Cartier if and only if blowing it up is an isomorphism. This makes it useful for detecting the Gorenstein property.

In [19], it was shown that there exists a degeneration from a Kazhdan-Lusztig variety to a Stanley-Reisner scheme associated to a subword complex. Subword complexes were introduced in [20] and provide a geometric way of viewing subwords of a word written in elements of Coxeter group. Here in the toric case, we can identify the boundary divisor as the topological boundary of a convex region in the character lattice, and it is especially easy to visualize the blow-up along this divisor.

**Theorem 1.0.1.** *Let  $Y = \text{Spec}(k[x_1, \dots, x_n]/I)$  where  $I$  is a square-free monomial ideal. Then the blow-up along  $\partial Y$  is Gorenstein.*

In fact, the blow-up of  $Y$  from Theorem 1.0.1 has an exceptional divisor whose reduction is the natural candidate for the boundary divisor of  $\tilde{Y}$ . We make use of the Frobenius splitting on  $Y$  and  $\tilde{Y}$  to determine what an anticanonical divisor for a reducible scheme should be. In fact  $\partial\tilde{Y}$  will be anticanonical under this definition. We check that it is Cartier by observing that the blow-up of  $\tilde{Y}$  along its boundary is an isomorphism. Hence  $\tilde{Y}$  is Gorenstein. Since being Gorenstein is open in flat families (even with this generalized version – see Section 5), the blow-up of the Kazhdan-Lusztig variety is also Gorenstein, proving our main theorem.

**Theorem 1.0.2.** *The blow-up of a Schubert variety  $X^w$  along  $\partial X^w$  is Gorenstein.*

This last implication requires a degeneration of the total transform of the Kazhdan-Lusztig variety to the total transform of the degeneration. Blow-ups however do not in general commute with degenerations. We show that we can choose a Gröbner degeneration of  $X^w \widetilde{\cap} X_v^o$  that commutes with blowing-up along the boundary divisor. This result is a more general tool that defines

a monomial weighting on a blow-up algebra that degenerates to the blow-up algebra of a given degeneration.

**Theorem 1.0.3.** *Let  $>_\lambda$  be a term order on  $S = k[x_1, \dots, x_k]$  defined by an integral weight function  $\lambda : \mathbb{Z}^k \rightarrow \mathbb{Z}$ . Let  $I = \langle g_1, \dots, g_m \rangle \subset J = \langle g_1, \dots, g_{m+n} \rangle$  be two ideals in  $S$  each generated by a Gröbner basis (with respect to  $\lambda$ ). Then there exists an integral weight function  $\tilde{\lambda} : \mathbb{Z}^{n+m+k} \rightarrow \mathbb{Z}$  defining a term order  $>_{\tilde{\lambda}}$  on the blow-up algebra of  $S/I$  along  $J/I$  which degenerates to the blow-up algebra of  $S/\text{init}_{>_\lambda}(I)$  along  $\text{init}_{>_\lambda}(J)/\text{init}_{>_\lambda}(I)$ .*

In summary, we show that the blow-up of a Schubert variety along its boundary is Gorenstein. This Gorensteinization will prove even more useful if we can explicitly describe the total transform  $\widetilde{X}^w$ . On a local level, equations for the blow-up algebra requires an understanding of syzygies. The boundary divisor is the union of Schubert varieties, which are each defined using determinantal polynomial equations. The union is therefore defined by the intersection of these conditions. This local question reduces to understanding what the syzygies are between products of determinants. While the syzygies between the  $k \times k$  minors of a given matrix are well-understood (and defined in terms of determinants – see [25]), it is not known whether the syzygies between products of these minors are also determinantal in nature. It is more likely that finding  $\widetilde{X}^w$  through other means is easier and would shed light on this syzygy problem. It turns out that Bott-Samelson varieties provide a possible remedy.

Bott-Samelson varieties are commonly used as a desingularization for Schubert varieties. Bott-Samelson maps are described in Section 2.5. They have combinatorial properties whose structure is well understood. Using the universal property of blow-up maps, we show that there exists a surjective map from a generalized Bott-Samelson variety  $BS^\mathcal{Q}$  to  $\widetilde{X}^w$ . A topic for future research is to

determine to what extent  $\widetilde{X}^w$  is isomorphic to some piece of  $BS^Q$ . The fact that  $\widetilde{X}^w$  is weakly normal simplifies the problem considerably.

**Proposition 1.0.4.** *A Kazhdan-Lusztig variety, its degeneration to a Stanley-Reisner scheme, and the blow-ups along their respective boundaries are weakly normal.*

This extension of Frobenius splittings to the total transform was studied in [24] when considering blow-ups of smooth varieties along smooth centers. Whether the Frobenius splitting extends in our case is not known, so the result in the proposition uses a different method. The weakly normal property however allows for a critical simplification (using a Zariski's main theorem type of argument).

**Theorem 1.0.5.** *There exists a surjective birational  $B$ -equivariant morphism  $\psi_Q : BS^Q \rightarrow \widetilde{X}^w$  which is an isomorphism iff it is a bijection on  $T$ -fixed points.*

Finally, let us justify why it is worth studying Gorensteinizations in general. We offer two applications which are topics for future study.

**Then Gorenstein Locus of  $X^w$ :** In the  $GL_n$  case, we can describe which  $X^w$  are Gorenstein using pattern interval avoidance. The Gorenstein locus is only conjecturally described in such a way (see [28]). Pattern avoidance conditions for  $X^w$  other than type  $A$  are not known.

The singular locus of  $X^w$  on the other hand is well-understood. One approach is using the quasi-resolutions of Cortez in [7] and [8]. Here a family of these quasi-resolutions  $\pi_i : Y_i \rightarrow X^w$  are chosen, where each  $Y_i$  is similar to the Bott-Samelson construction. The singular locus of  $X^w$  can then be described as

$$\bigcap_i (\pi_i(\text{Sing}(Y_i)) \cup \text{Br}(\pi_i))$$

where  $\text{Br}(\pi_i)$  is the branch locus of  $\pi_i$  and  $\text{Sing}(Y_i)$  is the singular locus of  $Y_i$ .

Using this as motivation, we observe that in our case we have a Gorensteinization  $\pi : \widetilde{X}^w \rightarrow X^w$  for which we hope that the non-Gorenstein locus can be described in terms of the non-Gorenstein locus of  $\widetilde{X}^w$  and the branch locus of  $\pi$ . Of course the former is empty, so understanding the branch locus of  $\pi$  is key (and difficult at this point).

We thank Alexander Woo and Alexander Yong for observing this connection.

**Resolving singularities via Nash Blow-ups:** Given an  $m$ -dimensional quasiprojective variety  $X \subset \mathbb{P}^n$ , we can define a Nash blow-up as the closure of the graph of the Gauss map,

$$X \rightarrow \text{Gr}(m+1, n+1)$$

taking a smooth point to its tangent space. The question is whether repeating this process terminates after finitely many steps (in which case we have resolved the singularities of  $X$ ). The problem remains open for toric varieties, although work has been done on this problem with all experimental results converging in finite time (see [2]).

In the Gorensteinization process of  $X^w$ , we degenerate to the toric case and consider the method of blowing-up boundary divisors until we have an isomorphism. While we don't know whether this process would terminate for toric varieties in general, we develop ideas towards this result in the sections that

follow.

The interesting connection is that this Gorensteinization process is linked to the Nash blow-up problem. We thank John Moody for the observation that the Gorensteinization process converging to a smooth variety implies that the Nash Blow-up process also converges. The convergence of this Gorensteinization process corresponds to a torus equivariant sheaf of finite type that also contains a sheaf which determines the convergence of the Nash blow-up process. These are both subsheaves of the Grauert-Riemenschneider sheaf.

This exciting link between Gorensteinizations for toric varieties and resolving their singularities via Nash blowups is certainly worth pursuing in future research.

## CHAPTER 2

### PRELIMINARIES

For us, a scheme will be separated of finite type over an algebraically closed field  $k$  of characteristic 0, unless specified otherwise. A variety will be an integral scheme.

Let  $G$  be a simple complex Lie group, and fix a maximal torus  $T$  as well as a Borel subgroup  $B$  containing  $T$ . By the Bruhat decomposition,  $B$  acts on  $G/B$  with finitely many orbits indexed by  $W$ , where  $W = N_G(T)/T$  is the Weyl group of  $G$ , and

$$G/B = \bigsqcup_{w \in W} BwB/B.$$

For each  $w \in W$ , let  $X_o^w = BwB/B$ , called a Schubert cell. Then its Zariski closure is the Schubert variety  $X^w$  (note the difference in the notation  $X_w$  used by some other authors). The left  $T$  action on  $X^w$  has finitely many  $T$ -fixed points  $e_v := vB/B$  for  $v \leq w$ . Every point in  $X^w$  is contained in the  $B$ -orbit of some  $e_v$ .

We can define the **boundary** of  $X^w$  using  $v$  which cover  $w$  in strong Bruhat order as

$$\partial X^w = \bigcup_{v < w} X^v.$$

**Example 2.0.1.** In the  $G = GL_4(\mathbb{C})$  case, it is not hard to see that the boundary of  $X^{4231}$  has four components given by

$$\partial X^{4231} = X^{4213} \cup X^{4132} \cup X^{3241} \cup X^{2431}$$

Recall that the dimension is computed by counting the number of inversions in the permutation, so here

$$\dim X^{4231} = 5 \text{ and } \dim \partial X^{4231} = 4.$$

□

Finally, we state a well-known result for reference.

**Lemma 2.0.2.** *The Schubert variety  $X^w$  is Cohen-Macaulay and normal.*

## 2.1 Kazhdan-Lusztig varieties

To get local equations for  $X^w$ , we resort to computing Kazhdan-Lusztig ideals (see [28, §3]) using local coordinates for  $G/B$ . There is an isomorphism between a neighborhood of any point in  $G/B$  with a neighborhood of a  $T$ -fixed point (using the  $B$ -action on  $G/B$ ). Let  $X_w^o := B_w B/B$  denote the opposite Schubert cell. To find local equations for  $X^w$  at  $e_v$ , it is enough to study  $X^w \cap vX_{\text{id}}^o$ , where  $vX_{\text{id}}^o$  is an affine neighborhood of  $e_v$ .

**Lemma 2.1.1** (Kazhdan-Lusztig Lemma).  $X^w \cap vX_{\text{id}}^o \cong (X^w \cap X_v^o) \times \mathbb{A}^{l(v)}$ .

*Proof.* See 3.2 in [28].

□

The  $X^w \cap X_v^o$  are called Kazhdan-Lusztig varieties. We justify the notation  $X^w$  instead of  $X_w$  because of the fact that  $X^w \cap X_v^o \neq \emptyset$  iff  $w \geq v$  (and conventionally posets are drawn with larger elements on the top).



Since  $X^w$  is covered by open sets of the form  $X^w \cap \nu X_{\text{id}}^o$ , it will be enough to study  $X^w \cap X_v^o$  to deduce information about  $X^w$ . We will denote the defining ideal for the Kazhdan-Lusztig variety  $X^w \cap X_v^o$  by  $I_{w,v}$ . A simple topological computation is needed to restrict  $\partial X^w$  to the local case:

**Lemma 2.1.2.**  $\partial((X^w \cap X_v^o) \times \mathbb{A}^{l(v)}) = (\partial X^w \cap X_v^o) \times \mathbb{A}^{l(v)}$ .

We will later see that the divisor class  $[\partial X^w \cap X_v^o] = [-K_{X^w \cap X_v^o}]$ . That is,  $\partial X^w \cap X_v^o$  is an anticanonical divisor for  $X^w \cap X_v^o$ .

**Proposition 2.1.3.** *The blow-up of  $X^w$  along  $\partial X^w$  is Gorenstein iff the blow-up of  $X^w \cap X_v^o$  along  $\partial X^w \cap X_v^o$  is Gorenstein for each  $v \in W$ .*

*Proof.* Being Gorenstein is a local property, so the result can be checked on the open sets  $X^w \cap \nu X_{\text{id}}^o$ . By the Kazhdan-Lusztig Lemma, we can restrict to checking  $(X^w \cap X_v^o) \times \mathbb{A}^{l(v)}$ . Since the blow-up of  $(X^w \cap X_v^o) \times \mathbb{A}^{l(v)}$  along  $(\partial X^w \cap X_v^o) \times \mathbb{A}^{l(v)}$  is just  $(\widetilde{X^w \cap X_v^o}) \times \mathbb{A}^{l(v)}$  (ie. the product of the total transform of  $X^w \cap X_v^o$  with affine space),  $\widetilde{X^w}$  is Gorenstein iff  $(\widetilde{X^w \cap X_v^o})$  is Gorenstein, and the result follows.  $\square$

**Example 2.1.4.** Let us compute local equations for the variety  $X^{53241}$  at  $e_{\text{id}}$ . To do this we need to intersect the Schubert conditions from the permutation matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(which provide defining equations for  $\pi^{-1}(X^{53241})$  where  $\pi : G \rightarrow G/B$ ) with the coordinates from the open cell (which is isomorphic to  $\mathbb{A}^{10}$ )

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ z_{41} & 1 & 0 & 0 & 0 \\ z_{31} & z_{32} & 1 & 0 & 0 \\ z_{21} & z_{22} & z_{23} & 1 & 0 \\ z_{11} & z_{12} & z_{13} & z_{14} & 1 \end{pmatrix}$$

The Schubert conditions say that the southwest-most  $2 \times 3$  matrix has rank  $< 2$ . Therefore the Kazhdan-Lusztig ideal is generated by the three  $2 \times 2$  minors:

$$I_{w,v} = \langle z_{11}z_{22} - z_{12}z_{21}, z_{11}z_{23} - z_{13}z_{21}, z_{12}z_{23} - z_{13}z_{22} \rangle.$$

□

## 2.2 The Gorenstein property

The Gorenstein property is defined using the canonical bundle, so we must take care in discussing this property for singular varieties. Let  $X$  be a normal variety of dimension  $n$  and  $U$  its regular locus. We can define the sheaf of differential forms of degree  $n$  on  $U$  without issues. This sheaf is invertible, and hence is of the form  $\mathcal{O}_U(D)$  for some Cartier divisor  $D$  on  $U$ . Since  $\text{codim}(X \setminus U) \geq 2$ , by normality we can extend  $D$  to a Weil divisor on all of  $X$  which we call a canonical divisor  $K_X$ . A similar definition works for the anticanonical divisor. These divisors are unique up to linear equivalence.

**Definition 2.2.1.** *An normal algebraic variety  $X$  is Gorenstein if  $\omega_X$  is invertible. That is,  $X$  is Gorenstein if  $K_X$  is a Cartier divisor.*

Since Schubert varieties are normal (see Lemma 2.0.2), Kazhdan-Lusztig varieties are also normal, so this definition makes sense for our purposes.

**Example 2.2.2.** In the  $GL_n(\mathbb{C})$  case we can detect which  $X^w$  are Gorenstein using interval pattern avoidance (see [28]). For  $n \leq 5$ , all  $X^w$  are Gorenstein except for

$$X^{53241}, X^{35142}, X^{42513}, X^{52431}.$$

□

Frobenius splittings will be heavily used in later arguments which is why we work with anticanonical divisors instead of canonical ones. This still works for the purposes of detecting the Gorenstein property. We thank Michel Brion for the discussion regarding this topic.

**Lemma 2.2.3.** *Let  $X$  be a normal algebraic variety such that the anticanonical divisor  $-K_X$  is Cartier. Then  $X$  is Gorenstein.*

*Proof.* If  $-K_X$  is Cartier, then there exists a collection  $\{(U_i, f_i)\}$  where the open subsets  $U_i$  cover  $X$  and  $-K_X$  is the divisor of the rational function  $f_i$ . Then  $K_X$  is the divisor associated to  $\{(U_i, \frac{1}{f_i})\}$  and is also Cartier. □

### 2.3 The subword complex

We will see in subsequent sections that a Kazhdan-Lusztig variety  $X^w \cap X_v^o$  can be degenerated to a Stanley-Reisner scheme associated to a subword complex. Subword complexes were first introduced by Knutson and Miller in [21]. Given an ordered list  $Q$  of simple reflections in a Coxeter group  $\Pi$  and a fixed element

$\pi \in \Pi$ , one can ask about what structure can be placed on collection of subwords  $Q$  that are also reduced expressions for  $\pi$ .

**Definition 2.3.1.** *Given a word  $Q = (\sigma_1, \dots, \sigma_m)$  where  $\sigma_i$  a simple reflection in  $\Pi$  and some fixed element  $\pi \in \Pi$ , we can define  $\Delta(Q, \pi)$  to be the set of subwords  $Q \setminus P$  where  $P$  is a subword that contains a subsequence which is a reduced expression for  $\pi$ .*

The subword complex has some useful properties that are worth mentioning here (although we will only be concerned with the case where  $\Pi$  is a Weyl group).

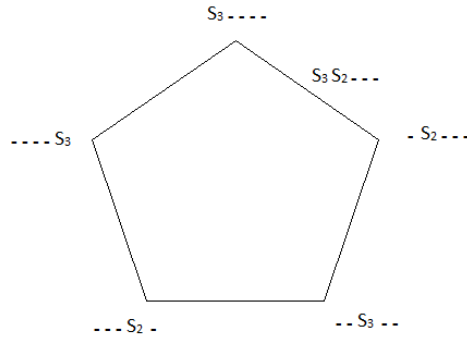
**Proposition 2.3.2.**  *$\Delta(Q, \pi)$  is a pure simplicial complex whose facets are the subwords  $Q \setminus P$  such that  $P \subset Q$  represents  $\pi$ .*

*Proof.* See Lemma 2.2 in [20]. □

**Proposition 2.3.3.** *The subword complex  $\Delta(Q, \pi)$  is either a ball or sphere. A face  $Q \setminus P$  is in the boundary of  $\Delta(Q, \pi)$  iff  $P$  has Demazure product  $\neq \pi$ .*

*Proof.* See Theorem 3.7 in [20]. □

**Example 2.3.4.** Let  $\Pi = S_4$  and consider  $\Delta(s_3 s_2 s_3 s_2 s_3, 1432)$  where  $s_i$  is the simple reflection swapping  $i$  and  $i + 1$  and  $\pi = 1432$  is a permutation. Here  $\pi$  has two reduced expressions in terms of the  $s_i$ , namely  $s_3 s_2 s_3$  and  $s_2 s_3 s_2$ . The facets will correspond to subwords  $P$  of  $Q$  such that  $Q \setminus P$  is a reduced expression for  $\pi$ . For example, the subsequence chosen as the first and second elements of  $Q$  is denoted by  $s_3 s_2 - -$  in the diagram below. Its complement is  $- - s_3 s_2 s_3$  which provides a reduced expression for  $\pi$ .



□

## 2.4 Frobenius splittings

As motivation for the definition below, consider a commutative ring  $R$ , and let us observe that  $R$  is reduced if the map  $x \rightarrow x^n$  only sends 0 to 0 for  $n > 0$ . We would like to write this condition as  $\ker(x \rightarrow x^n) = 0$ , but we cannot because the map is not linear in general. It does however make sense if we restrict to the case that  $n$  is prime and  $R$  contains the field  $\mathbb{F}_p$ , i.e. the map is the Frobenius endomorphism. Then,  $R$  being reduced says that there exists a one-sided inverse to the Frobenius map.

A **Frobenius splitting** of an  $\mathbb{F}_p$ -algebra  $R$  is a map  $\varphi : R \rightarrow R$  which satisfies:

$$(i) \varphi(a + b) = \varphi(a) + \varphi(b)$$

$$(ii) \varphi(a^p b) = a\varphi(b)$$

$$(iii) \varphi(1) = 1$$

An ideal  $I \subset R$  is compatibly split if  $\varphi(I) \subset I$ . See [19] for a more detailed exposition on the subject.

### 2.4.1 The trace map

Let  $R = \mathbb{F}_p[x_1, \dots, x_n]$ . A simple example of a splitting on  $R$  is the **standard splitting** defined on monomials  $m$  (and extended linearly). It is the  $p^{\text{th}}$  root map when  $m$  is a  $p^{\text{th}}$  power and 0 otherwise. The ideals that are compatibly split by the standard splitting are precisely the Stanley-Reisner ideals.

More generally define the trace map  $\text{Tr}(\cdot)$  first on monomials  $m$  (and then extend linearly):

$$\text{Tr}(m) = \begin{cases} \frac{\sqrt[p]{m \prod_i x_i}}{\prod_i x_i} & \text{if } m \prod_i x_i \text{ is a } p^{\text{th}} \text{ power} \\ 0 & \text{otherwise} \end{cases}$$

In general, given  $f \in R$ ,  $\varphi(g) = \text{Tr}(f^{p-1}g)$  defines a near splitting (taking  $f = \prod_i x_i$  gives the standard splitting). If in addition we have  $\text{Tr}(f^{p-1}) = 1$ , then we have a Frobenius splitting. For example,  $xyz$  always defines a splitting on  $\mathbb{F}_p[x, y, z]$  and  $x^2y + z^3 + y^3 + w^7$  defines a splitting on  $\mathbb{F}_7[w, x, y, z]$ .

## 2.4.2 Frobenius splittings for schemes

More generally, let  $X$  be scheme (separated of finite type over an algebraically closed field of characteristic  $p > 0$ ). The absolute Frobenius morphism  $F_X : X \rightarrow X$  is the identity on  $X$  and the  $p^{\text{th}}$  power map on  $\mathcal{O}_X$ . We say that  $X$  is Frobenius split if the  $\mathcal{O}_X$ -linear map  $F^\# : \mathcal{O}_X \rightarrow F_*\mathcal{O}_X$  splits (i.e. there exists a map  $\varphi$  such that  $\varphi \circ F^\#$  is the identity map).

**Lemma 2.4.1.** *Frobenius split schemes are reduced.*

*Proof.* See Proposition 1.2.1 in [6]. □

Just as we used  $(p-1)^{st}$  powers of  $f \in R[x_1, \dots, x_n]$  to define splittings on  $\mathbb{A}^n$ , we can define Frobenius splittings in general using  $(p-1)^{st}$  powers of sections of the anticanonical bundle of  $X$ .

Following Section 1.3 of [6], suppose that  $X$  is smooth with dimension  $n$ . A splitting  $\sigma : F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$  is a global section of the sheaf  $\mathcal{H}om(F_*\mathcal{O}_X, \mathcal{O}_X) = (F_*\mathcal{O}_X)^*$ . Then

$$\begin{aligned}
 H^0(X, (F_*\mathcal{O}_X)^*) &= H^n(X, \omega_X \otimes F_*\mathcal{O}_X)^* \\
 &= H^n(X, \omega_X \otimes F_*F^*\mathcal{O}_X)^* \\
 &= H^n(X, F_*F^*\omega_X)^* \\
 &= H^n(X, \omega_X^p)^* \\
 &= H^0(X, \omega_X^{1-p})
 \end{aligned}$$

For example, nonsingular projective irreducible curves of genus  $g \geq 2$  are not split. See Sections 1.3 and 1.4 in [6] for more details regarding spitting using

$\omega_X^{-1}$  and refer to Section 1.6 for instruction on how to pass from characteristic  $p$  statements to characteristic 0 results.

**Proposition 2.4.2.** *There exists a Frobenius splitting of  $G/B$  which compatibly splits all Schubert subvarieties  $X^w$  and and opposite Schubert subvarieties  $X_w$ .*

*Proof.* See Section 2.3 in [6]. □

**Corollary 2.4.3.** *There is a Frobenius splitting of the Kazhdan-Lusztig variety  $X^w \cap X_v^o$  which compatibly splits  $\partial X^w \cap X_v^o$ .*

*Proof.* The intersection and union of compatibly split subvarieties is again compatibly split, so the result follows by Proposition 2.4.2. □

## 2.5 Bott-Samelson resolutions

Let  $Q = (w_1, \dots, w_k)$ , where  $w_i \in W$ . A generalized Bott-Samelson variety  $BS^Q$  is the quotient of  $\overline{Bw_1B} \times \dots \times \overline{Bw_kB}$  by the  $B^k$  action given by:

$$(b_1, \dots, b_k) \cdot (p_1, \dots, p_k) := (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \dots, b_{k-1} p_k b_k^{-1}).$$

If we let  $\times^B$  denote the quotient by the above action for the  $k = 2$  case, we can write  $BS^Q$  as  $\overline{Bw_1B} \times^B \dots \times^B \overline{Bw_kB}/B$ .



We can define  $\text{Dem}(Q)$  inductively as

$$\text{Dem}((Q, s_i)) = \begin{cases} \text{Dem}(Q) \cdot s_i & l(\text{Dem}(Q \cdot s_i)) > l(\text{Dem}(Q)) \\ \text{Dem}(Q) & \text{otherwise} \end{cases}$$

where the Demazure product of the empty word is the identity. Then the  $B$ -equivariant map

$$\varphi_Q : \overline{Bw_1B} \times^B \dots \times^B \overline{Bw_kB} / B \rightarrow G/B$$

defined by  $(p_1, \dots, p_k) \rightarrow p_1 \cdot \dots \cdot p_k B/B$  maps onto  $X^{\text{Dem}(Q)}$ .

Sub Bott-Samelson varieties  $BS^R \subset BS^Q$  are naturally defined by taking  $R = (v_1, \dots, v_k)$  where  $v_i \leq w_i$ .

**Lemma 2.5.1.** *Let  $Q = (w_1, \dots, w_k)$  where  $w_i$  is a simple reflection  $s_{k_i} \in W$ . Then  $BS^Q$  is smooth.*

*Proof.* As seen in [10]: In this case,  $\overline{Bw_iB} \cong P_{k_i}$  is a minimal parabolic subgroup of  $G$ . Then  $BS^Q$  is just an iterated  $\mathbb{P}^1$ -bundle and  $\varphi_Q$  defines a resolution of singularities for  $X^w$ . □

See [13] for an example of how to visualize the  $BS^Q$  in Lemma 2.5.1 in the context of flag varieties. Although  $BS^Q$  is not smooth in general, it is Cohen-Macaulay and normal.

**Lemma 2.5.2.** *The Bott-Samelson variety  $BS^Q$  is both Cohen-Macaulay and normal.*

*Proof.* As seen in [10]: Using the projection map

$$\overline{Bw_1B} \times^B \dots \times^B \overline{Bw_kB}/B \rightarrow \overline{Bw_1B} \times^B \dots \times^B \overline{Bw_{k-1}B}/B$$

it is easy to see that  $BS^Q$  is just an iterated Schubert variety bundle over  $X^w$ . It is well known that  $X^w$  is both Cohen-Macaulay and normal. Since both of these conditions are local, and since the bundle map above is locally trivial, it follows by induction that  $BS^Q$  also has these properties.  $\square$

**Lemma 2.5.3.** *Let  $\varphi_Q$  be a Bott-Samelson map with  $Q = (w_1, \dots, w_k)$ . Then  $\varphi_Q$  is always a proper map. Furthermore,  $\varphi_Q$  is birational if and only if  $\text{Dem}(Q) = \prod w_i$  (in which case we say  $Q$  is reduced).*

*Proof.* As seen in [10]: Since  $BS^Q$  and  $X^w$  are projective, they are both proper over  $\mathbb{C}$  so  $\varphi_Q$  must be proper also. The second statement follows from the fact that  $\varphi_Q$  is an isomorphism away from the sub Bott-Samelson varieties  $BS^R$  where  $R$  is not reduced. Then  $Q$  is reduced iff  $R \neq Q$ .  $\square$

## 2.6 Geometric vertex decomposition

Before we describe the degeneration of  $X^w \cap X_v^o$  to the Stanley-Reisner scheme, let us define the **geometric vertex decomposition** (see the work of Knutson in [18] and [19]).

We start with an integral variety  $X \subset \mathbb{A}^n$ . Let us write  $\mathbb{A}^n = H \times L$ , with  $H$  a hyperplane and  $L$  a line. Consider the action of  $\mathbb{G}_m$  on  $\mathbb{A}^n$  by

$$z \cdot (x, l) = (x, zl)$$

and let us define

$$X' = \lim_{t \rightarrow 0} t \cdot X.$$

Here  $X'$  is a limit scheme. That is, it is the zero fibre of

$$\overline{\bigcup_{z \in \mathbb{G}_m} z \times (z \cdot X)} \subset \mathbb{A}^1 \times (H \times L).$$

There is a simple way to compute  $X'$  set-theoretically. Let  $\Pi \subset H$  be the closure of the image of the projection of  $X$  to  $H$ . We will denote the closure of  $X$  in  $H \times (L \cup \{\infty\})$  by  $\bar{X}$ . Finally, define  $\Lambda \subset H$  by  $\Lambda \times \{\infty\} = \bar{X} \cap (H \times \{\infty\}) = \bar{X} \setminus X$ .

**Lemma 2.6.1.** *The limit scheme  $X'$  of the geometric vertex decomposition can be described as a set by*

$$X' = (\Pi \times \{0\}) \cup_{\Lambda \times 0} (\Lambda \times L).$$

**Example 2.6.2.** *Let us write  $\mathbb{A}^2 = H \times L$  where  $H$  is the  $x$ -axis and  $L$  is the  $y$ -axis. Consider the variety  $X$  defined by  $xy - 1 = 0$ . The closure of the projection of  $X$  onto  $H$  is exactly  $H$ . Similarly, it is not hard to see that  $\Lambda \times L$  is the  $y$ -axis, and by the Lemma,  $X'$  is the union of the two axes.  $\square$*

This definition generalizes if we take  $H$  to be some general scheme, as long as  $L$  remains  $\mathbb{A}^1$ .

The degeneration that appears in the next section can be shown inductively by using geometric vertex decomposition applied to a Kazhdan-Lusztig variety  $KL_1$  where  $H$  can be chosen so that  $\Pi = KL_2$  and  $\Lambda = KL_3$  are two other Kazhdan-Lusztig varieties. More precisely, let  $X_w|_v := X_w \cap (vN_-B_+/B_+)$  where  $vN_-B_+/B_+$  is the permuted big cell (called a Schubert patch on  $X_w$ ).

**Theorem 2.6.3** (Theorem 2, [18]). *Let  $v, w$  be elements of  $W$ . If  $X_{w|v} \neq \emptyset$ , then  $v \geq w$  in the Bruhat order. Assume this hereafter.*

*If  $v = 1$ , then  $w = 1$  and  $X_{w|v} = N_-B_+/B_+$ . Otherwise, there exists a simple root  $\alpha$  such that  $vr_\alpha < v$  in the Bruhat order. Let  $X'$  be the degeneration of  $X_{w|v}$  described above.*

- *If  $wr_\alpha > w$ , then  $X' = X_{w|v}$  (the limiting process is trivial), and*

$$X_{w|v} \cong \Pi \times \mathbb{A}_{-v\cdot\alpha}^1, X_{w|vr_\alpha} \cong \Pi \times \mathbb{A}_{v\cdot\alpha}^1$$

*for the same  $\Pi$ .*

- *If  $wr_\alpha < w$  but  $w \not\leq vr_\alpha$ , then  $X' = X_{w|v}$  (again, the limiting process is trivial), and*

$$X_{wr_\alpha|vr_\alpha} \cong X_{w|v} \times \mathbb{A}_{v\cdot\alpha}^1.$$

- *If  $wr_\alpha < w \leq vr_\alpha$ , then  $X'$  is reduced, and has two components:*

$$X' = (\Pi \times 0) \cup_{\Lambda \times \{0\}} (\Lambda \times \mathbb{A}_{-v\cdot\alpha}^1)$$

*where  $\Pi \times \mathbb{A}_{v\cdot\alpha}^1 \cong X_{wr_\alpha|vr_\alpha}$  and  $\Lambda \times \mathbb{A}_{v\cdot\alpha}^1 \cong X_{w|vr_\alpha}$ .*

Simply put, we can degenerate  $KL_1$  as:

$$KL_1 \rightsquigarrow KL_2 \cup_{0 \times KL_3} (\mathbb{A}^1 \times KL_3).$$

We want to emphasize that this limit is in fact reduced. This inductive version of the degeneration can be found in greater detail in [18] and [19].

## CHAPTER 3

### DEGENERATING TO THE TORIC CASE

We will now consider the degeneration of the Kazhdan-Lusztig variety  $X^w \cap X_v^o$  to a Stanley-Reisner scheme. To give a picture of what will occur later in this section, we provide a motivational example.

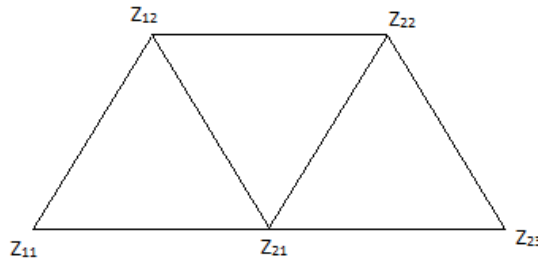
**Example 3.0.1.** In the  $GL_n(\mathbb{C})$  case, consider  $X^{53241} \cap X_{12345}^o$ . Then by Example 2.1.4 we have

$$I_{w,v} = \langle z_{11}z_{22} - z_{12}z_{21}, z_{11}z_{23} - z_{13}z_{21}, z_{12}z_{23} - z_{13}z_{22} \rangle$$

Consider the term order where  $z_{11} > z_{21} > \dots > z_{12} > \dots > z_{55}$ . Then

$$\text{init}(I_{w,v}) = \langle z_{11}z_{22}, z_{11}z_{23}, z_{12}z_{23} \rangle$$

This is the Stanley-Reisner ideal of the subword complex  $\Delta(Q, 12345)$  (the vertex  $z_{13}$  is a cone vertex and has not been drawn below).



Note: The fact that these “Woo-Yong” coordinates in [28] are equivalent to Bott-Samelson coordinates in this case follows from [29].  $\square$

**Theorem 3.0.2.** *Given  $\varphi_Q : BS^Q \rightarrow G/B$ , consider  $\varphi_Q^{-1}(X_w) \subset \mathbb{A}^{|Q|}$  and denote its ideal by  $I_w$ . Then  $J_w := \text{init}(I_w)$  is a squarefree monomial ideal. Its prime components are coordinate ideals  $C_F$  where  $Q \setminus F$  is a reduced word for  $w$ .*

*Proof.* See [19, §7.3].  $\square$

This theorem is stated in terms of  $X_w \cap X_v$  with  $Q$  a reduced word for  $v$ . Multiplying by the long Weyl group element  $w_0$  gives us our form of Kazhdan-Lusztig variety introduced in Section 2.1. We therefore denote the simplicial complex associated to  $I_{w,v}$  by  $\Delta(Q, w)$  where  $Q$  is reduced word for the permutation  $v$ . This is the **subword complex** introduced in Section 2.3

**Theorem 3.0.3.** *The subword complex  $\Delta(Q, w)$  is homeomorphic to a ball and is shellable. Its boundary sphere is*

$$\bigcup_{w' \succ w} \Delta(Q, w').$$

*We will denote its defining ideal by  $\partial I_{w,v} := \bigcap_{w' \succ w} I_{w',v}$ .*

*Proof.* See [20].  $\square$

**Corollary 3.0.4.** *Properties of  $I_{w,v}$ :*

- *Each  $I_{w,v}$  is Cohen-Macaulay.*
- *Each  $I_{w,v}$  is normal and  $\partial I_{w,v}$  defines an anticanonical divisor.*
- *$I_{w,v} = \bigcap_{w' \succ w, \text{biGrassmannian}} I_{w',v}$ . The concatenation of Grobner bases for the maximal biGrassmannians  $\leq w$  provides a Grobner basis for  $I_{w,v}$ .*

- Consider the poset morphism  $m : 2^Q \rightarrow W$  given by  $F \rightarrow \text{Dem}(Q \setminus F)$ . Then  $\Delta(Q, w) = m^{-1}([w, w_0])$ . That is,  $m$  defines a “Bruhat decomposition for  $\Delta(Q, w)$ ”.

*Proof.* A combination of results from [19] and [20]. □

### 3.1 Combinatorics

We wish to understand what happens when we blow-up the toric scheme (coming from the degeneration) along its boundary.

First recall that a normal (irreducible) variety containing a torus  $T \cong (\mathbb{G}_m)^n$  as an open subset, where the action of  $T$  on itself extends to an action of  $X$  is called a toric variety. The **boundary** of  $X$  is then defined as  $\partial X = X \setminus T$ . Denote the coordinate ring of  $X$  by  $R[X] = k[x_1, \dots, x_n]/I_X$ . To each affine toric variety  $X$  we can associate a convex rational polyhedral cone  $\sigma_X$  contained in the character lattice  $M_{\mathbb{R}}$ . This cone defines a saturated affine semigroup  $S_{\sigma_X}$  such that  $X = \text{Spec}(k[S_{\sigma_X}])$  (we are using the notation in [9, §1]). The irreducible components  $D_1, \dots, D_n$  of  $\partial X$  are defined as the vanishing of a primitive character in  $M_{\mathbb{R}}$  and correspond to the facets of  $\sigma_X$ . Here  $\partial X$  is the anticanonical divisor (up to linear equivalence). See [9] for more information about the correspondence between  $X$  and a cone or polytope in the character lattice.

A blow-up of an affine  $X$  along a (reduced) torus invariant subvariety of  $\partial X$  can be viewed in the character lattice as “planing off” the corresponding faces of  $\sigma_X$ . That is, it can be viewed by taking the convex hull of the lattice points that remain after removing components of the boundary. For now we will consider the combinatorics of such an operation and save the geometric questions for the

next section.

We will be working with reducible toric schemes which can be associated to convex polytopal complexes. A **lattice polytopal complex**  $\Delta$  is a union of lattice polytopes such that the intersection of any two is also a lattice polytope contained in  $\Delta$ . The boundary  $\partial\Delta$  of  $\Delta$  is the union of polytopes which form the boundary of the convex region  $\Delta$ . Affine toric schemes can then be associated to the cone on  $\Delta$ , denoted by  $\text{Cone}(\Delta)$  or  $C(\Delta)$ .

Just as in the irreducible case, the **blow-up of a lattice polytopal complex**  $\Delta$  along a subcomplex  $\Delta' \subset \partial\Delta$  is the polytopal complex  $\widetilde{\Delta}$  resulting from taking the convex hull of the lattice points in  $\Delta \setminus \Delta'$ .

We need to show that this operation in the character lattice  $M_{\mathbb{R}}$  is indeed the correct picture to visualize the Rees algebra  $R[It] = \bigoplus_{k \geq 0} I^k t^k$  for the blow-up of  $X$  along the reduced subscheme  $(\partial X)'$  (corresponding to  $\Delta$  and  $\Delta'$  respectively).

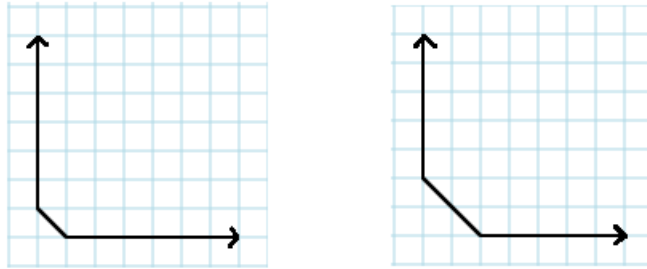
**Lemma 3.1.1.** *Let  $X$  be an affine toric scheme associated to a polyhedral complex  $\Delta \subset M_{\mathbb{R}}$  (so that  $X = \text{Spec}(k[S_{C(\Delta)}])$ ). Then the blow-up of  $X$  along a torus invariant subscheme  $(\partial X)' \subset \partial X$  is again a toric scheme that can be associated to the blow-up of  $\Delta$  along  $\Delta'$ .*

*Proof.* Let  $I$  be the defining ideal for  $(\partial X)'$ . The blow-up algebra  $R[It]$  is generated by  $I$  in degree one over  $R[X]$ . In the character lattice,  $R[X]$  can be viewed as  $C(\Delta)$  and  $I$  as  $C(\widetilde{\Delta})$ . We observe that the weights of  $I^k$  scaled by  $\frac{1}{k}$  form the same cone  $C(\widetilde{\Delta})$  in degree one (except using a denser lattice). Therefore it suffices to compute  $\widetilde{X}$  using the weights in the blow-up of  $\Delta$  along  $\Delta'$ . In fact  $\widetilde{X} = \text{Proj}(k[S_{C(\widetilde{\Delta})}])$ .  $\square$



After the first blow-up, our scheme is no longer affine. To continue viewing the blow-up as in Lemma 3.1.1, we just need to cover the total transform with affine charts and patch together the results.

**Example 3.1.2.** The blow-up of  $\mathbb{C}^2$  at the origin can be viewed in the character lattice as the first region in the figure below. The origin of  $\mathbb{C}^2$  corresponds to the origin of the first quadrant. After removing it and taking the convex hull of the lattice points that remain, we are left with the first picture below corresponding to  $I = \langle x, y \rangle$ . We can obtain the diagram for  $I^2$  by planing off the diagonal an extra step inward (the second diagram below). This new diagram scaled by  $\frac{1}{2}$  is the original one with a denser lattice.



□

Given the cone on  $\Delta$ , let  $\pi^1(C(\Delta)) = \text{Conv}(C(\Delta) \setminus \partial C(\Delta))$ . We can perform the same operation on  $\pi^1(C(\Delta))$  by taking the convex hull after removing the boundary, and we will denote this by  $\pi^2(C(\Delta))$ . Continuing this process, we have the following sequence:

$$\dots \rightarrow \pi^3(C(\Delta)) \rightarrow \pi^2(C(\Delta)) \rightarrow \pi^1(C(\Delta)) \rightarrow \pi^0(C(\Delta)) = C(\Delta)$$

We will say that  $C(\Delta)$  is **stable** if there exists a  $k > 0$  such that the toric scheme

associated to  $\pi^k(C(\Delta))$  is isomorphic to the toric scheme associated to  $\pi^{k-1}(C(\Delta))$ . In this case we will simply say that  $\pi^k(C(\Delta))$  and  $\pi^{k-1}(C(\Delta))$  are **equivalent**.

The most basic simplicial complex, just a single standard simplex, stabilizes very quickly, even if we only remove a portion of the boundary. This single example is the only case we actually need to extend the result to Stanley-Reisner complexes.

**Lemma 3.1.3.** *Let  $\Delta$  be the standard simplex in  $\mathbb{A}^n$ . Let  $\Delta'$  be some proper simplicial subcomplex of  $\Delta$ . Let  $\tilde{\Delta} = \text{Conv}(C(\Delta) \setminus C(\Delta'))$  and let  $\tilde{\Delta}' \subset \tilde{\Delta}$  be the exceptional facets (those facets in  $\tilde{\Delta}$  not originally in  $\Delta$ ). Then the blow-up of  $\tilde{\Delta}$  along  $\tilde{\Delta}'$  is equivalent to  $\tilde{\Delta}$ .*

*Proof.* First consider the case that  $\Delta'$  is just one face of  $\Delta$ . We may assume by a change of coordinates that it is defined by

$$\{x_1 = \dots = x_k = 0\} \cap \Delta \text{ for } 1 < k < n$$

Taking the convex hull of  $C(\Delta) \setminus C(\Delta')$  is equivalent to intersecting  $C(\Delta)$  with the half space  $H_1^+ = \{x_1 + \dots + x_k \geq 1\}$  (since this has removed the portion where  $\{x_1 = \dots = x_k = 0\}$  leaving only lattice points where at least one  $x_i > 0$ ). The new facet of the boundary  $\tilde{\Delta}'$  consists of the polytope  $C(\Delta) \cap \{x_1 + \dots + x_k = 1\}$ .

Blowing-up once more would result in a region given by intersecting  $C(\Delta)$  with half space  $H_2^+ = \{x_1 + \dots + x_k \geq 2\}$  and boundary defined by  $x_1 + \dots + x_k = 2$ .

It is clear that both correspond to isomorphic copies (using a Veronese map giving different embeddings) of the blow-up of  $\mathbb{A}^n$  along  $\{x_1 = \dots = x_k = 0\}$ , hence we have stability.

The case for general  $\Delta'$  easily follows. □

The simplicial complexes we will be concerned with are those that come from Stanley-Reisner ideals. A **Stanley-Reisner complex** is a simplicial complex whose face ideal is generated by square-free monomials. The advantage to working with such complexes is that their simplices are stable.

**Lemma 3.1.4.** *Let  $I \subset k[x_1, \dots, x_n]$  be an ideal generated by square-free monomials. Then the affine semigroup  $S_{C(\Delta_I)}$  in the character lattice  $M_{\mathbb{R}}$  associated to the reducible affine toric scheme*

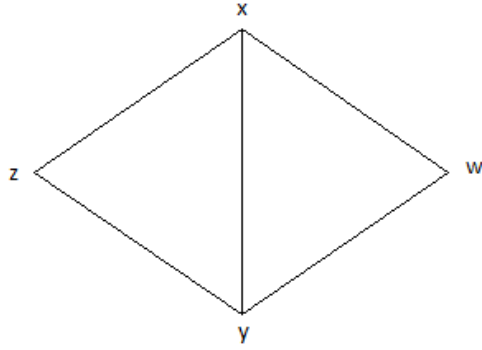
$$X = \text{Spec}(k[x_1, \dots, x_n]/I)$$

*is the cone on the Stanley-Reisner complex  $\Delta_I$ . Furthermore, each simplicial cone in  $C(\Delta_I)$  is isomorphic to the cone on a standard simplex.*

*Proof.* Since  $I$  is a square free monomial ideal, each component of  $I$  corresponds to a coordinate subspace of  $\mathbb{A}^n$ . It is an easy exercise to check that the cones in  $M_{\mathbb{R}}$  corresponding to each component are isomorphic to some orthant  $\mathbb{A}^k$ , which is the cone on some standard simplex in the appropriate coordinates. The gluing of these simplices must occur along faces which are not in the face ideal  $I$ .

□

**Example 3.1.5.** Consider  $\mathbb{C}[w, x, y, z]/\langle wz \rangle$ . There are two components in this affine scheme defined by  $w = 0$  and  $z = 0$ . The  $w = 0$  component is given by  $\text{Spec}(\mathbb{C}[x, y, z])$  which corresponds to the orthant  $\mathbb{C}_{\geq 0}^3$  in the character lattice. A similar statement holds for the  $z = 0$  component. Since  $wz = 0$ , adding the generator of the  $w$ -axis and the generator for the  $z$ -axis results in 0, a relation demonstrated by the gluing of one facet from each simplex (corresponding to the gluing along the  $xy$ -plane).

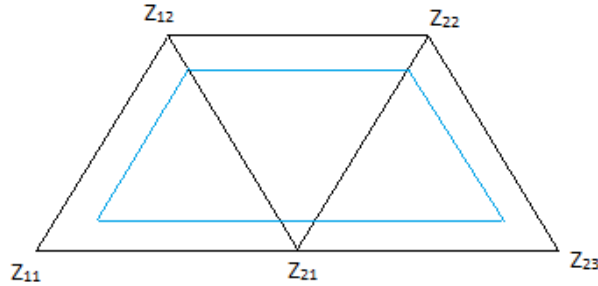


□

**Theorem 3.1.6.** *Let  $\Delta_I$  be as in Lemma 3.1.4. Then the blow-up of  $\Delta_I$  along  $\partial\Delta_I$  is stable. Even more,  $\pi^2(C(\Delta_I)) = \pi^1(C(\Delta_I))$ .*

*Proof.* It suffices to show that each simplex  $\sigma \in \Delta_I$  blown up along  $\sigma \cap \partial\Delta_I$  is stable. By Lemma 3.1.4, each simplex  $\sigma$  is isomorphic to the standard simplex. Therefore the result holds by Lemma 3.1.3. □

**Example 3.1.7.** We observe that the subword complex from Example 3.0.1 is stable. Removing the boundary and taking the convex hull would produce the blue polytopal complex. It is clear that doing this process once more produces an isomorphism.



□

### 3.2 Boundary divisors and Gorenstein toric varieties

In Section 3.1, we showed that blowing-up a Stanley-Reisner scheme along the boundary twice yields an isomorphism. Let us recall the following fact that follows immediately from the universal property of blow-ups (see Section 6.0.2):

**Lemma 3.2.1.** *The blow-up of a scheme  $X$  along some closed subscheme  $Y$  is an isomorphism iff  $Y$  is an effective Cartier divisor of  $X$ .*

This means that the (reduced) boundary divisor is Cartier after the first blow-up in the sequence described above.

**Proposition 3.2.2.** *Let  $X$  be a Stanley-Reisner scheme, and consider the sequence*

$$X = X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} X_2$$

where  $\pi_i$  is the blow-up of  $X_i$  along the reduction of the boundary  $\partial X_i$  and  $X_1$  and  $X_2$  are the total transforms. Then  $\partial X_1$  is an effective Cartier divisor.

*Proof.* By Theorem 3.1.6, the polytopal complexes  $\Delta_1$  and  $\Delta_2$  associated to  $X_1$  and  $X_2$  are equivalent. Therefore  $X_1$  is isomorphic to  $X_2$ . By Lemma 3.2.1,  $\partial X_1$  is a Cartier divisor.  $\square$

Note that the process in Section 3.1 defines the boundary  $\partial X_1$  as the reduction of the exceptional divisor. The components of the exceptional divisor could a priori have multiplicity  $> 1$ , which means that the blow-up along  $\partial X_1$  would correspond to a weighted removal of the boundary followed by taking the convex hull of what remains. While the exceptional divisor is Cartier, its reduction might not be, and in general this might not produce an isomorphism like in Proposition 3.2.2.

In the Stanley-Reisner case however, the reduced boundary divisor of the total transform is Cartier, but it is not immediately clear that it is *the* boundary divisor in the sense that we want it to be. That is, we do not know whether  $\partial X_1$  is anticanonical. We will use Frobenius splittings to guide our understanding of anticanonical for reducible schemes. First note that irreducible normal toric varieties are easy to Frobenius split.

**Lemma 3.2.3.** *Let  $X$  be a toric variety (irreducible and normal). Let  $t_1, \dots, t_n$  be the coordinates on the torus  $T$  coming from  $\mathbb{G}_m$ . Then*

$$\sigma = \frac{dt_1 \wedge \dots \wedge dt_n}{t_1 \dots t_n}$$

is a rational section of  $\omega_X$  and  $\sigma^{1-p}$  defines the unique  $T$ -invariant splitting of  $X$  that compatibly splits  $\partial X$ .

*Proof.* See Exercise 1.3.6 in [6]. □

In fact weakly normal toric schemes (see Section 6.2) are also Frobenius split by the standard splitting. Furthermore, the blow-up of a weakly normal scheme along a weakly integrally closed ideal is again weakly normal. This is enough to show that the total transform is again Frobenius split. A second proof is provided below, one that is easier to visualize.

**Proposition 3.2.4.** *Let  $X$  be a weakly normal affine toric scheme. Then the blow-up of  $X$  along  $\partial X$  is also Frobenius split.*

*Proof.* Weakly normal toric schemes are Frobenius split by the standard splitting. Since  $X$  is affine, there is a polyhedral complex  $\Delta$  such that  $\text{Cone}(\Delta)$  defines an affine semigroup in the character lattice  $M_{\mathbb{R}}$  associated to  $X$ .

One can check that  $X$  is Frobenius split by the standard splitting iff the following condition holds:

$$(pa_1, \dots, pa_n) \in \text{Cone}(\Delta) \cap \mathbb{Z}^n \Rightarrow (a_1, \dots, a_n) \in \text{Cone}(\Delta) \cap \mathbb{Z}^n$$

In other words, the standard splitting acts as the multiplication by  $\frac{1}{p}$  map on lattice points.

The blow-up of  $X$  along  $\partial X$  can be viewed in the character lattice by taking the convex hull of the lattice points in  $\text{Cone}(\Delta) \setminus \text{Cone}(\partial(\Delta))$  by Section 3.1. Suppose the total transform was not Frobenius split by the standard splitting. Then there would be a lattice points  $L = (pa_1, \dots, pa_n) \in \widetilde{\text{Cone}}(\Delta)$  such that  $\frac{1}{p}L \notin \widetilde{\text{Cone}}(\Delta)$ . Since  $\frac{1}{p}L \in \text{Cone}(\Delta)$ , we conclude that  $\frac{1}{p}L \in \text{Cone}(\partial(\Delta))$ . But then  $L \in \text{Cone}(\partial(\Delta))$  (since it is a cone) which means that  $L$  couldn't be in  $\widetilde{\text{Cone}}(\Delta)$ , a contradiction.  $\square$

**Corollary 3.2.5.** *Let  $X$  be a Stanley-Reisner scheme. Then  $X$  is Frobenius split by the standard splitting.*

*Proof.* The components of  $X$  and the components of their intersections are coordinate subspaces, so each component is normal and each split by the standard splitting.  $\square$

In the (irreducible) toric variety case, the blow-up along the boundary is Frobenius split by the standard splitting. By Section 2.4, the standard splitting is defined using the unique toric invariant section of the anticanonical bundle, which has to be the boundary divisor. In this way we know that the boundary of the total transform is anticanonical.

We are dealing with reducible toric schemes however, so we need to take greater care in how we show that the boundary divisor is anticanonical. We have shown that the total transform of a Frobenius split toric variety is again split. This Frobenius splitting however does not necessarily come from a section of the anticanonical bundle as in Section 2.4. At this point we need to decide how we want to define the anticanonical divisor for reducible schemes.

As motivation, recall that by Theorem 3.0.3 we know that the  $\partial X^w \cap X_o^v$  de-



generates to the boundary of a subword complex. We would like to say that the anticanonical divisor  $\partial X^w \cap X_v^o$  has degenerated to an anticanonical divisor. Consider Example 3.1.5. In this case, we don't want to include the interior facet corresponding to the gluing of two orthants along the coordinate subspace  $w = z = 0$ . Let us also adopt the philosophy that anticanonical sections should tell us what is Frobenius split. We know that  $w = z = 0$  is Frobenius split if  $w = 0$  and  $z = 0$  are, further showing that interior components used for gluing shouldn't be included in our definition of anticanonical.

By Section 2.2, we know how to define the anticanonical bundle for normal varieties. Given a reduced equidimensional algebraic variety  $X$ , let us denote its normalization by  $\nu : \bar{X} \rightarrow X$ . In this normalization process, irreducible components of  $X$  have become disjoint. Suppose also that the irreducible components of  $X$  are normal. Then the ramification locus  $R \subset \bar{X}$  of  $\nu$  is just the fibre over the intersection of the components (in our previous example,  $w = z = 0$ ). This ramification locus together with the components coming from the boundary of  $X$  should be anticanonical (in the usual sense) in  $\bar{X}$ . This motivates the following definition:

**Definition 3.2.6.** *Let  $X$  be a reduced equidimensional scheme whose irreducible components are normal. Let  $\nu : \bar{X} \rightarrow X$  be its normalization and  $R \subset \bar{X}$  the ramification locus. A divisor  $D \subset X$  is called **anticanonical** if:*

- $(\nu^{-1}(D) \cup R) \cap \bar{X}_{reg}$  is anticanonical in  $\bar{X}_{reg}$
- $\text{codim}(\nu^{-1}(D) \cap R) > 1$ .

The last condition ensures that invariant toric divisors in a toric scheme  $X$  are

either part of the branch locus or part of the anticanonical divisor of  $X$  (indeed it forces the branch locus to be a divisor). We get the following immediate results from the definition:

**Lemma 3.2.7.** *Let  $X$  be a reduced equidimensional toric scheme associated to the polytopal complex  $\Delta$ . Suppose that  $\Delta$  is homeomorphic to a ball or sphere. Also assume that the components of  $X$  are normal. Then  $\partial\Delta$  is anticanonical.*

*Proof.* Let  $\nu : \bar{X} \rightarrow X$  be the normalization map. Since the components of  $X$  are normal, the ramification locus  $R$  of  $\nu$  is just the fibre over facets of  $\Delta \setminus \partial\Delta$ . Then  $R \cup \nu^{-1}(\partial\Delta)$  is just the union of the toric invariant divisors corresponding to the boundary of each irreducible component, so is anticanonical in  $\bar{X}$  by 3.2.3.

Since  $\Delta$  is a ball or sphere, the intersection of the branch locus and  $D$  has codimension  $> 1$ . □

**Corollary 3.2.8.** *Let  $X$  be an affine toric scheme associated to the subword complex  $\Delta = \Delta(Q, w)$ . Then  $\partial X$  is anticanonical.*

*Proof.* By Section 2.3,  $\Delta$  is a ball or sphere. □

**Proposition 3.2.9.** *Let  $X$  be as in Lemma 3.2.8. Then the boundary divisor of the blow-up of  $X$  along  $\partial X$  is an anticanonical divisor.*

*Proof.* Each irreducible component in the total transform remains normal. Indeed the affine semigroup  $S$  in the character lattice corresponding to each component remains saturated. Seen another way, the blow-up of a normal scheme along an integrally closed ideal is again normal. Each component was a coordinate subspace, and their boundaries and intersections were again coordinate subspaces.

It is clear that the blow-up of  $\Delta$  along  $\partial\Delta$  remains a ball or sphere.  $\square$

**Corollary 3.2.10.** *Let  $X$  be a Stanley-Reisner scheme. The blow-up of  $X$  along its boundary  $\partial X$  is Gorenstein.*

*Proof.* By Proposition 3.2.9, the boundary divisor of the blow-up is an anticanonical divisor. By Proposition 3.2.2, it is also an effective Cartier divisor. By Lemma 2.2.3, the blow-up is Gorenstein.  $\square$

We have shown that the blow-up of the degeneration is Gorenstein. What remains is to show that the blow-up of  $X^w \cap X_v^o$  along  $\partial X^w \cap X_v^o$  can be degenerated to this Gorenstein variety. This is the content of the next section.

CHAPTER 4  
DEGENERATING BLOW-UP ALGEBRAS

To understand under what conditions the degeneration of the Rees algebra is the Rees algebra of the degeneration, we utilize some well-known results involving Gröbner bases and syzygies. Given a finite set of  $\{p_1, \dots, p_r\} \subset S = k[x_1, \dots, x_k]$ , we define

$$\text{Syz}(\{p_i\}) = \left\{ \sum a_i \epsilon_i \in \langle \epsilon_1, \dots, \epsilon_r \rangle \leq S[\epsilon_1, \dots, \epsilon_r] : \sum (a_i \epsilon_i)|_{\epsilon_i=p_i} = 0 \right\}.$$

When dealing with syzygies, we usually require the coefficients  $a_i \in S$ , which is why we talk about syzygy modules over  $S$ , viewing the relations as the kernel of a map  $S^r \rightarrow S$ . In our case it is easier to deal with ideals, so we allow the coefficients  $a_i$  to be in  $S[\epsilon_1, \dots, \epsilon_r]$ . Note that  $\text{Syz}(\{p_i\})$  is just the ideal generated by the first syzygy equations (see Lemma 4.0.2).

The first lemma rewrites the Rees algebras we are concerned with in a convenient form to utilize results about syzygies.

**Lemma 4.0.1.** *Let  $I = \langle g_1, \dots, g_m \rangle \subset J = \langle g_1, \dots, g_{m+n} \rangle$  be two ideals in  $S = k[x_1, \dots, x_k]$ . Denote  $S/I$  and  $J/I$  by  $\bar{S}$  and  $\bar{J}$  respectively. Then*

$$\bar{S}[t\bar{J}] \cong S[\epsilon_1, \dots, \epsilon_{m+n}]/(I + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{g_i\})).$$

*Proof.* Consider the map

$$\varphi : S[\epsilon_1, \dots, \epsilon_{m+n}] \rightarrow \bar{S}[t\bar{J}]$$

defined on generators by  $\epsilon_i \rightarrow t\bar{g}_i$  (note that  $tI \leq tJ$ ). Let us compute  $\ker(\varphi)$ .

Given  $f \in S[\epsilon_1, \dots, \epsilon_{m+n}]$ , we can write  $f = h_0 + \dots + h_k$  using the standard grading on  $S[\epsilon_1, \dots, \epsilon_{m+n}]$ , generated in degree one by the  $\epsilon_i$ . It then suffices to show the result for  $f = f_0 + \sum_{i=1}^n f_i \epsilon_i$  where  $f_i \in S$  and extend to other degrees as needed. Now

$$\varphi(f) = \bar{f}_0 + t \sum_{i=1}^{m+n} \bar{f}_i \bar{g}_i.$$

Then  $\varphi(f) \in I$  iff  $f_0 \in I$  and  $\sum_{i=m+1}^{m+n} f_i g_i \in I$  (the  $g_i$  for  $i = 1, \dots, m$  are already in  $I$ ).

Now  $\sum_{i=m+1}^{m+n} f_i g_i \in I$  means that there exists  $h_i \in S$  such that  $\sum_{i=m+1}^{m+n} f_i g_i - \sum_{j=1}^m h_j g_j = 0$  in  $S$ . This relation is contained in  $\text{Syz}(\{g_i\})$ . Then

$$f = f_0 + \sum_{i=m+1}^{m+n} f_i \epsilon_i - \sum_{j=1}^m h_j \epsilon_j + \sum_{j=1}^m (f_j + h_j) \epsilon_j \in I + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{g_i\}).$$

The reverse containment is easy to check. □

Writing the Rees algebra in this way allows us to work in  $S$  instead of  $S/I$  and to then use standard results about computing syzygies using Gröbner bases. To properly rewrite the Rees algebra in this form however, we needed to consider the syzygies on the generators of both  $I$  and  $J$  instead of just  $J$  (which gives the extra relations on generators of  $J$  that land in  $I$ ). The phrasing of the next lemma in using both  $I$  and  $J$  would otherwise seem a little strange without viewing it in this context.

**Lemma 4.0.2.** *Using the notation of Lemma 4.0.1, suppose that the generators for  $I$  and  $J$  are also Gröbner bases with respect to some monomial ordering on  $S$ . Define*

$$\sigma_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j$$

$$\tau_{ij} = m_{ji}\epsilon_i - m_{ij}\epsilon_j - \sum_u f_u^{(ij)}\epsilon_u$$

where  $m_{ij} = \text{init}(g_i)/\text{GCD}(\text{init}(g_i), \text{init}(g_j)) \in S$  and the  $f_u^{(ij)}$  come from the division algorithm applied to  $m_{ji}g_i - m_{ij}g_j$ . Then  $\text{Syz}(\{g_i\})$  is generated by the  $\tau_{ij}$  and  $\text{Syz}(\{\text{init}(g_i)\})$  is generated by the  $\sigma_{ij}$ .

*Proof.* Note that since  $I \subset J$ ,  $\{g_1, \dots, g_m, g_{m+1}, \dots, g_{m+n}\}$  is a Gröbner basis for  $J = I+J$ . The result follows from Lemma 15.1, Theorem 15.8 (Buchberger's Criterion), and Theorem 15.10 (Schreyer's theorem) in [12].  $\square$

We now define the crucial ordering on the extra variables  $\epsilon_i$  to introduce a meaningful degeneration of the blow-up algebra.

**Lemma 4.0.3.** *Let  $>_\lambda$  be a term order on  $S = k[x_1, \dots, x_k]$  defined by an integral weight function  $\lambda : \mathbb{Z}^k \rightarrow \mathbb{Z}$ . Using the notation of Lemma 4.0.2, consider the integral weighting  $\tilde{\lambda}$  defined on generators by*

$$\tilde{\lambda}(z) = \begin{cases} \lambda(\text{init}_{>_\lambda}(g_i)) & \text{if } z = \epsilon_i, \\ \lambda(x_j) & \text{if } z = x_j \end{cases}$$

*Then,  $\{g_1, \dots, g_m, \epsilon_1, \dots, \epsilon_m\} \cup \{\tau_{ij}\}$  is a Gröbner basis with respect to  $<_{\tilde{\lambda}}$  for  $I + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{g_i\})$  and its initial ideal is*

$$\text{init}_{>_\lambda}(I) + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{\text{init}_{>_\lambda}(g_i)\}).$$

*Proof.* This is a variation of the proof in Schreyer's theorem (see Theorem 15.10 in [12]). Observe that

$$\text{init}(\tau_{ij}) = m_{ji}\epsilon_i - m_{ij}\epsilon_j$$

since  $m_{ji}\text{init}(g_i) = m_{ij}\text{init}(g_j)$  and because these terms are greater than any that appear in the  $f_u^{(ij)}\epsilon_u$  as a result of the division algorithm.

First, we will show that the  $\tau_{ij}$  don't just generate  $\text{Syz}(\{g_i\})$  but are a Gröbner basis for it. Let  $\tau = \sum h_v\epsilon_v$  be a syzygy of  $\{g_i\}$  (with terms that can be cancelled already eliminated). For each index  $v$ , set  $n_v\epsilon_v = \text{init}(h_v\epsilon_v)$ . Let  $q$  be the maximum weight of a term in  $\tau$ . Then let  $\sigma = \sum n_v\epsilon_v$  such that  $\tilde{\lambda}(n_v\epsilon_v) = q$ . That is,  $\text{init}(\tau) = \sigma$ .

Notice that since  $\tau$  was a syzygy of the  $g_i$ ,  $\sigma$  must be a syzygy of the  $\text{init}(g_i)$ . Indeed  $\tau$  gives rise to the relation

$$\sum_{\text{weight } q} n_v\epsilon_v + \sum_u a_u\epsilon_u = 0$$

by separating the terms from the relation provided by  $\sigma$  with the remaining terms. Then

$$\sum_{\text{weight } q} n_v\epsilon_v = - \sum_u a_u\epsilon_u.$$

If  $\sum_{\text{weight } q} n_v\epsilon_v \neq 0$ , then the left hand side would contain a term with weight  $q$ . The right hand side necessarily has weight strictly less than  $q$ , a contradiction.

Thus  $\sigma$  is a syzygy of the  $\text{init}(g_i)$  and therefore is in the ideal generated by the  $\sigma_{ij}$  by Lemma 4.0.2. Since  $\text{init}(\tau_{ij}) = \sigma_{ij}$ , we are done.

Next, let  $f \in I + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{g_i\})$ . The worry at this point should be that

cancellation results in an  $\text{init}(f)$  not in our target ideal. For example, perhaps some element in  $I + \langle \epsilon_1, \dots, \epsilon_m \rangle$  can be used to eliminate the  $m_{ji}\epsilon_i - m_{ij}\epsilon_j$  in some  $\tau_{ij}$  leaving only  $\sum_u f_u^{(ij)} \epsilon_u$  terms not found in  $\text{Syz}(\{\text{init}(g_i)\})$ .

Let  $m$  be a term in  $\text{init}(f)$ . Clearly if  $m$  was a leading term for an element in  $I + \langle \epsilon_1, \dots, \epsilon_m \rangle$ , then  $m \in \text{init}(I) + \langle \epsilon_1, \dots, \epsilon_m \rangle$ . Otherwise,  $m$  is a leading term of some combination involving a  $\tau_{ij}$ , and there are 2 cases to consider. Fix  $(i, j) = (i_0, j_0)$ .

If  $m$  is a multiple of a term in  $\tau_{i_0 j_0}$  where at least one  $\epsilon_{i_0}$  and  $\epsilon_{j_0}$  are in  $\langle \epsilon_{m+1}, \dots, \epsilon_{m+n} \rangle$ , we are done since no cancellation is possible to eliminate these leading terms, so  $m \in \text{init}(I) + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{\text{init}(g_i)\})$

If both  $\epsilon_{i_0}$  and  $\epsilon_{j_0}$  are in  $\langle \epsilon_1, \dots, \epsilon_m \rangle$ , then the division algorithm applied to  $(m_{j_0 i_0} \epsilon_{i_0} - m_{i_0 j_0} \epsilon_{j_0})|_{\epsilon_i = g_i} \in I$  would have produced  $f_u^{(i_0 j_0)}$  also contained in  $I$  since the  $g_1, \dots, g_m$  are a Gröbner basis for  $I$ , and such a decomposition for ideal containment is unique. Then  $m \in \text{init}(I) + \langle \epsilon_1, \dots, \epsilon_m \rangle$ , completing the proof.

□

**Theorem 4.0.4.** *Let  $>_\lambda$  be a term order on  $S = k[x_1, \dots, x_k]$  defined by an integral weight function  $\lambda : \mathbb{Z}^k \rightarrow \mathbb{Z}$ . Let  $I = \langle g_1, \dots, g_m \rangle \subset J = \langle g_1, \dots, g_{m+n} \rangle$  be two ideals in  $S$  each generated by a Gröbner basis (with respect to  $\lambda$ ). Then there exists an integral weight function  $\tilde{\lambda} : \mathbb{Z}^{n+m+k} \rightarrow \mathbb{Z}$  defining a term order  $>_{\tilde{\lambda}}$  on the blow-up algebra of  $S/I$  along  $J/I$  which Gröbner degenerates to the blow-up algebra of  $S/\text{init}_{>_\lambda}(I)$  along  $\text{init}_{>_\lambda}(J)/\text{init}_{>_\lambda}(I)$ .*

*Proof.* By Lemma 4.0.1, the blow-up algebra  $\bar{S}[t\bar{J}]$  is isomorphic to

$$S[\epsilon_1, \dots, \epsilon_{m+n}]/(I + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{g_i\}))$$



We know that this degenerates to

$$S[\epsilon_1, \dots, \epsilon_{m+n}]/(\text{init}_{>\lambda}(I) + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{\text{init}_{>\lambda}(g_i)\}))$$

using the integral weighting  $\tilde{\lambda}$  defined in Lemma 4.0.3.

On the other hand,  $S/I$  and  $J/I$  degenerate to  $S/\text{init}_{>\lambda}(I)$  and  $\text{init}_{>\lambda}(J)/\text{init}_{>\lambda}(I)$ . By Lemma 4.0.1, the blow-up algebra of  $S/\text{init}_{>\lambda}(I)$  along  $\text{init}_{>\lambda}(J)/\text{init}_{>\lambda}(I)$  is

$$S[\epsilon_1, \dots, \epsilon_{m+n}]/(\text{init}_{>\lambda}(I) + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{\text{init}_{>\lambda}(g_i)\}))$$

as required. □

**Corollary 4.0.5.** *The blow-up of a Kazhdan-Lusztig variety along its boundary commutes with the degeneration in Section 3 using the integral weight function  $\tilde{\lambda}$  in Theorem 4.0.4.*

*Proof.* Let  $I$  be the generating ideal for the Kazhdan-Lusztig variety and  $J$  the generating ideal for its boundary. Choose an integral weight function  $\lambda$  that produces the lexicographic ordering for the equations in  $I$  and  $J$  (ie. the  $\lambda$  needed for the degeneration in Section 3). The result follows. □

CHAPTER 5  
THE BLOW-UP OF  $X^W$  ALONG  $\partial X^W$

We need to show that the general definition of Gorenstein presented in 3.2.6 is an uppersemicontinuous property. This means that it is an open condition in flat families so that checking the Gorenstein property on the special fibre is sufficient for showing it on the general fibre. To achieve this, we need to check that the property of a sheaf being dualizing is open.

Let  $X$  be a proper scheme of dimension  $n$  over  $k$ . Recall that  $\kappa$  is said to be a dualizing sheaf if:

- $H^n(X, \kappa) \cong k$
- For all coherent sheaves  $\mathcal{F}$ , the following map is a perfect pairing:

$$H^i(X, \mathcal{F}) \times H^{n-i}(X, \mathcal{F}^* \otimes \kappa) \rightarrow H^n(X, \kappa).$$

Suppose we are given a flat family  $M$  over  $\mathbb{A}^1$  (where  $M$  is a proper scheme of dimension  $n$  over  $k$ ) such that  $M_t$  Gröbner degenerates to  $M_0$ . Suppose we also have a coherent sheaf  $\kappa_t$  over  $M_t$  that degenerates to  $\kappa_0$  over  $M_0$ . Then the pair  $(M_t, \kappa_t)$  has been Gröbner degenerated to  $(M_0, \kappa_0)$ . We would like to know whether  $\kappa_0$  being a dualizing sheaf implies that  $\kappa_t$  is dualizing as well.

The perfect pairing property from the definition is already an open condition since it can be rephrased as a map into the dual space being an isomorphism. The first condition however may not be open in general. In fact  $H^n(M_t, \kappa_t)$  can jump in dimension on the special fibre. In particular, if  $H^n(M_0, \kappa_0) \cong k$ , then

$H^n(M_t, \kappa_t) \cong k$  or 0. What is true however, is that

$$H^{n-1}(M_0, \kappa_0) = 0 \Rightarrow H^n(M_t, \kappa_t) \cong k$$

Indeed,  $H^{n-1}(M_0, \kappa_0) = 0$  implies that no such jump in dimension occurred at the special fibre, therefore  $H^n(M_t, \kappa_t) \cong k$  as required.

**Proposition 5.0.1.** *Let  $(M_t, \kappa_t)$  Gröbner degenerate to  $(M_0, \kappa_0)$  as above. Suppose also that  $H^{n-1}(M_0, \kappa_0) = 0$ . Then  $\kappa_t$  is a dualizing sheaf if  $\kappa_0$  is.*

In the case that concerns us, we know that  $X^w \widetilde{\cap} X_v^o$  Gröbner degenerates to  $M_0 = \text{Proj}(R[t(\text{init}J_{w,v})])$  (the total transform of a Stanley-Reisner scheme), extending the degeneration from Section 3. In particular, the exceptional divisor  $E$  in  $X^w \widetilde{\cap} X_v^o$  must degenerate to the boundary divisor  $E_0 = \partial M_0$  (which was the exceptional divisor in the toric case).

We know that  $E_0$  is a Cartier anticanonical divisor (as defined in 3.2.6) which we can use to define our prospective  $\kappa_0$ . For each  $M_t$  in the degeneration above, we have a reduced Cartier exceptional divisor  $E_t$  such that  $-E_t$  is also Cartier. We can then define  $\kappa_t = \mathcal{O}(-E_t)$ . Since  $\kappa_1$  is coherent, we can give it a finite presentation, and take the Gröbner limit in the Quot scheme to define  $\kappa_0$ .

We need only check that it is dualizing and that  $H^{n-1}(M_0, \kappa_0) = 0$ . In fact we can deduce these cohomology conditions on  $M_0$  from its normalization.

We know that  $M_0$  is equidimensional and has normal components. We will first show that  $\kappa_0$  is a dualizing sheaf when  $M_0$  has two components whose intersection has codimension one and is also normal. Say  $M_0 = X \cup Y$ . Given a coherent sheaf  $\mathcal{F}$  on  $M_0$ , we have the short exact sequence:

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_X \oplus \mathcal{F}|_Y \rightarrow \mathcal{F}|_{X \cap Y} \rightarrow 0$$

which gives rise to the long exact sequence in sheaf cohomology:

$$\dots \rightarrow H^{i-1}(\mathcal{F}|_{X \cap Y}) \rightarrow H^i(\mathcal{F}|_{X \cup Y}) \rightarrow H^i(\mathcal{F}|_{X \sqcup Y}) \rightarrow \dots$$

Let  $\mathcal{F} = \kappa_0$ . Since the normalization of  $M_0$  is just  $X \sqcup Y$  (which is normal), and since  $X \cap Y$  has dimension  $n - 1$  and is also normal, we have that

$$0 = H^{n-2}(X \cap Y, \mathcal{F}|_{X \cap Y}) \rightarrow H^{n-1}(X \cup Y, \mathcal{F}|_{X \cup Y}) \rightarrow H^{n-1}(X \sqcup Y, \mathcal{F}|_{X \sqcup Y}) = 0$$

forcing  $H^{n-1}(X \cup Y, \mathcal{F}|_{X \cup Y}) = 0$  as required.

The perfect pairing condition also follows from the normal cases  $X \sqcup Y$  and  $X \cap Y$ . We have a natural map  $\pi_Z : H^i(Z, \mathcal{F}|_Z) \rightarrow H^{n-i}(Z, \mathcal{F}^*|_Z)^*$  which is an isomorphism for  $Z = X \sqcup Y, X \cap Y$ . We can draw the commuting diagram:

$$\begin{array}{ccccccccc} H^{i-1}(\mathcal{F}|_{X \cup Y}) & \longrightarrow & H^i(\mathcal{F}|_{X \cap Y}) & \longrightarrow & H^i(\mathcal{F}|_{X \cup Y}) & \longrightarrow & H^i(\mathcal{F}|_{X \sqcup Y}) & \longrightarrow & H^{i+1}(\mathcal{F}|_{X \cap Y}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{n-i+1}(\mathcal{F}^*|_{X \cup Y})^* & \rightarrow & H^{n-i}(\mathcal{F}^*|_{X \cap Y})^* & \rightarrow & H^{n-i}(\mathcal{F}^*|_{X \cup Y})^* & \rightarrow & H^{n-i}(\mathcal{F}^*|_{X \sqcup Y})^* & \rightarrow & H^{n-i-1}(\mathcal{F}^*|_{X \cap Y})^* \end{array}$$

By the five lemma,  $\pi_{X \cup Y}$  will also be an isomorphism.

We can now prove the result for a general union using the inductive version of the degeneration of a Kazhdan-Lusztig variety  $KL_1$  to the Stanley-Reisner scheme (see Section 2.6). Recall that  $KL_1$  can be degenerated as

$$KL_1 \rightsquigarrow KL_2 \cup_{0 \times KL_3} (\mathbb{A}^1 \times KL_3)$$

as shown in [18].

We have proven the base case above. If we now assume that the result holds for  $X = KL_2, Y = KL_3$ , then the same argument shows that the result holds for  $KL_1$  too.

In the blow-up along the boundary divisor, the intersection of components (corresponding to interior facets of the subword complex) satisfy the same conditions as the Stanley-Reisner case. Therefore the sheaf associated to the Cartier boundary divisor of the total transform is dualizing.

We are now ready to state our main result:

**Theorem 5.0.2.** *Let  $I_{w,v} \leq k[z_{i,j}]$  be the defining ideal for Kazhdan-Lusztig variety  $X^w \cap X_v^o$ . The boundary divisor is defined by  $J_{w,v} = \bigcap_{u < w} I_{u,v}$ . Then the blow-up of  $S = k[z_{ij}]/I_{w,v}$  along  $J_{w,v}$  is Gorenstein.*

*Proof.* By Theorem 3.0.2 and Corollary 3.0.4, there exists a degeneration of the pair  $(S, J_{w,v})$  to  $(R, \text{init}(J_{w,v}))$  where  $R = k[z_{ij}]/\text{init}(I_{w,v})$ . Here  $R$  is the coordinate ring for a Stanley-Reisner scheme. In fact  $\text{init}(I_{w,v})$  is the Stanley-Reisner ideal of a subword complex  $\Delta(Q, w)$ . Viewing  $\text{Spec}(R)$  as a toric scheme, we can associate to it an affine semigroup in the character lattice which can be viewed as the cone on the polytope  $\Delta(Q, w)$  (in appropriate coordinates). Here  $\text{init}(J_{w,v})$  corresponds to the boundary of this convex region.

By Corollary 3.2.10, the blow-up of  $R$  along  $\text{init}(J_{w,v})$  is Gorenstein. Furthermore, by Corollary 4.0.5, there exists a degeneration of the blow-up algebra  $S[tJ_{w,v}]$  to  $R[t(\text{init}J_{w,v})]$ . We have the following diagram:

$$\begin{array}{ccc}
S[tJ_{w,v}] & \rightsquigarrow & R[t(\text{init}J_{w,v})] \\
\uparrow & & \uparrow \\
(S, J_{w,v}) & \xrightarrow{\text{lexinit}} & (R, \text{init}J_{w,v})
\end{array}$$

Finally, we need to determine whether being Gorenstein (in the more general sense mentioned in Section 3) is open in flat families. The beginning of Section 5 shows the semicontinuity of the Gorenstein property.

□

**Corollary 5.0.3.** *The blow-up of  $X^w$  along  $\partial X^w$  is Gorenstein.*

*Proof.* See Proposition 2.1.3.

□

**Example 5.0.4.** Consider  $X^{53241} \cap X_{12345}^o$  with

$$I_{w,v} = \langle z_{11}z_{22} - z_{12}z_{21}, z_{11}z_{23} - z_{13}z_{21}, z_{12}z_{23} - z_{13}z_{22} \rangle$$

as in Example 3.0.1, defined by vanishing of all  $2 \times 2$  minors of

$$\begin{pmatrix} z_{21} & z_{22} & z_{23} \\ z_{11} & z_{12} & z_{13} \end{pmatrix}$$

The blow-up along the boundary satisfies the following equations (with  $\epsilon_1, \epsilon_2$  in degree one):

$$I = \langle z_{11}z_{22} - z_{12}z_{21}, z_{11}z_{23} - z_{13}z_{21}, z_{12}z_{23} - z_{13}z_{22},$$

$$\epsilon_1 z_{11} - \epsilon_2 z_{21}, \epsilon_1 z_{12} - \epsilon_2 z_{22}, \epsilon_1 z_{13} - \epsilon_2 z_{23} \rangle$$

Those are the  $2 \times 2$  minors of

$$\begin{pmatrix} z_{21} & z_{22} & z_{23} & \epsilon_1 \\ z_{11} & z_{12} & z_{13} & \epsilon_2 \end{pmatrix}.$$

These equations look like the ones defining  $X^{643251} \cap X_{123456}^o$ . A quick check of the Gorenstein criteria for Schubert varieties shows that  $X^{643251}$  is not Gorenstein, and indeed we need to account for  $\epsilon_1$  and  $\epsilon_2$  being in degree one. Therefore we take the GIT quotient  $X^{643251} \cap X_{123456}^o // \mathbb{C}^\times$  where  $\mathbb{C}^\times$  acts diagonally on  $(\epsilon_1, \epsilon_2)$ .

Now we should check that the non-Gorenstein locus is contained in the subvariety where  $\epsilon_1 = \epsilon_2 = 0$  (the subset quotiented out). For example  $X^{643251}$  is not Gorenstein at  $e_{432165}$ , but can be defined by the vanishing of all of the coordinates in

$$\begin{pmatrix} z_{21} & z_{22} & z_{23} & \epsilon_1 \\ z_{11} & z_{12} & z_{13} & \epsilon_2 \end{pmatrix}$$

and hence is not present in  $X^{643251} \cap X_{123456}^o // \mathbb{C}^\times$ . □

The other three  $n = 5$  examples in type A are also of this form. While we don't expect  $\widetilde{X}^w$  to always be a GIT quotient of a Kazhdan-Lusztig variety, this example serves as a good transition to the next important question - how can we best interpret  $\widetilde{X}^w$ ? Is it isomorphic to some well-known variety? Can we determine the exceptional locus of the blow-up map? Does  $\widetilde{X}^w$  enjoy any other useful properties?

The first approach is to study  $\widetilde{X}^w$  locally, through local equations for the blow-up algebra using syzygies. This is the content found in the remainder of this section. The second approach is a more global approach which deduces information about  $\widetilde{X}^w$  using Bott-Samelson maps. This is the content of Section 6.

Focusing on the local question for now, we recall that the boundary of a Kazhdan-Lusztig variety  $X^w \cap X_v^o$  is defined by the ideal

$$\bigcap_{u \triangleleft w} I_{u,v}$$

By the work of Bertiger in [3], we know that  $\cap I_{u,v}$  is generated by certain products of determinants. Understanding the equations for the blow-up is the same as understanding the syzygies of this intersection. What are the syzygies of  $\cap I_{u,v}$ ? Are they determinantal?

We know that the equations in  $I_{u,v}$  can be written as the sum of similar ideals where  $u$  is **bigrassmanian**. Recall that  $\pi$  is bigrassmannian if  $\pi$  and  $\pi^{-1}$  admit a unique descent, and this is the same as  $X^\pi$  being defined using one rank condition. Can we solve this simpler problem of understanding the syzygies of  $I_{u,v}$  if  $u$  is bigrassmanian?

## 5.1 Syzygies of products of ideals

Let  $I_1, \dots, I_r \subset R$  be ideals of a Noetherian integral domain  $R$ . Suppose that  $I_k = \langle g_{k,1}, \dots, g_{k,n_k} \rangle$  and consider the ring  $S = R[z_{1,1}, \dots, z_{r,n_r}] / \langle z_{i,j} - z_{k,l} \rangle$  with one  $z_{i,j}$  for each generator of each ideal  $I_k$  and with equal generators identified. We will denote the ideal of syzygies of  $\{g_{k,i}\}_{i=1}^{n_k}$  by  $\text{Syz}(I_k)$ . That is,  $\text{Syz}(I_k) \subset S$  will consist of all expressions of the form

$$\sum_{i=1}^{n_k} a_i z_{k,i}, \text{ where } a_i \in S \text{ such that } \sum_{i=1}^{n_k} (a_i z_{k,i}) \Big|_{z_{i,j}=g_{i,j}} = 0$$



Similarly,

$$\text{Syz}\left(\prod I_k\right) \subset R[\{z_{1,i_1} \cdot \dots \cdot z_{r,i_r}\}]/\langle z_{i,j} - z_{k,l} \rangle \subset S$$

consists of the syzygies of the  $N = \prod_{i=1}^r n_i$  products of the form  $g_{1,1}, \dots, g_{r,n_r}$ .

We would like to understand  $\text{Syz}(\prod I_k)$  in terms of the  $\text{Syz}(I_k)$ . Let  $I = \{(i_1, \dots, i_r), 1 \leq i_j \leq n_j\}$ . An element  $0 \neq z \in \text{Syz}(\prod I_k)$  has the form

$$\sum_{(i_1, \dots, i_r) \in I} a_{(i_1, \dots, i_r)} z_{1,i_1} \cdot \dots \cdot z_{r,i_r}$$

By viewing this as an equation written in the  $\{z_{1,j}\}_{j=1}^{n_1}$ , we see that either all the  $z_{1,j}$  are equal so that we can factor them out, or at least 2 are distinct and  $z \in \text{Syz}(I_1)$ . By next looking at the  $\{z_{2,j}\}_{j=1}^{n_2}$ , we see that either all of the  $z_{2,j}$  are equal so that we can factor them out or  $z \in \text{Syz}(I_2)$ .

Continuing in this way, we see that  $z$  is contained in the intersection of some  $\text{Syz}(I_k)$  (it must be contained in at least one of these, otherwise all the  $z_{1,i_1} \cdot \dots \cdot z_{r,i_r}$  were equal and  $z = 0$ ).

Let us denote the ideal  $\langle z_{k,1}, \dots, z_{k,n_k} \rangle$  by  $\bar{I}_k$ . The above argument shows that

$$\begin{aligned} z \in & \text{Syz}(I_1) \cap \dots \cap \text{Syz}(I_r) + \bar{I}_1(\text{Syz}(I_2) \cap \dots \cap \text{Syz}(I_r)) + \dots \\ & + \bar{I}_r(\text{Syz}(I_1) \cap \dots \cap \text{Syz}(I_{r-1})) + \dots + \bar{I}_1 \cdot \dots \cdot \bar{I}_{r-1} \text{Syz}(I_r) \end{aligned}$$

Put in a more compact form, we have shown that

$$\text{Syz}\left(\prod_{k=1}^r I_k\right) \subset \sum_{\emptyset \neq M \subset \{1, \dots, r\}} \left( \prod_{k \in M^c} \bar{I}_k \right) \left( \bigcap_{k \in M} \text{Syz}(I_k) \right)$$

For the reverse containment, observe that for  $z \in \text{Syz}(I_k) \cap \text{Syz}(I_j)$  we can write  $z$  in two ways

$$a_1 z_{k,1} + \dots + a_{n_k} z_{k,n_k} = b_1 z_{j,1} + \dots + b_{n_j} z_{j,n_j}$$

We will assume that any cancellation on each side has already taken place. This means that each  $a_m \in \bar{I}_j$  as a result. In particular both sides contain products of the form  $z_{k,i_k} z_{j,i_j}$ . This shows that  $z \in \text{Syz}(I_j I_k)$ . Generalizing this we get the reverse containment, as required.

**Theorem 5.1.1.**  $\text{Syz}(\prod_{k=1}^r I_k) = \sum_{\emptyset \neq M \subset \{1, \dots, r\}} (\prod_{k \in M^c} \bar{I}_k) (\bigcap_{k \in M} \text{Syz}(I_k))$ .

## 5.2 Viewing syzygies of intersections in terms of products

Building on the ideas from the previous section, our goal is to ultimately understand the syzygies of  $\cap I_k$  in terms of the syzygies of  $\prod I_k$  which we studied in the last section.

Let  $T = R[w_1, \dots, w_n]$  (simplified from the  $z_{i,j}$  coordinates used in the previous section) and observe that

$$\text{Syz}(\prod I_k), \text{Syz}(\cap I_k) \subset T$$

where  $w_i$  is associated to  $g_i \in R$ .

Consider  $z \in \text{Syz}(\cap I_k)$ . Then

$$z = a_1 y_1 + \dots + a_p y_p$$

where  $a_i \in R$  and  $y_i$  corresponds to a generator of  $\cap I_k$  (using [3] to fix generators). Then  $z^r$  involves products of  $r$  choices of  $y_i$  which when restricted to  $g_j$  must be in  $\prod I_k$  (indeed  $\cap I_k \subset \sum I_k$ ). That is to say that  $z^r \in \text{Syz}(\prod I_k)$ . Finally, since  $\prod I_k \subset \cap I_k$ , we have the same containment regarding syzygies.

We have thus shown

$$\text{Syz}(\cap I_k) \subset \sqrt{\text{Syz}(\prod I_k)} \subset \sqrt{\text{Syz}(\cap I_k)}$$

The next lemma provides the last piece to proving the main result for this section.

**Lemma 5.2.1.** *If  $\cap I_k$  is reduced, then so is  $\text{Syz}(\cap I_k)$ .*

*Proof.* Given an element  $z = a_0 + a_1 w_1 + \dots + a_n w_n$  of  $R[w_1, \dots, w_n]$ , suppose that

$$(a_0 + a_1 w_1 + \dots + a_n w_n)^m \in \text{Syz}(\cap I_k)$$

Then  $a_0^m$  and  $(a_i w_i)^m$  must be in  $\cap I_k$  after evaluating at  $w_i = g_i$ . Since  $\cap I_k$  is reduced, the restriction of each  $a_i w_i$  must also be in  $\cap I_k$ . Therefore  $z \in \text{Syz}(\cap I_k)$ .

□

Using Frobenius splittings to show this result might also prove useful. Finally, we have:

**Proposition 5.2.2.**  $\sqrt{\text{Syz}(\prod I_w)} = \text{Syz}(\cap I_w)$ .

This shows that understanding the syzygies of the product is the same as understanding the syzygies of the intersection.

### 5.3 Specializing to bigrassmannians

We know that a Schubert variety  $X^w$  can be decomposed as

$$X^w = \bigcup_{\substack{\nu \geq w, \\ w_0\nu \text{ bigrassmannian}}} X^\nu.$$

Bigrassmannians are especially easy to work with since they involve only one rank condition.

In our problem, we are blowing-up along the ideal  $\cap J^\sigma$  where each  $J^\sigma$  is the defining ideal for a component of the boundary divisor of  $X^w \cap X_\tau^o$  (which is defined by  $I$ ). Since the boundary has dimension 1 less than  $X^w \cap X_\tau^o$ , we can write

$$J^\sigma = I + I^\nu$$

where  $\nu$  is a bigrassmannian.

Since a bigrassmannian  $\nu$  is defined by the vanishing of all  $k \times k$  minors in an  $m \times n$  box, we can define a simple bigrassmannian  $\nu_s$  as the case when  $m = k$  and  $n = k + 1$ . Such simple bigrassmannians have a special property.

**Lemma 5.3.1.** *The syzygy ideal of  $I^{\nu_s}$ , with  $\nu_s$  a simple bigrassmannian, is a determinantal variety.*

In fact the equations for  $I^{\nu_s}$  can be found in the same way as Example 5.0.4 by adding an extra row and column to the matrix for  $\nu_s$ . This corresponds to the defining equations for a Kazhdan-Lusztig variety. When  $\nu$  is a bigrassmannian,

we know exactly what the generators are for the syzygy ideal of  $I^v$  thanks to [25].

To make use of these facts, we notice that since

$$\sum_{\emptyset \neq M \subset \{1, \dots, r\}} \left( \prod_{k \in M^c} \bar{J}^{\sigma_k} \right) \left( \bigcap_{k \in M} \text{Syz}(J^{\sigma_k}) \right) = \text{Syz} \left( \prod J^{\sigma} \right) \subset \text{Syz}(\cap J^{\sigma})$$

there are a number of ideals for Kazhdan-Lusztig varieties corresponding to simple bigrassmannians contained in this sum of ideals, some of which might be interesting to consider. In particular, with some work we could make some mild conjectures: perhaps the blow-up of a Kazhdan-Lusztig variety along its boundary is contained in some product of Kazhdan-Lusztig varieties (with some GIT quotient to account for certain variables in degree one). This would generalize the results we observed in low dimensions. However the containment at this time doesn't seem to be explicit enough to get a useful form for the total transform. Extending the  $B$ -action to the total transform might identify it as a  $B$ -invariant subvariety of this product.

The fact that a product of Kazhdan-Lusztig varieties is appearing in our local computations suggests that perhaps globally we can describe the blowup as some product of Schubert-like varieties. This is where generalized Bott-Samelson varieties come into the picture, further described in the next section.

## CHAPTER 6

### UNDERSTANDING THE TOTAL TRANSFORM

To get an explicit modular interpretation of blow-up of  $X^w$  along  $\partial X^w$ , we seek a way to compare  $\widetilde{X}^w$  to other well-studied varieties. We will exploit Bott-Samelson resolutions of  $X^w$  to get convenient maps from Bott-Samelson varieties onto  $\widetilde{X}^w$ . To obtain this map, we first recall the universal property of blow-up maps.

#### 6.0.1 Universal property of blow-ups

The blow-up of a scheme  $X$  along some closed subscheme  $Y$  is a fibre diagram:

$$\begin{array}{ccc} E & \longrightarrow & Bl_Y(X) \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where  $E$  is an effective Cartier divisor, such that any other map to  $X$  with an effective Cartier divisor as the fibre over  $Y$  factors through the diagram. The only Cartier divisors considered in the sections that follow are effective.

**Lemma 6.0.1.** *Let  $\pi : \widetilde{X}^w \rightarrow X^w$  be the blow-up of  $X^w$  along  $\partial X^w$ . Suppose  $\varphi_Q : BS^Q \rightarrow X^w$  is a generalized Bott-Samelson resolution such that  $\varphi_Q^{-1}(\partial X^w)$  is a Cartier divisor in  $BS^Q$ . Then there exists a surjective birational proper map  $\psi_Q : BS^Q \rightarrow \widetilde{X}^w$  such that*

$$\begin{array}{ccc} BS^Q & \xrightarrow{\psi_Q} & \widetilde{X}^w \\ & \searrow \varphi_Q & \downarrow \pi \\ & & X^w \end{array}$$

*Proof.* By the universal property of blow-up maps,  $\varphi_Q$  factors through the blow-up of  $X^w$  along  $\partial X^w$ . The map  $\psi_Q$  is birational since  $\varphi_Q$  and  $\pi$  are (see 2.5.3). Similarly,  $\psi_Q$  is proper since  $\varphi_Q$  is proper and  $\pi$  is separated. Finally, since the universal property uses fibre diagrams,  $\pi$  and  $\varphi_Q$  surjective implies that  $\psi_Q$  is surjective as well.  $\square$

**Corollary 6.0.2.** *Suppose  $\varphi_Q : BS^Q \rightarrow X^w$  is a generalized Bott-Samelson resolution where  $BS^Q$  is factorial. Then there exists surjective proper birational map  $\psi_Q : BS^Q \rightarrow \widetilde{X}^w$  which factors through the blow-up of  $X^w$  along  $\partial X^w$ . In particular, choosing  $Q = (w_1, \dots, w_k)$  such that each  $w_i$  is a simple reflection provides one such map.*

*Proof.* By dimension considerations,  $\psi_Q^{-1}(\partial X^w)$  is a divisor in  $BS^Q$ . Since Cartier and Weil divisors are equivalent in factorial varieties, the result follows. Furthermore, choosing  $Q$  so that the  $w_i$  are simple reflections results in  $BS^Q$  being smooth (by Lemma 2.5.1) and hence factorial.  $\square$

Understanding when  $BS^Q$  is factorial requires an understanding of the same question for Schubert varieties. There is a pattern embedding description for this in [28], at least in the  $GL_n$  case. One key takeaway from Corollary 6.0.2 is that there exist many such maps  $\psi_Q$  when we choose  $Q$  appropriately. The existence of these maps already provides us with new information regarding the  $T$ -fixed points of  $\widetilde{X}^w$ .

**Lemma 6.0.3.** *Let  $\pi : \widetilde{X}^w \rightarrow X^w$  be the blow-up of  $X^w$  along  $\partial X^w$ . Then the  $B$ -action on  $X^w$  extends to the total transform  $\widetilde{X}^w$ . Furthermore,  $\psi_Q$  is a  $B$ -equivariant map.*

*Proof.* Since  $\partial X^w$  is  $B$ -invariant, the  $B$ -action extends to  $\widetilde{X}^w$  (see Proposition 3.9.1 of [22]). Furthermore,  $\pi$  is  $B$ -equivariant. Since  $\varphi_Q$  is also  $B$ -equivariant, then  $\psi_Q$  is too.  $\square$

**Corollary 6.0.4.**  $\widetilde{X}^w$  has finitely many  $T$ -fixed points.

*Proof.* By Lemma 6.0.3,  $T$ -fixed points must map onto  $T$ -fixed points. The result follows from the fact that  $BS^Q$  has finitely many  $T$ -fixed points.  $\square$

Our end goal is to show that understanding  $\widetilde{X}^w$  restricts to an understanding of its  $T$ -fixed points. Since the  $T$ -fixed points of  $BS^Q$  are easy to describe, the hope is that  $\psi_Q$  can provide us with an explicit statement about the structure of  $\widetilde{X}^w$ .

## 6.1 Extending the Frobenius splitting

Our proof that  $\widetilde{X}^w$  is Gorenstein utilized the notion of Frobenius splittings and their relation to sections of the anticanonical bundle. It is natural to ask whether  $\widetilde{X}^w$  is also Frobenius split.

There do exist results about extending Frobenius splittings to the total transform of a blow-up, such as in the work of Lakshmibai, Mehta, and Parameswaran in [24]. This result however only works for the blow-up of a smooth Frobenius split variety along a smooth center (under certain mild vanishing conditions). In Proposition 3.2.4, we extended the Frobenius splitting in the toric case.

There are two natural approaches to extending the Frobenius splitting to the total transform. One perspective is to directly define a Frobenius splitting on the blow-up algebra



$$S[\epsilon_1, \dots, \epsilon_{m+n}]/(I + \langle \epsilon_1, \dots, \epsilon_m \rangle + \text{Syz}(\{g_i\}))$$

by defining a splitting map on monomials of the form

$$C\epsilon_1^{a_1} \cdot \dots \cdot \epsilon_{m+n}^{a_{m+n}}$$

A second approach is to observe that  $\widetilde{X}^w$  degenerates to a Frobenius split variety. That is, there is a flat morphism  $F : \mathcal{Y} \rightarrow \mathbb{A}^1$  where  $F^{-1}(0) = \mathcal{Y}_0$  is the toric scheme found in Section 3, and  $F^{-1}(t) = \mathcal{Y}_t$  is the general fibre which is isomorphic to  $\widetilde{X}^w$ . But is being Frobenius split open in flat families?

One might hope that there is a Frobenius splitting of the whole family  $\mathcal{Y}$  which compatibly splits each  $\mathcal{Y}_t$ . However, there can only ever be finitely many compatibly split subvarieties (see [23]). On the other hand, it might be possible to extend the splitting of  $\mathcal{Y}_0$  to a splitting of  $\mathcal{Y}$  where  $\mathcal{Y}_t$  is not necessarily compatibly split. One could then view the general fibre  $\mathcal{Y}_t$  as a GIT quotient of  $\mathcal{Y}$ . Indeed, there is a  $\mathbb{G}_m$  action on  $\mathcal{Y}$ , and if  $\mathcal{Y}$  is Frobenius split, then so is  $\mathcal{Y}_t \cong \mathcal{Y}/\mathbb{G}_m$  (using a different splitting – see Theorem 7.1 in [26]). At present however, we don't know if  $\widetilde{X}^w$  is Frobenius split.

## 6.2 Weakly normal varieties

Recall that a variety  $X$  is normal if every finite birational map  $f : Y \rightarrow X$  is an isomorphism. We say that  $X$  is **weakly normal** if every finite birational *bijective* map is an isomorphism. Bijective here means that the reduction of the fibre over a point is again a point. In characteristic 0, the notions of weakly normal and seminormal are equivalent (see [27] for more information).

A convenient way to show that a variety is weakly normal is to show that it is Frobenius split.

**Lemma 6.2.1.** *Frobenius split varieties are weakly normal.*

*Proof.* See Proposition 1.2.5 in [6]. □

We would like to know that  $\widetilde{X}^w$  is weakly normal in order to reduce questions involving  $\psi_Q$  to information about the restricted map on  $T$ -fixed points. We rely on the fact that  $\widetilde{X}^w$  degenerates to a weakly normal scheme, and that being weakly normal is an open condition in flat families.

**Proposition 6.2.2.** *Let  $\mathcal{Y}$  be a flat family over  $\mathbb{A}^1$  such that  $\mathcal{Y}_0$ , the special fibre over the origin, is weakly normal. Then  $\mathcal{Y}_t$  is also weakly normal.*

*Proof.* See [27]. □

**Corollary 6.2.3.** *The blow-up of  $X^w$  along  $\partial X^w$  is weakly normal.*

*Proof.* By Corollary 4.0.5, the total transform  $X^w \widetilde{\cap} X_v^o$  Gröbner degenerates to a toric scheme, which by Lemma 3.2.4 is Frobenius split. The result now holds by Lemma 6.2.1 and Proposition 6.2.2, and using that being weakly normal can be checked locally. □

We note that there is another proof of this result. Indeed the blow-up of  $X^w$  (a normal variety) along an integrally closed boundary ideal is again normal, and hence weakly normal. Therefore  $\widetilde{X}^w$  being weakly normal can be deduced using the blow-up map, the degeneration, or Frobenius splittings.

### 6.3 Reducing to $T$ -fixed points

The next theorem will finally reduce understanding  $\psi_Q$  to a question of  $T$ -fixed points. First we need a small lemma:

**Lemma 6.3.1.** *Suppose that  $X \hookrightarrow \mathbb{P}^n$  is an equivariant inclusion with respect to a torus action  $T$ . If  $\dim X > 0$  (and  $X \neq \emptyset$ ), then  $|X^T| > 1$ .*

*Proof.* If  $X = X^T$ , then the result is obvious. Otherwise, choose an  $x \in X \setminus X^T$ . Let  $Y = \overline{T \cdot x}$  and  $\nu : \tilde{Y} \rightarrow Y$  be the normalization of  $Y$ . Then  $\tilde{Y}$  is a projective toric variety. Furthermore, since  $x$  is not a fixed point by assumption,  $\dim \tilde{Y} \geq 1$ .

We wish to use the  $T$ -fixed points of  $\tilde{Y}$  to show the existence of at least two  $T$ -fixed points for  $Y$ . Let  $\mathcal{L}$  be the pullback of the sheaf  $\mathcal{O}(1)$  in the diagram:

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ \tilde{Y} & \xrightarrow{\nu} & Y \end{array}$$

Since  $\nu$  is a finite map,  $\mathcal{L}$  is an ample line bundle over  $\tilde{Y}$ . Then  $\tilde{Y}$  is a toric variety associated to a lattice polytope  $P$  (see [9, Thm 6.2.1]). In fact,  $P$  is the convex hull of the weights of the action of  $T$  on the fibres  $\mathcal{L}|_f$  for  $f \in \tilde{Y}^T$ .

Note that the weight of the  $T$ -action on  $\mathcal{L}|_f$  is the same as the weight of  $T$  acting on  $\mathcal{O}(1)|_{\nu(f)}$ . Therefore, distinct  $T$ -fixed points of  $\tilde{Y}$  correspond to distinct weights for the  $T$ -action on the line bundles, which in turn corresponds to distinct  $T$ -fixed points for  $Y$ . It is enough then to show that  $|\tilde{Y}^T| > 1$  to imply that  $|Y^T| > 1$ .

By the orbit-cone correspondence [9, Thm 3.2.6], since  $\dim \tilde{Y} > 0$ ,  $|\tilde{Y}^T| > 1$  as required. □

**Theorem 6.3.2.** *Let  $\psi_Q : BS^Q \rightarrow \widetilde{X}^w$  be as in Lemma 6.0.2. Then  $\psi_Q$  is an isomorphism iff it is a bijection on  $T$ -fixed points.*

*Proof.* One direction is obvious. For the other direction, observe that  $\psi_Q$  is birational by Lemma 6.0.2 and  $\widetilde{X}^w$  is weakly normal by Corollary 6.2.3, so it is enough to show that  $\psi_Q$  is finite and bijective to conclude that it is an isomorphism. By assumption,  $\psi_Q$  is a bijection on  $T$ -fixed points, so by  $B$ -equivariance we can use the  $B$ -action to show that it is bijective at all points. Since  $\psi_Q$  is also proper, we only need that  $\psi_Q$  is quasi-finite for the result to hold.

Consider the fibre  $Z_v = \psi_Q^{-1}(e_v)$ , which is a projective subvariety of  $BS^Q$ . The embedding of  $Z_v$  into projective space is equivariant with respect to the torus action on  $Z_v$  (since  $\psi_Q$  is  $B$ -equivariant). Under these conditions,  $\dim(Z_v) > 0 \Rightarrow |Z_v^T| > 1$  by Lemma 6.3.1. Since  $|Z_v^T| = 1$  by assumption, we conclude that  $\dim(Z_v) = 0$ .

Since  $\psi_Q$  is  $B$ -equivariant, the fibre of a point in  $Be_vB/B$  is a  $B$ -translate of  $Z_v$ . These Bruhat cells cover  $X^w$ , so  $\psi_Q$  is quasi-finite. □

An understanding of when the Bott-Samelson resolution  $\varphi_Q$  is an isomorphism over the smooth locus might be useful in determining information about  $\psi_Q$ . A discussion about these strict Bott-Samelson maps as well as examples on how reducing to  $T$ -fixed points can be a powerful tool can be found in Appendix A.

While these results suggest that  $\widetilde{X}^w$  is somewhere between a Bott-Samelson variety and a Schubert variety, the hope is that  $\widetilde{X}^w$  is itself a Bott-Samelson variety or some simple quotient of one. This will be the topic of future research.

## APPENDIX A

### STRICT BOTT-SAMELSON VARIETIES

In 1958, Bott and Samelson introduced certain spaces [5] that provided convenient desingularizations of Schubert varieties (which were later generalized by Hansen [16] and Demazure [11]). Around the same time, Hironaka published his famous result on the resolution of singularities for algebraic varieties in characteristic zero [17]. His result could of course be applied to Schubert varieties to obtain a second, very different sort of desingularization. While Hironaka's method is a stronger result in general, the fact that Bott-Samelson resolutions are the preferred desingularization method would speak to the combinatorial conveniences they possess. What if we could get the best of both methods?

An important feature of a Hironaka desingularization (such as the algorithm in [4]) is the fact that the desingularization map is an isomorphism over the smooth locus. One calls such a desingularization a **strict resolution** of singularities. In general, the Bott-Samelson resolution is not an isomorphism over the smooth locus of a Schubert variety. Even more, while Hironaka's method utilizes blow-ups, a Bott-Samelson resolution is not in general a blow-up map (see Section A.3). Nevertheless, the resolution has many combinatorial properties that are natural for working with Schubert varieties. There is for example an action of a torus  $T$  with isolated fixed points, and the map on  $T$ -fixed points is especially easy to utilize.

Similar constructions to Bott-Samelson varieties can be considered for the purpose of strict resolutions. For example, in [8], Cortez introduces quasi-resolutions of Schubert varieties and studies their singularities in terms of intersections of the exceptional loci. This approach extends the work in [7] where

strict resolutions are provided for covexillary Schubert varieties. For the general case however, these quasi-resolutions provide a partial desingularization that may not necessarily be strict.

We will explore both desingularization algorithms in more detail, and suggest a more general method for resolving singularities which is close to the Bott-Samelson construction, and yet is a strict desingularization. We can do this by taking advantage of a generalized Bott-Samelson resolution (see Section 2.5). In Section A.4 we implement a computer program to verify which Schubert varieties can be resolved strictly using this generalized method. We will verify the following result:

**Theorem A.0.1.** *Let  $X^w$  be a singular Schubert variety. If  $w \in S_5$ , then there exists a strict Bott-Samelson resolution of singularities for  $X^w$  iff  $w \neq 45312$ . If  $w \in S_6$ , then there exists a strict Bott-Samelson resolution of singularities for  $X^w$  if  $w \not\in \{156423, 453126, 632541\}$ .*

Unfortunately the converse of the theorem for the  $n = 6$  case does not hold (see Example A.4.6). The longer list of exceptions in the  $n = 6$  case mostly relate to a pattern embedding of 45312, which suggested a recursive construction for such a list in general. The conjecture for  $n > 4$  in Section A.4 was formulated based on these results. Since there are other reasonable interpretations of the  $n = 5, 6$  computer results, it seemed necessary to verify that the conjecture was true at least for  $n = 7, 8$  and indeed this is the case.

The curiosity of whether a Bott-Samelson resolution could be viewed as a blow-up (bringing the technique closer to Hironaka's version) led to the work in Section A.3. While no general theorems are provided in this section, a new viewpoint of the resolution is explored and some groundwork is provided for

further research. The material in this appendix is the content presented in [10].

## A.1 Schubert variety singularities

There is a well-known way to tell whether a Schubert variety is singular:  $X^w$  is smooth iff it avoids the patterns 3412 and 4231. While this answers the question about which Schubert varieties are singular, we need to know where they are singular if we are going to talk about  $\varphi_Q$  being strict.

This question can be answered using interval pattern avoidance [28, §2]. Theorem 6.1 in the same article provides the necessary intervals for finding the singular locus. Roughly put, given an interval  $[u, v]$  on the list from the theorem ( $u \leq v$  are both permutations), if  $v$  occurs as a pattern in  $w$ , then a similarly embedded  $u$  provides a piece of the singular locus of  $X^w$ .

As an example,  $[2143, 4231]$  is on the list mentioned above. Since 4231 is clearly a subpattern of itself, the singular locus of  $X^{4231}$  contains the Schubert subvariety  $X^{2143}$ . The only other applicable interval on that list is  $[1324, 3412]$ , and since 3412 is not a subpattern of 4231, then  $X^{2143}$  is in fact the entire singular locus of  $X^{4231}$ .

The details about interval pattern avoidance are not important for the purpose of the sections that follow. These details were necessary for implementing computer code which checked which  $X^w$  could be resolved using strict  $\varphi_Q$ .

## A.2 Hironaka's desingularization

Given a variety  $X$  defined over an algebraically closed field of characteristic 0, there exists a sequence of blow-ups

$$X = X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} \dots \xleftarrow{\sigma_k} X_k \xleftarrow{\sigma_{k+1}} \dots \xleftarrow{\sigma_t} X_t$$

such that  $X_t$  is smooth (here  $X_{j+1}$  denotes the strict transform of  $X_j$ ) and each  $\sigma_{j+1}$  has smooth center  $C_j \subset \text{Sing}(X_j)$  (see [4]).

The statement of a Hironaka desingularization usually includes conditions on the exceptional divisors (such as being simple normal crossings with the strict transform). We will ignore these for now and instead focus on two main features. First, the center  $C_j$  is contained in the singular locus of  $X_j$ . Second, the desingularization map is a sequence of blow-up maps. The reader will probably notice that the Bott-Samelson resolution has neither of these properties, so a comparison between each desingularization method might be difficult to arrange. Instead we will focus on finding a suitable subvariety of  $X^w$  to call the center of  $\varphi_Q$ .

Let us consider a motivational example to observe the similarities between each desingularization technique. Consider  $X^{4231}$ , one of the two simplest singular Schubert varieties. The permutation 4231 can be written as the word  $s_1 s_2 s_3 s_2 s_1$  in simple reflections. Let  $Q = (s_1, s_2 s_3 s_2 s_1)$ . It is not hard to show that



$$\varphi_Q : \overline{BS_1B} \times^B \overline{BS_2S_3S_2S_1B}/B \rightarrow X^{4231}$$

is a strict Bott-Samelson resolution. As we will see later in Section A.4, it is enough to check that this is true at  $T$ -fixed points. In other words, we are examining the product map on Bruhat intervals

$$[e, s_1] \times [e, s_2s_3s_2s_1] \rightarrow [e, s_1s_2s_3s_2s_1].$$

We observe that this map is 2-to-1 over  $\{e, s_1, s_3, s_1s_3\}$  (for example,  $(e, s_3)$  and  $(s_1, s_3s_1)$  are the fibre over  $s_3$ ) and hence one-to-one everywhere else by a cardinality check. By [28],  $X^{4231}$  is singular precisely at the  $T$ -fixed points  $\{e, s_1, s_3, s_1s_3\}$ . Therefore  $\varphi_Q$  is a strict Bott-Samelson resolution of singularities (notice that  $BS^Q$  is smooth).

To observe what a Hironaka resolution would do (at least locally), we should work in an affine neighborhood on the Schubert variety, and compute some Kazhdan-Lusztig varieties. We can show that the Kazhdan-Lusztig ideal of  $X^{4231}$  near the identity is generated by  $z_{21}z_{12} - z_{22}z_{11} = 0$  (see Section 2.1). The center of blowing-up in a Hironaka resolution must then be the origin. It is also easy to see that the strict transform is smooth.

How does this relate back to the Bott-Samelson construction? Observe that the  $T$ -fixed points for which the map was not injective were  $\{1, s_1, s_3, s_1s_3\}$ . This means that  $\varphi_Q$  is injective away from  $X^w$  for  $w \leq 2143$ . We can compute the Kazhdan-Lusztig ideal for the singular locus  $X^{2143}$  (in the same chart  $X_{\text{id}}^0$ ) to observe that it is locally defined by  $z_{11} = z_{12} = z_{21} = z_{22} = 0$ , the origin. This corresponds to the same center as a Hironaka desingularization algorithm!

Although this local picture is encouraging, we have to remember that  $\varphi_Q$  is not the same thing as the blow-up map at the origin. In fact, the preimage of the singular locus in  $\varphi_Q$  is not even a divisor in  $BS^Q$ !

This suggests that if we want to make the Bott-Samelson resolution look more like a blow-up, we will have to also blow-up points where the map is already one-to-one in order to get a higher dimensional fibre. If we can find a subvariety  $Y$  in  $X^w$  which contains the singular locus such that the fibre of  $\varphi_Q$  over  $Y$  is a divisor, then the Bott-Samelson resolution will be one step closer to behaving like a blow-up.

### A.3 Comparing Bott-Samelson resolutions to blow-ups

To make a fair comparison between the desingularization maps we are considering, we begin by asking what relationship a blow-up map has with the Bott-Samelson resolution. A simple application of the universal property for blow-ups gives us a starting point.

**Lemma A.3.1.** *Let  $\varphi : X \rightarrow Z$  be a proper birational map. Then there exists a Cartier divisor  $D \subset X$  such that  $\varphi$  factors through a blow-up of  $Z$  along  $Y = \varphi(D)$ .*

*Proof.* Since  $\varphi$  is birational, it is generically one-to-one. Therefore there is a subvariety  $X_{ni} \subset X$  such that  $\varphi$  is an isomorphism everywhere except on  $X_{ni}$ . We can find a Cartier divisor  $D$  containing  $X_{ni}$  (the product of the local equations for  $X_{ni}$  provide a local defining equation for  $D$ ). Since  $\varphi$  is proper,  $Y$  is a closed subvariety of  $Z$ . Now  $\varphi$  is an isomorphism on  $D \setminus X_{ni}$ , and  $X_{ni}$  is the full preimage of its image; therefore  $\varphi^{-1}(Y) = D$ . By the universal property of blow-ups,  $\varphi$  factors

through the blow-up of  $Z$  along  $Y$ , as required.  $\square$

**Proposition A.3.2.** *Let  $\varphi_Q : BS^Q \rightarrow X^w$  be a Bott-Samelson resolution of  $X^w$  where  $Q$  is a reduced word for  $w$ . Then there exists a Cartier divisor  $D \subset BS^Q$  such that  $\varphi_Q$  factors through a blow-up of  $X^w$  along  $\varphi_Q(D)$ . Furthermore,  $\text{codim}(\varphi_Q(D)) \geq 2$  iff  $D = BS_{ni}^Q$ .*

*Proof.* The existence of  $D$  follows from Lemma 2.5.3 and Lemma A.3.1.

Suppose  $D = BS_{ni}^Q$ . Notice that  $\text{codim}(\varphi_Q(BS^R)) \geq 2$  if  $R$  is not reduced. Since  $BS_{ni}^Q$  is the union of such  $BS^R$ ,  $\text{codim}(\varphi_Q(D)) \geq 2$ .

Next suppose that  $\text{codim}(\varphi_Q(D)) \geq 2$ . If  $D \setminus BS_{ni}^Q \neq \emptyset$ , then  $\varphi_Q(D)$  has codimension one in  $Y$  since  $\varphi_Q$  is an isomorphism away from  $BS_{ni}^Q$  (and  $BS^Q$  and  $X^w$  have the same dimension). This contradicts our assumption, hence  $D = BS_{ni}^Q$ .  $\square$

Unfortunately, we cannot claim that  $\varphi_Q$  is itself the blow-up of  $X^w$  along  $Y$  since  $\varphi_Q$  may not satisfy the universal property. One barrier to this is when  $D$  from Proposition A.3.2 is not irreducible. In this case, it might be possible to blow-down one of the irreducible components of  $D$  to get another map where the inverse image of  $Y$  is a Cartier divisor. To put the mind of the reader at ease as to whether such a situation occurs, the next proposition provides such an example. Therefore, when  $D$  necessarily has more than one irreducible component, it is not obvious under what conditions  $\varphi_Q$  is a blow-up. In the proposition that follows, recall that  $Q = 132312$  is a shorthand for  $Q = (s_1, s_3, s_2, s_3, s_1, s_2)$  (see Section 2.5).

**Proposition A.3.3.** *The Bott-Samelson resolution  $\varphi_Q$  for  $Q = 132312$  is not a blow-up map.*

*Proof.* It can be checked that  $BS_{ni}^Q$  is a union of three codimension 1 sub Bott-Samelson varieties, namely

$$BS^{13-312}, BS^{132-12} \text{ and } BS^{1323-2}.$$

Suppose that  $\varphi_Q$  is a blow-up of  $X^{4321}$  along some center  $C$ . Since any blow-up is an isomorphism away from its center,  $C$  necessarily contains  $\varphi_Q(BS_{ni}^Q)$ . Then  $\varphi_Q^{-1}(C)$  is the union of the three  $BS^R$  listed above and possibly some other divisor  $E$ . Now consider the following commuting diagram

$$\begin{array}{ccc} BS^Q & \xrightarrow{\psi} & BS^{Q'} \\ & \searrow \varphi_Q & \downarrow \varphi_{Q'} \\ & & X^{4321} \end{array}$$

where  $Q' = (s_1 s_3 s_2 s_3, s_1, s_2)$ . It's not hard to check that  $\varphi_{Q'}^{-1}(C)$  is the union of two codimension 1 sub Bott-Samelson varieties, namely  $BS^{R'}$  for  $R' = (s_1 s_3 s_2, s_1, s_2), (s_1 s_3 s_2 s_3, e, s_2)$ , as well as  $\psi(E)$  (the map  $\psi$  has collapsed down the  $BS^{13-312}$  component of  $BS_{ni}^Q$ ). Note that both  $BS^Q$  and  $BS^{Q'}$  are smooth, so all the above divisors are automatically Cartier. By the universal property of blow-up maps, there exists a fibre diagram:

$$\begin{array}{ccc} BS_{ni}^{Q'} \cup \psi(E) & \longrightarrow & BS^{Q'} \\ \downarrow & & \downarrow \pi \\ BS_{ni}^Q \cup E & \longrightarrow & BS^Q \\ \downarrow & & \downarrow \varphi_Q \\ C & \longrightarrow & X^w \end{array}$$

Since  $BS^Q$  and  $BS^{Q'}$  have the same dimension, the preimage of a divisor under  $\pi$  must again be a divisor (it clearly cannot be all of  $BS^{Q'}$ ). Therefore, the

preimage of each irreducible component of  $BS_{ni}^Q$  must be a union of irreducible components in  $BS_{ni}^{Q'}$ . There are however 3 components in  $BS_{ni}^Q$  and only 2 components in  $BS_{ni}^{Q'}$ , a contradiction.

□

Note that  $BS_{ni}^Q \cup E$  from the proposition is an example of a choice for  $D$  from the beginning of the section. Interestingly enough, choosing another reduced expression  $Q$  for 4321 will avoid many of the problems mentioned in Proposition A.3.2.

Since we cannot draw a similar factorization diagram for  $BS^{Q'}$  as we did for  $BS^Q$ , it might be true that  $\varphi_{Q'}$  is a blow-up, and that choosing our Cartier divisor  $D$  with more than one component is acceptable (although this is not immediately clear).

Let us now consider those cases for which an irreducible  $D$  can be chosen. If  $Y = \varphi_Q(D)$  is a Cartier divisor in  $X^w$ , the blow-up of  $X^w$  along  $Y$  is an isomorphism and clearly won't be the Bott-Samelson resolution (except in the rare instance that  $\varphi_Q$  was already an isomorphism). The next example shows how obvious choices of  $D$  can potentially lead to this problem.

**Example A.3.4.** In Section A.2, we saw that  $\varphi_Q$  for  $Q = (r_1, r_2 r_3 r_2 r_1)$  was a strict Bott-Samelson resolution for  $X^{4231}$  (whose singular locus is  $X^{2143}$ ). The natural choice for any  $D$  from Lemma A.3.1 is some combination of codimension 1 sub Bott-Samelson varieties. These  $B$ -invariant divisors after all generate the effective cone of  $BS^Q$  since  $Q$  is reduced (see [1] for commentary on divisor classes of Bott-Samelson varieties). It is not hard to check however that  $R$  is reduced for any codimension 1  $BS^R$  that contains the preimage of  $X^{2143}$ . In other words,

$\text{codim}(BS_{ni}^Q) > 1$  and any choice of  $D$  will result in  $\text{codim}(\varphi_Q(D)) = 1$  by Lemma A.3.1. Since  $X^{4231}$  is not factorial, we must consider whether  $\varphi_Q(D)$  is a Cartier divisor. One can verify that the image of any codimension 1  $BS^R$  is in fact not Cartier. We have thus found an irreducible  $D$  containing  $BS_{ni}^Q$  for which  $\varphi_Q(D)$  is a viable center.  $\square$

On the other hand, if  $Y$  is not Cartier then the blow-up map is not trivial. By the proposition, we have the following fibre diagram:

$$\begin{array}{ccc}
 D & \longrightarrow & BS^Q \\
 \downarrow & & \downarrow \pi \\
 E & \longrightarrow & Bl_Y(X^w) \\
 \downarrow & & \downarrow \\
 Y & \longrightarrow & X^w
 \end{array}$$

Then both  $D$  and  $E$  are irreducible Cartier divisors. Since  $\varphi_Q$  and the blow-up map are proper and birational,  $\pi$  must also be proper and birational. If we know that  $Bl_Y(X^w)$  is normal (which is equivalent to  $\mathcal{I}(Y)^k$  being integrally closed for all  $k > 0$ ), then any finite birational map to it must be an isomorphism. Under these conditions, the problem has been reduced to checking whether  $\pi$  is quasi-finite to conclude that  $\varphi_Q$  is isomorphic to a blow-up. This boils down to understanding the fibres of  $\varphi_Q$  and comparing that to the fibres of the blow-up. Even this question proves difficult for all but the simplest cases. See [14] and [15] for more information on the fibres of a Bott-Samelson resolution.

Let us conclude this section with a final thought. If  $D$  can be chosen to be  $BS^R$  where  $R$  is not reduced, then  $Y = \varphi_Q(D)$  is a Schubert subvariety since  $\varphi_Q$  is  $B$ -equivariant and  $BS^R$  is  $B$ -invariant. The question of blowing-up  $X^w$  along a  $B$ -invariant subspace might provide extra information by extending the  $B$ -action

to the total transform. This prompts the question, when is a Hironaka resolution of  $X^w$  actually  $B$ -equivariant? Such an equivariant resolution of singularities exists and is discussed in Proposition 3.9.1 of [22].

## A.4 Finding strict Bott-Samelson resolutions

In this section we will drop our previous attempts to view  $\varphi_Q$  as a blow-up, and instead focus on whether it is possible to resolve the singularities of  $X^w$  by a sequence of strict  $\varphi_Q$  maps. Let us first show that it is enough to restrict our attention to generalized  $BS^Q$  maps for  $Q = (w_1, w_2)$ .

Let  $R = (v_1, v_2, v_3)$ . We can view  $\varphi_R$  as  $\varphi_{R''} \circ (\varphi_{R'} \widetilde{\times} \text{id}_{v_3})$  where  $R' = (v_1, v_2)$  and  $R'' = (v_1 * v_2, v_3)$  (each is surjective). The map  $\varphi_{R'} \widetilde{\times} \text{id}_{v_3}$  takes  $[g_1, g_2, g_3]$  to  $[g_1 g_2, g_3]$  (the notation  $\widetilde{\times}$  is taken from the theory of affine Grassmannians which we will not stop to define here). Then  $\varphi_R$  being strict implies that  $\varphi_{R'}$  and  $\varphi_{R''}$  are also strict. The same is true if  $R' = (v_2, v_3)$  and  $R'' = (v_1, v_2 * v_3)$ . By induction we get the following result:

**Lemma A.4.1.** *Let  $X^w$  be a singular Schubert variety. Any strict Bott-Samelson resolution of  $X^w$  by a sequence of  $\varphi_{Q_k}$  can be factored into a sequence of strict  $\varphi_{R_i}$  maps where each  $R_i = (v_1^i, v_2^i)$ .*

In other words, it is enough to check strictness for  $\varphi_Q$  where  $Q$  is a 2-tuple. Given a Schubert variety  $X^w$ , and given a reduced  $w = s_{i_1} s_{i_2} \dots s_{i_p}$ , we ask whether we can write  $w = w_1 * w_2$  where  $w_1 = s_{i_1} \dots s_{i_q}$  and  $w_2 = s_{i_{q+1}} \dots s_{i_p}$  such that

$$\overline{Bw_1B} \times^B \overline{Bw_2B}/B \rightarrow X^w$$

is strict? Each  $\overline{Bw_iB}/B$  is a Schubert variety, and if it is not smooth we hope that we can continue this process until we have a smooth Bott-Samelson variety. Therefore, we wish to find

$$\varphi_{Q_i} : BS^{Q_i} \rightarrow \overline{Bw_iB}/B$$

for each  $i$ , where the product of these two maps provides a strict map onto  $\overline{Bw_1B} \times^B \overline{Bw_2B}/B$ . Notice however that the  $Q_i$  might use a different reduced form than that used in  $w_i$ . This process is highly dependent on how we write the word in simple reflections as only certain reduced words will provide strict resolutions. For this reason finding a strict resolution of singularities is not just about finding some  $\overline{Bw_1B} \times^B \dots \times^B \overline{Bw_kB}/B$  with the usual map onto  $X^w$ , but rather some sequence of Bott-Samelson maps whose composition may not be the standard product map  $\varphi_Q$ .

Let us make one final reduction which will make it possible to use a computer search to find strict  $\varphi_Q$ .

**Lemma A.4.2.** *Let  $\varphi_Q$  be a Bott-Samelson resolution onto  $X^w$  with  $Q$  reduced. Then  $\varphi_Q$  is an isomorphism iff it is a bijection on  $T$ -fixed points.*

*Proof.* This is a variation of Theorem 6.3.2. □

It is therefore enough to check strictness on  $T$ -fixed points. The map

$$\overline{Bw_1B} \times^B \overline{Bw_2B}/B \rightarrow X^w$$

will be strict iff the product map on Bruhat intervals is strict:



$$[e, w_1] \times [e, w_2] \rightarrow [e, w].$$

**Example A.4.3.** Let us try and strictly resolve  $X^{3412}$ , the other singular Schubert variety for the  $n = 4$  case (the variety  $X^{4231}$  was discussed in Section A.2). The word  $s_2s_1s_3s_2$  is a reduced expression for this permutation. Taking  $Q = (s_2, s_1s_3s_2)$ , we want to know if  $\varphi_Q$  is strict. By Lemma A.4.2, it is enough to check this on  $T$ -fixed points, so consider the product map

$$[e, s_2] \times [e, s_1s_3s_2] \rightarrow [e, s_2s_1s_3s_2].$$

The map is obviously 2-to-1 over  $\{e, s_2\}$ . Checking the cardinalities of the intervals, it is not hard to see that the map is a bijection everywhere else. By [28],  $X^{3412}$  is only singular on  $X^{1324}$ , or  $\{e, s_2\}$  in terms of reduced words for the  $T$ -fixed points. Therefore  $X^{3412}$  can be resolved strictly.  $\square$

The problem of finding strict Bott-Samelson resolutions has its own difficulties, and like the previous section, the answer is not very satisfying. We would like to say that there always exists a  $Q = (w_1, w_2)$  such that  $\varphi_Q$  is strict. This is not the case, however, since there are permutations for which any choice of  $Q$  will not be strict (see below). This leads to a further complication. What if initially there exists a  $Q = (w_1, w_2)$  such that  $\varphi_Q$  is strict, but then  $w_2$  is a reduced word for one of these non-strict resolutions?

Another fundamental barrier to proving general results in this section is the lack of tools for identifying the singularities of  $X^w$  based on a reduced word for the permutation  $w$ . Indeed interval pattern embeddings for permutations don't

translate very well into criteria involving reduced words. Therefore, the verification of which permutations possess a strict desingularization had to proceed using a computer search.

#### A.4.1 The $n = 5$ case

Consider  $X^w$  for  $w \in S_5$ . There are certainly more cases to consider than the two from  $n = 4$ . A computer check through all the possibilities (this potentially means every division of every reduced word for  $w$ ) reveals that the only problematic Schubert variety is  $X^{45312}$ . The permutation 45312 can be written in simple reflections as  $s_2s_3s_2s_1s_4s_2s_3s_2$ , and no division will provide a strict resolution. We can check that  $X^{45312}$  is smooth at  $X^{21345}$  and  $X^{12354}$  (in reduced words this is  $s_1$  and  $s_4$  respectively).

Any division of  $s_2s_3s_2s_1s_4s_2s_3s_2$  into two will produce a  $\varphi_Q$  with a fibre over  $s_4$ . This is simply because any choice of  $Q = (w_1, w_2)$  must satisfy  $s_2 \leq w_1, w_2$ . Therefore,  $(s_2, s_4s_2)$  or  $(s_2s_4, s_2)$  will be a fibre over  $s_4$  (in addition to the obvious  $(e, s_2)$  or  $(s_2, e)$ ). A computer check shows that no other reduced word works. In fact, it is not too hard to convince yourself of this by hand.

A priori, it is possible for some other permutation  $v$  to be divided in such a way that the only resolution is

$$\overline{Bv_1B} \times^B \overline{Bv_2B}/B \rightarrow X^v$$

where  $\overline{Bv_1B}/B = X^{45312}$  (and therefore no subsequent divisions are possible). However, this would mean that some subword of  $v$  was a reduced word for 45312. That is, it would require  $v > v_1$  in the Bruhat order.

Luckily, the only permutations above  $w$  in Bruhat order are 54312, 45321, 54321 which are all smooth already (and hence wouldn't be subject to this division process). We have thus shown:

**Proposition A.4.4.** *Let  $X^w$  be a singular Schubert variety for  $w \in S_5$ . There exists a strict Bott-Samelson resolution of singularities for  $X^w$  iff  $w \neq 45312$ .*

#### A.4.2 The $n = 6$ case

This case is substantially harder than the last, and even a computer check can take a long time to verify all possibilities. The list of Schubert varieties  $X^w$  for which no division of any reduced word for  $w$  works in defining a strict  $\varphi_Q$  are for

$$w = 156423, 453126, 456312, 465132, 465312, 546213, 546312, \\ 564123, 564132, 564213, 564312, 632541, 653421.$$

Unlike the  $n = 5$  case, there are indeed  $v \geq w$  for which  $X^v$  is not smooth. Hence if we started subdividing a word for  $v$ , we may eventually encounter a  $w$  on the list that could not be strictly resolved. To avoid this complication, we will only consider permutations which are not above any of the  $w$  on the list.

Because of this simplification, we can now reduce our list to

$$u = 156423, 453126, 632541$$

since any other  $w$  in the list above satisfies  $w \geq u$  for some  $u$ . We therefore have a weaker result than that for  $n = 5$ :

**Proposition A.4.5.** *Let  $X^w$  be a singular Schubert variety for  $w \in S_6$ . There exists a strict Bott-Samelson resolution of singularities for  $X^w$  if  $w \not\geq 156423, 453126$  or  $632541$ .*

A natural question to ask is whether the condition in the proposition is a closed one. In other words, is it true that  $w \geq 156423, 453126$  or  $632541$  results in  $X^w$  not having a strict Bott-Samelson resolution? Unfortunately this is not the case.

**Example A.4.6.** Consider the singular Schubert variety  $X^{456123}$ . Even though  $156423 \leq 456123$ ,  $X^{456123}$  still admits a strict Bott-Samelson resolution of singularities starting with  $\varphi_Q$  for  $Q = (s_3 s_4 s_5 s_2 s_3, s_4 s_1 s_2 s_3)$ .

While the second factor  $\overline{BS_{s_4 s_1 s_2 s_3} B} / B$  of  $BS^Q$  is smooth, the first is not, but can be strictly resolved using  $\varphi_R$  with  $R = (s_3, s_4 s_5 s_2 s_3)$ .

There are of course  $w > 156423, 453126$  or  $632541$  for which  $X^w$  cannot be strictly resolved. For example, the only strict Bott-Samelson map onto  $X^{256413}$  using  $\varphi_Q$  is with  $Q = (s_1, w')$  where  $w'$  is some reduced expression for  $X^{156423}$  (which we know cannot be strictly resolved).  $\square$

As the feasibility of continuing this process diminishes with each increase of  $n$ , it might be prudent to step back and try to find a pattern.

One observation is that  $156423$  and  $453126$  both have embedded versions of  $45312$ . Indeed, given a permutation  $\pi$  in  $S_m$ , we can define a new permutation  $\tau(k) \in S_n$  (for  $n > m$ ) by pattern embedding  $\pi$  into  $123 \cdots n$  starting at position  $k$  and ending in position  $k + m$ . We will call this a translation of  $\pi$  in  $S_n$ . For example, if  $\pi = 45312$ , then  $\tau(1) = 453126$  and  $\tau(2) = 156423$ . It is not a surprise

that translations of problematic permutations are also a problem for higher  $n$  as their reduced expressions in  $S_n$  still possess the same division problems as they did in  $S_m$ .

In actuality, the only new permutation for  $n = 6$  is 632541. This permutation could be formed by inverting the first and last value, the second and third value, and the third-last and second-last values (of course this is just one interpretation). In  $S_n$  this can be written as  $\pi_n = n324 \cdots (n-3)(n-1)(n-2)1$  for  $n > 6$  with  $\pi_6 = 632541$ . Let us now define a list  $\Gamma_n$  of permutations we want to avoid. We know  $\Gamma_5 = \{45312\}$ . Then in general let

$$\Gamma_n = \{\pi_n\} \cup \Gamma_{n-1}^T$$

where  $\Gamma_{n-1}^T$  are the translations of permutations in  $\Gamma_{n-1}$ . For example,

$$\Gamma_7 = \{1267534, 1564237, 1743652, 4531267, 6325417, 7324651\}$$

It is worth noting that  $|\Gamma_{n+4}| = \frac{n(n+1)}{2}$ . Indeed,  $\Gamma_{n+4}$  is just the union of the translations of 45312 and  $\pi_6$  up to  $\pi_{n+4}$ . There are  $n$  translations of 45312 and  $n + 5 - k$  translations of  $\pi_k$ . This gives a new interpretation of the triangular numbers. Many thanks to the reviewer of this article for making this observation.

We can now formulate a conjecture to generalize our work so far

**Conjecture A.4.7.** *The singular Schubert variety  $X^w$  for  $w \in S_n$  with  $n > 4$  can be desingularized using a sequence of strict Bott-Samelson resolutions if  $w \not\prec v$  for  $v \in \Gamma_n$ .*

**Theorem A.4.8.** *The conjecture is true for  $5 \leq n \leq 8$ .*

Future work towards this problem includes developing techniques for verifying the conjecture without the need for a computer program. This would mean a deeper understanding of how singularities can be more easily read from a reduced word for the permutation.

The precise geometric description of the singular locus given in [8] might prove useful in studying strict resolutions. In fact, strict resolutions for permutations in  $\Gamma_n$  might exist using the constructions in [7] and [8]. It is also interesting to note that some of the permutations in  $\Gamma_n$  appear in the interval pattern embedding description of the singular locus in [28].

Furthermore, a better understanding of how Bott-Samelson resolutions compare with a Hironaka resolution may provide a clearer path to creating hybrid desingularization algorithms for similar combinatorial objects.

## BIBLIOGRAPHY

- [1] D. Anderson. Effective divisors on Bott-Samelson varieties. *arXiv:1501.00034*, 2014.
- [2] A. Atanasov, C. Lopez, A. Perry, N. Proudfoot, and M. Thaddeus. Resolving toric varieties with Nash blowups. *Exp. Math.*, **20** (3):288–303, 2011.
- [3] A. Bertiger. Generating the ideals defining unions of Schubert varieties. *Inter. Math. Research Notices*, **2015** (21):10847–10858, 2015.
- [4] E. Bierstone and P.D. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.*, **128**:207–302, 1997.
- [5] R. Bott and H. Samelson. Applications of the theory of Morse to symmetric spaces. *American Journal of Mathematics*, **80**:964–1029, 1958.
- [6] M. Brion and S. Kumar. *Frobenius Splitting Methods in Geometry and Representation Theory*. 2005.
- [7] A. Cortez. Singularités génériques des variétés de Schubert covexillaires. *Ann. Inst. Fourier (Grenoble)*, **51** (2) :375–393, 2001.
- [8] A. Cortez. Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire. *C. R. Acad. Sci. Paris*, **333** :561–566, 2001.
- [9] D. Cox, J. Little, and H. Schenck. *Toric Varieties: Graduate Studies in Mathematics Volume 124*. 2011.
- [10] S. Da Silva. Strict bott-samelson resolutions of schubert varieties. *Exp. Math*, 2017.
- [11] M. Demazure. Désingularisation des variétés de Schubert généralisées. *Annales Scientifiques de l'École Normale Supérieure*, **7** (4):53–88, 1974.
- [12] D. Eisenbud. *Commutative algebra with a view towards algebraic geometry*. 1995.
- [13] L. Escobar. Brick manifolds and toric varieties of brick polytopes. *Electronic Journal of Combinatorics*, **23** (2):2–25, 2016.

- [14] S. Gaussent. The fibre of the Bott-Samelson resolution. *Indag. Math. (N.S.)*, **12** (4):453–468, 2001.
- [15] S. Gaussent. Erratum to: The fibre of the Bott-Samelson resolution. *Indag. Math. (N.S.)*, **16** (2):179–180, 2005.
- [16] H.C. Hansen. On cycles in flag manifolds. *Mathematica Scandinavica*, **33**:269–274, 1973.
- [17] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero: I. *Ann. of Math. (2)*, **79** (1):109–203, 1964.
- [18] A. Knutson. Schubert patches degenerate to subword complexes. *Transform. Groups*, **13**:715–726, 2008.
- [19] A. Knutson. Frobenius splitting, point-counting, and degeneration. *arxiv:0911.4941*, 2009.
- [20] A. Knutson and E. Miller. Subword complexes in coxeter groups. *Adv. Math.*, **184** (1):161–176, 2004.
- [21] A. Knutson and E. Miller. Gröbner geometry of schubert polynomials. *Ann. of Math.*, **161** (3):1245–1318, 2005.
- [22] J. Kollár. *Lectures on Resolution of Singularities*. Princeton NJ: Princeton University Press, 2007.
- [23] S. Kumar and V. Mehta. Finiteness of the number of compatibly split subvarieties. *Int. Math. Research Notices*, **19**:3595–3597, 2009.
- [24] V. Lakshmibai, V. Mehta and A. Parameswaran. Frobenius splittings and blow-ups. *J. Algebra*, **208**:101–128, 1998.
- [25] Y. Ma. The first syzygies of determinantal ideals. *J. Pure and Applied Alg.*, **85**:73–103, 1993.
- [26] K. Smith. Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties. *Michigan Math. J.*, **48**:553–572, 2000.
- [27] M. Vitulli. Weak normality and seminormality. *arxiv:0906.3334*, 2009.



- [28] A. Woo and A. Yong. Governing singularities of Schubert varieties. *J. Algebra*, **320** (2):495–520, 2008.
- [29] A. Woo and A. Yong. A Gröbner basis for Kazhdan-Lusztig ideals. *American J. of Math*, **134** (4):1089–1137, 2012.