# ON THE MINIMALITY OF NON-$\sigma$-SCATTERED ORDERS

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In this dissertation we study the minimality of non-$\sigma$-scattered orders. While there are insightful theorems, due to Laver, about $\sigma$-scattered orders, we will show the class of non-$\sigma$-scattered orders tend to be more chaotic by a number of consistency results. For instance, we show if there is a supercompact cardinal, there is a forcing extension in which there is no minimal non-$\sigma$-scattered linear order. This shows that Laver's theorem regarding $\sigma$-scattered linear orders is sharp. Our work also includes results concerning trees. For instance, we show it is consistent that there is a Kurepa tree which is minimal with respect to club embeddings. Moreover, we show it is consistent that there is a minimal non-$\sigma$-scattered linear order which does not contain any real or Aronszajn type. Working on these problems resulted in a few byproduct theorems as well.

## BIOGRAPHICAL SKETCH

Hossein Lamei Ramandi was born in July 31, 1985, in Karaj, Iran, to parents Hassan Lamei Ramandi and Akram Mirzabaygi. After finishing high school in Karaj, Hossein pursued his education in Mathematics at University of Tehran. In 2012, he joined Cornell University to study Set Theory.

Dedicated to my parents.

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CHAPTER 1

# INTRODUCTION

## 1.1 History and motivation

In this dissertation we study the structure of the class of linear orders that are not a countable union of scattered suborders. Recall that a linear order is *scattered* if it does not contain a copy of a countable dense linear order with at least two elements. Also by a theorem due to Cantor a linear order is scattered if and only if it does not have a copy of the rationals. If a linear order can be represented as a countable union of scattered suborders we call it $\sigma$*-scattered*. After some history — and motivation which is in the fabric of history — we present some technical prerequisites.

### 1.1.1 Early studies

Scattered linear orders were first studied by Hausdorff. His characterization states that the class of all scattered orders is the smallest class of linear orders which contains the singleton orders and is closed under well-ordered and reverse well-ordered sums. Hausdorff also showed that any scattered linear order of cardinality $\kappa$ has to contain a copy of $\kappa$ or $\kappa^*$, the reverse of $\kappa$. In [11], Laver confirmed a long standing conjecture due to Fraïssé asserting that the class of countable linear orders is *well quasiordered* — every descending chain and antichian of countable linear orders is finite.

Here the class of linear orders is ordered by the embeddability relation. In fact, Laver proved the broader class of all *$\sigma$-scattered* linear orders is well quasiordered.

In the final paragraph of [11], Laver writes, "Finally, the question arises as to how the order types outside of the class $\mathfrak{M}$ of $\sigma$-scattered linear orders, behave under embeddability." For instance, is it possible to prove the conclusion of Laver's theorem for a broader class of linear orders? In other words:

**Question 1.1.1.** *Does there always exist a non-$\sigma$-scattered linear order which embeds into all of its non-$\sigma$-scattered suborders?*

This question was in some ways suggested by Fraïssé. After he articulated his conjecture for the class of countable linear orders, he revised his conjecture by extending it to the class of scattered linear orders [1].

### 1.1.2 Minimal real and Aronszajn types

Not very long after Laver's theorem, Baumgartner proved in [3] that PFA — the proper forcing axiom — implies that all $\aleph_1$-dense sets of reals are isomorphic. Note that by Hausdorff's result, any uncountable $\sigma$-scattered linear order has to contain a copy of $\omega_1$ or $\omega_1^*$. Therefore Baumgartner's result shows that it is consistent to have minimal non-$\sigma$-scattered linear orders. In fact, in the same paper Baumgartner mentions that his result shows that it is possible to extend the class $\mathfrak{M}$ of $\sigma$-scattered linear orders and still consistently have the conclusion of Laver's theorem for that class.

Later, Abraham and Shelah showed in [1] that under PFA every two $\aleph_1$-dense nonstationary *Countryman lines* are either isomorphic or reverse isomorphic. An

[1]See the first page of [11].

Aronszajn line $A$ is said to be *nonstationary* if there is a continuous $\subset$-sequence $\langle D_\xi : \xi \in \omega_1 \rangle$ of countable subsets of $A$ which cover $A$ and which has the property that for all $\xi \in \omega_1$, no maximal interval of $A \setminus D_\xi$ has a greatest or least element. A linear order is called Countryman if its square is a countable union of chains. Galvin had noticed that Countryman lines do not have separable uncountable suborders or copies of $\omega_1$ and $\omega_1^*$. Moreover, every Countryman line contains an $\aleph_1$-dense nonstationary suborder. Therefore their result also introduces new non-$\sigma$-scattered linear orders that can consistently be minimal. It should be mentioned that later Todorcevic showed that $MA_{\omega_1}$ implies that every two Countryman lines are either isomorphic or reverse isomorphic [21]. Also a stronger statement appeared in [13], asserting that under PFA the class of Aronszajn lines is well quasiordered. The results mentioned so far imply that there are various non-$\sigma$-scattered linear orders that are consistently minimal.

Abraham and Shelah's work also contained theorems involving isomorphism types of Aronszajn trees.

**Theorem 1.1.2.** *[1]* PFA *implies that every two Aronszajn trees are club isomorphic.*

Aronszajn trees were first introduced by Kurepa [20, page 246]. Aside from Aronszajn trees, Kurepa introduced the notion of Kurepa trees, $\omega_1$-trees which have at least $\omega_2$ many branches. While Aronszajn trees exist in all models of $ZFC$, Kurepa trees consistently do not exist if there are some large cardinals. Silver proved from the existence of an inaccessible cardinal that it is consistent that they do not exist, and Solovay showed if $\mathbf{V} = \mathbf{L}$ then there are Kurepa trees[6, page 53].

Although there is a powerful structural theorem regarding the club isomorphisms of Aronszajn trees, similar questions regarding Kurepa trees do not seem to be addressed. For instance,

**Question 1.1.3.** *Is it consistent that there is a Kurepa tree which is minimal with respect to club embeddings?*

Baumgartner's result, as he puts it, is a generalization of Cantor's theorem about countable dense linear orders. Regarding higher densities, [2] Kurepa lines are one of the natural order types of density $\aleph_1$ to study. Here a Kurepa line is a linear order of cardinality at least $\aleph_2$ which does not contain a copy of any uncountable subset of the reals and which has a dense subset of cardinality $\aleph_1$.[3] Baumgartner's theorem motivated us to ask the following question.

**Question 1.1.4.** *Is it consistent that there is a minimal Kurepa line?*

The results mentioned so far are all in the direction of enlarging the class of $\sigma$-scattered linear orders and obtaining a class which is closed under taking suborders and which is well quasiordered. Although the class of *real types* — $\aleph_1$-sized subsets of the reals — is not well quasiordered under CH [7], PFA implies that the class of $\sigma$-scattered linear orders union with the class of real types is well quasi ordered [3]. Almost similar statements are valid for the class of Countryman lines. If $2^{\aleph_0} < 2^{\aleph_1}$ the class of Countryman lines is not well quasiordered [1], [12, page 4]. On the other hand, under PFA one can add these linear orders to the class of $\sigma$-scattered linear orders and obtain a broader class which is closed under taking suborders and which is well quasiordered.

In [4], Baumgartner introduced a new class of non-$\sigma$-scattered linear orders which are never minimal with respect to being non-$\sigma$-scattered. More precisely,

[2]The density of a linear order $L$, is the minimum cardinality of $X \subset L$ such that $X$ is dense in $L$.

[3]See [20], page 251.

the class of all linear orders that are of the form of Baumgartner's example has elements which are not $\sigma$-scattered and which are not minimal with respect to not being $\sigma$-scattered in any outer model with the same $\aleph_1$. His construction can be described as the lexicographic ordering on a family $\{C_\alpha : \alpha \in S\}$ where $S \subset \lim(\omega_1)$ is stationary and $C_\alpha$ is a strictly increasing $\omega$-sequence in $\alpha$ for each $\alpha \in S$. We refer to such a linear ordering as *Baumgartner type*. Although he introduces new type of linear order — non-$\sigma$-scattered linear orders which do not contain any of the previously known non-$\sigma$-scattered linear orders — he did not study to what extent his result is sharp. More precisely,

**Question 1.1.5.** *Does there exist a non-$\sigma$-scattered linear order which does not contain any real type, Aronszajn type, Baumgartner type or the reverse of a Baumgartner type?*

### 1.1.3 Order types with no real or Aronszajn suborders

Recall that we would like to understand to what extent Laver's theorem is sharp in the sense of Question 1.1.1. Therefore the first — and perhaps the main — starting point for such a study is to determine what the main witnesses are if the answer to Question 1.1.1 is affirmative. As mentioned above Baumgartner figured out there are non-$\sigma$-scattered linear orders which will remain non-minimal non-$\sigma$-scattered as long as they are non-$\sigma$-scattered. This property is completely different from the ones for real and Countryman lines, as they can be made minimal with forcings which preserve their non-$\sigma$-scatteredness.

Motivated by the structure of Baumgartner types, Ishiu and Moore characterized — under the forcing axiom $\mathrm{PFA}^+$ — when a non-$\sigma$-scattered linear order is minimal with respect to being not $\sigma$-scattered.

**Theorem 1.1.6.** *[8] Assume* PFA$^+$. *Every minimal non-$\sigma$-scattered linear order has to be a real or an Aronszajn type.*

Their work is both significant — in terms of the strength of the result — and innovative because of the machinery they created for the study of non-$\sigma$-scattered linear orders. This result is also different from all other results about the non-$\sigma$-scattered linear orders that existed up to that point. Until the work in [8], all results either indicated that some non-$\sigma$-scattered linear order which is constructable in $ZFC$ can be minimal, or else they introduced some specific example of a non-$\sigma$-scattered linear order which is not minimal. In other words, there were no results which — even with some additional acceptable hypothesis — could specify all of the non-$\sigma$-scattered linear orders which are not minimal. Note that the only non-$\sigma$-scattered linear orders, other than the ones containing real or Aronszajn types, contain Baumgartner types, which have no chance of being minimal. This motivates us to ask whether the set theoretic hypothesis in the theorem above can be eliminated:

**Question 1.1.7.** *Must every minimal non-$\sigma$-scattered linear order be of real or Aronszajn type?*

Moore also studied the extent to which Laver's theorem is sharp in [16], where he proves it is consistent with $CH$ that $\omega_1$ and $\omega_1^*$ are the only linear orders that are minimal with respect to being uncountable. These two works provide powerful machinery and suggest interesting questions in studying the structure of the class of non-$\sigma$-scattered linear orders.

Note that an affirmative answer to Question 1.1.7 together with Moore's result in [16], would imply that Laver's theorem is sharp which would answer Question 1.1.1.

### 1.1.4 Our results

In chapter 2, based on the machinery provided in [8] and [16], we answer questions 1.1.1 and 1.1.5. In other words, we provide a rough classification theorem for the class of non-$\sigma$-scattered linear order and show that Laver's theorem is sharp. While working on these problems we noticed our work answers the following question which was asked by Baumgartner in [5].

**Question 1.1.8.** *Is it consistent with CH that no Aronszajn tree has a base of cardinality* $\aleph_1$*?*

Here a base for an Aronszajn tree $T$ is a collection $\mathcal{B}$ of uncountable downward closed subtrees of $T$ such that for all uncountable downward closed subtrees $U$ of $T$ there is a $V \in \mathcal{B}$ with $V \subset U$. Chapter 3 is devoted to question 1.1.3, and the last chapter answers questions 1.1.7 and 1.1.4.

## 1.2 Technical prerequisites

This section is devoted to some background and conventions on trees, linearly ordered sets and forcing axioms. More discussion can be found in [8], [14], [16], [19] and [20]. We will also introduce two set-theoretic axioms $(*)$ and $(\dagger)$ which will play an important role in the proofs of Theorem 2.1.2 and 2.1.3.

Recall that a *tree* $T$ is a partially ordered set such that for all $x \in T$ the set

$\{t \in T : t < x\}$ is well ordered by the ordering on $T$. For an ordinal $\alpha$ which is less than the height of $T$, $T_\alpha$ denotes the $\alpha$'th level of $T$. If $T$ is a tree and $t \in T$, $T(t)$ denotes the tree of all $t' \in T$ that are compatible with $t$ and $T^{[2]} = \bigcup_{\xi < ht(T)} T_\xi^2$. For a regular cardinal $\kappa$, a *$\kappa$-tree* $T$ is a tree which has levels of size $< \kappa$ and does not branch at limit heights, i.e. there are no distinct pair $s, t \in T$ which have the same height and predecessors. A chain $b \subset T$ is called a *branch* of $T$ if it intersects all levels of $T$. If $x$ is an element of $T$ and $\alpha \leq ht(x)$, $x \restriction \alpha$ is the $s \leq x$ which has height $\alpha$. If $x$ is a branch of $T$ and $\alpha < ht(T)$, $x \restriction \alpha$ is the element of the branch which has height $\alpha$. If $x, y$ are elements or branches of $T$ , then $x \Delta y$ is the first ordinal $\alpha$ such that either $x \restriction \alpha \neq y \restriction \alpha$ or one of them is not defined. If the levels of a tree $T$ are equipped with total orderings the *lexicographic* — or *lex* — order on $T$ is defined as follows. Assuming $x, y$ are elements in or branches through $T$, $x \leq_{lex} y$ if either for some $\alpha \leq ht(x)$, $x \restriction \alpha = y$ or $x \restriction (x \Delta y)$ is less than $y \restriction (x \Delta y)$ in the order corresponding to the $(x \Delta y)$'th level of $T$.

An $\omega_1$-tree $T$ is called *Aronszajn* if it has no branches. It is called *Kurepa* if it has at least $\omega_2$ many branches. For $C \subset \omega_1$, $T \restriction C = \{t \in T : \text{height of } t \text{ is in } C\}$. If $S, T$ are trees, $f : T \longrightarrow S$ is called a *tree embedding* if for all $t, s \in T$, $t <_T s$ iff $f(t) <_S f(s)$. A tree $T$ is said to be everywhere Kurepa if for all $t \in T$ the tree of all elements that are compatible with $t$ is Kurepa. A linear order $L$ is said to be *$\kappa$-dense* if for all $x, y$ in $L$ there are exactly $\kappa$ elements strictly in between $x, y$.

The Proper Forcing Axiom (PFA) was first introduced and proved to be consistent by Baumgartner, relative to the existence of a supercompact cardinal. Baumgartner's proof uses two important theorems one of which is due to Laver and the the other is due to Shelah. We present Baumgartner's proof here. We will also give a brief introduction to PFA.

**Definition 1.2.1.** *A poset $\mathcal{P}$ is proper if for every uncountable set $X$, whenever $S \subset [X]^\omega$ is stationary in the ground model, $\Vdash_{\mathcal{P}}$ "$\check{S}$ is stationary".*

Recall that $H_\theta$ is the set of $x$ with $|Tc(x)| < \theta$, where $Tc(x)$ means the transitive closure of $x$. For a set — or a structure — $X$, we say a countable elementary submodel $M$ of $H_\theta$ is suitable for $X$ if $X$ — together with the structure — is in $M$ and $2^{|X|} < \theta$. The following proposition is usually used in order to check properness of forcings.

**Proposition 1.2.2.** *(Shelah; see [2]) A poset $\mathcal{P}$ is proper iff for some regular cardinal $\theta$ there is a club of countable suitable models $M \prec H_\theta$ such that whenever $p \in \mathcal{P} \cap M$ there is a $q \leq p$ that is $(M, \mathcal{P})$-generic. $q \in \mathcal{P}$ is said to be $(M, \mathcal{P})$-generic, if for all $\bar{q} \leq q$ and dense $D \subset \mathcal{P}$ with $D \in M$ there is a condition $r \in D \cap M$ which is compatible with $\bar{q}$. Equivalently $q$ forces that $\dot{G}$ is $M$-generic.*

**Definition 1.2.3.** *The proper forcing axiom asserts that if $\mathcal{P}$ is proper and $\mathcal{D}$ is an $\aleph_1$-sized collection of dense subsets of $\mathcal{P}$ then there is a filter that intersects all elements of $\mathcal{P}$. We refer to this axiom as* PFA. PFA$^+$ *asserts that for proper posets $\mathcal{P}$ and $\mathcal{D}$ as above and a $\mathcal{P}$-name $\tau$ for a stationary subset of $\omega_1$ there is a filter $G$ which meets all the dense sets in $\mathcal{D}$ and such that the set $\{\xi \in \omega_1 : \exists q \in G$ $q \Vdash \check{\xi} \in \tau\}$ is stationary.*

Now we present the proof of the consistency of PFA$^+$.

**Theorem 1.2.4.** *(Shelah) A countable support iteration of proper forcings is proper.*

**Definition 1.2.5.** *A cardianl $\kappa$ is said to be $\gamma$-supercompact if there is an elementary embedding $j : \boldsymbol{V} \prec M$ such that crit$(j) = \kappa$, $\gamma < j(\kappa)$, and $M^\gamma \subset M$. $\kappa$ is called supercompact if it is $\gamma$-supercompact for all $\gamma \geq \kappa$.*

If $U$ is a normal ultrafilter on $\kappa$, $Ult(\mathbf{V}, U)$ is the ultrapower structure obtained by $U$. Then $j_U : \mathbf{V} \longrightarrow M = Ult(\mathbf{V}, U)$ — defined by $j_U(x)$ being the equivalence class of the constant function $x$ — is an elementary embedding. Note that by normality, $U$ is countably complete and $Ult(\mathbf{V}, U)$ is well founded. Therefore we can assume $M$ is transitive.

**Theorem 1.2.6.** *(Laver) Assume $\kappa$ is a supercompact cardinal. There is a function $f : \kappa \longrightarrow \boldsymbol{V}_\kappa$ such that for all $x \in \boldsymbol{V}$ and $\lambda \geq |Tc(x)| + \kappa$ there is a normal ultrafilter $U$ on $P_\kappa(\lambda)$ such that $(j_U(f))(\kappa) = x$.*

**Theorem 1.2.7.** *Assume there is a supercompact cardinal. There is a forcing extension satisfying* PFA$^+$.

*Proof.* Assume $\kappa$ is a supercompact cardinal and fix a function $\psi : \kappa \longrightarrow \mathbf{V}_\kappa$ as in Theorem 1.2.6. Let $P$ be the countable support iteration of all proper posets in $\mathbf{V}_\kappa$ of length $\kappa$, described as follows. Given all iterands of $P$ up to $\alpha$, if $\psi(\alpha) = (\dot{Q}, \dot{\Delta})$ where $\dot{Q}$ is a $P_\alpha$-name for a proper poset and $\dot{\Delta}$ is a $P_\alpha$-name for a $\theta$-sized collection of dense subsets of $\dot{Q}$ for some $\theta < \kappa$, then the $\alpha$'th iterand is $\dot{Q}$. Otherwise the $\alpha$'th iterand is the trivial forcing. Since $P$ is proper it preserves $\omega_1$. Since each iterand in $P$ has size $< \kappa$, $P$ preserves all cardinals $\geq \kappa$ as well. But every cardinal in between $\omega_1$ and $\kappa$ is obviously collapsed to $\omega_1$. Hence in the extension $\kappa = 2^{\aleph_0} = \aleph_2$. In order to see PFA$^+$ holds in any extension by $P$, let $G$ be $\mathbf{V}$-generic for $P$. Let $K$ be a proper poset in $\mathbf{V}[G]$, $\mathcal{D} = \{D_\alpha : \alpha \in \gamma\}$ be a collection of dense sets in $K$ with $\gamma \in \kappa$, and $\tau$ be a $K$-name for a stationary subset of $\omega_1$.

Assume $\dot{K}, \dot{D}$ be $P$-names for $K, \mathcal{D}$. Let $\lambda > 2^{2^{|Tc(K)|}}$. By the previous theorem there is $j : \mathbf{V} \prec M$ with $crit(j) = \kappa$ such that $j(\kappa) > \lambda$, $M^\lambda \subset M$ and $(jf)(\kappa) = (\dot{K}, \dot{\mathcal{D}})$.

$P$ has the $\kappa$-cc, and $M^\lambda \subset M$, therefore in $\mathbf{V}[G]$, $M[G]^\lambda \subset M[G]$. Hence $K$ is proper in $\mathbf{V}[G]$ and consequently $K$ is proper in $M[G]$. Note that $j(P)$ is a countable support iteration of proper forcings of length $j(\kappa)$ with the book keeping function $j(f)$. Also note that $crit(j) = \kappa$ so the iteration of the first $\kappa$ iterands of $j(P)$ is the same as $P$. Since $(j(f))(\kappa) = (\dot{K}, \dot{D})$, for some $\dot{R}$ we have $j(P) = P * \dot{K} * \dot{R}$. Now let $\bar{G}$ be an extension of $G$ which is generic for $j(P)$ and $j^* : \mathbf{V}[G] \longrightarrow M[\bar{G}]$ be an extension of $j$ defined by

$$j^*(\pi_G) = j(\pi)_{\bar{G}}.$$

Since $j^*$ is an elementary embedding, the filter $H := \bar{G} \upharpoonright \{\kappa\}$ is $\mathbf{V}[G]$-generic. So $H$ interprets $\tau$ to be a stationary subset of $\omega_1$ and meets $D_\alpha$ for all $\alpha \in \gamma$. But $j[H]$, the image of $H$ under $j$, is in $M[\bar{G}]$ and $j^*(\mathcal{D})$ generic for $j^*(K)$. Therefore by elementarity of $j^*$ there is a $\mathcal{D}$-generic filter in $\mathbf{V}[G]$ for the poset $K$ which also interprets $\tau$ to be a stationary subset of $\omega_1$. $\square$

We now recall the notion of a forcing axiom associated to a class of partial orders.

**Notation 1.2.8.** *If $\mathfrak{P}$ is a class of partial orders, then by* $\mathrm{FA}(\mathfrak{P})$ *we mean the forcing axiom for the class $\mathfrak{P}$: whenever $P$ is in $\mathfrak{P}$ and $\mathcal{D}$ is a collection of $\aleph_1$-many dense subsets of $P$, there is a filter $G \subset P$ which intersects all of the dense sets in $\mathcal{D}$.* $\mathrm{FA}^+(\mathfrak{P})$ *is the assertion that whenever $P$ is in $\mathfrak{P}$, $\mathcal{D}$ is a collection of $\aleph_1$-many dense subsets of $P$, and $\dot{S}$ is a name for a stationary subset of $\omega_1$, then there is a filter $G \subset P$ which intersects all the dense sets in $\mathcal{D}$ and satisfies that the set*

$$\{\xi \in \omega_1 : \exists p \in G (p \Vdash \check{\xi} \in \dot{S})\}$$

*is stationary.*

The following axiom is a consequence of $\mathrm{FA}^+(\sigma\text{-closed})$ and will play an important role in our analysis of non-$\sigma$-scattered linear orders of cardinality $\aleph_1$.

**Definition 1.2.9.** *(†) is the assertion that if $S \subset \omega_1$ is stationary and for each $\alpha \in S$, $U_\alpha \subset \alpha$ is open, then there is a club $E \subset \omega_1$ such that for stationarily many $\alpha \in S \cap E$ there is an $\bar{\alpha} < \alpha$ such that either $E \cap (\bar{\alpha}, \alpha) \subset U_\alpha$ or $E \cap (\bar{\alpha}, \alpha) \cap U_\alpha = \emptyset$.*

Let $P$ be the poset consisting of all countable closed subsets of $\omega_1$, ordered by end extension and let $\dot{E}$ be the $P$-name for the union of the generic filter. By using the arguments of [14], it is possible to show that if $S \subset \omega_1$ is stationary and $\langle U_\alpha : \alpha \in S\rangle$ is as in the formulation of (†), then every condition forces that $\dot{E}$ satisfies the conclusion of (†) for $\langle U_\alpha : \alpha \in S\rangle$. In particular $\mathrm{FA}^+(\sigma\text{-closed})$ implies (†). Moreover, (†) holds in the model obtained by adding $\aleph_2$ Cohen subsets of $\omega_1$ to a model of GCH.

It will often be convenient to let, for each set $X$, $\theta_X$ denote the least regular cardinal such that all finite iterates of the power set applied to $X$ are in $H(\theta_X)$, the collection of sets of hereditary cardinality less than $\theta_X$. Let $\mathcal{E}(X)$ denote the collection of all countable elementary submodels of $H(\theta_X)$ which have $X$ as an element.

We will now recall a number of definitions from [8]. For a linearly ordered set $L$ we will use $\hat{L}$ to denote the completion of $L$. Formally this is the set of all Dedekind cuts of $L$ with $z$ identified with the cut $\{x \in L : x < z\}$. The purpose of the following definitions is to abstractly recover the set of indices from a Baumgartner type, purely from its order-theoretic properties.

**Definition 1.2.10.** *Whenever $L$ is a linearly ordered set and $Z$ is some arbitrary set we say that $Z$ captures $x \in L$ if there is a $z \in Z \cap \hat{L}$ such that there is no* *element of $Z \cap L$ which is strictly in between $z$ and $x$.*

**Fact 1.2.11.** *Assume $L' \subset L$ are linear orders, $x \in L'$ and $M$ is a countable elementary submodel of $H_\theta$ that has $L, L'$ as elements, where $\theta > 2^{|\hat{L}|}$ is regular. Then $M$ captures $x$ as an element of $L$ iff $M$ captures $x$ as an element of $L'$.*

The following fact shows that after adding embeddings and making many non-$\sigma$-scattered suborders $\sigma$-scattered, the linear order to which we added embeddings has still non-$\sigma$-scattered dense suborders, which are in fact the witnesses for Theorem 4.1.2.

**Fact 1.2.12.** *Assume $L$ is a linear order which has size $\aleph_2$, all elements of $L$ have cofinality and coinitiality $\omega_1$, and $L' \subset L$ is dense and has cardinality $\aleph_1$. Then $L'$ is not $\sigma$-scattered.*

*Proof.* Assume not. Since all $x \in L$ have cofinality and coinitiality $\omega_1$, there is a scattered suborder $L_0$ of $L'$ whose closure has cardinality $\aleph_2$. For $x, y \in L_0$ let $x \sim y$ if there are at most $\aleph_1$ many elements of $\bar{L_0}$ in between $x, y$. Note that there are exactly $\aleph_1$ many equivalence classes and between every two distinct equivalence classes there are infinitely many, in fact $\aleph_1$ many, equivalence classes. Now let $L_1$ be a suborder of $L_0$ which intersects each equivalence class at exactly one point. $L_1$ is an infinite dense linear order which contradicts scatteredness of $L_0$. □

**Fact 1.2.13.** *[8] Suppose $L$ is a linear order and let $\lambda$ be a regular cardinal such that $\hat{L}$ is in $H(\lambda)$. If $M$ is a countable elementary submodel of $H(\lambda)$ with $L \in M$ and $x \in \hat{L} \setminus M$, then $M$ captures $x$ if and only if there is a unique $z \in \hat{L} \cap M$ such that there is no element of $M \cap L$ which is strictly in between $x$ and $z$. In this case we say $M$ captures $x$ via $z$.*

**Definition 1.2.14.** *[8] If $L$ is a linear order, define $\Gamma(L)$ to be the set of all countable subsets $Z$ of $\hat{L}$ such that for some $x \in L$, $Z$ does not capture $x$. (This is the relative complement of the set $\Omega(L)$ in [8].)*

If $B = \langle x_\alpha : \alpha \in S \rangle$ is a Baumgartner type and $M$ is a countable elementary submodel of $H(\theta)$ for some regular cardinal $\theta \geq \omega_2$ with $B \in M$, then $M \in \Gamma(B)$ if and only if $M \cap \omega_1 \in S$. This is because $M$ captures all elements of $B$ except $x_\delta$, where $\delta = M \cap \omega_1$. So $\Gamma(B)$ is equivalent to $S$ modulo the equivalence induced by the following quasiorder.

**Definition 1.2.15.** *(Woodin; see [10]) Let $A, B$ be two collections of countable sets and $X = \bigcup A$, $Y = \bigcup B$. we say $B \leq A$ if there is an injection $\iota : X \to Y$ such that for club many $M$ in $[Y]^\omega$, if $M \in B$ then $\iota^{-1}M$ is in $A$. We let $B < A$ if $B \leq A$ but not $A \leq B$; $A$ and $B$ are* equivalent *if $A \leq B$ and $B \leq A$.*

The following results summarize the properties of the map $L \mapsto \Gamma(L)$ and the quasiorder $\leq$.

**Proposition 1.2.16.** *If $L_0$ and $L$ are linear orders and $L_0$ embeds into $L$ then $\Omega(L_0) \geq \Omega(L)$.*

**Theorem 1.2.17.** *[8] A linear order $L$ is $\sigma$-scattered if and only if $\Gamma(L)$ is not stationary.*

**Proposition 1.2.18.** *[8] If $L_0$ and $L$ are linearly ordered sets and $L_0$ embeds into $L$, then $\Gamma(L_0) \leq \Gamma(L)$.*

A key feature of Baumgartner types $L$ is that it is always possible to find a non-$\sigma$-scattered suborder $L_0$ such that $\Gamma(L_0) < \Gamma(L)$. This is not always possible in the more general class of non-$\sigma$-scattered orders as the next proposition shows.

**Proposition 1.2.19.** *[8] If a linear order $L$ contains a real or Aronszajn type, then $\Gamma(L)$ contains a club.*

**Definition 1.2.20.** *If $L$ is a linear order and $M$ is in $\mathcal{E}(L)$, then we say that an element $x$ of $L$ is* internal *(respectively* external*) to $M$, if there is a club $E \subset [\hat{L}]^{\omega}$ in $M$ such that every (respectively no) element of $E \cap M$ captures $x$.*

The next definition will play a central role in the proofs of our results. It abstracts the property of Baumgartner types needed to allow us to decrease $\Gamma$ by thinning out the linear order.

**Definition 1.2.21.** *A linear order $L$ is said to be* amenable *if whenever $M$ is in $\mathcal{E}(L)$ and $x \in L$, then $x$ is internal to $M$.*

Observe that by Theorem 1.2.17, $\sigma$-scattered linear orders are amenable. It is also true that Baumgartner types are amenable.

**Proposition 1.2.22.** *[8] If $L$ is a non-$\sigma$-scattered amenable linear order of cardinality $\aleph_1$ and $S \subset \Gamma(L)$ is stationary, then there is a non-$\sigma$-scattered $L_0 \subset L$ such that $\Gamma(L_0) \leq S$.*

In particular, non-$\sigma$-scattered amenable linear orders of cardinality $\aleph_1$ are not minimal. The next theorem shows that the existence of external elements of a linear order characterizes the presence of either a real or Aronszajn suborder. In particular amenable linear orders do not contain real or Aronszajn types.

**Theorem 1.2.23.** *[8] The following are equivalent for a linear order $L$:*

- *$L$ contains a real or Aronszajn type.*
- *There are $M$ in $\mathcal{E}(L)$ and $x \in L$ such that $x$ is external to $M$.*

We are now ready to formulate the other set-theoretic hypothesis which will be needed in our analysis.

**Definition 1.2.24.** $(*)$ *is the assertion that for every non-$\sigma$-scattered linear order $L$ there is a continuous $\in$-chain $\langle M_\xi : \xi \in \omega_1 \rangle$ in $\mathcal{E}(L)$ such that:*

- *the set of all $\xi \in \omega_1$ such that $M_\xi \cap \hat{L} \in \Gamma(L)$ is a stationary set,*
- *$\hat{L_0} \subset \bigcup_{\xi\in\omega_1} M_\xi$, where $L_0 = L \cap (\bigcup_{\xi\in\omega_1} M_\xi)$,*
- *for every $\xi$ if $M_\xi \cap \hat{L} \in \Gamma(L)$ then there is an $x \in L_0$ such that $M_\xi$ does not capture $x$.*

Observe that if $L_0 \subset L$ are as in the statement of $(*)$, then $L_0$ is also non-$\sigma$-scattered. Thus $(*)$ implies every non-$\sigma$-scattered linear order contains a non-$\sigma$-scattered suborder of cardinality $\aleph_1$. Also, if we apply $(*)$ to a linear order of cardinality at most $\aleph_1$, then $L \subset \bigcup_{\xi\in\omega_1} M_\xi$ and consequently $\hat{L} \subset \bigcup_{\xi\in\omega_1} M_\xi$. This gives the following fact.

**Fact 1.2.25.** *Assume $(*)$. If $L$ is a linear order of cardinality at most $\aleph_1$ which does not contain a real type, then $|\hat{L}| \leq \aleph_1$.*

In particular $(*)$ implies that CH is true. A consequence of the work in [7] and [16] is that by iterating certain forcings over a model of CH, it is possible to obtain a generic extension in which there is no minimal real or Aronszajn type. We briefly review this result and recall some of the relevant definitions and terminology. If $T$ is an Aronszajn tree, then a *subtree* of $T$ is an uncountable downward closed subset of $T$.

**Notation 1.2.26.** *If $T$ is a tree, $t \in T$ and $\alpha$ is an ordinal, then $t \restriction \alpha$ is defined to be $t$ if $\alpha$ is greater than the height of $t$ otherwise it is the unique $s \leq t$ with height $\alpha$.*

**Definition 1.2.27.** *A sequence* $\langle f_\alpha : \alpha \in \lim(\omega_1)\rangle$ *is called a ladder system coloring if the* $\langle \mathrm{dom}(f_\alpha) : \alpha \in \omega_1\rangle$ *forms a ladder system and the range of each* $f_\alpha$ *is contained in* $\omega$*.*

**Definition 1.2.28.** *If* $T$ *is an* $\omega_1$*-tree, then a ladder system coloring* $\langle f_\alpha : \alpha \in \lim(\omega_1)\rangle$ *can be* $T$-uniformized *if there is a subtree* $U$ *of* $T$ *and function from* $\phi : U \to \omega$ *such that whenever height of* $u \in U$ *is a limit ordinal* $\alpha$*,* $f_\alpha$ *agrees with* $\xi \mapsto \phi(u \restriction \xi)$ *at all except for finitely many* $\xi \in \mathrm{dom}(f_\alpha)$*.*

**Definition 1.2.29.** (*A*) *is the assertion that every ladder system coloring can be* $T$*-uniformized for every Aronszajn tree* $T$*.*

The significance of (A) lies in the following theorem, along with the fact that it is consistent with CH.

**Theorem 1.2.30.** *[16] Assume* (*A*) *and* $2^{\aleph_0} < 2^{\aleph_1}$*. There are no minimal Aronszajn lines.*

In [16], a forcing $Q_{T,\bar{f}}$ was introduced which $T$-uniformizes a given ladder system coloring $\bar{f}$. We will recall the definition of this poset in Section 2.4 when we need to analyze it, but for now we will simply summarize its important properties.

While $(< \omega_1)$-properness and complete properness play a role in the proof of the main result of this chapter, they can be treated as black boxes via the following results, along with the straightforward fact that $\sigma$-closed posets are both $(<\omega_1)$-proper and completely proper.

**Lemma 1.2.31.** *[16] For every ladder system coloring* $\bar{f}$ *and Aronszajn tree* $T$*, the forcing* $Q_{T,\bar{f}}$ *is completely proper and* $(<\omega_1)$*-proper.*

**Theorem 1.2.32.** *[19] A countable support iteration of* $(<\omega_1)$*-proper, completely proper forcing is proper and does not introduce new real numbers.*

We will also need the following iteration theorem of Shelah.

**Theorem 1.2.33.** *[19, III.8.5] If the iterands of a countable support iteration are proper and don't add new uncountable branches to* $\omega_1$*-trees, then the iteration is proper and does not add uncountable branches to* $\omega_1$*-trees.*

CHAPTER 2

# THERE MAY BE NO MINIMAL NON-$\sigma$-SCATTERED LINEAR ORDER

## 2.1 Introduction

In [11], Laver verified a longstanding conjecture of Fraïssé: the countable linear orders are well quasiordered by embeddability. That is to say if $L_i$ $(i < \infty)$ is an infinite sequence of countable linear orderings, then there is an $i < j$ such that $L_i$ is embeddable into $L_j$. In fact, Laver proved the following stronger result.

**Theorem 2.1.1.** *[11] The class $\mathscr{M}$ of $\sigma$-scattered linear orders is well quasiordered by embeddability.*

Recall that a linear order is *scattered* if it does not contain an isomorphic copy of the linear order $(\mathbb{Q}, \leq)$ and is *$\sigma$-scattered* if it is a union of countably many scattered suborders.

In the final paragraph of [11], Laver writes, "Finally, the question arises as to how the order types outside of $\mathscr{M}$ behave under embeddability." For instance, is there a class of linear orders which is closed under taking suborders, which properly includes the class of $\sigma$-scattered linear orders, and which is well quasiordered by embeddability? Cast in another way, is there a non-$\sigma$-scattered linear order which embeds into all of its non-$\sigma$-scattered suborders?

Already in [3], Baumgartner proved that it is consistent that any two $\aleph_1$-*dense* sets of reals are isomorphic; in fact this conclusion is a consequence PFA. Here a linear order is $\kappa$-*dense* if all of its intervals have cardinality $\kappa$. It is not difficult to show that any suborder of $\mathbb{R}$ of cardinality $\aleph_1$ is biembeddable with an $\aleph_1$-dense set of reals and thus in Baumgartner's model, any set of reals of cardinality $\aleph_1$ is minimal with respect to being non-$\sigma$-scattered. On the other hand, it follows easily from work of Dushnik and Miller [7] that the Continuum Hypothesis (CH) implies that there are no minimal uncountable linear orders which are separable. (In fact Dushnik and Miller show in ZFC that there is no minimal separable linear order of cardinality of the continuum.)

The main result of this chapter is that Theorem 2.1.1 is sharp.

**Theorem 2.1.2.** *If there is a supercompact cardinal, then there is a forcing extension which satisfies CH in which there are no minimal non-$\sigma$-scattered linear orders.*

This result builds on work of Moore [16] and Ishiu-Moore [8]. In [16] it was proved that it is consistent with CH that $\omega_1$ and $\omega_1^*$ are the only minimal uncountable linear orderings. In fact, this conclusion is derived from the conjunction of CH and a certain combinatorial consequence (A) of PFA. Notice that if $\omega_1$ and $\omega_1^*$ are the only minimal uncountable linear orders, then any minimal non-$\sigma$-scattered linear order must have the property that it does not contain an uncountable separable suborder or an *Aronszajn suborder.* Here an Aronszajn line is an uncountable linear order which does not contain uncountable separable or scattered suborders.

In [8] it was proved that $PFA^+$, a strengthening of PFA, implies that every minimal non-$\sigma$-scattered linear order is either isomorphic to a set of reals of cardinality $\aleph_1$ or else is an *Aronszajn line.* Moreover, Martinez-Ranero [13], building on work of Moore [15] [17] proved that that PFA implies that the class of Aronszajn lines is well quasiordered by embeddability. In [8], it was pointed out that if the consequences of $PFA^+$ needed to carry out the analysis in that paper were consistent with the conjunction of (A) and CH, then one could establish the consistency of “there are no minimal non-$\sigma$-scattered linear orders.” In fact these consequence of $\mathrm{PFA}^+$ followed from a weaker axiom $\mathrm{CPFA}^+$ which had been expected to be consistent with CH; this was later refuted in [18]. The strategy here for proving Theorem 2.1.2 also utilizes the combination of (A) and CH, but involves a re-examination of the hypotheses sufficient to obtain the results of [8].

In addition to proving Theorem 2.1.2, we will also establish a result concerning the structure of non-$\sigma$-scattered linear orders under the assumption of $\mathrm{PFA}^+$. Baumgartner proved in [4] that there exist non-$\sigma$-scattered linear orders which do not contain real or Aronszajn types. His construction can be described as the lexicographic ordering on a family $\{x_\alpha : \alpha \in S\}$ where $S \subset \omega_1$ is stationary and $x_\alpha$ is a cofinal strictly increasing $\omega$-sequence in $\alpha$ for each $\alpha$ in $S$. We will refer to such a linear ordering as a *Baumgartner type* and we will refer to $S$ as its *index set.*

**Theorem 2.1.3.** *Assume* $\mathrm{PFA}^+$ *and let* $X \subset \mathbb{R}$ *have cardinality* $\aleph_1$ *and* $C$ *be a Countryman line. If* $L$ *is a non-*$\sigma$*-scattered linear order, then* $L$ *contains an isomorphic copy of one of the following linear orders:* $X$, $C$, $C^*$, *a Baumgartner type or its reverse.*

The proof of Theorem 2.1.2 immediately yields the following result.

**Theorem 2.1.4.** *It is consistent with CH that no Aronszajn tree has a base of cardinality* $\aleph_1$.

Here a collection $\mathcal{B}$ of uncountable downward closed subtrees of an Aronszajn tree $T$ is called a *base* if whenever $U$ is an uncountable downward closed subtree of $T$, there is $V \in \mathcal{B}$ such that $V \subset U$. This answers a problem posed in [5], where it is proved that every Aronszajn tree has a base of cardinality $\aleph_1$ after Levy collapsing an inaccessible cardinal to $\aleph_2$.

This chapter will be organized as follows. In Section 2.2 we will prove Theorem 2.1.3. Section 2.3 contains the analysis needed to derive the conclusion of Theorem 2.1.2 from a list of axioms. Section 2.4 gives a proof that the collection of axioms used in Section 2.3 is consistent. This section also includes a proof of theorem 2.1.4 as a remark.

## 2.2 A rough classification of non-$\sigma$-scattered orders

In [8] it was shown that under PFA$^+$, every non-$\sigma$-scattered linear order contains an amenable non-$\sigma$-scattered suborder of cardinality $\aleph_1$. In this section we prove that under a fragment of PFA$^+$ every non-$\sigma$-scattered amenable linear order contains a copy of a Baumgartner type or its reverse. Taken together, these results determines a basis for the class of non-$\sigma$-scattered linear orders under PFA$^+$: if $X$ is any set of reals of cardinality $\aleph_1$ and $C$ is any Countryman type, then any non-$\sigma$-scattered linear order must contain an isomorphic copy of either $X$, $C$, $C^*$, or a Baumgartner type of cardinality $\aleph_1$ or its reverse. Recall that $MA_{\omega_1}$ and (†) are both consequences of PFA$^+$.

**Theorem 2.2.1.** *Assume the conjunction of* $\mathrm{MA}_{\omega_1}$ *and* (†)*. If $L$ is an amenable non-$\sigma$-scattered linear order of size $\aleph_1$, then it contains a copy of a Baumgartner type or its reverse.*

First we will prove the following lemma.

**Lemma 2.2.2.** *Suppose that $L$ is an amenable linear order of cardinality $\aleph_1$. If*

*$\langle M_\xi : \xi \in \omega_1\rangle$ is a continuous $\in$-chain of elements of $\mathcal{E}(L)$ which is in $N \in \mathcal{E}(L)$ and $N \cap \omega_1 = \delta$, then $M_\delta$ and $N$ capture the same elements of $L$.*

*Proof.* First observe that by continuity of the $\in$-chain and the fact that $\{\nu \in \omega_1 : M_\nu \cap \omega_1 = \nu\}$ is a club in $N$, $M_\delta \subset N$ and $M_\delta \cap \omega_1 = \delta$. Next observe that since $L$ has cardinality $\aleph_1$, $N \cap L = M_\delta \cap L$ and thus any element of $L$ captured by $M_\delta$ is captured by $N$. Now suppose that $N$ captures $x \in L$ and let $z \in \hat{L} \cap N$ be such that there is no element of $N \cap L$ which is strictly in between $x$ and $z$. Since $L$ is amenable, there is a club $E \subset [\hat{L}]^\omega$ in $M_\delta$ such that for all $Z \in M_\delta \cap E$, $Z$ captures $x$. Let $\lambda \in \theta_L \cap M_0$ be a regular cardinal such that the powerset of $[\hat{L}]^\omega$ is in $H(\lambda)$. Let $\overline{M} \in N$ be a countable elementary submodel of $H(\lambda)$ such that $\langle M_\xi \cap H(\lambda) : \xi \in \omega_1\rangle$, $E$, and $z$ are in $\overline{M}$. Observe that for sufficiently large $\xi < \delta$, $M_\xi \cap \hat{L}$ is in $E$ and if $\nu = \overline{M} \cap \omega_1$ then $L \cap \overline{M} = M_\nu \cap L$. Notice that $\overline{M}$ captures $x$ via $z$. Since $M_\nu \cap \hat{L}$ is in $E \cap M_\delta$, $M_\nu$ also captures $x$. By Fact 1.2.13, it must be that $z$ is in $M_\nu$ and hence $M_\delta$. $\square$

*Proof of Theorem 2.2.1.* Now let $\langle M_\xi : \xi \in \omega_1\rangle$ be a continuous $\in$-chain of elements of $\mathcal{E}(L)$. Since $L$ is amenable it does not contain any real types, there is a countable set $X_\xi \subset L$ such that if $M_\xi \cap L \subset X_\xi$ and if $y \in L \backslash M_\xi$, there is a unique $x \in X_\xi \backslash M_\xi$ such that there is no element of $M_\xi \cap L$ strictly in between $x, y$. Let $x : \omega \times \omega_1 \to L$ be such that for all $\xi \in \omega_1$, $X_\xi = \{x(n, M_\xi \cap \omega_1) : n \in \omega\}$. Now let $\langle N_\xi : \xi \in \omega_1\rangle$ be a continuous $\in$-chain of elements of $\mathcal{E}(L)$ such that $\langle M_\xi : \xi \in \omega_1\rangle$ and $x$ are in $N_0$. Note that there is a club of $\xi$ in $\omega_1$ such that $M_\xi \cap \omega_1 = \xi = N_\xi \cap \omega_1$ and hence $M_\xi$ and $N_\xi$ capture the same elements of $L$. Since $\Gamma(L)$ is stationary, then by applying the pressing down lemma there is a stationary set $S_0 \subset \omega_1$, an $n \in \omega$, and a club $E \subset [\hat{L}]^\omega$ such that if $\xi \in S_0$:

- $M_\xi \cap \omega_1 = \xi = N_\xi \cap \omega_1$;
- $x(n,\xi)$ is not captured by $N_\xi$;
- $E$ is in $N_\xi$ and if $Z$ is in $N_\xi \cap E$, then $Z$ captures $x(n,\xi)$.

Set $y_\xi = x(n,\xi)$ for all $\xi \in S_0$. Now it is easy to see that for all $\xi$ and $\eta$ in $S_0$, $N_\xi$ captures $y_\eta$ if and only if $\xi \neq \eta$.

Let $z_\xi$ ($\xi \in \omega_1$) be an enumeration of all $z \in \hat{L}$ for which there are $\eta \in \omega_1$ and $\alpha \in S_0$ such that $N_\eta$ captures $y_\alpha$ via $z$. We can assume without loss of generality that this enumeration is in $N_0$. For every $\alpha \in S_0$ define $g_\alpha : \alpha \to \{z_\xi : \xi \in \omega_1\}$ by letting $g_\alpha(\xi)$ be the unique $z \in N_\xi$ such that $N_\xi$ captures $y_\alpha$ via $z$. Note that if $g_\alpha(\xi) = z_\eta$ then $\eta \in \alpha$.

**Claim 2.2.3.** *The following are true for $\alpha, \beta \in S_0$:*

1. *$\{\xi \in \alpha : g_\alpha(\xi) \neq g_\alpha(\xi+1)\}$ has order type $\omega$ and supremum $\alpha$*
2. *If $y_\alpha < y_\beta$, then $g_\alpha(\xi) \leq g_\beta(\xi)$ for all $\xi < \min(\alpha,\beta)$.*
3. *If $\alpha < \beta$, then there is a $\xi < \alpha$ such that $g_\alpha(\xi) \neq g_\beta(\xi)$.*
4. *If $\xi < \eta < \min(\alpha,\beta)$ and $g_\alpha(\xi) \neq g_\beta(\xi)$, then $g_\alpha(\eta) \neq g_\beta(\eta)$.*

*Proof.* First observe that for each $\alpha \in S_0$ and limit $\eta \in \alpha$ there is an $\bar{\eta} \in \eta$ such that $g_\alpha \restriction (\bar{\eta}, \eta]$ is constant. On the other hand $N_\alpha$ does not capture $y_\alpha$ and therefore the set $\{\xi \in \alpha : g_\alpha(\xi) \neq g_\alpha(\xi+1)\}$ must have ordertype $\omega$ and supremum $\alpha$. This proves (1); the remainder of the items follow easily from (1) and the definitions. □

Define $C_\alpha$ to be the set of all $\xi \in \alpha$ such that $z_\xi$ is in the range of $g_\alpha$ and equip the set $\{C_\alpha : \alpha \in S_0\}$ with the lexicographic order. For each $\alpha \in S_0$ let

$U_\alpha = \{\xi \in \alpha : g_\alpha(\xi) < y_\alpha\}$ and observe that $U_\alpha$ is an open subset of $\alpha$. So by (†) there is a stationary set $S \subset S_0$ such that either

- for every $\alpha \in S$ and $\xi \in S \cap \alpha$, $g_\alpha(\xi) > y_\alpha$ or,
- for every $\alpha \in S$ and $\xi \in S \cap \alpha$, $g_\alpha(\xi) < y_\alpha$.

Without loss of generality assume that for every $\alpha \in S$ and $\xi \in \alpha \cap S$, $g_\alpha(\xi) > y_\alpha$. Define $S'$ to be the set of all elements of $S$ which are limit points of elements of $S$.

Let $Q$ be the set of all finite $p \subset S'$ such that whenever $\alpha \neq \beta$ are in $p$, $C_\alpha <_{lex} C_\beta$ if and only if $y_\alpha < y_\beta$. We will prove that $Q$ is c.c.c..

Suppose for a contradiction that $X$ is an uncountable antichain in $Q$. By applying the $\Delta$-System Lemma and removing the root if necessary, we may assume that $X$ is pairwise disjoint and consists of elements of some fixed cardinality $n$. Let $M$ be an element of $\mathcal{E}(Q)$ such that $X$, $L$, $x$, $\langle N_\xi : \xi \in \omega_1\rangle$, $\langle y_\xi : \xi \in S\rangle$, and $\langle z_\xi : \xi \in \omega_1\rangle$ are all in $M$. Set $\delta = M \cap \omega_1$ and let $p = \{\alpha_1, ..., \alpha_n\}$ be in $X$ such that $\delta < \alpha_i$ for all $i \leq n$. Let $\zeta \in \delta \cap S$ be such that:

- if $i, j \leq n$, then $g_{\alpha_i} \restriction \delta \neq g_{\alpha_j} \restriction \delta$ implies $g_{\alpha_i}(\zeta) \neq g_{\alpha_j}(\zeta)$;
- the range of $g_{\alpha_i} \restriction \zeta + 1$ coincides with the range of $g_{\alpha_i} \restriction \delta$ for each $i \leq n$ (i.e. $C_{\alpha_i} \cap \delta \subset \zeta + 1$ for each $i \leq n$).

Notice that the existence of $\zeta$ follows from the observation that if $g_\alpha(\xi) \neq g_\beta(\xi)$, then $g_\alpha(\eta) \neq g_\beta(\eta)$ for all $\eta > \xi$. By elementarity of $M$ there exists a $p' = \{\alpha'_1, ..., \alpha'_n\}$ in $M \cap X$ such that:

- for all $i, j \leq n$, $y_{\alpha_i} < y_{\alpha_j}$ if and only if $y_{\alpha'_i} < y_{\alpha'_j}$;

- if $i \leq n$, then $g_{\alpha_i}(\zeta) = g_{\alpha'_i}(\zeta)$.

We will now show that $p \cup p' \in Q$. Let $i, j \leq n$. There are two cases, depending on whether $g_{\alpha_i}(\zeta)$ and $g_{\alpha_j}(\zeta)$ are the same. If $g_{\alpha_i}(\zeta) \neq g_{\alpha_j}(\zeta)$, then observe that $g_{\alpha_j}(\zeta) = g_{\alpha'_j}(\zeta)$ and

$$C_{\alpha_i} \cap (\zeta + 1) \neq C_{\alpha_j} \cap (\zeta + 1) = C_{\alpha'_j} \cap (\zeta + 1).$$

Since $p$ and $p'$ are both in $Q$, it follows that $y_{\alpha_i} < y_{\alpha'_j}$ is equivalent to $C_{\alpha_i} <_{lex} C_{\alpha'_j}$

If $g_{\alpha_i}(\zeta) = g_{\alpha_j}(\zeta)$, then observe that $g_{\alpha_i} \restriction \delta = g_{\alpha_j} \restriction \delta$ and thus that $C_{\alpha_i} \cap \delta = C_{\alpha_j} \cap \delta$. Observe that

$$g_{\alpha'_j} \restriction \zeta = g_{\alpha_j} \restriction \zeta = g_{\alpha_i} \restriction \zeta$$

and that $g_{\alpha_i}$ is constant on the interval $[\zeta, \delta)$. Also, $g_{\alpha'_j}$ is not constant on $[\zeta, \delta)$ by Claim 2.2.3. Observe that there is a $\xi \in S$ such that $\zeta < \xi < \alpha'_j$ and

$$g_{\alpha'_j}(\xi) < g_{\alpha'_j}(\zeta) = g_{\alpha_j}(\zeta) = g_{\alpha_j}(\xi) = g_{\alpha_i}(\xi)$$

It follows that $y_{\alpha_i} > y_{\alpha'_j}$. On the other hand,

$$C_{\alpha_i} \cap \delta = C_{\alpha_i} \cap (\zeta + 1) = C_{\alpha'_j} \cap (\zeta + 1) \neq C_{\alpha'_j} \cap \delta$$

and consequently $C_{\alpha'_j} <_{lex} C_{\alpha_i}$. Since $i, j \leq n$ were arbitrary, $p$ and $p'$ are compatible and thus $Q$ is c.c.c..

By applying $\mathrm{MA}_{\aleph_1}$ to the finite support product $Q^{<\omega}$ of countably many copies of $Q$, it is possible to find a partition of $S$ into countably many pieces such that whenever $\alpha$ and $\beta$ are in the same piece of the partition, $C_\alpha <_{lex} C_\beta$ if and only if $y_\alpha < y_\beta$. Since there is a piece of this partition which is stationary, it shows that $L$ contains a Baumgartner type. $\square$

We finish this section by noting if we add a Cohen real $r$ to a model of ZFC, then Theorem 2.2.1 will not hold in the resulting generic extension. To see this, suppose that $r \in 2^\omega$ and $\langle x_\xi : \xi \in \lim(\omega_1)\rangle$ is such that $x_\xi : \omega \to \xi$ is increasing and has cofinal range for each $\xi$. Define a linear ordering on $\lim(\omega_1)$ by $\xi <_r \eta$ if and only if

$$x_\xi(n) < x_\eta(n) \text{ is equivalent to } r(n) = 0$$

where $n$ is minimal such that $x_\xi(n) \neq x_\eta(n)$. It is left to the reader to check that if $S \subset \lim(\omega_1)$ is stationary, then there is a comeager set of $r$ such that $(S, <_r)$ contains both a copy of $\omega_1$ and of $\omega_1^*$. Furthermore, if $S$ is non-stationary, then $(S, <_r)$ is $\sigma$-scattered and thus not a Baumgartner type. On the other hand, it is not hard to show that every uncountable subset of a Baugmartner type contains a copy of $\omega_1$; in particular, Baumgartner types do not contain $\omega_1^*$. Since every stationary subset of $\omega_1$ in the generic extension by a Cohen real contains a ground model stationary set, this proves the claim.

## 2.3 An axiomatic analysis of non-$\sigma$-scattered orders

In this section we will prove the following proposition.

**Proposition 2.3.1.** *Assume* (†) *and* (∗). *If $L$ is a non-$\sigma$-scattered linear order which does not contain a real or Aronszajn type, then there is a non-$\sigma$-scattered suborder $L' \subset L$ with $\Gamma(L') < \Gamma(L)$.*

*Proof.* As noted in the previous chapter, (∗) implies that $L$ contains a non-$\sigma$-scattered suborder $L_0$ such that $\hat{L}_0$ has cardinality $\aleph_1$. We may therefore assume without loss of generality that $|L| = |\hat{L}| = \aleph_1$. This implies, in particular that if

$M$ and $N$ are in $\mathcal{E}(L)$ and $M \cap \omega_1 = N \cap \omega_1$, then $M \cap \hat{L} = N \cap \hat{L}$. If $Z \subset \hat{L}$ is countable, let $\{x(n, Z) : n \in \omega\} \subset L$ be such that $Z \cap L \subset \{x(n, Z) : n \in \omega\}$ and if $y \in L \setminus Z$, then there is an $n$ such that no element of $L \cap Z$ is between $x(n, Z)$ and $y$. This is possible since $L$ does not contain a real type. Let $\langle N_\xi : \xi \in \omega_1 \rangle$ be a continuous $\in$-chain in $\mathcal{E}(L)$ with the map $Z \mapsto \{x(n, Z) : n \in \omega\}$ in $N_0$. Since $L$ is not $\sigma$-scattered, there is an $n \in \omega$ such that

$$S_0 = \{\xi \in \omega_1 : N_\xi \cap \omega_1 = \xi \text{ and } N_\xi \text{ does not capture } x(n, N_\xi \cap \hat{L})\}$$

is stationary. Fix such an $n$ and set $x_\xi = x(n, N_\xi \cap \hat{L})$. For each $\alpha \in S_0$ let $U_\alpha$ be the set of all $\xi \in \alpha$ such that $N_\xi$ captures $x_\alpha$. Clearly $U_\alpha$ is an open subset of $\alpha$ so by (†) there is a stationary subset $S \subset S_0$ and a club $E \subset \omega_1$ such that for every $\alpha \in S$ there is an $\bar{\alpha} \in \alpha$ such that either $E \cap (\bar{\alpha}, \alpha) \subset U_\alpha$ or $E \cap (\bar{\alpha}, \alpha) \cap U_\alpha = \emptyset$. The second alternative can only happen for at most nonstationary many $\alpha \in S$, because $L$ has no external element by Theorem 1.2.23. By applying the Pressing Down Lemma and thinning $S$ down if necessary, we can assume that for every $\alpha, \beta \in S$, $N_\alpha$ captures $x_\beta$ if and only if $\alpha \neq \beta$.

Now let $S' \subset S$ be stationary such that $S \setminus S'$ is also stationary and define $L' = \{x_\xi : \xi \in S'\}$. We will show that $L'$ is not $\sigma$-scattered and that $\Gamma(L') < \Gamma(L)$.

Fix an $M \in \mathcal{E}(L)$ which has $\langle N_\xi : \xi \in \omega_1 \rangle$, $S$, and $S'$ as elements and has the property that $M \cap \omega_1 = \delta \in S'$. To show that $L'$ is not $\sigma$-scattered we prove that $M$ does not capture $x_\delta$ in $L'$. Suppose for contradiction that $M$ captures $x_\delta$ in $L'$ via $z \in \hat{L} \cap M$. By replacing $L$ with $L^*$ if necessary, we may assume that $z < x_\delta$. Let

$$A = \{x_\xi : \xi \in S' \text{ and } z < x_\xi\}.$$

Observe that $A$ is in $M$ and hence $\inf(A)$ is also in $M$. Since $\inf(A) \leq x_\delta$ and $x_\delta$ is not in $M$, it follows that $\inf(A) < x_\delta$. Since $M$ does not capture $x_\delta$ in $L$, there is $y \in L \cap M$ such that $z \leq \inf(A) < y < x_\delta$. By elementarity of $M$, there is a $\xi \in S' \cap M$ such that $z < x_\xi < y < x_\delta$. But this contradicts our assumption that $M$ captures $x_\delta$ in $L'$ via $z$.

To see that $\Gamma(L) \not\leq \Gamma(L')$ it suffices to show that the set of all $M \in \mathcal{E}(L)$ which capture all elements of $L'$ but does not capture some elements of $L$ forms a stationary set. To this end let $M \in \mathcal{E}(L)$ with $L' \in M$ and $M \cap \omega_1 \in S \setminus S'$ and observe that $M$ does not capture $x_\delta$ in $L$ but it captures all elements of $L'$. □

## 2.4 The consistency of the axioms

In this section we will prove that if there is a supercompact cardinal, then there is a forcing extension with the same reals which satisfies $(*)$, $(\dagger)$, and (A). By results of the previous section this will finish the proof of Theorem 2.1.2. Our forcing construction will resemble the consistency proof of PFA$^+$ and will involve a countable support iteration of forcings which are completely proper, $(<\omega_1)$-proper, and which do not add new uncountable branches through $\omega_1$-trees. By results of Shelah discussed in the introduction, the resulting iteration will not introduce new reals or uncountable branches through $\omega_1$-trees.

All of the iterands used in building the iteration will either be $\sigma$-closed or else be of the following form.

**Definition 2.4.1.** *[16] For an Aronszajn tree $T$ and ladder system coloring $\bar{f}$ let $Q_{T,\bar{f}}$ be the set of all conditions $q = (\phi_q, \mathcal{U}_q)$ such that:*

- *$\phi_q$ is a function from $X_q \subset T$ into $\omega$ such that $X_q$ is a countable downward closed subset of $T$ which has a last level of height $\alpha_q$,*

- *if $t \in X_q$ has limit height $\delta$, $f_\delta$ agrees with $\xi \mapsto \phi_q(t \restriction \xi)$ at all $\xi \in C_\alpha$ except for finitely many $\xi \in C_\alpha$.*
- *$\mathcal{U}_q$ is a non-empty countable collection of pruned subtrees of $T^{[n]}$ for some $n$.*
- *for every $U \in \mathcal{U}_q$ there is some $\sigma \in U$ which is a subset of the last level of $X_q$.*

*($T^{[n]}$ is the collection of all weakly increasing $n$-tuples from some level of $T$, regarded as a tree with the coordinatewise order.) Define $\leq$ on $Q$ by $p \leq q$, in $Q$ if $X_p \restriction \alpha_q = X_q$, $\mathcal{U}_q \subset \mathcal{U}_p$, and $\phi_p \restriction X_q = \phi_q$.*

**Remark 2.4.2.** *A simplification of this type of forcings can be used to prove Theorem 2.1.4. We start with a model $\boldsymbol{V}$ satisfying $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. For an Aronszajn tree $T$ let $Q_T$ be the set of all conditions as above forgetting the information about the ladder system coloring. More precisely $Q_T$ consists of all conditions $q = (X_q, \mathcal{U}_q)$ such that,*

- *$X_q$ is a countable downward closed subset of $T$ which has a last level of height $\alpha_q$,*
- *$\mathcal{U}_q$ is a non-empty countable collection of pruned subtrees of $T^{[n]}$ for some $n$.*
- *for every $U \in \mathcal{U}_q$ there is some $\sigma \in U$ which is a subset of the last level of $X_q$.*

*Define $\leq$ on $Q$ by $p \leq q$, in $Q$ if $X_p \restriction \alpha_q = X_q$, $\mathcal{U}_q \subset \mathcal{U}_p$. It is easy to see that the forcing $Q_{T,\bar{f}}$ projects onto $Q_T$ for every Aronszajn tree $T$, so by the work in [16], $Q_T$ is completely proper, $< \omega_1$-proper and satisfies the proper isomorphism condition. Now let $\mathcal{P}$ be the countable support iteration of all posets of $Q_T$ of length $\omega_2$ such that whenever $T$ is an Aronszajn tree in some intermediate model, $Q_T$ is* *repeated in the iteration cofinally often. Let $G$ be $\mathcal{P}$-generic over $\boldsymbol{V}$. Then it is easy to see that $\omega_2$ is preserved and in $\boldsymbol{V}[G]$*

1. $2^\omega = \omega_1 + 2^{\omega_1} = \omega_2$,
2. *if $T$ is an Aronszajn tree, there is a sequence $\langle V_i : i \in \omega_2 \rangle$ of uncountable downward closed subtrees of $T$ such that whenever $i \in j$, $V_i$ contains no subtree of $V_j$.*

*Note that in general $Q_T$ adds a downward closed subtree $U$ of $T$ which does not contain any ground model subtrees $W$ of $T$. This implies 2 and proves Theorem 2.1.4.*

The following lemma asserts that these forcings $Q_{T,\bar{f}}$ do not add new uncountable branches to $\omega_1$-trees.

**Lemma 2.4.3.** *Suppose $T$ is Aronszajn and $S$ is an $\omega_1$-tree, $\bar{f}$ is a ladder system coloring. The forcing $Q_{T,\bar{f}}$ does not add new uncountable branches to $S$. Consequently, if $L$ is a linear order of size $\aleph_1$, then forcing with $Q_{T,\bar{f}}$ does not introduce new elements to $\hat{L}$.*

*Proof.* Let $Q$ denote $Q_{T,\bar{f}}$ and let $\dot{b}$ be a $Q$-name which is forced by some $p \in Q$ to be an uncountable branch in $S$ which is not in the ground model. If $q$ is in $Q$ and $\sigma$ is in $T^{[n]}$ for some $n$, then we say that $\sigma$ is *consistent* with $q$ if the range of $\sigma \restriction \alpha_q$ is contained in $X_q$.

Let $M \in \mathcal{E}(Q)$ with $p, \dot{b} \in M$ and set $\delta = M \cap \omega_1$.

**Claim 2.4.4.** *If $\sigma \in T_\delta^{[n]}$ is consistent with $p$ and $s \in S_\delta$, then there is a condition $q \leq p$ in $M \cap Q$ such that $q \Vdash \check{s} \notin \dot{b}$ and such that $\sigma$ is consistent with $q$.*

*Proof.* By Lemma 5.5 in [16] we can find a decreasing sequence $\langle p_k : k \in \omega\rangle$ in $M$ such that:

- $p_0 = p$,
- $p_{k+1}$ decides $\dot{b} \restriction \alpha_{p_k}$,
- $\sigma$ is consistent with $p_k$ for all $k$,
- $\langle p_k : k \in \omega\rangle$ has a lower bound in $M$.

Thus without loss of generality we can assume that $p$ forces $\check{s} \restriction \check{\alpha}_p \in \dot{b}$.

Suppose for contradiction that for every $q \leq p$ in $M \cap Q$, if $q \Vdash \check{s} \notin \dot{b}$, then $\sigma$ is not consistent with $q$. Define $W$ to be the set of all $\tau \in T^{[n]}$ which are compatible with $\sigma \restriction \alpha_p$ and such that there exists an $\bar{s} \in S_{\mathrm{ht}(\tau)}$ compatible with $s \restriction \alpha_p$ and for all $q \leq p$, if $q \Vdash \bar{s} \notin \dot{b}$ and $\alpha_q \leq \mathrm{ht}(\tau)$, then $\mathrm{range}(\tau \restriction \alpha_q) \not\subset X_q$. Since $W$ is definable from parameters in $M$, it is in $M$. Observe that $W$ is downwards closed and that $s$ witnesses that $\sigma$ is in $W$. Hence by elementarily of $M$, $W$ is uncountable. Let $U$ be the set of all $\tau \in W$ which have uncountably many extensions in $W$. Notice that $\sigma \restriction \alpha_p$ is in $U$ and thus $p' = (\varphi_p, X_p, \mathcal{U}_p \cup \{U\})$ is a condition in $Q$.

For each $\tau \in U_\delta$ and $t \in S_\delta$, let $\varphi(\tau, t)$ be the assertion: *"whenever $r \leq p'$ is $(M, Q)$-generic with* $\mathrm{range}(\tau) \subset X_r$*, $r \Vdash \check{t} \in \dot{b}$."* Notice that if $r$ is $(M, Q)$-generic, then so is $r \restriction \delta$. It is easy to see that for every $\tau \in W_\delta$ there exists a unique $t \in S_\delta$ which extends $s \restriction \alpha_p$ such that $\varphi(\tau, t)$. Moreover, observe that if $\tau_1$ and $\tau_2$ are in $U_\delta$ and $s_1, s_2$ are such that $\phi(\tau_1, s_1)$ and $\phi(\tau_2, s_2)$, then we can find an $(M, Q)$-generic condition $r \leq p'$ which is consistent to both $\tau_1$ and $\tau_2$. This implies $r \Vdash \check{s}_1 = \check{s}_2$. Thus $s \in S_\delta$ satisfies that $\phi(\tau, s)$ holds for every $\tau \in U_\delta$.

We now claim that $p' \Vdash \check{s}' \in \dot{b}$ for all $s' < s$. Since such an $s'$ is necessarily in $M$, by elementarity it suffices to show that if $p'' \leq p'$ is in $M$, then $p''$ has an extension $r$ which forces that $\check{s}' \in \dot{b}$. Because $Q$ is proper, $p''$ has an $(M, Q)$-generic extension $r$. Let $\tau \in U_\delta$ be such that range$(\tau) \subseteq X_r$. Since $r \leq p'$ and $\varphi(\tau, s')$ holds, $r \Vdash \check{s}' \in \dot{b}$. Thus we have established that $p' \Vdash \check{s}' \in \dot{b}$ for all $s' < s$. By elementarily, $\{t \in S : p' \Vdash \check{t} \in \dot{b}\}$ is uncountable, which implies that $p'$ decides $\dot{b}$, a contradiction. □

Returning to the main proof, by the claim we can find a condition $\bar{p} \leq p$ such that $\bar{p} \Vdash S_\delta \cap \dot{b} = \emptyset$. □

**Theorem 2.4.5.** *Assume there is a supercompact cardinal. Then there is forcing extension in which (A), (†), and (∗) hold.*

*Proof.* Let $\mathbf{V}$ be a ground model with a supercompact cardinal $\kappa$. By performing some preparatory forcing if necessary, we may assume that CH is true. Mimicking the consistency proof of PFA, use a Laver function $\psi$ to build a countable support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha \in \kappa \rangle$ such that:

- $\dot{Q}_\alpha$ is a $P_\alpha$-name in $V_\kappa$ for a partial order which is either $\sigma$-closed or of the form $Q_{\dot{T}, \bar{f}}$;
- if $\psi(\alpha)$ is a $P_\alpha$-name and $p \in P_\alpha$ forces that $\psi(\alpha)$ either $\sigma$-closed or of the form $Q_{\dot{T}, \bar{f}}$, then $p$ forces $\dot{Q}_\alpha = \psi(\alpha)$.

Let $G \subset P_\kappa$ be a $\mathbf{V}$-generic filter. By a similar argument as in the proof of the consistency of PFA$^+$ from the existence of a supercompact cardinal, $\mathbf{V}[G]$ satisfies (A). By Lemma 1.2.31 and Theorem 1.2.32 the iteration does not add new reals and thus $\mathbf{V}[G]$ satisfies CH. By Lemmas 1.2.33 and 2.4.3, every final segment of the iteration does not add new uncountable branches to $\omega_1$-trees. Arguing as in the proof of the consistency of $\mathrm{PFA}^+$ from a supercompact cardinal, $\mathbf{V}[G]$ satisfies $\mathrm{FA}^+(\sigma\text{-closed})$ and in particular (†).

We will now show that $(*)$ holds in $\mathbf{V}[G]$. Fix for a moment a non-$\sigma$-scattered linear order $L$ in $\mathbf{V}[G]$ and let $Q$ be the set of all countable continuous $\in$-chains in $\mathcal{E}(L)$ ordered by end extension. It is obvious that $Q$ is $\sigma$-closed and easily verified that

$$\dot{S} = \{(\check{\xi}, q) : \xi \in \mathrm{dom}(q) \text{ and } q(\xi) \cap \hat{L} \in \Gamma(L)\}$$

is a $Q$-name for a stationary subset of $\omega_1$. Since $Q$ is countably closed, it does not add new elements to $\hat{L}$. Thus if $H \subset Q$ is a $\mathbf{V}[G]$-generic filter, then $\mathbf{V}[G][H]$ contains the desired witness to $(*)$ for $L$. Moreover, this witness is preserved in any further generic extension by a proper forcing in which $\hat{L}$ does not gain new elements.

The proof that $\mathrm{FA}^+(\sigma\text{-closed})$ holds in $\mathbf{V}[G]$ can now be applied in this situation to show that $(*)$ holds in $\mathbf{V}[G]$. The only difference is that while in the verification of $\mathrm{FA}^+(\sigma\text{-closed})$ it is sufficient to know that the factor forcings are proper, in our setting it is necessary to know that, additionally, the factor forcings do not add new elements to the completions of linear orders. As noted already, this follows from Lemmas 1.2.33 and 2.4.3. □

CHAPTER 3

# A MINIMAL KUREPA TREE WITH RESPECT TO CLUB EMBEDDINGS

## 3.1 Introduction

In this chapter we study some specific $\omega_1$-trees with respect to isomorphisms restricted to a closed unbounded subset of $\omega_1$. Abraham and Shelah first introduced club embeddings and club isomorphisms of $\omega_1$-trees and proved the following theorem.

**Theorem 3.1.1.** *[1]* PFA *implies that every two Aronszajn trees are club isomorphic.*

Here two $\omega_1$-trees $S, T$ are *club isomorphic*, if there is a club $C \subset \omega_1$ such that $T \restriction C$ is isomorphic to $S \restriction C$. This theorem may be regarded as an evidence that under some reasonable forcing axioms, like PFA or some strengthening of it, Aronszajn trees behave like non-atomic countable trees. For instance, considering the fact that $2^{<\omega}$ is a minimal countable non-atomic tree, one might ask whether or not there are minimal Aronszajn trees.

Although there is a powerful structural theorem regarding the club isomorphisms of Aronszajn trees, similar questions regarding Kurepa trees do not seem to have been considered in the literature. In this chapter we will prove:

**Theorem 3.1.2.** *It is consistent with $GCH$ that there is a Kurepa tree $T$ which is club isomorphic to all of its downward closed everywhere Kurepa subtrees. Moreover $T$ has no Aronszajn subtrees.*

An $\omega_1$-tree $T$ is said to be *everywhere Kurepa* if for all $x \in T$, the tree of all $y \in T$ that are compatible with $x$, is Kurepa. Since any Kurepa tree contains an everywhere Kurepa subtree, this theorem implies that the tree in the theorem is actually club minimal with respect to being Kurepa, i.e. for every downward closed Kurepa subtree $U \subset T$ there is a club $C \subset \omega_1$ and a one to one, level and order preserving function $f : T \restriction C \longrightarrow U \restriction C$.

We need the following definition which is related to the invariant defined in the previous chapter.

**Definition 3.1.3.** *$\Omega(T)$ is the set of all countable $Z \subset \mathcal{B}(T)$ with the property that for all $t \in T_{\alpha_Z}$ there is a $b \in Z$ with $t \in b$, where $\alpha_Z = \sup\{b\Delta b' : b, b' \in Z\}$.*

The relation between the $\Omega$ defined here and the one in [8] can be described as follows. Assume $T$ is an $\omega_1$-tree which is equipped with a lexicographic order. Let $L$ be the linear order consisting of the elements of $T$ with the lexicographic order. $\Omega(T)$ defined above is equivalent to $\Omega(L)$ defined in before, in the sense of 1.2.14 if we identify $\hat{L}$ with $\mathcal{B}(T)$.

The forcings we use to add embeddings are not proved to be proper, but their behavior towards suitable models $M$ are similar to proper posets often enough. This property of posets is called $\mathcal{E}$-completeness, and is shown to be a sufficient criterion for preserving $\omega_1$ in [19]. The notion $S$-completeness here essentially coincides with Shelah's $\mathcal{E}$-completeness.

In section 3.2, based on the work in [19], and the notion of proper isomorphism condition for proper posets we will prove the lemmas needed for certain chain conditions which are not included in [19]. We have also included the proof of the fact that $S$-complete forcings are closed under countable support iterations although this is proved in [19]. This makes the proof of the lemmas needed for chain condition properties clearer. Section 3.3 is devoted to the proof of Theorem 3.1.2.

## 3.2 $S$-Completeness, Iteration and Chain Condition

In this section we provide the machinery to iterate these posets and sufficient criteria for verifying the chain conditions of the forcings we will use. Everything in this section is built on the material in [19].

**Definition 3.2.1.** *Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. A poset $\mathcal{P}$ is said to be $S$-complete, if every descending $(M, \mathcal{P})$-generic sequence, $\langle p_n : n \in \omega \rangle$ has a lower bound, for all $M$ with $M \cap X \in S$ and $M$ suitable for $X, \mathcal{P}$.*

First note that $S$-complete forcings preserve the stationarity of all stationary subsets of $S$. Although it is clear from the definition we emphasize that $S$-complete is not stronger than properness unless $S$ is a club. If $S$ contains a club, then being $S$-complete is very close to being $\sigma$-complete and the forcing axiom for this class of forcings is a theorem of $ZFC$. Next fact follows vacuously from the definition.

**Fact 3.2.2.** *Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. If $\mathcal{P}$ is an $S$-complete forcing then it preserves $\omega_1$ and adds no new countable sequences of ordinals.*

Now we prove that for a given stationary $S \subset [X]^\omega$ where $X$ is uncountable, the class of all $S$-complete forcings is closed under countable support iterations.

We follow the same strategy as in the proof of the analogous lemma for proper posets in [19].

**Fact 3.2.3.** *Assume $S, X$ are as above, $\mathcal{P}$ is $S$-complete, and $\Vdash_{\mathcal{P}}$ "$\dot{\mathcal{Q}}$ is $\check{S}$-complete". Then $\mathcal{P} * \dot{\mathcal{Q}}$ is $S$-complete.*

*Proof.* Assume $M$ is suitable for $\mathcal{P} * \dot{\mathcal{Q}}$ and $M \cap X \in S$. Let $\langle p_n * \dot{q}_n : n \in \omega \rangle$ be a descending $(M, \mathcal{P} * \dot{\mathcal{Q}})$-generic sequence. Since $\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{P})$-generic sequence, it has a lower bound $p \in \mathcal{P}$. Moreover

$$p \Vdash_{\mathcal{P}} \text{"}\langle \dot{q}_n : n \in \omega \rangle \text{ is an } (M[\dot{G}_{\mathcal{P}}], \dot{\mathcal{Q}})\text{-generic."}$$

On the other hand, the $(M, \mathcal{P})$-generic condition $p$ forces that $M[\dot{G}_{\mathcal{P}}] \cap \mathbf{V} = M$ and consequently $M[\dot{G}_{\mathcal{P}}] \cap \check{X} \in \check{S}$. So it forces that the sequence $\langle \dot{q}_n : n \in \omega \rangle$ has a lower bound as well. Let $\dot{q}$ be a $\mathcal{P}$-name for such a condition, then $p * \dot{q}$ is a lower bound for $\langle p_n * \dot{q}_n : n \in \omega \rangle$. □

**Lemma 3.2.4.** *Assume $X$ is uncountable, $S \subset [X]^{\omega}$ is stationary, $\langle \mathcal{P}_i, \dot{\mathcal{Q}}_j : i \leq \delta, j < \delta \rangle$ is a countable support iteration of $S$-complete forcings, $N$ is suitable for $\mathcal{P}_\delta$, $N \cap X \in S$, $\langle p_n : n \in \omega \rangle$ is an $(N, \mathcal{P}_\delta)$-generic descending sequence of conditions, $\alpha < \delta$ is in $N$ and $q \in \mathcal{P}_\alpha$ is a lower bound for $\langle p_n \restriction \alpha : n \in \omega \rangle$. Then there is a lower bound $q' \in \mathcal{P}_\delta$ for $\langle p_n : n \in \omega \rangle$, such that $q' \restriction \alpha = q$.*

*Proof.* We use induction on $\delta$. If $\delta$ is a successor ordinal the lemma follows from the induction hypothesis and the argument in the proof of the previous fact. If $\delta$ is limit, let $\langle \alpha_n : n \in \omega \rangle$ be an increasing cofinal sequence in $N \cap \delta$ such that $\alpha_0 = \alpha$, and for all $i$, $\alpha_i \in N$. Note that for all $i$, $\langle p_n \restriction \alpha_i : n \in \omega \rangle$ is a descending $(N, \mathcal{P}_{\alpha_i})$-generic sequence. So by the induction hypothesis there is a sequence $q_i$, $i \in \omega$, such that

- $q_0 = q$
- $q_i \in \mathcal{P}_{\alpha_i}$ is a lower bound for $\langle p_n \restriction \alpha_i : n \in \omega \rangle$
- If $i < j$ then $q_j \restriction \alpha_i = q_i$

Now $q' = \bigcup_{i \in \omega} q_i$ works. □

**Corollary 3.2.5.** *Assume $X$ is uncountable and $S \subset [X]^\omega$ is stationary. Then the class of $S$-complete forcings are closed under countable support iterations.*

We will use the following fact in the next section which follows vacuously from the last definition.

**Fact 3.2.6.** *Assume $T$ is an $\omega_1$-tree which has no Aronszajn subtree in the ground model $\mathbf{V}$, $\Omega(T) \subset [\mathcal{B}(T)]^\omega$ is stationary, and $\mathcal{P}$ is an $\Omega(T)$-complete forcing. Then $T$ has no Aronszajn subtree in $\mathbf{V}^{\mathcal{P}}$.*

*Proof.* Assume $\dot{U}$ is a $\mathcal{P}$-name for a downward closed Aronszajn subtree of $T$. Let $p \in \mathcal{P}$, $M$ be suitable with $M \cap \mathcal{B}(T) \in \Omega(T)$ and $p, \dot{U} \in M$. Also let $\delta = M \cap \omega_1$. For all $b \in M \cap \mathcal{B}(T)$ the set $D_b$ consisting of all conditions $q \in \mathcal{P}$ which forces that $b(\check{\alpha}) \notin \dot{U}$ for some $\alpha \in \omega_1$ is dense and in $M$. Note that if $q \in D_b$, it decides the minimum $\alpha \in \omega_1$, which witnesses that $q \in D_b$. Now let $\langle p_n : n \in \omega \rangle$ be a decreasing $(M, \mathcal{P})$-generic sequence, with $p_0 = p$, and $\bar{p}$ be a lower bound for this sequence. Then $\bar{p}$ forces that $\dot{U}$ has no element in $\{b(\delta) : b \in M \cap \mathcal{B}(T)\} = T_\delta$. This implies that $U$ is a countable set which is a contradiction. □

Now we deal with the chain condition issue for $S$-complete forcings. The following definition is a modification of the $\kappa$-properness isomorphism condition.

**Definition 3.2.7.** *Assume $S, X$ are as above. We say that $\mathcal{P}$ satisfies the $S$-closedness isomorphism condition for $\kappa$, or $\mathcal{P}$ has the $S$-cic for $\kappa$ , where $\kappa$ is an ordinal, if whenever*

- *$M, N$ are suitable models for $\mathcal{P}$,*
- *both $M \cap X, N \cap X$ are in $S$,*
- *$h : M \to N$ is an isomorphism such that $h \restriction (M \cap N) = id$,*
- *$min((N \setminus M) \cap \kappa) > \sup(M \cap \kappa)$, and*
- *$\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{P})$-generic sequence,*

*then there is a common lower bound $q \in \mathcal{P}$ for $\langle p_n : n \in \omega \rangle$ and $\langle h(p_n) : n \in \omega \rangle$.*

**Lemma 3.2.8.** *Assume $2^{\aleph_0} < \kappa$, $\kappa$ is a regular cardinal and that $S, X$ are as above. If $\mathcal{P}$ satisfies the $S$-cic for $\kappa$ then it has the $\kappa$-c.c.*

*Proof.* Let $\langle p_\xi : \xi \in \kappa \rangle$ be a collection of conditions in $\mathcal{P}$, and for each $\xi \in \kappa$, $M_\xi$ be a suitable model for $\mathcal{P}$ such that $M \cap X \in S$, $\kappa, \xi$, and $\langle p_\xi : \xi \in \kappa \rangle$ are in $M$. Consider the function $f : \kappa \longrightarrow \kappa$ defined by $\xi \mapsto \sup(M_\xi \cap \xi)$. Obviously for all $\xi$ with $cf(\xi) > \omega$, $f(\xi) < \xi$. So there is a stationary $W \subset \kappa$ such that the function $f \restriction W$ is a constant. Now find $U \subset W$ of size $\kappa$ such that for all $\xi < \eta$ in $U$, $\sup(M_\xi \cap \kappa) < \eta$ and $M_\xi \cap \xi = M_\eta \cap \eta$.

For each $\xi \in U$ Let $\langle {p_\xi}^n : n \in \omega \rangle$ be descending and $(M_\xi, \mathcal{P})$-generic with ${p_\xi}^0 = p_\xi$. Since $2^{\aleph_0} < \kappa$ we can thin down $U$ if necessary so that for all $\xi, \eta$ in $U$, $M_\xi$ is isomorphic to $M_\eta$ via the map $h_{\xi\eta} : M_\xi \to M_\eta$, induced by the transitive collapse maps.

Now consider models $M_\xi$ together with $\langle {p_\xi}^n : n \in \omega \rangle$ as constants. There are at most continuum many of the isomorphism types of these models and by extensionality the isomorphism between $M_\xi$ and $M_\eta$ is unique if it exists. So we can thin down the collection $\langle p_\xi : \xi \in \kappa\rangle$ again so that for all $\xi, \eta$ and $n \in \omega$, $h_{\xi\eta}({p_\xi}^n) = {p_\eta}^n$, in addition to what we had so far.

Now since $\mathcal{P}$ satisfies $S$-cic, for every pair of distinct $\xi, \eta$ in $U$, there is a condition $q \in \mathcal{P}$ which is a common lower bound for sequences $\langle {p_\xi}^n : n \in \omega\rangle$ and $\langle {p_\eta}^n : n \in \omega\rangle$, meaning in particular that $p_\xi$ is compatible with $p_\eta$ which was desired. $\square$

We are now ready to state and prove the lemma we need for the chain condition issues.

**Lemma 3.2.9.** *Suppose $\langle \mathcal{P}_i, \dot{\mathcal{Q}}_j : i \leq \delta, j < \delta\rangle$ is a countable support iteration of $S$-complete forcings, where $S \subset [X]^\omega$ is stationary and $X$ is uncountable. Assume in addition that*

$$\Vdash_{\mathcal{P}_i} \text{“}\dot{\mathcal{Q}} \textit{ has the } \check{S}\textit{-cic for } \kappa\text{”},$$

*for all $i \in \delta$. Then $\mathcal{P}_\delta$ has the $S$-cic for $\kappa$.*

*Proof.* First note that if $\mathcal{P}$ is any forcing, $M, N$ are suitable for $\mathcal{P}$, $h : M \to N$ is an isomorphism, $p$ is both $(M, \mathcal{P})$-generic and $(N, \mathcal{P})$-generic, and $G \subset \mathcal{P}$ is $\mathbf{V}$-generic with $p \in G$, then $h[G] : M[G] \to N[G]$ defined by $\tau_G \mapsto (h(\tau))_G$ is an isomorphism as well.

Before we deal with the general case, we prove the lemma for $\mathcal{P} * \dot{\mathcal{Q}}$. Let $M, N, h$ be as in definition 3.2.7 for $\mathcal{P} * \dot{\mathcal{Q}}$, and let $\langle p_n * \dot{q}_n : n \in \omega\rangle$ be a descending $(M, \mathcal{P} * \dot{\mathcal{Q}})$-generic such that the sequences $\langle p_n : n \in \omega\rangle$ and $\langle h(p_n) : n \in \omega\rangle$ have a common lower bound $p \in \mathcal{P}$. Since $p$ is both $(M, \mathcal{P})$-generic and $(N, \mathcal{P})$-generic, it forces the hypotheses of the definition 3.2.7 for $M[\dot{G_{\mathcal{P}}}], N[\dot{G_{\mathcal{P}}}], \dot{\mathcal{Q}}, h[\dot{G_{\mathcal{P}}}]$, and $\langle \dot{q}_n : n \in \omega \rangle$. By the assumption on $\dot{\mathcal{Q}}$, there is a $\mathcal{P}$-name $\dot{q}$ which is forced by $p$ to be a common lower bound for $\langle \dot{q}_n : n \in \omega \rangle$ and $\langle h[\dot{G_{\mathcal{P}}}](\dot{q}_n) : n \in \omega \rangle$. So $p * \dot{q}$ is a common lower bound for $\langle p_n * \dot{q}_n : n \in \omega \rangle$ and its image under $h$.

Now let $(**)$ be the assertion that:

"suppose $\alpha < \delta < \kappa$, $M, N, h$, and $\langle p_n : n \in \omega \rangle$, are as in definition 3.2.7 for $\mathcal{P} = \mathcal{P}_\delta$, with $\alpha \in M$. Moreover $r \in \mathcal{P}_\alpha$ and $r_h \in \mathcal{P}_{h(\alpha)}$ are lower bounds for $\langle p_n \restriction \alpha : n \in \omega \rangle$ and $\langle h(p_n \restriction \alpha) : n \in \omega \rangle$, respectively such that

- *supp*$(r) \subset M$, and *supp*$(r_h) \subset N$, and
- $r(\xi) = r_h(\xi)$ for all $\xi$ in $M \cap N$.

Then there are lower bounds $r \in \mathcal{P}_\delta$, $\bar{r} \in \mathcal{P}_\delta$ for $\langle p_n : n \in \omega \rangle$ and $\langle h(p_n) : n \in \omega \rangle$ respectively such that

- *supp*$(\bar{r}) \subset M$, and *supp*$(\bar{r_h}) \subset N$,
- $\bar{r}(\xi) = \bar{r_h}(\xi)$ for all $\xi$ in $M \cap N$,
- $\bar{r} \restriction \alpha = r$ and $\bar{r_h} \restriction h(\alpha) = r_h$."

First note that $(**)$ is stronger than the lemma by letting $\alpha = 0$ and gluing $\bar{r}$ and $\bar{r_h}$ together, in order to get the desired condition.

We use induction to show $(**)$. The successor step is trivial by what we just proved, and if $\delta$ is limit the proof is exactly the same as the Lemma 3.2.4. This is by considering sequences $\langle \alpha_i : i \in \omega \rangle$ which is cofinal in $M \cap \delta$ with $\alpha_0 = \alpha$ as well as its image under $h$ which is cofinal in $N \cap \delta$, because $h$ fixes the intersection. $\square$

We finish this section with a few remarks.

**Remark 3.2.10.** *Unlike Lemma 2.4 of chapter VIII of [19], in the last lemma there is no hypothesis on the length of the iteration. In other words by the lemmas in this section, as long as $\kappa$ is regular and greater than the continuum, any countable support iteration of posets that have the $S$-cic for $\kappa$ has the $\kappa$-cc.*

**Remark 3.2.11.** *It is possible to define $S$-proper posets to be the ones which have $M$-generic condition $q$ below $p$, whenever $M$ comes from a stationary set $S$, and $p$ is a condition inside $M$. These posets inherit many nice properties of proper posets. For instance, they preserve stationarity of all stationary subsets of $S$, and their countable support iterations do not add new branches to $\omega_1$-trees provided that the iterands have this property.*

## 3.3 Minimal Kurepa Trees

In this section I will prove Theorem 4.1.2. A notion of fastness for closed unbounded subsets of $\omega_2$ is used in the definition of the forcings which add embeddings. A club $C_U \subset \omega_2$ is fast enough for $U, T$ if it is the set of all $\sup(M_\xi \cap \omega_2)$ where $\langle M_\xi : \xi \in \omega_2 \rangle$ is a continuous $\in$-chain of $\aleph_1$-sized elementary submodels of $H_\theta$ such that $\xi \cup \omega_1 \subset M_\xi$, $U, T$ are in $M_0$, and $\langle M_\eta : \eta \leq \xi \rangle$ is in $M_{\xi+1}$

**Definition 3.3.1.** *Suppose $T$ is an everywhere Kurepa tree with $\mathcal{B}(T) = \langle b_\xi ; \xi \in \omega_2 \rangle$, $U$ a downward closed everywhere Kurepa subtree of $T$, and $C_U \subset \omega_2$ a club that is fast enough. The forcing $\mathcal{Q}_{T,U}$ is the set of all conditions $p = (f_p, \phi_p)$ such that:*

1. *$f_p : T \restriction A_p \longrightarrow U \restriction A_p$ is a level preserving tree isomorphism, where $A_p \subset \omega_1$ is countable and closed with $max A_p = \alpha_p$,*

2. *$\phi_p$ is a countable partial injection from $\omega_2$ to $\omega_2$ such that*

   (a) *for all $\xi \in \mathrm{dom}(\phi_p)$, $b_{\phi_p(\xi)} \in \mathcal{B}(U)$,*

   (b) *$\phi_p$ respects $C_U$, i.e. for all $\alpha \in C_U$, $\xi \in \mathrm{dom}(\phi_p)$, $\xi \leq \alpha$ if and only if $\phi_p(\xi) \leq \alpha$,*

3. *for each $t \in T_{\alpha_p}$ there are at most finitely many $\xi \in \mathrm{dom}(\phi_p)$ with $t \in b_\xi$, and*

4. *if $\xi \in \mathrm{dom}(\phi_p)$ then $f_p(b_\xi(\alpha_p)) = b_{\phi_p(\xi)}(\alpha_p)$.*

*We let $p \leq q$ if $f_q \subset f_p$ and $\phi_q \subset \phi_p$, and for simplicity we use $\mathcal{Q}$ in order to refer to $\mathcal{Q}_{T,U}$.*

**Lemma 3.3.2.** *The set of all conditions $q$ with $\alpha_q \geq \alpha$ is dense for all $\alpha \in \omega_1$.*

*Proof.* Let $q \in \mathcal{Q}$ and $\alpha > \alpha_q$, and $\beta > \alpha$ such that the following hold.

- for every $t \in T_{\alpha_q}$ there are infinitely many elements in $T_\beta$ that are greater than $t$,
- for every $t \in U_{\alpha_q}$ there are infinitely many elements in $U_\beta$ that are greater than $t$,
- for all $\xi, \eta$ in range$(\phi_q)$, $\beta > b_\xi \Delta b_\eta$, and
- for all $\xi, \eta$ in dom$(\phi_q)$, $\beta > b_\xi \Delta b_\eta$.

Now let $A_p = A_q \cup \{\beta\}$, $\phi_p = \phi_q$. Define $f_p$ to be an extension of $f_q$ whose restriction to $T_\beta$ is a bijection to $U_\beta$, which preserves the tree order, and which satisfies the condition 4 of the definition above. In order to see why the extension from $f_q$ to $f_p$ is possible note that $q$ satisfies the condition 3. So for each $t$ of height $\alpha_q$ the condition

4 puts restriction on only finitely many $s$ of height $\beta$ that are greater than $t$. For each $t \in T_{\alpha_q}$ let $f_p^t$ be any bijection from $\{s > t : s \in T_\beta\}$ to $\{u > f_q(t) : u \in U_\beta\}$, such that for all $\xi \in \mathrm{dom}(\phi_q)$ with $t \in b_\xi$, $f_p^t(b_\xi(\beta)) = b_{\phi_q(\xi)}(\beta)$. Now $f_p = \bigcup_{t \in T_\beta} f_p^t$ is as desired. $\square$

We need the fastness of the club $C_U$ in order to obtain the following lemma.

**Lemma 3.3.3.** *For all $\xi \in \omega_2$ the set of all conditions $q$ with $\xi \in \mathrm{dom}(\phi_q)$ is dense.*

*Proof.* Let $q$ be a condition, $\eta \in \omega_2$, $t = b_\eta(\alpha_q)$, and $u = f_q(t)$. Also let $\xi \in \omega_2$ be the smallest ordinal with $\eta \in M_\xi$. Since for all $\alpha \in \omega_2$, $\alpha \subset M_\alpha$, such an ordinal exists. Since for all $\alpha \in \omega_2$, $\omega_1 \subset M_\alpha$, $M_\alpha \cap \omega_2 \in \omega_2$. Let $\delta_\alpha = M_\alpha \cap \omega_2$, for each $\alpha \in \omega_2$. Note that $\xi$ is a successor ordinal, so let $\xi = \xi^- + 1$. Now by elementarity of $M_\xi$ and the fact that $\delta_{\xi^-} \in M_\xi$ there exists $\eta' \in (\delta_{\xi^-}, \delta_\xi)$ such that $u \in b_{\eta'}$. Now extend $q$ to $p$ by $f_p = f_q$ and $\phi_p = \phi_q \cup \{(\eta, \eta')\}$. $\square$

Similarly we have the following lemma.

**Lemma 3.3.4.** *For all $\xi \in \omega_2$, the set of all conditions $q$ with $\xi \in ran(\phi_p)$ is dense.*

**Lemma 3.3.5.** *Suppose $T$ is an everywhere Kurepa tree with $\Omega(T) = S$ stationary. The following hold.*

1. *$\mathcal{Q}_{T,U}$ is $S$-complete.*
2. *$\mathcal{Q}_{T,U}$ has the $S$-cic for $\omega_2$.*

*Proof.* To see (1) assume $M$ is suitable for $\mathcal{Q}_{T,U}$ and $M \cap \mathcal{B}(T) \in S$. Also let $\langle p_n = (f_n, \phi_n) : n \in \omega \rangle$ be a descending $(M, \mathcal{Q}_{T,U})$-generic sequence, and $\delta = M \cap \omega_1$. By elementarity and density argument,

- $\bigcup_{n\in\omega} \mathrm{dom}(\phi_n) = M \cap \mathcal{B}(T)$, and
- $\bigcup_{n\in\omega} \mathrm{dom}(f_n) = T \restriction A$, for some $A$ which is cofinal in $\delta$.

Let $\phi_p = \bigcup_{n\in\omega} \phi_n$ and for each $\xi \in M \cap \omega_2$ define $f_p(b_\xi(\delta)) = b_{\phi_p(\xi)}(\delta)$. This makes $p$ a condition in the poset and a lower bound for the sequence $\langle p_n : n \in \omega \rangle$, since $T_\delta \subset \bigcup(\mathcal{B}(T) \cap M)$.

For (2), let $M, N, \langle p_n = (f_n, \phi_n) : n \in \omega \rangle$, and $h$ be as in definition 3.2.7, with $M \cap \omega_1 = N \cap \omega_1 = \delta$ and $min((N \setminus M) \cap \kappa) > \sup(M \cap \kappa)$. Also let $\alpha_M = min((M \setminus N) \cap \kappa)$. We let $h(\phi_n) = \psi_n$ and since $h$ fixes the the elements of $M \cap N$, $h(f_n) = f_n$. Note that $b(\delta) = [h(b)](\delta)$, for all $b \in \mathcal{B}(T) \cap M$. Let $\phi = \bigcup_{n\in\omega}(\phi_n \cup \psi_n)$ and $f(b_\xi(\delta)) = b_{\phi(\xi)}(\delta)$.

We need to show that $\phi$ is one to one. Obviously, if $\xi \neq \eta \neq h(\xi)$ then $\phi(\xi) \neq \phi(\eta)$. So assume for a contradiction that $h(\xi) \neq \xi$ and $\phi(\xi) = \phi(h(\xi))$. Fix $n \in \omega$ with $\xi \in \mathrm{dom}(\phi_n)$. Then $\phi_n(\xi) = \psi_n(h(\xi)) = [h(\phi_n)](h(\xi)) = h(\phi_n(\xi))$. Since $h$ fixes $\phi_n(\xi)$, $\phi_n(\xi) \in M \cap N$. On the other hand $\alpha_M \leq \xi$, since $h$ is an isomorphism which fixes the elements of $M \cap N$. Therefore

$$\phi_n(\xi) < \sup(M \cap N \cap \omega_2) < \alpha_M \leq \xi.$$

But $C_U \in M \cap N$ and $\sup(M \cap N \cap \omega_2) \in C_U$ which contradicts the fact that $\phi_n$ respects $C_U$. $\square$

It follows immediately from the definition that if $S$ is stationary in $[X]^\omega$ for some uncountable set $X$, then $S$-complete forcings preserve the stationarity of $S$ and do not add new branches to $\omega_1$-trees. Now by Lemmas 3.3.5, 3.2.4, 3.2.8, and 3.2.9, the following proposition is obvious.

**Proposition 3.3.6.** *Assume GCH. If $T$ is an everywhere Kurepa tree with $\omega_2$ many branches such that $\Omega(T) \subset [\mathcal{B}(T)]^\omega$ is stationary , then there is a forcing extension in which GCH is still true and $T$ is club isomorphic to all of its downward closed everywhere Kurepa subtrees.*

In order to prove Theorem 3.1.2, it suffices to show that there is a Kurepa tree that satisfies the hypothesis of the proposition. Let $\mathcal{K}$ be the poset consisting of conditions of the form $p = (T_p, b_p)$ where

- $T_p$ is a countable tree of height $\alpha_p + 1$ such that for all $t \in T_p$ there exists $s \in (T_p)_{\alpha_p}$ with $t < s$,
- $b_p$ is countable partial function from $\omega_2$ to the last level of $T_p$.

$p \leq q$ in $\mathcal{K}$ if

- $(T_p)_{\leq \alpha_q} = T_q$
- $\text{dom}(b_p) \supset \text{dom}(b_q)$
- for all $\xi \in \text{dom}(b_q)$, $b_q(\xi) \leq b_P(\xi)$

It is well known that $\mathcal{K}$ is countably closed and under $CH$, has the $\omega_2$-chain condition. Let $T$ be the $\mathcal{K}$-generic tree, then $\Omega(T)$ is stationary in $[\mathcal{B}(T)]^\omega$. To see that let $p$ be a condition that forces the contrary and $\dot{E}$ be a $\mathcal{K}$-name for a club in

$[\mathcal{B}(T)]^\omega$ which is forced by $p$ to be disjoint from $\Omega(T)$. Let $M$ be suitable for $\mathcal{K}$ with $p, \dot{E}$, etc in $M$. Then for any sequence $\langle p_n : n \in \omega \rangle$ which is $(M, \mathcal{K})$-generic and $p_0 \leq p$ we can form a lower bound $\bar{p}$ for the sequence such that $\mathrm{dom}(\bar{p}) = M \cap \omega_2$. Note that such a condition forces that $M \cap \omega_2 = M[\dot{G}] \cap \omega_2$, where $\dot{G}$ is the canonical name for the $\mathcal{K}$-generic filter. On the other hand $\bar{p}$ forces that $M[\dot{G}] \cap \mathcal{B}(T) \in \dot{E}$, because it is $M$-generic. So $\bar{p}$ forces that $M[\dot{G}] \cap \mathcal{B}(T) \in \dot{E} \cap \Omega(T)$ which is a contradiction.

In order to show that $T$ does not have any Aronszajn subtree in the final model, after embeddings added, we will show that

$$\Vdash_{\mathcal{K}} \dot{T} \text{ has no Aronszajn subtrees.}$$

Note that this suffices by Fact 3.2.6. Let $\dot{U}$ be a $\mathcal{K}$-name for an uncountable downward closed subtree of $\dot{T}$, where $\dot{T}$ is a $\mathcal{K}$-name for the tree $T$. Let $M$ be a suitable model for $\mathcal{K}$ with $\dot{U} \in M$. By the assumptions, for all $\xi \in \omega_2 \cap M$, and $p \in M \cap \mathcal{P}$ there is an extension $q \in M \cap \mathcal{P}$ of $p$ such that for some $\alpha \in \omega_1$, $q$ forces that $b_\xi \restriction \alpha \notin \dot{U}$, where $b_\xi = \bigcup_{p \in \dot{G}} b_p(\xi)$ and $\dot{G}$ is the canonical name for the generic filter of the forcing $\mathcal{K}$. Note that by elementarity $\alpha$ is in $M \cap \omega_1$. Now let $\langle p_n : n \in \omega \rangle$ be an $(M, \mathcal{K})$-generic sequence such that for all $\xi \in \omega_2 \cap M$ there is an $n \in \omega$ such that $p_n \Vdash b_\xi(M \cap \omega_1) \notin \dot{U}$. Let $q$ be a lower bound for this sequence such that $\mathrm{dom}(b_q) = M \cap \omega_2$. Then $q$ forces that $\dot{U}$ is countable, which is a contradiction. We showed that every downward closed subtree of $T$ contains $b_\xi$ for some $\xi \in \omega_2$. This shows that $T$ has no Aronszajn subtree and $\langle b_\xi : \xi \in \omega_2 \rangle$ is the collection of all branches of $T$.

**Remark 3.3.7.** *The Kurepa tree constructed in [9] has no Aronszajn subtrees and satisfies the hypothesis of the last proposition. So it can be made minimal in the* *same way.*

CHAPTER 4

# A NEW MINIMAL NON-$\sigma$-SCATTERED LINEAR ORDER

## 4.1 Introduction

All of the results proving that Laver's theorem is consistently not sharp were based on the consistency of the minimality of real types or Aronszajn types. So it is natural to ask:

**Question 4.1.1.** *Does every minimal non-$\sigma$-scattered linear order have to be real or Aronszajn type?*

Note that an affirmative answer to this question would imply that the assumption $\mathrm{PFA}^+$ was needed in order to obtain the results in [8]. Consequently, the model Moore came up with in [16] would satisfy "Laver's thorem is sharp." Therefore the work in the previous chapter as well as the large cardinal assumption would not be needed to prove Laver's theorem is sharp. In this chapter we will provide a negative answer to this question. In particular real and Aronszajn types are not the only possible obstructions to the sharpness of Laver's theorem.

**Theorem 4.1.2.** *It is consistent with GCH that there is a non-$\sigma$-scattered linear order $L$ which contains no real or Aronszajn type and is minimal with respect to not being $\sigma$-scattered.*

Moreover, Theorem 4.1.2 is related the following question which is due to Galvin.

**Question 4.1.3.** *[4, problem 4] Is there a linear order which is minimal with respect to not being $\sigma$-scattered and which has the property that all of its uncountable* *suborders contain a copy of $\omega_1$?*

Note that a consistent negative answer is already given by Ishiu and Moore in [8]. Theorem 4.1.2 does not answer Galvin's question because the linear order we introduce, contains many copies of $\omega_1^*$.

This chapter is organized as follows. Section 4.2 is devoted to constructing a specific Kurepa tree that is a suitable candidate for having suborders that witness Theorem 4.1.2. In Section 4.3 we introduce the posets that add isomorphisms we need. Section 4.4 finishes the proof of Theorem 4.1.2.

## 4.2 The generic homogeneous Kurepa tree

**Definition 4.2.1.** *Assume $C \subset \omega_2$ consists of all $\sup(M_\xi \cap \omega_2)$ where $\langle M_\xi : \xi \in \omega_2 \rangle$ is a continuous $\in$-chain of $\aleph_1$-sized elementary submodels of $H_{(2^{\omega_2})^+}$, such that $\xi \cup \omega_1 \subset M_\xi$ and $\langle M_\eta : \eta \leq \xi \rangle$ is in $M_{\xi+1}$. The poset $\mathcal{H}$ consists of conditions $q = (T_q, b_q, \Pi_q)$ for which the following statements hold:*

1. *$T_q$ is a countable tree of height $\alpha_q + 1$ which is equipped with a lexicographic order such that for all $t \in (T_q)_{<\alpha_q}$, the set $t^+$, consisting of all immediate successors of $t$, is isomorphic to the rationals when considered with the lexicographic order.*
2. *$b_q$ is a bijective function from a countable subset of $\omega_2$ to the last level of $T_q$.*
3. *The collection , $\Pi_q = \langle \pi^q_{t,s} : (t,s) \in T_q^{[2]} \rangle$ such that $\pi^q_{t,s}$ is a tree isomorphism from $T_q(t)$ to $T_q(s)$, which preserves the lexicographic order.*
4. *The collection $\Pi_q$ is coherent, in the sense that if $t' > t$ and $\pi^q_{t,s}(t') = s'$ then $\pi^q_{t',s'} = \pi^q_{t,s} \restriction T_q(t')$.*

5. *The collection* $\Pi_q$ *is symmetric in the sense that* $\pi^q_{s,t} = (\pi^q_{t,s})^{-1}$.

6. *The collection* $\Pi_q$ *respects the club* $C$*, in the sense that if* $\alpha \in C$*,* $t, s$ *are in* $T_q$ *and have the same height then,* $\xi < \alpha$ *iff* $b_q^{-1}(\pi^q_{t,s}(b_q(\xi))) < \alpha$.

7. *The collection* $\Pi_q$ *respects the composition operation, in the sense that if* $t, s, u$ *are in* $(T_q)_\xi$ *and* $\xi < \alpha_q$ *then* $\pi^q_{s,u} \circ \pi^q_{t,s} = \pi^q_{t,u}$.

*For* $p, q \in \mathcal{H}$ *we let* $q \leq p$ *if*

1. $(T_q)_{\leq \alpha_p} = T_p$ *and the lex order on* $T_p$ *is the same as the one on* $T_q$*,*

2. $\mathrm{dom}(b_p) \subset \mathrm{dom}(b_q)$*,*

3. *for all* $\xi \in \mathrm{dom}(b_p)$*,* $b_p(\xi) \leq b_q(\xi)$*,*

4. *for all* $(t, s) \in \bigcup_{\xi \in \alpha_q} (T_p)^2_\xi$*,* $\pi^p_{t,s}$ *is equal to* $\pi^q_{t,s} \restriction T_p$*, and*

5. *for all* $(t, s) \in \bigcup_{\xi \in \alpha_q} (T_p)^2_\xi$*, and* $\xi, \eta \in \mathrm{dom}(b_p)$*, if* $\pi^p_{t,s}(b_p(\xi)) = b_p(\eta)$ *then* $\pi^q_{t,s}(b_q(\xi)) = b_q(\eta)$.

**Notation 4.2.2.** *Assume* $G$ *is a generic filter for* $\mathcal{H}$*. Define* $T_G$ *to be* $\bigcup_{q \in G} T_q$ *and* $b_\xi$ *to be the branch* $\{b_q(\xi) : q \in G\}$*. If* $t, s$ *are in* $T_G$ *and have the same height* $\pi_{t,s}$ *denotes* $\bigcup_{q \in G} \pi^q_{t,s}$.

**Lemma 4.2.3.** $\mathcal{H}$ *is* $\sigma$*-closed.*

*Proof.* Let $\langle p_n : n \in \omega \rangle$ be a decreasing sequence in $\mathcal{H}$ and $\sup(\alpha_{p_n})_{n \in \omega} = \alpha$. Let $T = \bigcup_{n \in \omega} T_{p_n}$. Note that $\langle b_{p_n}(\xi) : n \in \omega \rangle$ is a cofinal chain in $T$ for all $\xi \in \bigcup_{n \in \omega} \mathrm{dom}(b_{p_n})$. Let $T_q$ be a countable tree of height $\alpha + 1$ such that

- $(T_q)_{<\alpha} = T$,
- for all $\xi \in \bigcup_{n \in \omega} \mathrm{dom}(b_{p_n})$, $\langle b_{p_n}(\xi) : n \in \omega \rangle$ has an upper bound in $T_q$, and

- every element of height $\alpha$ is an upper bound for $\langle b_{p_n}(\xi) : n \in \omega \rangle$, for some $\xi \in \bigcup_{n\in\omega} \mathrm{dom}(b_{p_n})$.

Now let $q$ be the condition with $\alpha_q = \alpha$ and $T_q$ as above. Let $b_q$ be the function form $\bigcup_{n\in\omega} \mathrm{dom}(b_{p_n})$ to the last level of $T_q$ such that for all $\xi$ in the domain, $b_q(\xi)$ is the upper bound for the chain $(b_{p_n}(\xi) : n \in \omega)$. Similarly for all $t, s$ that are of the same height and are in $T$, $\bigcup_{n\in\omega} \pi^{p_n}_{t,s}$ can be extended to the last level of $T_q$ as follows. For such $t, s$ let $t', s'$ are in the last level of $T_q$ with $t < t'$ and $s < s'$. Also let $b_t, b_s$ be branches through $T$ which have $t', s'$ as their top elements respectively. By the last condition on how the last level of $T_q$ is chosen, these two branches are indexed by some $\xi, \eta$ in $\bigcup_{n\in\omega} \mathrm{dom}(b_{p_n})$. Since $\langle p_n : n \in \omega \rangle$ is decreasing, by condition 5 of the definition of the order on $\mathcal{H}$, $\bigcup_{n\in\omega} \pi^{p_n}_{t,s}$ maps the elements of $b_t$ to the elements of $b_s$. Therefore $\pi^q_{t,s}(t') = s'$, makes $q$ satisfy the condition 5 of the definition of the order of $\mathcal{H}$ for all $p_n$'s. It is easy to see that the condition $q$ described above satisfies the other conditions of the definition of the order on $\mathcal{H}$. □

**Lemma 4.2.4.** *GCH implies that $\mathcal{H}$ has the $\aleph_2$-cc.*

*Proof.* Let $\langle q_\xi : \xi \in \omega_2 \rangle$ be a collection of conditions in $\mathcal{H}$. Since there are $\aleph_1$-many possibilities for $T_q$ and $\Pi_q$, we can thin down this collection to a subset of the same cardinality so that $T_{q_\xi}$ and $\Pi_{q_\xi}$ do not depend on $\xi$. Now define $f : C \longrightarrow \omega_2$ by $f(\xi) = \sup(\mathrm{dom}(b_{q_\xi}) \cap \xi)$, where $C$ is the club that all elements of $\mathcal{H}$ respect. Note that for all $\xi \in C$ with $cf(\xi) > \omega$, $f$ is regressive. So there is a stationary $S \subset C$, and $\alpha \in \omega_2$ such that $f \restriction S$ is the constant $\alpha$. We can thin down $S$ to a stationary subset $S'$ if necessary, so that in $\langle q_\xi : \xi \in S' \rangle$, $\mathrm{dom}(b_{q_\xi}) \cap \alpha$ and $b_{q_\xi} \restriction \alpha$ do not depend on $\xi$. Let $S'' \subset S' \setminus (\alpha + 1)$ be of size $\aleph_2$ and whenever $\xi < \eta$ are in $S''$, $\sup(\mathrm{dom}(b_{q_\xi})) < \eta$. Note that $\langle b_{q_\xi} : \xi \in S'' \rangle$ forms a $\Delta$-system with root $r$

such that the $\mathrm{dom}(r) \subset \alpha$. Moreover for all $\xi \in S''$, $min(\mathrm{dom}(b_{q_\xi}) \smallsetminus r) \geq \xi$. Since $S'' \subset C$, every two conditions in $\langle q_\xi : \xi \in S'' \rangle$ are compatible. □

**Fact 4.2.5.** *The following sets are dense in $\mathcal{H}$.*

1. $H_\alpha := \{q \in \mathcal{H} : \alpha_q > \alpha\}$.
2. *For* $\xi \in \omega_2$, $I_\xi := \{q \in \mathcal{H} : \xi \in \mathrm{dom}(b_q)\}$.

*Proof.* 1. We use induction on the height of the conditions. Note that the limit case is trivial as $\mathcal{H}$ is countably closed. Let $p \in \mathcal{H}$, we show $p$ has an extension $q$ with $\alpha_q = \alpha_p + 1$. For each $t$ in $T_{\alpha_p}$ we add a copy of the rationals to $T_p$ as the immediate successors of $t$. We let $(T_q)_{\alpha_q}$ be the union of the elements that are added to the tree in this way. Now find $\langle \pi^q_{t,s} : (t,s) \in ((T_q)_{\alpha_p})^2 \rangle$ such that 3,5,7 of the definition of conditions in 4.2.1 are satisfied. Note that this only requires a system of isomorphisms of copies of the rationals which are symmetric and which respect the composition — in the sense of 5,7 of 4.2.1. For $t, s$ in lower levels, i.e. $t, s \in (T_q)_{<\alpha_p}$, extend $\pi^p_{t,s}$ to the last level of $T_q$ to obtain $\pi^q_{t,s}$ such that 4 of the definition of order in 4.2.1 holds for $p, q$.

Now we need to index the last level of $T_q$ by ordinals in $\omega_2$ such that 2,6 of the definition of the conditions in 4.2.1 and 2,3,5 of the definition of the order in 4.2.1 hold. Define the equivalence relation $\sim$ on $\mathrm{dom}(b_p)$ by $\xi \sim \eta$ if and only if there are $t, s$ of height $< \alpha_p$ such that $\pi^p_{t,s}(b_p(\xi)) = b_p(\eta)$. Here we use the fact that $p$ satisfies conditions 5,7 of the definition of conditions in 4.2.1. Let $A \subset \mathrm{dom}(b_p)$ which intersects each equivalence class at exactly one point. For each $\xi \in A$ let $u_\xi$ be an immediate successor of $b_p(\xi)$ and let $b_q(\xi) = u_\xi$. Now for each $\eta \in \mathrm{dom}(b_p)$ find $\xi \in A$ such that $\eta \sim \xi$ as well as $t, s$ of height $< \alpha_p$ such that $\pi^p_{t,s}(b_p(\xi)) = b_p(\eta)$. Now let $b_q(\eta) = \pi^q_{st,s}(u_\xi)$. Since $q$ satisfies 4 of the definition of conditions in 4.2.1, the definition of $b_q(\eta)$ does not depend on the choice of $t, s$ and the condition 5 of the definition of the order in 4.2.1 holds for $p, q$.

In order to index the rest of the elements of the last level of $T_q$, let $X$ be a countable subset of $\omega_2$ such that for all $\xi, \eta \in X$, $C \cap (\xi, \eta) = \emptyset$. Now index the rest of the elements by the elements of $X$ in a one to one fashion. This makes $b_q$ a bijection from a countable subset of $\omega_2$ onto the last level of $T_q$ as desired.

2. Let $p \in \mathcal{H}$ and $\xi \notin \mathrm{dom}(b_p)$ as in the lemma. By revising $X$ so that $\xi \in X$, extend $p$ to $q$ of height $\alpha_p + 1$ as above.

□

The proof of the following lemma is the same as the proof of Lemma 4.2.3.

**Lemma 4.2.6.** *If $M$ is suitable for $\mathcal{H}$ and $\langle p_n : n \in \omega \rangle$ is a decreasing $(M, \mathcal{H})$-generic sequence, then there is a lower bound $q$ for $\langle p_n : n \in \omega \rangle$ such that* $\mathrm{dom}(b_q) = M \cap \omega_2$, *and $\alpha_q = M \cap \omega_1$.*

**Fact 4.2.7.** *The following are true if $G$ is a generic filter for $\mathcal{H}$.*

- *The generic tree $T := \bigcup_{q \in G} T_q$ is a Kurepa tree such that $\langle \{b_q(\xi) : q \in G\} : \xi \in \omega_2 \rangle$ is an enumeration of the set of all branches.*
- *$T$ has no Aronszajn subtree. Moreover, any uncountable downward closed subtree of $T$ contains a branch $b_\xi$ for some $\xi \in \omega_2$.*
- *Assume $L$ is the linear order consisting of all branches of $T$, $\mathcal{B}(T)$, ordered by the lexicographic order of the tree $T$. Then $\Omega(L)$ is stationary.*

*Proof.* The first two statements follow vacuously from the last lemma. For the last statement, let $M$ be suitable for $\mathcal{H}$ and $p \in M \cap \mathcal{H}$. Then the $(M, \mathcal{H})$-generic condition from the last lemma forces that $M[\dot{G}] \cap \mathcal{B}(L) \in \Omega(L)$. □

From now on $T$ is the generic Kurepa tree generated by $\mathcal{H}$ unless otherwise mentioned. Also $K$ is the linear order $\mathcal{B}(T)$ ordered by the lexicographic order of the tree $T$. We fix an enumeration of of $\mathcal{B}(T) = \langle b_\xi : \xi \in \omega_2 \rangle$.

The rest of this section is devoted to showing that $K$ has many non-$\sigma$-scattered suborders that are amenable. These facts are not used in the proof of the results in the next sections but show some possible obstruction for the minimality of suborders of $K$. In the next section these non-$\sigma$-scattered suborders are forced to be $\sigma$-scattered by an improper forcing. Here we say a countable sequence of conditions in $\mathcal{H}$ forces a statement if every lower bound of that sequence forces that statement.

**Definition 4.2.8.** *Let $T$ be the $\mathcal{H}$-generic tree. Then $t \in T$ is said to be simple if whenever $M$ is a countable elementary submodel of $H_\theta$ containing $T$, $M$ captures $t \in T$. Otherwise $t$ is said to be complex.*

**Lemma 4.2.9.** *Assume GCH holds in $\mathbf{V}$, $M$ is suitable for $\mathcal{H}$, $\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{H})$-generic sequence, $t \in T_0 := \bigcup_{n\in\omega} T_{p_n}$, $\langle p_n \rangle_{n\in\omega} \Vdash$ "$t$ is simple", $b$ is a branch in $T_0$ and $ht(t) < \alpha < \delta := M \cap \omega_1$. Then there exists $s \in T_0$ such that $ht(s) = \alpha$, $t < s$, $s \notin b$ and $\langle p_n \rangle_{n\in\omega} \Vdash$ "$s$ is simple".*

*Proof.* First note that if $G$ is $\mathcal{H}$-generic over $\mathbf{V}$ then $H_{\omega_3}[G] = H_{\omega_3}^{\mathbf{V}[G]}$ has a well ordering $\lhd$. Let $\dot{\lhd}$ be an $\mathcal{H}$-name for $\lhd$. Since $\langle p_n \rangle_{n\in\omega}$ is $M$-generic it decides $\dot{\lhd} \cap (M[\dot{G}])^2$, in the sense that, if $\tau$ and $\pi$ are two $\mathcal{H}$-names that are in $M$ then there is an $n \in \omega$ such that $p_n \Vdash$ "$\tau \dot{\lhd} \pi$" or $p_n \Vdash$ "$\pi \dot{\lhd} \tau$".

Also note that if $t$ is simple then so is every $t' \in t^+$. Now let $\sigma \in M$ be an $\mathcal{H}$-name for a branch of the $\mathcal{H}$-generic tree such that $\langle p_n \rangle_{n\in\omega}$ forces that

- $t \in \sigma$,

- $\sigma(ht(t)+1) \neq b(ht(t)+1)$, and
- $\sigma$ is the $\dot{\lhd}$-minimum branch of $\dot{T}$ with the properties above.

Let $s \in T_0$ such that $\langle p_n \rangle_{n\in\omega}$ forces that $s = \sigma(\alpha)$. We will show that $\langle p_n \rangle_{n\in\omega} \Vdash$ "$s$ is simple". Let $G$ be an $\mathcal{H}$-generic filter containing $\langle p_n \rangle_{n\in\omega}$ and in $\mathbf{V}[G]$, $N$ be a countable elementary submodel of $H_{\omega_3}$. If $N \cap \omega_1 \leq ht(t)$, by simplicity of $t$, $N$ captures $s$. if $ht(t) < N \cap \omega_1$ then $t^+ \subset N$ so $\sigma_G$ which is $min_{\lhd}\{b \in \mathcal{B}(T) : b(ht(t) = 1) = s(ht(t)+1)\}$. So by elementarity $\sigma_G \in N$ and $N$ captures $s$. □

**Proposition 4.2.10.** *Assume GCH holds in* $\mathbf{V}$ *and* $G$ *is* $\mathbf{V}$*-generic for* $\mathcal{H}$*. Then* $K$ *has an amenable non-*$\sigma$*-scattered suborder.*

*Proof.* Let $L = \{t \in T : t \text{ is minimal complex}\}$ ordered by the lexicographic order of the $\mathcal{H}$-generic tree $T$. To see $L$ is amenable, let $t \in L$ and $M$ be countable elementary submodel of $H_\theta$ with $T, L \in M$, where $\theta$ is a regular large enough cardinal. Let $E = \{N \cap \mathcal{B}(T) : N \text{ is a countable elementary submodel of } H_{\omega_3} \text{ with } T, L \in N\}$. Since $t$ is a minimal complex element of $T$ every $N \in E \cap M$ captures $t$. So $t$ is internal to $M$ and $L$ is amenable.

In order to see $L$ is not $\sigma$-scattered we will show that $\Gamma(L)$ is stationary in $[\hat{L}]^\omega$. Assume $\dot{E}$ is an $\mathcal{H}$-name for for a club in $[\hat{L}]^\omega$ and $q \in \mathcal{H}$. In $\mathbf{V}$, let $M$ be countable elementary submodel of $H_\theta$ and $\theta$ be a regular large enough cardinal. Let $\langle p_n : n \in \omega \rangle$ be an $M$-generic sequence such that $p_0 = q$. Also let $\langle b_n : n \in \omega \rangle$ be an enumeration of all branches of $T_0 = \bigcup_{n\in\omega} T_{p_n}$ which are downward closure of $\{b_{p_n}(\xi) : n \in \omega\}$ for some $\xi \in M \cap \omega_2$. By the previous lemma there is a sequence $\langle t_k : k \in \omega \rangle$ of elements in $T_0$ such that for all $k \in \omega$, $\langle p_n \rangle_{n\in\omega}$ forces that $t_k$ is simple, $t_k < t_{k+1}$, $t_k \notin b_k$ and $\sup\{ht(t_k) : k \in \omega\} = \delta := M \cap \omega_1$.

Now let $T_p = T_0 \cup (T_p)_\delta$ where is a minimal set such that

- for each $\xi \in M \cap \omega_2$, $\{b_{p_n}(\xi) : n \in \omega\}$ has a unique upper bound in $T_\delta$,
- the sequence $t_k$ has a unique upper bound for in $T_\delta$, and
- for each $u, v \in T_0$ and $t \in T_\delta$, $(\pi^{p_n}_{u,v})[\{s \in T_0 : s < t\}]$ has a unique upper bound in $T_\delta$.

It is easy to see that there are $b_p$ which is a from a countable subset of $\omega_2$ to $T_\delta$ and $\Pi_p$ consisting of natural extensions of the maps $\pi^{p_n}_{u,v}$ where $u, v$ are in $T_0$, such that $p = (T_p, b_p, \Pi_p)$ is a lower bound for $p$.

On the other hand $p$ forces the following statement.

- There are minimal complex elements at the $\delta$'th level of the $\mathcal{H}$-generic tree $T$.
- $M[\dot{G}] \cap \tau \in \dot{E}$, where $\tau$ is an $\mathcal{H}$-name for $\hat{L}$ in $M$.
- $M[\dot{G}] \cap \tau$ does not capture all elements of $\dot{L}$.

Therefore $\nVdash_{\mathcal{H}} \Vdash$ "$\dot{L}$ is not $\sigma$-scattered." Note that the elements of $L$ form an antichain in $T$. Let $L' \subset K$ such that for every $t \in L$ there is a unique branch $b \in L'$ with $t \in b$. Then $L'$ is isomorphic to $L$, hence $K$ has an amenable non-$\sigma$-scattered suborder.

$\square$

## 4.3 Adding embeddings

In the previous section we introduced a forcing which generates a Kurepa tree $T$ equipped with a lexicographic order which also has some homogeneity properties.

In this section we use the homogeneity of $T$ to prove the countable support iteration of some forcings that add embeddings among the $\aleph_1$-sized dense subsets of the linear order $K = (\mathcal{B}(T), <_{lex})$ does not collapse cardinals. We fix an enumeration $\langle b_\xi : \xi \in \omega_2 \rangle$ of the branches of the tree $T$ for the rest of the paper, and recall that for each $t \in T$, the set $t^+$, consisting of all immediate successors of $t$ with respect to $<_T$, is isomorphic to the rationals when considered with the lex order inherited from the tree $T$. Here homogeneity of $T$ means that there is a collection $\Pi = \langle \pi_{t,s} : t, s \in T \text{ and } ht(t) = ht(s) \rangle$ with the following properties.

1. for all $t, s$ in $T$ which have the same height $\pi_{t,s}$ is a tree and lex order isomorphism from the tree of all elements that are compatible with $t$ to the tree of all elements that are compatible with $s$.
2. $\Pi$ is symmetric, in the sense that $\pi_{t,s} = (\pi_{s,t})^{-1}$.
3. $\Pi$ is coherent in the sense that if $t, s, t', s'$ are in $T$, $ht(t) = ht(s)$, $t < t'$, $s < s'$ and $\pi_{ts}(t') = s'$, then $\pi_{t,s} \restriction T(t') = \pi_{t',s'}$, where $T(t') = \{u \in T : t' \text{ is compatible with } u\}$.

**Definition 4.3.1.** *Assume $T$ is as above and $X, Y$ are two $\aleph_1$-sized subsets of $\omega_2$ such that both $\langle b_\xi ; \xi \in X \rangle$ and $\langle b_\xi ; \xi \in Y \rangle$ are dense in $K$. $\mathcal{F}_{XY} (= \mathcal{F})$ is the poset consisting of all conditions $p = (f_p, \phi_p)$ for which the following holds:*

1. *$f_p : T \restriction A_p \longrightarrow T \restriction A_p$ is a lex order and level preserving tree isomorphism where $A_p \subset \omega_1$ is countable and closed with $max A_p = \alpha_p$.*
2. *$\phi_p$ is a countable partial injection from $\omega_2$ to $\omega_2$ such that:*
    - *(a) for all $\xi \in \mathrm{dom}(\phi_p)$, if $\xi \in X$ then $\phi_p(\xi) \in Y$,*
    - *(b) for all $\xi \in \mathrm{dom}(\phi_p) \setminus X$, $b_{\phi_p(\xi)} = \pi_{t,s}[b_\xi]$, where $t$ is an immediate successor of $b_\xi(\alpha_p)$ and $s$ is a successor of $f_p(b_\xi(\alpha_p))$, and*

*(c) the map $b_\xi \mapsto b_{\phi_p(\xi)}$ is lexicographic order preserving.*

3. *For all $t \in T_{\alpha_p}$ there are at most finitely many $\xi \in \mathrm{dom}(\phi_p)$ with $t \in b_\xi$.*

4. *For all $\xi \in \mathrm{dom}(\phi_p)$, $f_p(b_\xi(\alpha_p)) = b_{\phi_p(\xi)}(\alpha_p)$.*

*We let $q \leq p$ if $f_p \subset f_q$ and $\phi_p \subset \phi_q$.*

It is obvious that the sets $\{q \in \mathcal{F} : \alpha_q > \beta\}$ and $\{q \in \mathcal{F} : \xi \in \mathrm{dom}(\phi_q)\}$ are dense for all $\beta \in \omega_1$ and $\xi \in \omega_2$. Therefore the forcing $\mathcal{F}$ adds a lexicographic order embedding from $X$ to $Y$ via the map $\Phi \restriction X$ where $\Phi = \bigcup_{p \in G} \phi_p$ and $G$ is the generic filter for $\mathcal{F}$. We will show that countable support iterations of these forcings do not collapse cardinals.

**Lemma 4.3.2.** *Assume $X, Y$ are $\aleph_1$-sized subsets of $\omega_2$ such that $\{b_\xi : \xi \in X\}$ and $\{b_\xi : \xi \in Y\}$ are dense subsets of $K$. Then*

- *$\mathcal{F}_{XY}$ is $\Omega(T)$-complete, and*
- *$\mathcal{F}_{XY}$ has the $\Omega(T)$-cic for $\omega_2$.*

*Proof.* To see (1), assume $M$ is suitable for $\mathcal{F}$ and $M \cap K \in S$. Also let $\langle p_n = (f_n, \phi_n) : n \in \omega\rangle$ be a descending $(M, \mathcal{F})$-generic sequence and $\delta = M \cap \omega_1$. Note that $M \cap \omega_2 = \bigcup_{n\in\omega} \mathrm{dom}(\phi_n)$ and $\bigcup_{n\in\omega} A_{p_n}$ is cofinal in $\delta$. Now let $\phi_p = \bigcup_{n\in\omega} \phi_n$, and $f_p = \bigcup_{n\in\omega} f_n \cup f$, where $f(b_\xi(\delta)) = b_{\phi_p(\xi)}(\delta)$. This makes $p$ a lower bound for $\langle p_n : n \in \omega\rangle$, since $M \cap K \in S$, and $\{b_\xi(\delta) : \xi \in M \cap \omega_2\} = T_\delta$.

For (2), let $M, N, \langle p_n = (f_n, \phi_n) : n \in \omega\rangle$, and $h$ be as in definition 3.2.7 with $M \cap \omega_1 = N \cap \omega_1 = \delta$. Since $h$ fixes the intersection $h(f_n) = f_n$ and $b(\delta) = [h(b)](\delta)$, for all $b \in M \cap \mathcal{B}(T)$. Let $\phi_p = \bigcup_{n\in\omega}(\phi_n \cup h(\phi_n))$ and $f_p = \bigcup_{n\in\omega} f_n \cup f$, where $f(b_\xi(\delta)) = b_{\phi_p(\xi)}(\delta)$. Note that by part 2.b. of the definition 4.3.1, hypothesis on

$M, N$, and the fact that all $\pi_{t,s}$ in $\Pi$ preserve the lexicographic order of $T$, $\phi_p$ is one to one and the map $b_\xi \mapsto b_{\phi_p(\xi)}$ is lex order preserving. This shows that $p$ is a condition which is also a lower bound for $\langle p_n : n \in \omega \rangle$ and its image under $h$. $\square$

## 4.4 Proof of the main theorem

In this section we will finish the proof of Theorem 4.1.2. The strategy is to show that if two $\aleph_1$-sized $L, L' \subset K$ have closure of cardinality $\aleph_2$, then they are biembeddable. Note that by lemma 4.2.10, $K$ has non-$\sigma$-scattered suborders whose closure have cardinality $\aleph_1$. So in order to use the strategy mentioned above, we need to make these suborders $\sigma$-scattered by forcings for which the analogue of Lemma 4.3.2 holds.

**Definition 4.4.1.** *Assume $L \subset K$, $|\bar{L}| \leq \aleph_1$. $\mathcal{P}_L(= \mathcal{P})$ is the poset consisting of all conditions $p : \alpha_p + 1 \longrightarrow [\bar{L}]^\omega \cap \Omega(L)$ that are $\subset$-increasing and continuous.*

**Lemma 4.4.2.** *Assume $L$ is a suborder of $K$ whose closure has size $\leq \aleph_1$. Then $\mathcal{P}_L$ is $\Omega(K)$-complete and has the $\Omega(K)$-cic for $\omega_2$.*

*Proof.* Let $M$ be suitable for $\mathcal{P}_L$ and $M \cap K \in \Omega(K)$. By 1.2.16 $M \cap \bar{L} \in \Omega(L)$, so $\mathcal{P}_L$ is $\Omega(K)$-complete.

To see $\mathcal{P}_L$ is $\Omega(K)$-cic for $\omega_2$, note that if $h : M \longrightarrow N$ is an isomorphism that fixes $M \cap N$, then $h$ fixes $\bar{L} \cap M$ because $|\bar{L}| = \aleph_1$. So any lower bound for an $M$-generic sequence is a lower bound for it image under $h$. $\square$

Now we are ready to prove Theorem 4.1.2. Assume $GCH$ holds in $\mathbf{V}$ and $T$ is the generic Kurepa tree from the forcing $\mathcal{H}$ in $\mathbf{V}^{\mathcal{H}}$. By Facts and Lemmas

3.2.6, 4.2.7, 4.3.2, 4.4.2, and the work in the previous chapter there is a countable support iteration of forcings of length $\omega_2$ which is $\Omega(T)$-complete and extends $\mathbf{V}^{\mathcal{H}}$ to a model in which the following hold.

1. $T$ is club isomorphic to all of its everywhere Kurepa subtrees and has no Aronszajn subtree.
2. If $X, Y$ are two dense suborders of $K = (\mathcal{B}(T), <_{lex})$ and $|X| = |Y| = \aleph_1$ then $X$ embeds into $Y$ as a linear order.
3. If $X \subset K$ and $|\bar{X}| \leq \aleph_1$ then $X$ is $\sigma$-scattered.

Note that if $L \subset K, |L| = \aleph_1, |\bar{L}| = \aleph_2$, then there is $L_0 \subset L$ such that $\bar{L}_0$ is $\aleph_2$-dense. To see this, for $b, b' \in L$, let $b \sim b'$ if there are at most $\aleph_1$ many elements of $\bar{L}$ in between $b, b'$. It is obvious that there are at least two distinct equivalence classes and the set of equivalence classes is $\aleph_2$-dense. Here the equivalence classes are ordered by the order of their elements and since the equivalence classes are convex subsets of $L$ this order is well defied. Now let $L_0$ be a suborder that intersects each equivalence class at exactly one point. $\bar{L}_0$ is $\aleph_2$-dense since $\bar{L} \setminus \bar{L}_0 \subset \{\bar{L} \cap [b] : b \in L\}$, and $|\{\bar{L} \cap [b]_\sim : b \in L\}| \leq \aleph_1$.

Note that for such an $L_0$, the tree $\bigcup \bar{L}_0$ is an everywhere Kurepa subtree of $T$. So $L_0$ is isomorphic to an $\aleph_1$-sized dense suborder of $K$. This finishes the proof because all $\aleph_1$-sized dense suborders of $K$ are biembeddable.

We will finish with some remarks about the iteration of the forcings we used. The most important features of the forcings we used are $\Omega(T)$-completeness and $\aleph_2$-chain conditions. These forcings preserve the stationarity of stationary subsets of $\Omega(T)$, but they do not need to preserve the stationarity of stationary subsets of $\Gamma(T)$. In fact some of the iterands we considered, shoot clubs into the complement of some stationary subsets of $\Gamma(T)$. On the other hand the set $\Gamma(T)$ itself remains stationary in the final model we obtain. Our observation asserting $\Gamma(T)$ is stationary is based on 1.2.17 and the fact that $\omega_2$ is preserved. This is because any $\aleph_2$-sized linear order of density $\aleph_2$ is non-$\sigma$-scattered. The phenomenon that only preserving $\omega_2$ — without any control on countable structures which come from $\Gamma(T)$ — guarantees that $\Gamma(T)$ remains stationary seems to be new and mysterious. For instance, assume $S \subset \Gamma(T)$ is stationary and is not in the form of $\Omega$ or $\Gamma$ of any suborders of $K$. Is there any way to determine whether or not $S$ remains stationary in the extension under countable support iterations of these forcings?

## 4.5 A minimal Kurepa line

Now we are ready to answer Question 1.1.4. Here $K$ is the same as the previous section. For $L \subset K$ we let $C_L \subset \omega_2$ be a fast club in the sense of section 3.3.

**Definition 4.5.1.** *Assume $L \subset K$, $\bar{L} = K$, and $L$ is $\aleph_2$-dense. $Q_L$ is the poset consisting of all conditions $p = (g_p, A_p)$ such that*

1. *$g_p$ is a bijection from a countable subset of $\omega_2$ to $\omega_2$ which respects the club $C_L$,*
2. *$A_p$ is a countable antichain of $T$ such that $A_p \cap (\bigcup\{b_\xi : \xi \in image(g_p)\}) = \emptyset$, and*
3. *if $g_p = \emptyset$ then $A_p = \emptyset$.*

*We let $p \leq q$ if $g_p \supset g_q$ and $A_p \supset A_q$.*

Note that $Q_L$ is countably closed. Moreover, by a similar argument to the one given in the proof of Lemma 3.3.5, $Q_L$ satisfies $\Omega(K)$-cic for $\omega_2$. Therefore $Q_L$ preserves $\omega_1$ and $\omega_2$. Also note that for a generic $G \subset Q_L$,

$$\{b_\xi : \exists q \in G\ \xi \in image(g_q)\} \cup \bigcup \{b_\xi : \xi \in \omega_2, \exists p \in G\ b_\xi \cap A_p \neq \emptyset\} = K.$$

Therefore $Q_L$ adds an $\aleph_2$-dense closed subset of $L$. Now by including the forcings $Q_L$ in the iteration of length $\omega_2$ that is used in the previous section we reach a model in which

1. if $X, Y$ are suborders of $K$ with $|\bar{X}| = |\bar{Y}| = \aleph_2$ and $|X| = |Y| = \aleph_1$ then $X, Y$ are biembeddable, and
2. if $L \subset K$ is $\aleph_2$-dense then there is an $\aleph_1$-sized $X \subset L$ with $\bar{X} \subset L$.

Since every Kurepa suborder of $K$ contains an $\aleph_2$-dense suborder, $K$ embeds into all of it Kurepa suborders. Therefore we have established the following theorem.

**Theorem 4.5.2.** *It is consistent that there is a minimal Kurepa line.*

# BIBLIOGRAPHY


[1] U. Abraham and S. Shelah. Isomorphism types of Aronszajn trees. *Israel J. Math.*, 50(1-2):75–113, 1985.

[2] James Baumgartner. Applications of the Proper Forcing Axiom. In K. Kunen and J. Vaughan, editors, *Handbook of Set-Theoretic Topology*. North-Holland, 1984.

[3] ——— All $\aleph_1$-dense sets of reals can be isomorphic. *Fund. Math.*, 79(2):101–106, 1973.

[4] ——— A new class of order types. *Ann. Math. Logic*, 9(3):187–222, 1976.

[5] ——— Bases for Aronszajn trees. *Tsukuba J. Math.*, 9(1):31–40, 1985.

[6] Keith J. Devlin. *Aspects of constructibility.* Lecture Notes in Mathematics, Vol. 354. Springer-Verlag, Berlin-New York, 1973.

[7] Ben Dushnik and E. W. Miller. Concerning similarity transformations of linearly ordered sets. *Bull. Amer. Math. Soc.*, 46:322–326, 1940.

[8] Tetsuya Ishiu and Justin Tatch Moore. Minimality of non $\sigma$-scattered orders. *Fund. Math.*, 205(1):29–44, 2009.

[9] Borisa Kuzeljevic and Stevo Todorcevic. Forcing with matrices of countable elementary submodels. *Proc. Amer. Math. Soc.*, 145(5):2211–2222, 2017.

[10] Paul B. Larson. *The stationary tower*, volume 32 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004. Notes on a course by W. Hugh Woodin.

[11] Richard Laver. On Fraïssé's order type conjecture. *Ann. of Math. (2)*, 93:89–111, 1971.

[12] Carlos Martinez-Ranero. *Contributions towards a fine structure theory of Aronszajn orderings*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–University of Toronto (Canada).

[13] ——— Well-quasi-ordering Aronszajn lines. *Fund. Math.*, 213(3):197–211, 2011.

[14] Justin Tatch Moore. Set mapping reflection. *J. Math. Log.*, 5(1):87–97, 2005.

[15] ——— A five element basis for the uncountable linear orders. *Ann. of Math. (2)*, 163(2):669–688, 2006.

[16] ——— $\omega_1$ and $-\omega_1$ may be the only minimal uncountable order types. *Michigan Math. Journal*, 55(2):437–457, 2007.

[17] ——— A universal Aronszajn line. *Math. Res. Lett.*, 16(1):121–131, 2009.

[18] ——— Forcing axioms and the continuum hypothesis. Part II: transcending $\omega_1$-sequences of real numbers. *Acta Math.*, 210(1):173–183, 2013.

[19] Saharon Shelah. *Proper and Improper Forcing*. Springer-Verlag, Berlin, second edition, 1998.

[20] Stevo Todorcevic. Trees and linearly ordered sets. In *Handbook of set-theoretic topology*, pages 235–293. North-Holland, Amsterdam, 1984.

[21] ——— *Walks on ordinals and their characteristics*, volume 263 of *Progress in Mathematics*. Birkhäuser, 2007.

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