DERIVED CHARACTER MAPS OF LIE REPRESENTATIONS AND CHERN–SIMONS FORMS

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We study the derived representation scheme $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ parametrizing the representations of a Lie algebra $\mathfrak{a}$ in a reductive Lie algebra $\mathfrak{g}$. We define two canonical maps $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{HC}^{(r)}_{\bullet}(\mathfrak{a}) \to \mathrm{H}_{\bullet}[\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})]^G$ and $\Phi_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{H}_{\bullet}[\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})]^G \to \mathrm{H}_{\bullet}[\mathrm{DRep}_{\mathfrak{h}}(\mathfrak{a})]^{\mathbf{W}}$, called the Drinfeld trace and the derived Harish-Chandra homomorphism, respectively. The Drinfeld trace is defined on the r-th Hodge component of the cyclic homology of the universal enveloping algebra $\mathcal{U}\mathfrak{a}$ of the Lie algebra $\mathfrak{a}$ and depends on the choice of a G-invariant polynomial $P \in \mathrm{Sym}^r(\mathfrak{g}^*)^G$ on the Lie algebra $\mathfrak{g}$. The Harish Chandra homomorphism $\Phi_{\mathfrak{g}}(\mathfrak{a})$ is a graded algebra homomorphism extending to representation homology the natural restriction map $k[\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a})]^G \to k[\mathsf{Rep}_{\mathfrak{h}}(\mathfrak{a})]^{\mathbf{W}}$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathbf{W}$ is the associated Weyl group. We give general formulas for these maps in terms of Chern–Simons forms. As a consequence, we show that, if $\mathfrak{a}$ is an abelian Lie algebra, the composite map $\Phi_{\mathfrak{g}}(\mathfrak{a}) \circ \mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})$ is given by a canonical differential operator defined on differential forms on $A = \mathrm{Sym}(\mathfrak{a})$ and depending only on the Cartan data $(\mathfrak{h}, \mathbf{W}, P)$, where $P \in \mathrm{Sym}(\mathfrak{h}^*)^{\mathbf{W}}$. We derive a combinatorial formula for this operator that plays a key role in the study of derived commuting schemes in [4].

BIOGRAPHICAL SKETCH

Aliaksandr Patotski was born in Baranovichy, Belarus on September 13, 1990. Because of the very strange spelling of his name, he usually goes by Sasha. After graduating from Lyceum# 1 in 2007, he attended Belarussian State University, and was a student in the Department of Applied Mathematics and Computer Science. In 2012 he was accepted to Cornell University as a graduate student in mathematics. Starting Spring 2013 and until his graduation in 2018 he was a student of Yuri Berest.

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CHAPTER 1

INTRODUCTION

1 Background and motivation

1.1 Motivation

If $A = k[\Gamma]$ is the group algebra of a finite group Γ (or more generally, a semi-simple Artinian algebra) over a field k of characteristic 0, any n-dimensional representation $\rho\colon A \to \mathrm{Mat}_n(k)$ is determined, up to isomorphism, by its *character* $\chi_\rho\colon A \to k$, $\chi_\rho\colon a \mapsto \mathrm{tr}(\rho(a))$, where $\mathrm{tr}\colon \mathrm{Mat}_n(k) \to k$ is the usual trace on $n \times n$-matrices. For each fixed $n \geq 0$, there are only finitely many isomorphism classes of such representations, and we can view the characters as a single map $\mathrm{Tr} := \mathrm{Tr}_n(A)$ assigning to an element $a \in A$ a GL_n-invariant *function* $\mathrm{Tr}(a)$ on the set of n-dimensional representations of A, given by the formula $\mathrm{Tr}(a)(\rho) := \chi_\rho(a)$.

This picture has a natural geometric generalization to an arbitrary algebra A. Namely, the set of n-dimensional representations of A can be given the structure of an affine scheme called *the n-th representation scheme* and denoted by $\mathsf{Rep}_n(A)$. The group GL_n acts naturally on $\mathsf{Rep}_n(A)$ by conjugating the representations. Because the characters of representations are GL_n-invariant, the set of all characters can be assembled into a single linear map $\mathrm{Tr}\colon A \to \mathcal{O}[\mathsf{Rep}_n(A)]^{\mathrm{GL}}$ from A to the GL_n-invariant elements in the coordinate ring of the scheme $\mathsf{Rep}_n(A)$. We call $\mathrm{Tr} := \mathrm{Tr}_n(A)$ the *(classical) character map* of n-dimensional representations of A. Since $\mathrm{tr}[X, Y] = 0$ for any two matrices X, Y, the character map factors through commutators defining the map

$$\mathrm{Tr}\colon A/[A,A] \to \mathcal{O}[\mathsf{Rep}_n(A)]^{\mathrm{GL}} \tag{1.1}$$

The importance of this character map can be illustrated by a theorem of Procesi (see [37]), which says that the image of Tr generates the algebra $\mathcal{O}[\mathsf{Rep}_n(A)]^{\mathrm{GL}}$ of GL_n-invariant functions on $\mathsf{Rep}_n(A)$.

The construction of the scheme $\mathsf{Rep}_n(A)$ is natural in A, i.e. the assignment $A \mapsto \mathcal{O}[\mathsf{Rep}_n(A)]$ defines a functor $\mathcal{O}[\mathsf{Rep}_n(-)]\colon \mathsf{Alg}_k \to \mathsf{ComAlg}_k$ called the *representation functor*. The main object of this study is the *derived representation scheme* $\mathrm{DRep}_n(A)$ of n-dimensional representations of an algebra A. This scheme was constructed in [6] in an abstract way by extending the classical representation functor $\mathcal{O}[\mathsf{Rep}_n(-)]$ to the category of differential graded (DG) algebras, and deriving it in the sense of non-abelian homological algebra [39, 13]. The derived scheme $\mathrm{DRep}_n(A)$ is represented by a commutative DG algebra viewed as an object in the homotopy category of DG algebras; its homology $\mathrm{H}_\bullet[\mathrm{DRep}_n(A)]$ depends only on A (and n) and is called the n-dimensional *representation homology* of A and denoted by $\mathrm{HR}_\bullet(A,n)$. The action of GL_n on $\mathsf{Rep}_n(A)$ extends by functoriality to $\mathrm{HR}_\bullet(A,n)$. We define $\mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$ to be the GL_n-invariant part of $\mathrm{HR}_\bullet(A,n)$.

The derived scheme $\mathrm{DRep}_n(A)$ is indeed an extension of the classical representation scheme $\mathsf{Rep}_n(A)$ as the 0-th representation homology group $\mathrm{HR}_0(A,n) \simeq \mathcal{O}[\mathsf{Rep}_n(A)]$ is isomorphic to the coordinate ring of the scheme $\mathsf{Rep}_n(A)$.

Although the approach of [6] allows for explicit calculations, the homology groups $\mathrm{HR}_\bullet(A,n)$ are very hard to compute in practice, with few examples currently known. It is therefore natural to ask how to relate $\mathrm{HR}_\bullet(A,n)$ to other, more computable invariants of A.

One such invariant is the cyclic homology $\mathrm{HC}_\bullet(A)$ of the algebra A introduced by A. Connes in the context of index theory in noncommutative geometry, see [11] and Section 3 for details. Cyclic homology can be thought of as an additive analog of algebraic K-theory in the sense that it can be constructed from the Lie algebra $\mathfrak{gl}_\infty(A)$ in exactly the same way as the algebraic K-theory of A is made from the general linear group $\mathrm{GL}_\infty(A)$ (see [27] and [41]). In particular, the 0-th cyclic homology is given by the abelianization of $\mathfrak{gl}_\infty(A)$:

$$\mathrm{HC}_0(A) = \mathfrak{gl}_\infty(A)/[\mathfrak{gl}_\infty(A), \mathfrak{gl}_\infty(A)] \simeq A/[A, A]$$

which is the additive analog of the classical Bass definition of algebraic K_1:

$$\mathrm{K}_1(A) = \mathrm{GL}_\infty(A)/[\mathrm{GL}_\infty(A), \mathrm{GL}_\infty(A)]$$

In contrast to algebraic K-theory, however, the cyclic homology groups $\mathrm{HC}_\bullet(A)$ of an algebra A can be computed as the homology of a simple chain complex called the *Connes cyclic complex*:

$$A \xleftarrow{b} A^{\otimes 2}/(1-\tau) \xleftarrow{b} A^{\otimes 3}/(1-\tau) \xleftarrow{b} A^{\otimes 4}/(1-\tau) \longleftarrow \dots$$

where b is the classical Hochschild differential defined by

$$b(a_0, \dots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \dots, a_i a_{i+1}, \dots a_n) + (-1)^n (a_n a_0, a_1, \dots, a_{n-1})$$

and $\tau\colon A^{\otimes(n+1)} \to A^{\otimes(n+1)}$ is

$$\tau(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1}),$$

where we denote $(a_0, \dots, a_n) := a_0 \otimes \cdots \otimes a_n \in A^{\otimes n}$.

It turns out the classical character maps (1.1) can be extended naturally to the *derived character maps*

$$\mathrm{Tr}_n(A)_p\colon\ \mathrm{HC}_p(A) \to \mathrm{HR}_p(A, n)^{\mathrm{GL}},\ \forall p \geq 0\,, \tag{1.2}$$

defined in [6, Section 4] and relating the higher cyclic homology to representation homology of the algebra A. For notational brevity we re-denote $\mathrm{Tr}_\bullet := \mathrm{Tr}_n(A)_\bullet$. The 0-th character map

$$\mathrm{Tr}_0 := \mathrm{Tr}_n(A)_0 \colon\ \mathrm{HC}_0(A) \simeq A/[A,A] \to \mathrm{HR}_0(A,n)^{\mathrm{GL}} \simeq \mathcal{O}[\mathsf{Rep}_n(A)]^{\mathrm{GL}}$$

coincides with the classical character map Tr defined above.

There is a general (chain level) formula derived in [6] which computes Tr_n for any A and n. However, this formula is *not quite* explicit: it is given in terms of a certain A_∞-quasi-isomorphism, which is hard to construct in practice. One of the main goals of this thesis is to derive an *explicit* (homology level) formula for the derived character maps in the special case of universal enveloping algebras of arbitrary (DG) Lie algebras.

We illustrate these formulas in the simplest case when $n = 1$ and $A = \mathrm{Sym}(V)$ is the symmetric algebra of a d-dimensional k-vector space V. After certain natural identifications, one can show the character map $\mathrm{Tr}_\bullet$ is induced by a linear map

$$\mathrm{Tr}_\bullet \colon \Omega^\bullet(V) \to \mathbf{Sym}\left(V \oplus \Lambda^2 V \oplus \cdots \oplus \Lambda^d V\right) \tag{1.3}$$

where $\Omega^\bullet(V)$ is the space of algebraic (polynomial) differential forms on the vector space V, and the exterior power $\Lambda^i V$ on the R.H.S. has homological degree $i-1$. Let us denote by $\lambda(v_1, \dots, v_i)$ the element of $\mathbf{Sym}(V \oplus \cdots \oplus \Lambda^d V)$ of degree $i-1$ corresponding to $v_1 \wedge \cdots \wedge v_i \in \Lambda^i V$.

Let $d = 2$, and fix a basis $x, y \in V$. Then $A \simeq k[x,y]$ and the R.H.S. of (1.3) is isomorphic to $\mathbf{Sym}(x, y, \lambda)$ with $\lambda = \lambda(x,y)$ of degree 1. The only non-trivial character map is $\mathrm{Tr}_1 \colon \overline{\mathrm{HC}}_1(A) \to \mathrm{HR}_1(A,1)$ in homological degree 1, and it is given by the following formula:

$$\mathrm{Tr}_1(P dx + Q dy) = (P_y - Q_x)\lambda(x,y)$$

for $P, Q \in A = k[x, y]$, and P_y means partial derivative of P w.r.t. y, and similar for Q_x. In other words, Tr_1 is just given by the de Rham differential, combined with the (degree-shifting) map $dx\,dy \mapsto \lambda$. We will show that this is true for algebra $A = \mathrm{Sym}(V)$ for any finite-dimensional V: Tr_1 is always given by the de Rham differential, see Section 2.2.

When $d = 3$, apart from Tr_1, the only other non-trivial character map is Tr_2 in homological degree 2. Fix a basis $x, y, z \in V$. For a differential form $\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx \in \Omega^2(V)$, denote by $\mathrm{div}(\omega) := P_z + Q_x + R_y$ its divergence. The character map $\mathrm{Tr}_2 \colon \Omega^2(V) \to \mathrm{HR}_2(A, 1)$ turns out to be a *second order* linear differential operator on differential forms. Its value on ω is given by the following beautiful formula:

$$\begin{aligned}\mathrm{Tr}_2(\omega) = {} & \mathrm{div}(\omega)\,\lambda(x, y, z) + \\ & + \mathrm{div}(\omega)_x\ \lambda(x, z)\lambda(y, x) + \mathrm{div}(\omega)_y\ \lambda(y, x)\lambda(z, y) + \mathrm{div}(\omega)_z\ \lambda(z, y)\lambda(x, z)\ .\end{aligned}$$

Note that, as in homological degree 1, the first summand is just the image of the de Rham differential. But somewhat unexpectedly there also appear second order terms. We will give a general formula for $\mathrm{Tr}_p(\mathrm{Sym}\,V)$ for any V and any $p \geq 0$ in Section 2.

It is surprising that the derived character maps are given by *differential* operators on $\Omega^\bullet(V)$. This should be compared to classical trace functionals in differential geometry $\int \colon \Omega^n(M) \to \mathbf{R}$ given by *integrating* differential forms over submanifolds (or geometric cycles) of appropriate dimension (see [7]). The fact that the derived trace maps are represented by differential rather than integral operators still lacks a conceptual explanation. Nevertheless, we will see that our explicit formulas for Tr are given in terms of Chern–Simons forms which also play a fundamental role in differential geometry.

1.2 Lie representation homology and Drinfeld traces

The symmetric algebra $\mathrm{Sym}(V)$ can be thought of as the universal enveloping algebra $A = \mathcal{U}\mathfrak{a}$ of an *abelian* Lie algebra $\mathfrak{a} = V$. For an arbitrary Lie algebra $\mathfrak{a}$, in place of the derived representation scheme $\mathrm{DRep}_n(A)$ one can construct its Lie analog — the derived representation scheme $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ — and define the natural analogs of the derived character maps $\mathrm{Tr}_\bullet$.

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra defined over a field k of characteristic zero. For an arbitrary Lie algebra $\mathfrak{a}$, the set of all representations of $\mathfrak{a}$ in $\mathfrak{g}$ has a natural structure of an affine k-scheme called the representation scheme $\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a})$. Parallel to the case of associative algebras, in [4] we constructed a derived version of $\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a})$ by extending the representation functor $\mathsf{Rep}_{\mathfrak{g}}$ to the category of differential graded (DG) Lie algebras and taking its non-abelian derived functor. Following [4], the homology $\mathrm{H}_\bullet[\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})]$ of this derived scheme is called the *representation homology* of $\mathfrak{a}$ in $\mathfrak{g}$, and is denote by $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})$. The algebraic group G associated with $\mathfrak{g}$ acts naturally on $\mathsf{Rep}_{\mathfrak{g}}$ via the adjoint representation. This action extends to representation homology, and we define $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^G$ to be the G-invariant part of $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})$ (see Section 2 for details).

In [4], we defined two maps relating $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})$ to other invariants:

$$\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{HC}_\bullet^{(r)}(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^G \tag{1.4}$$

$$\Phi_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^G \to \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{h})^{\mathbf{W}} \tag{1.5}$$

which we called the *Drinfeld trace* and the *derived Harish Chandra homomorphism*, respectively. The Drinfeld trace is an analog of the character map (1.2), and is defined on the r-th Hodge component of the cyclic homology of the universal enveloping algebra $\mathcal{U}\mathfrak{a}$ of the Lie algebra $\mathfrak{a}$. Drinfeld trace depends on the choice of a

G-invariant polynomial $P \in \mathbf{Sym}^r(\mathfrak{g}^*)^G$ on the Lie algebra $\mathfrak{g}$: it should be thought of as a derived extension of the character map for Lie representations. The Harish-Chandra homomorphism $\Phi_{\mathfrak{g}}(\mathfrak{a})$ is a graded algebra homomorphism extending to representation homology the natural restriction map $k[\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a})]^G \to k[\mathsf{Rep}_{\mathfrak{h}}(\mathfrak{a})]^{\mathbf{W}}$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathbf{W}$ is the associated Weyl group.

The maps (1.4) and (1.5) are particularly interesting when $\mathfrak{a}$ is a two-dimensional abelian Lie algebra. In this case, $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ represents the derived commuting scheme of the Lie algebra $\mathfrak{g}$, a higher homological extension of the classical commuting scheme $\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a}) = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} \,:\, [x, y] = 0\}$ parametrizing the pairs of commuting elements in $\mathfrak{g}$. It turns out that, if $\mathfrak{a}$ is (homologically) graded with generators having opposite parities, then $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^G$ is a free graded commutative algebra generated by the (images of) Drinfeld traces (1.4) corresponding to free polynomial generators $\{P_1, \ldots, P_l\}$ of the invariant algebra $\mathbf{Sym}(\mathfrak{g}^*)^G$. (As shown in [4], this result is equivalent to the strong Macdonald conjecture proved in [17].) On the other hand, when both generators of $\mathfrak{a}$ are even (e.g., have homological degree 0), it is conjectured in [4] (with some evidence provided) that the Harish-Chandra homomorphism $\Phi_{\mathfrak{g}}(\mathfrak{a})$ is actually an algebra isomorphism.

Here, we study the maps (1.4) and (1.5) for arbitrary DG Lie algebras. We give a general formula for the Drinfeld trace in terms of Chern-Simons forms in a convolution DG algebra canonically attached to the pair $(\mathfrak{a}, \mathfrak{g})$ (see Theorem 1.2.1). Our construction is inspired by an idea of Beilinson [2] who suggested that Chern-Simons classes of canonical $\mathfrak{g}$-torsors on convolution algebras should give additive analogues of Borel regulator maps. As a consequence, we show that the composite map

$$\Phi_{\mathfrak{g}}(\mathfrak{a}) \circ \mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{HC}_\bullet^{(r)}(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^G \to \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{h})^{\mathbf{W}} \tag{1.6}$$

which we refer to as the *reduced Drinfeld trace* $\mathrm{Tr}_{\mathrm{H}}(\mathfrak{a})$, depends only on the Cartan data $(\mathrm{H}, \mathbf{W}, P)$ (provided $\mathbf{Sym}(\mathfrak{g}^*)^G$ is identified with $\mathbf{Sym}(\mathfrak{h}^*)^{\mathbf{W}}$ via the Chevalley isomorphism). If $\mathfrak{a}$ is an abelian Lie algebra (of any dimension), the cyclic homology groups $\mathrm{HC}_{\bullet}^{(r)}(\mathfrak{a})$ can be expressed in terms of (algebraic) differential forms on the vector space $\mathfrak{a}$, and the reduced Drinfeld trace $\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a})$ is given by a canonical $\mathbb{W}$-invariant differential operator defined on de Rham algebra of $\mathbf{Sym}(\mathfrak{a})$. We will give an explicit combinatorial formula for this operator, computing thus the higher character maps for all symmetric algebras (see Theorem 1.2.1 and Examples 2.4). We should mention that this formula plays a crucial role in [4], where it is used to verify the Harish Chandra quasi-isomorphism conjecture for the classical Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$ and $\mathfrak{sp}_{2n}$ in the stable limit $n \to \infty$; however, it is given in [4] without a proof.

1.3 References

The results of this thesis are published in the following papers.

[i] Representation homology, Lie algebra cohomology and derived Harish-Chandra homomorphism. *J. Eur. Math. Soc. (JEMS)*, 19(9): 2811—2893, 2017. (with Yu. Berest, G. Felder, A. C. Ramadoss and T. Willwacher).

[ii] Chern-Simons forms and higher character maps of Lie representations. *Int. Math. Res. Not. (IMRN)*, (1): 158—212, 2017. (with Yu. Berest, G. Felder, A. C. Ramadoss and T. Willwacher).

[iii] Character varieties as a tensor product. *Journal of Algebra*, to appear (with M. Kassabov).

In Section 2 of Chapter 3 we describe a construction of Lie representation schemes and character varieties as functor tensor product over a certain small category. The results of that section have been published in [iii]. In that paper we also discuss the case of group representation schemes, which is only outlined in Section 2.4.

Most of the results of Chapter 4 are published in [i], including the construction of Lie representation homology, its relation to Chevalley–Eilenberg homology of current Lie algebras, and the construction of the derived Harish-Chandra map and the Drinfeld trace maps. The only exception is Section 1 which describes derived representation schemes for associative algebras. The results of this section appeared in [6] and serve as the main motivation for the current work. The content of Sections 2, 3 and 4 is contained in Sections 6 and 7 of [i].

Finally, Chapter 5 appears in [ii] and very closely follow Sections 3–5 *loc. cit.* It contains the main results of this work, including Theorem 1.2.1 that relates Drinfeld traces to Chern–Simons forms and the explicit formulas of Section 2 for symmetric algebras $\mathbf{Sym}(W)$. The only new result in this section is the computation of reduced Drinfeld traces for the Lie algebras $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$, which has not been published before.

2 Overview of contents and main results

In Chapter 2 we briefly recall the necessary background and fix the notation. This Chapter has two unrelated parts: Section 2 reviews basic results from the theory of model categories needed for this work, and Section 3 is a brief introduction to cyclic homology. Specifically, Section 2 contains axiomatics of model categories, definitions of homotopy categories and derived functors, as well as examples of algebraic model categories we will be using throughout. This material is standard, and most of the results in this section may be found, for example, in the survey [13]. Section 3 collects the basics of cyclic homology, including its definition as a (non-abelian) derived functor (Section 3.2), as well as calculation for universal enveloping algebras. Section 3.3 reviews some well-known results on cyclic homology of commutative DG algebras, and Section 3.4 is a formal dualization of these results for cocommutative DG coalgebras. It is known that for an algebra A and its Koszul dual coalgebra C there is a natural isomorphism $\overline{\mathrm{HC}}_\bullet(A) \xrightarrow{\sim} \overline{\mathrm{HC}}_{\bullet+1}(C)$ induced by the HKR maps. The new result here is Theorem 3.5.5 relating this isomorphism to the isomorphism induced by a certain A_∞-quasi-isomorphism T described in [6, Theorem 4.2].

Chapter 3 deals with the classical representation and character varieties. Section 1 reviews main constructions and definitions for associative and Lie algebras. The results of Section 2, although mostly known, provide a new way of looking at representation varieties: as generalized tensor products (coends) of functors over a PROP of cocommutative Hopf algebras. The main theorem in this section is Theorem 2.4.1.

In Chapter 4 we define Lie representation homology, the derived Harish-Chandra homomorphism and the Drinfeld trace maps. We start Section 1 by recalling the construction of representation homology for associative algebras, since the case of Lie algebras is parallel to it, and we used it as a motivation. Section 2 describes Lie representation homology $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})$ of a Lie algebra $\mathfrak{a}$ into a Lie algebra $\mathfrak{g}$, see Theorem 2.2.2. Next, we describe the precise relation between Lie representation homology defined in Section 2.3 and the classical Chevalley–Eilenberg homology of Lie algebras of currents. This relation is given in terms of Koszul duality, see Theorem 3.2.2. Next we recall the construction of derived Harish–Chandra homomorphism $\Phi_{\mathfrak{g}}(\mathfrak{a})$. We only use it to define *reduced* Drinfeld traces so we omit a lot of details. We refer the reader to [4] for an in-depth discussion. Next, we define Drinfeld homology $\mathrm{H}^{(d)}_{\bullet}(\mathsf{Lie}, \mathfrak{a})$ of a Lie algebra $\mathfrak{a}$ (see Definition 4.2.2). We clarify its relation to the (reduced) cyclic homology of the universal enveloping algebra $\mathcal{U}\mathfrak{a}$ in Theorem 4.3.1, which is the main result of the section. Finally, Section 4.5 describes Drinfeld trace maps, which are Lie analogs of the derived character maps $\mathrm{Tr}_{\bullet}$ for associative algebras.

The main part of the thesis is Chapter 5. It defines *reduced* Drinfeld traces, which turn out to coincide with the character maps at the end of Section 1.1. The main result is Theorem 1.2.1 which gives an explicit formula for the reduced Drinfeld traces in terms of Chern–Simons forms on a certain convolution DG algebra (see Sections 1.1 and 1.2). We use this formula to give a formula for the trace maps of polynomial algebras $A = \mathrm{Sym}(W)$. We are able to link Lie and associative traces using the results of Section 1.3 and the fact that $\mathrm{Sym}(W) \simeq \mathcal{U}\mathfrak{a}$ for the abelian Lie algebra $\mathfrak{a} = W$. The formula allows to compute examples of reduced Drinfeld traces for $\mathfrak{a} = \mathfrak{sl}_2$ and $\mathfrak{a} = \mathfrak{sl}_3$, see Section 3. Finally, comparing the trace formula of Theorem 1.2.1 with the formula from [6, Theorem 4.2] describing derived character in terms of components of a certain A_∞-quasi-isomorphism, we obtain a combinatorial identity describing sums of these components over binary trees in terms of explicit Chern–Simons forms (see Corollary 4.2.3).

CHAPTER 2

PRELIMINARIES

1 Notation and conventions

We always work over a field k of characteristic 0. All tensor products are over k unless otherwise stated, and we will write $\otimes$ instead of $\otimes_k$. The same convention applies to the free product $*$.

The word "variety" in this text will always means "scheme," and more precisely an affine finitely generated scheme over k. We do not require it to be reduced, i.e. the coordinate ring can have nilpotents.

The word "algebra" usually means associative unital algebra over k. We will often think of algebras as graded, concentrated in homological degree 0. The same convention also applies to coalgebras and Lie algebras.

Unless otherwise stated, all differential graded (DG) algebras are assumed to be $\mathbf{Z}$-graded unital algebras over k and have *homological grading*, i.e. the differential d is of degree $|d| = -1$. We use the word "commutative" to mean "graded commutative," i.e. $ab = (-1)^{|a||b|}ba$ for any two homogeneous elements a, b. This is an instance of a more general Koszul sign rule which we adopt throughout this text. The rule says that whenever two symbols are transposed, a sign appears corresponding to the symbols' degrees. For any DG object R, i.e. algebra, coalgebra or a Lie algebra, we denote by $R_\#$ the underlying graded object (i.e. we "forget" the differential).

For a graded vector space V, the shifted space $V[1]$ is a graded vector space with $V[1]_n = V_{n-1}$, and for $v \in V$ the same element v but in $V[1]$ is denoted by sv. Therefore, $|sv| = |v| + 1$ for a homogeneous v. If V is a complex with the differential d, the shifted complex $V[1]$ has the differential $d[1]$ with $d[1]_n = -d_{n-1}$.

The free graded (associative) algebra on a graded vector space V is denoted TV. The functor **Sym** that takes a graded vector space V to the free commutative algebra $\mathbf{Sym}(V)$ is always assumed to be graded, and the algebra is graded commutative. We will also use functors Sym and Λ. If V is a graded vector space, $\mathrm{Sym}(V)$ and $\Lambda(V)$ denote the usual (ungraded) symmetric and alternating powers, respectively. In particular, if V is concentrated in degree 0, there is an isomorphism of graded vector spaces $\mathbf{Sym}^n(V[1]) \simeq \Lambda^n(V)[n]$.

For an algebra with an augmentation $\varepsilon\colon A \to k$ or a coalgebra C with a counit $\varepsilon\colon C \to k$ we denote by $\overline{A}$ and $\overline{C}$ the subspaces $\mathrm{Ker}(\varepsilon)$. Moreover, if $\eta\colon k \to A$ is a unit or $\eta\colon k \to C$ is a coaugmentation, we will use the same notation $\overline{A}$ and $\overline{C}$ to denote $\mathrm{Coker}(\eta)$.

2 Model categories

The goal of this section is mostly to fix the notation, and not to give an introduction to the subject of model categories and non-abelian derived functors. See the paper [13] for a well-written detailed introduction to model categories. There is a well-written concise exposition in [6, Appendix A].

2.1 Definition of model categories

A *(closed) model category* C is a category with three distinguished classes of morphisms, denoted $\mathrm{Fib}, \mathrm{Cof}, \mathrm{WE}$ and called the *fibrations, cofibrations and weak equivalences* respectively. We will usually denote the fibrations by $\twoheadrightarrow$, the cofibrations by $\hookrightarrow$ and the weak equivalences by $\xrightarrow{\sim}$. The fibrations that are at the same time weak equivalences are called *acyclic fibrations* and will be denoted by $\overset{\sim}{\twoheadrightarrow}$. Similarly, the cofibrations which are weak equivalences will be called *acyclic cofibrations*, and will be denoted by $\overset{\sim}{\hookrightarrow}$. We usually denote model categories omitting $\mathrm{Fib}, \mathrm{Cof}, \mathrm{WE}$ from the notation, but they are always understood.

The classes $\mathrm{Fib}, \mathrm{Cof}, \mathrm{WE}$ are closed under compositions, contain all the isomorphisms, and moreover the following five axioms are satisfied.

MC1 The category C has all finite limits and colimits. In particular, C has both initial and terminal objects, denoted by e and $*$ respectively.

MC2 If $f\colon X \to Y$ and $g\colon Y \to Z$ are two composable morphisms in C, then if two out of the three morphisms $f, g, g \circ f$ are in WE, then so is the third morphism. This axiom is called "*2-out-of-3 axiom.*"

MC3 All three classes $\mathrm{WE}, \mathrm{Fib}, \mathrm{Cof}$ are closed under taking retracts. Recall that $f\colon X \to Y$ is called a *retract* of $g\colon X' \to Y'$ if there is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

in which the rows compose to the identity morphisms id_X and id_Y, respectively.

MC4 Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

with $f \in$ Cof and $g \in$ Fib. If one of the maps f or g is in addition a *weak equivalence*, then there exists a lifting $h\colon B \to X$ making the diagram commutative.

MC5 Every morphism $f\colon A \to X$ in C can be factored as a composition $A \overset{\sim}{\hookrightarrow} B \twoheadrightarrow X$ of an acyclic cofibraion followed by a fibration, and as a composition $A \hookrightarrow Y \overset{\sim}{\twoheadrightarrow} X$ of a cofibration followed by an acyclic fibration.

Whenever a lifting in Axiom MC4 exists, we say that f has a *left lifting property* (LLP) with respect to g, and g has a *right lifting property* (RLP) with respect to f. We denote the class of all morphisms that have LLP w.r.t. the acyclic fibrations by LLP(Fib $\cap$ WE). Similarly, the class of morphisms having RLP w.r.t. the acyclic cofibrations will be denoted by RLP(Cof $\cap$ WE). Then Axiom MC4 says that Cof $\subseteq$ LLP(Fib $\cap$ WE) and Fib $\subseteq$ RLP(Cof $\cap$ WE).

Proposition 2.1.1. *Let C be a model category.*

1. *Cofibrations are the morphisms which have the LLP w.r.t. acyclic fibrations.*
2. *Fibrations are the morphisms which have the RLP w.r.t. acyclic cofibrations.*
3. *Acyclic cofibrations are the morphisms which have the LLP w.r.t. fibrations.*
4. *Acyclic fibrations are the morphisms which have the RLP w.r.t. cofibrations.*

In particular, this proposition implies that once WE and one of Cof, Fib are defined, the third class is uniquely determined by the other two. We will use this implication a lot when giving examples of model categories, see Section 2.3 below.

Definition 2.1.2. *We say that an object $A \in \mathrm{Ob}(\mathsf{C})$ is* cofibrant *if the unique morphism $e \to A$ is a cofibration. Similarly, we say that A is* fibrant *if the morphism $A \to *$ is a fibration. We say that a model category C is* cofibrant *(resp.* fibrant*) if every object $A \in \mathrm{Ob}(\mathsf{C})$ is cofibrant (resp. fibrant).*

For a pair of objects $R, A \in \mathrm{Ob}(\mathsf{C})$ with R cofibrant, we say that R is a *cofibrant replacement*, or a *cofibrant resolution* if there exists an acyclic fibration $R \overset{\sim}{\twoheadrightarrow} A$. Axiom MC5 ensures that for any object $A \in \mathrm{Ob}(\mathsf{C})$ there exists a cofibrant replacement $R \overset{\sim}{\twoheadrightarrow} A$ in C. We define fibrant replacements in an analogous way.

Definition 2.1.3. *For a model category C, define its* homotopy category $\mathsf{Ho}(\mathsf{C})$ *to be the localization $\mathsf{C}[\mathrm{WE}^{-1}]$ of C along the class of weak equivalences, with the localization functor $\gamma\colon \mathsf{C} \to \mathsf{Ho}(\mathsf{C})$.*

This is not the standard definition, and definitely not the most useful one. However, it will suffice to make the exposition coherent. For a detailed construction of the homotopy category see [13, Section 5].

2.2 Derived functors and Quillen pairs

If C, D are model categories, a functor $F\colon \mathsf{C} \to \mathsf{D}$ is called *homotopy invariant* (or *homotopical*) if $F(\mathrm{WE}_{\mathsf{C}}) \subseteq \mathrm{WE}_{\mathsf{D}}$. In this case, it extends (or descends) to a functor $\mathsf{Ho}(F)$ between the homotopy categories

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{C}}"'] & \mathsf{D} \arrow[d, "\gamma_{\mathsf{D}}"] \\ \mathsf{Ho}(\mathsf{C}) \arrow[r, "\mathsf{Ho}(F)"] & \mathsf{Ho}(\mathsf{D}) \end{tikzcd}$$

This rarely happens in practice: interesting functors are usually *not* homotopical. The idea is that we should define derived functors

$$\mathbf{L}F, \mathbf{R}F \colon \mathsf{Ho}(\mathsf{C}) \to \mathsf{Ho}(\mathsf{D})$$

which give the "best possible" homotopical approximation to $\mathsf{Ho}(F)$.

Definition 2.2.1. *Let C be a model category, $F \colon \mathsf{C} \to \mathsf{D}$ a functor from C to any category D. A* left derived functor $\mathbf{L}F$ *of the functor F is a* right *Kan extension of F along γ_{C}, $\mathbf{L}F := \mathrm{Ran}_{\gamma_{\mathsf{C}}}(F)$.*

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "F"] \arrow[dr, "\gamma_{\mathsf{C}}"'] & \Uparrow & \mathsf{D} \\ & \mathsf{Ho}(\mathsf{C}) \arrow[ur, dashed, "\mathbf{L}F"'] & \end{tikzcd}$$

In other words, the left derived functor of a functor F is a functor $\mathbf{L}F \colon \mathsf{Ho}(\mathsf{C}) \to \mathsf{D}$ equipped with a natural transformation

$$t \colon \mathbf{L}F \circ \gamma_{\mathsf{C}} \Rightarrow F$$

which is universal among all pairs $(G \colon \mathsf{Ho}(\mathsf{C}) \to \mathsf{D}, s \colon G \circ \gamma_{\mathsf{C}} \Rightarrow F)$. In other words, for any such pair there exists a unique natural transformation $\overline{s} \colon G \Rightarrow \mathbf{L}F$ making the following diagram commute:

$$\begin{tikzcd} G \circ \gamma_{\mathsf{C}} \arrow[rr, "s"] \arrow[dr, "\overline{s} \circ \gamma_{\mathsf{C}}"'] & & F \\ & \mathbf{L}F \circ \gamma_{\mathsf{C}} \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Similarly one defines *right derived functor* of a functor F to be a *left* Kan extension $\mathbf{R}F = \mathrm{Lan}_{\gamma_{\mathsf{C}}}(F)$ of F along γ_{C}.

Definition 2.2.2. *If $F \colon \mathsf{C} \to \mathsf{D}$ is a functor between two model categories, then its* (total) left derived fucntor $\mathbf{L}F$ *is the left derived functor of the composition $\gamma_{\mathsf{D}} \circ F \colon \mathsf{C} \to \mathsf{Ho}(\mathsf{D})$, i.e. $\mathbf{L}F = \mathrm{Ran}_{\gamma_{\mathsf{C}}}(\gamma_{\mathsf{D}} \circ F)$. Similarly,* (total) right derived functor *of F is the right derived functor of the composition $\gamma_{\mathsf{D}} \circ F$.*

Remark 2.2.3. Just like with Kan extensions in general, the derived functor of a functor F may or may not exist. Theorem 2.2.5 gives one condition that guarantees the existence of the derived functor.

Definition 2.2.4. *Let $F\colon \mathsf{C} \leftrightarrows \mathsf{D}\colon G$ be a pair of adjoint functors between two model categories. We say (F, G) is a* Quillen pair *if F preserves weak equivalences between cofibrant objects, and G preserves weak equivalences between fibrant objects. We call F a* left Quillen *and G a* right Quillen functor *respectively.*

Theorem 2.2.5 (Quillen's Adjunction Theorem)**.** *If (F, G) is a Quillen pair, then $\mathbf{L}F$ and $\mathbf{R}G$ exist and form an adjoint pair*

$$\mathbf{L}F\colon \mathsf{Ho}(\mathsf{C}) \leftrightarrows \mathsf{Ho}(\mathsf{D}) : \mathbf{R}G$$

The functors $\mathbf{L}F$ and $\mathbf{R}G$ are given by the formulas

$$\mathbf{L}F(A) = \gamma_{\mathsf{D}} F(R)$$
$$\mathbf{R}G(X) = \gamma_{\mathsf{C}} G(Q)$$

where $p_A\colon R \overset{\sim}{\twoheadrightarrow} A$ is any cofibrant replacement of A in C and $i_X\colon X \overset{\sim}{\hookrightarrow} Q$ is any fibrant replacement of X in D.

Remark 2.2.6. The condition for a functor F (resp., G) to be a left Quillen (resp., right Quillen) functor is a sufficient, but not a *necessary* condition for $\mathbf{L}F$ (resp. $\mathbf{R}G$) to exist. In fact, there are many interesting functors, which are *not* Quillen but still have derived functors (see Section 3.2 and in particular Remark 3.2.3).

Lemma 2.2.7. *Let $F, G\colon \mathsf{C} \to \mathsf{D}$ be two functors between model categories s.t. $\mathbf{L}F$ and $\mathbf{L}G\colon \mathsf{Ho}(\mathsf{C}) \to \mathsf{Ho}(\mathsf{D})$ exist. Then, any natural transformation $\varphi\colon F \Rightarrow G$ induces a (unique) natural transformation $\mathbf{L}\varphi\colon \mathbf{L}F \Rightarrow \mathbf{L}G$.*

Theorem 2.2.8 (Quillen's Equivalence Theorem)**.** *Let* $F\colon \mathsf{C} \leftrightarrows \mathsf{D}\colon G$ *be a Quillen pair. Denote by* C_c *the full subcategory of cofibrant objects in* C*, and by* D_f *the full subcategory of fibrant objects in* D*. Assume that*

(∗) *for every* $A \in \mathrm{Ob}(\mathsf{C}_c)$ *and* $X \in \mathrm{Ob}(\mathsf{D}_f)$*,*

$$(f\colon A \to G(X)) \in \mathrm{WE}_{\mathsf{C}} \Leftrightarrow (f^*\colon F(A) \to X) \in \mathrm{WE}_{\mathsf{D}}$$

Then $\mathbf{L}F\colon \mathsf{Ho}(\mathsf{C}) \simeq \mathsf{Ho}(\mathsf{D})\colon \mathbf{R}G$ *are mutually inverse equivalences of categories.*

2.3 Some examples

Complexes

Let A be an algebra, and let $\mathsf{C} = \mathsf{Com}^+(A)$ be the category of *non-negatively graded* chain complexes of A-modules. Let WE be the class of all quasi-isomorphisms, i.e. morphisms $f_\bullet\colon X_\bullet \to Y_\bullet$ of complexes inducing isomorphism in homology.

Let Fib be the class if maps $f_\bullet\colon X_\bullet \to Y_\bullet$ such that $f_n\colon X_n \to B_n$ is a surjective map of A-modules, in every strictly positive degree $n > 0$. We put $\mathrm{Cof} = \mathrm{LLP}(\mathrm{Fib} \cap \mathrm{WE})$.

Theorem 2.3.1. *The class* Cof *consists of all maps* $f_\bullet\colon X_\bullet \to Y_\bullet$ *of complexes such that* $f_n\colon X_n \to Y_n$ *is* injective *and* $\mathrm{Coker}(f_n)$ *is a projective* A*-module, for all* $n \geqslant 0$*.*

Corollary 2.3.2. *The category* $\mathsf{Com}^+(A)$ *with the model structure described above is a fibrant model category. Cofibrant objects are precisely the complexes* $X_\bullet$ *with all components* X_n *being projective* A*-modules.*

This model structure is called *projective* model structure on $\mathsf{Com}^+(A)$. There is a dual model structure, called *injective*, defined analogously. We will only consider the projective model structure.

The category $\mathsf{Com}(A)$ of all chain complexes, not necessarily non-negatively graded, also admits a projective model structure. Weak equivalences are quasi-isomorphisms and fibrations are degree-wise surjections. By Proposition 2.1.1, this will completely determine the class of cofibrations. It is true that every cofibration is a degree-wise injection with projective cokernel, but not all such maps are cofibrations, see [21].

Remark 2.3.3. Weak equivalences in both $\mathsf{Com}(A)$ and $\mathsf{Com}^+(A)$ are given by quasi-isomorphisms. Therefore, there are natural equivalences of categories $\mathsf{Ho}(\mathsf{Com}(A)) \simeq \mathcal{D}(A)$ and $\mathsf{Ho}(\mathsf{Com}^+(A)) \simeq \mathcal{D}^+(A)$, where $\mathcal{D}(A)$ and $\mathcal{D}^+(A)$ denotes the usual derived and bounded derived category, respectively.

Over and under categories

Let $d \in \mathrm{Ob}(\mathsf{C})$ be an object of a category C.

Definition 2.3.4. *The* over category C/d *is a category, whose objects are all arrows* $f\colon c \to d$ *in* C*, for all* $c \in \mathsf{C}$*. The set of morphisms between two arrows* $f\colon c \to d$ *and* $f'\colon c' \to d$ *are all arrows* $g\colon c \to c'$ *such that* $f' \circ g = f$*.*

Similarly, the under category $d\backslash\mathsf{C}$ *is defined as the category of arrows* $f\colon d \to c$*, with morphisms between* $f\colon d \to c$ *and* $f'\colon d \to c'$ *being all arrows* $g\colon c \to c'$ *such that* $g \circ f = f'$*.*

Now suppose the category C has a model structure. Then both categories $d\backslash\mathsf{C}$

and C/d have a natural model structure. Namely, a morphism

$$\begin{tikzcd} & d \arrow[dl] \arrow[dr] & \\ A \arrow[rr] & & B \end{tikzcd}$$

in $d\backslash\mathsf{C}$ is in $\mathrm{WE}, \mathrm{Fib}, \mathrm{Cof}$ if and only if $A \to B$ is in $\mathrm{WE}, \mathrm{Fib}, \mathrm{Cof}$ in C, respectively. Similarly for the category C/d.

Associative and commutative DG algebras

Let DGA_k denote the category of associative unital DG algebras over k, and let DGCA_k be its full subcategory of graded commutative algebras.

Theorem 2.3.5. *The categories* DGA_k *and* DGCA_k *have model structures in which*

1. *the weak equivalences are the quasi-isomorphisms,*
2. *the fibrations are the maps which are surjective in all degrees,*
3. *the cofibrations are the morphisms having LLP with respect to acyclic fibrations.*

Both categories DGA_k *and* DGCA_k *are fibrant, with the initial object* k *and terminal* 0.

We will also consider the categories $\mathsf{DGA}_{k/k}$ and $\mathsf{DGCA}_{k/k}$ of *augmented* DG algebras, i.e. morphisms of algebras $A \to k$ to the ground field k. These are a particular case of over categories, and so they naturally have a model structure induced by the model structure on DGA_k, see Section 2.3.

Sometimes it is convenient to work with the categories DGA_k^+ and DGCA_k^+ of *non-negatively* graded DG algebras. For these categories, there is a more explicit description of cofibrations that we will often use. First we need to introduce a notion of an *almost free* extension.

A DG algebra R is called *almost free* if the underlying graded algebra $R_\#$ is a free graded algebra. In other words, R is almost free if $R \simeq TV$ is isomorphic to the tensor algebra of a graded vector space. More generally, we say a map of DG algebras $f\colon A \to B$ in DGA_k^+ is an *almost free extension* if there is an isomorphism $B_\# \simeq A_\# * TV$ of graded algebras such that the composition of $f_\#$ with this isomorphism is just the inclusion $A_\# \hookrightarrow A_\# * TV$.

Similarly, a commutative DG algebra $S \in \mathsf{DGCA}_k^+$ is called *almost free* if $S_\# \simeq \mathbf{Sym}(V)$ for a graded vector space V. A morphism $f\colon A \to B$ in DGCA_k^+ is an *almost free extension* if $f_\#$ is isomorphic to the inclusion $A_\# \hookrightarrow A_\# \otimes \mathbf{Sym}(V)$.

Theorem 2.3.6 ([34, 8])**.** *The categories DGA_k^+ and DGCA_k^+ have model structures in which*

1. *the weak equivalences are the quasi-isomorphisms,*
2. *the fibrations are the maps which are surjective in all degrees,*
3. *the cofibrations are the retracts of almost free extensions (cf. axiom MC3).*

Both categories DGA_k^+ and DGCA_k^+ are fibrant, with the initial object k and terminal 0.

Proposition 2.3.7. *The inclusion functor $\mathsf{DGA}_k^+ \hookrightarrow \mathsf{DGA}_k$ preserves the cofibrations and the weak equivalences. In particular, it descends to the corresponding homotopy categories.*

This proposition allows us to use almost free resolutions of non-negatively graded algebras as cofibrant resolutions in both DGA_k^+ and DGA_k.

DG Lie algebras and cocommutative coalgebras

In this section we will describe model structures on the categories of DG Lie algebras DGLA_k, coaugmented cocommutative DG coalgebras $\mathsf{DGCC}_{k/k}$, augmented commutative DG algebras $\mathsf{DGCA}_{k/k}$ and DG Lie coalgebas DGLC_k. Before we proceed we need to review the bar/cobar formalism in the Lie setting.

There is a pair of adjoint functors

$$\mathbf{\Omega}_{\mathsf{Comm}} \,:\, \mathsf{DGCC}_{k/k} \rightleftarrows \mathsf{DGLA}_k \,:\, \mathbf{B}_{\mathsf{Lie}}\ , \tag{2.1}$$

where $\mathbf{B}_{\mathsf{Lie}}$ is defined by the classical Chevalley-Eilenberg complex of a DG Lie algebra. Namely, to a DG Lie algebra $\mathfrak{a}$ it assigns the cocommutative DG coalgebra $\mathbf{B}_{\mathsf{Lie}}(\mathfrak{a}) = (\mathbf{Sym}^c(\mathfrak{a}[1]), d_1 + d_2)$ where d_1 is induced by the intrinsic differential δ on $\mathfrak{a}$ and d_2 is induced by the shifted Lie bracket of $\mathfrak{a}$:

$$\mathbf{Sym}^2(\mathfrak{a}[1]) \simeq \Lambda^2(\mathfrak{a}) \otimes k[1] \otimes k[1] \xrightarrow{[\,,]\otimes m[-1]} \mathfrak{a} \otimes k[1] \simeq \mathfrak{a}[1]$$

The functor $\mathbf{\Omega}_{\mathsf{Comm}}$ assigns to a cocommutative coalgebra C the free graded Lie algebra $\mathfrak{L}(\overline{C}[-1])$ on the vector space $\overline{C}[-1]$. The differential on $\mathbf{\Omega}_{\mathsf{Comm}}(C)$ is given by $d_1 + d_2$, where d_1 is induced by the inner differential on C and d_2 is the lift of the linear map

$$k[-1] \otimes \overline{C} \xrightarrow{\Delta_{-1} \otimes \Delta} k[-1] \otimes k[-1] \otimes \mathbf{Sym}^2(\bar{C}) \;\cong\; \Lambda^2(\bar{C}[-1])$$

where $\Delta_{-1} \,:\, k[-1] \to k[-1] \otimes k[-1]$ takes $1_{k[-1]}$ to $-1_{k[-1]} \otimes 1_{k[-1]}$ and Δ is the coproduct on $\overline{C}$.

Dually, there is a pair of adjoint functors

$$\mathbf{\Omega}_{\mathsf{Lie}} : \mathsf{DGLC}_k \rightleftarrows \mathsf{DGCA}_{k/k} : \mathbf{B}_{\mathsf{Comm}}\ , \tag{2.2}$$

where $\mathbf{\Omega}_{\mathsf{Lie}}$ is defined by the Chevalley-Eilenberg complex of a DG Lie coalgebra. Namely, if $\mathfrak{C}$ is a DG Lie coalgebra with Lie cobracket $]-[\,:\ \mathfrak{C} \to \wedge^2\mathfrak{C}$, the Chevalley-Eilenberg complex of $\mathfrak{C}$ is defined to be the (augmented) commutative DG algebra

$$\mathbf{C}^c(\mathfrak{C}; k) \;:=\; (\mathbf{Sym}(\mathfrak{C}[-1]),\, d_1 + d_2)\ ,$$

where d_1 is induced by the differential on $\mathfrak{C}$ and $d_2|_{\mathfrak{C}[-1]}$ is given by the composite map

$$\mathfrak{C}[-1] \cong k[-1] \otimes \mathfrak{C} \xrightarrow{\Delta_{-1}\otimes]-[} k[-1] \otimes k[-1] \otimes \Lambda^2\mathfrak{C} \;\cong\; \mathbf{Sym}^2(\mathfrak{C}[-1])\,.$$

Here, $\Delta_{-1}\,:\,k[-1] \to k[-1] \otimes k[-1]$ takes $1_{k[-1]}$ to $-1_{k[-1]} \otimes 1_{k[-1]}$.

The functor $\mathbf{B}_{\mathsf{Comm}}$ takes $R \in \mathsf{DGCA}_{k/k}$ to the cofree DG Lie coalgebra $\mathfrak{L}^c(\overline{R}[1])$ equipped with (co)differential $d_1 + d_2$ where d_1 is induced by the differential on R and d_2 is determined by the linear map

$$\Lambda^2(\overline{R}[1]) \;\cong\; k[1] \otimes k[1] \otimes \mathbf{Sym}^2(\overline{R}) \xrightarrow{\mu_1\otimes\mu} k[1] \otimes \overline{R} \;\cong\; \overline{R}[1]$$

(Here, μ_1 identifies $1_{k[1]} \otimes 1_{k[1]}$ with $1_{k[1]}$ and $\mu : \mathbf{Sym}^2(\overline{R}) \to \overline{R}$ is induced by the multiplication map on R).

Notation. If there is no danger of confusion, we will use the notation $\mathrm{CE} := \mathbf{B}_{\mathtt{Lie}}$ and $\mathrm{CE}^c := \mathbf{\Omega}_{\mathtt{Lie}}$ for the Chevalley-Eilenberg functors on Lie algebras and Lie coalgebras, respectively.

The following theorem collects basic facts about the model structures and Quillen equivalences for Lie (co)algebras: part (i) is well known (essentially, due to Quillen [38]); for part (ii) and (iii), see, for example, [20, Theorems 3.1 and 3.2] and [40, Corollary 4.15].

Theorem 2.3.8. (i) *The categories $\mathsf{DGCA}_{k/k}$ and DGLA_k have model structures where the weak-equivalences are the quasi-isomorphisms, and the fibrations are the degreewise surjective maps.*

(ii) *The category* DGLC_k *(resp.,* $\mathsf{DGCC}_{k/k}$*) admits a model structure, where the weak equivalences are the maps* f *such that* $\mathbf{\Omega}_{\mathsf{Lie}}(f)$ *(resp.,* $\mathbf{\Omega}_{\mathsf{Comm}}(f)$*) is a quasi-isomorphism, and the cofibrations are degreewise monomorphisms.*

(iii) *For the above model structures, the pairs of functors* (2.1) *and* (2.2) *are Quillen equivalences.*

Note that part (iii) says that the functors (2.1) and (2.2) induce derived equivalences

$$\mathbf{L}\,\mathbf{\Omega}_{\mathsf{Comm}} \,:\, \mathsf{Ho}(\mathsf{DGCC}_{k/k}) \rightleftarrows \mathsf{Ho}(\mathsf{DGLA}_k) \,:\, \mathbf{R}\mathbf{B}_{\mathsf{Lie}}\,,$$

$$\mathbf{L}\,\mathbf{\Omega}_{\mathsf{Lie}} \,:\, \mathsf{Ho}(\mathsf{DGLC}_k) \rightleftarrows \mathsf{Ho}(\mathsf{DGCA}_{k/k}) \,:\, \mathbf{R}\mathbf{B}_{\mathsf{Comm}}\,.$$

3 Cyclic homology

3.1 Basics of cyclic homology

All results of this section are well-known, and we refer the reader to the wonderful book [26] for a very detailed exposition.

Let A be an associative unital algebra over a field k of $\operatorname{char} k = 0$. Then the cyclic homology groups $\mathrm{HC}_\bullet(A)$ are by definition the homology groups of the total complex $\mathrm{Tot}[\mathrm{CC}(A)]$ of the bicomplex $\mathrm{CC}(A)$:

$$\begin{tikzcd}
 & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] & \vdots \arrow[d, "b"'] & \vdots \arrow[d, "-b'"'] & \vdots \arrow[d, "b"'] & \\
\cdots & A^{\otimes 3} \arrow[l] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-\tau"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 3} \arrow[l, "1-\tau"'] \arrow[d, "-b'"'] & A^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
\cdots & A^{\otimes 2} \arrow[l] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-\tau"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-\tau"'] \arrow[d, "-b'"'] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"'] & \cdots \arrow[l] \\
\cdots & A \arrow[l] \arrow[d] & A \arrow[l, "1-\tau"'] \arrow[d] & A \arrow[l, "N"'] \arrow[d] & A \arrow[l, "1-\tau"'] \arrow[d] & A \arrow[l, "N"'] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

The map b is the Hochschild differential given by

$$b(a_0,\ldots,a_n)=\sum_{i=0}^{n-1}(-1)^i a_0,\ldots,a_ia_{i+1},\ldots a_n+(-1)^n a_na_0,a_1,\ldots,a_{n-1}$$

and b' is the reduced Hochschild differential

$$b'(a_0,\ldots,a_n)=\sum_{i=0}^{n-1}(-1)^i a_0,\ldots,a_ia_{i+1},\ldots a_n.$$

The maps τ, N are defined by

$$\tau(a_0\otimes\cdots\otimes a_n)=(-1)^n a_n\otimes a_0\otimes\cdots\otimes a_{n-1}$$

and $N=1+\tau+\cdots+\tau^n$ on $A^{\otimes n+1}$.

Let $\mathrm{C}^\lambda(A)$ be the *Connes complex*

$$A \xleftarrow{b} A^{\otimes 2}/(1-\tau) \xleftarrow{b} A^{\otimes 3}/(1-\tau) \xleftarrow{b} A^{\otimes 4}/(1-\tau) \longleftarrow \cdots$$

Projection onto the first column $\mathrm{CC}(A)\twoheadrightarrow \mathrm{CC}(A)_{0\bullet}$ induces a natural surjective map of complexes $\mathrm{Tot}[\mathrm{CC}(A)]\twoheadrightarrow \mathrm{C}^\lambda(A)$.

Theorem 3.1.1 ([26], Theorem 2.1.5)**.** *The map* $\mathrm{Tot}[\mathrm{CC}(A)] \overset{\sim}{\twoheadrightarrow} \mathrm{CC}^\lambda(A)$ *is a quasi-isomorphism, and therefore* $\mathrm{HC}_\bullet(A)\simeq \mathrm{H}_\bullet[\mathrm{C}^\lambda(A)]$.

Definition 3.1.2. *A* mixed complex (M, b, B) *is a non-negatively graded vector space M equipped with a degree -1 endomorphism b and a degree $+1$ endomorphism B satisfying*

$$b^2 = 0 \qquad B^2 = 0 \qquad [b, B] = bB + Bb = 0$$

Let (M, b, B) be a mixed complex. The bicomplex $\mathcal{B}(M)$ *associated* to M is the bicomplex given by:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
M_2 \arrow[d, "b"'] & M_1 \arrow[l, "B"] \arrow[d, "b"'] & M_0 \arrow[l, "B"] \\
M_1 \arrow[d, "b"'] & M_0 \arrow[l, "B"] & \\
M_0 & &
\end{tikzcd}$$

In other words, it is a bicomplex with $\mathcal{B}(M)_{pq} = \begin{cases} M_{q-p}\,, & p \geq 0 \\ 0\,, & p < 0 \end{cases}$ with the horizontal differential $\mathcal{B}(M)_{pq} \to \mathcal{B}(M)_{p-1,q}$ given by B, and the vertical differential $\mathcal{B}(M)_{pq} \to \mathcal{B}(M)_{p,q-1}$ given by b.

Define the *cyclic homology* of a mixed complex (M, b, B) to be

$$\mathrm{HC}_\bullet(M) := \mathrm{H}_\bullet\left[\mathrm{Tot}\,\mathcal{B}(M)\right] .$$

Let $(\mathrm{C}(A), b, B)$ be the mixed complex given by $\mathrm{C}(A)_n = A^{\otimes n+1}$, the differential b is the Hochschild differential, and B of degree $+1$ is the Connes differential defined by

$$B(a_0, \ldots, a_n) = \sum_{i=0} (-1)^{ni}(1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}) -$$
$$- (-1)^{n(i-1)}(a_{i-1}, 1, a_i, \ldots, a_n, a_0, \ldots, a_{i-2})$$

Denote the bicomplex $\mathcal{B}[\mathrm{C}(A)]$ associated to the mixed complex $(\mathrm{C}(A), b, B)$ simply by $\mathcal{B}(A)$. There is a natural inclusion $\mathrm{Tot}[\mathcal{B}(A)] \hookrightarrow \mathrm{Tot}[\mathrm{CC}(A)]$ sending $x \in \mathcal{B}(A)_{pq} = \mathrm{C}(A)_{q-p} = A^{\otimes q-p+1}$ to the element

$$x \mapsto x \oplus sN(x) \in \mathrm{C}(A)_{q-p} \oplus \mathrm{C}(A)_{q-p+1} = \mathrm{CC}_{2p,q-p} \oplus \mathrm{CC}_{2p-1,q-p+1} \subset \mathrm{Tot}[\mathrm{CC}(A)]_{p+q}$$

where s denotes the extra degeneracy

$$s(a_0, \dots, a_n) = (1, a_0, \dots, a_n)$$

Theorem 3.1.3 ([26], Theorem 2.1.8)**.** *For any associative unital algebra A the inclusion map* $\mathrm{Tot}[\mathcal{B}(A)] \hookrightarrow \mathrm{Tot}[\mathrm{CC}(A)]$ *is a quasi-isomorphism and therefore*

$$\mathrm{HC}_\bullet(A) \simeq \mathrm{H}_\bullet\left[\mathrm{Tot}\,\mathcal{B}(A)\right]$$

Let $(\overline{\mathrm{C}}(A), b, \overline{B})$ be the following mixed complex. Since A is a unital algebra, one can define $\overline{A} := \mathrm{Coker}(k \to A)$ and $\overline{\mathrm{C}}(A)_n = A \otimes \left(\overline{A}\right)^{\otimes n}$. The Hochschild differential b descends to a differential on $\overline{\mathrm{C}}(A)$ of degree -1 which we still denote b. The *reduced Connes differential* $\overline{B}$ of degree $+1$ is defined by

$$\overline{B}(a_0, \dots, a_n) = \sum_{i=0} (-1)^{ni} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1})$$

We denote by $\overline{\mathcal{B}}(A)$ the bicomplex associated to the mixed complex $(\overline{\mathrm{C}}(A), b, \overline{B})$.

Lemma 3.1.4. *The obvious projection* $(\mathrm{C}(A), b, B) \twoheadrightarrow (\overline{\mathrm{C}}(A), b, \overline{B})$ *is a morphism of mixed complexes, inducing an isomorphism* $\mathrm{H}_\bullet\left[\mathrm{Tot}\,\overline{\mathcal{B}}(A)\right] \simeq \mathrm{HC}_\bullet(A)$.

To summarize, we have the following sequence of bicomplexes with all the maps inducing isomorphisms on homology:

$$\overline{\mathcal{B}}(A) \twoheadleftarrow \mathcal{B}(A) \hookrightarrow \mathrm{CC}(A) \twoheadrightarrow \mathrm{C}^\lambda(A) \tag{2.3}$$

Let A be an unital algebra, and $\eta\colon k \hookrightarrow A, 1 \mapsto 1_A \in A$ be the unit map. It induces a natural inclusion of bicomplexes $\overline{\mathcal{B}}(k) \hookrightarrow \overline{\mathcal{B}}(A)$. Define the *reduced cyclic homology* $\overline{\mathrm{HC}}_\bullet(A)$ to be $\overline{\mathrm{HC}}_\bullet(A) := \mathrm{H}_\bullet\left[\mathrm{Tot}\,\overline{\mathcal{B}}(A)_{\mathrm{red}}\right]$ where the bicomplex $\overline{\mathcal{B}}(A)_{\mathrm{red}}$ is the quotient bicomplex $\overline{\mathcal{B}}(A)_{\mathrm{red}} := \overline{\mathcal{B}}(A)/\overline{\mathcal{B}}(k)$. Short exact sequence of bicomplexes

$$0 \longrightarrow \overline{\mathcal{B}}(k) \longrightarrow \overline{\mathcal{B}}(A) \longrightarrow \overline{\mathcal{B}}(A)_{\mathrm{red}} \longrightarrow 0$$

induces a long exact sequence of the homology of the corresponding total complexes:

$$\ldots \longrightarrow \mathrm{HC}_n(k) \longrightarrow \mathrm{HC}_n(A) \longrightarrow \overline{\mathrm{HC}}_n(A) \longrightarrow \mathrm{HC}_{n-1}(k) \longrightarrow \ldots$$

If A is augmented, i.e. if $A \simeq k \oplus \overline{A}$ is an algebra isomorphism, then the long exact sequence above splits and there is a natural isomorphism $\mathrm{HC}_\bullet(A) \simeq \mathrm{HC}_\bullet(k) \oplus \overline{\mathrm{HC}}_\bullet(A)$. Note also that

$$\mathrm{HC}_n(k) \simeq \begin{cases} 0, & n \text{ is odd,} \\ k, & n \text{ is even.} \end{cases}$$

Define *reduced Connes complex* $\overline{\mathrm{C}}^\lambda(A)$ of an algebra A as the quotient $\mathrm{C}^\lambda(A)/D$ of the Connes complex $\mathrm{C}^\lambda(A)$ by the subcomplex $D \subset C^\lambda(A)$, with $D_n \subset \mathrm{C}^\lambda(A)_n = A^{\otimes n+1}/(1-\tau)$ spanned by elements $(a_0, a_1, \ldots, a_n)$ having at least one $a_i = 1$.

Proposition 3.1.5 ([26], Proposition 2.2.14)**.** *There exists a canonical isomorphism*

$$\overline{\mathrm{HC}}_\bullet(A) \simeq \overline{\mathrm{H}}_\bullet\left[\overline{\mathrm{C}}^\lambda(A)\right].$$

Theorem 3.1.6 ([26], Theorem 4.6.7)**.** *For any commutative algebra A, the bicomplex $\mathcal{B}(A)$ breaks up naturally into a direct sum of subcomplexes $\mathcal{B}(A)^{(i)}$ for $i \geq 0$ whose homology are denoted by $\mathrm{HC}^{(i)}_\bullet(A)$. Therefore,*

$$\mathrm{HC}_0(A) = \mathrm{HC}^{(0)}_0(A), \text{ and}$$
$$\mathrm{HC}_n(A) = \mathrm{HC}^{(1)}_n(A) \oplus \cdots \oplus \mathrm{HC}^{(n)}_0(A), \text{ when } n \geq 1$$

We refer to the decomposition of Theorem 3.1.6 as *Hodge decomposition* of cyclic homology.

Remark 3.1.7. The natural projection $\mathcal{B}(A) \twoheadrightarrow \mathrm{C}^{\lambda}(A)$ induces a decomposition $\mathrm{C}^{\lambda}(A) = \bigoplus \mathrm{C}^{\lambda,(i)}(A)$ of the Connes complex into a sum of subcomplexes whose homology are the groups $\mathrm{HC}^{(i)}_{\bullet}(A)$.

Remark 3.1.8. All the results and constructions generalize almost verbatim to the case of *graded* algebras. The only difference is the use of Koszul sign rule.

3.2 Cyclic homology as a derived functor

In this section we provide yet another definition of cyclic homology, this time using the non-abelian derived functors technique described in Section 2. We will use this approach to cyclic homology to define trace maps in Section 1.3 of Chapter 4, and a very similar approach in Section 4.2 of Chapter 5 to define Drinfeld trace maps.

Let A be a DG algebra, and define $A_{\natural} := A/[A, A]$ where $[A, A]$ is the *vector subspace* of A spanned by all the commutators of elements of A. The assignment $A \mapsto A_{\natural}$ defines a functor $\mathsf{DGA}_k \to \mathsf{Com}(k)$. Thus, any morphism $f\colon S \to A$ of DG algebras induces a natural map $f_{\natural}\colon S_{\natural} \to A_{\natural}$ of complexes. Fixing S and taking the mapping cone of $f_{\natural}$ gives a functor from the under category DGA_S to $\mathsf{Com}(k)$

$$\mathcal{C}\colon \mathsf{DGA}_S \to \mathsf{Com}(k), \qquad (S \xrightarrow{f} A) \mapsto \mathrm{cone}(f_{\natural}). \tag{2.4}$$

Recall from Remark 2.3.3 that there is an equivalence of categories $\mathsf{Ho}(\mathsf{Com}(k)) \simeq \mathcal{D}(k)$.

Theorem 3.2.1. *The functor $\mathcal{C}$ above has a (total) left derived functor*

$\mathbf{L}\colon \mathsf{Ho}(\mathsf{DGA})_k \to \mathcal{D}(k)$ *given by*

$$\mathbf{L}\mathcal{C}(S\backslash A) = \mathrm{cone}(S_\natural \to Q(S\backslash A)_\natural)$$

where $S \to Q(S\backslash A)$ *is a cofibrant resolution of* $S \to A$ *in* DGA_S*.*

Theorem 3.2.1 was implicitly used in [14, 15] without a proof.

Proposition 3.2.2 ([6], Proposition 5)**.** *For any* $A \in \mathsf{Alg}_k$*, there are canonical isomorphisms*

1. $\mathrm{HC}_n(A) \simeq \mathrm{H}_{n+1}\,[\mathbf{L}\mathcal{C}(A\backslash 0)]$ *for all* $n \geq 0$
2. $\overline{\mathrm{HC}}_n(A) \simeq \mathrm{H}_n\,[\mathbf{L}\mathcal{C}(k\backslash A)]$ *for all* $n \geq 1$.

This gives yet another definition of cyclic and reduced cyclic homologies.

Remark 3.2.3. The functor $\mathcal{C}$ provides an example of a functor which does have a total derived functor, but which is not a part of a Quillen pair.

3.3 Cyclic homology of commutative DG algebras

Recall that, for any commutative DG algebra $A \in \mathsf{CDGA}_k$, the DG module Ω^1_A of Kähler differentials of A is defined as the free DG A-module generated by the symbols da (for $a \in A$) modulo the relations

$$d(\delta\, a) \,=\, \delta\, da\,, \quad d(ab) \,=\, da.b + a.db\,.$$

Here, δ denotes the differentials intrinsic to A and Ω^1_A. For $a \in A$ homogeneous, da has the same homological degree in Ω^1_A as a has in A.

Consider the (homologically) graded algebra $\mathbf{Sym}_A\,\Omega^1_A[1]$. Let d denote the (unique) degree 1 derivation on $\mathbf{Sym}_A\,\Omega^1_A[1]$ satisfying $d(a) = da\,,\ \ d(da) = 0\,,\ \ \forall\, a \in A$. Let δ denote the (unique) degree -1 derivation on $\mathbf{Sym}_A\,\Omega^1_A[1]$ induced by the differential intrinsic to A. It is easy to verify that d and δ are square 0 and (anti)commute. Hence, $(\mathbf{Sym}_A\,\Omega^1_A[1], \delta, d)$ is an algebra object in the category of mixed complexes: we refer to it as the *mixed algebra* of A. Note that $k \hookrightarrow \mathbf{Sym}_A\,\Omega^1_A[1]$ via the unit map $k \hookrightarrow A$. We call the mixed complex $(\mathbf{Sym}_A\,\Omega^1_A[1]/k, \delta, d)$ the *mixed de Rham complex* of A and denote it by $\mathrm{DR}^\bullet(A)$.

On the other hand, A may be viewed as a *cohomologically graded* algebra by inverting degrees $A^i = A_{-i}$. In this case, $d+\delta$ can also be viewed as a degree $+1$ differential on $\mathbf{Sym}_A\,\Omega^1_A[-1]$. We call the *cochain algebra* $(\mathbf{Sym}_A\,\Omega^1_A[-1], d+\delta)$ the *de Rham algebra* of A and denote it by $\mathrm{DR}_\bullet(A)$.

Note that $\mathbf{Sym}^q_A(\Omega^1_A[1]) \cong \Lambda^q_A\,\Omega^1_A[q]$: explicitly, this isomorphism is given by

$$a_0 da_1 \ldots da_q \mapsto (-1)^{|a_2|+2|a_3|+\ldots+(q-1)|a_q|} a_0 da_1 \ldots da_q\,.$$

We refer to the complex $(\Lambda^q_A\,\Omega^1_A, \delta)$ as the *complex of de Rham q-forms of A* and denote it by Ω^q_A. Let $\Omega^q_{\bar{A}}$ denote Ω^q_A when $q > 0$ and A/k when $q = 0$.

The following theorem relates the cyclic and de Rham homologies of a commutative DG algebra.

Theorem 3.3.1 ([26], Theorem 5.4.7)**.** *Assume that $A \in \mathsf{DGCA}_k$ is smooth as a graded algebra. Then,* $\mathrm{HC}_\bullet(A)$ *is canonically isomorphic to the cyclic homology of the mixed de Rham complex* $\mathrm{DR}^\bullet(A)$*.*

There is a natural direct sum decomposition

$$\mathrm{CC}[\mathrm{DR}^\bullet(A)] \,=\, \bigoplus_{i\geq 0} \mathrm{CC}^{(i)}[\mathrm{DR}^\bullet(A)]\ ,$$

where $\mathrm{CC}^{(i)}[\mathrm{DR}^\bullet(A)] := \oplus_{n=i}^{2i} \Omega_{\bar{A}}^{2i-n}[n]$ is the total complex of the double complex $\mathrm{C}^{(i)}$, where

$$\mathrm{C}_{p,q}^{(i)} = \begin{cases} [\Omega_{\bar{A}}^{i-p}]_{q-i}\,, & p \geq 0 \\ 0\,, & p < 0 \end{cases}$$

The horizontal differential $\mathrm{C}_{p,q}^{(i)} \to \mathrm{C}_{p-1,q}^{(i)}$ is d and the vertical differential $\mathrm{C}_{p,q}^{(i)} \to \mathrm{C}_{p,q-1}^{(i)}$ is δ.

Proposition 3.3.2 ([26], Proposition 5.4.9)**.** *The isomorphism of Theorem 3.3.1 is compatible with Hodge decompositon of Theorem 3.1.6. In other words, it induces a canonical isomorphism*

$$\overline{\mathrm{HC}}^{(i)}(A) \cong \mathrm{H}_\bullet(\mathrm{CC}^{(i)}[\mathrm{DR}^\bullet(A)])\,.$$

Let $A = (\mathbf{Sym}(V), \delta)$ where V is a finite-dimensional graded k-vector space. Then, the de Rham algebra of A is acyclic with respect to the de Rham differential. In other words, $\mathrm{CC}^{(i)}[\mathrm{DR}^\bullet(A)]$ is quasi-isomorphic to $\Omega_{\bar{A}}^i / d\Omega_{\bar{A}}^{i-1}[i]$, with the quasi-isomorphism being induced by the projection

$$\mathrm{p} : \mathrm{CC}^{(i)}[\mathrm{DR}^\bullet(A)] = \mathrm{Tot}(\mathrm{C}^{(i)}) \twoheadrightarrow \mathrm{C}_{0,\bullet}^{(i)} = \Omega_{\bar{A}}^i[i] \twoheadrightarrow \Omega_{\bar{A}}^i / d\Omega_{\bar{A}}^{i-1}[i] \qquad (2.5)$$

As a consequence of this and Theorem 3.3.1, we obtain

Theorem 3.3.3 ([26], Theorem 5.4.12)**.** *For $A = (\mathbf{Sym}(V), \delta)$, there is a canonical isomorphism*

$$\overline{\mathrm{HC}}_n(A) \cong \bigoplus_{i \geq 0} \mathrm{H}_{n-i}[(\Omega_{\bar{A}}^i / d\Omega_{\bar{A}}^{i-1}, \delta)]\,.$$

In other words, $\overline{\mathrm{HC}}_\bullet(A)$ is canonically isomorphic to $\mathrm{H}_\bullet[\mathrm{DR}^\bullet(A)/d\mathrm{DR}^\bullet(A), \delta]$.

Recall from Proposition 3.1.5 that the reduced cyclic homology of A is isomorphic to the homology of the reduced Connes' complex $\overline{C}^\lambda(A)$. Since $A = (\mathbf{Sym}(V), \delta)$ is commutative, there is a Hodge decomposition $\overline{C}^\lambda(A) = \bigoplus_{i=1}^\infty \overline{C}^{\lambda,(i)}(A)\,.$

The isomorphism in Theorem 3.3.3 is compatible with the Hodge decompositon. To be precise, consider the antisymmetrization map

$$\begin{aligned} \varepsilon \,:\, \bar{A} \otimes \mathbf{Sym}^i(\bar{A}[1]) &\to \bar{A} \otimes \bar{A}[1]^{\otimes i} \\ a_0 \otimes a_1 \wedge \ldots \wedge a_i &\mapsto \sum_{\sigma \in \mathbf{S}_i} (-1)^{f(\sigma, a_1, \ldots a_i)} (a_0, a_{\sigma(1)}, \ldots, a_{\sigma(i)}) \end{aligned}$$

where $(-1)^{f(\sigma, a_1, \ldots a_i)}$ is obtained from the Koszul sign rule when permuting elements with degrees $|a_1| + 1, \ldots, |a_i| + 1$ via σ. By [26, Section 2.3.5], ε induces a map of complexes

$$\varepsilon \,:\, \Omega^i_{\bar{A}} / d\Omega^{i-1}_{\bar{A}}[i] \to \overline{\mathrm{C}}^{\lambda,(i)}(A)\,, \tag{2.6}$$

which is known to be a quasi-isomorphism when $A = (\mathbf{Sym}(V), \delta)$. Its inverse is given by the composite map

$$I_{\mathrm{HKR}} \,:\, \overline{\mathrm{C}}^{\lambda,(i)}(A) \hookrightarrow \overline{\mathrm{C}}^{\lambda}(A) \twoheadrightarrow A \otimes \bar{A}[1]^{\otimes i}/\mathrm{Im}(1-\tau) \twoheadrightarrow \Omega^i_{\bar{A}} / d\Omega^{i-1}_{\bar{A}}[i]\ , \tag{2.7}$$

where the last arrow is defined by $(a_0, \ldots, a_i) \mapsto \frac{1}{i!}\, a_0 da_1 \ldots da_i$. The quasi-isomorphism (2.7) is induced by the classical Hochschild-Kostant-Rosenberg map on Hochschild chain complex of A; we therefore call I_{HKR} the Hochschild-Kostant-Rosenberg map.

3.4 The mixed de Rham coalgebra of a cocommutative DG coalgebra

We now formally dualize the constructions of Section 3.3 replacing algebras by coalgebras. Let $C \in \mathsf{DGC}_k$ be a counital DG coalgebra. The notion of comodule Ω^1_C of Kähler codifferentials is dual to that of the module of Kähler differentials (cf. [38, Section 4]). More precisely, Ω^1_C is the set of symmetric elements in the C-bicomodule $C \otimes C / \Delta(C)$. There is a universal (degree 0) coderivation $d :\, \Omega^1_C \to C$,

$\omega \mapsto d\omega$. The tensor coalgebra $\mathrm{T}^c_C(\Omega^1_C[-1])$ has a unique degree 1 coderivation d such that $d(\omega) := d\omega$ for all $\omega \in \Omega^1_C$ and $d(\alpha) = 0$ for all $\alpha \in C$. It also has a degree -1 coderivation δ induced by the (co)differential on C.

It is easy to check that d and δ are square zero and (anti)commute. Further, both d and δ restrict to (co)differentials on $\mathbf{Sym}^c_C\,\Omega^1_C[-1]$, the largest cocommutative sub-coalgebra of $T_C\,(\Omega^1_C[-1])$. This makes $(\mathbf{Sym}^c_C\,\Omega^1_C[-1], \delta, d)$ a coalgebra object in the category of mixed complexes. Via the counit of C, the mixed coalgebra $(\mathbf{Sym}^c_C\,\Omega^1_C[-1], \delta, d)$ acquires a counit. We call the kernel of this counit the *mixed de Rham complex* of C and denote it by $\mathrm{DR}^\bullet(C)$ and refer to $(\mathbf{Sym}^c_C\,\Omega^1_C[-1], \delta, d)$ as the *mixed coalgebra* of C. On the other hand, one can view $d + \delta$ as a degree -1 (co)differential on $\mathbf{Sym}_C\,\Omega^1_C[1]$: we refer to the resulting cocommutative DG coalgebra as the *de Rham coalgebra* of C and denote it by $\mathrm{DR}_\bullet(C)$. The image of $\mathbf{Sym}^c_C\,\Omega^1_C[-1] \cap [\Omega^1_C[-1]]^{\otimes_C q}$ in $[\Omega^1_C]^{\otimes_C q}[-q]$ will be denoted by $\Omega^q_C[-q]$. We refer to the complex (Ω^q_C, δ) as the *complex of de Rham q forms of* C. For $C \in \mathsf{DGC}_k$, let $\Omega^q_{\bar{C}} := \Omega^q_C$ for $q > 0$, with $\Omega^0_{\bar{C}}$ being the kernel of the counit from Ω^0_C to k.

Recall that if (M, b, B) is a mixed complex, its *negative cyclic homology* is the homology of the complex $\mathrm{Tot}\,\mathcal{B}^-(M)$ where the bicomplex $\mathcal{B}^-(M)$ is defined by

$$\mathcal{B}^-_{p,q}(M) \;=\; \begin{cases} M_{q-p}\,, & p \le 0 \\ 0\,, & p > 0 \end{cases}$$

The horizontal differential $\mathcal{B}^-_{p,q}(M) \to \mathcal{B}^-_{p-1,q}(M)$ is B and the vertical differential $\mathcal{B}^-_{p,q}(M) \to \mathcal{B}^-_{p,q-1}(M)$ is b. The next theorem is dual to Theorem 3.3.1.

Theorem 3.4.1. *Let $C := (\mathbf{Sym}^c(V), \delta)$ where V is a graded k-vector space of finite (total) dimension. Then there is a canonical isomorphism*

$$\overline{\mathrm{HC}}_\bullet(C) \;\cong\; \mathrm{H}_\bullet(\mathcal{B}^-[\mathrm{DR}^\bullet(C)])\,.$$

There is a natural direct sum decomposition

$$\mathcal{B}^{-}[\mathrm{DR}^{\bullet}(C)] \,=\, \bigoplus_{i\geq 0} \mathcal{B}^{-,(i)}[\mathrm{DR}^{\bullet}(C)]\,,$$

where $\mathcal{B}^{-,(i)}[\mathrm{DR}^{\bullet}(C)] \;:=\; \bigoplus_{n=i}^{2i} \Omega^{2i-n}_{\bar{C}}[-n]$ is the total complex of the double complex $\mathcal{B}^{(i)}$, where

$$\mathcal{B}^{(i)}_{p,q} \,=\, \begin{cases} [\Omega^{p+i}_{\bar{C}}]_{q+i}\,, & p \leq 0 \\ 0\,, & p > 0 \end{cases}$$

The horizontal differential $\mathcal{B}^{(i)}_{p,q} \to \mathcal{B}^{(i)}_{p-1,q}$ is d and the vertical differential $\mathcal{B}^{(i)}_{p,q} \to \mathcal{B}^{(i)}_{p,q-1}$ is δ. Dual to Proposition 3.3.2, we have

Proposition 3.4.2. *The isomorphism in Theorem 3.4.1 is compatible with Hodge decomposition: in other words, it induces an isomorphism*

$$\overline{\mathrm{HC}}^{(i)}_{\bullet}(C) \,\cong\, \mathrm{H}_{\bullet}(\mathcal{B}^{-,(i)}[\mathrm{DR}^{\bullet}(C)])\,, \quad \forall\, i \geq 0\,.$$

Note that when $C = (\mathbf{Sym}^c(V), \delta)$ where V is a finite-dimensional graded vector space, $\mathcal{B}^{-,(i)}(\mathrm{DR}^{\bullet}(C))$ is quasi-isomorphic to $\mathrm{Ker}(d\,:\,\Omega^i_{\bar{C}} \to \Omega^{i-1}_{\bar{C}})[-i]$. This quasi-isomorphism is induced by the natural inclusion

$$\iota\,:\, \mathrm{Ker}(d\,:\,\Omega^i_C \to \Omega^{i-1}_C)[-i] \hookrightarrow \mathcal{B}^{-,(i)}(\mathrm{DR}^{\bullet}(C))\,.$$

We thus obtain the following statement (which is dual to Theorem 3.3.3).

Theorem 3.4.3. *There is a canonical isomorphism*

$$\overline{\mathrm{HC}}_n(C) \,\cong\, \bigoplus_{q\geq 0} \mathrm{H}_{n+q}([\mathrm{Ker}(d\,:\,\Omega^q_C \to \Omega^{q-1}_C), \delta])\,.$$

In other words, $\overline{\mathrm{HC}}_{\bullet}(C)$ *is canonically isomorphic to* $\mathrm{H}_{\bullet}[\mathrm{Ker}(d\,:\,\mathrm{DR}^{\bullet}(C) \to \mathrm{DR}^{\bullet}(C)), \delta]$.

Dually to Theorem 3.3.3, the isomorphism in Theorem 3.4.3 respects Hodge decomposition. Further, $\overline{\mathrm{HC}}_\bullet(C)$ is the homology of Connes' reduced complex $\overline{\mathrm{C}}^\lambda(C)$ of C. Dually to [38, Lemma 1.2], $\overline{\mathrm{C}}^\lambda(C) = \mathbf{\Omega}(C)_\natural[1]$, where $\mathbf{\Omega}(C)$ is the cobar construction of C and $R_\natural := R/(k+[R,R])$ for any $R \in \mathsf{DGA}_{k/k}$. Dually to Theorem 3.3.3, the isomorphism in Theorem 3.4.3 is by the map

$$\varepsilon : \overline{\mathrm{C}}^\lambda(C) \xrightarrow{\sim} \bigoplus_i \mathrm{Ker}(d : \Omega^i(C) \to \Omega^{i-1}(C))[-i] \tag{2.8}$$

obtained by taking (graded) linear dual of (2.6) applied to $E := \mathrm{Hom}_k(C,k)$. The inverse map is given by the (graded) dual of the Hochschild-Kostant-Rosenberg map. We refer to this last map as the *co-HKR* map and denote it by I^c_{HKR}.

3.5 The mixed Hopf algebra of a vector space

In this subsection, we let $A := \mathbf{Sym}(V)$ and $C := \mathbf{Sym}^c(V[1])$ where V is a finite-dimensional graded vector space (with trivial differential). Note that C is Koszul dual to A in the sense of [28]. The following lemma is easy to verify.

Lemma 3.5.1. *Ω^1_C is isomorphic to the cofree C-comodule cogenerated by $V[1]$, i.e, $\Omega^1_C \cong V[1] \otimes_k C$. Under this isomorphism, the universal coderivation $d : \Omega^1_C \to C$ becomes the map*

$$C \otimes_k V[1] \to C\,, \quad v \otimes \alpha \mapsto v \cdot \alpha\,,$$

where $\cdot$ denotes the product in $\mathbf{Sym}(V[1])$.

Let $\mathcal{H}(V) := \mathbf{Sym}(V \oplus V[1])$, equipped with the graded Hopf algebra structure of the symmetric algebra of the graded vector space $V \oplus V[1]$.

Proposition 3.5.2. *The de Rham differential d on $\mathcal{H}(V)$ makes $(\mathcal{H}(V), 0, d)$ a*

Hopf algebra object in the category of mixed complexes. As an algebra, $(\mathcal{H}(V), 0, d)$ *is the mixed algebra of* A*. As a coalgebra,* $(\mathcal{H}(V), 0, d)$ *is the mixed coalgebra of* C*.*

Proof. Indeed, as a graded coalgebra, $\mathcal{H}(V) = \mathbf{Sym}^c(V) \otimes \mathbf{Sym}^c(V[1])$. Futher, by Lemma 3.5.1, for any $v_1, \ldots, v_q \in V$ and $\alpha \in \mathbf{Sym}^c(V[1])$, we have

$$d_C(v_1 \ldots v_q \otimes \alpha) = \sum_i \mathrm{sgn}_i \; v_1 \ldots \hat{v}_i \ldots v_q \otimes s(v_i)\alpha \,,$$

where d_C is the de Rham codifferential on the mixed coalgebra of C, $s : V \to V[1]$ is the operator increasing homological degree by 1, and sgn_i is the sign

$$\mathrm{sgn}_i = (-1)^{|v_i|(|v_{i+1}|+\ldots+|v_q|)}(-1)^{|v_1|+\ldots+|\widehat{v_i}|+\ldots|+|v_q|}$$

From this, one sees that d_C *is equal to* the de Rham differential d. It follows that the de Rham differential is a differential of degree 1 on the graded Hopf algebra $\mathcal{H}(V)$ and that $(\mathcal{H}(V), 0, d)$ viewed as a DG coalgebra is equal to the mixed coalgebra of C. This verifies the first and third assertions in the desired proposition. The second assertion is obvious. $\square$

Remark 3.5.3. By Proposition 3.5.2, the identity map is an isomorphism of mixed complexes between the mixed algebra of A and the mixed coalgebra of C. However, under this isomorphism, the p-forms on A having coefficients of polynomial degree q are identified with q-forms on C having coefficients of polynomial degree p.

Let d_A denote the differential d on the mixed de Rham complex $\mathrm{DR}^\bullet(A)$ of A and let d_C denote the differential d on $\mathrm{DR}^\bullet(C)$. Combining the isomorphisms of Theorem 3.3.3 and Theorem 3.4.3 with the identifications of Proposition 3.5.2, we have

Proposition 3.5.4. *There is a natural isomorphism* $\overline{\mathrm{HC}}_\bullet(A) \xrightarrow{\sim} \overline{\mathrm{HC}}_{\bullet+1}(C)$ *given by the composite map*

$$\overline{\mathrm{HC}}_n(A) \cong [\mathrm{Coker}(d_A)]_n \xrightarrow{d_A} [\mathrm{Ker}(d_A)]_{n+1} = [\mathrm{Ker}(d_C)]_{n+1} \cong \overline{\mathrm{HC}}_{n+1}(C)\ .$$

On the other hand, for any associative algebra $A \in \mathsf{Alg}_k$ and any cofibrant resolution $R \xrightarrow{\sim} A$ in DGA_k, there is a quasi-isomorphism of complexes[1]

$$T\,:\,\overline{\mathrm{C}}^\lambda(A) \to R_\natural \tag{2.9}$$

determined by a twisting cochain $f : \mathbf{B}(A) \to R[1]$ whose components $f_n : A^{\otimes n} \to R_{n-1}$ are the Taylor components of an A_∞-inverse to the quasi-isomorphism $R \xrightarrow{\sim} A$. Explicitly, (2.9) is induced on n chains by the map (*cf.* [6, Theorem 4.2])

$$T_n\,:\,A^{\otimes n+1} \to R_\natural\,, \qquad (a_0,\ldots,a_n) \mapsto \sum_{p\in\mathbf{Z}_{n+1}} (-1)^{nk}[f_n(a_p,a_{1+p},\ldots,a_{n+p})]\,,$$

where $[f]$ is the image of $f \in R$ in $R_\natural$.

Now, if $A = \mathbf{Sym}(V)$ and $C = \mathbf{Sym}^c(V[1])$, we take $R = \mathbf{\Omega}(C)$ to be the cobar construction of C, so that $R_\natural = \overline{\mathrm{C}}^\lambda(C)[-1]$. In this case, we have the following result which refines Proposition 3.5.4.

Theorem 3.5.5. *The isomorphism* $\overline{\mathrm{HC}}_\bullet(A) \xrightarrow{\sim} \overline{\mathrm{HC}}_{\bullet+1}(C)$ *induced by* T *is given by*

$$\overline{\mathrm{HC}}_\bullet(A) \xrightarrow[\cong]{I_{\mathrm{HKR}}} [\mathrm{Coker}(d_A)]_\bullet \xrightarrow[\cong]{d} [\mathrm{Ker}(d_C)]_{\bullet+1} \xrightarrow[\cong]{I^c_{\mathrm{HKR}}} \overline{\mathrm{HC}}_{\bullet+1}(C)$$

where I_{HKR} *and* I^c_{HKR} *are the Hochschild-Kostant-Rosenberg maps defined in Section 3.3 and Section 3.4, respectively.*

A detailed proof of Theorem 3.5.5 is given in [3, Appendix B].

[1]This quasi-isomorphism is constructed in [6, Section 4.3], where it is denoted $s\tau^\natural(\theta)$.

3.6 The cyclic homology of universal enveloping algebras

In this section we review the calculation of cyclic homology groups for universal enveloping algebras. We will use the results of this section to compute examples of trace maps in Section 3. The main references are the papers [16, 23] and the book [26]. Throughout this section we assume $A = \mathcal{U}\mathfrak{a}$ is the universal enveloping for a Lie algebra $\mathfrak{a}$ (where $\mathfrak{a}$ is concentrated in homological degree 0).

Recall that A is a quadratic algebra[2], whose Koszul dual coalgebra C is the Chevalley-Eilenberg coalgebra, see Section 2.3. It is a semi-free cocommutative DG coalgebra $C = \mathrm{CE}(\mathfrak{a}) = \mathbf{Sym}^c(\mathfrak{a}[1])$ cogenerated by $\mathfrak{a}[1]$ of homological degree 1, and equipped with the differential $\delta = \delta^{\mathrm{CE}}$ given by

$$\delta^{\mathrm{CE}}(x_1, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j+1}([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n)$$

Let $(\Omega^\bullet(\mathfrak{a}), \delta^A, d^A_{dR})$ be the following mixed complex. As a graded vector space, $\Omega^\bullet(\mathfrak{a}) \simeq \mathbf{Sym}(\mathfrak{a} \oplus \mathfrak{a}[1])$ where the elements $sa \in \mathfrak{a}[1]$ are usually denoted by da. The differential d^A_{dR} (of degree $+1$) is given by

$$d^A_{dR}(a_1 \ldots a_p da_{p+1} \ldots da_{p+q}) = (-1)^q \sum_{i=1}^{p} a_1 \ldots \widehat{a_i} \ldots a_p da_i da_{p+1} \ldots da_{p+q}$$

and the differential δ^A of degree -1 evaluated on a q-form $\omega = a_1 \ldots a_p da_{p+1} \ldots da_{p+q}$ is

$$\begin{aligned}\delta^A(\omega) = &\sum_{i=1}^{p} \sum_{j=1}^{q} (-1)^{j+1} [a_i, a_j] a_1 \ldots \widehat{a_i} \ldots a_p \, da_{p+1} \ldots \widehat{da}_{p+j} \ldots da_{p+q} \\ &+ \sum_{i<j} (-1)^{i+j} a_1 \ldots a_p \, d[a_{p+i}, a_{p+j}] da_{p+1} \ldots \widehat{da}_{p+i} \ldots \widehat{da}_{p+j} \ldots da_{p+q}\end{aligned}$$

Theorem 3.6.1 ([23], Theorem 3). $\mathrm{HC}_\bullet(\mathcal{U}\mathfrak{a})$ *is isomorphic to the cyclic homology of the mixed complex* $(\Omega^\bullet(\mathfrak{a}), d^A_{dR}, \delta^A)$.

[2] Strictly speaking, $\mathcal{U}\mathfrak{a}$ is not quadratic but *quadratic-linear* algebra (or augmented nonhomogeneous quadratic algebra) in the sense of Positselski, see [36].

Recall from Section 3.4 the construction of the mixed de Rham complex $\mathrm{DR}^\bullet(C)$. For $C = \mathrm{CE}(\mathfrak{a})$, it is isomorphic to $(\overline{\mathbf{Sym}}^c(\mathfrak{a}[1] \oplus \mathfrak{a}), \delta, d)$, where d is the de Rham differential and δ is induced by δ^{CE}.

The following result can be found in [16, Section 3] (see, in particular, Lemma 3.2.2 *loc. cit.*).

Proposition 3.6.2. *The de Rham differential d^A_{dR} induces a quasi-isomorphism of mixed complexes $d^A_{dR}\colon (\overline{\Omega^\bullet}(\mathfrak{a}), d^A_{dR}, \delta^A) \to \mathrm{DR}^\bullet(C)$. In particular, it induces an isomorphism of reduced cyclic homology groups*

$$\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a}) \simeq \overline{\mathrm{HC}}_{\bullet+1}(C)$$

Remark 3.6.3. This isomorphism is analogous to the isomorphism of Theorem 3.5.5. However Theorem 3.5.5 does not directly apply in this case because A is noncommutative.

To compute $\mathrm{HC}_\bullet(\mathcal{U}\mathfrak{a})$ one uses the cyclic bicomplex associated to the mixed complex $(\Omega^\bullet(\mathfrak{a}), d^A_{dR}, \delta^A)$ and then applies a spectral sequence argument. There are two homological spectral sequences. If we first apply d^A_{dR}, since $\mathfrak{a}$ is a vector space, its de Rham cohomology vanish in positive degrees. Therefore, the first page E^1 of such spectral sequence will be concentrated in the column $p = 0$ (except the modules k along the diagonal, which don't matter since they are precisely the terms killed in $\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})$, and thus $E^2 = E^\infty$. This implies the isomorphism of Theorem 3.4.3

$$\overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a}) \simeq \bigoplus_{q\geq 0} \mathrm{H}_\bullet\left[\mathrm{Coker}\left(\Omega^{q-1}(\mathfrak{a}) \to \Omega^q(\mathfrak{a})\right)[q],\, \delta^A\right]$$

where $\Omega^q(\mathfrak{a}) \simeq \mathbf{Sym}(\mathfrak{a}) \otimes \mathbf{Sym}^q(\mathfrak{a}[1])$.

On the other hand, if one starts with the differential δ^A, one gets the spectral sequence described in [16, Prop. 3.1.3]:

$$E^1_{pq} = \mathrm{H}^{CE}_{q-p}(\mathfrak{a}, \mathbf{Sym}(\mathfrak{a})) \Rightarrow \mathrm{HC}_{p+q}(\mathcal{U}\mathfrak{a}) \tag{2.10}$$

This spectral sequence does not degenerate on the E^2 page. However, at least in the case when $\mathfrak{a}$ is semi-simple, all the higher differentials can be explicitly calculated. We will use this spectral sequence in Section 3.

Namely, if $\mathfrak{a}$ is a semi-simple Lie algebra of rank l with exponents $m_1, \dots, m_l$, then

$$\mathrm{H}^{\mathrm{CE}}_{\bullet}(\mathfrak{a}; \mathbf{Sym}(\mathfrak{a})) \simeq \mathrm{H}^{\mathrm{CE}}_{\bullet}(\mathfrak{a}; k) \otimes \mathbf{Sym}(\mathfrak{a})^{\mathfrak{a}} \simeq \mathbf{Sym}(\eta_1, \dots, \eta_l) \otimes \mathbf{Sym}(\sigma_1, \dots, \sigma_l)$$

with $\deg(\eta_i) = 2m_i + 1$ and σ_i is a homogeneous polynomial on $\mathfrak{a}$ of degree $m_i + 1$ concentrated in homological degree 0. The higher differential d^i on page E^i is defined by $d^i(\sigma_i) = \eta_i$, $d^i(\sigma_j) = 0$ for $j \neq i$ and $d^i(\eta_j) = 0$ for all j, see [16, Section 3].

One can actually describe the E^∞ page of the spectral sequence explicitly. First we introduce some notation. For a monomial $\sigma_{a_1}^{s_1} \dots \sigma_{a_p}^{s_p} \eta_{b_1} \dots \eta_{b_q}$, we say that it is *$i$-admissible* if $i \leq m_{a_1} \leq m_{b_1}$. So for instance $\sigma_2^3 \sigma_4 \eta_2$ is not an i-admissible monomial for any i, while $\sigma_1^3 \sigma_4 \eta_2$ is 0- and 1-admissible but not 2-admissible.

We say that a monomial $\sigma_{a_1}^{s_1} \dots \sigma_{a_p}^{s_p} \eta_{b_1} \dots \eta_{b_q}$ is *of order j* if $j = \sum_{i=1}^{q} (2m_{b_i} + 1)$. So, the monomial $\sigma_2^3 \sigma_4 \eta_1 \eta_3$ is of order $2m_1 + 1 + 2m_3 + 1 = 2(m_1 + m_3) + 2$. The following result appears in [16, Theorem 3.5.1].

Theorem 3.6.4. *The i-admissible monomials of order j form a basis of $E^\infty_{i,i+j}$. In particular, $E^\infty_{pq} = 0$ for $p > q$ and for $p > m_l$, the largest exponent of $\mathfrak{a}$.*

Example 3.6.5. Let $\mathfrak{a} = \mathfrak{sl}_2$. The homology $\mathrm{H}^{\mathrm{CE}}_{\bullet}(\mathfrak{a}; \mathbf{Sym}(\mathfrak{a}))$ is isomorphic to the free graded commutative algebra $\mathbf{Sym}(\sigma, \eta)$ generated by elements σ, η of

$\deg(\sigma) = 2$ and $\deg(\eta) = 3$. The element $\sigma \in \mathbf{Sym}(\mathfrak{sl}_2)^{\mathfrak{sl}_2} \simeq k[c]$ corresponds to the Casimir element c, which in the standard basis $\{e, f, h\}$ is $c = 2ef + \frac{h^2}{2}$.

The E^1 page of the spectral sequence above will look as follows:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
4 & 0 & k[\sigma]\eta & 0 & 0 & k[\sigma] & \cdots \\
3 & k[\sigma]\eta & 0 & 0 & k[\sigma] \arrow[ull, "d^2"'] & 0 & \cdots \\
2 & 0 & 0 & k[\sigma] \arrow[ull, "d^2"'] & 0 & 0 & \cdots \\
1 & 0 & k[\sigma] & 0 & 0 & 0 & \cdots \\
0 & k[\sigma] & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 1 & 2 & 3 & \cdots &
\end{tikzcd}$$

Differential d^1 is clearly zero so $E^1 = E^2$. Because $d^2(\sigma) = \eta$, the spectral sequence degenerates on the third page, $E^3 = E^\infty$ and the cyclic homology is isomorphic to

$$\overline{\mathrm{HC}}_i(\mathcal{U}\mathfrak{sl}_2) \simeq \begin{cases} k[w]/k, & i = 0 \\ k[u]/k, & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

This coincides with the result of Kassel, see [23, Example 9.4].

Consider $\overline{\mathrm{HC}}_2(\mathcal{U}\mathfrak{sl}_2) \simeq k[u]/k$. We can choose a representative $u \in \mathrm{Sym}^2(\mathfrak{sl}_2) = E^\infty_{1,1}$ to be $u = 2ef + \frac{h^2}{2}$. For any $n \geq 1$ let us compute the element in $\overline{\mathrm{HC}}_3(C)$ of $C = \mathrm{CE}(\mathfrak{sl}_2)$ corresponding to u^n under the isomorphism 3.6.2.

First we compute $u_n \in E^\infty_{0,2} \simeq \mathrm{Sym}(\mathfrak{sl}_2) \otimes \Lambda^2(\mathfrak{sl}_2)$ such that $u_n + u^n$ represents the corresponding cycle in $\overline{\mathrm{HC}}_2(\mathcal{U}\mathfrak{sl}_2)$. For that, we should have $\delta(u_n) = -d(u^n)$.

One can check that the element

$$u_n = -\frac{n}{3}\left(2ef + \frac{h^2}{2}\right)^{n-1} (hde\, df + edf\, dh + f dh\, de)$$

will do the job. Finally, applying d_{dR} to u_n we get the element

$$u_n' := -\frac{n(2n+1)}{3}\left(2ef + \frac{h^2}{2}\right)^{n-1} de\, df\, dh \in \mathbf{Sym}(\mathfrak{sl}_2[1] \oplus \mathfrak{sl}_2)$$

of degree 3 that represents the class $u_n' \in \overline{\mathrm{HC}}_3(C)$ corresponding to $u^n \in \overline{\mathrm{HC}}_2(\mathcal{U}\mathfrak{sl}_2)$. If we are only interested in a basis for $\overline{\mathrm{HC}}_3(C)$ we can drop the constant.

CHAPTER 3

CLASSICAL REPRESENTATION VARIETIES

1 Representation and character schemes

In this section we define representation and character schemes of associative and Lie algebras and describe some very basic facts and properties. We will omit almost all the details, they will be provided in Sections 1 and 2 for a more general DG case, as well as their derived analogs. The goal of this section is to introduce the main objects of our study, and also provide the necessary background for the Section 2.

1.1 Representation functor

Associative algebras

Fix an algebra $A \in \mathsf{Alg}_k$ and a vector space $V = k^n$, $n \geq 1$. We are interested in the moduli space of representations of A into k^n. Consider the moduli functor

$$\mathsf{Rep}_n(A)\colon \mathsf{ComAlg}_k \to \mathsf{Sets} \qquad B \mapsto \mathrm{Hom}_{\mathsf{Alg}_k}(A, \mathrm{Mat}_n(B)) \ , \tag{3.1}$$

where we write $\mathrm{Mat}_n(B) := \mathrm{Mat}_n(k) \otimes B$. Note that the assignment $B \mapsto \mathrm{Mat}_n(B)$ gives a functor $\mathrm{Mat}_n(-) : \mathsf{ComAlg}_k \to \mathsf{Alg}_k$ on the category of commutative algebras.

Theorem 1.1.1. *There exists a commutative algebra $A_n \in \mathsf{ComAlg}_k$ corepresenting the functor $\mathsf{Rep}_n(A)$. In other words, there is a natural isomorphism of sets*

$$\mathrm{Hom}_{\mathsf{ComAlg}_k}(A_n, B) \simeq \mathrm{Hom}_{\mathsf{Alg}_k}(A, \mathrm{Mat}_n(B)) \tag{3.2}$$

Moreover, the assignment $A \mapsto A_n$ is natural in A, i.e. it defines a functor

$$(-)_n \colon \mathsf{Alg}_k \to \mathsf{ComAlg}_k \ . \tag{3.3}$$

The isomorphism (3.2) *defines an adjunction between the functors* $(\,\text{–}\,)_n$ *and* $\mathrm{Mat}_n(\,\text{–}\,)$.

Definition 1.1.2. *By Theorem 1.1.1, the functor* $\mathsf{Rep}_n(A)$ *is the functor of point of an affine algebraic scheme, which we will also denote* $\mathsf{Rep}_n(A)$*. We call it the* representation scheme*. By definition,* $\mathsf{Rep}_n(A) = \mathrm{Spec}(A_n)$.

Definition 1.1.3. *Functor* (3.3) *is called the* n-th representation functor on Alg_k*, or simply* representation functor.

Letting $B = A_n$ in (3.2), we have a canonical algebra map $\pi_n : A \to \mathrm{Mat}_n(A_n)$, called the *universal n-dimensional representation* of A.

The group $\mathrm{GL}_n(k)$ acts naturally on A_n by algebra automorphisms. Precisely, each $g \in \mathrm{GL}_n(k)$ defines a unique automorphism of A_n corresponding under (3.2) to the composite map

$$A \xrightarrow{\pi_n} \mathrm{Mat}_n(A_n) \xrightarrow{\mathrm{Ad}(g)} \mathrm{Mat}_n(A_n) \ . \tag{3.4}$$

This action is natural in A and thus defines the functor

$$(-)_n^{\mathrm{GL}} : \ \mathsf{Alg}_k \to \mathsf{ComAlg}_k \ , \quad A \mapsto A_n^{\mathrm{GL}} \ , \tag{3.5}$$

which is a subfunctor of the representation functor $(-)_n$.

Finally, note that the assignment $B \mapsto \mathrm{Mat}_n(B)$ is defined even for noncommutative $B \in \mathsf{Alg}_k$. It is therefore natural to ask if the representation functor $(-)_n \colon \mathsf{Alg}_k \to \mathsf{ComAlg}_k$ can be lifted to a functor $\mathsf{Alg}_k \to \mathsf{Alg}_k$ left adjoint to $\mathrm{Mat}_n(-)$.

Theorem 1.1.4 ([6], Theorem 2.1)**.** *The functor*

$$\sqrt[n]{-}\ :\ \mathsf{Alg}_k \to \mathsf{Alg}_k\ ,\quad A \mapsto [A *_k \mathrm{Mat}_n(k)]^{\mathrm{Mat}_n(k)}\ , \tag{3.6}$$

is left adjoint to $\mathrm{Mat}_n(\,-\,)$*. Moreover, the representation functor is the abelianization of* $\sqrt[n]{-}$*:*

$$(\,-\,)_n :\ \mathsf{Alg}_k \to \mathsf{ComAlg}_k\ ,\quad A \mapsto A_n := (\sqrt[n]{A})_{\mathrm{ab}}\ , \tag{3.7}$$

where $A_{\mathrm{ab}} := A/\langle[A,A]\rangle$ *is the quotient by the two-sided ideal generated by the commutators.*

Lie algebras

In this section we describe a natural generalization of the representation functor construction from associative algebras to the case of Lie algebras.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For any Lie algebra $\mathfrak{a}$, there is an affine scheme $\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a})$ parametrizing representations of $\mathfrak{a}$ in $\mathfrak{g}$. More precisely, the functor

$$\mathfrak{g}(-)\ :\ \mathsf{ComAlg}_k \to \mathsf{LieAlg}_k\,,\quad B \mapsto \mathfrak{g}(B)\ :=\ \mathfrak{g}\otimes B$$

has a left adjoint $(-)_{\mathfrak{g}}\ :\ \mathsf{LieAlg}_k \to \mathsf{ComAlg}_k\,,\quad \mathfrak{a}\mapsto \mathfrak{a}_{\mathfrak{g}}$. For the explicit construction see Section 2.2 below where we extend this to the case of DG Lie algebras. In particular, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{ComAlg}_k}(\mathfrak{a}_{\mathfrak{g}}, B) \simeq \mathrm{Hom}_{\mathsf{LieAlg}_k}(\mathfrak{a}, \mathfrak{g}(B)). \tag{3.8}$$

for any $B \in \mathsf{ComAlg}_k$ and $\mathfrak{a} \in \mathsf{LieAlg}_k$. Taking $B = \mathfrak{a}_{\mathfrak{g}}$ in the above adjunction gives the *universal representation* $\pi\colon \mathfrak{a} \to \mathfrak{g}(\mathfrak{a}_{\mathfrak{g}})$.

For a fixed Lie algebra $\mathfrak{a}$, the commutative algebra $\mathfrak{a}_{\mathfrak{g}}$ represents the functor

$$\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a})\ :\ \mathsf{ComAlg}_k \to \mathsf{Sets}\,,\quad B \mapsto \mathrm{Hom}_{\mathsf{LieAlg}_k}(\mathfrak{a}, \mathfrak{g}(B))\,. \tag{3.9}$$

Definition 1.1.5. *The* Lie representation scheme $\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a})$ *of a Lie algebra* $\mathfrak{a}$ *into a Lie algebra* $\mathfrak{g}$ *is the affine scheme* $\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a}) = \mathrm{Spec}(\mathfrak{a}_{\mathfrak{g}})$.

Note that the Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}(B)$ for any commutative algebra B by $x \cdot (y \otimes b) = [x, y] \otimes b$. This action is obviously functorial in B, and therefore induces an action of $\mathfrak{g}$ on $\mathfrak{a}_{\mathfrak{g}}$ by derivations via the adjunction (3.8). One can, therefore, form the functor

$$(-)_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}} \,:\, \mathsf{LieAlg}_k \to \mathsf{ComAlg}_k\,, \quad \mathfrak{a} \mapsto \mathfrak{a}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}}\,. \tag{3.10}$$

Definition 1.1.6. *We define* Lie character variety *to be* $\mathcal{X}(\mathfrak{a}, \mathfrak{g}) := \mathrm{Spec}\left(\mathfrak{a}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}}\right)$.

Remark 1.1.7. We will extend this definition to the case of DG Lie algebras in Section 2.

Remark 1.1.8. If G is a Lie group with $\mathcal{L}ie(G) \simeq \mathfrak{g}$ then of course G also acts on $\mathfrak{a}_{\mathfrak{g}}$, and one can define $\mathsf{Rep}(\mathfrak{a}, \mathfrak{g}) /\!\!/ G := \mathrm{Spec}\left(\mathfrak{a}_{\mathfrak{g}}^{G}\right)$. However, we will only consider $\mathcal{X}(\mathfrak{a}, \mathfrak{g})$ since we will be mostly interested in the DG setting. Besides, for connected Lie groups the two are isomorphic.

1.2 Character maps

Associative algebras

Let A be an algebra, fix $n > 0$ and let $\mathsf{Rep}_n(A)$ be the corresponding representation variety. Recall from 1.1 the universal representation $\pi_n :\, A \to \mathrm{Mat}_n(A_n)$.

Composing π_n with the map

$$\mathrm{Mat}_n(A_n) = \mathrm{Mat}_n(k) \otimes A_n \xrightarrow{\mathrm{tr} \otimes \mathrm{id}} k \otimes A_n \simeq A_n$$

we get the canonical map $\mathrm{Tr}_n \colon A \to \mathrm{Mat}_n(A_n) \to A_n$.

Because for any two matrices X, Y we have $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, Tr_n vanishes on all commutators $[x, y] \in [A, A]$ and therefore factors through $A_\natural := A/[A, A]$. Notice that $[A, A]$ is not an ideal in A, just a subspace. Moreover, because tr is GL_n-invariant, so is Tr_n, and so the image of Tr_n is a subspace of $A_n^{\mathrm{GL}_n}$. We abuse the notation and denote the resulting map $A_\natural \to A_n^{\mathrm{GL}}$ again simply by Tr_n:

$$\begin{tikzcd} A \arrow[r, "\mathrm{Tr}_n"] \arrow[d] & \mathcal{O}[\mathsf{Rep}_n(A)] \\ A/[A,A] \arrow[r, "\mathrm{Tr}_n"] & \mathcal{O}[\mathsf{Rep}_n(A)]^{\mathrm{GL}} \arrow[u, hook, "i"'] \end{tikzcd}$$

Definition 1.2.1. *The map* $\mathrm{Tr}_n \colon A_\natural \to A_n^{\mathrm{GL}}$ *defined above is called* the n-th trace map.

The map Tr_n induces an *algebra* homomorphism

$$\mathrm{Sym}\,\mathrm{Tr}_n \colon \mathrm{Sym}(A_\natural) \to \mathcal{O}[\mathsf{Rep}_n(A)]^{\mathrm{GL}_n} = \mathcal{O}[\mathcal{X}(A, n)].$$

Theorem 1.2.2 (Procesi)**.** *The algebra homomorphism*

$$\mathrm{Sym}\,\mathrm{Tr}_n \colon \mathrm{Sym}(A_\natural) \to \mathcal{O}[\mathsf{Rep}_n(A)]^{GL_n}$$

constructed above is surjective.

Traces for Lie character varieties

We will now define an analog of the trace map 1.2.1 for the case of Lie representation scheme.

First we need to introduce some notation. For any Lie algebra $\mathfrak{a}$, let

$$\lambda^{(r)}(\mathfrak{a}) := \mathrm{Sym}^r(\mathfrak{a})/[\mathfrak{a}, \mathrm{Sym}^r(\mathfrak{a})]$$

be the space of coinvariants of the adjoint action of $\mathfrak{a}$ on $\mathrm{Sym}^r(\mathfrak{a})$. Note that the vector space $\lambda^{(r)}(\mathfrak{a})$ comes with a natural map $\mathfrak{a} \times \mathfrak{a} \times \ldots \times \mathfrak{a} \to \lambda^{(r)}(\mathfrak{a})$, which is the universal symmetric ad-invariant multilinear form on $\mathfrak{a}$ of degree r. The assignment $\mathfrak{a} \mapsto \lambda^{(r)}(\mathfrak{a})$ defines a functor $\mathsf{LieAlg}_k \to \mathsf{Vect}$.

Let $\pi\colon \mathfrak{a} \to \mathfrak{g}(\mathfrak{a}_\mathfrak{g})$ be the universal representation defined in Section 1.1. Fix an integer $r \geq 0$. Let $I^r(\mathfrak{g})$ denote the space of $\mathfrak{g}$-invariant polynomials of degree r on $\mathfrak{g}$. Fix a polynomial $P \in I^r(\mathfrak{g})$, $P\colon \lambda^{(r)}(\mathfrak{g}) \to k$. Apply the functor $\lambda^{(r)}$ to the representation π to get the map $\lambda^{(r)}(\mathfrak{a}) \to \lambda^{(r)}(\mathfrak{g}(\mathfrak{a}_\mathfrak{g}))$. There is a canonical map $\lambda^{(r)}(\mathfrak{g}(\mathfrak{a}_\mathfrak{g})) \to \lambda^{(r)}(\mathfrak{g}) \otimes \mathfrak{a}_\mathfrak{g}$ induced by the multiplication map $(\mathfrak{a}_\mathfrak{g})^{\otimes r} \to \mathfrak{a}_\mathfrak{g}$. Finally, compositing with P we get

$$\lambda^{(r)}(\mathfrak{a}) \xrightarrow{\lambda^{(r)}(\pi)} \lambda^{(r)}(\mathfrak{g}(\mathfrak{a}_\mathfrak{g})) \longrightarrow \lambda^{(r)}(\mathfrak{g}) \otimes \mathfrak{a}_\mathfrak{g} \xrightarrow{P\otimes\mathrm{id}} k \otimes \mathfrak{a}_\mathfrak{g} \simeq \mathfrak{a}_\mathfrak{g}$$

Note that because P is $\mathfrak{g}$-invariant, the image of the above composition is a subspace of $\mathfrak{a}_\mathfrak{g}^{\mathrm{ad}\,\mathfrak{g}} \subset \mathfrak{a}_\mathfrak{g}$. Therefore, we can think of this map as being to $\mathfrak{a}_\mathfrak{g}^{\mathrm{ad}\,\mathfrak{g}}$. We will denote this composition by Tr_P to emphasize its dependence on P, or simply by Tr when it won't cause confusion. The map $\mathrm{Tr}_P\colon \lambda^{(r)}(\mathfrak{a}) \to \mathfrak{a}_\mathfrak{g}^{\mathrm{ad}\,\mathfrak{g}}$ is called the 0-*th Drinfeld trace map associated to* P. The reason it's called "0-th" will become clear when we will be talking about derived Drinfeld traces, see Section 1.

2 Representation schemes as a tensor product

In this section we provide a different approach to representation and character varieties. Namely, we construct them as functor tensor products over a certain small monoidal category H. The strict *monoidal* functors $\mathsf{H} \to \mathsf{Vect}$ will correspond precisely to the cocommutative Hopf algebras. The main result of this section is Theorem 2.3.1. For more details we refer to [22].

The first part of Theorem 2.3.1 is essentially equivalent to a result of Massuyeau–Turaev, obtained independently in [31]. One of the advantages of our construction is that it can be applied both to monoidal and non-monoidal functors. In particular, this construction allows us to view the *character* variety as the tensor product of functors over H (see Theorem 2.3.1 below).

Another advantage of this approach is that it gives an alternative way of defining Lie representation homology using derived functors in the classical abelian sense as opposed to the non-abelian derived functors between Quillen's model categories, see Section 2.3.1 for more details.

2.1 Basic category theory

This section serves two purposes. The first is to recall some basic definitions on about monoidal categories and functors and fix the notation. We will some details and technicalities in most definitions, just stating the idea, and we refer the reader to the textbook [30] for a thorough exposition. The second is to describe the construction of the functor tensor product and prove several properties that will be used later for the functors $\mathsf{H} \to \mathsf{Vect}_k$.

Monoidal categories and functors

Definition 2.1.1. *A* monoidal category C *is a category equipped with the following data:*

1. *a functor* $\odot\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ *called the* tensor product*;*
2. *a* unit object $1_{\mathsf{C}} \in \mathrm{Ob}(\mathsf{C})$*;*

3. *a natural isomorphism* $\alpha\colon ((-)\odot(-))\odot(-) \xrightarrow{\sim} (-)\odot((-)\odot(-))$ *called the* associator*;*

4. *two natural isomorphisms* $\lambda\colon 1_{\mathsf{C}}\odot(-) \xrightarrow{\sim} \mathrm{id}_{\mathsf{C}}$ *and* $\rho\colon (-)\odot 1_{\mathsf{C}} \xrightarrow{\sim} \mathrm{id}_{\mathsf{C}}$ *called the* left *and* right unitors*, respectively.*

These data satisfy some natural compatibility conditions, see [30, Sect. VII.1] for details. A monoidal category is called strict *if the three structure isomorphisms* α, λ, ρ *are the* identities*.*

Definition 2.1.2. *A monoidal category is called* symmetric *if for any pair of objects* $c, c' \in \mathrm{Ob}(\mathsf{C})$ *there is a natural isomorphism* $s_{cc'}\colon c\odot c' \xrightarrow{\sim} c'\odot c$ *called the* braiding*, compatible with the rest of the structure and satisfying* $s_{cc'}\circ s_{c'c} = \mathrm{id}_{c\odot c'}$*.*

Example 2.1.3. The category Vect_k of vector spaces over k is symmetric monoidal with the monoidal structure given by the usual tensor product $\otimes_k$. We will always assume this monoidal structure on Vect_k and will not specify it in the future.

Definition 2.1.4. *A* ***monoidal functor*** $F\colon (\mathsf{C},\odot)\to(\mathsf{D},\otimes)$ *between monoidal categories* C, D *is a triple* $(F, \varepsilon_F, \varphi)$ *consisting of a usual functor* $F\colon \mathsf{C}\to\mathsf{D}$*, a morphism* $\varepsilon_F\colon 1_{\mathsf{D}}\to F(1_{\mathsf{C}})$ *and a natural transformation* $F(c)\otimes F(c') \xrightarrow{\varphi_{cc'}} F(c\odot c')$ *for any* $c, c' \in \mathrm{Ob}(\mathsf{C})$ *satisfying certain* associativity *and* unitality *conditions, see [30, Section XI.2]. The naturality means that for any* $f\colon c\to c'$ *morphism in* C*, the following diagram commutes:*

$$\begin{tikzcd}
F(c)\otimes F(c'') \arrow[r, "\varphi_{cc''}"] \arrow[d, "F(f)\otimes \mathrm{id}_{c''}"'] & F(c\odot c'') \arrow[d, "F(f\odot \mathrm{id}_{c''})"] \\
F(c')\otimes F(c'') \arrow[r, "\varphi_{c'c''}"] & F(c'\odot c'')
\end{tikzcd} \tag{3.11}$$

A monoidal functor is called strict *if the morphisms* ε_F *and* φ *are identities.*

Definition 2.1.5. *If $(\mathsf{C}, \odot)$ and $(\mathsf{D}, \otimes)$ are symmetric monoidal categories, a monoidal functor $F\colon \mathsf{C} \to \mathsf{D}$ is called* symmetric *if it commutes with the braiding maps, i.e. for any two objects $c, c' \in \mathrm{Ob}(\mathsf{C})$ the following diagram commutes:*

$$\begin{tikzcd} F(c) \otimes F(c') \arrow[r, "s_{cc'}"] \arrow[d, "\varphi_{cc'}"'] & F(c') \otimes F(c) \arrow[d, "\varphi_{c'c}"] \\ F(c \odot c') \arrow[r, "F(s_{cc'})"] & F(c' \odot c) \end{tikzcd}$$

Functor tensor product

Let C be a small monoidal k-category. Let $F\colon \mathsf{C} \to \mathsf{Vect}_k$ and $G\colon \mathsf{C}^{op} \to \mathsf{Vect}_k$ be two functors. Define the tensor product $G \otimes_{\mathsf{C}} F$ as the coequalizer

$$G \otimes_{\mathsf{C}} F := \mathrm{coeq}\left[\bigoplus_{c,c' \in \mathrm{Ob}(\mathsf{C})} \mathrm{Hom}_{\mathsf{C}}(c, c') \otimes G(c') \otimes F(c) \rightrightarrows \bigoplus_{c \in \mathrm{Ob}(\mathsf{C})} G(c) \otimes F(c)\right]$$

where the two arrows are defined as follows. On the component $\mathrm{Hom}_{\mathsf{C}}(c, c') \otimes G(c') \otimes F(c)$ the arrows are given by

$$f \otimes v \otimes w \mapsto G(f)v \otimes w \in G(c) \otimes F(c)$$

$$f \otimes v \otimes w \mapsto v \otimes F(f)w \in G(c') \otimes F(c')\,,$$

where $f \in \mathrm{Hom}_{\mathsf{C}}(c, c'), v \in G(c'), w \in F(c)$. In other words,

$$G \otimes_{\mathsf{C}} F = \bigoplus_{c \in \mathrm{Ob}(\mathsf{C})} G(c) \otimes F(c) \Big/ (G(f)v \otimes w = v \otimes F(f)w) \qquad (3.12)$$

for all $f \in \mathrm{Hom}_{\mathsf{C}}(c', c'')$, $v \in G(c'')$, $w \in F(c')$, $c', c'' \in \mathsf{C}$.

Lemma 2.1.6. *Let $x \in \mathrm{Ob}(\mathsf{C})$. If $F = h^x := \mathrm{Hom}_{\mathsf{C}}(x, -)\colon \mathsf{C} \to \mathsf{Vect}_k$ is a corepresented functor and $G\colon \mathsf{C}^{op} \to \mathsf{Vect}_k$ is any functor, then $G \otimes_{\mathsf{C}} h^x \simeq G(x)$.*

Proof. In this case $G \otimes_{\mathsf{C}} h^x$ is the coequalizer of the diagram

$$\bigoplus_{c,c'} \mathrm{Hom}_{\mathsf{C}}(x, c) \otimes \mathrm{Hom}_{\mathsf{C}}(c, c') \otimes G(c') \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bigoplus_{c} \mathrm{Hom}_{\mathsf{C}}(x, c) \otimes G(c)$$

where α is the map

$$\alpha = (- \circ -) \otimes \mathrm{id}\colon \mathrm{Hom}_{\mathsf{C}}(x,c) \otimes \mathrm{Hom}_{\mathsf{C}}(c,c') \otimes G(c') \to \mathrm{Hom}_{\mathsf{C}}(x,c') \otimes G(c')$$

and β is the map

$$\beta = \mathrm{id} \otimes G(\mathrm{ev})\colon \mathrm{Hom}_{\mathsf{C}}(x,c) \otimes \mathrm{Hom}_{\mathsf{C}}(c,c') \otimes G(c') \to \mathrm{Hom}_{\mathsf{C}}(x,c) \otimes G(c)$$

where $G(\mathrm{ev})\colon \mathrm{Hom}_{\mathsf{C}}(c,c') \otimes G(c') \to G(c)$ is given by $f \otimes v \mapsto G(f)v$.

Consider the map

$$\varphi = G(\mathrm{ev})\colon \bigoplus_c \mathrm{Hom}_{\mathsf{C}}(x,c) \otimes G(c) \to G(x) \tag{3.13}$$

To prove the desired isomorphism, we need to check that φ satisfies the coequalizer universal property. In other words, we need to check that for any map $\psi\colon \bigoplus_c \mathrm{Hom}_{\mathsf{C}}(x,c) \otimes G(c) \to Y$ coequalizing α and β there exists a unique $\overline{\psi}$ making the following diagram commute:

$$\begin{tikzcd}
\bigoplus_{c,c'} \mathrm{Hom}_{\mathsf{C}}(x,c) \otimes \mathrm{Hom}_{\mathsf{C}}(c,c') \otimes G(c') \arrow[r, shift left] \arrow[r, shift right] & \bigoplus_c \mathrm{Hom}_{\mathsf{C}}(x,c) \otimes G(c) \arrow[r, "\varphi"] \arrow[d, "\psi"'] & G(x) \arrow[dl, dashed, "\exists ! \overline{\psi}"] \\
& Y &
\end{tikzcd}$$

We define $\overline{\psi}(v) := \psi(\mathrm{id}_x \otimes v)$ for $v \in G(x)$, where $\mathrm{id}_x \in \mathrm{Hom}_{\mathsf{C}}(x,x)$ is the identity morphism. It is a direct check that the resulting diagram commutes. $\square$

Proposition 2.1.7. *Let $F\colon \mathsf{C} \to \mathsf{Vect}_k$ and $G\colon \mathsf{C}^{op} \to \mathsf{Vect}_k$ be two lax monoidal functors. Then $G \otimes_{\mathsf{C}} F$ has a natural structure of a commutative algebra over k.*

Proof. Define

$$(\psi_{cc'} \otimes \varphi_{cc'}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})\colon G(c) \otimes F(c) \otimes G(c') \otimes F(c') \to G(c \odot c') \otimes F(c \odot c')$$

using the usual braiding τ in Vect_k. The naturality conditions (3.11) imply that these maps induce a well-defined multiplication map on $G \otimes_{\mathsf{C}} F$. The unit $k \to G \otimes_{\mathsf{C}} F$ is induced by the natural map

$$k \simeq k \otimes k \xrightarrow{\varepsilon_G \otimes \varepsilon_F} G(1_{\mathsf{C}}) \otimes F(1_{\mathsf{C}})$$

The associativity and unitality for $G \otimes_{\mathsf{C}} F$ follow from the associativity and unitality axioms for lax monoidal functors. □

Lemma 2.1.8. *Suppose that $x \in \mathrm{Ob}(\mathsf{C})$ is a* counital coalgbra object *in C, i.e. there are morphisms $\Delta_x \colon x \to x \odot x$ and $\varepsilon_x \colon x \to 1_{\mathsf{C}}$ satisfying the usual compatibility axioms for a coalgebra. Then $F := h^x \colon \mathsf{C} \to \mathsf{Vect}_k$ is a lax monoidal functor.*

Proof. The coproduct $\Delta_x \colon x \to x \odot x$ induces a natural map

$$\mathrm{Hom}_{\mathsf{C}}(x, y) \otimes \mathrm{Hom}_{\mathsf{C}}(x, y') \to \mathrm{Hom}_{\mathsf{C}}(x, y \odot y')$$

sending $f \otimes g$ to the composition $x \to x \odot x \xrightarrow{f \odot g} y \odot y'$. The map $\varepsilon_x \colon x \to 1_{\mathsf{C}}$ induces a natural map $1_{\mathsf{Vect}_k} = k \to F(1_{\mathsf{C}}) = \mathrm{Hom}(x, 1_{\mathsf{C}})$ sending $1 \in K$ to $\varepsilon_x \in \mathrm{Hom}(x, 1_{\mathsf{C}})$. These two maps give the lax monoidal structure on $F = h^x$. □

Lemma 2.1.9. *Let x be a counital coalgebra object in C, $F = h^x \colon \mathsf{C} \to \mathsf{Vect}_k$ is the corresponding functor, and let $G \colon \mathsf{C}^{op} \to \mathsf{Vect}$ be a lax monoidal contravariant functor. Then the isomorphism $G \otimes_{\mathsf{C}} h^x \simeq G(x)$ of Lemma 2.1.6 is an* isomorphism of algebras.

Proof. The functor $F = h^x$ is monoidal by Lemma 2.1.8, and so by Proposition 2.1.7, the vector space $G \otimes_{\mathsf{C}} F \in \mathrm{Ob}(\mathsf{D})$ is naturally an algebra. The algebra structure on $G(x)$ is induced by $G(x) \otimes G(x) \to G(x \odot x) \to G(x)$ and $1_{\mathsf{D}} \to G(1_{\mathsf{C}}) \to G(x)$.

Since the algebra structure on the tensor product $G \otimes_{\mathsf{C}} h^x$ is induced by the algebra structure on $\bigoplus_c \mathrm{Hom}_{\mathsf{C}}(x, c) \otimes G(c)$, it is enough to verify that the map φ in (3.13) is an algebra homomorphism. This is a direct verification. $\square$

Lemma 2.1.10. *The functor category* $\mathsf{Fun}(\mathsf{C}, \mathsf{Vect}_k)$ *has limits and colimits (in particular equalizers and coequalizers), which can be computed pointwise.*

Proof. It's a well-known fact. See, for example, [24, Section 3.3]. $\square$

Let $F\colon \mathsf{C} \to \mathsf{Vect}_k$ be a functor and let $\gamma \in \mathrm{Hom}_{\mathsf{Vect}_k}(V, F(c))$ be a fixed linear map, for some $V \in \mathsf{Vect}_k$ and $c \in \mathrm{Ob}(\mathsf{C})$. Then γ can be extended to a natural transformation $V \otimes \mathrm{Hom}_{\mathsf{C}}(c, -) \Rightarrow F$,

$$V \otimes \mathrm{Hom}_{\mathsf{C}}(c, c') \xrightarrow{\gamma \otimes \mathrm{id}} F(c) \otimes \mathrm{Hom}_{\mathsf{C}}(c, c') \xrightarrow{F(\mathrm{ev})} F(c') \,.$$

Let $\{\gamma_i\}$ be a collection of morphisms in $\mathrm{Hom}_{\mathsf{Vect}_k}(V, F(c))$. Denote by $F/\{\gamma_i\}$ the coequalizer in the functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{Vect}_k)$ of the natural transformations $V \otimes \mathrm{Hom}_{\mathsf{C}}(c, -) \Rightarrow F$ induced by γ_i.

Lemma 2.1.11. *The tensor product* $G \otimes_{\mathsf{C}} (F/\{\varphi_i\})$ *is isomorphic to the coequalizer in* Vect_k *of the maps*

$$\overline{\gamma}_i\colon G(c) \otimes V \xrightarrow{\mathrm{id} \otimes \gamma_i} G(c) \otimes F(c) \subset \bigoplus_{c \in \mathrm{Ob}(\mathsf{C})} G(c) \otimes F(c) \twoheadrightarrow G \otimes_{\mathsf{C}} F$$

Proof. The proof is quite technical, and we refer the reader to [22, Lemma A.9]. $\square$

2.2 Hopf algebras as monoidal functors

The category $(\mathsf{H}, \odot)$ is a strict symmetric monoidal category generated by a single object $[1]$. We denote $[1]^{\odot n}$ by $[n]$ and the unit of H by $[0]$. Thus we have

$[n] \odot [m] = [n+m]$. The set of morphisms is generated by "multiplication" $\mu\colon [2] \to [1]$, "comultiplication" $\Delta\colon [1] \to [2]$, "antipode" $S\colon [1] \to [1]$, "unit" $\eta\colon [0] \to [1]$, "counit" $\varepsilon\colon [1] \to [0]$ and "transposition" $\tau\colon [2] \to [2]$ subject to the usual relations satisfied by cocommutative Hopf algebras, see [22, Appendix A] for the full list. This category is called the *PROP of cocommutative Hopf algebras*.

Remark 2.2.1. We will often work with the *k-linearized* version of H, where the space of morphisms is the span of the set $\mathrm{Hom}_{\mathsf{H}}([n],[m])$. We will still denote the linearized category by H.

Remark 2.2.2. From the definition, there is an equivalence between the category of cocommutative Hopf algebras and the category of *strict symmetric monoidal* functors $\mathsf{H} \to \mathsf{Vect}$, assigning to a Hopf algebra H the functor $[H]\colon [n] \mapsto H^{\otimes n}$. Similarly, the category of *commutative* Hopf algebras is equivalent to the category of *contravariant* strict symmetric monoidal functors $\mathsf{H}^{op} \to \mathsf{Vect}$.

There is an alternative description of H, given by the following folklore statement:

Theorem 2.2.3. *There is an equivalence of categories $\mathsf{H} \simeq \mathsf{G}_{ff}^{op}$, where G_{ff} is the category of finitely generated free groups.*

This result was mentioned in [35, Section 5] and [42, Section 2], with no proof. A proof can be found, for example, in [19, Theorem 8]. A detailed proof of the following Corollary (which is equivalent to Theorem 2.2.3) can be found in [10, Thm. 4.5].

Corollary 2.2.4. *For any integers n, m we have $\mathrm{Hom}_{\mathsf{H}}([n],[m]) \simeq F_n^{\times m}$, where F_n denotes the free group on n-generators (which comes with a specified choice of generators).*

Corollary 2.2.5. *The functor $h^{[n]} = \mathrm{Hom}_{\mathsf{H}}([n], -)$ corepresented by $[n] \in \mathrm{Ob}(\mathsf{H})$ is isomorphic to the functor $[F_n]\colon \mathsf{H} \to \mathsf{Sets}$ sending $[m] \mapsto F_n^{\times m}$. If working with the linearized H, then $h^{[n]}\colon \mathsf{H} \to \mathsf{Vect}_k$ is given by $[m] \mapsto k[F_n^{\times m}] \simeq k[F_n]^{\otimes m}$.*

Example 2.2.6. Let $\mathfrak{a}$ be a Lie algebra, there is a strict monoidal functor $[\mathcal{U}\mathfrak{a}]\colon \mathsf{H} \to \mathsf{Vect}_k$ associated to it, sending $[n]$ to $\mathcal{U}\mathfrak{a}^{\otimes n}$. If A is a Lie group with $\mathfrak{L}(A) = \mathfrak{a}$, there are natural functors $[\mathcal{U}\mathfrak{a}]^{\mathfrak{a}}$ and $[\mathcal{U}\mathfrak{a}]^{A}$ sending $[n]$ to the invariants $(\mathcal{U}\mathfrak{a}^{\otimes n})^{\mathfrak{a}}$ and $(\mathcal{U}\mathfrak{a}^{\otimes n})^{A}$ of the diagonal action of $\mathfrak{a}$ and A, respectively. The latter two functors are only lax monoidal, and so do not correspond to a Hopf algebra. The lax monoidal structure is given by the inclusion

$$\varphi_{nm}\colon \left(\mathcal{U}\mathfrak{a}^{\otimes n}\right)^{\mathfrak{a}} \otimes \left(\mathcal{U}\mathfrak{a}^{\otimes m}\right)^{\mathfrak{a}} \hookrightarrow \left(\mathcal{U}\mathfrak{a}^{\otimes n+m}\right)^{\mathfrak{a}}$$

and the identity map $\varepsilon\colon 1_{\mathcal{V}ect} = k \to (\mathcal{U}\mathfrak{a}^{\otimes 0})^{\mathfrak{a}} = k$.

Example 2.2.7. If G is an affine algebraic group, its algebra of regular functions $\mathcal{O}(G)$ defines a contravariant strict monoidal functor which we denote simply by $[\mathcal{O}(G)]\colon \mathsf{H}^{op} \to \mathsf{Vect}_k$ given by $[n] \mapsto \mathcal{O}(G)^{\otimes n} \simeq \mathcal{O}(G^n)$. Once again, taking invariants of the adjoint action gives a lax monoidal functor $[\mathcal{O}(G)]^G\colon [n] \mapsto \mathcal{O}(G^n)^G$, with the lax monoidal structure given by the maps analogous to the case of $[\mathcal{U}\mathfrak{a}]^{\mathfrak{a}}$ and $[\mathcal{U}\mathfrak{a}]^{A}$.

Because $[\mathcal{O}(G)]$ is contravariant and $[\mathcal{U}\mathfrak{a}]$ is covariant, we can form their tensor product $[\mathcal{O}(G)] \otimes_{\mathsf{H}} [\mathcal{U}\mathfrak{a}]$. Since both $[\mathcal{O}(G)]$ and $[\mathcal{U}\mathfrak{a}]$ are symmetric monoidal, by Proposition 2.1.7 the vector space $[\mathcal{O}(G)] \otimes_{\mathsf{H}} [\mathcal{U}\mathfrak{a}]$ has the structure of a commutative algebra. Moreover, since the functor $[\mathcal{O}(G)]^G$ is lax monoidal, the tensor product $[\mathcal{O}(G)]^G \otimes_{\mathsf{H}} [\mathcal{U}\mathfrak{a}]$ is naturally a commutative algebra, too.

2.3 Representation and character varieties

Theorem 2.3.1. *There are natural isomorphisms of commutative algebras*

1. $[\mathcal{O}(G)] \otimes_{\mathsf{H}} [\mathcal{U}\mathfrak{a}] \simeq \mathcal{O}(\mathsf{Rep}(\mathfrak{a}, \mathfrak{g}))$, *where* $\mathfrak{a}$ *is a Lie algebra,* $\mathfrak{g} = \mathfrak{L}(G)$ *and* $\mathsf{Rep}(\mathfrak{a}, \mathfrak{g})$ *is the Lie representation variety.*
2. $[\mathcal{O}(G)]^G \otimes_{\mathsf{H}} [\mathcal{U}\mathfrak{a}] \simeq \mathcal{X}(\mathfrak{a}, \mathfrak{g})$, *where* $\mathfrak{a}$ *and* G *are as above and* $\mathcal{X}(\mathfrak{a}, \mathfrak{g})$ *is the Lie character variety, see Definition 1.1.5.*

Proof. We will only sketch the proof. Since any Lie algebra $\mathfrak{a}$ is a quotient of a free Lie algebra $\mathfrak{L}(V)$, it is enough to prove the isomorphism for free Lie algebras. In that case, if $\mathfrak{a} = \mathfrak{L}(V)$ is a free Lie algebra, then $\mathcal{U}\mathfrak{a} \simeq TV$.

Lemma 2.3.2. *Let* V *be a finite-dimensional vector space. There is an isomorphism of functors from* H *to* Vect_k*:*

$$[TV] \simeq \bigoplus_{n=0}^{\infty} V^{\otimes n} \otimes_{S_n} \left(h^{[n]} \Big/ \{\varphi_i, \psi_j, \nu_j\} \right)$$

where the morphisms $\varphi_i, \psi_j, \nu_j \colon k \to \mathrm{Hom}_{\mathsf{H}}([n], [n+1])$, $i \geq 0$ *are given by* $\varphi_0 = 0$ *and*

$$\varphi_i = \mathrm{id}^{\otimes i-1} \otimes \Delta \otimes \mathrm{id}^{\otimes n-i} - \mathrm{id}^{\otimes i-1} \otimes \eta \otimes \otimes \mathrm{id}^{n-i+1} - \mathrm{id}^{\otimes i} \otimes \eta \otimes \mathrm{id}^{n-i}$$

$$\psi_i = \mathrm{id}^{\otimes i-1} \otimes S \otimes \mathrm{id}^{\otimes n-i} + \mathrm{id}^{\otimes n}$$

$$\nu_i = \mathrm{id}^{\otimes i-1} \otimes \varepsilon \otimes \mathrm{id}^{\otimes n-i}$$

Proof of Lemma. For simplicity, let's only prove the isomorphism in the case $m = 1$. In this case, $[TV]([1]) = TV = \bigoplus_n V^{\otimes n} \simeq \bigoplus_n V^{\otimes n} \otimes_{S_n} k[S_n]$. Therefore, it is enough to show that $k[S_n] \simeq h^{[n]}/\{\varphi_i\}([1])$. We emphasize that this is only isomorphism between vector spaces, since the right-hand side does not have natural algebra structure.

Any morphism $\alpha\colon [n] \to [1]$ in H can be represented as the composition of a certain number of comultiplications, followed by antipodes, then a permutation, and finally followed by the multiplication.

The relations $\varphi_i \sim 0$ allow us to remove all comultiplications, and after that all antipodes. Thus, a morphism in $\mathrm{Hom}_{\mathsf{H}}([n],[1])/\{\varphi_i\}$ is represented (up to sign) by a permutation, followed by the multiplication, i.e. by an element of $k[S_n]$. It is easy to see that this is an isomorphism. $\square$

Thus, by Lemma 2.3.2, we have

$$[k(G)] \otimes_{\mathsf{H}} [TV] \simeq \bigoplus_{n\geqslant 0} k(G) \otimes_{\mathsf{H}} \left(V^{\otimes n} \otimes_{S_n} h^{[n]} \Big/ \sim \right)$$

By Lemma 2.1.11 and Lemma 2.1.6, the latter is isomorphic to the quotient $\bigoplus_{n\geqslant 0} \left(k(G)^{\otimes n} \otimes_{S_n} V^{\otimes n}\right) / \sim$. The chain of equivalences in $[\mathcal{O}(G)] \otimes_{\mathsf{H}} [\mathcal{U}\mathfrak{a}]$

$$\begin{aligned} 1 \otimes x = 1 \cdot 1 \otimes x &\sim 1 \otimes 1 \otimes (x \otimes 1 + 1 \otimes x) \\ &\sim 2 \cdot 1 \otimes 1 \otimes 1 \otimes x \sim 2 \cdot 1 \otimes x \end{aligned}$$

implies that $1 \otimes x = 0$ in the tensor product $[\mathcal{O}(G)] \otimes_{\mathsf{H}} [\mathcal{U}\mathfrak{a}]$. Therefore

$$\bigoplus_{n\geqslant 0} \left(\mathcal{O}(G)^{\otimes n} \otimes_{S_n} V^{\otimes n}\right) \Big/ \sim \quad \cong \quad \bigoplus_{n\geqslant 0} I^{\otimes n} \otimes_{S_n} V^{\otimes n} \Big/ \sim ,$$

where $I := \mathrm{Ker}(\mathcal{O}(G) \to k)$ is the kernel of the counit map. The relations imply that $1 \otimes v = 0$ for any $v \in V^{\otimes n}$, $1 \in k(G)^{\otimes n}$. Moreover, for any $f \in I^2 \subset k(G)$, $f \otimes v = 0$ in the quotient. Indeed, it is enough to see it for the element X of the form $X = (f_1 - f_1(e))(f_2 - f_2(e)) \otimes_{\mathsf{H}} v$. We have

$$\begin{aligned} X &\sim \big((f_1 - f_1(e)) \otimes (f_2 - f_2(e)) \big) \otimes_{\mathsf{H}} \Delta(v) \\ &\sim \big(f_1 \otimes f_2\big) \otimes_{\mathsf{H}} \big((\mathrm{id} - \eta\varepsilon) \otimes (\mathrm{id} - \eta\varepsilon) \circ \Delta(v)\big) \\ &\sim \big(f_1 \otimes f_2\big) \otimes_{\mathsf{H}} \big((\mathrm{id} - \eta\varepsilon) \otimes (\mathrm{id} - \eta\varepsilon)(1 \otimes v + v \otimes 1)\big) = 0 \end{aligned}$$

Thus, the quotient is isomorphic to $\bigoplus_{n\geqslant 0}(I/I^2)^{\otimes n}\otimes_{S_n} V^{\otimes n}$, which is isomorphic to $\mathrm{Sym}(\mathfrak{g}^*\otimes V)\simeq k(\mathsf{Rep}(\mathfrak{L}(V),\mathfrak{g}))$ since $\mathfrak{g}^*\simeq I/I^2$. This finishes the proof.

□

2.4 Remarks on the tensor product construction

Character varieties of groups

The constructions of Section 1.1 can be repeated almost verbatim in the case of representations of groups. Let us briefly spell it out.

Let Γ be a discrete group and G a finite dimensional affine algebraic group over k. Consider the functor $\mathsf{ComAlg}_k\to\mathsf{Sets}$ sending $B\mapsto \mathrm{Hom}_{\mathsf{Groups}}(\Gamma, G(B))$, where $G(B)$ are the B-points of the scheme G. It is corepresentable by a commutative algebra $\Gamma_G\in\mathrm{Ob}(\mathsf{ComAlg})$. This algebra can be thought of as the coordinate algebra of the affine scheme $\mathsf{Rep}_G(\Gamma)=\mathrm{Spec}(\Gamma_G)$. In other words, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{ComAlg}_k}(\Gamma_G, B)\simeq \mathrm{Hom}_{\mathsf{Groups}}(\Gamma, G(B))$$

Fixing G, the assignment $\Gamma\mapsto\Gamma_G$ defines a functor $\mathsf{Groups}\to\mathsf{ComAlg}_k$ which is called *representation functor*.

The group G acts on $\mathsf{Rep}_G(\Gamma)$, and the categorical quotient $\mathsf{Rep}_G(\Gamma)\,/\!\!/\,G = \mathrm{Spec}(\Gamma_G^G)$ is called the *character variety (scheme)* and is denoted by $\mathcal{X}_G(\Gamma)$.

Any group Γ has a natural cocommutative Hopf algebra associated to it, namely its group algebra $k[\Gamma]$. It gives rise to a strict monoidal functor $k[\Gamma]\colon \mathsf{H}\to\mathsf{Vect}_k$ sending $[n]\mapsto k[\Gamma]^{\otimes n}\simeq k[\Gamma^{\times n}]$. On the other hand, the regular ring $\mathcal{O}(G)$ is a commutative Hopf algebra, and thus defines a strict monoidal functor $[\mathcal{O}(G)]\colon \mathsf{H}^{op} \to \mathsf{Vect}_k$ sending $[n] \mapsto \mathcal{O}(G)^{\otimes n} \simeq \mathcal{O}(G^{\times n})$. There is an analog of Theorem 2.3.1 for group representation varieties. For full proof see [22, Theorem 4.1].

Theorem 2.4.1. *There are natural isomorphisms of commutative algebras*

1. $[\mathcal{O}(G)] \otimes_{\mathsf{H}} k[\Gamma] \simeq \mathcal{O}(\mathsf{Rep}_G(\Gamma))$;
2. $[\mathcal{O}(G)]^G \otimes_{\mathsf{H}} k[\Gamma] \simeq \mathcal{O}(\mathcal{X}_G(\Gamma))$ *where* $[\mathcal{O}(G)]^G\colon \mathsf{H} \to \mathsf{Vect}_k$ *is the functor sending* $[n]$ *to the algebra* $\mathcal{O}(G^{\times n})^G$ *of* $\operatorname{ad} G$*-invariant function on* $G^{\times n}$.

Remark 2.4.2. Notice that the functor $[\mathcal{O}(G)]^G$ is not strict monoidal, and therefore does not correspond to a Hopf algebra. However, it is lax monoidal which in particular implies $[\mathcal{O}(G)]^G \otimes_{\mathsf{H}} k[\Gamma]$ has a structure of a (commutative) algebra.

Connection with the associative case

Notice that the technique of this section cannot be directly applied to the case of associative algebras because they do not give functors on the category H.

The Sweedler product of Anel-Joyal (see [1, Section 3.4]) can also be interpreted as a tensor product over a the category of so-called *non-commutative sets* $\mathsf{F(as)}$. This is the category with $\mathrm{Ob}(\mathsf{F(as)}) = \mathbf{N} = \{[0], [1], \dots\}$. A morphism from $[m]$ to $[n]$ is a map of sets $f\colon [m] = \{1, \dots, m\} \to \{1, \dots, n\} = [n]$ with a total ordering on $f^{-1}(i)$ for all $i \in [n]$. This category is the PROP of unital associative algebras, i.e. the category of strict monoidal functors $\mathsf{F(as)} \to \mathsf{Vect}$ is equivalent to the category of unital associative algebras, see [35, Section 3].

If A is an associative unital algebra, and C is a coassociative counital coalgebra, then $[C] \otimes_{\mathsf{F(as)}} [A] \simeq C \triangleright A := T(C \otimes A)/\langle c \otimes ab - (c^{(1)} \otimes a) \otimes (c^{(2)} \otimes b), c \otimes 1 - \varepsilon(c) \cdot 1\rangle$

is the Sweedler product of [1, Section 3.4].

In particular, if $C = \mathrm{Mat}_n(k)^*$ is the linear dual coalgebra of the algebra of $n \times n$ matrices, then $[C] \otimes_{\mathcal{F}(as)} [A] \simeq \sqrt[n]{A}$ where $\sqrt[n]{A}$ was defined in Chapter 4 Section 1.1. Therefore, the abelianization $([C] \otimes_{\mathcal{F}(as)} [A])_{ab}$ is isomorphic to the algebra $\mathcal{O}[\mathsf{Rep}_n(A)]$ of regular functions on the representation scheme of the algebra A.

CHAPTER 4

LIE REPRESENTATION HOMOLOGY

1 Representation homology and derived characters for associative algebras

In this section we briefly recall the construction of derived representation schemes of associative algebras, as described in [6]. There are several reasons to include this material here. First of all, the constructions of Chapter 4 are generalizations of the associative case to Lie algebras. Second, the main calculations in Chapter 4 are that of the reduced Drinfeld traces, which coincide with the derived character maps of 1-dimensional representations, see below.

1.1 The representation functor

We already introduced the notion of a representation functor in Section 1.1. We start this section by quickly spelling out the extension of this construction to DG algebras, as well as fill in some details. We then proceed to define representation homology.

The functor $\mathrm{Mat}_n(-)\colon \mathsf{Alg}_k \to \mathsf{Alg}_k$ sending $B \mapsto \mathrm{Mat}_n(B) = \mathrm{Mat}_n(k) \otimes B$ extends in the obvious way to the functor $\mathrm{Mat}_n(-)\colon \mathsf{DGA}_k \to \mathsf{DGA}_k$.

The functor $\sqrt[n]{-}\colon \mathsf{Alg}_k \to \mathsf{Alg}_k$ left adjoint to $\mathrm{Mat}_n(\sqrt[n]{-})$ also extends naturally to a functor $\sqrt[n]{-}\colon \mathsf{DGA}_k \to \mathsf{DGA}_k$ using the same formula $A \mapsto [A *_k \mathrm{Mat}_n(k)]^{\mathrm{Mat}_n(k)}$ as in Section 1.1 for associative algebras. The extended functor $\sqrt[n]{-}$ is still left adjoint to $\mathrm{Mat}_n(-)$. Finally, we can define the representation functor $(-)_n\colon \mathsf{DGA}_k \to \mathsf{DGCA}_k$ by $A \mapsto (\sqrt[n]{A})_{\mathrm{ab}}$ as in Theorem 1.1.1.

The theorem analogous to Theorem 1.1.1 holds.

Theorem 1.1.1. *The functor* $(\,-)_n$ *is left adjoint to* $\mathrm{Mat}_n(\,-)$ *on the category of commutative DG algebras. Thus, for any* $A \in \mathsf{DGA}_k$ *, the DG algebra* A_n *(co)represents the functor of points of the affine DG scheme*

$$\mathsf{Rep}_n(A):\ \mathsf{DGCA}_k \to \mathsf{Sets}\ ,\quad B \mapsto \mathrm{Hom}_{\mathsf{DGA}_k}(A,\, \mathrm{Mat}_n(B))\ , \tag{4.1}$$

parametrizing the n*-dimensional representations of* A*. In particular, there is a bijection*

$$\mathrm{Hom}_{\mathsf{DGA}_k}(A, \mathrm{Mat}_n(B))\ \cong\ \mathrm{Hom}_{\mathsf{DGCA}_k}(A_n, B)\ , \tag{4.2}$$

functorial in $A \in \mathsf{DGA}_k$ *and* $B \in \mathsf{DGCA}_k$*.*

Letting $B = A_n$ in (4.2), we have a canonical DG algebra map $\pi_n : A \to \mathrm{Mat}_n(A_n)$, called the *universal* n-dimensional representation of A.

The algebra A_n has the following canonical presentation described in [6, Section 2.4]. Let $\{e_{ij}\}_{i,j=1}^n$ be the basis of elementary matrices in $\mathrm{Mat}_n(k)$. For each $a \in A$, define the 'matrix' elements of a in $A *_k \mathrm{Mat}_n(k)$ by

$$a_{ij} := \sum_{k=1}^{n} e_{ki}\, a\, e_{jk}\ .$$

Then $a_{ij} \in \sqrt[n]{A}$ for all $i, j = 1, 2, \ldots, n$, and we also write a_{ij} for the corresponding elements in A_n.

Lemma 1.1.2. *The algebra* A_n *is generated by the elements* $\{a_{ij}\ :\ a \in A\}$ *satisfying the relations*

$$(a+b)_{ij} = a_{ij} + b_{ij}\ ,\quad (ab)_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}\ ,\quad \lambda_{ij} = \delta_{ij}\, \lambda\ ,\quad \forall\, a, b \in A\ ,\ \lambda \in k\ .$$

The differential on A_n is determined by the formula: $d(a_{ij}) = (da)_{ij}$ *. The universal* n*-dimensional representation of* A *is given by*

$$\pi_n : A \to \mathrm{Mat}_n(A_n)\ , \quad a \mapsto \|a_{ij}\|\ .$$

Later on we will work with augmented DG algebras. Note that, if A is augmented, then A_n has a natural augmentation $\varepsilon_n : A_n \to k$ coming from (3.7) applied to the augmentation map of A. This defines a functor $\mathsf{DGA}_{k/k} \to \mathsf{DGCA}_{k/k}$, which we again denote by $(-)_n$. On the other hand, the matrix algebra functor can be modified in the following way:

$$\mathrm{Mat}'_n(\text{–}) : \mathsf{DGCA}_{k/k} \to \mathsf{DGA}_{k/k}\ , \qquad \mathrm{Mat}'_n(B) := k \oplus \mathrm{Mat}_n(\overline{B})\ , \tag{4.3}$$

With this modification, Theorem 1.1.1 holds for augmented DG algebras. Moreover, as a special case of [6, Theorem 2.2], we have

Theorem 1.1.3. (a) *The adjoint functors* $(\,\text{–}\,)_n\ :\ \mathsf{DGA}_{k/k} \rightleftarrows \mathsf{DGCA}_{k/k}\ :\ \mathrm{Mat}'_n(\text{–})$ *form a Quillen pair.*

(b) *The functor* $(\,\text{–}\,)_n\ :\ \mathsf{DGA}_{k/k} \to \mathsf{DGCA}_{k/k}$ *has a total left derived functor defined by*

$$\mathbf{L}(\,\text{–}\,)_n : \mathsf{Ho}(\mathsf{DGA}_{k/k}) \to \mathsf{Ho}(\mathsf{DGCA}_{k/k})\ , \quad A \mapsto R_n\ ,$$

where $R \overset{\sim}{\twoheadrightarrow} A$ *is any cofibrant resolution of* A *in* $\mathsf{DGA}_{k/k}$*.*

(c) *For any* A *in* $\mathsf{DGA}_{k/k}$ *and* B *in* $\mathsf{DGCA}_{k/k}$*, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathsf{Ho}(\mathsf{DGCA}_{k/k})}(\mathbf{L}(A)_n\,,\,B)\ \cong\ \mathrm{Hom}_{\mathsf{Ho}(\mathsf{DGA}_{k/k})}(A\,,\,\mathrm{Mat}'_n(B))\,.$$

Definition 1.1.4. *Given* $A \in \mathsf{Alg}_{k/k}$*, we define* $\mathrm{DRep}_n(A) := \mathbf{L}(R)_n$*, where* $R \overset{\sim}{\twoheadrightarrow} A$ *is a cofibrant resolution of* A *in* $\mathsf{DGA}_{k/k}$*. The homology of* $\mathrm{DRep}_n(A)$ *is* *an augmented (graded) commutative algebra, which is independent of the choice of resolution (by Theorem 1.1.3). We set*

$$\mathrm{HR}_\bullet(A,n) \;:=\; \mathrm{H}_\bullet[\mathrm{DRep}_n(A)] \tag{4.4}$$

and call (4.4) *the n-dimensional representation homology of A.*

By [6, Theorem 2.5], for any $A \in \mathsf{Alg}_k$, there is a natural isomorphism of algebras

$$\mathrm{HR}_0(A,n) \cong A_n \ .$$

Hence $\mathrm{DRep}_n(A)$ may indeed be viewed as a 'higher' derived version of the representation functor (3.7).

1.2 GL-invariants

Recall from Section 1.1 the action of the group $\mathrm{GL}_n(k)$ on A_n. This action extends verbatim to the case of DG algebras, thus defining a subfunctor $(-)_n^{\mathrm{GL}}$ of the representation functor $(-)_n$ by

$$(-)_n^{\mathrm{GL}}:\ \mathsf{DGA}_{k/k} \to \mathsf{DGCA}_{k/k}\ ,\quad A \mapsto A_n^{\mathrm{GL}}\ . \tag{4.5}$$

On the other hand, there is a natural action of $\mathrm{GL}_n(k)$ on the n-th representation homology of A so we can form the invariant subalgebra $\mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}$. The next theorem, which is a consequence of [6, Theorem 2.6], shows that these two constructions agree.

Theorem 1.2.1. (*a*) *The functor $(-)_n^{\mathrm{GL}}$ has a total left derived functor*

$$\mathbf{L}[(-)_n^{\mathrm{GL}}]:\ \mathsf{Ho}(\mathsf{DGA}_{k/k}) \to \mathsf{Ho}(\mathsf{DGCA}_{k/k})\ ,\quad A \mapsto R_n^{\mathrm{GL}}\ .$$

(*b*) *For any* $A \in \mathsf{DGA}_{k/k}$, *there is a natural isomorphism of graded algebras*

$$\mathrm{H}_\bullet[\mathbf{L}(A)_n^{\mathrm{GL}}] \cong \mathrm{HR}_\bullet(A,n)^{\mathrm{GL}} \ .$$

Abusing notation we often write $\mathrm{DRep}_n(A)^{\mathrm{GL}}$ instead of $\mathbf{L}(A)_n^{\mathrm{GL}}$ for any DG algebra A.

The following Theorem describes the relation between representation homology and Lie cohomology.

Theorem 1.2.2 ([4], Theorem 3.1)**.** *Let* $A \in \mathsf{DGA}k/k$ *and let* $C \in \mathsf{DGC}_{k/k}$ *be a Koszul dual coalgebra of* A. *Then, for any* $n \geq 1$, *there are isomorphisms in* $\mathsf{Ho}(\mathsf{CDGA}_{k/k})$ *:*

$$\mathrm{DRep}_n(A) \;\cong\; \mathbf{C}^c(\mathfrak{gl}_n^*(\overline{C});k) \ , \quad \mathrm{DRep}_n(A)^{\mathrm{GL}} \;\cong\; \mathbf{C}^c(\mathfrak{gl}_n^*(C),\, \mathfrak{gl}_n^*(k);k)\, . \qquad (4.6)$$

Consequently,

$$\mathrm{HR}_\bullet(A,n) \;\cong\; \mathrm{H}_\bullet(\mathfrak{gl}_n^*(\overline{C});k) \ , \quad \mathrm{HR}_\bullet(A,n)^{\mathrm{GL}} \;\cong\; \mathrm{H}_\bullet(\mathfrak{gl}_n^*(C),\, \mathfrak{gl}_n^*(k);k)\, . \qquad (4.7)$$

1.3 Derived character maps

The category $\mathsf{DGA}_{k/k}$ of augmented DG algebras is equivalent to the under category $k\backslash\mathsf{DGA}_k$. The cyclic functor $\mathcal{C}\colon k\backslash\mathsf{DGA}_k \to \mathsf{Com}(k)$ defined in Section 3.2 sends

$$R \mapsto \bar{R}/[\overline{R},\overline{R}] \;\cong\; R/(k+[R,R]) \ ,$$

where for an augmented DG algebra $R \in \mathsf{DGA}_{k/k}$, we denote by $\overline{R} \subset R$ the kernel of the augmentation map of R. To distinguish this from the general case of under categories, we denote this functor by $(-)_\natural$, i.e. $R_\natural := \overline{R}/[\overline{R},\overline{R}]$.

The following theorem is a special case of Theorem 3.2.1 and Proposition 3.2.2.

Theorem 1.3.1 ([15]). *(a) The functor* $(\,-\,)_\natural$ *has a total left derived functor*

$$\mathbf{L}(\,-\,)_\natural : \mathsf{Ho}(\mathsf{DGA}_{k/k}) \to \mathsf{Ho}(\mathsf{Com}_k)\ , \quad A \mapsto R_\natural\ ,$$

where $R \stackrel{\sim}{\twoheadrightarrow} A$ *is a(ny) cofibrant resolution of* A *in* $\mathsf{DGA}_{k/k}$.

(b) For $A \in \mathsf{Alg}_{k/k}$*, there is a natural isomorphism of graded vector spaces*

$$\mathrm{H}_\bullet[\mathbf{L}(A)_\natural] \cong \overline{\mathrm{HC}}_\bullet(A)\ ,$$

where $\overline{\mathrm{HC}}_\bullet(A)$ *denotes the reduced cyclic homology of* A.

Now, fix $n \geq 1$ and, for $R \in \mathsf{DGA}_k$, consider the composite map

$$R \xrightarrow{\pi_n} \mathrm{Mat}_n(R_n) \xrightarrow{\mathrm{Tr}} R_n$$

where π_n is the universal representation of R and Tr is the usual matrix trace. This map factors through $R_\natural$ and its image lies in R_n^{GL}. Hence, we get a morphism of complexes

$$\mathrm{Tr}_n(R)_\bullet :\ R/[R,R] \to R_n^{\mathrm{GL}}\ , \tag{4.8}$$

which extends by multiplicativity to a map of graded commutative algebras

$$\mathbf{Sym}\ \mathrm{Tr}_n(R)_\bullet :\ \mathbf{Sym}(R/[R,R]) \to R_n^{\mathrm{GL}}\ . \tag{4.9}$$

If $R \in \mathsf{DGA}_{k/k}$ is augmented, the natural inclusion $\bar{R} \hookrightarrow R$ induces a morphism of complexes $R_\natural \to R/[R,R]$. Composed with (4.8) this defines a morphism of functors that extends to a morphism of the derived functors from $\mathsf{Ho}(\mathsf{DGA}_{k/k})$ to $\mathsf{Ho}(\mathsf{Com}\,k)$:

$$\mathbf{L}\,\mathrm{Tr}_n :\ \mathbf{L}(\,-\,)_\natural \to \mathbf{L}(-)_n^{\mathrm{GL}}\,. \tag{4.10}$$

Now, for an ordinary k-algebra $A \in \mathsf{Alg}_{k/k}$, applying (4.10) to a cofibrant resolution R of A in $\mathsf{DGA}_{k/k}$, taking homology and using the identification of Theorem 1.3.1(b), we get natural maps

$$\mathrm{Tr}_n(A)_\bullet :\ \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_\bullet(A,n)^{\mathrm{GL}}\ , \quad \forall\, n \geq 0\ . \tag{4.11}$$

In degree zero, $\mathrm{Tr}_n(A)_0$ is induced by the obvious linear map $A \to \mathcal{O}[\mathsf{Rep}_n(A)]^{\mathrm{GL}}$ defined by taking characters of representations. Thus, the higher components of (4.11) may be thought of as derived (or higher) characters of n-dimensional representations of A. For each $n \geq 1$, these characters assemble to a single homomorphism of graded commutative algebras which we denote

$$\mathbf{Sym}\ \mathrm{Tr}_n(A)_\bullet : \mathbf{Sym}\,[\overline{\mathrm{HC}}_\bullet(A)] \to \mathrm{H}_\bullet(A,n)^{\mathrm{GL}} .$$

An explicit formula evaluating $\mathrm{Tr}_n(A)_\bullet$ on cyclic chains is given in [6, Section 4.3].

2 Lie representation functor

2.1 Convolution Lie algebras

For two complexes $X, Y \in \mathsf{Com}(k)$ a map $f\colon X \to Y$ is said to be *of degree* m if $f(X_i) \subset Y_{i+m}$ for all $i \in \mathbf{Z}$. Note that f is not required to respect the differentials δ_X, δ_Y on X and Y. Denote by $\mathbf{Hom}(X,Y)_m$ the space of morphisms $X \to Y$ of degree m. The graded vector space $\mathbf{Hom}(X,Y) := \bigoplus_{m\in\mathbf{Z}} \mathbf{Hom}(X,Y)_m$ has a natural structure of a complex, with the differential δ given by

$$\delta(f) = f \circ \delta_X - (-1)^{|f|}\delta_Y \circ f$$

for any homogeneous $f \in \mathbf{Hom}(X,Y)$. In particular, $\delta(f) = 0$ if and only if f commutes with the differentials, i.e. if and only if f is a morphisms of complexes.

For a fixed $\mathfrak{C} \in \mathsf{DGLC}_k$ and $A \in \mathsf{DGCA}_k$, we define a Lie bracket on $\mathbf{Hom}(\mathfrak{C}, A)$ by

$$[f,\, g] := m \,\circ\, (f \otimes g) \circ]-[$$

where $m : A\otimes A \to A$ is the multiplication map on A, and $]-[: \mathfrak{C} \to \mathfrak{C}\otimes\mathfrak{C}$ is the Lie cobracket on $\mathfrak{C}$. For $\mathfrak{C} \in \mathsf{DGLC}_k$ fixed, this gives a functor

$$\mathbf{Hom}(\mathfrak{C}, -) : \mathsf{DGCA}_k \to \mathsf{DGLA}_k \,. \tag{4.12}$$

which we call a convolution functor.

2.2 The left adjoint functor

For arbitrary elements ξ, η in a DG Lie algebra $\mathfrak{a}$ and for $x \in \mathfrak{C}$, let $(]x[, \xi, \eta)$ denote the image of $x \otimes \xi \otimes \eta$ under the composite map

$$\mathfrak{C}\otimes\mathfrak{a}\otimes\mathfrak{a} \xrightarrow{]-[} \mathfrak{C}\otimes\mathfrak{C}\otimes\mathfrak{a}\otimes\mathfrak{a} \xrightarrow{\cong} (\mathfrak{C}\otimes\mathfrak{a})^{\otimes 2} \twoheadrightarrow \mathbf{Sym}^2(\mathfrak{C}\otimes\mathfrak{a}) \hookrightarrow \mathbf{Sym}(\mathfrak{C}\otimes\mathfrak{a})$$

The following proposition describes the left adjoint for the Lie convolution algebra functor $\mathbf{Hom}(\mathfrak{C}, -)$.

Proposition 2.2.1. *The functor* (4.12) *has a left adjoint* $(-)_{\mathfrak{C}} : \mathsf{DGLA}_k \to \mathsf{DGCA}_k$ *, which is given by*

$$\mathfrak{a} \mapsto \mathfrak{a}_{\mathfrak{C}} := \mathbf{Sym}_k(\mathfrak{C}\otimes\mathfrak{a})/\langle x\otimes[\xi,\eta] - (]x[, \xi, \eta)\rangle \ ,$$

where $(]x[, \xi, \eta)$ *is defined above.*

Proof. For $B \in \mathsf{DGCA}_k$ consider natural maps

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{DGLA}_k}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{C}, B)) \arrow[r, hook] \arrow[dd, "\simeq"] & \mathrm{Hom}_{\mathsf{Com}(k)}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{C}, B)) \arrow[d, "="] \\ & \mathrm{Hom}_{\mathsf{Com}(k)}(\mathfrak{C}\otimes\mathfrak{a}, B) \arrow[d, "\simeq"] \\ \mathrm{Hom}_{\mathsf{DGCA}_k}(\mathfrak{a}_{\mathfrak{C}},\ B) \arrow[r, hook] & \mathrm{Hom}_{\mathsf{DGCA}_k}(\mathbf{Sym}_k(\mathfrak{C}\otimes\mathfrak{a}), B) \end{tikzcd}$$

The map

$$\mathrm{Hom}_{\mathsf{DGA}_k}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{C}, B)) \hookrightarrow \mathrm{Hom}_{\mathsf{DGCA}_k}(\mathbf{Sym}_k(\mathfrak{C} \otimes \mathfrak{a}), B) \tag{4.13}$$

obtained by following the upper right part of the above diagram is explicitly given by

$$(f \,:\, \mathfrak{a} \to \mathbf{Hom}(\mathfrak{C}, B)) \mapsto \left[\hat{f} \,:\, \mathbf{Sym}_k(\mathfrak{C} \otimes_k \mathfrak{a}) \to B\,, \quad x \otimes \xi \mapsto (-1)^{|x||\xi|} f(\xi)(x)\right]\,.$$

Further, by a straightforward calculation, f is a DG Lie algebra homomorphism iff $f([\,\xi\,,\,\eta\,]) = [\,f(\xi)\,,\,f(\eta)]$ iff for all $\xi, \eta \in \mathfrak{a}$ and $x \in \mathfrak{C}$, $\hat{f}\,(x \otimes [\xi\,,\,\eta] - (]x[,\xi,\eta)) = 0$. This shows that the image of the map (4.13) is precisely $\mathrm{Hom}_{\mathsf{DGCA}_k}(\mathfrak{a}_{\mathfrak{C}}, B)$. This proves the desired proposition. □

As a consequence of Proposition 2.2.1, we obtain:

Theorem 2.2.2. *The pair of functors* $(\text{–})_{\mathfrak{C}} \,:\, \mathsf{DGLA}_k \rightleftarrows \mathsf{DGCA}_k \,:\, \mathbf{Hom}(\mathfrak{C}, \text{–})$ *is a Quillen pair. As a result, the functor* $(\text{–})_{\mathfrak{C}}$ *has a (total) left derived functor*

$$\mathbf{L}(\text{–})_{\mathfrak{C}} \,:\, \mathsf{Ho}(\mathsf{DGLA}_k) \to \mathsf{Ho}(\mathsf{DGCA}_k)\,, \quad \mathfrak{a} \mapsto \mathfrak{L}_{\mathfrak{C}}$$

where $\mathfrak{L} \stackrel{\sim}{\to} \mathfrak{a}$ *is any cofibrant resolution in* DGLA_k.

Proof. By Proposition 2.2.1 and [13, Remark 9.8], it suffices to check that $\mathbf{Hom}(\mathfrak{C}, \text{–})$ preserves degree-wise surjections and quasi-isomorphisms. This is obvious. □

The functor $\mathbf{Hom}(\mathfrak{C}, \text{–})$ can be modified naturally to give a functor on *augmented* commutative DG algebras

$$\mathbf{Hom}(\mathfrak{C}, \text{–}) \,:\, \mathsf{DGCA}_{k/k} \to \mathsf{DGLA}_k\,, \quad A \mapsto \mathbf{Hom}(\mathfrak{C}, \overline{A})\,.$$

The left adjoint $(-)_{\mathfrak{C}}$ of $\mathbf{Hom}(\mathfrak{C},–) : \mathsf{DGCA}_{k/k} \to \mathsf{DGLA}_k$ is the functor assigning to each $\mathfrak{a} \in \mathsf{DGLA}_k$ the commutative DG algebra $\mathfrak{a}_{\mathfrak{C}}$ equipped with the canonical augmentation

$$\varepsilon : \mathfrak{a}_{\mathfrak{C}} \to k$$

corresponding to the $0 \in \mathrm{Hom}_{\mathsf{DGLA}_k}(\mathfrak{a}, \mathbf{Hom}(\mathfrak{C}, k))$ under the adjunction (4.14). As in Theorem 2.2.2, it is easy to verify that the pair of functors $(-)_{\mathfrak{C}} : \mathsf{DGLA}_k \rightleftarrows \mathsf{DGCA}_{k/k} : \mathbf{Hom}(\mathfrak{C},–)$ is Quillen. Hence, $(-)_{\mathfrak{C}}$ has a left derived functor

$$\mathbf{L}(–)_{\mathfrak{C}} : \mathsf{Ho}(\mathsf{DGLA}_k) \to \mathsf{Ho}(\mathsf{DGCA}_{k/k})\,,$$

which, after applying the forgetful functor $\mathsf{Ho}(\mathsf{DGCA}_{k/k}) \to \mathsf{Ho}(\mathsf{DGCA}_k)$ coincides with $\mathbf{L}(-)_{\mathfrak{C}} : \mathsf{Ho}(\mathsf{DGLA}_k) \to \mathsf{Ho}(\mathsf{DGCA}_k)$.

2.3 Derived Lie representation schemes

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k. Let $\mathfrak{C} := \mathfrak{g}^*$, the dual Lie coalgebra. Then,

$$\mathbf{Hom}(\mathfrak{C}, A) = \mathbf{Hom}(\mathfrak{g}^*, A) \cong \mathfrak{g} \otimes A =: \mathfrak{g}(A)\,.$$

Thus, the commutative DG algebra $\mathfrak{a}_{\mathfrak{g}} := \mathfrak{a}_{\mathfrak{C}}$ represents the functor

$$\mathsf{Rep}_{\mathfrak{g}}(\mathfrak{a}) : \mathsf{DGCA}_k \to \mathsf{Sets}\,, \quad A \mapsto \mathrm{Hom}_{\mathsf{DGLA}_k}(\mathfrak{a}, \mathfrak{g}(A))\;,$$

that is, there is a natural isomorphism of sets

$$\mathrm{Hom}_{\mathsf{DGCA}_k}(\mathfrak{a}_{\mathfrak{g}}, A) \cong \mathrm{Hom}_{\mathsf{DGLA}_k}(\mathfrak{a}, \mathfrak{g}(A))\,. \tag{4.14}$$

As in the associative case, we now define

$$\mathrm{DRep}_{\mathfrak{g}}(-) := \mathbf{L}(-)_{\mathfrak{g}} : \mathsf{Ho}(\mathsf{DGLA}_k) \to \mathsf{Ho}(\mathsf{DGCA}_k)\,.$$

We call $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ *the derived representation scheme parametrizing representations of* $\mathfrak{a}$ *in* $\mathfrak{g}$. Further, if G is a Lie group whose Lie algebra is $\mathfrak{g}$, G acts (via the adjunction (4.14)) on $\mathfrak{a}_{\mathfrak{g}}$ by automorphisms for any $\mathfrak{a} \in \mathsf{DGLA}_k$. One can therefore, form the subfunctor

$$(-)^G_{\mathfrak{g}} \,:\, \mathsf{DGLA}_k \to \mathsf{DGCA}_k\,, \quad \mathfrak{a} \mapsto \mathfrak{a}^G_{\mathfrak{g}}$$

of $(-)_{\mathfrak{g}}$. An argument using Brown's lemma similar to the proof of [6, Theorem 2.6] shows that the functor $(-)^G_{\mathfrak{g}}$ has a total left derived functor

$$\mathrm{DRep}_{\mathfrak{g}}(-)^G \,:=\, \mathbf{L}(-)^G_{\mathfrak{g}} \,:\, \mathsf{Ho}(\mathsf{DGLA}_k) \to \mathsf{Ho}(\mathsf{DGCA}_k)\,.$$

We define the representation homologies

$$\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g}) \,:=\, \mathrm{H}_\bullet\left[\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})\right] \;\;,\;\; \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^G \,:=\, \mathrm{H}_\bullet\left[\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G\right]\,.$$

More generally, $\mathfrak{g}$ acts (via the adjunction (4.14)) on $\mathfrak{a}_{\mathfrak{g}}$ by derivations. One can therefore form the functor

$$(-)^{\mathrm{ad}\,\mathfrak{g}}_{\mathfrak{g}} \,:\, \mathsf{DGLA}_k \to \mathsf{DGCA}_k\,, \quad \mathfrak{a} \mapsto \mathfrak{a}^{\mathrm{ad}\,\mathfrak{g}}_{\mathfrak{g}}$$

Again, an argument paralleling the proof of [6, Theorem 2.6] shows that the functor $(-)^{\mathrm{ad}\,\mathfrak{g}}_{\mathfrak{g}}$ has a total left derived functor

$$\mathrm{DRep}_{\mathfrak{g}}(-)^{\mathrm{ad}\,\mathfrak{g}} \,:=\, \mathbf{L}(-)^{\mathrm{ad}\,\mathfrak{g}}_{\mathfrak{g}} : \mathsf{Ho}(\mathsf{DGLA}_k) \to \mathsf{Ho}(\mathsf{DGCA}_k)\,.$$

We define

$$\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \,:=\, \mathrm{H}_\bullet\left[\mathrm{DRep}^{\mathrm{ad}\,\mathfrak{g}}_{\mathfrak{g}}(\mathfrak{a})\right]\,.$$

Note that if $\mathfrak{g}$ is the Lie algebra of a reductive Lie group G, the functors $(-)^G_{\mathfrak{g}}$ and $(-)^{\mathrm{ad}\,\mathfrak{g}}_{\mathfrak{g}}$ coincide. Hence, in this situation, their derived functors coincide as well.

The discussion in Section 2.2 points out that $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ can be viewed as an object in $\mathsf{Ho}(\mathsf{DGCA}_{k/k})$ (rather than $\mathsf{Ho}(\mathsf{DGCA}_k)$). In the same way, $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad}\,\mathfrak{g}}$

can be viewed as an object in $\mathsf{Ho}(\mathsf{DGCA}_{k/k})$. Similarly, if G is a Lie group whose Lie algebra is $\mathfrak{g}$, one can consider $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G$ as an object in $\mathsf{Ho}(\mathsf{DGCA}_{k/k})$. In this case, $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \cong \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad}\,\mathfrak{g}}$.

3 Representation homology and Lie cohomology

3.1 Relation to DG algebras and linear duality

We now introduce the following functors on coalgebras: the co-abelianization functor $(-)^{\mathrm{ab}} : \mathsf{DGC}_{k/k} \to \mathsf{DGCC}_{k/k}$ assigning to a DG coalgebra C its maximal cocommutative DG subcoalgebra $C^{\mathrm{ab}} \subseteq C$ (this functor is dual to the abelianization functor defined in Theorem 1.1.4); the universal co-enveloping coalgebra functor $\mathcal{U}^c : \mathsf{DGLC}_k \to \mathsf{DGC}_{k/k}$ dual to the universal enveloping algebra functor $\mathcal{U} : \mathsf{DGLA}_k \to \mathsf{DGA}_{k/k}$; the Lie coalgebra functor $\mathcal{L}ie^c : \mathsf{DGC}_{k/k} \to \mathsf{DGLC}_k$ assigning to each C the co-augmentation coideal $\bar{C}$ viewed as a DG Lie coalgebra (this functor is dual to the Lie algebra functor $\mathcal{L}ie : \mathsf{DGA}_{k/k} \to \mathsf{DGLA}_k$ assigning to $A \in \mathsf{DGA}_{k/k}$ the augmentation ideal $\bar{A}$ viewed as a DG Lie algebra. Their relationship to Quillen equivalences (2.1) and (2.2) is summarized by the following theorem.

Theorem 3.1.1. *In each of the following diagrams the square subdiagrams obtained by starting at any corner and mapping to the opposite corner commute (up to isomorphism).*

$$
\begin{tikzcd}
\mathsf{DGC}_{k/k} \arrow[r, shift left, "\mathbf{\Omega}"] \arrow[d, shift left, "(-)^{\mathrm{ab}}"] & \mathsf{DGA}_{k/k} \arrow[l, shift left, "\mathbf{B}"] \arrow[d, shift left, "\mathcal{L}ie"] \\
\mathsf{DGCC}_{k/k} \arrow[u, hook, shift left, "in"] \arrow[r, shift left, "\mathbf{\Omega}_{\mathsf{Comm}}"] & \mathsf{DGLA}_k \arrow[l, shift left, "\mathbf{B}_{\mathsf{Lie}}"] \arrow[u, shift left, "\mathcal{U}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{DGC}_{k/k} \arrow[r, shift left, "\mathbf{\Omega}"] \arrow[d, shift left, "\mathcal{L}ie^c"] & \mathsf{DGA}_{k/k} \arrow[l, shift left, "\mathbf{B}"] \arrow[d, shift left, "(-)_{\mathrm{ab}}"] \\
\mathsf{DGLC}_k \arrow[u, shift left, "\mathcal{U}^c"] \arrow[r, shift left, "\mathbf{\Omega}_{\mathsf{Lie}}"] & \mathsf{DGCA}_{k/k} \arrow[l, shift left, "\mathbf{B}_{\mathsf{Comm}}"] \arrow[u, hook, shift left, "in"]
\end{tikzcd}
\tag{4.15}
$$

Proof. Let $C \in \mathsf{DGCC}_{k/k}$. Let $\mathfrak{L}V$ denote the free Lie algebra generated by a graded vector space V. Since $T_k V \cong \mathcal{U}(\mathfrak{L}V)$ as graded k-algebras, we have an isomorphism of graded algebras

$$\mathbf{\Omega}(C) \cong \mathcal{U}[\mathbf{\Omega}_{\mathsf{Comm}}(C)] .$$

The fact that this isomorphism commutes with differentials follows from the fact that the coalgebra C is cocommutative. Hence, on the category of cocommutative DG coalgebras, we have an isomorphism of functors

$$\mathbf{\Omega} \cong \mathcal{U} \circ \mathbf{\Omega}_{\mathsf{Comm}} . \tag{4.16}$$

By adjunction, this gives an isomorphism

$$\mathbf{B}_{\mathsf{Lie}} \circ \mathcal{L}ie \cong (-)^{\mathrm{ab}} \circ \mathbf{B} . \tag{4.17}$$

which proves the result for the first diagram in (4.15). A similar argument shows that

$$(-)_{\mathrm{ab}} \circ \mathbf{\Omega} \cong \mathbf{\Omega}_{\mathsf{Lie}} \circ \mathcal{L}ie^{c} . \tag{4.18}$$

which by adjunction (together with (4.18)) gives an isomorphism of functors on commutative DG algebras:

$$\mathbf{B} \cong \mathcal{U}^{c} \circ \mathbf{B}_{\mathsf{Comm}} . \tag{4.19}$$

This establishes the desired result for the second diagram in (4.15). $\square$

The following theorem is an immediate consequence of [40, Theorem 4.17], which explains the canonical linear dualities relating Lie algebraic and coalgebraic Quillen functors.

Theorem 3.1.2. (i) *For $\mathfrak{g} \in \mathsf{DGLA}_k$, there is a natural isomorphism*

$$\mathbf{\Omega}_{\mathsf{Lie}}(\mathfrak{g}^*) \cong \mathbf{B}_{\mathsf{Lie}}(\mathfrak{g})^* . \tag{4.20}$$

(ii) *For* $\mathfrak{C} \in \mathsf{DGLC}_k$*, we have a natural isomorphism*

$$\mathbf{B}_{\mathsf{Lie}}(\mathfrak{C}^*) \;\cong\; \mathbf{\Omega}_{\mathsf{Lie}}(\mathfrak{C})^*\,. \tag{4.21}$$

Remark. Note that $\mathbf{B}_{\mathsf{Lie}}(\mathfrak{g})^*$ is precisely the complex of Lie *cochains* of $\mathfrak{g}$ with trivial coefficients. In particular, $\mathrm{H}_\bullet[\mathbf{B}_{\mathsf{Lie}}(\mathfrak{g})^*] \;=\; \mathrm{H}^{-\bullet}(\mathfrak{g};k)$ (where $\mathrm{H}^\bullet$ denotes Lie *cohomology*). Thus, the isomorphism 4.20 explicitly relates the homological and cohomological Chevalley-Eilenberg complexes. In addition, Theorem 3.1.1 and Theorem 3.1.2 hold in the bigraded setting. In that setting, $(-)^*$ means taking the bigraded dual.

3.2 Representation homology and Lie cohomology

Proposition 3.2.1. *For any* $\mathfrak{C} \in \mathsf{DGLC}_k$*, the following diagram commutes (upto isomorphism of functors):*

$$\begin{tikzcd}
\mathsf{DGCC}_{k/k} \arrow[r, shift left, "\mathbf{\Omega}_{\mathsf{Comm}}"] \arrow[d, "\mathfrak{C}\bar{\otimes}-"'] & \mathsf{DGLA}_k \arrow[l, shift left, "\mathbf{B}_{\mathsf{Lie}}"] \arrow[d, shift right, "(-)_{\mathfrak{C}}"'] \\
\mathsf{DGLC}_k \arrow[r, "\mathbf{\Omega}_{\mathsf{Lie}}"] & \mathsf{DGCA}_{k/k} \arrow[u, shift right, "\mathbf{Hom}(\mathfrak{C},-)"']
\end{tikzcd}$$

where $\mathfrak{C}\bar{\otimes}C \;:=\; \mathfrak{C}\otimes\overline{C}$ *for any* $C \in \mathsf{DGCC}_{k/k}$*. Thus, there is an isomorphism of functors*

$$(\,-)_{\mathfrak{C}} \circ \mathbf{\Omega}_{\mathsf{Comm}} \;\cong\; \mathbf{\Omega}_{\mathsf{Lie}} \circ (\mathfrak{C}\bar{\otimes}\,-) \;=\; \mathrm{CE}^c \circ (\mathfrak{C}\bar{\otimes}\,-)\,. \tag{4.22}$$

Proof. For any $C \in \mathsf{DGCC}_{k/k}$ and $\mathfrak{L} \in \mathsf{DGLA}_k$, let $\mathsf{Tw}(C,\mathfrak{L})$ denote the set of Maurer-Cartan elements (i.e, elements satisfying $d\alpha + \frac{1}{2}[\alpha,\alpha] = 0$) in the DG Lie algebra $\mathbf{Hom}(\overline{C}, L)$. Similarly, for $\mathfrak{L}^c \in \mathsf{DGLC}_k$ and $A \in \mathsf{DGA}_{k/k}$, $\mathsf{Tw}(\mathfrak{L}^c, A)$ shall denote the set of Maurer-Cartan elements in the DG Lie algebra $\mathbf{Hom}(\mathfrak{L}^c, \overline{A})$. Now, for any $A \in \mathsf{DGCA}_{k/k}$ and $C \in \mathsf{DGCC}_{k/k}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{DGCA}_{k/k}}(\boldsymbol{\Omega}_{\mathsf{Comm}}(C)_{\mathfrak{C}}, A) &\cong \mathrm{Hom}_{\mathsf{DGLA}_k}(\boldsymbol{\Omega}_{\mathsf{Comm}}(C), \mathbf{Hom}(\mathfrak{C}, \overline{A})) \\
&\cong \mathsf{Tw}(C\,,\, \mathbf{Hom}(\mathfrak{C}, \overline{A})) \\
&\cong \mathsf{Tw}(\mathfrak{C} \bar{\otimes} C, A) \\
&\cong \mathrm{Hom}_{\mathsf{DGCA}_{k/k}}(\mathrm{CE}^c(\mathfrak{C} \bar{\otimes} C), A)\,.
\end{aligned}$$

The first isomorphism above is Proposition 2.2.1, the second is from [20, Theorem 2.2.5], the third is because the DG Lie algebras $\mathbf{Hom}(\overline{C}, \mathbf{Hom}(\mathfrak{C}, A))$ and $\mathbf{Hom}(\mathfrak{C} \bar{\otimes} C, \overline{A})$ are isomorphic by the standard hom-tensor duality, and the fourth is from arguments similar to those in [20] proving the second. (4.22) follows from this by Yoneda's lemma. The rest of the desired proposition follows from (4.22) by adjunction. $\square$

Theorem 3.2.2. *(a) Suppose that $\mathfrak{C} = \mathfrak{g}^*$ for some Lie algebra $\mathfrak{g}$. If $\boldsymbol{\Omega}_{\mathsf{Comm}}(C) \xrightarrow{\sim} \mathfrak{a}$ is a quasi-isomorphism for some $C \in \mathsf{DGCC}_{k/k}$, then*

$$\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathrm{CE}^c(\mathfrak{g}^*(\overline{C}); k)\,, \quad \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad}\,\mathfrak{g}} \cong \mathrm{CE}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k)\,.$$

In particular,

$$\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g}) \cong \mathrm{H}_\bullet(\mathfrak{g}^*(\overline{C}); k)\,, \quad \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \cong \mathrm{H}_\bullet(\mathfrak{g}^*(C), \mathfrak{g}^*; k)\,.$$

(b) Suppose, in addition, that $\mathfrak{g}$ and C are finite-dimensional. Let $A = C^*$. Then,

$$\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathrm{CE}(\mathfrak{g}(\overline{A}); k)^*\,, \quad \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad}\,\mathfrak{g}} \cong \mathrm{CE}(\mathfrak{g}(A), \mathfrak{g}; k)^*\,.$$

In particular,

$$\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g}) \cong \mathrm{H}^{-\bullet}(\mathfrak{g}(\overline{A}); k)\,, \quad \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \cong \mathrm{H}^{-\bullet}(\mathfrak{g}(A), \mathfrak{g}; k)\,.$$

Proof. That $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}) \cong \mathrm{CE}^c(\mathfrak{g}^*(\overline{C}); k)$ is immediate from Proposition 3.2.1. That $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad}\,\mathfrak{g}} \cong \mathrm{CE}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k)$ then follows from the fact that $\mathrm{CE}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) = \mathrm{CE}^c(\mathfrak{g}^*(\overline{C}); k)^{\mathrm{ad}\,\mathfrak{g}}$. This proves (a). Part (b) follows from (a) and (4.20). $\square$

Remark. Recall that, by (4.16), $\mathbf{\Omega}(C) \cong \mathcal{U}(\mathbf{\Omega}_{\mathsf{Comm}}(C))$ for any $C \in \mathsf{DGCC}_{k/k}$.

Now, take $\mathfrak{g} = \mathfrak{gl}_n(k)$ and $\mathfrak{C} = \mathfrak{g}^* \in \mathsf{DGLC}_k$ and $M = \mathrm{Mat}_n(k)^* \in \mathsf{DGC}_k$. The following proposition clarifies the relation between derived representation schemes of Lie algebras and their associative counterparts.

Proposition 3.2.3. *Let $C \in \mathsf{DGCC}_{k/k}$. Then there is a natural isomorphism of commutative DG algebras*

$$\mathbf{\Omega}(C)_n \cong \mathbf{\Omega}_{\mathsf{Comm}}(C)_{\mathfrak{gl}_n} . \tag{4.23}$$

Hence, for any DG Lie algebra $\mathfrak{a}$,

$$\mathrm{DRep}_n(\mathcal{U}(\mathfrak{a})) \cong \mathrm{DRep}_{\mathfrak{gl}_n}(\mathfrak{a}) . \tag{4.24}$$

Proof. For any $B \in \mathsf{DGCA}_k$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{DGCA}_k}(\mathbf{\Omega}(C)_n, B) &\cong \mathrm{Hom}_{\mathsf{DGA}_k}(\mathbf{\Omega}(C), \mathrm{Mat}_n(B)) \\
&\cong \mathrm{Hom}_{\mathsf{DGA}_k}(\mathcal{U}(\mathbf{\Omega}_{\mathsf{Comm}}(C)), \mathrm{Mat}_n(B)) \\
&\cong \mathrm{Hom}_{\mathsf{DGLA}_k}(\mathbf{\Omega}_{\mathsf{Comm}}(C), \mathfrak{gl}_n(B)) \\
&\cong \mathrm{Hom}_{\mathsf{DGCA}_k}(\mathbf{\Omega}_{\mathsf{Comm}}(C)_{\mathfrak{gl}_n}, B)
\end{aligned}$$

The isomorphism (4.23) now follows from Yoneda's lemma. To prove (4.24), note that for any $\mathfrak{a} \in \mathsf{DGLA}_k$, one can find $C \in \mathsf{DGCC}_{k/k}$ such that $\mathbf{\Omega}_{\mathsf{Comm}}(C) \to \mathfrak{a}$

is a cofibrant resolution in DGLA_k (for example, $C = \mathbf{B}_{\mathsf{Lie}}(\mathfrak{a})$). Then, $\mathbf{\Omega}(C) \cong \mathcal{U}(\mathbf{\Omega}_{\mathsf{Comm}}(C)) \to \mathcal{U}(\mathfrak{a})$ is a cofibrant resolution of $\mathcal{U}(\mathfrak{a})$ in DGA_k. Hence,

$$\mathrm{DRep}_n(\mathcal{U}(\mathfrak{a})) \cong \mathbf{\Omega}(C)_n \cong \mathbf{\Omega}_{\mathsf{Comm}}(C)_{\mathfrak{gl}_n} \cong \mathrm{DRep}_{\mathfrak{gl}_n}(\mathfrak{a}) .$$

This completes the proof of the proposition. □

4 Drinfeld homology and Drinfeld trace map

Throughout this section, $\mathfrak{g}$ will denote a finite-dimensional reductive Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ its Cartan subalgebra and $\mathbf{W}$ the Weyl group of $\mathfrak{g}$. Recall that in this case, if V is any k-vector space with an action of the reductive group G whose Lie algebra is $\mathfrak{g}$, $V^G \cong V^{\mathrm{ad}\,\mathfrak{g}}$.

The aim of this section is to construct the derived Harish-Chandra homomorphism $\Phi_{\mathfrak{g}}(\mathfrak{a})$ and Drinfeld trace maps $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})$. These two maps relate representation homology $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})$ to more computable invariants. Namely, $\Phi_{\mathfrak{g}}(\mathfrak{a})$ gives relation to the representation homology $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{h})^{\mathbf{W}}$ of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ invariant under the Weyl group action. Drinfeld traces $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})$ relate $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{g})$ to something called Drinfeld homology (see below) which is an analog of cyclic homology for associative algebras.

4.1 The derived Harish-Chandra homomorphism

The natural inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ defines a homomorphism of Lie coalgebras $\mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$ and hence, for any $C \in \mathsf{DGCC}_{k/k}$, a morphism of commutative DG algebras

$$\Phi_{\mathfrak{g}}(C) \,:\, \mathrm{CE}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \to \mathrm{CE}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k) . \tag{4.25}$$

Proposition 4.1.1. *The image of $\Phi_{\mathfrak{g}}(C)$ lies in the DG subalgebra $\mathrm{CE}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^{\mathbf{W}}$ of chains that are invariant under the action of the Weyl group $\mathbf{W}$ of $\mathfrak{g}$.*

Proof. As $\mathfrak{h}$ is an abelian Lie algebra, we have an isomorphism

$$\mathrm{CE}^c(\mathfrak{h}^*(C), \mathfrak{h}^*(k); k) \cong \mathrm{CE}^c(\mathfrak{h}^*(\overline{C}); k)$$

The map $\Phi_{\mathfrak{g}}(C)$ is thus the restriction of the natural map $\mathrm{CE}^c(\mathfrak{g}^*(C); k) \to \mathrm{CE}^c(\mathfrak{h}^*(\overline{C}); k)$ to $\mathrm{CE}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \cong \mathrm{CE}^c(\mathfrak{g}^*(\overline{C}); k)^{\mathrm{ad}\,\mathfrak{g}} \cong \mathrm{CE}^c(\mathfrak{g}^*(\overline{C}); k)^G$, where G is the Lie group attached to $\mathfrak{g}$. Now, let N denote the normalizer of $\mathfrak{h}$ in G, so that there is a surjective group homomorphism $N \twoheadrightarrow \mathbf{W}$. Since $\mathbf{W}$ acts naturally on $\mathfrak{h}^*$, so does N. Thus, N acts on $\mathfrak{h}^*(\overline{C})$ as well, making $\mathfrak{h}^*(\overline{C})$ a DG-Lie coalgebra with N-action. This, in turn, induces an N-action on the commutative DG algebra $\mathrm{CE}^c(\mathfrak{h}^*(\overline{C}); k)$. On the other hand, the adjoint action of G on $\mathfrak{g}$ makes $g^*(\overline{C})$ a DG Lie coalgebra with G- (and hence, N-)action. Thus, the commutative DG algebra $\mathrm{CE}^c(g^*(\overline{C}); k)$ acquires a G- (and hence, N-) action. Since the map $\mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$ is N-equivariant, the map $\mathfrak{g}^*(\overline{C}) \twoheadrightarrow \mathfrak{h}^*(\overline{C})$ is N-equivariant as well. Therefore, the map $\mathrm{CE}^c(\mathfrak{g}^*(\overline{C}); k) \to \mathrm{CE}^c(\mathfrak{h}_n^*(\overline{C}); k)$ is N-equivariant. Since any element of $\mathrm{CE}^c(\mathfrak{g}^*(C), \mathfrak{g}^*(k); k)$ is G-invariant (and hence, N-invariant), any element in the image of $\Phi_{\mathfrak{g}}(C)$ is N-invariant (and hence, $\mathbf{W}$-invariant). $\square$

Thus, we have a morphism of commutative DG algebras

$$\Phi_{\mathfrak{g}}(C) \,:\, \mathrm{CE}^c(\mathfrak{g}^*(C), \mathfrak{g}^*; k) \to \mathrm{CE}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^{\mathbf{W}},$$

which we call the *derived Harish-Chandra homomorphism.* Suppose that there exists a quasi-isomorphism $\boldsymbol{\Omega}_{\mathsf{Comm}}(C) \overset{\sim}{\twoheadrightarrow} \mathfrak{a}$ for some Lie algebra $\mathfrak{a}$. By Theorem 3.2.2, the derived Harish-Chandra homomorphism can be viewed as a map in $\mathsf{Ho}(\mathsf{DGCA}_{k/k})$

$$\Phi_{\mathfrak{g}}(C) \colon \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^G \to \mathrm{CE}^c(\mathfrak{h}^*(C), \mathfrak{h}^*; k)^{\mathbf{W}}.$$

It follows that the derived Harish-Chandra homomorphism induces the map

$$\mathrm{H}_\bullet(\Phi_{\mathfrak{g}}) \,:\, \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})^G \to \mathrm{H}_\bullet(\mathfrak{h}^*(C),\mathfrak{h}^*;k)^{\mathbf{W}}\,.$$

If C has zero differential, then $\mathrm{CE}^c(\mathfrak{h}^*(C),\mathfrak{h}^*;k)$ has also zero differential, and hence in this case, $\mathrm{H}_\bullet(\Phi_{\mathfrak{g}})$ maps $\mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})^G$ to $\mathbf{Sym}_k(\mathfrak{h}^*\otimes\overline{C}[-1])^{\mathbf{W}}$.

Example 4.1.2. Let $k=\mathbf{C}$ and let $\mathfrak{a} := \mathbf{C}.x$ be the one-dimensional Lie algebra with x having weight 1 and homological degree 0. The cocommutative coalgebra $C := \mathbf{Sym}^c(\mathfrak{a}[1])$ is Koszul dual to $\mathfrak{a}$. In this case, $\mathfrak{g}^*(C) \cong \mathfrak{g}^*.(sx)\oplus\mathfrak{g}^*$ and $\mathrm{CE}^c(\mathfrak{g}^*(C),\mathfrak{g}^*;\mathbf{C}) = \mathrm{Sym}(\mathfrak{g}^*)^G$. In this case, the derived Harish-Chandra homomorphism becomes the Chevalley isomorphism

$$\mathrm{Sym}(\mathfrak{g}^*)^G \xrightarrow{\sim} \mathrm{Sym}(\mathfrak{h}^*)^{\mathbf{W}}\,.$$

4.2 Drinfeld functor and Drinfeld homology

In [12], Drinfeld introduced the functor

$$\lambda:\,\mathsf{DGLA}_k\to\mathsf{Com}(k)\;,\quad \mathfrak{a}\mapsto\mathrm{Sym}^2(\mathfrak{a})/\langle[x,y]\cdot z - x\cdot[y,z]\,:\,x,y,z\in\mathfrak{a}\rangle$$

that assigns to a Lie algebra $\mathfrak{a}$ (the target of) the universal invariant bilinear form on $\mathfrak{a}$. As shown in [18], this functor plays a role of the cyclic functor (2.4) on the category of Lie algebras: its left derived $\mathbf{L}\lambda$ exists and defines the analogue of cyclic homology for Lie algebras (*cf.* [18, Theorem (5.3)]).

More generally, extending Drinfeld's construction, for an integer $d\geq 1$, we define

$$\lambda^{(d)}\,:\,\mathsf{DGLA}_k\to\mathsf{Com}(k)\;,\quad \mathfrak{a}\mapsto\mathrm{Sym}^d(\mathfrak{a})/[\mathfrak{a},\mathrm{Sym}^d(\mathfrak{a})]\,.$$

This functor assigns to a Lie algebra $\mathfrak{a}$ (the target of) the universal invariant multilinear form of degree d on $\mathfrak{a}$; in particular, for $d=2$, we have $\lambda^{(2)}=\lambda$.

These are the extension of the functors from Section 1.2 to the category of DG Lie algebras.

Note that the symmetric invariant d-multilinear forms $\mathfrak{g} \times \ldots \times \mathfrak{g} \to k$ are in one-to-one correspondence with linear maps $\lambda^{(d)}(\mathfrak{g}) \to k$. To be precise, the nondegenerate pairing

$$\mathrm{Sym}^d(\mathfrak{g}) \times \mathrm{Sym}^d(\mathfrak{g}^*) \to k$$

induces a nondegenerate pairing

$$\lambda^{(d)}(\mathfrak{g}) \times \mathrm{Sym}^d(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}} \to k\,.$$

The next theorem generalizes the result of [18, Theorem (5.3)] in the case of the Lie operad.

Theorem 4.2.1. *For each $d \geq 1$, the functor $\lambda^{(d)}$ has a (total) left derived functor given by*

$$\mathbf{L}\lambda^{(d)} : \mathsf{Ho}(\mathsf{DGLA}_k) \to \mathsf{Ho}(\mathsf{Com}_k)\,, \quad \mathfrak{a} \mapsto \lambda^{(d)}(\mathfrak{L})\,,$$

where $\mathfrak{L} \xrightarrow{\sim} \mathfrak{a}$ is a cofibrant resolution of $\mathfrak{a}$ in DGLA_k.

Proof. Suppose that $\mathfrak{L} \in \mathsf{DGLA}_k$ is cofibrant and that $f, g : \mathfrak{L} \to \mathfrak{a}$ are homotopic. Then, there exists $h : \mathfrak{L} \to \mathfrak{a} \otimes k[t, dt]$ such that $h(0) = f$ and $h(1) = g$. Here, $\deg(t) = 0$ and $d(t) := dt$ and $h(a)$ denotes postcomposition of h with the map $\mathrm{id}_{\mathfrak{a}} \otimes \mathrm{ev}_a$ where $\mathrm{ev}_a : k[t, dt] \to k$ is the map taking t to a and dt to 0 for any $a \in k$. Note that for any $B \in \mathsf{DGCA}_k$, one has natural maps in Com_k

$$\lambda^{(d)}(\mathfrak{a} \otimes B) \to \lambda^{(d)}(\mathfrak{a}) \otimes B\,. \tag{4.26}$$

One therefore, has a map $H : \lambda^{(d)}(\mathfrak{L}) \to \lambda^{(d)}(\mathfrak{a}) \otimes k[t, dt]$ given by the composition

$$\lambda^{(d)}(\mathfrak{L}) \xrightarrow{\lambda^{(d)}(h)} \lambda^{(d)}(\mathfrak{a} \otimes k[t, dt]) \xrightarrow{(4.26)} \lambda^{(d)}(\mathfrak{a}) \otimes k[t, dt]$$

Clearly, $H(0) = \lambda^{(d)}(f)$ and $H(1) = \lambda^{(d)}(g)$. It follows that the maps $\lambda^{(d)}(f)$ and $\lambda^{(d)}(g)$ are homotopic in Com_k.

Now, if $f : \mathfrak{L} \to \mathfrak{L}'$ is a weak equivalence between cofibrant objects in DGLA_k, there exists a $g : \mathfrak{L}' \to \mathfrak{L}$ in DGLA_k such that fg and gf are homotopic to the respective identities. It follows that $\lambda^{(d)}(fg)$ and $\lambda^{(d)}(gf)$ are homotopic to the respective identities as well. Hence, $\lambda^{(d)}(f)$ is a quasi-isomorphism. In other words, the functors $\lambda^{(d)}$ take weak equivalences between cofibrant objects to weak equivalences. The desired theorem now follows from Brown's lemma (*cf.* [13, Lemma 9.9]). $\square$

Definition 4.2.2. *Drinfeld homology is the homology* $\mathrm{H}_\bullet[\mathbf{L}\lambda^{(d)}(\mathfrak{a})]$ *, and is denoted by* $\mathrm{HC}^{(d)}_\bullet(\mathsf{Lie}, \mathfrak{a})$ *.*

Remark 4.2.3. Note that $\lambda^{(1)}$ is just the abelianization functor: $\mathfrak{a} \mapsto \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}]$, and hence $\mathrm{HC}^{(1)}_\bullet(\mathsf{Lie}, \mathfrak{a}) \cong \mathrm{H}_{\bullet+1}(\mathfrak{a}; k)$ for any Lie algebra $\mathfrak{a}$ (see, e.g., [5, Example 1, Sect.2.6]).

For $d = 2$, $\mathrm{HC}^{(2)}_\bullet(\mathsf{Lie}, \mathfrak{a})$ is precisely the Lie cyclic homology introduced in [18] and denoted $\mathrm{HA}_\bullet(\mathsf{Lie}, \mathfrak{a})$ in that paper.

4.3 Relation to cyclic homology

In general, the meaning of the homology groups $\mathrm{HC}^{(d)}_\bullet(\mathsf{Lie}, \mathfrak{a})$ is clarified by the following theorem.

Theorem 4.3.1. *Let* $\mathfrak{a} \in \mathsf{DGLA}_k$*. The reduced cyclic homology of the universal enveloping algebra* $\mathcal{U}(\mathfrak{a})$ *of the Lie algebra* $\mathfrak{a}$ *has a natural Hodge-type decomposition*

$$\overline{\mathrm{HC}}_\bullet[\mathcal{U}(\mathfrak{a})] \cong \bigoplus_{d=1}^{\infty} \mathrm{HC}^{(d)}_\bullet(\mathsf{Lie}, \mathfrak{a}) . \tag{4.27}$$

Proof. Let $C \in \mathsf{DGCC}_{k/k}$ be a coalgebra Koszul dual to the Lie algebra $\mathfrak{a}$ (for example, $C = \mathbf{B}_{\mathsf{Lie}}(\mathfrak{a})$). Then we have a cofibrant resolution $\boldsymbol{\Omega}_{\mathsf{Comm}}(C) \stackrel{\sim}{\to} \mathfrak{a}$ in DGLA_k. For a graded k-vector space V, there are natural isomorphisms of graded vector spaces

$$T_k(V)_\natural \;\cong\; T_k(V)/(k + [V, T_k(V)]) \cong T_k(V)/(k + [L_k(V), T_k(V)]) \;,$$

where $L_k(V) \subset T_k(V)$ is the free (graded) Lie algebra generated by V. It follows that

$$\boldsymbol{\Omega}(C)_\natural \;\cong\; \boldsymbol{\Omega}(C)/(k + [\boldsymbol{\Omega}_{\mathsf{Comm}}(C), \boldsymbol{\Omega}(C)]) \tag{4.28}$$

as complexes of k-vector spaces. By (4.16), $\boldsymbol{\Omega}(C) \;\cong\; \mathcal{U}(\boldsymbol{\Omega}_{\mathsf{Comm}}(C))$. On the other hand, since $\boldsymbol{\Omega}_{\mathsf{Comm}}(C)$ is a DG Lie algebra, we have an isomorphism of DG $\boldsymbol{\Omega}_{\mathsf{Comm}}(C)$-modules

$$\mathbf{Sym}_k[\boldsymbol{\Omega}_{\mathsf{Comm}}(C)] \stackrel{\sim}{\to} \mathcal{U}_k[\boldsymbol{\Omega}_{\mathsf{Comm}}(C)] \tag{4.29}$$

given by the symmetrization map. Therefore, writing

$$\mathrm{Sym}^d[\boldsymbol{\Omega}_{\mathsf{Comm}}(C)]_\natural \;:=\; \frac{\mathrm{Sym}^d(\boldsymbol{\Omega}_{\mathsf{Comm}}(C))}{[\boldsymbol{\Omega}_{\mathsf{Comm}}(C), \mathrm{Sym}^d(\boldsymbol{\Omega}_{\mathsf{Comm}}(C))]} \;,$$

we get the following decomposition

$$\boldsymbol{\Omega}(C)_\natural \;\cong\; \bigoplus_{d=1}^{\infty} \mathrm{Sym}^d[\boldsymbol{\Omega}_{\mathsf{Comm}}(C)]_\natural \;. \tag{4.30}$$

Note that $\boldsymbol{\Omega}(C) \stackrel{\sim}{\to} \mathcal{U}(\mathfrak{a})$ is a cofibrant resolution in $\mathsf{DGA}_{k/k}$. By [6, Proposition 3.1]), we have

$$\mathrm{H}_\bullet[\boldsymbol{\Omega}(C)_\natural] \;\cong\; \overline{\mathrm{HC}}_\bullet(\mathcal{U}(\mathfrak{a}))\,.$$

On the other hand,

$$\mathrm{H}_\bullet[\mathrm{Sym}^d(\boldsymbol{\Omega}_{\mathsf{Comm}}(C))_\natural] \;\cong\; \mathrm{HC}_\bullet^{(d)}(\mathsf{Lie}, \mathfrak{a})$$

since C is Koszul dual to $\mathfrak{a}$. This proves the desired theorem. $\square$

4.4 Lie-Hodge decomposition

Let $\mathfrak{a} \in \mathsf{DGLA}_k$ and let $C \in \mathsf{DGCC}_{k/k}$ be Koszul dual to $\mathfrak{a}$. By (4.16), $\boldsymbol{\Omega}(C) \cong \mathcal{U}(\boldsymbol{\Omega}_{\mathsf{Comm}}(C))$. From this isomorphism, $\boldsymbol{\Omega}(C)$ acquires the structure of a (primitively generated) cocommutative DG Hopf algebra whose DG Lie algebra of primitives is $\boldsymbol{\Omega}_{\mathsf{Comm}}(C)$. Let $m_p : \boldsymbol{\Omega}(C)^{\otimes p} \to \boldsymbol{\Omega}(C)$ denote the p-fold product and let $\Delta^p : \boldsymbol{\Omega}(C) \to \boldsymbol{\Omega}(C)^{\otimes p}$ denote the p-fold coproduct. For each $p \geq 2$, define the Adams operation

$$\psi^p := m_p \circ \Delta^p : \boldsymbol{\Omega}(C) \to \boldsymbol{\Omega}(C)\,.$$

Note that $\psi^p \circ \psi^q = \psi^{pq}$. The following proposition is dual to [15, Propositions 5.3.4—5.3.6].

Proposition 4.4.1. *The Adams operations $\psi^p\,,\ p \geq 2$ descend to Adams operations*

$$\psi^p : \boldsymbol{\Omega}(C)_\natural \to \boldsymbol{\Omega}(C)_\natural\,,\ p \geq 2\,.$$

It is verified without difficulty that on the image of $\mathrm{Sym}^d(\boldsymbol{\Omega}_{\mathsf{Comm}}(C))$ in $\boldsymbol{\Omega}(C)$ under the symmetrization map (4.29), ψ^p coincides with multiplication by p^d. Therefore,

Proposition 4.4.2. *ψ^p acts on the direct summand $\mathrm{Sym}^d[\boldsymbol{\Omega}_{\mathsf{Comm}}(C)]_\natural$ of (4.30) by multiplication by p^d.*

Corollary 4.4.3. *There are Adams operations*

$$\psi^p : \overline{\mathrm{HC}}_\bullet\,[\mathcal{U}(\mathfrak{a})] \to \overline{\mathrm{HC}}_\bullet\,[\mathcal{U}(\mathfrak{a})]\ ,\ p \geq 2\,.$$

Further, $\mathrm{HC}^{(d)}_\bullet(\mathsf{Lie}, \mathfrak{a})$ is precisely the (graded) eigenspace corresponding to the eigenvalue p^d of ψ^p for each $p \geq 2$.

Corollary 4.4.3 justifies referring to (4.27) as a Hodge decomposition. When C is the k-linear dual of a smooth commutative algebra A, the decomposition (4.27) can be related to the Hodge decomposition of the cyclic cohomology $\mathrm{HC}^\bullet(A)$ by the following proposition, which is an immediate consequence of [15, Corollary 6.5.1].

Proposition 4.4.4. *Let $\mathfrak{a} \in \mathsf{DGLA}_k$, and let $C \in \mathsf{DGCC}_{k/k}$ be a cocommutative coalgebra Koszul dual to $\mathfrak{a}$. Assume that $C \cong A^*$ for some* smooth *commutative k-algebra A. Then*

$$\mathrm{HC}_\bullet^{(d)}(\mathsf{Lie}, \mathfrak{a}) \;\cong\; \mathrm{HC}^{-\bullet}_{(d-1)}(A)[-1]\,.$$

In particular, $\mathrm{HC}_\bullet^{(2)}(\mathsf{Lie}, \mathfrak{a})$ is isomorphic (up to a shift) to the Harrison cohomology of A. More generally, for $C \in \mathsf{DGCC}_{k/k}$, one can define a Hodge decomposition for $\mathrm{HC}_\bullet(C)$ dual to the decomposition defined in [26, Theorem 4.6.7]. When $C = A^*$, this Hodge decomposition coincides with that on $\mathrm{HC}^{-\bullet}(A)$ (after the obvious identification is made). Proposition 4.4.4 therefore shows that the homology of the direct summand $\mathrm{Sym}^d(\boldsymbol{\Omega}_{\mathsf{Comm}}(C))_\natural[1]$ of $\boldsymbol{\Omega}(C)_\natural[1]$ should be denoted by $\mathrm{HC}_\bullet^{(d-1)}(C)$. Thus,

$$\mathrm{HC}_\bullet^{(d)}(\mathsf{Lie}, \mathfrak{a}) \;\cong\; \mathrm{HC}_\bullet^{(d-1)}(C)[-1]\,. \tag{4.31}$$

4.5 Drinfeld trace maps

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra over k. The adjunction (4.14) gives a universal representation

$$\pi_{\mathfrak{g}} \,:\, \mathfrak{a} \to \mathfrak{g} \otimes \mathfrak{a}_{\mathfrak{g}}$$

for any $\mathfrak{a} \in \mathsf{DGLA}_k$. Let $\mathfrak{L} \xrightarrow{\sim} \mathfrak{a}$ be a cofibrant resolution in DGLA_k. For any $d \geq 1$, consider the composite map

$$\lambda^{(d)}(\mathfrak{L}) \xrightarrow{\lambda^{(d)}(\pi_{\mathfrak{g}})} \lambda^{(d)}(\mathfrak{g} \otimes \mathfrak{L}_{\mathfrak{g}}) \xrightarrow{(4.26)} \lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}} \tag{4.32}$$

Proposition 4.5.1. *The image of the composite map* (4.32) *lies in* $\lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}}$.

Proof. Equip $\mathfrak{L}$ with the trivial $\mathfrak{g}$-action. Then, $\pi_{\mathfrak{g}} : \mathfrak{L} \to \mathfrak{g} \otimes \mathfrak{L}_{\mathfrak{g}}$ is $\mathfrak{g}$-equivariant. It follows that $\lambda^{(d)}(\pi_{\mathfrak{g}})$ is $\mathfrak{g}$-equivariant as well. On the other hand, it is easy to verify that the map (4.26) from $\lambda^{(d)}(\mathfrak{g} \otimes \mathfrak{L}_{\mathfrak{g}})$ to $\lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}$ is also $\mathfrak{g}$-equivariant. Since $\mathfrak{g}$ acts trivially on $\lambda^{(d)}(\mathfrak{g})$, the desired proposition follows. □

One therefore obtains the maps

$$\mathrm{Tr}_{\mathfrak{g}} : \lambda^{(d)}(\mathfrak{L}) \to \lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}} .$$

For $d = 2$ and $\mathfrak{g}$ simple, the vector space $\lambda^{(d)}(\mathfrak{g})$ is one-dimensional: indeed, there is a unique (up to a scalar factor) invariant bilinear form on $\mathfrak{g}$ (the Cartan-Killing form). Hence, at the level of homology, the map $\mathrm{Tr}_{\mathfrak{g}}$ induces a *canonical* trace map

$$\mathrm{Tr}_{\mathfrak{g}} : \mathrm{HC}_{\bullet}(\mathsf{Lie}, \mathfrak{a}) \to \mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} . \tag{4.33}$$

More generally, let $P \in \mathrm{Sym}^d(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}$. Then, one has the trace map

$$\mathrm{Tr}_{\mathfrak{g}}^{P} : \lambda^{(d)}(\mathfrak{L}) \xrightarrow{\mathrm{Tr}_{\mathfrak{g}}} \lambda^{(d)}(\mathfrak{g}) \otimes \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}} \xrightarrow{P(-)\otimes \mathrm{id}} \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}}$$

Recall that we have the Chevalley isomorphism

$$\mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}} \cong \mathrm{Sym}(\mathfrak{h}^*)^W \cong k[\bar{P}_1, \ldots, \bar{P}_l]$$

where $\deg(\bar{P}_i) = d_i$ for $1 \leq i \leq l$ and the d_i are the fundamental degrees of $\mathfrak{g}$. Choosing $P_i \in \mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}$ corresponding to $\bar{P}_i$ under Chevalley's isomorphism, we get a family of trace maps

$$\mathrm{Tr}_{\mathfrak{g}}^{(d_i)} := \mathrm{Tr}_{\mathfrak{g}}^{P_i} : \lambda^{(d_i)}(\mathfrak{L}) \to \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}} ,$$

which yields a homomorphism of commutative DG algebras

$$\mathbf{Sym}_k[\mathrm{Tr}_\bullet(\mathfrak{L})] \,:\, \mathbf{Sym}_k[\oplus_{i=1}^l \lambda^{(d_i)}(\mathfrak{L})] \to \mathfrak{L}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}}\,. \tag{4.34}$$

We refer to (4.34) as the *Drinfeld trace map*. We shall sometimes abuse this terminology and use it for closely related maps as well. At the level of homology, (4.34) gives

$$\mathbf{Sym}_k[\mathrm{Tr}_\bullet(\mathfrak{a})] \,:\, \mathbf{Sym}_k[\oplus_{i=1}^l \mathrm{HC}_\bullet^{(d_i)}(\mathsf{Lie},\mathfrak{a})] \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}}\,. \tag{4.35}$$

In particular, if $\mathfrak{L} = \mathbf{\Omega}_{\mathsf{Comm}}(C)$, then by (4.31), the above map becomes

$$\mathbf{Sym}_k[\mathrm{Tr}_\bullet(\mathfrak{a})] \,:\, \mathbf{Sym}_k[\oplus_{i=1}^l \mathrm{HC}_\bullet^{(d_i-1)}(C)[-1]] \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}}\,. \tag{4.36}$$

Remark. The Drinfeld trace map depends on the choice of the $P_i\,,\ 1 \le i \le l$. This choice in turn depends precisely on the choice of an isomorphism

$$\mathrm{Sym}(\mathfrak{h}^*)^W \,\cong\, k[\bar{P}_1,\ldots,\bar{P}_l]\,.$$

Example 4.5.2. Let $\mathfrak{a} := k.x$ be a one-dimensional Lie algebra over k with generator x having weight 1 and homological degree 0. Note that $\mathfrak{a}$ is a free (and therefore, cofibrant) DG Lie algebra. Since $\mathfrak{a}$ is also abelian,

$$\mathbf{L}\lambda^{(d)}(\mathfrak{a}) \,\cong\, \mathrm{Sym}^d(\mathfrak{a}) \,=\, k.x^d\,.$$

In this case, $\mathfrak{a}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}} \,=\, \mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}$ and the map

$$\lambda^{(d)}(\mathfrak{a}) \to \mathfrak{a}_{\mathfrak{g}}^{\mathrm{ad}\,\mathfrak{g}} \otimes \lambda^{(d)}(\mathfrak{g})$$

becomes the map dual to the nondegenerate pairing

$$\lambda^{(d)}(\mathfrak{g}) \otimes \mathrm{Sym}^d(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}} \to k\,.$$

It follows that for a fixed choice of isomorphism

$$\mathrm{Sym}(\mathfrak{h}^*)^W \,\cong\, k[\bar{P}_1,\ldots,\bar{P}_l]\,,$$

the Drinfeld trace becomes the map

$$k[\bar{P}_1, \ldots, \bar{P}_l] \to \mathrm{Sym}(\mathfrak{g}^*)^{\mathrm{ad}\,\mathfrak{g}}, \quad \bar{P}_i \mapsto P_i$$

where the variable $\bar{P}_i$ is identified with x^{d_i}. Since P_i corresponds to $\bar{P}_i$ under the Chevalley restriction isomorphism, the Drinfeld trace is indeed a generalization of the map inverse to the Chevalley restriction isomorphism. Combining this observation with Example 4.1.2, we conclude that when $\mathfrak{a}$ is a one-dimensional Lie algebra, the derived Harish-Chandra homomorphism and the Drinfeld trace are mutually inverse (quasi-)isomorphisms.

CHAPTER 5

DERIVED TRACE MAPS

1 Drinfeld trace maps and Chern-Simons forms

1.1 Chern-Simons forms

In this section, all DG algebras will be *cohomologically graded*. Let $\mathcal{A}$ be a commutative DG algebra, and let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Recall that a $\mathfrak{g}$-valued *connection* on $\mathcal{A}$ is an element $\theta \in \mathcal{A}^1 \otimes \mathfrak{g}$; its *curvature* is given by $\Omega := d\theta + \frac{1}{2}[\theta, \theta]$ in $\mathcal{A}^2 \otimes \mathfrak{g}$, and it is easy to verify that $d\Omega = [\Omega, \theta]$, which is usually called the Bianchi identity.

Now, fix $P \in I^{r+1}(\mathfrak{g})$, a homogeneous ad-invariant polynomial of degree $r+1$. Given $\alpha \in \mathcal{A} \otimes \mathrm{Sym}^{r+1}(\mathfrak{g})$, regard P as a linear map $\mathrm{Sym}^{r+1}(\mathfrak{g}) \to k$ and define $P(\alpha)$ by applying to α the evaluation map $\mathrm{id}_{\mathcal{A}} \otimes \mathrm{ev}_P : \mathcal{A} \otimes \mathrm{Sym}^{r+1}(\mathfrak{g}) \to \mathcal{A}$. Thus, $P(\alpha)$ is an element of $\mathcal{A}$ having the same cohomological degree as α. A simple calculation, using the Bianchi identity, shows that $dP(\Omega^{r+1}) = 0$ for any $\theta \in \mathcal{A}^1 \otimes \mathfrak{g}$. Thus $P(\Omega^{r+1})$ is a cocycle in $\mathcal{A}$ of degree $2r+2$. In fact, this cocycle is always exact, and among all coboundaries representing $P(\Omega^{r+1})$, one can specify a natural element $\mathrm{TP}(\theta) \in \mathcal{A}^{2r+1}$ called the Chern-Simons form [9]. Explicitly, this form is defined by the formula (cf. [9, (3.1)])

$$TP(\theta) := (r+1) \int_0^1 P(\theta.\Omega_t^r)dt \ ,$$

where $\Omega_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta]$ is the curvature for the family of connections $\theta_t := t\theta$, $t \in [0, 1]$, contracting θ to 0. We refer to $\mathrm{TP}(\theta)$ as the *Chern-Simons form* of P and θ. A classical calculation (see [9, Prop. 3.2]) gives

Proposition 1.1.1. $d\,\mathrm{TP}(\theta) = P(\Omega^{r+1})$.

We remark that the Chern-Simons form can be also defined directly, without integration, by the following formula (cf. [9, (3.5)])

$$\mathrm{TP}(\theta) = \sum_{i=0}^{r} A_i\, \Psi_{i,P} \,, \tag{5.1}$$

where $A_i := \frac{(-1)^i (r+1)!r!}{2^i (r-i)!(r+1+i)!}$ and $\Psi_{i,P} := P(\theta[\theta,\theta]^i\Omega^{r-i})$.

An example of a commutative DG algebra with $\mathfrak{g}$-valued connection is the Weil algebra $\mathcal{W}(\mathfrak{g})$ of $\mathfrak{g}$. Recall (*cf.* [32, Section 6.9]) that $\mathcal{W}(\mathfrak{g}) := \mathbf{Sym}(\mathfrak{g}^*[1] \oplus \mathfrak{g}^*[2]) \cong \Lambda(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*)$, where the generators $\mathfrak{g}^*$ of $\Lambda(\mathfrak{g}^*)$ are in cohomological degree 1 and the generators $\mathfrak{g}^*$ of $\mathrm{Sym}(\mathfrak{g}^*)$ are in cohomological degree 2. The differential is given by the identity map on the generators of cohomological degree 1 and vanishes on the generators of cohomological degree 2. It is therefore, easy to see that $\mathcal{W}(\mathfrak{g})$ is acyclic, i.e, quasi-isomorphic to k. The identity map $\mathfrak{g}^* \to \mathfrak{g}^* = \mathcal{W}^1(\mathfrak{g})$ gives a $\mathfrak{g}$-valued connection $\theta_{\mathcal{W}}$ on $\mathcal{W}(\mathfrak{g})$. There is a second isomorphism $\mathcal{W}(\mathfrak{g}) \cong \Lambda(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*)$ such that $\theta_{\mathcal{W}}$ is still the identity from $\mathfrak{g}^*$ to the $\mathfrak{g}^*$ viewed as the space of generators of $\mathfrak{g}^*$ and the curvature $\Omega_{\mathcal{W}}$ of $\theta_{\mathcal{W}}$ is the identity map from $\mathfrak{g}^*$ to the space of generators of $\mathrm{Sym}(\mathfrak{g}^*)$. Thus, under this second isomorphism, the element $P(\Omega_{\mathcal{W}}^{r+1})$ is identified with the degree $2r+2$ element $P \in \mathrm{Sym}(\mathfrak{g}^*) \subset \Lambda(\mathfrak{g}^*) \otimes \mathrm{Sym}(\mathfrak{g}^*)$ for any $P \in I^{r+1}(\mathfrak{g})$. It follows from Proposition 1.1.1 that $P = d\,\mathrm{TP}(\theta_{\mathcal{W}})$.

The Weil algebra is the *universal* commutative DG algebra with a $\mathfrak{g}$-valued connection. Indeed, a connection θ on a commutative DG algebra $\mathcal{A}$ can be viewed as a map of vector spaces $\theta : \mathfrak{g}^* \to \mathcal{A}^1$. This defines a (characteristic) homomorphism $c : \mathcal{W}(\mathfrak{g}) \to \mathcal{A}$ by $c(\mu) = \theta(\mu), c(\hat{\mu}) = d_{\mathcal{A}}\theta(\mu)$ for all $\mu \in \mathfrak{g}^*$. Here each $\mu \in \mathfrak{g}^*$ is viewed as a degree 1 element of $\Lambda(\mathfrak{g}^*)$ and $\hat{\mu} := d_{\mathcal{W}}\mu$, i.e, μ viewed as a degree 2 element of $\mathrm{Sym}(\mathfrak{g}^*)$. Clearly, $\theta = c(\theta_{\mathcal{W}}), \Omega = c(\Omega_{\mathcal{W}})$, etc. From this, it follows that for any $P \in I^{r+1}(\mathfrak{g})$,

$$P(\Omega^{r+1}) = c(P)\,, \quad \mathrm{TP}(\theta) = c(\mathrm{TP}(\theta_{\mathcal{W}}))\,.$$

Thus, $\mathrm{TP}(\theta_{\mathcal{W}})$ is the universal Chern-Simons form.

1.2 Main theorem

Let $C := (\mathbf{Sym}^c(V), \delta)$ be a semi-cofree, cocommutative, conilpotent coaugmented DG coalgebra cogenerated by a finite-dimensional graded vector space V. Assume further that the corestriction of δ to V vanishes on $\mathbf{Sym}^r(V)$ for $r \gg 0$. Consider the convolution DG algebra

$$\mathcal{A} := \mathbf{Hom}(\mathrm{DR}_{\bullet}(C), \mathrm{CE}^c(\mathfrak{g}^*(\overline{C}); k))\,.$$

Note that taking bigraded linear duals[1] gives an isomorphism of convolution algebras

$$\mathcal{A} \cong \mathcal{A}_E := \mathbf{Hom}(\mathrm{CE}(\mathfrak{g}(\overline{E}); k), \mathrm{DR}_{\bullet}(E))\,, \tag{5.2}$$

where $E := (\mathbf{Sym}(V^*), \delta^*)$. Equip $\mathcal{A}$ with a *cohomological* grading by inverting all homological degrees.

The DG algebra $\mathcal{A}$ has a natural $\mathfrak{g}$-valued connection $\theta \in \mathcal{A}^1 \otimes \mathfrak{g}$ given by

$$\theta(c) = \sum_{\alpha} \xi_{\alpha}^*(s^{-1}\bar{c}) \otimes \xi_{\alpha}\,, \tag{5.3}$$

where $\{\xi_{\alpha}\}$ is a basis of $\mathfrak{g}$ and $\{\xi_{\alpha}^*\}$ is the dual basis of $\mathfrak{g}^*$ and $\xi_{\alpha}^*(\bar{c}) := \xi_{\alpha}^* \otimes \bar{c}$ ($\bar{c}$ being the image of c in $\overline{C}$). Similarly, the curvature $\Omega \in \mathcal{A}^2 \otimes \mathfrak{g}$ of the connection

[1] Equipping V and V^* with weight 1 makes the coalgebras $\mathrm{DR}_{\bullet}(C), \mathrm{CE}(\mathfrak{g}(\bar{E}); k)$ and the algebras $\mathrm{CE}^c(\mathfrak{g}^*(\bar{C}); k), \mathrm{DR}_{\bullet}(E)$ bigraded. While taking bigraded duals, we stick to the convention that homological degrees are inverted while weights are preserved. Thus, $\mathrm{DR}_{\bullet}(C)$ is the bigraded dual of $\mathrm{DR}_{\bullet}(E)$, etc. (even though the differentials are not necessarily weight-preserving).

θ vanishes on Ω^j_C for all $j \neq 1$ and satisfies

$$\Omega(\omega) = \sum_\alpha \xi^*_\alpha(s^{-1}d\omega) \otimes \xi_\alpha\,, \tag{5.4}$$

for $\omega \in \Omega^1_C$. Further, $\mathcal{A}$ is a commutative DG algebra, making $\mathcal{A} \otimes \mathfrak{g}$ a DG Lie algebra. Hence, $[\theta, \theta] \in \mathcal{A}^2 \otimes \mathfrak{g}$. Thus, for any $P \in I^{r+1}(\mathfrak{g})$, the Chern-Simons form $\mathrm{TP}(\theta)$ associated with θ arises as an element of $\mathcal{A}^{2r+1}$.

Recall that the map s^{2r} increasing homological degree by $2r$ gives a map of graded vector spaces from $\mathrm{CC}^{-,(r)}[\mathrm{DR}^\bullet(C)]$ to $\mathrm{DR}_\bullet(C)$. Dually, s^{2r} gives a map of graded vector spaces from $\mathrm{DR}_\bullet(E)$ to $\mathrm{CC}^{(r)}[\mathrm{DR}^\bullet(E)]$. Let θ_E denote the image of the connection θ of $\mathcal{A}$ under (5.2). It is known (see [3, Theorem A.1]) that the (shifted) Chern-Simons form $s^{2r}\,\mathrm{TP}(\theta_E)$ gives a map of complexes

$$s^{2r}\,\mathrm{TP}(\theta_E)\,:\,\mathrm{CE}(\mathfrak{g}(\bar{E});k) \to \mathrm{CC}^{(r)}[\mathrm{DR}^\bullet(E)][1]\,.$$

Taking (bigraded) linear duals, we see that the (shifted) Chern-Simons form $\mathrm{TP}(\theta)s^{2r}$ gives a map of complexes

$$\mathrm{TP}(\theta)s^{2r}\,:\,\mathrm{CC}^{-,(r)}[\mathrm{DR}^\bullet(C)][-1] \to \mathrm{CE}^c(\mathfrak{g}^*(\bar{C});k)\,. \tag{5.5}$$

By Theorem 3.4.3, the inclusion

$$\iota\,:\,\mathrm{Ker}(d\,:\,\Omega^r_C \to \Omega^{r-1}_C)[-r] \hookrightarrow \mathrm{CC}^{-,(r)}[\mathrm{DR}^\bullet(C)]$$

is a quasi-isomorphism. Let $\varepsilon\,:\,\overline{\mathrm{C}}^{\lambda,(r)}(C) \to \mathrm{Ker}(d\,:\,\Omega^r_C \to \Omega^{r-1}_C)[-r]$ be as in (2.8). By Theorem 3.2.2,

$$\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a}) \,\cong\, \mathrm{CE}^c(\mathfrak{g}^*(\bar{C});k)\,, \tag{5.6}$$

where $\mathfrak{a}$ is the DG Lie algebra Koszul dual to C. On the other hand, recall from [4, Prop. 7.4] that

$$\overline{\mathrm{HC}}_{\bullet+1}(C) \,\cong\, \overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{a})\,. \tag{5.7}$$

[4, Prop. 7.4] further implies that the isomorphism (5.7) respects the Hodge decomposition of Theorem 4.3.1 to give an isomorphism

$$\overline{\mathrm{HC}}^{(r)}_{\bullet+1}(C) \;\cong\; \overline{\mathrm{HC}}^{(r+1)}_{\bullet}(\mathfrak{a}) \tag{5.8}$$

With these identifications, $\mathrm{TP}(\theta)s^{2r}$ can be interpeted as a map from $\overline{\mathrm{HC}}^{(r+1)}_{\bullet}(\mathfrak{a})$ to $\mathrm{H}_{\bullet}(\mathfrak{a},\mathfrak{g})$.

Theorem 1.2.1. *For any invariant polynomial* $P \in I^{r+1}(\mathfrak{g})$*, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{HC}^{(r+1)}_{\bullet}(\mathfrak{a}) \arrow[r, "(5.8)", "\cong"'] \arrow[dr, "\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})"'] & \overline{\mathrm{HC}}^{(r)}_{\bullet+1}(C) \arrow[r, "\iota\circ\varepsilon", "\cong"'] & \mathrm{H}_{\bullet+1}(\mathrm{CC}^{-,(r)}[\mathrm{DR}^{\bullet}(C)]) \arrow[d, "\frac{1}{(r+1)!}\mathrm{TP}(\theta)s^{2r}"] \\
& \mathrm{HR}_{\bullet}(\mathfrak{a},\mathfrak{g}) \arrow[r, "\text{Thm. 3.2.2}", "\cong"'] & \mathrm{H}_{\bullet}(\mathfrak{g}^*(\overline{C});k)
\end{tikzcd}$$

Thus, the map $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})$ *is given by the formula*

$$\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) \;=\; \frac{1}{(r+1)!}\,\mathrm{TP}(\theta)s^{2r}\,,$$

where θ *is defined in* (5.3)*.*

Proof. For brevity, we denote the maps induced on homologies by a given map of complexes by the same symbol as that of the original map of complexes. Let $\mathcal{L} := \mathbf{\Omega}_{\mathtt{Comm}}(C)$. Note that $\overline{C}[-1] \hookrightarrow \mathfrak{L}$ as graded k-vector spaces. Recall from Section 4.2 that the Drinfeld trace was constructed as a map of complexes

$$\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L}) \,:\, \lambda^{(r+1)}(\mathcal{L}) \to \mathbf{C}^c(\mathfrak{g}^*(\overline{C});k)\,.$$

Dually, one obtains a map of complexes

$$\mathbf{C}(\mathfrak{g}(\overline{E});k) \to \mathbf{Sym}^{r+1}(\mathfrak{L}^c(\overline{E})) \cap \mathbf{B}(E)^{\natural}\;,. \tag{5.9}$$

where $\mathbf{B}(E)^{\natural}$ denotes the cocommutator subspace of the coalgebra $\mathbf{B}(E)$. The isomorphism

$$\mathrm{C}^{\lambda,(r)}(C)[-1] \;\cong\; \lambda^{(r+1)}(\mathfrak{L}) \tag{5.10}$$

inducing (5.8) on homologies is obtained by taking k-linear duals on the isomorphism

$$\mathbf{Sym}^{r+1}(\mathcal{L}^c(\bar{E})) \cap \mathbf{B}(E)^\natural \;\cong\; \mathrm{C}^{\lambda,(r)}(E)[1]\,,$$

whose inverse is explicitly given by the map N which acts on $E[1]^{\otimes n}$ by $1+\tau+\ldots+\tau^{n-1}$ where τ is the n-cycle $(0,1,\ldots,n-1)$. Let φ_P denote the composite map

$$\mathbf{C}(\mathfrak{g}(\overline{E});k) \xrightarrow{(5.9)} \mathbf{Sym}^{r+1}(\mathfrak{L}^c(\overline{E})) \cap \mathbf{B}(E)^\natural \xrightarrow{\cong} \mathrm{C}^{\lambda,(r)}(E)[1]\,.$$

The composite map

$$\mathrm{C}^{\lambda,(r)}(C)[-1] \xrightarrow{(5.10)} \lambda^{(r+1)}(\mathcal{L}) \xrightarrow{\mathrm{Tr}_{\mathfrak{g}}(\mathcal{L})} \mathbf{C}^c(\mathfrak{g}^*(\overline{C});k)$$

is thus equal to the map obtained by applying φ_P to E and taking (bigraded) linear duals.

It is known (see Theorem [3, Theorem A.3]) that the diagram

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{g}(\overline{E});k) \arrow[d, "\frac{1}{(r+1)!}s^{2i}\,\mathrm{TP}(\theta_E)"'] \arrow[dr, "\varphi_P"] & \\
\mathrm{CC}^{(r)}[\mathrm{DR}^\bullet(E)][1] \arrow[r, "\varepsilon\circ\mathrm{p}"] & \overline{\mathrm{C}}^{\lambda,(r)}(E)[1]
\end{tikzcd}$$

commutes on homologies, where p is as in (2.5) and ε is as in (2.6). The desired result follows immediately from the above fact by taking bigraded linear duals. $\square$

It is easy to verify that

Proposition 1.2.2. $\mathrm{TP}(\theta)s^{2r}\circ\iota \;=\; P(\theta.\Omega^r)s^{2r}$.

As a consequence of Theorem 1.2.1 and Proposition 1.2.2, we have

Corollary 1.2.3. *For any* $P \in I^{r+1}(\mathfrak{g})$, *the Drinfeld trace map* $\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) : \overline{\mathrm{HC}}_\bullet^{(r+1)}(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}}$ *is given by*

$$\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) \;=\; \frac{1}{(r+1)!}P(\theta.\Omega^r)s^{2r}\;, \tag{5.11}$$

where θ is defined in (5.3).

We now specialize to the case $\mathfrak{g} = \mathfrak{gl}_V$ and use formula (5.11) to express the derived character maps for associative algebras.

1.3 The case of $\mathfrak{gl}_n$

Recall from Section 1.3, for any k-algebra A and finite-dimensional vector space $V = k^n$, there is a derived scheme $\mathrm{DRep}_n(A)$ and natural maps

$$\mathrm{Tr}_n(A) \,:\, \overline{\mathrm{HC}}_\bullet(A) \to \mathrm{HR}_\bullet(A, n) \tag{5.12}$$

relating the reduced cyclic homology of A to its representation homology $\mathrm{HR}_\bullet(A, n) := \mathrm{H}_\bullet[\mathrm{DRep}_n(A)]$.

By Theorem 1.2.2, there is a natural isomorphism of algebras

$$\mathrm{HR}_\bullet(A, n) \simeq \mathrm{H}_\bullet(\mathfrak{gl}_n^*(\overline{C}); k)\,, \tag{5.13}$$

where $C \in \mathsf{DGC}_{k/k}$ is a Koszul dual coalgebra of A. Moreover, since the cobar construction $R := \boldsymbol{\Omega}(C)$ provides a cofibrant resolution of A in $\mathsf{DGA}_{k/k}$, by Proposition 3.2.2 there is an isomorphism $\overline{\mathrm{HC}}_\bullet(A) \cong \mathrm{H}_\bullet[R_\natural]$, where $R_\natural := \overline{R}/[\overline{R}, \overline{R}]$. With these identifications, the character map $\mathrm{Tr}_n(A)$ becomes

$$\mathrm{Tr}_n(A) \,:\, \mathrm{H}_\bullet[R_\natural] \to \mathrm{H}_\bullet(\mathfrak{gl}_n^*(\overline{C}); k)\ .$$

Now, let $A = \mathcal{U}\mathfrak{a}$ be the universal enveloping algebra of a Lie algebra $\mathfrak{a}$. In this case, the Koszul dual coalgebra C can be chosen to be cocommutative, with $\mathcal{L} := \boldsymbol{\Omega}_{\mathsf{Comm}}(C)$, giving a cofibrant resolution of the Lie algebra $\mathfrak{a}$ in DGLA_k. By Theorem 3.2.2, $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{gl}_n) \cong \mathrm{H}_\bullet(\mathfrak{gl}_n^*(\overline{C}); k)$. Combining it with the isomorphism (5.13) we get

$$\mathrm{HR}_\bullet(A,n) \cong \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{gl}_n)\ . \tag{5.14}$$

Identifying $\mathfrak{gl}_n = \mathrm{Mat}_n(k)$, we define polynomials $\mathrm{Tr} \in I^{q+1}(\mathfrak{gl}_n)$ by $\mathrm{Tr}(X,\ldots,X) := \mathrm{tr}_n(X^{q+1})$, where $\mathrm{tr}_n : \mathrm{Mat}_n(k) \to k$ is the usual matrix trace on $V = k^n$. With this choice of invariant polynomials, the direct sum of the Drinfeld traces gives a map $\bigoplus_{q\geq 1} \overline{\mathrm{HC}}^{(q)}_\bullet(\mathfrak{a}) \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{gl}_n)$, which upon isomorphism of Theorem 4.3.1, coincides with the character map (5.12). By Corollary 1.2.3, we now conclude

$$\mathrm{Tr}_n(A) \;=\; \sum_{q=0}^{\infty} \frac{1}{(q+1)!}\,\mathrm{Tr}(\theta.\Omega^q)\, s^{2q}\,. \tag{5.15}$$

1.4 Reduced Drinfeld trace maps

We keep the assumption that $\mathfrak{g}$ is a finite-dimensional reductive Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\mathbf{W}$ be the corresponding Weyl group.

In Section 4.1 we constructed a natural map $\Phi_{\mathfrak{g}}(\mathfrak{a}) : \mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})^{\mathrm{ad}\,\mathfrak{g}} \to \mathrm{DRep}_{\mathfrak{h}}(\mathfrak{a})^{\mathbf{W}}$ called derived Harish-Chandra homomorphism. It is induced by the morphism of complexes $\mathrm{CE}^c(\mathfrak{g}^*(C),\, \mathfrak{g}^*;\, k) \twoheadrightarrow \mathrm{CE}^c(\mathfrak{h}^*(C),\, \mathfrak{h}^*;\, k)^{\mathbf{W}}$ corresponding to the projection $\mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$ dual to the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. We abuse notation and denote by $\Phi_{\mathfrak{g}}(\mathfrak{a})$ the induced map on homology:

$$\Phi_{\mathfrak{g}}(\mathfrak{a})\colon\ \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \to \mathrm{HR}_\bullet(\mathfrak{a},\mathfrak{h})^{\mathbf{W}}$$

Now, let $I(\mathfrak{h})^{\mathbf{W}} := \mathbf{Sym}(\mathfrak{h}^*)^{\mathbf{W}}$ denote the space of $\mathbf{W}$-invariant polynomials on $\mathfrak{h}$. Recall that $\mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*$ extends to an isomorphism of algebras $I(\mathfrak{g}) \xrightarrow{\sim} I(\mathfrak{h})^{\mathbf{W}}$ which is called the Chevalley isomorphism for $\mathfrak{g}$.

Lemma 1.4.1. *For every integer $r \geq 1$, the following diagram commutes*

$$\begin{tikzcd}
I^r(\mathfrak{g}) \otimes \overline{\mathrm{HC}}_{\bullet}^{(r)}(\mathfrak{a}) \arrow[r, "\cong"] \arrow[d, "\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})"'] & I^r(\mathrm{H})^{\mathbf{W}} \otimes \overline{\mathrm{HC}}_{\bullet}^{(r)}(\mathfrak{a}) \arrow[d, "\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})"] \\
\mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \arrow[r, "\Phi_{\mathfrak{g}}(\mathfrak{a})"] & \mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{h})^{\mathbf{W}}
\end{tikzcd}$$

Proof. Let $P \in I^r(\mathfrak{g})$ and let $P_{\mathbf{W}} \in I(\mathfrak{h})^{\mathbf{W}}$ denote the image of P under the Chevalley isomorphism. Similarly, let $\theta_{\mathfrak{h}}$ denote the connection (5.3) on the convolution algebra $\mathcal{A}_{\mathfrak{h}} := \mathbf{Hom}(\mathrm{DR}_{\bullet}(C), \mathrm{CE}^c(\mathfrak{h}^*(\bar{C}); k))$ and let $\Omega_{\mathfrak{h}}$ denote the curvature of $\theta_{\mathfrak{h}}$. The inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ induces an inclusion $\mathcal{A}_{\mathfrak{h}} \otimes \mathrm{Sym}(\mathfrak{h}) \hookrightarrow \mathcal{A}_{\mathfrak{h}} \otimes \mathrm{Sym}(\mathfrak{g})$ of commutative DG algebras. Clearly, for any element $\alpha \in \mathcal{A}_{\mathfrak{h}} \otimes \mathrm{Sym}(\mathrm{H})$, $P_{\mathbf{W}}(\alpha) = P(\alpha)$, where α on the right hand side is viewed as an element of $A_{\mathfrak{h}} \otimes \mathrm{Sym}(\mathfrak{g})$. By (5.11),

$$\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a}) = \frac{1}{r!} P_{\mathbf{W}}(\theta_{\mathfrak{h}}.\Omega_{\mathfrak{h}}^{r-1}), \qquad \mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a}) = \frac{1}{r!} P(\theta.\Omega^{r-1}).$$

It therefore suffices to verify that $\Phi_{\mathfrak{g}}(\theta.\Omega^{r-1}) = \theta_{\mathfrak{h}}.\Omega_{\mathfrak{h}}^{r-1}$ as elements of $A_{\mathrm{H}} \otimes \mathrm{Sym}(\mathfrak{g})$. This follows from the fact that $\Phi_{\mathfrak{g}}$ is a DG algebra homomorphism and $\Phi_{\mathfrak{g}}(\theta) = \theta_{\mathfrak{h}}$, which is easy to verify by direct calculation. $\square$

Lemma 1.4.1 shows that, modulo the Chevalley isomorphism, the composite map

$$\overline{\mathrm{HC}}_{\bullet}^{(r)}(\mathfrak{a}) \xrightarrow{\mathrm{Tr}_{\mathfrak{g}}(\mathfrak{a})} \mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}} \xrightarrow{\Phi_{\mathfrak{g}}(\mathfrak{a})} \mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{h})^{\mathbf{W}} \tag{5.16}$$

equals $\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a})$, which depends only on $\mathfrak{h}$, $\mathbf{W}$ and the choice of a invariant polynomial $P \in I(\mathfrak{h})^{\mathbf{W}}$ but not on the Lie algebra $\mathfrak{g}$. We call $\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a})$ the *reduced Drinfeld trace map*, or simply *reduced trace*. This map is more accessible than the Drinfeld trace, since $\mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{h})^{\mathbf{W}}$ is easy to compute in many cases.

In fact, the computation of $\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a})$ reduces to the rank one case. To be precise, let $\mathfrak{h} = k$ be a one-dimensional Lie algebra with a preferred basis. Denote by $\overline{\mathrm{Tr}}(\mathfrak{a})$

the Drinfeld trace map for $\mathfrak{a}$ corresponding to the canonical element in $\mathrm{Sym}(\mathfrak{h}^*)$ of degree r. Then, for an arbitrary $\mathfrak{h}$, the map $\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a})$ factors through $\overline{\mathrm{Tr}}(\mathfrak{a})$. To see this, choose a Koszul dual coalgebra $C \in \mathsf{DGCC}_{k/k}$ for $\mathfrak{a}$, and let $R := \boldsymbol{\Omega}(C)$. Then, for a given $\mathfrak{h}$, choose a linear basis $\{\xi_\alpha\} \subset \mathfrak{h}$ and define a DG algebra homomorphism

$$\vartheta_{\mathfrak{h}} : R_{\mathrm{ab}} \to \mathrm{CE}^c(\mathfrak{h}^*(\overline{C}); k) \otimes \mathrm{Sym}(\mathfrak{h})$$

by sending the canonical generators $s^{-1}c$ of R_{ab} to the elements

$$\vartheta_{\mathfrak{h}}(s^{-1}c) = \sum_\alpha \xi^*_\alpha(s^{-1}c) \otimes \xi_\alpha ,$$

where $\{\xi^*_\alpha\} \subset \mathfrak{h}^*$ is the dual basis to $\{\xi_\alpha\}$. Note that the map $\vartheta_{\mathfrak{h}}$ thus defined is independent on the choice of basis $\{\xi_\alpha\}$. Now, for any $P \in I^r(\mathfrak{h})^{\mathbf{W}}$, the evaluation at P on the second factor gives a map $\mathrm{CE}^c(\mathfrak{h}^*(\overline{C}); k) \otimes \mathrm{Sym}(\mathfrak{h}) \to \mathrm{CE}^c(\mathfrak{h}^*(\overline{C}); k)^{\mathbf{W}}$. Write $P(\vartheta_{\mathfrak{h}})$ for the composition of this map with $\theta_{\mathfrak{h}}$:

$$P(\vartheta_{\mathfrak{h}}) : R_{\mathrm{ab}} \to \mathrm{CE}^c(\mathfrak{h}^*(\overline{C}); k)^{\mathbf{W}} .$$

On homology, this induces a map $\mathrm{HR}_\bullet(\mathfrak{a}, k) \to \mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{h})^W$.

Lemma 1.4.2. *For any $P \in I(\mathfrak{h})^{\mathbf{W}}$, the trace map* (5.16) *factors as*

$$\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a}) = P(\vartheta_{\mathfrak{h}}) \circ \overline{\mathrm{Tr}}(\mathfrak{a}) .$$

Proof. Let θ_0 denote $\theta_{\mathfrak{h}}$ for $\mathfrak{h} := k$ and let Ω_0 denote the curvature of θ_0. By (5.11),

$$\overline{\mathrm{Tr}}(\mathfrak{a}) = \frac{1}{r!}[\theta_0.\Omega_0^{r-1}] , \qquad \mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a}) = \frac{1}{r!}P(\theta_{\mathfrak{h}}.\Omega_{\mathfrak{h}}^{r-1}) .$$

The desired result now follows from the observation that $\theta_{\mathfrak{h}} = \vartheta_{\mathfrak{h}}(\theta_0)$. □

Example 1.4.3. Let $\mathfrak{h} := \mathfrak{h}_n$ be the subalgebra of diagonal matrices in $\mathfrak{gl}_n$ and let $\mathbf{W} = \mathbf{S}_n$ be the symmetric group acting on $\mathfrak{h}_n$ by permuting the diagonal entries. Let $P_q \in I^{q+1}(\mathfrak{h}_n)^{\mathbf{S}_n}$ be the symmetric polynomial given by the $(q+1)$-th power sum. Then $\mathrm{CE}(\mathfrak{h}^*(C);k)^{\mathbf{W}} = [R_{\mathrm{ab}}^{\otimes n}]^{\mathbf{S}_n}$, where $\mathbf{S}_n$ acts on $R_{\mathrm{ab}}^{\otimes n}$ by permuting the factors. Writing $\mathbf{S}^n[R_{\mathrm{ab}}] := [R_{\mathrm{ab}}^{\otimes n}]^{\mathbf{S}_n}$, we see that $\sum_{q=0}^{\infty} P_q(\vartheta_{\mathrm{H}}) : R_{\mathrm{ab}} \to \mathbf{S}^n[R_{\mathrm{ab}}]$ is precisely the symmetrization map

$$\mathrm{sym} : R_{\mathrm{ab}} \to \mathbf{S}^n[R_{\mathrm{ab}}] , \qquad r \mapsto \sum_{i=1}^{n} (1, \ldots, r, \ldots, 1) ,$$

where r in the i-th factor is in the i-th summand. Now, let $A := \mathcal{U}\mathfrak{a}$. With the above choice of invariant polynomials, the direct sum of the reduced Drinfeld traces $\mathrm{Tr}_{\mathfrak{h}_n}(\mathfrak{a})$ gives a map $\bigoplus\limits_{q\geq 1} \overline{\mathrm{HC}}_{\bullet}^{(q)}(\mathfrak{a}) \to \mathbf{S}^n[R_{\mathrm{ab}}]$, which upon isomorphism (4.27), coincides with the character map

$$\overline{\mathrm{Tr}}_n(A) : \overline{\mathrm{HC}}_{\bullet}(A) \to \mathbf{S}^n[\mathrm{H}_{\bullet}(R_{\mathrm{ab}})]$$

constructed in Section 1.3. For $n = 1$, the reduced character map

$$\overline{\mathrm{Tr}}(A) : \overline{\mathrm{HC}}_{\bullet}(A) \to \mathrm{H}_{\bullet}(R_{\mathrm{ab}}) \tag{5.17}$$

coincides with the character map Tr_1 in (5.12).

Thus, thanks to Lemma 1.4.2, computing the trace map $\mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a})$ for any $\mathfrak{h}$ and any invariant polynomial $P \in I(\mathfrak{h})^{\mathbf{W}}$ reduces to computing the map $\overline{\mathrm{Tr}}(\mathfrak{a}) = \mathrm{Tr}_{\mathfrak{h}}(\mathfrak{a})$ for $\mathfrak{h}$ being one-dimensional. In the next section, we will given an explicit formula for $\overline{\mathrm{Tr}}(\mathfrak{a})$ for an arbitrary abelian Lie algebra $\mathfrak{a}$ in terms of differential forms on $\mathrm{Sym}(\mathfrak{a})$.

1.5 Lie representation homology of 1-dimensional representations

From now on we will be mostly interested in the *reduced* Drinfeld traces, whose target is the space $\mathrm{HR}_{\bullet}(\mathfrak{a}, \mathfrak{gl}_1)$. Let us compute it for the case $\mathfrak{a}$ is a Lie algebra (not

DG) since this is the main case of interest for us. Note that by the isomorphism 5.14, this is equivalent to computing representation homology $\mathrm{HR}_\bullet(\mathcal{U}\mathfrak{a}, 1)$.

Let $C := \mathrm{CE}(\mathfrak{a}; k)$ be the Chevalley–Eilenberg coalgebra of $\mathfrak{a}$. As a graded coalgebra, $C = \mathbf{Sym}^c(\mathfrak{a}[1])$ is the semi-cofree cocommutative coalgebra cogenerated by the vectors space $\mathfrak{a}[1]$ concentrated in homological degree 1. The differential (of degree -1) is given by

$$\delta_{\mathrm{CE}}(sa_1 \dots sa_n) = \sum_{i<j} (-1)^{i+j+1} s[a_i, a_j] sa_1 \dots \widehat{sa_i} \dots \widehat{sa_j} \dots sa_n$$

The coalgebra C is Koszul dual to $\mathfrak{a}$, which implies that the natural map $\boldsymbol{\Omega}(C) \twoheadrightarrow \mathcal{U}(\mathfrak{a})$ is a quasi-isomorphism. Thus, $R := \Omega(C)$ is a cofibrant resolution of $\mathcal{U}(\mathfrak{a})$. It is a semi-free DG algebra with the differential $\delta = \delta_1 + \delta_2$ where δ_1 is induced by the comultiplication Δ on C, and δ_2 is induced by the differential δ_{CE} on C.

If $\mathfrak{g} = \mathfrak{gl}_1 \simeq k$, then the derived representation scheme $\mathrm{DRep}_{\mathfrak{g}}(\mathfrak{a})$ is isomorphic to the abelianization R_{ab} of R. Thus $\mathrm{DRep}_{\mathfrak{gl}_1}(\mathfrak{a})$ is a semi-free graded commutative DG algebra generated by the elements

$$\lambda(a_1, a_2, \dots, a_k) := s^{-1}(sa_1 \dots sa_k) \in \mathbf{Sym}^k(\mathfrak{a}[1])[-1] \,. \tag{5.18}$$

Lemma 1.5.1. *The differential δ_1 on the standard resolution R of $A = \mathbf{Sym}(V)$ is defined by*

$$\delta_1 \lambda(v_1, \dots, v_n) = \sum_{\substack{p+q=n \\ 1 \le p \le q}} (-1)^p \sum_{\sigma \in \mathrm{Sh}(p,q)} (-1)^\sigma \left[\lambda(v_{\sigma(1)}, \dots, v_{\sigma(p)}),\, \lambda(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}) \right] \,,$$

where $\mathrm{Sh}(p,q)$ denotes the set of (p,q)-shuffles. Hence, $\delta_1 R \subset [R\,,\, R]$.

Thus δ_1 is zero on R_{ab}, and so $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{gl}_1) \simeq \mathrm{H}_\bullet \left[\mathbf{Sym}(\overline{C}[-1]), \delta = \delta_2 \right]$ where δ_2 is the differential induced by δ_{CE}. Since symmetric power commutes with taking homology, we get

Proposition 1.5.2. $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{gl}_1) \simeq \mathbf{Sym}\left[\overline{\mathrm{H}}^{\mathrm{CE}}_\bullet(\mathfrak{a}; k)[-1]\right]$.

Example 1.5.3. Let $\mathfrak{a} = V$ is a finite dimensional abelian Lie algebra. Since $\mathfrak{a}$ has zero bracket, the differential $\delta_2 = 0$ and so $\mathrm{H}_\bullet(\mathfrak{a}, \mathfrak{gl}_1)$ is isomorphic simply to the abelianization R_{ab} of the standard cofibrant resolution $R \overset{\sim}{\twoheadrightarrow} A$, which is the tensor algebra generated by the vector space $\mathbf{Sym}(V[1])[-1]$. The Chevalley-Eilenberg homology of $\mathfrak{a}$ is simply $\mathrm{H}^{\mathrm{CE}}_\bullet(\mathfrak{a}; k) \simeq \mathbf{Sym}(\mathfrak{a}[1])$. As a graded vector space, $\overline{\mathrm{H}}^{\mathrm{CE}}_\bullet(\mathfrak{a}; k)[-1]$ is spanned by the elements $\lambda(v_1, \ldots, v_i)$ for $i = 1, \ldots, N$ of degree $i-1$. By Proposition 1.5.2, the homology $\mathrm{HR}_\bullet(\mathfrak{a}, \mathfrak{gl}_1) \simeq R_{\mathrm{ab}} \simeq \mathbf{Sym}(\overline{\mathrm{H}}^{\mathrm{CE}}_\bullet(\mathfrak{a}; k)[-1])$ is a free graded commutative algebra generated by the elements $\lambda(v_1, \ldots, v_i)$ of degree $i - 1$ for $i = 1, \ldots, N$.

Example 1.5.4. Let $\mathfrak{a}$ be a semi-simple Lie algebra of rank l with exponents $m_1, \ldots, m_l$. It is well-known that Chevalley–Eilenberg homology is given by

$$\mathrm{H}^{\mathrm{CE}}_\bullet(\mathfrak{a}; k) \simeq \mathbf{Sym}(\eta_1, \ldots, \eta_l)$$

where $|\eta_i| = 2m_i + 1$. For example, for $\mathfrak{a} = \mathfrak{sl}_{l+1}$ of rank l the exponents are $m_i = i$ and so $|\eta_i| = 2i + 1$ for $i = 1, \ldots, l$. Finally, by Proposition 1.5.2 the representation homology is a free graded algebra generated by elements which we denote $\lambda(\eta_{b_1}, \ldots, \eta_{b_p})$ of degree $\sum_{i=1}^{p} 2m_{b_i} + 1$, $p \geq 1$.

2 Traces of abelian Lie algebras

In this section we give explicit formulas for *reduced* Drinfeld traces in the case $\mathfrak{a}$ is an abelian Lie algebra, i.e. a vector space concentrated in degree 0 with zero Lie bracket. Because of the isomorphism 5.14, this is equivalent to computing the character maps for associative algebra $A = \mathrm{Sym}(V)$, where $V = \mathfrak{a}$ viewed as a vector space. So in particular the results of Section 1.3 will apply.

Throughout this section, A will denote the algebra $A = \mathrm{Sym}(V)$. We will write $\overline{\mathrm{Tr}}(\mathfrak{a})$ for $\mathfrak{a} = V$ as $\overline{\mathrm{Tr}}(A)$ or simply as $\overline{\mathrm{Tr}}$ when there is no danger of confusion.

2.1 Symmetric algebras

Recall that $\mathfrak{gl}_1$-valued representation homology of $\mathfrak{a}$ is isomorphic to the abelianization R_{ab} of the standard cofibrant resolution $R = \mathbf{\Omega}(C)$ of A given by the (associative) cobar construction of the Koszul dual coalgebra $C = \mathbf{Sym}^c(V[1])$, *cf.* Example 1.5.3.

Omitting the shifts, we can write

$$R_{\mathrm{ab}} = \mathrm{Sym}(V) \otimes \mathbf{Sym}(\Lambda^2 V \oplus \Lambda^3 V \oplus \ldots \oplus \Lambda^N V) \tag{5.19}$$

with understanding that the elements of $\wedge^k V$ have (homological) degree $k-1$. Note that the de Rham algebra of A can be identified as $\Omega_A^\bullet = \mathbf{Sym}(V \oplus V[1]) = \mathrm{Sym}(V) \otimes \Lambda(V)$, and for each $k \geq 1$, there is a canonical (injective) map

$$s^{-1} : \Omega_A^k \to R_{\mathrm{ab}} \ , \quad a\, dv_1 \ldots dv_k \mapsto a\, \lambda(v_1, \ldots, v_k) \ . \tag{5.20}$$

This map shifts homological degree by -1, whence its notation.

Next, we recall that there is a canonical isomorphism (*cf.* Theorem 3.3.3)

$$\overline{\mathrm{HC}}_\bullet(A) \cong \Omega_{\bar{A}}^\bullet / d\Omega_{\bar{A}}^{\bullet-1} \ . \tag{5.21}$$

On the other hand, regarding $A = \mathrm{Sym}(V)$ as the universal enveloping algebra of the abelian Lie algebra $\mathfrak{a} = V$, we have the (dual) Hodge decomposition (4.27) of $\overline{\mathrm{HC}}_\bullet(A)$. Under the isomorphism (5.21), the image of the direct summand $\overline{\mathrm{HC}}_\bullet^{(r)}(\mathfrak{a})$ in (4.27) is precisely $\mathrm{Sym}^r(V) \otimes \Lambda^\bullet(V)/d[\mathrm{Sym}^{r+1}(V) \otimes \Lambda^{\bullet-1}(V)]$. Thus, for the abelian Lie algebra $\mathfrak{a} = V$, we have an isomorphism

$$\overline{\mathrm{HC}}^{(r)}_{\bullet}(\mathfrak{a}) \;\cong\; \mathrm{Sym}^r(V) \otimes \Lambda^{\bullet}(V)/d[\mathrm{Sym}^{r+1}(V) \otimes \Lambda^{\bullet-1}(V)]\,. \tag{5.22}$$

Upon the isomorphism (4.27), the direct sum of the reduced Drinfeld traces $\overline{\mathrm{Tr}}(\mathfrak{a}) \,:\, \overline{\mathrm{HC}}^{(r)}(\mathfrak{a}) \to R_{\mathrm{ab}}$ becomes the reduced character map $\overline{\mathrm{Tr}}(A) \,:\, \overline{\mathrm{HC}}_{\bullet}(A) \to R_{\mathrm{ab}}$ in (5.17). Let ε be as in (2.6) and let T be the quasi-isomorphism in (3.5.4). By Theorem 3.5.5, the isomorphisms $T \circ \varepsilon\,,\, \varepsilon^{-1}d \,:\, \Omega^{\bullet}_{\bar{A}}/d\Omega^{\bullet-1}_{\bar{A}} \to \mathrm{H}_{\bullet}(R_{\natural})$ coincide. It follows that $\overline{\mathrm{Tr}}(A)$ is identified with the composite map

$$\Omega^{\bullet}_{\bar{A}}/d\Omega^{\bullet-1}_{\bar{A}} \xrightarrow{\varepsilon^{-1}d} \mathrm{H}_{\bullet}(R_{\natural}) \xrightarrow{\overline{\mathrm{Tr}}(A)} R_{\mathrm{ab}}$$

Let $\omega \,\in\, \Omega^p_A$ be a form whose polynomial coefficients are homogeneous of degree $q+1$. The homological degree of $d\omega$ in $\mathrm{DR}^{\bullet}(C)$ is $p+1$. On the other hand, $d\omega$ can also be viewed as an element of $\mathrm{DR}_{\bullet}(C)$, where its homological degree is $2q+p+1$. We may therefore, suppress s^{2q} from the notation when we apply (5.15) to $[d\omega] \,\in\, \mathrm{H}_{p+1}[\mathrm{DR}^{\bullet}(C)]$ by reinterpreting $d\omega$ as an element of $\mathrm{DR}_{\bullet}(C)$. With these conventions, formula (5.15) immediately implies:

Theorem 2.1.1. *The reduced character map* $\overline{\mathrm{Tr}}(A) \,:\, \Omega^{\bullet}_{\bar{A}}/d\Omega^{\bullet-1}_{\bar{A}} \to R_{\mathrm{ab}}$ *is given by*

$$\overline{\mathrm{Tr}}(A)(\omega) \;=\; \sum_{q=0}^{\infty} \frac{1}{(q+1)!}\,[\theta.\Omega^q](d\omega)\,. \tag{5.23}$$

We now illustrate the formula of Theorem 2.1.1 in concrete examples.

2.2 Traces in low homological degrees

Let $x_1, \ldots, x_N$ be a basis for V and let $dx_1, \ldots, dx_N$ denote the corresponding basis elements in $V[1]$ (i.e, $dx_i \,:=\, sx_i$). In this case, the de Rham coalgebra of C is

$\mathbf{Sym}^c(V[1]) \otimes \mathbf{Sym}^c(V[2])$ equipped with the de Rham differential of A, with the difference between $\mathrm{DR}_\bullet(C)$ and $\mathrm{DR}_\bullet(A)$ being the interpretation of the generators of A as degree 2 cogenerators of $\mathrm{DR}_\bullet(C)$ rather than degree 0 generators of A. Let $f(x_1, \dots, x_N)$ be a homogenous polynomial of degree r in $x_1, \dots, x_N$. For notational brevity, the element $f(x_1, \dots, x_N)dx_{i_1} \dots dx_{i_p}$ of $\mathrm{DR}_\bullet(A)$ (which is of cohomological degree p and is in Ω^p_A) shall continue to be denoted by $f(x_1, \dots, x_N)dx_{i_1} \dots dx_{i_p}$ when viewed as an element of $\Omega^r_C \subset \mathrm{DR}_\bullet(C)$ (where its homological degree is $2r+p$). Let $R := \mathbf{\Omega}(C)$. Recall that we denote the element $s^{-1}(dv_1 \dots dv_p) \in R_{\mathrm{ab}}$ by $\lambda(v_1, \dots, v_p)$ for $v_1, \dots, v_p \in V$. In what follows, let $\mathfrak{g} := \mathfrak{gl}_1 = k$. Choose the element 1 as the basis as well as the dual basis of $\mathfrak{g}$. With these choices, (5.3) becomes

$$\theta(f(x_1, \dots, x_N)dx_{i_1} \dots dx_{i_p}) \,=\, f(0, \dots, 0)\lambda(x_{i_1}, \dots, x_{i_p})\,. \tag{5.24}$$

Similarly, (5.4) becomes

$$\Omega(f(x_1, \dots, x_N)dx_{i_1} \dots dx_{i_p}) \,=\, \begin{cases} \lambda(f, x_{i_1}, \dots, x_{i_p}), & \text{if } f \in V \\ 0, & \text{if } f \notin V \end{cases} \tag{5.25}$$

Homological degree 0

Let $f(x_1, \dots, x_N)$ be a homogenous polynomial of degree $r+1$ in A. Note that the summands of df are of the form $u_1 \dots u_r du_{r+1}$, where $u_1, \dots, u_{r+1} \in V$. By (5.23), we need to evaluate $[\theta.\Omega^q](u_1 \dots u_r du_{r+1})$. Note that

$$\Delta^{q+1}(u_1 \dots u_r du_{r+1}) \,=\, \sum \pm u_{S_1} du_{T_1} \otimes \dots \otimes u_{S_{q+1}} du_{T_{q+1}}\,,$$

where the summation above runs over $S_1 \sqcup \ldots \sqcup S_{q+1} = \{1, \ldots, r\}$ and $T_1 \sqcup \ldots \sqcup T_{q+1} = \{r+1\}$. Hence,

$$[\theta.\Omega^q](u_1 \ldots u_r du_{r+1}) = \sum \pm\theta(u_{S_1} du_{T_1})\Omega(u_{S_2} du_{T_2}) \ldots \Omega(u_{S_{q+1}} du_{T_{q+1}}) . \quad (5.26)$$

It follows from (5.24) and (5.25) that the only summands contributing to the R.H.S of (5.26) are those for which $S_1 = \emptyset$, $|S_2| = \ldots = |S_{q+1}| = 1$ and $T_1 \neq \emptyset$. Hence, the R.H.S of (5.26) is nonzero only when $q = r$, in which case it equals

$$r!u_1 \ldots u_{r+1} = r!\iota_\epsilon(u_1 \ldots u_r du_{r+1}) ,$$

where ι_ϵ denotes contraction with the Euler vector field $\epsilon := \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. Therefore, for f homogenous in A of degree $r+1$, $r \geq 0$,

$$\overline{\mathrm{Tr}}(A)(f) = \frac{1}{(r+1)!}[\theta.\Omega^r](df) = \frac{1}{r+1}\iota_\epsilon(df) = f . \quad (5.27)$$

Homological degree 1

Let $\omega = \sum_{i=1}^N f_i dx_i$ where the coefficients f_i are homogenous of degree $r+1$. Note that the summands of $d\omega$ are of the form $u_1 \ldots u_r du_{r+1} du_{r+2}$, where $u_1, \ldots, u_{r+2} \in V$. By (5.23), we need to evaluate $[\theta.\Omega^q](u_1 \ldots u_r du_{r+1} du_{r+2})$. Note that

$$\Delta^{q+1}(u_1 \ldots u_r du_{r+1} du_{r+2}) = \sum \pm u_{S_1} du_{T_1} \otimes \ldots \otimes u_{S_{q+1}} du_{T_{q+1}} ,$$

where the summation above runs over $S_1 \sqcup \ldots \sqcup S_{q+1} = \{1, \ldots, r\}$ and $T_1 \sqcup \ldots \sqcup T_{q+1} = \{r+1, r+2\}$. Hence,

$$[\theta.\Omega^q](u_1 \ldots u_r du_{r+1} du_{r+2}) = \sum \pm\theta(u_{S_1} du_{T_1})\Omega(u_{S_2} du_{T_2}) \ldots \Omega(u_{S_{q+1}} du_{T_{q+1}}) . \quad (5.28)$$

It follows from (5.24) and (5.25) that the only summands contributing to the R.H.S of (5.28) are those for which $S_1 = \emptyset$, $|S_2| = \ldots = |S_{q+1}| = 1$ and $T_1 \neq \emptyset$. Hence, the R.H.S of (5.28) is nonzero only when $q = r$. In this case, the summands for which $T_1 = \{r+1, r+2\}$ contribute

$$r!\lambda(u_{r+1}, u_{r+2})u_1 \dots u_r \;=: \; r!s^{-1}(u_1 \dots u_r du_{r+1} du_{r+2})$$

to $[\theta.\Omega^r](u_1 \dots u_r du_{r+1} du_{r+2})$. The summands for which $|T_1| = 1$ together add up to

$$\begin{aligned} r!(\sum_{p=1}^{i} u_1 \dots \hat{u}_p \dots u_{r+1}\lambda(u_p, u_{r+2}) - u_1 \dots \hat{u}_p \dots u_r u_{r+2}\lambda(u_p, u_{r+1}) \\ = r!s^{-1}(d\iota_\epsilon - 2)(u_1 \dots u_r du_{r+1} du_{r+2}))\,. \end{aligned}$$

Hence,

$$\begin{aligned} \overline{\mathrm{Tr}}(A)(\omega) &= \frac{1}{(r+1)!}[\theta.\Omega^r](d\omega) \\ &= \frac{1}{(r+1)!}[r!s^{-1}d\omega + r!s^{-1}(d\iota_\epsilon - 2)(d\omega)] \\ &= \frac{1}{(r+1)!}[r!s^{-1}d\omega + r!s^{-1}(d\iota_\epsilon + \iota_\epsilon d - 2)(d\omega)] \\ &= \frac{1}{(r+1)!}[r!s^{-1}d\omega + r!s^{-1}r(d\omega)] \\ &= s^{-1}d\omega\,. \end{aligned}$$

Homological degree 2

Let $\omega = \sum_{i<j} f_{ij} dx_i dx_j$ be a two-form whose coefficients f_{ij} are homogenous polynomials of degree $r+1$. Note that the summands of $d\omega$ are of the form $u_1 \dots u_r du_{r+1} du_{r+2} du_{r+3}$, where $u_1, \dots, u_{r+3} \in V$. By (5.23), we need to evaluate $[\theta.\Omega^q](u_1 \dots u_r du_{r+1} du_{r+2} du_{r+3})$. Note that

$$\Delta^{q+1}(u_1 \dots u_r du_{r+1} du_{r+2} du_{r+3}) \;=\; \sum \pm u_{S_1} du_{T_1} \otimes \dots \otimes u_{S_{q+1}} du_{T_{q+1}}\,,$$

where the summation above runs over $S_1 \sqcup \ldots \sqcup S_{q+1} = \{1, \ldots, r\}$ and $T_1 \sqcup \ldots \sqcup T_{q+1} = \{r+1, r+2, r+3\}$. Hence,

$$[\theta.\Omega^q](u_1 \ldots u_r du_{r+1} du_{r+2} du_{r+3}) = \sum \pm \theta(u_{S_1} du_{T_1}) \Omega(u_{S_2} du_{T_2}) \ldots \Omega(u_{S_{q+1}} du_{T_{q+1}}). \tag{5.29}$$

It follows from (5.24) and (5.25) that the only summands contributing to the R.H.S of (5.29) are those for which $S_1 = \emptyset$, $|S_2| = \ldots = |S_{q+1}| = 1$ and $T_1 \neq \emptyset$. Hence, the R.H.S of (5.29) is nonzero only when $q = r$. The summands for which $T_1 = \{r+1, r+2, r+3\}$ contribute

$$r! s^{-1}(u_1 \ldots u_r du_{r+1} du_{r+2} du_{r+3})$$

to the R.H.S of (5.29). Similarly, the summands for which $|T_1| = 1$ and one of $|T_2|, \ldots, |T_{r+1}|$ is 2 add up to

$$r! s^{-1} \sum_{p=1}^{r} u_1 \ldots \hat{u}_p \ldots u_r [u_{r+1} \lambda_{p,r+2,r+3} - u_{r+2} \lambda_{p,r+1,r+3} + u_{r+3} \lambda_{p,r+1,r+2}]$$
$$= r! s^{-1}(d\iota_\epsilon - 3)(u_1 \ldots u_r du_{r+1} \ldots du_{r+3}) \, .$$

where for brevity we denoted $\lambda_{pqr} := \lambda(u_p, u_q, u_r)$. Similarly, the summands for which $|T_1| = 2$ add up to $-2r! \cdot D^{(2,2)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+3})$, where $D^{(2,2)}(u_1 \ldots u_r du_{r+1} \ldots du_{r+3})$ is given by

$$-\frac{1}{2} \sum_{j=1}^{r} u_1 \ldots \hat{u}_j \ldots u_r [-\lambda_{j,r+1} \lambda_{r+2,r+3} + \lambda_{j,r+2} \lambda_{r+1,r+3} - \lambda_{j,r+3} \lambda_{r+1,r+2}]$$

Finally the summands for which $|T_1| = 1$ and two of $|T_2|, \ldots, |T_{r+1}|$ are 1 add up to $r! \hat{D}^{(2,2,1)}(u_1 \ldots . u_r du_{r+1} \ldots du_{r+3})$ where

$$\hat{D}^{(2,2,1)}(u_1 \ldots . u_r du_{r+1} \ldots du_{r+3}) :=$$

$$\sum_{1 \leq k \neq l \leq r} u_1 \ldots \hat{u}_k \ldots \hat{u}_l \ldots u_r [u_{r+3} \lambda_{k,r+1} \lambda_{l,r+2} + u_{r+2} \lambda_{k,r+1} \lambda_{l,r+3} + u_{r+1} \lambda_{k,r+2} \lambda_{l,r+3}]$$

A direct computation shows that

$$\hat{D}^{(2,2,1)} \circ d = -(r-1) D^{(2,2)} \circ d \tag{5.30}$$

on 2-forms with polynomial coefficients that are homogenous of degree $r+1$. Hence,

$$\begin{aligned}
\overline{\mathrm{Tr}}(\omega) &= \frac{1}{(r+1)!}[\theta.\Omega^r](d\omega) \\
&= \frac{1}{(r+1)!}[r!s^{-1}d\omega + r!s^{-1}(d\iota_\epsilon - 3)(d\omega) - 2r!D^{(2,2)}(d\omega) + r!\hat{D}^{(2,2,1)}(d\omega)] \\
&= \frac{1}{(r+1)!}[r!s^{-1}d\omega + r!s^{-1}r(d\omega) - (r+1)r!D^{(2,2)}(d\omega)] \\
&= s^{-1}d\omega - D^{(2,2)}(d\omega)\,.
\end{aligned}$$

2.3 Traces as differential operators

We aim to provide a formula for reduced characters of $A = \mathrm{Sym}(V)$ in arbitrary homological degree. To this end, for each $1 \leq p \leq k \leq \dim(V)$, let us define a linear map

$$V \otimes \Lambda^k(V) \to \Lambda^p(V) \otimes \Lambda^{k+1-p}(V) \tag{5.31}$$

which sends $u \otimes v_1 \wedge \ldots \wedge v_k \in V \otimes \Lambda^k(V)$ to

$$\frac{1}{p!}\sum_{j_1<\ldots<j_{p-1}} \varepsilon(p,s)\,(u \wedge v_{j_1} \wedge \ldots \wedge v_{j_{p-1}}) \,\otimes\, (v_1 \wedge \ldots \wedge \hat{v}_{j_1} \wedge \ldots \wedge \hat{v}_{j_{p-1}} \wedge \ldots \wedge v_k)$$

where $\varepsilon(p,s) = (-1)^{\sum j_s - \frac{(p-1)(p-2)}{2}}$, and with the convention that this is the identity map if $p = 1$. By duality, (5.31) gives a canonical map

$$\Delta_k^{(p,k+1-p)} :\ \Lambda^k(V) \to V^* \otimes \Lambda^p(V) \otimes \Lambda^{k+1-p}(V)\,. \tag{5.32}$$

Now, for any multi-index $(i_1, \ldots, i_m) \in \mathbf{N}^m$ such that $i_1 + \ldots + i_m = k + m - 1$, we can construct

$$\Delta_k^{(i_1,\ldots,i_m)} :\ \Lambda^k(V) \to \mathrm{Sym}^{m-1}(V^*) \otimes \Lambda^{i_1}(V) \otimes \ldots \otimes \Lambda^{i_m}(V) \tag{5.33}$$

by iterating (5.32):

$$\Lambda^k(V) \to V^* \otimes \Lambda^{k+1-i_m}(V) \otimes \Lambda^{i_m}(V) \to \ldots \to (V^*)^{\otimes m-1} \otimes \Lambda^{i_1}(V) \otimes \ldots \otimes \Lambda^{i_m}(V)$$

and then projecting $(V^*)^{\otimes m-1} \twoheadrightarrow \mathrm{Sym}^{m-1}(V^*)$.

Finally, interpreting the elements $\mathrm{Sym}(V^*)$ as constant coefficient differential operators on $\mathrm{Sym}(V)$, we define differential operator $D_k^{(i_1,\ldots,i_m)}$ on forms as follows:

$$\begin{array}{c} \mathrm{Sym}(V)\otimes\Lambda^k V \\ \Big\downarrow {\scriptstyle 1\otimes\Delta_k^{(i_1,\ldots,i_m)}} \\ \mathrm{Sym}(V)\otimes\mathrm{Sym}^{m-1}(V^*)\otimes\Lambda^{i_1}(V)\otimes\ldots\otimes\Lambda^{i_m}(V) \\ \Big\downarrow {\scriptstyle \mathrm{act}\,\otimes\,(s^{-1})^{\otimes m}} \\ R_{\mathrm{ab}} \end{array} \tag{5.34}$$

where $\mathrm{act}: \mathrm{Sym}^{m-1}(V^*)\otimes\mathrm{Sym}(V)\to\mathrm{Sym}(V)\hookrightarrow R_{\mathrm{ab}}$ is the action map and s^{-1} is the embedding defined by $s^{-1}(v_1\wedge\ldots\wedge v_k) := \lambda(v_1,\ldots,v_k)$.

For example, the first order differential operator $D_k^{(p,\,k+1-p)}: \Omega^k(A)\to R_{\mathrm{ab}}$ whose value on a differential form $\omega=(u_1\ldots u_n)\,dv_1\ldots dv_k$ is given by

$$\frac{1}{p!}\sum_{i=1}^{n}(u_1\ldots\hat{u}_i\ldots u_n)\sum_{j_1<\ldots<j_{p-1}}\pm\lambda(u_i,v_{j_1},\ldots,v_{j_{p-1}})\lambda(v_1,\ldots,\hat{v}_{j_1},\ldots,\hat{v}_{j_{p-1}},\ldots,v_k),$$

where the sign $\pm$ is given by $(-1)^{\sum j_s-\frac{(p-1)(p-2)}{2}}$ for $j_1<\ldots<j_{p-1}$. In particular, for $p=2$, $k=3$ and $\omega=(u_1\ldots u_n)dv_1dv_2dv_3$ one has $D^{(2,2)}[\omega]=$

$$\frac{1}{2}\sum_{i=1}^{n}(u_1\ldots\hat{u}_i\ldots u_n)[\lambda(u_i,v_1)\lambda(v_2,v_3)-\lambda(u_i,v_2)\lambda(v_1,v_3)+\lambda(u_i,v_3)\lambda(v_1,v_2)]$$

The character formula

Theorem 2.3.1. *Let $\omega\in\mathrm{Sym}^{r+1}(V)\otimes\Lambda^l(V)\subset\Omega^l_A$ be an l-form with homogeneous polynomial coefficients of degree $r+1$. Then,*

$$\overline{\mathrm{Tr}}(A)(\omega) = s^{-1}(d\omega)+\sum_{i_1+\ldots+i_m=l+m}c^{(i_1,\ldots,i_m)}D^{(i_1,\ldots,i_m)}(d\omega),$$

where the sum runs over all tuples $(i_1,\ldots,i_m)$ such that $i_1,\ldots,i_{m-1}\geq 2$ and $i_m\geq 1$ adding up to $l+m$ and where $(r+1)c^{(i_1,\ldots,i_m)}$ depends only on $i_1,\ldots,i_m$.

Proof. By (5.23),

$$\overline{\mathrm{Tr}}(A)(\omega) \,=\, \sum_{q=0}^{\infty} \frac{1}{(q+1)!}[\theta.\Omega^q](d\omega)\,.$$

Note that $d\omega$ is a k-linear combination of summands of the form $u_1 \dots u_r du_{r+1} \dots du_{r+l+1}$, where $u_1, \dots, u_{r+l+1} \in W$. Further observe that

$$\Delta^{q+1}(u_1 \dots u_r du_{r+1} \dots du_{r+l+1}) \,=\, \sum \pm u_{S_1} du_{T_1} \otimes \dots \otimes u_{S_{q+1}} du_{T_{q+1}}\,,$$

where the summation above runs over $S_1 \sqcup \dots \sqcup S_{q+1} \,=\, \{1, \dots, r\}$ and $T_1 \sqcup \dots \sqcup T_{q+1} = \{r+1, \dots, r+l+1\}$. Hence,

$$[\theta.\Omega^q](u_1 \dots u_r du_{r+1} \dots du_{r+l+1}) \,=\, \sum \pm\theta(u_{S_1} du_{T_1})\Omega(u_{S_2} du_{T_2}) \dots \Omega(u_{S_{q+1}} du_{T_{q+1}})\,. \tag{5.35}$$

It follows from (5.24) and (5.25) that the only summands contributing to the R.H.S of (5.35) are those for which $S_1 \,=\, \emptyset$, $|S_2| = \dots = |S_{q+1}| = 1$ and $T_1 \neq \emptyset$. Hence, the R.H.S of (5.35) is nonzero only when $q = r$.

Given any tuple $(i_1, \dots, i_m)$ with $i_1, \dots, i_{m-1} \,\geq\, 2$ and $i_m \,\geq\, 1$ such that $i_1 + \dots + i_m \,=\, l + m$, the summands on the R.H.S of (5.35) with $|T_1| = i_m$ and $m-1$ among $|T_2|, \dots, |T_{r+1}|$ being equal to $i_1, \dots i_{m-1}$ contribute $r!\hat{D}^{(i_1,\dots,i_m)}(u_1 \dots u_r du_{r+1} \dots du_{r+l+1})$ equal to the sum

$$\sum_{\underline{T}} \sum_{j_1 \neq \dots \neq j_{m-1}} \pm u_1 \dots \hat{u}_{j_1} \dots \hat{u}_{j_{m-1}} \dots u_r s^{-1}(du_{T_1}) s^{-1}(du_{j_1} du_{T_2}) \dots s^{-1}(du_{j_{m-1}} du_{T_m})$$

where the summation runs over all m-tuples $\underline{T} = (T_1, \dots, T_m)$ of sets with cardinalities $(i_m, i_1 - 1, \dots, i_{m-1} - 1)$ that partition the set $\{r+1, \dots, r+l+1\}$, i.e. $T_1 \sqcup \dots \sqcup T_m = \{r+1, \dots, r+l+1\}$. Note that $\hat{D}^{(i_1,\dots,i_m)} \,=\, \hat{c}^{(i_1,\dots,i_m)} D^{(i_1,\dots,i_m)}$, where the constant $\hat{c}^{(i_1,\dots,i_m)}$ depends only on $(i_1, \dots, i_m)$.

The summands on the R.H.S for which $|T_1| \,=\, l+1$ contribute an element

$r!s^{-1}(u_1 \dots u_r du_{r+1} \dots du_{r+l+1})$. Similarly, for $\omega = u_1 \dots u_r du_{r+1} \dots du_{r+l+1}$

$$\begin{aligned}
\hat{D}^{(l,1)}(\omega) &= \sum_{i=1}^{r} \sum_{j=1}^{l+1} (-1)^{j-1} u_1 \dots \hat{u_i} \dots u_r u_{r+j} s^{-1}(du_i du_{r+1} \dots \hat{du}_{r+j} \dots du_{r+l+1}) \\
&= s^{-1}(d\iota_\epsilon - l - 1)(u_1 \dots u_r du_{r+1} \dots du_{r+l+1})\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{1}{(r+1)!}[\theta.\Omega^r](\eta) = &\frac{1}{r+1}s^{-1}(\eta) + \frac{1}{r+1}s^{-1}(d\iota_\epsilon - l - 1)(\eta) + \\
&+ \sum_{i_1+\ldots+i_m = l+m} \frac{1}{r+1}\hat{D}^{(i_1,\ldots,i_m)}(\eta) \\
= &\frac{1}{r+1}s^{-1}(\eta) + \frac{1}{r+1}s^{-1}(d\iota_\epsilon - l - 1)(\eta) + \\
&+ \sum_{i_1+\ldots+i_m = l+m} \frac{1}{r+1}\hat{c}^{(i_1,\ldots,i_m)} D^{(i_1,\ldots,i_m)}(\eta)
\end{aligned}$$

for any $\eta \in \mathrm{Sym}^r(V) \otimes \Lambda^{l+1}(V) \subset \Omega_A^{l+1}$. Hence, for $\omega \in \mathrm{Sym}^{r+1}(V) \otimes \Lambda^l(V)$,

$$\begin{aligned}
\frac{1}{(r+1)!}[\theta.\Omega^r](d\omega) = &\frac{1}{r+1}s^{-1}(d\omega) + \frac{1}{r+1}s^{-1}(d\iota_\epsilon - l - 1)(d\omega) + \\
&+ \sum_{i_1+\ldots+i_m = l+m} \frac{1}{r+1}\hat{c}^{(i_1,\ldots,i_m)} D^{(i_1,\ldots,i_m)}(d\omega) \\
= &s^{-1}(d\omega) + \sum_{i_1+\ldots+i_m = l+m} \frac{1}{r+1}\hat{c}^{(i_1,\ldots,i_m)} D^{(i_1,\ldots,i_m)}(d\omega)\,.
\end{aligned}$$

This proves the desired result. $\square$

Remark 2.3.2. Comparing the formula in Theorem 2.3.1 with the computation in Section 2.2 when $l = 2$,we see that $c^{(2,2)} = \frac{-2}{r+1}$ and $c^{(2,2,1)} \neq 0$. However, since $\hat{D}^{(2,2,1)} \circ d = -(r-1)D^{(2,2)} \circ d$, the summands $\frac{1}{r+1}\hat{D}^{(2,2)}(d\omega)$ and $\frac{1}{r+1}\hat{D}^{(2,2,1)}(d\omega)$ of $\overline{\mathrm{Tr}}(A)(\omega)$ add up to $-D^{(2,2)}(d\omega)$.

2.4 Examples

We illustrate the formulas of Section 2.2 for polynomial algebras in two and three variables.

Polynomials of two variables

Let $\dim(V) = 2$. Choose a basis in V and identify $A = k[x, y]$. Then $R = k\langle x,\, y,\, t\rangle$ with $\deg x = \deg y = 0$ and $\deg t = 1$. The differential on R is defined by $\delta t = [x, y]$, so that $t = -\lambda(x, y)$. Section 2.2 says that $\overline{\mathrm{Tr}}(A) : \Omega^1_A \to R_{\mathrm{ab}}$ is given by

$$\overline{\mathrm{Tr}}(A)[P\,dx + Q\,dy] \;=\; s^{-1}[(Q_x - P_y)dxdy] = (Q_x - P_y)\,\lambda(x, y) = (P_y - Q_x)\,t\ .$$

This formula can also be obtained directly from the explicit formulas of [6, Ex. 4.1].

Polynomials of three variables

Let $A = k[x, y, z]$. Using the notation of [5, Ex. 6.3.2], we write the minimal resolution of A in the form $R = k\langle x, y, z, \xi, \theta, \lambda, t\rangle$, where $\deg x = \deg y = \deg z = 0$, $\deg \xi = \deg \theta = \deg \lambda = 1$ and $\deg t = 2$. The differential on R is defined by

$$\delta\xi = [y, z], \quad \delta\theta = [z, x], \quad \delta\lambda = [x, y]; \quad \delta t = [x, \xi] + [y, \theta] + [z, \lambda]$$

Comparing with (5.18) we see that

$$\xi = \lambda(z, y)\ , \quad \theta = \lambda(x, z)\ , \quad \lambda = \lambda(y, x)\ , \quad t = \lambda(x, y, z)\ .$$

By Section 2.2, thee value of $\overline{\mathrm{Tr}}(A) : \Omega^1_A \to R_{\mathrm{ab}}$ on a form $\omega = P\,dx + Q\,dy + R\,dz$ is given by

$$\begin{aligned}
\overline{\mathrm{Tr}}(A)[\omega] &= s^{-1}[(Q_x - P_y)dxdy + (R_y - Q_z)dydz + (P_z - R_x)dzdx] \\
&= (Q_x - P_y)\,\lambda(x,y) + (R_y - Q_z)\,\lambda(y,z) + (P_z - R_x)\,\lambda(z,x) \\
&= (P_y - Q_x)\,\lambda + (Q_z - R_y)\,\xi + (R_x - P_z)\,\theta\ .
\end{aligned}$$

Next, to compute $\overline{\mathrm{Tr}}(A)_2$ we take $\omega \in \Omega^2(A)$ in the form

$$\omega = P\,dxdy + Q\,dydz + R\,dzdx\ .$$

The trace formula in Section 2.2 implies (after a tedious but straightforward calculation) that

$$\overline{\mathrm{Tr}}[\omega] = (P_z{+}Q_x{+}R_y)\,t + (P_z{+}Q_x{+}R_y)_x\,\theta\lambda{+}(P_z{+}Q_x{+}R_y)_y\,\lambda\xi{+}(P_z{+}Q_x{+}R_y)_z\,\xi\theta\ .$$

3 More examples: reduced Drinfeld traces of semi-simple Lie algebras

In this section we provide a few examples of calculations of reduced Drinfeld traces in the case of semi-simple Lie algebras. Our main tool will be the Feigin–Tsygan spectral sequence described in 3.6. We combine the results of Theorem 3.6.4 with Theorem 1.2.1 and the description of the connection θ and its curvature Ω from Section 2.2. Let us do a quick recap of the main formulas.

Recall from Section 1.2 that the reduced Drinfeld traces $\overline{\mathrm{Tr}}_\bullet$ are given in terms of the connection θ and its curvature Ω on the convolution algebra $\mathcal{A} := \mathrm{Hom}(\mathrm{DR}_\bullet(C), R_{\mathrm{ab}})$.

Explicitly, θ and Ω are given by

$$\theta(f(x_1,\ldots,x_N)dx_{i_1}\ldots dx_{i_p}) \;=\; f(0,\ldots,0)\lambda(x_{i_1},\ldots,x_{i_p})$$

$$\Omega(f(x_1,\ldots,x_N)dx_{i_1}\ldots dx_{i_p}) \;=\; \begin{cases} \lambda(f,x_{i_1},\ldots,x_{i_p}), & \text{if } f\in\mathfrak{a} \\ 0, & \text{if } f\notin\mathfrak{a} \end{cases}$$

Since we are working with $\mathfrak{gl}_1$-valued representation homology, the formula (5.15) becomes

$$\mathrm{Tr}_n \;=\; \sum_{q=0}^{\infty}\frac{1}{(q+1)!}[\theta.\Omega^q]\, s^{2q}\,. \tag{5.36}$$

To evaluate $[\theta.\Omega^q]$ on a form $u_1\ldots u_r du_{r+1}\ldots du_{r+s}$, let

$$\Delta^{q+1}(u_1\ldots u_r du_{r+1}\ldots du_{r+s}) \;=\; \sum \pm u_{S_1}du_{T_1}\otimes\ldots\otimes u_{S_{q+1}}du_{T_{q+1}}\,,$$

where Δ is the comultiplication in $\mathrm{DR}_\bullet(C)$ and the summation above runs over $S_1\sqcup\ldots\sqcup S_{q+1} \;=\; \{1,\ldots,r\}$ and $T_1\sqcup\ldots\sqcup T_{q+1}=\{r+1,\ldots,r+s\}$. Then,

$$[\theta.\Omega^q](u_1\ldots u_r du_{r+1}\ldots du_{r+s}) \;=\; \sum \pm\theta(u_{S_1}du_{T_1})\Omega(u_{S_2}du_{T_2})\ldots\Omega(u_{S_{q+1}}du_{T_{q+1}})\,. \tag{5.37}$$

It follows that a necessary condition for $[\theta.\Omega^q](u_1\ldots u_r du_{r+1}\ldots du_{r+s})$ to be non-zero is that $S_1 \;=\; \emptyset$, $|S_2|=\ldots=|S_{q+1}|=1$ and $T_1\neq\emptyset$.

Lie algebra $\mathfrak{a}=\mathfrak{sl}_2$

The goal is to compute the reduced Drinfeld traces

$$\overline{\mathrm{Tr}}_\bullet\colon \overline{\mathrm{HC}}_\bullet(\mathcal{U}\mathfrak{sl}_2)\to \mathrm{HR}_\bullet(\mathfrak{sl}_2,\mathfrak{gl}_1)$$

Fix the standard basis e,f,h of $\mathfrak{sl}_2$. Recall from Example 1.5.4 that $\mathrm{HR}_\bullet(\mathfrak{sl}_2,\mathfrak{gl}_1)\simeq$ $\mathbf{Sym}(\lambda(\eta))$ is a polynomial algebra generated by the element $\lambda(\eta)$ of degree 2. The element $\eta\in\mathrm{H}^{\mathrm{CE}}_\bullet(\mathfrak{sl}_2;k)$ is represented by the volume form $dedfdh\in\Lambda^3(\mathfrak{sl}_2)[3]$.

In Example 3.6.5 we computed the reduced cyclic homology of $\mathcal{U}\mathfrak{sl}_2$. The homology is only non-zero in degrees 0 and 2 with

$$\overline{\mathrm{HC}}_0(\mathcal{U}\mathfrak{sl}_2) \simeq k[w]/k, \ \deg(w) = 0 \qquad \overline{\mathrm{HC}}_2(\mathcal{U}\mathfrak{sl}_2) \simeq k[u]/k, \ \deg(u) = 2 \tag{5.38}$$

The generator w in (5.38) can be represented by the symmetrized Casimir element $2ef+\frac{h^2}{2} \in \mathrm{Sym}(\mathfrak{sl}_2)$. More precisely, the class $w^n \in \overline{\mathrm{HC}}_0(\mathcal{U}\mathfrak{sl}_2)$ can be represented by $w_n = \left(2ef + \frac{h^2}{2}\right)^n \in \Omega^0_{\mathfrak{sl}_2} \simeq \mathrm{Sym}(\mathcal{U}\mathfrak{sl}_2)$. The trace map $\overline{\mathrm{Tr}}_\bullet$ is naturally a map from $\overline{\mathrm{HC}}_{\bullet+1}(C)$ for the Koszul dual coalgebra $C = \mathrm{CE}(\mathcal{U}\mathfrak{sl}_2)$. The element $w_n' \in \overline{\mathrm{HC}}_1(C)$ corresponding to $w_n \in \overline{\mathrm{HC}}_0(\mathcal{U}\mathfrak{sl}_2)$ under the isomorphism of Proposition 3.6.2 is represented by

$$w_n' = d\left(2ef + \frac{h^2}{2}\right)^n = n\left(2ef + \frac{h^2}{2}\right)^{n-1} (2edf + 2fde + hdh) \in \mathrm{DR}_1(C)$$

If we want to compute an element $w_n \in \Omega^2_{\mathfrak{sl}_2}$ that represents the class $u^n \in \overline{\mathrm{HC}}_2(\mathcal{U}\mathfrak{sl}_2)$, we need to use the transgression as in Example 3.6.5. Using the results of that example, $\overline{\mathrm{HC}}_2(\mathcal{U}\mathfrak{sl}_2)$ is spanned by the homology classes of the elements

$$u_n = \left(2ef + \frac{h^2}{2}\right)^{n-1} (hde\, df + edf\, dh + fdh\, de) \in \Omega^2_{\mathfrak{sl}_2} \simeq \mathrm{Sym}(\mathfrak{sl}_2) \otimes \Lambda^2(\mathfrak{sl}_2)[2]$$

whose image in $\overline{\mathrm{HC}}_3(C)$ under the isomorphism of Proposition 3.6.2 is represented by

$$u_n' = (2n+1)\left(2ef + \frac{h^2}{2}\right)^{n-1} dedf dh \in \mathrm{DR}_3(C)$$

Finally, we need to compute the value of the trace $\overline{\mathrm{Tr}}_\bullet$ on the elements above. The 0-th reduced trace

$$\overline{\mathrm{Tr}}_0 \colon \overline{\mathrm{HC}}_0(\mathcal{U}\mathfrak{sl}_2) \simeq \overline{\mathrm{HC}}_1(C) \to \mathrm{HR}_0(\mathfrak{sl}_2, \mathfrak{gl}_1)$$

sends a basis element $w_n = \left(2ef + \frac{h^2}{2}\right)^n \in \overline{\mathrm{HC}}_0(\mathcal{U}\mathfrak{sl}_2)$, $n \geq 1$ to the homology class of the same polynomial in e, f, h but viewed as an element of $\boldsymbol{\Omega}(C)_{\mathrm{ab}} \simeq \mathbf{Sym}(\overline{C}[-1])$.

All these polynomials are homologous to 0, since $\mathrm{HR}_0(\mathfrak{sl}_2, \mathfrak{gl}_1) \simeq k$ is spanned by the constant polynomial 1. Therefore, $\overline{\mathrm{Tr}}_0 = 0$.

When we apply $[\theta.\Omega^q]$ to the element u'_n, if $q \geq 1$ the resulting class in $\mathrm{HR}_2(\mathfrak{sl}_2, \mathfrak{gl}_1)$ will be zero. Indeed, $\mathrm{HR}_2(\mathfrak{sl}_2, \mathfrak{gl}_1)$ is a 1-dimensional vector space spanned by $\lambda(\eta)$, $\deg(\eta) = 3$. If $q \geq 1$, the comultiplication Δ is applied at least once to u'_n resulting in a tensor product of differential forms of rank < 3 and no such form produces a non-zero class in $\mathrm{HR}_2(\mathfrak{sl}_2, \mathfrak{gl}_1)$ (since it is spanned by $\lambda(\eta)$ for a rank 3 form η). So $q = 0$, and $\overline{\mathrm{Tr}}_2 = [\theta]$. Because of the formula for θ above, if $n \geq 1$ the term $f = \left(2ef + \frac{h^2}{2}\right)^{n-1}$ is a homogeneous polynomial of positive degree so $f(0, \ldots, 0) = 0$. Thus, the only potentially non-zero value of $\overline{\mathrm{Tr}}_2$ comes from evaluating $[\theta]$ on $u'_0 = dedfdh$, and this value is by definition $\lambda(\eta)$.

To summarize,

$$\overline{\mathrm{Tr}}_2(u_0) = \lambda(\eta)$$

and all the other $\overline{\mathrm{Tr}}_i$ for $i \neq 2$ and $\overline{\mathrm{Tr}}_2(u_n)$ for $n \neq 0$ are zero.

Lie algebra $\mathfrak{a} = \mathfrak{sl}_3$

Let us quickly sketch the formula for reduced Drinfeld traces in the case $\mathfrak{a} = \mathfrak{sl}_3$. The line of reasoning is similar as for $\mathfrak{sl}_2$ above, so we will skip details and freely use formulas and notations of the example of $\mathfrak{sl}_2$ above. What is surprising is that almost the entire calculation is just comparing various degrees and eliminating the cases where $\overline{\mathrm{Tr}}$ is forced to be zero.

Before we proceed, let us introduce some notation. Let $v_0 \in E^{\infty}_{i,i+j}$ be represented by an i-admissible monomial $v = \sigma_{a_1}^{s_1} \ldots \sigma_{a_p}^{s_p} \eta_{b_1} \ldots \eta_{b_q} \in E^0_{i,i+j}$ of order j, *cf.* Section 3.6. We denote by $\deg_\sigma(v) := \deg\left(\sigma_{a_1}^{s_1} \ldots \sigma_{a_p}^{s_p}\right)$ the weight degree of the symmetric part, and by $\deg_\eta(v) := j$ the order. When we obtain the corresponding element $v_i \in \mathrm{DR}_\bullet(C)$, it has degrees $\deg_\sigma(v_i) = \deg_\sigma(v_0) - i - 1$ and $\deg_\eta(v_i) = j + 2i + 1$.

Lie algebra $\mathfrak{sl}_3$ has exponents $m_1 = 1$ and $m_2 = 2$. The algebra $\mathrm{Sym}(\mathfrak{sl}_3)^{\mathfrak{sl}_3}$ is generated by two elements σ_1 of degree 2 and σ_2 of degree 3. Thus, for any monomial $v = \sigma_1^a \sigma_2^b \eta_1^{\varepsilon_1} \eta_2^{\varepsilon_1} \in E^\infty$ we have $\deg_\sigma(v) = 2a + 3b$.

The Chevalley–Eilenberg homology $\mathrm{H}_\bullet^{\mathrm{CE}}(\mathfrak{sl}_3; k) \simeq \mathbf{Sym}(\eta_1, \eta_2)$ with $\deg(\eta_1) = 3$ and $\deg(\eta_2) = 5$. Thus, by Proposition 1.5.2 algebra $\mathrm{HR}_\bullet(\mathfrak{sl}_3, \mathfrak{gl}_1) = \mathbf{Sym}(\lambda(\eta_1), \lambda(\eta_2), \lambda(\eta_1, \eta_2))$ is generated by three elements of degrees $2, 4, 7$ respectively. There are the following admissible monomials:

- 0- and 1-admissible monomials $\sigma_1^a \sigma_2^b$ with $a, b \geq 0$ not all zero of order $j = 0$;
- 0- and 1-admissible monomials $\sigma_1^a \sigma_2^b \eta_2$ with $a \geq 1$ of order $j = 5$;
- 2-admissible monomials are σ_2^b for $b \geq 1$.

We now go case-by-case and see which trace maps can potentially be non-zero. We will use notation from formula (5.37) and the discussion underneath it.

Case $i = 0$ and $j = 0$. Monomial $\sigma_1^a \sigma_2^b \in E_{00}^\infty$ with $a, b \geq 0$ not all zero. The corresponding element u_0 of $\mathrm{DR}_\bullet(C)$ has degrees $\deg_\eta = 1$ and $\deg_\sigma = 2a + 3b - 1$. Because $\mathrm{H}_\bullet(\mathfrak{sl}_3, \mathfrak{gl}_1)$ has no elements of degree 1, the trace is zero on such monomials.

Case $i = 1$ and $j = 0$. Monomial $\sigma_1^a \sigma_2^b \in E_{11}^\infty$ with $a, b \geq 0$ not all zero. The corresponding element u_0 of $\mathrm{DR}_\bullet(C)$ has degrees $\deg_\eta = j + 2i + 1 = 3$ and $\deg_\sigma = 2a + 3b - i - 1 = 2a + 3b - 2$. When could applying $[\theta.\Omega^q]$ to such elements be non-zero? In the notation of 5.37, since $T_1 \neq \emptyset$ and the smallest degree of a generator in $\mathrm{H}_\bullet(\mathfrak{sl}_3, \mathfrak{gl}_1)$ is 2, the set T_1 should have three elements. Thus,

$q = 2a + 3b - 2$ should equal 0 (since there will be no other sets $T_2, \ldots, T_{q+1}$ besides T_1). This can only be is $a = 1$ and $b = 0$. As a result, for the case $i = 1, j = 0$ the only value that could potentially be non-zero is $[\theta](\sigma_1)$. We already know from $\mathfrak{sl}_2$ that $[\theta](\sigma_1) = \lambda(\eta_1)$.

Case $i = 0$ **and** $j = 5$**.** Monomial $\sigma_1^a \sigma_2^b \eta_2 \in E_{11}^{\infty}$ with $a \geq 1$. The degrees are $\deg_\eta = 6$ and $\deg_\sigma = 2a + 3b - 1$. Once again, we are applying $[\theta.\Omega^q]$ and see when it can potentially be non-zero. Since $T_1 \neq \emptyset$ and $\theta(u_{S_1} du_{T_1})$ should be non-zero, $T_1 = \{\eta_1\}$ or $T_1 = \{\eta_2\}$. Note that every other set $T_2, \ldots, T_{q+1}$ should have at least 2 elements in it. Moreover, $q = 2a + 3b - 1$ so all together $T_2, \ldots, T_{q+1}$ have at least $2 \cdot q$ elements, but we know that this number equals $6 - |T_1| = 3$. So, q must equal to $q = 1$. But then, there is only one other set, T_2, having 3 elements in it, and when computed $\Omega(u_{S_2} du_{T_2}) = \lambda($ 4 elements of $\mathfrak{sl}_3)$ has degree 3. Since there is no generator in $\mathrm{HR}_\bullet(\mathfrak{sl}_3, \mathfrak{gl}_1)$ of degree 3, the map $\overline{\mathrm{Tr}}$ is zero in this case. The case $T_1 = \{\eta_2\}$ is treated the same.

Other cases are considered in the same vein (they are shorter, we discussed the longest ones already). As a result, the following are the only non-zero values:

$$\overline{\mathrm{Tr}}_2(\sigma_1) = \lambda(\eta_1), \quad i = 1, j = 0$$

$$\overline{\mathrm{Tr}}_7(\sigma_1 \eta_2) = \lambda(\eta_1, \eta_2), \quad i = 1, j = 5$$

and all other trace maps are zero on all other monomials.

4 Reduced derived characters: a combinatorial description

In this section, we will give another formula for reduced trace maps of symmetric algebras in terms of binary trees. Throughout, we will keep the notation and assumptions of the previous section.

Our starting point is Theorem 4.2 of [6] that gives a general formula for the derived character maps in terms of Taylor components $f_{k+1} : A^{\otimes(k+1)} \to R$ of an A_∞-quasi-isomorphism $f : A \to R$ inverting a DG algebra resolution of A. We apply this formula to the minimal resolution $R := \mathbf{\Omega}(C)$ of the symmetric algebra $A = \mathbf{Sym}(V)$; as a consequence, we get the following

Proposition 4.0.1. *The map* $\overline{\mathrm{Tr}}(A) : \Omega_A^\bullet/d\Omega_A^{\bullet-1} \to R_{\mathrm{ab}}$ *is given by the formula*

$$\overline{\mathrm{Tr}}(A)[a_0\, da_1 \dots da_k] = \sum_{\sigma\in\mathbf{S}_{k+1}} (-1)^\sigma \, \bar{f}_{k+1}(a_{\sigma^{-1}(0)}, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(k)}) \ , \qquad (5.39)$$

where $\bar{f}_{k+1} : A^{\otimes(k+1)} \to R \twoheadrightarrow R_{\mathrm{ab}}$ *are the components of the* A_∞*-morphism* $A \xrightarrow{f} R \twoheadrightarrow R_{\mathrm{ab}}$.

Proof. $\overline{\mathrm{Tr}}(A)$ is induced by the composition $\bar{T} : \Omega_A^k/d\Omega_A^{k-1} \xrightarrow{\varepsilon_k} \overline{\mathrm{C}}_k^\lambda(A) \xrightarrow{\mathrm{can}\circ T} R_{\mathrm{ab}}$. Here, T is as in (2.9). By [6, (4.21)], $\bar{T}$ is given by

$$\sum_{\tau\in\mathbf{Z}_{k+1}} (-1)^\tau \sum_{\sigma\in\mathbf{S}_k} (-1)^\sigma \, \bar{f}_{k+1}(a_{\tau^{-1}\sigma^{-1}(0)}, a_{\tau^{-1}\sigma^{-1}(1)}, \dots, a_{\tau^{-1}\sigma^{-1}(k)}) \ ,$$

where σ ranges over the subgroup of permutations of $\{0, 1, \dots, k\}$ preserving 0 and τ ranges over the cyclic subgroup $\mathbf{Z}_{k+1}$ of $\mathbf{S}_{k+1}$ generated by $i \mapsto i+1$. Since $\mathbf{S}_{k+1}$ is the product of its subgroups $\mathbf{S}_k$ and $\mathbf{Z}_{k+1}$, the above sum equals the right-hand side of (5.39). $\square$

4.1 Merkulov's construction

Let $\pi\colon R \overset{\sim}{\twoheadrightarrow} A$ be a fixed semi-free resolution. Choose a linear section f_1 of π and identify A with its image in R under f_1. Since R is quasi-isomorphic to A, the complex $R_\bullet$ is acyclic in all degrees ≥ 1. For each $i \geq 0$, we fix a decomposition of R_i such that $R_0 = A \oplus B_0$ and $R_i = B_i \oplus L_i$ for $i \geq 1$. Here $B_i = d_{i+1}(R_{i+1}) \subseteq R_i$, and $L_i = s_i\,(R_i/B_i) \subseteq R_i$, where $s_i \colon R_i/B_i \hookrightarrow R_i$ is a section of the canonical projection $p_i \colon R_i \twoheadrightarrow R_i/B_i$.

Next, we pick a homotopy $h \colon R \to R[1]$ between the maps id_R and $f_1 \circ \pi$ satisfying $h_i|_{L_i} = 0, h_0|_A = 0$ and $h_i|_{B_i} \colon B_i \xrightarrow{\sim} L_{i+1}$. One can construct the components $h_i : R_i \to R_{i+1}$ of h inductively:

Since $d_0 = 0$, the equation for h_0 simply is

$$d_1 h_0 = \mathrm{id}_R - f_1 \pi \ . \tag{5.40}$$

For $n \geq 1$, we have $\pi|_{R_n} \equiv 0$. Hence h_n is defined by

$$d_{n+1} h_n = \mathrm{id}_R - h_{n-1} d_n \tag{5.41}$$

Now, given $h : R \to R[1]$, for $i \geq 1$ we define the operations $\mu_i \colon R^{\otimes i} \to R$ by

- There is no μ_1, but we formally set $h\mu_1 := -\mathrm{id}_R$;
- $\mu_2 : R \otimes R \to R$ is the multiplication map $\mu_2(a_1 \otimes a_2) = a_1 a_2$;
- For $i \geq 2$, μ_i is a map of degree $i-2$ defined by

$$\mu_i := \sum_{\substack{s+t=i \\ s,t\geq 1}} (-1)^{s+1} \mu_2(h\mu_s \otimes h\mu_t) \ . \tag{5.42}$$

Finally, for $k \geq 1$, we define $f_{k+1} : A^{\otimes(k+1)} \to R$ by

$$f_{k+1} := -h_{k-1} \circ \mu_{k+1} \circ f_1^{\otimes(k+1)} \ . \tag{5.43}$$

The following observation is due to Merkulov [33] (see also [29, Prop. 2.3, Lemma 2.5]).

Theorem 4.1.1. *The maps* (5.43) *define an* A_∞*-quasi-isomorphism* $f : A \to R$ *inverse to* π *.*

Remark 4.1.2. In general, if R is any DG algebra, the above construction also yields a (minimal) A_∞ structure on $\mathrm{H}_\bullet(R)$. The corresponding higher multiplications are defined by $m_k := \pi \circ \mu_k \circ f_1^{\otimes(k+1)}$, $k \geq 2$. In the case when $\mathrm{H}_\bullet(R) = A$ is an ordinary algebra, the operations $m_3, m_4, \dots$ are trivial for degree reasons, while m_2 coincides with the induced multiplication on A, since $m_2(a_1, a_2) = \pi(f_1(a_1)f_1(a_2)) = \pi(f_1(a_1))\pi(f_1(a_2)) = a_1a_2$.

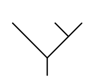
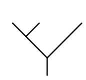
4.2 Traces and binary trees

Substituting (5.43) into formula (5.39) we get

$$\overline{\mathrm{Tr}}(a_0 da_1 \dots da_k) = \sum_{\sigma \in \mathbf{S}_{k+1}} (-1)^{1+\sigma}\, \bar{h}_{k-1}\, \mu_{k+1}\left(f_1(a_{\sigma(0)}), \dots, f_1(a_{\sigma(k)})\right) \qquad (5.44)$$

Merkulov's construction provides us with the recursive formula (5.42) for μ_{k+1}, and hence for f_{k+1}, in terms of operations μ_i with $i < k+1$ and homotopy h.

By $\mathfrak{T}_k$ we denote the set of rooted planar binary trees with $k{+}1$ leaves. Operation f_T for a tree $T \in \mathfrak{T}_k$ is defined in the following way. First of all, we will label all the leaves and internal vertices in the following way. Every leaf we will label by 0. After that, if a vertex v has left son with label l and right son with label r, then we label v by $l + r + 1$. After that, we insert f_1 into each leaf, and if a vertex v was labeled by some number l, we insert $h_{l-1}\mu_2$ into v. In the very last vertex (the one that is adjacent with the root) we insert $-h_{k-1}\mu_2$. Then moving along the tree down to the root we can read off the map f_{k+1}. For example, f_T for the trees

$T_1 =$  $T_2 =$ 

will be just $f_{T_1}(a_0,a_1,a_2) = -h_1\mu_2(f_1(a_0)\otimes h_0\mu_2(f_1(a_1)\otimes f_1(a_2))) = -h_1(\tilde{a}_0\cdot h_0(\tilde{a}_1\cdot \tilde{a}_2))$ and $f_{T_2}(a_0,a_1,a_2) = -h_1(h_0(\tilde{a}_0\cdot\tilde{a}_1)\cdot\tilde{a}_2)$.

There is an algorithm how to determine the sign $(-1)^T$ that corresponds to a tree T. First we label leaves by '+1'. After that, for any vertex v that has left son labeled by a sign l and right son labeled by r, we label v by $l\cdot r\cdot(-1)^{s+1}$, where s is the number of leaves to the left from v. Then the sign of the tree $(-1)^T$ is by definition the sign of the last vertex (the one that is adjacent with the root). For example, for the trees T_1 and T_2 above we will have $(-1)^{T_1} = 1$ and $(-1)^{T_2} = -1$.

The construction defined above (almost) coincides with the construction given in [25, Sect. 6.4]. The only difference is that we expanded all higher multiplications μ_i with $i > 2$.

Lemma 4.2.1 (*cf.* [25])**.**

$$f_{k+1} = \sum_{T\in\mathfrak{T}_k}(-1)^T f_T \tag{5.45}$$

Proof. The proof can be obtained by easy induction on the number of vertices. $\square$

Now if we apply the result of the Lemma 4.2.1 to the formula (5.44) we will get the following expression for the reduced trace:

$$\overline{\mathrm{Tr}}(a_0da_1\dots da_k) \;=\; \sum_{\sigma\in\mathbf{S}_{k+1}}(-1)^\sigma\sum_{T\in\mathfrak{T}_k}(-1)^T\,\bar{f}_T\left(\tilde{a}_{\sigma(0)},\dots,\tilde{a}_{\sigma(k)}\right). \tag{5.46}$$

Two *labeled* planar rooted binary trees (σ, T) and (σ', T') are *equivalent* if there exists a rooted tree isomorphism $\varphi : T \to T'$ that preserves the labels on leaves (labeling is given by a choice of $\sigma \in \mathbf{S}_{k+1}$ which we think of as labels on $k+1$ leaves). Let's denote the set of equivalence classes of pairs (σ, T) by $\langle\mathfrak{T}\rangle_k$.

For any tree T define $[f]_T$ to be a map, obtained from f_T by replacing any $\mu_2(a \otimes b)$ appearing in f_T by $[a, b]$. For example, if $f_T(a_0, a_1, a_2) = -h_1(h_0(\tilde{a}_0 \cdot \tilde{a}_1) \cdot \tilde{a}_2)$, then $[f]_T = -h_1[h_0[\tilde{a}_0, \tilde{a}_1], \tilde{a}_2]$.

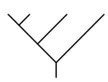
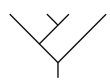
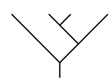
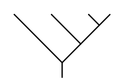
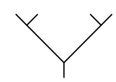
Lemma 4.2.2.

$$\overline{\mathrm{Tr}}(a_0 da_1 \dots da_k) = \sum_{[\sigma, T] \in \langle \mathfrak{T} \rangle_k} (-1)^{\sigma_0} \cdot (-1)^{T_0} \, [\, \bar{f} \,]_{T_0} \left(\tilde{a}_{\sigma_0(0)}, \dots, \tilde{a}_{\sigma_0(k)} \right). \qquad (5.47)$$

Here (σ_0, T_0) *is a representative of the class* $[\sigma, T]$*.*

Proof. Straightforward induction on the number of vertices. □

As an example, let us consider the case $k = 3$. There are 5 elements in $\mathfrak{T}_3$:

$T_1 =$  $T_2 =$  $T_3 =$  $T_4 =$  $T_5 =$ 

Their signs are going to be $(-1)^{T_1} = -1$, $(-1)^{T_2} = +1$, $(-1)^{T_3} = -1$, $(-1)^{T_4} = +1$ and $(-1)^{T_5} = -1$.

There are 15 equivalence classes in $\langle \mathfrak{T} \rangle_3$. There are 3 classes $[(\sigma, T)]$ with $T = T_5$ and $\sigma \in \Sigma_1 := \{(0123), (0213), (0312)\}$. There are 12 classes $[(\sigma, T)]$ with $T = T_4$ and $\sigma \in \Sigma_2 = \{(\sigma(0), \sigma(1), \sigma(2), \sigma(3)) \in \mathbf{S}_4 \mid \sigma(2) < \sigma(3)\}$. So for $\overline{\mathrm{Tr}}_3$, we will have the following explicit formula

$$\begin{aligned} \overline{\mathrm{Tr}}(a_0 da_1 da_2 da_3) &= \sum_{\sigma \in \Sigma_1} (-1)^{\sigma} \, \bar{h}_2 \left[h_0[\tilde{a}_{\sigma(0)}, \tilde{a}_{\sigma(1)}], h_0[\tilde{a}_{\sigma(2)}, \tilde{a}_{\sigma(3)}] \right] \qquad (5.48) \\ &+ \sum_{\sigma \in \Sigma_2} (-1)^{1+\sigma} \, \bar{h}_2 \left[\tilde{a}_{\sigma(0)}, h_1[\tilde{a}_{\sigma(1)}, h_0[\tilde{a}_{\sigma(2)}, \tilde{a}_{\sigma(3)}]] \right] \end{aligned}$$

More explicitly, $\overline{\mathrm{Tr}}$ evaluated on $\omega = a_0 da_1 da_2 da_3$ is given by

$$\begin{aligned}
\overline{\mathrm{Tr}} &= \bar{h}_2\left[h_0[\tilde{a}_0,\tilde{a}_1],h_0[\tilde{a}_2,\tilde{a}_3]\right] - \bar{h}_2\left[h_0[\tilde{a}_0,\tilde{a}_2],h_0[\tilde{a}_1,\tilde{a}_3]\right] + \bar{h}_2\left[h_0[\tilde{a}_0,\tilde{a}_3],h_0[\tilde{a}_1,\tilde{a}_2]\right] \\
&- \bar{h}_2\left[\tilde{a}_0,h_1\left[\tilde{a}_1,h_0[\tilde{a}_2,\tilde{a}_3]\right]\right] + \bar{h}_2\left[\tilde{a}_0,h_1\left[\tilde{a}_2,h_0[\tilde{a}_1,\tilde{a}_3]\right]\right] - \bar{h}_2\left[\tilde{a}_0,h_1\left[\tilde{a}_3,h_0[\tilde{a}_1,\tilde{a}_2]\right]\right] \\
&+ \bar{h}_2\left[\tilde{a}_1,h_1\left[\tilde{a}_0,h_0[\tilde{a}_2,\tilde{a}_3]\right]\right] - \bar{h}_2\left[\tilde{a}_2,h_1\left[\tilde{a}_0,h_0[\tilde{a}_1,\tilde{a}_3]\right]\right] + \bar{h}_2\left[\tilde{a}_3,h_1\left[\tilde{a}_0,h_0[\tilde{a}_1,\tilde{a}_2]\right]\right] \\
&- \bar{h}_2\left[\tilde{a}_1,h_1\left[\tilde{a}_2,h_0[\tilde{a}_0,\tilde{a}_3]\right]\right] + \bar{h}_2\left[\tilde{a}_1,h_1\left[\tilde{a}_3,h_0[\tilde{a}_0,\tilde{a}_2]\right]\right] - \bar{h}_2\left[\tilde{a}_2,h_1\left[\tilde{a}_3,h_0[\tilde{a}_0,\tilde{a}_1]\right]\right] \\
&+ \bar{h}_2\left[\tilde{a}_2,h_1\left[\tilde{a}_1,h_0[\tilde{a}_0,\tilde{a}_3]\right]\right] - \bar{h}_2\left[\tilde{a}_3,h_1\left[\tilde{a}_1,h_0[\tilde{a}_0,\tilde{a}_2]\right]\right] + \bar{h}_2\left[\tilde{a}_3,h_1\left[\tilde{a}_2,h_0[\tilde{a}_0,\tilde{a}_1]\right]\right]
\end{aligned}$$

By Theorem 2.1.1 and Lemma 4.2.2, we have

Corollary 4.2.3.

$$\sum_{[\sigma,T]\in\langle\mathfrak{T}\rangle_k} (-1)^{\sigma_0}(-1)^{T_0}\,[\,\bar{f}\,]_{T_0}\left(\tilde{a}_{\sigma_0(0)},\ldots,\tilde{a}_{\sigma_0(k)}\right) = \sum_{q=0}^{\infty}\frac{1}{(q+1)!}[\theta.\Omega^q](da_0da_1\ldots da_k)$$

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