SURFACES IN THREE- AND FOUR-DIMENSIONAL TOPOLOGY

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SURFACES IN THREE- AND FOUR-DIMENSIONAL TOPOLOGY

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We investigate two ways in which a surface embedded or immersed in a manifold can reveal information about topology of the ambient space. In particular, we prove a special case of the Simple Loop Conjecture for 3-Manifolds and the study trisections of 4-manifolds from the perspectives of the mapping class group and the curve complex of a surface.

BIOGRAPHICAL SKETCH

Drew grew up in Red Hook, NY. He receieved a B.S. in Computation Mathematics from the Rochester Institute of Technology in 2012 and a Ph.D. in Mathematics from Cornell University in 2018.

To Riley, who is (unbeknownst to him) the best friend I've ever had.

ACKNOWLEDGEMENTS

First and foremost, the author is extremely grateful to Jason Manning for his thoughtful advice, friendly critique, and unending patience.

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An additional thanks is due to Alan Reid for pointing out the connection between Example 1.7.2 and Casson's construction in [19].

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Introduction

Most of the projects the author has pursued follow the theme of determining the structure of a 3- or 4-manifold from information contained in its two-dimensional submanifolds. This document contains a summary of two distinct directions of this research.

Chapter 1 studies immersed surfaces in 3-manifolds through the lens of the Simple Loop Conjecture for 3-manifolds. The main result is Theorem 1.2.2, which establishes the SLC for 3-manifolds that admit a particular type of geometric structure.

Chapter 2 introduces and studies trisections of 4-manifolds, which are decompositions of 4-manifolds into simple four-dimensional submanifolds that intersect along simple lower-dimensional submanifolds. Each trisection has a *central surface* that contains all of the information of the trisection and hence of the ambient 4-manifold. After establishing definitions, the chapter focuses on ways of capturing and analyzing the information contained in a trisected 4-manifold via structures built on the central surface, namely the mapping class group and the curve complex. (It is also worth noting that additional work in the spirit of Chapter 2 has been done in part by the author in [7], where the homology and intersection form of a trisected 4-manifold is computed from homological information contained in the central surface.)

CHAPTER 1

SURFACES IN 3-MANIFOLDS: THE SIMPLE LOOP CONJECTURE

Throughout this chapter, we call a loop in a manifold M *essential* if it is neither nullhomotopic nor homotopic into the boundary of M. Loops that are not essential are *inessential*. A loop is *simple* if it is embedded in the ambient manifold.

For a space X, we write $|X|$ to denote the number of connected components of X. For a compact surface Σ with $L \subset \Sigma$ an embedded closed 1-manifold, we will write $\Sigma\backslash\backslash L$ to denote the metric completion of $\Sigma \backslash L$ (with respect to some choice of complete metric on Σ). Thus $\Sigma\backslash\backslash L$ is the space obtained by gluing copies of S^1 onto the open ends of $\Sigma \backslash L$.

1.1 Motivation

1.1.1 Incompressible and π_1-Injective Surfaces

To motivate the topic of this chapter, we begin with two related notions from classical 3-manifold topology: *incompressibility* and π_1*-injectivity*. Throughout this section, let M be a 3-manifold and let $\Sigma \subset M$ be an embedded (connected) surface.

Definition 1.1.1. A surface Σ is *incompressible* in M if every embedded loop in Σ that bounds an embedded disk in M also bounds a disk in Σ. More formally, Σ is incompressible if whenever there is an embedded disk $D \subset M$ such that $D \cap \Sigma = \partial D$, there is an embedded disk $D' \subset \Sigma$ with $\partial D' = \partial D$.

Definition 1.1.2. A surface Σ is π_1*-injective* in M if the homomorphism $\pi_1 \Sigma \to \pi_1 M$ induced by the embedding is injective.

Thus an incompressible surface is one whose topology reflects that of the ambient 3-manifold, while a π_1-injective surface is one whose fundamental group reflects that of the ambient 3-manifold. One can immediately observe a relationship between the two notions:

Lemma 1.1.3. *If $\Sigma \subset M$ is π_1-injective, then it is incompressible.*

Proof. If $D \subset M$ is an embedded disk with $D \cap \Sigma = \partial D$ for which ∂D does *not* bound a disk Σ, then ∂D represents an nontrivial element of the kernel of $\pi_1\Sigma \to \pi_1 M$. Therefore this map is not injective. □

It is natural to ask if the converse is true: is every incompressible surface also π_1-injective? In [36], Stallings constructs an example of a nonorientable surface in a lens space that is incompressible but not π_1-injective. Thus if we seek an affirmative answer to the above question, we must first restrict that class of surfaces we are considering.

1.1.2 2-Sided Surfaces and the Loop Theorem

We can prove a partial converse to Lemma 1.1.3 if we restrict to a particular class of embeddings of surfaces in 3-manifolds.

Definition 1.1.4. If M is a connected manifold, the *orientation character* of M is the homomorphism $\rho_M : \pi_1 M \to \mathbb{Z}/2$ whose value on $b \in \pi_1 M$ is nontrivial if and only if some (and hence any) loop in M representing b is orientation reversing. (Equivalently, $\rho_M(b)$ is nontrivial if and only if b acts on the universal cover of M by an orientation reversing homeomorphism.)

Note that a manifold is orientable if and only if its orientation character is trivial.

Definition 1.1.5. If M and N are connected manifolds with orientation characters ρ_M and ρ_N, a map $F : M \to N$ is called *2-sided* if $\rho_N \circ F_* = \rho_M$. Otherwise F is *1-sided*. Hence F is

2-sided if and only if it takes orientation preserving loops in M to orientation preserving loops in N, and likewise for orientation reversing loops.

Remark 1.1.6. There are other (equivalent) definitions of 2-sidedness for immersions of manifolds, but since most of the arguments in this chapter involve the fundamental groups of the manifolds in question, the given definition will be more useful.

For 2-sided embeddings, the Loop Theorem of Papakyriakopoulos implies the desired converse.

Theorem 1.1.7 (Loop Theorem)**.** *Let M be a 3-manifold with nonempty boundary, and let $f : D^2 \to M$ be a continuous map such that $f(\partial D^2) \subset \partial M$ is an essential curve. Then there is an embedding $f' : D^2 \to M$ with $f(\partial D^2) \subset \partial M$ also essential.*

Corollary 1.1.8. *If $\Sigma \subset M$ is a connected, 2-sided, incompressible, embedded surface, then it is π_1-injective.*

Sketch of proof of corollary. We suppose that Σ is *not* π_1-injective, so there is a map $f : D^2 \to M$ such that $f|_{\partial D^2}$ is essential in Σ. We may properly homotope f so that it is transverse to Σ (using the fact that Σ is two-sided), and then make further modifications to f so that $f^{-1}(\Sigma)$ is a collection of loops in D^2 (including ∂D^2 itself) whose images are essential in Σ. Let ℓ be such a loop that is innermost in D^2, and let $D' \subset D^2$ be the disk bounded by ℓ.

Let M' be the result of cutting M open along Σ; since Σ is two sided, M' has two boundary components that are homeomorphic to Σ. By construction, the map $\bar{f} = f|_{D'}$ satisfies $\bar{f}^{-1}(\partial M') = \partial D'$. Applying the Loop Theorem to $\bar{f}$, we obtain an embedding $\bar{f}' : D' \to M'$ for which $\bar{f}'(\partial D')$ is essential in $\partial M'$. Composing this embedding with the natural projection $M' \to M$, we obtain an embedding of a disk in M whose boundary is an essential curve on Σ, thus showing that Σ is *not* incompressible. □

Thus we see that, if we consider only 2-sided surfaces, then incompressibility and π_1-injectivity are the same notion. See [12] for complete proofs of the Loop Theorem and Corollary 1.1.8.

1.2 The Simple Loop Conjecture for 3-Manifolds

It seems natural to try to extend the discussion of incompressible and π_1-injective surfaces to the context of *immersed* surfaces. To accomplish this, we need to extend the definition of *incompressible* to immersed surfaces.

Definition 1.2.1. Let $F : \Sigma \to M$ be an immersion of a surface Σ in a 3-manifold M. Then F is *incompressible* if every simple loop ℓ in Σ for which $F(\ell)$ bounds an *immersed* disk in M also bounds a disk in Σ.

As in the embedded case, π_1-injective implies incompressible. The converse is the content of the Simple Loop Conjecture for 3-Manifolds. Notice that we restrict to 2-sided immersions.

Conjecture (Problem 3.96 in [17])**.** *Let Σ be a closed surface and let M be a closed 3-manifold. If $F : \Sigma \to M$ is a 2-sided immersion for which the induced map $F_* : \pi_1\Sigma \to \pi_1 M$ is not injective, then there is an essential simple loop in Σ that represents an element of the kernel of F_*.*

1.2.1 Previous Results on the SLC

An analogous result for maps between surfaces (also called the Simple Loop Conjecture) is due to Gabai [8]. The Simple Loop Conjecture for 3-Manifolds is known to hold for a few broad classes of target 3-manifolds.

Seifert fibered 3-manifolds

In [11], Hass shows that the Simple Loop Conjecture holds when the target 3-manifold admits a *Seifert fibering*, which is a foliation by circles. The core of Hass' argument is that an incompressible immersed surface in 3-manifold is homotopoic to a minimal immersed surface, and furthermore in a a Seifert fibered space every such immersed surface is either vertical (a union of fiber circles) or horizontal (transverse to every circle). Hass shows that every horizontal surface in a Seifert fibered space is π_1-injective and that any vertical surface that is not π_1-injective is also not incompressible. Thus every incompressible immersed surface in a Seifert fibered space is π_1-injective.

Graph 3-manifolds

In [30], Rubinstein and Wang extend Hass' result to prove the Simple Loop Conjecture for *graph 3-manifolds*, which are 3-manifolds with a decomposition by incompressible tori into Seifert fibered spaces (referred to as *vertex manifolds*). An immersed surface in such a 3-manifold inherits a decomposition into subsurfaces by considering its intersection with each vertex manifold.

Rubinstein and Wang's argument proceeds by showing that an immersed surface in a graph 3-manifold is π_1-injective if and only if it can be homotoped so that the image of each of these subsurfaces is vertical or horizontal in the vertex manifold that it maps into. An immersion with nontrivial kernel must therefore be unable to be homotoped to be vertical or horizontal on some subsurface. The authors then claim that Hass' result[1] can be applied to find a simple loop in this subsurface with inessential image.

[1] In the Rubinstein-Wang paper, Hass' result is applied to a Seifert fibered 3-manifold *with boundary*, but Hass' statement of the SLC in [11] is stated and proved only for *closed* 3-manifolds.

1.2.2 Main Results

The goal of this chapter is the following result.

Theorem 1.2.2. *The Simple Loop Conjecture holds when the target 3-manifold admits a geometric structure modeled on Sol.*

If M is a 3-manifold that is finitely covered by a torus bundle over S^1, then M admits a geometric structure modeled on Euclidean 3-space, *Nil*, or *Sol*. Since all compact Euclidean and *Nil* manifolds are Seifert fibered (see [34]), we obtain the following corollary by combining the above theorem with Hass' solution to the SLC for Seifert fibered spaces.

Corollary 1.2.3. *The Simple Loop Conjecture holds when the target 3-manifold is finitely covered by a torus bundle over* S^1*.*

Remark 1.2.4. It is worth noting that while the results mentioned in Sections 1.2.1 rely heavily on the geometric structures of the target 3-manifolds, the approaches presented here are almost entirely topological and group-theoretic in nature. There is also a lingering question whether the techniques of [30] apply to *Sol* manifolds, although though the authors seem to be implicitly ruling them out (see for instance, [30, Lemma 1.0.2]).

The rest of this chapter is organized as follows. In Section 1.3 we give some definitions and notation for the objects that will be studied. Section 1.4 contains a brief survey of which compact 3-manifolds admit geometric structures modeled on *Sol*. This entails a refinement of a classification given by Scott in [34], and reduces the problem at hand to studying maps from closed surfaces into certain kinds of torus bundles over S^1 and orientable torus semi-bundles. In Sections 1.5 and 1.6 we give proofs of the Simple Loop Conjecture for these two types of 3-manifold, respectively. We conclude in Section 1.7 with some remarks regarding how the results presented here relate to a group-theoretic formulation of the Simple Loop Conjecture, and we show that it fails to hold for some metabelian targets.

1.3 Definitions

We refer the reader to [34] for an explanation of what it means for a manifold to admit a geometric structure, as well as some basic facts about the Euclidean, *Nil*, and *Sol* geometries. In particular, we will need the following two results.

Theorem 1.3.1 ([34, Theorem 5.2]). *If M is a closed 3-manifold which admits a geometric structure modeled on one of the eight geometries, then the geometry involved is unique.*

Corollary 1.3.2 (see [34, Theorem 5.3(ii)]). *If M is a closed 3-manifold that admits a Seifert fibering, then M does not admit a geometric structure modeled on Sol.*

1.3.1 Torus Bundles and Semi-Bundles

By *torus bundle* we mean a fiber bundle over S^1 whose fibers are tori. This can also be viewed as a quotient $T \times I/((p, 0) \sim (\varphi(p), 1))$ where T is a torus and $\varphi : T \to T$ is a homeomorphism.

For each $i \in \{1, 2\}$, let N_i be either a twisted I-bundle over a torus or a Klein bottle, so that $\partial N_i \cong T$. A *torus semi-bundle* $M = N_1 \cup_\varphi N_2$ is obtained by gluing N_1 and N_2 by a homeomorphism $\varphi : \partial N_1 \to \partial N_2$. Such a 3-manifold is orientable if and only if both N_1 and N_2 are twisted I-bundles over a Klein bottle.

If M is a torus semi-bundle, at times we will refer to the *middle torus* of M, which is the image of ∂N_1 and ∂N_2 after the gluing. We will also make use of maps $\rho_i : \pi_1 N_i \to \mathbb{Z}/2$, which are the quotients of $\pi_1 N_i$ by the index two subgroup corresponding to the double covers of N_i by the product $T \times I$. (This is sometimes called the *monodromy* of the I-bundle N_i.) Notice that, for $b \in \pi_1 N_i$, $\rho_i(b)$ is trivial if and only if b is represented by a loop that is homotopic into ∂N_i. Furthermore, when N_i is a twisted I-bundle over a torus (and is therefore nonorientable), ρ_i coincides with the orientation character of N_i.

If M is a torus semi-bundle, then there is a double cover of M that is the union of the two $T \times I$ double covers of N_1 and N_2 along their boundaries (via some homeomorphism of the torus). This is a torus bundle over a circle, and is in turn covered by $T \times \mathbb{R}$ with deck group $\mathbb{Z}$. Hence M is covered by $T \times \mathbb{R}$ with deck group the *infinite dihedral group* $D = \langle g_1, g_2 \mid g_1^2 = g_2^2 = 1 \rangle$. The induced action on $\mathbb{R}$ is the usual discrete action of D on $\mathbb{R}$, where g_1 and g_2 act by reflections about 0 and 1, respectively. The projection $T \times \mathbb{R} \to \mathbb{R}$ therefore induces a projection $M \to \mathrm{I}(2,2)$, where $\mathrm{I}(2,2)$ is a 1-dimensional orbifold called the *mirrored interval*. (See [4] for definitions and notation.) It follows that M can be viewed as an *orbifold fiber bundle* over $\mathrm{I}(2,2)$. The generic fibers of this bundle are 2-sided tori in M, and the fibers over the mirrored points are the 1-sided tori or Klein bottles of M.

1.4 Classification of Compact 3-Manifolds Modeled on *Sol*

In [34], Scott gives the following classification of closed 3-manifolds modeled on *Sol*. (Note that a homeomorphism $\varphi : T \to T$ of a torus is called *hyperbolic* if φ_* acts on $H_1(T;\mathbb{Z})$ with $\mathrm{tr}(T)^2 > 4$.)

Theorem 1.4.1 ([34, Theorem 5.3(i)]). *Let M be a closed 3-manifold. Then M possesses a geometric structure modeled on Sol if and only if M is a finitely covered by a torus bundle over S^1 with hyperbolic monodromy. In particular, M itself is either a bundle over S^1 with fibre the torus or Klein bottle or is the union of two twisted I-bundles over the torus or Klein bottle.*

We refine this classification as follows.

Theorem 1.4.2. *Let* M *be a closed 3-manifold. Then* M *possesses a geometric structure modeled on Sol if and only if one of the following holds:*

1. M *is a torus bundle over* S^1 *with hyperbolic monodromy, or*
2. M *is an orientable torus semi-bundle with gluing map (in canonical coordinates) given by* $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ *where* $rstu \neq 0$.

The notion of *canonical coordinates* on the middle torus of a torus semi-bundle is explained in the definition that precedes Proposition 1.5 of [37].

Proof. It is shown in [37] that an orientable torus semi-bundle admits a *Sol* structure if and only if its gluing map is of the form stated above. Hence to complete the proof we must show that the other types of 3-manifolds mentioned in Scott's classification do *not* admit geometric structures modeled on *Sol*.

CASE 1. M is a Klein bottle bundle over S^1. Let

$$B = \langle a, b \mid aba^{-1}b = 1 \rangle$$

be the fundamental group of a Klein bottle, and let $A = \langle a^2, b \rangle \approx \mathbb{Z} \oplus \mathbb{Z}$ be the normal subgroup of B corresponding to the double cover of the Klein bottle by a torus. The fundamental group of M has the form

$$\pi_1 M = \langle B, t \mid txt^{-1} = \varphi(x), \ \forall x \in B \rangle$$

for some automorphism φ of B coming from a homeomorphism of the Klein bottle.

We now show that every such automorphism of B preserves the subgroup A. We first observe that every element of B can be written uniquely as $a^i b^j$ for $i, j \in \mathbb{Z}$. Since φ

must preserve the commutator subgroup $[B, B] = \langle b^2 \rangle$, we have $\varphi(b^2) = b^{\pm 2}$, and a short computation shows that in fact $\varphi(b) = b^{\pm 1}$. It follows that $\varphi(a) = a^i b^j$ where $i, j \in \mathbb{Z}$ and i is odd, since otherwise φ has image in the proper subgroup A. We have

$$\varphi(a^2) = (a^i b^j)(a^i b^j) = (a^i a^i)(b^{-j} b^j) = a^{2i},$$

and similarly $\varphi^{-1}(a^2) = a^{2i'}$ for some $i' \in \mathbb{Z}$. From $a^2 = \varphi^{-1}(\varphi(a^2)) = a^{2i \cdot i'}$ we find that $i \cdot i' = 1$, and so $i = \pm 1$. In summary, $\varphi(b) = b^{\pm 1}$ and $\varphi(a^2) = a^{\pm 2}$, so φ preserves the subgroup A.

We therefore conclude that $\pi_1 M$ contains an index-2 subgroup of the form

$$H = \left\langle A, t \mid txt^{-1} = \varphi|_A(x),\ \forall x \in A \right\rangle.$$

Let $\hat{M}$ be the double cover of M corresponding to H, which is a torus bundle over S^1 with monodromy $\varphi|_A$. By the argument in the previous paragraph, there is a choice of basis for A so that

$$\varphi|_A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Therefore $\varphi|_A$ corresponds to a periodic homeomorphism of the torus, and so $\hat{M}$ admits a Euclidean structure by [34, Theorem 5.5]. It follows that M is not does not admit a *Sol* structure, for if it did the structure could be lifted to a *Sol* structure on $\hat{M}$, which would violate Theorem 1.3.1.

CASE 2. M is a Klein bottle semi-bundle. Then M is double covered by a Klein bottle bundle over S^1 and therefore has a degree-4 cover that is a torus bundle over S^1 that admits a Euclidean structure. As in the previous case, M does not admit a *Sol* structure.

CASE 3. M is a nonorientable torus semi-bundle. Then M is the union of two twisted I-bundles N_1 and N_2 over a torus or Klein bottle, at least one of which (say N_1) is an I-bundle over a torus. We will show that M admits a Seifert fibering, and therefore does not admit a *Sol* structure by Corollary 1.3.2.

Choose an arbitrary Seifert fibration for N_2; up to isomorphism there are precisely two of these when N_2 is an I-bundle over a Klein bottle (see [12], for instance) and infinitely many when N_2 is an I-bundle over a torus, as we will show.

If T is a torus, then for any $p/q \in \mathbb{Q} \cup \{\infty\}$, T can be foliated by p/q-curves. This foliation extends to the product Seifert fibration of $T \times I$ by p/q-curves in each torus $T \times \{t\}$. Finally, since the covering involution corresponding to the cover $T \times I \to N_1$ preserves the fibration on $T \times I$, it descends to a Seifert fibration of N_1 so that ∂N_1 is foliated by p/q curves. Note that this is the one of the "generalized" Seifert fibrations as defined in [34], as the critical fibers are not isolated. In fact, the one-sided torus in N_1 forms a subsurface of critical fibers.

It follows that a Seifert fibration on M can be constructed by choosing a Seifert fibration on N_1 so that the foliation of the boundary agrees with the image of the foliation of ∂N_2 under the gluing map. □

1.5 Torus Bundles

The first of the two main theorems that will imply Theorem 1.2.2 is the following.

Theorem 1.5.1. *If M is a torus bundle, then the Simple Loop Conjecture holds for M.*

In fact, a slightly stronger result holds for most surfaces.

Theorem 1.5.2. *Let Σ be a closed surface and let M be a torus bundle. If $\chi(\Sigma)$ is even and negative and $F : \Sigma \to M$ is a 2-sided map, then there is a essential simple loop in Σ that represents an element of* $\ker F_*$. *If $\chi(\Sigma)$ is odd then there is no 2-sided map $\Sigma \to M$.*

After we prove Theorem 1.5.2, to complete the proof of Theorem 1.5.1 it will remain to handle the two cases where $\chi(\Sigma) = 0$. The Simple Loop Conjecture is known to hold for

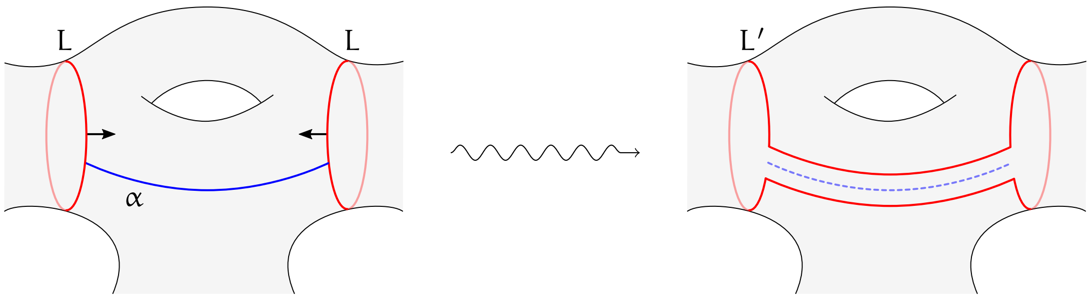


Figure 1.1: Surgery along α reduces the number of components of L by one.

maps $\Sigma \to M$ where Σ is a torus and M is any 3-manifold [11, Section 4.4], and Proposition 1.5.5 will deal with the case in which Σ is a Klein bottle.

Let L be a (not necessarily connected) 1-submanifold of a surface Σ and let α be an arc in Σ with endpoints on L and interior disjoint from L. Then *surgery* of L along α entails fattening α to a strip $I \times I$ with $L \cap (I \times I) = \partial I \times I$, deleting the interior of $\partial I \times I$ from L, and gluing in $I \times \partial I$ to L. Notice that if α is an arc between two distinct components of L, then the result of surgery along α is to connect the two components of L by a bridge, as shown in Figure 1.1.

The following can be established by a standard homomtopy argument.

Lemma 1.5.3. *Let Σ be a (not necessarily closed) surface, let J denote the open interval $(0, 1)$, and let $H : \Sigma \to J$ be a map that is transverse to a point $r \in J$. If α is an arc that connects two components of $L = H^{-1}(r)$ whose interior is disjoint from L, then H can be homotoped in a neighborhood of α so that the preimage of r changes by surgery along α.*

Lemma 1.5.4. *Let Σ be a closed surface, let $G : \Sigma \to S^1$ be a π_1-surjective map, and choose $q \in S^1$. Then G can be homotoped so that the preimage $L = G^{-1}(q)$ is a essential 2-sided simple loop in Σ.*

Proof. Choose G within its homotopy class so that q is a regular value of G and $L = G^{-1}(q)$ is a collection of disjoint simple loops in Σ with a *minimal* number of components. Observe that L is 2-sided but may not be connected. We shall show that the minimality assumption on L along with the assumption that G is π_1-surjective forces L to be connected.

Choose a co-orientation of $q \in S^1$ and pull it back to a co-orientation of L in Σ. We summarize this data by drawing a single arrow orthogonal to each component of L that indicates to which side of each component the co-orientation is pointing, as demonstrated in Figures 1.1 and 1.2. When we cut Σ along L to obtain $\Sigma\backslash\backslash L$, we label the boundary components of the resulting surface with the co-orientations of the components of the L that the boundary components correspond to.

We can homotope G to reduce the number of components of L whenever a component Σ_0 of $\Sigma\backslash\backslash L$ has two boundary loops that are either *both co-oriented into* or *both co-oriented out of* Σ_0. This happens, for instance, whenever Σ_0 has three or more boundary components. Start by choosing a simple arc $\alpha \subset \Sigma_0$ connecting the two boundary components of Σ_0 with coherent co-orientations, so that $G(\alpha)$ is a nullhomotopic loop in S^1 based at q. If U is a small neighborhood of α in Σ, then we can homotope G with support in U so that $G|_U$ is not surjective. Hence $G|_U$ has image in a subset of S^1 homeomorphic to $J = (0, 1)$, and so we may apply Lemma 1.5.3 to $G|_U$ to obtain a further homotopy of G supported in U. This has the effect of surgering L along α, which reduces of the number of components of L by one as shown in Figure 1.1.

Another reduction of L is possible if some component Σ_0 of $\Sigma\backslash\backslash L$ has only one boundary component. In this case, we homotope G by sending all of Σ_0 past q; this homotopy can be taken to be the identity outside of any neighborhood of Σ_0. If L' is the preimage of q after the homotopy, then L' consists of the same loops as L except for the loop that formed the boundary of Σ_0, which has been eliminated.

It follows that if G is chosen to minimize the number of components of L, then every component Σ_0 of $\Sigma\backslash\backslash L$ has exactly two boundary components: one co-oriented into Σ_0

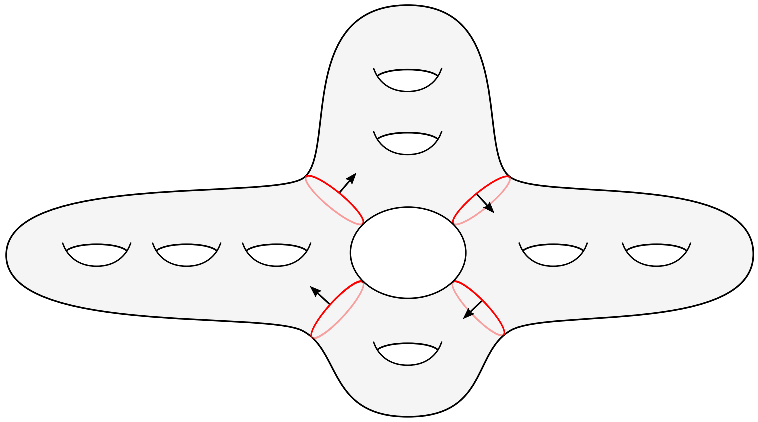


Figure 1.2: If L has more than one component, then no loop in Σ can have a signed intersection of ±1 with L.

and the other co-oriented out of Σ_0, as shown in Figure 1.2. We now observe that the homomorphism $G_* : \pi_1\Sigma \to \pi_1 S^1 \approx \mathbb{Z}$ is given by signed intersection with L, where the sign measures whether a loop in Σ agrees with the co-orientation of L. From the construction of the co-orientation we see that G_* must have image $|L|\mathbb{Z} \leq \mathbb{Z}$. Since G_* is surjective, we have $|L| = 1$, and so L is connected. This completes the proof. □

Proof of Theorem 1.5.2. Let $P : M \to S^1$ denote the bundle projection of M, and let $G = P \circ F : \Sigma \to S^1$.

CASE 1. The map G is π_1-surjective. Applying Lemma 1.5.4 to G, we may homotope G so that the preimage of a point $q \in S^1$ is a 2-sided simple loop $L \subset \Sigma$ for which any loop in $\Sigma \backslash\backslash L$ has inessential image under G. Since we have that $G(\Sigma\backslash\backslash L) \subset \{S^1 \backslash q\}$, we may use the homotopy lifting property of the fiber bundle $M \to S^1$ to homotope F so that $F(\Sigma\backslash\backslash L) \subset M\backslash M_q$, where M_q is the fiber of M lying above q.

Since $M\backslash M_q$ is homeomorphic to $T \times I$ and is therefore orientable, it follows from the 2-sidedness of F that $\Sigma\backslash\backslash L$ must be orientable. Therefore $\Sigma\backslash\backslash L$ is an orientable compact surface with two boundary components, and so $\chi(\Sigma\backslash\backslash L) = \chi(\Sigma)$ must be even. This

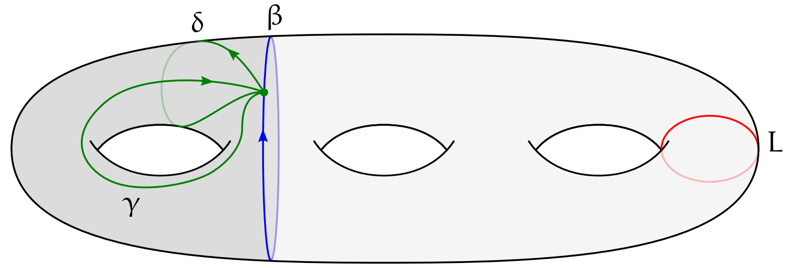


Figure 1.3: The simple loop β in $\ker F_*$ is the boundary of the punctured torus $\Sigma_0 \subset \Sigma$.

proves the claim that there is no 2-sided map $\Sigma \to M$ when $\chi(\Sigma)$ is odd.

We may now suppose that $\chi(\Sigma) = 2 - 2g$, where $g \geq 2$ is an integer. Then $\chi(\Sigma \backslash\backslash L) = 2 - 2g$, so $\Sigma \backslash\backslash L$ is the connect sum of a twice-punctured sphere with $g - 1$ tori. It follows that there is an embedded punctured torus Σ_0 in $\Sigma \backslash\backslash L$. The boundary loop β of Σ_0 is a separating simple loop in Σ whose corresponding element in $\pi_1 \Sigma$ is the commutator of the elements represented by loops γ and δ, as shown in Figure 1.3. The loops β, γ, and δ all have image in $M \backslash M_q$, and since $M \backslash M_q$ has abelian fundamental group it follows that $F_*[\beta]$ is trivial in $\pi_1 M$. Thus β is the desired essential simple loop in the kernel of F_*. (A similar argument shows that any essential separating loop in $\Sigma \backslash\backslash L$ must represent an element of $\ker F_*$.)

CASE 2. The map G is not π_1-surjective. In this case, either G_* is the zero map or it has image $n\mathbb{Z} \leq \mathbb{Z} \approx \pi_1 S^1$ for some $n \neq 0, \pm 1$.

If G_* is the zero map, then G is homotopic to a constant map, and the homotopy can be lifted to a homotopy of F so that the resulting image of Σ is contained in a torus fiber M_p of M. Since M_p is an orientable 2-sided submanifold of M, by the 2-sidedness of F we have that Σ is orientable, and so $\chi(\Sigma)$ cannot be odd. If $\chi(\Sigma) \leq -2$ then there is a essential separating loop in Σ, and we argue as above that such a loop represents an element of $\ker F_*$.

If instead G_* has image a finite index subgroup $n\mathbb{Z} \leq \mathbb{Z}$, then $p_*^{-1}(n\mathbb{Z})$ is a proper finite-index subgroup of $\pi_1 M$ and F lifts to the corresponding cover $\widetilde{M} \to M$. Since $\widetilde{M}$ must also be a torus bundle over a circle and the projection $\widetilde{M} \to M$ is π_1-injective, we may replace M by $\widetilde{M}$ and F by its lift and appeal to Case 1. $\square$

The following result will complete the proof of Theorem 1.5.1.

Proposition 1.5.5. *Let* K *be a Klein bottle and let* G *be an infinite torsion-free group. If* $f : \pi_1 K \to G$ *is a homomorphism with nontrivial kernel, then there is a essential simple loop in* K *that represents an element of* $\ker f$.

Proof. We proceed by reducing to the case in which f has image an infinite cyclic subgroup of G. Write the fundamental group of K as

$$\pi_1 K = \left\langle a, b \mid aba^{-1}b = 1 \right\rangle,$$

and let $H = \left\langle a^2, b \right\rangle \leq \pi_1 K$ be the index-2 subgroup of $\pi_1 K$ corresponding to the double cover of K by a torus. The kernel of $f|_H$ must be nontrivial: for if $x \in \ker f_*$ is not the identity then $x^2 \in H \cap \ker f_*$ is also not the identity. Hence $f|_H$ is a non-injective map from a rank-2 free-abelian group to a torsion free group, and so the image of $f|_H$ is either trivial or infinite cyclic. If $f(H) = 1$, then since $f(a)^2 = f(a^2) = 1$ and M is torsion-free, $f(a)$ must be trivial. In this case f is the trivial map and we're done. If $f(H)$ is infinite cyclic, then $f(\pi_1 K)$ is a virtually-infinite-cyclic torsion-free group, and so must be infinite cyclic (see, for instance, [35, Theorem 5.12]).

Therefore we may replace f by a surjective map $f' : \pi_1 K \to \mathbb{Z}$. Since S^1 is a $K(\mathbb{Z}, 1)$, there is a map $F : K \to S^1$ with $F_* = f'$, and so Lemma 1.5.4 can be applied to obtain a essential 2-sided simple loop $L \subset K$ such that every loop in $K \setminus L$ has inessential image in S^1. Hence we see that $K \setminus L$ is an annulus, the core of which is a essential simple loop in K that represents an element of $\ker f'$, and hence of $\ker f$. $\square$

1.6 Torus Semi-Bundles

The following theorem, together with Theorem 1.5.1, will establish Theorem 1.2.2.

Theorem 1.6.1. *If* M *is an orientable torus semi-bundle that admits a geometric structure modeled on Sol, then the Simple Loop Conjecture holds for* M.

As in the torus bundle case, we have a slightly stronger statement for maps from surfaces of sufficiently large genus into orientable torus semi-bundles.

Theorem 1.6.2. *Let* Σ *be a closed surface and let* M *be an orientable torus semi-bundle. If* $\chi(\Sigma) < -2$ *and* $F : \Sigma \to M$ *is a 2-sided map, then there is an essential simple loop in* Σ *that represents an element of* $\ker F_*$.

To prove the theorem, we will employ the following two lemmas, which allow us to homotope maps from surfaces to torus semi-bundles into a simplified position.

Lemma 1.6.3. *Let Let* M *be an orientable torus semi-bundle with middle torus* $S \subset M$, *let* Σ *be a (not necessarily closed) surface, and let* $F : \Sigma \to M$ *be a map that is transverse to* S. *Suppose that* $\alpha \subset \Sigma$ *is a simple arc that connects two distinct components of* $L = F^{-1}(S)$ *whose interior is disjoint from* L *and that* $F(\alpha)$ *is homotopic (rel endpoints) into* S. *Then* F *can be homotoped in a neighborhood of* α *so that the preimage of* S *changes by surgery along* α.

Proof. Let U be a tubular neighborhood of α in Σ that does not intersect any components of L except the two that are connected by α. Since $F(\alpha)$ is homotopic into S, after possibly shrinking U we can homotope F with support in U so that $F|_U$ has image that does not intersect either of the 1-sided surfaces that are the zero sections of the twisted I-bundles that were used to construct M.

It follows that $F|_U$ has image in a subset of M that is homeomorphic to $T \times J$, where T is a torus and $J = (0, 1)$. Let $P : T \times J \to J$ denote the projection onto the second factor, and

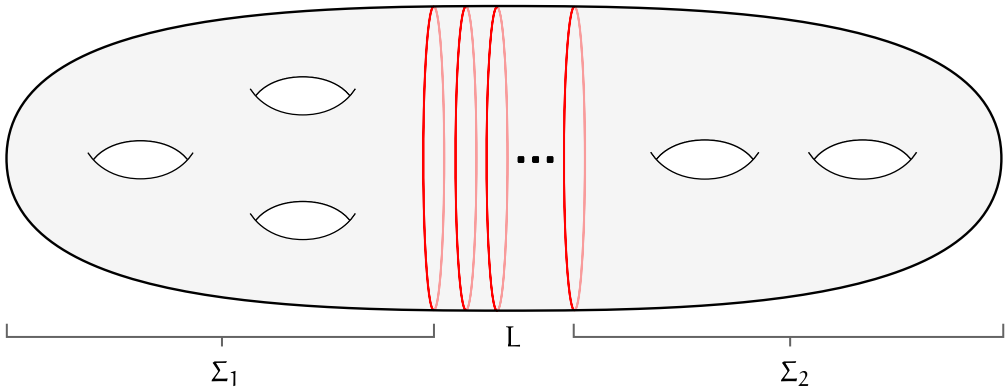


Figure 1.4: The multicurve L is a collection of parallel loops separating Σ into a collection of annuli along with two punctured surfaces, Σ_1 and Σ_2.

let $r \in J$ be the image of S. Then $P \circ F|_U : U \to J$ satisfies the assumptions of Lemma 1.5.3, so we may apply it to obtain a homotopy of $P \circ F|_U$ after which L has been surgered along α. Since $T \times J \to J$ is a fiber bundle, we can lift the homotopy of $P \circ F|_U$ to a homotopy of $F|_U$, and from that we obtain a homotopy of F supported in U, as desired. □

Lemma 1.6.4. *Let M be an orientable torus semi-bundle with middle torus $S \subset M$, let Σ be a closed surface with $\chi(\Sigma) < 0$, and let $F : \Sigma \to M$ be a (2-sided) map that injects on simple loops (that is, there are no elements represented by simple loops in the kernel of F_*). Then F can be homotoped so that $L = F^{-1}(S)$ is either empty or is a collection of parallel 2-sided separating essential simple loops in Σ.*

Figure 1.4 shows a typical picture of $L \subset \Sigma$ when $L \neq \emptyset$.

Proof. In the notation of Section 1.3.1, let $M = N_1 \cup_\varphi N_2$ with monodromies $\rho_i : \pi_1 N_i \to \mathbb{Z}/2$. Choose F within its homotopy class so that F is transverse to S and so that $L = F^{-1}(S)$ is a *minimal* collection of 2-sided simple loops in Σ.

STEP 1. First, suppose that some component Σ_0 of $\Sigma \backslash\backslash L$ has three or more boundary components. Let C_1, C_2, C_3 be three of the boundary components of Σ_0. (Since S separates

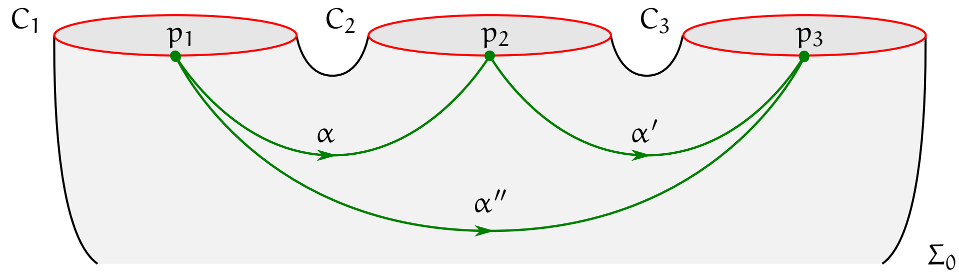


Figure 1.5: The arcs α, α', and α'' joining the boundary components of Σ_0.

M, no two of the C_i correspond to the same component of L.) Choose a basepoint $q \in S$; after a homotopy of F supported in a tubular neighborhood of the C_i, we may assume that each C_i contains a point p_i for which $F(p_i) = q$. In Σ_0 choose simple arcs α from p_1 to p_2, α' from p_2 to p_3, and α'' from p_1 to p_3 such that α'' is path-homotopic to the concatenation of α and α', as shown in Figure 1.5. By construction, each of $F(\alpha)$, $F(\alpha')$, and $F(\alpha'')$ are loops in M based at q, and without loss of generality all three lie in N_1. It follows that $\rho_1[F(\alpha)]$, $\rho_1[F(\alpha')]$, and $\rho_1[F(\alpha'')]$ are elements in $\mathbb{Z}/2$ with $\rho_1[F(\alpha)] + \rho_1[F(\alpha')] = \rho_1[F(\alpha'')]$, and so one of the three elements must be trivial in $\mathbb{Z}/2$. Hence one of the arcs (say α) in Σ_0 has image under F that is homotopic into $\partial N_1 = S$, and so by Lemma 1.6.3 we can homotope F so that the result on L is surgery along α, which reduces the number of components of L.

STEP 2. Next, suppose that some component Σ_0 of $\Sigma \backslash\backslash L$ has two boundary components and is not an annulus. As in the previous step, we can homotope F in a neighborhood of $\partial\Sigma_0$ so that each boundary component has a point p_i ($i = 1, 2$) that maps to the basepoint $q \in S$. Without loss of generality we assume that $F(\Sigma_0) \subset N_1$. There are two cases to consider.

CASE 2A. There is a simple loop $\alpha \subset \Sigma_0$ based at p_1 with $\rho_1[F(\alpha)]$ nontrivial in $\mathbb{Z}/2$. Homotope α in Σ_0 so that α becomes the concatenation of two simple arcs α' and α'' from p_1 to p_2, as shown in Figure 1.6. It follows that $F(\alpha')$ and $F(\alpha'')$ are loops in N_1 based

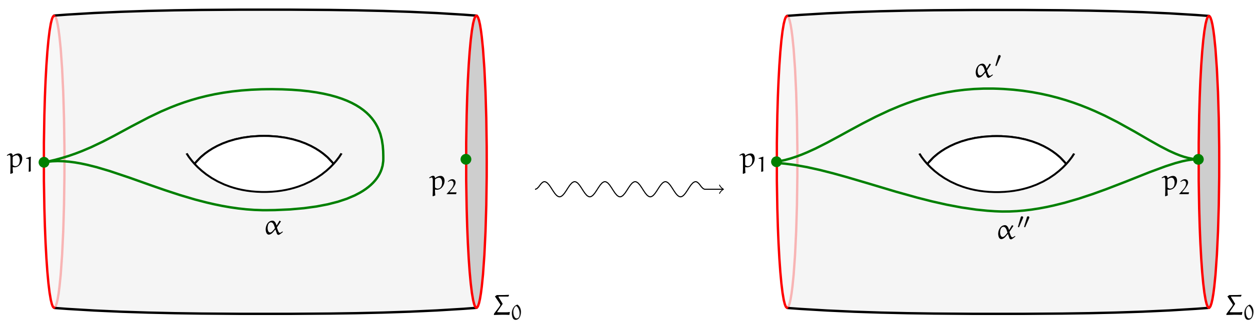


Figure 1.6: Pulling α towards p_2 and viewing it as two arcs.

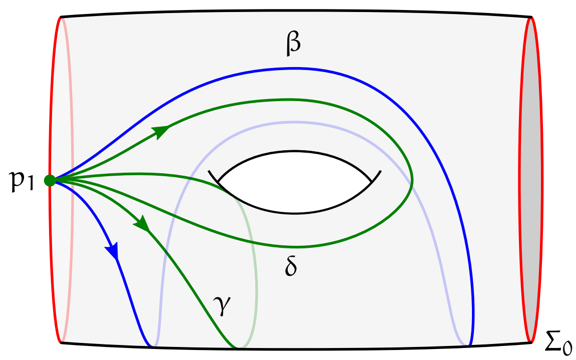


Figure 1.7: The simple loop β represents the commutator of $[\gamma]$ and $[\delta]$.

at q, and since $\rho_1[F(\alpha')] + \rho_1[F(\alpha'')] = \rho_1[F(\alpha)]$ is nontrivial in $\mathbb{Z}/2$, one of $\rho_1[F(\alpha')]$ and $\rho_1[F(\alpha'')]$ must be trivial. As before, an arc with trivial image can be used (Lemma 1.6.3) to homotope F surger L, which reduces the number of components of L by one.

CASE 2B. For every simple loop $\alpha \subset \Sigma_0$ based at p_1, $\rho_1[F(\alpha)]$ is trivial. Since we assumed Σ_0 is not an annulus, it is a twice-punctured orientable surface of genus greater than 0. It follows that we can find two simple loops γ and δ in Σ_0 whose commutator in $\pi_1\Sigma_0$ is represented by a simple loop β; see Figure 1.7. Since $[\beta], [\gamma], [\delta] \in \pi_1\Sigma_0$ all have trivial image under $\rho_1 \circ F_*$, $\rho_1[F(\beta)]$, $\rho_1[F(\gamma)]$, and $\rho_1[F(\delta)]$ must lie in the subgroup of $\pi_1 N_1$ corresponding to the boundary S. But since $\pi_1 S$ is abelian, the commutator $F_*[\beta]$ is trivial. This contradicts the assumption that F injects on simple loops, and so it is impossible that $\rho_1 \circ F_*$ is trivial on every simple loop in Σ_0.

We conclude that the number of components of L can be reduced whenever some component of $\Sigma\backslash\backslash L$ has exactly two boundary components and is not an annulus.

STEP 3. It follows from the previous two steps that if F is chosen in its homotopy class so that L has a minimal number of components, then L is either empty or every component of $\Sigma\backslash\backslash L$ is either an annulus or a surface with exactly one boundary component. The assumption that $\chi(\Sigma) < 0$ rules out the possibility that *every* component of $\Sigma\backslash\backslash L$ is an annulus, and so Σ consists of two punctured orientable surfaces connected by some number of annuli. □

Proof of Theorem 1.6.2. Let Σ be a closed surface with $\chi(\Sigma) < -2$, let $M = N_1 \cup_\varphi N_2$ be a torus semi-bundle, and let $F : \Sigma \to M$ be a 2-sided map. By Lemma 1.6.4, we may assume that F has been homotoped so that $L = F^{-1}(S)$ is either empty or is a collection of parallel curves as in Figure 1.4. (According to the lemma, if this is not possible then we can already find a simple loop in $\ker F_*$.)

If $L = \emptyset$ then without loss of generality F has image in N_1, which is homotopy equivalent to a Klein bottle. Since $\pi_1 N_1$ does not contain the fundamental group of any surface of negative Euler characteristic, the induced map $\pi_1\Sigma \to \pi_1 N_1$ has nontrivial kernel. Using Gabai's result [8], we conclude that there is a simple loop in the kernel of F_*.

We now consider the case in which $L \neq \emptyset$. If Σ_1 and Σ_2 are the two non-annular subsurfaces of Σ as shown in Figure 1.4, then

$$\chi(\Sigma_1) + \chi(\Sigma_2) = \chi(\Sigma).$$

It follows that either $\chi(\Sigma_1) < -1$ or $\chi(\Sigma_2) < -1$.

Without loss of generality, we will henceforth assume that $\chi(\Sigma_1) < -1$ and that $F(\Sigma_1) \subset N_1$.

If $f = \rho_1 \circ (F|_{\Sigma_1})_* : \pi_1(\Sigma_1) \to \mathbb{Z}/2$, then since F sends $\partial\Sigma_1$ (which is a component of L) into S, we have $f[\partial\Sigma_1] = 0$. It follows that f represents a class in $H^1(\Sigma_1, \partial\Sigma_1; \mathbb{Z}/2)$. If f represents the trivial class, then all of $F(\Sigma_1)$ is homotopic into S, and we can homotope F to send all of Σ_1 past S and reduce the number of components of L, contradicting the assumption that F has already been homotoped to minimize the number of components. Therefore f is nontrivial in $H^1(\Sigma_1, \partial\Sigma_1; \mathbb{Z}/2)$, and so by Lefschetz Duality, there is a nontrivial homology class $f_* \in H_1(\Sigma_1; \mathbb{Z}/2)$ for which the value of f on any loop α based on $\partial\Sigma_1$ is given by the signed intersection (mod 2) of α with any 1-chain representing of f_*.

Let ℓ be a simple loop in Σ_1 that represents f_*. (A simple loop representative exists by [25].) Since f_* is nontrivial, ℓ is essential and every loop in $\Sigma_1 \setminus \ell$ is in the kernel of f and therefore has image in N_1 that is homotopic into S. The fact that $\chi(\Sigma_1) < -1$ implies that $\Sigma_1 \setminus\setminus \ell$ is homeomorphic to a closed surface of genus at least one with three open discs removed. As in the proof of Theorem 1.5.2, we can find an embedded punctured torus P in $\Sigma_1 \setminus\setminus \ell$ whose boundary β represents the commutator of simple loops γ and δ contained in P. Since $[\beta]$, $[\gamma]$, and $[\delta]$ all have image under F_* in the abelian subgroup $\pi_1 S \leq \pi_1 M$, we conclude that β is the desired simple loop representing an element of $\ker F_*$. □

With Proposition 1.5.5 and the proof of the Simple Loop Conjecture when the domain is a torus given in [11], we will complete the proof of Theorem 1.6.1 with the following special case.

Lemma 1.6.5. *Let Σ denote the closed orientable surface with $\chi(\Sigma) = -2$. If M is an orientable torus semi-bundle and $F : \Sigma \to M$ is a (2-sided) map, then either there is a essential simple loop in $\ker F_*$ or M does not admit a geometric structure modeled on Sol.*

Proof. By Lemma 1.6.4, we can homotope F so that the preimage $L = F^{-1}(S)$ of the middle torus of M is a minimal collection of parallel curves in Σ as in Figure 1.4. As in the proof of Theorem 1.6.2 we may also assume that $L \neq \emptyset$, so L separates Σ into punctured tori Σ_1 and Σ_2 along with a collection of $n = |L| - 1$ annuli.

CASE 1: $n = 0$. In this case, L is connected and separates Σ into punctured tori Σ_1 and Σ_2. We can write the fundamental group of Σ as

$$\pi_1\Sigma = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] = [a_2, b_2] \rangle ,$$

where a_i and b_i are the generators of the fundamental group of Σ_i. The fundamental group of M has presentation

$$\pi_1 M = \left\langle x_1, y_1, x_2, y_2 \mid x_i y_i x_i^{-1} y_i = 1, x_1^2 = x_2^{2r} y_2^t, y_1 = x_2^{2s} y_2^u \right\rangle ,$$

where x_i and y_i are the generators of the fundamental group of the twisted I-bundle over a Klein bottle N_i, and M has been constructed by gluing N_1 to N_2 via a homeomorphism $\partial N_1 \to \partial N_2$ whose matrix is

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL_2(\mathbb{Z})$$

with respect to the bases $\langle x_i^2, y_i \rangle$ of the fundamental groups of the boundaries of the N_i. By the definition of L we see that F restricts to a proper map of Σ_i into N_i, and so $F_*(a_i)$ and $F_*(b_i)$ must lie in $\langle x_i, y_i \rangle$ for $i = 1, 2$. The subgroup $\langle x_i, y_i \rangle$ of $\pi_1 M$ is isomorphic to the fundamental group of a Klein bottle, and its commutator subgroup is infinite cyclic with generator y_i^2. Hence the commutators $[a_i, b_i]$ are mapped to even powers of y_i, and from the relation in $\pi_1\Sigma$ we obtain an equation

$$y_1^{2k_1} = y_2^{2k_2}$$

for some integers k_1 and k_2. Applying the rightmost relation of the presentation of $\pi_1 M$ given above, we have

$$x_2^{4sk_1} y_2^{2uk_1} = y_2^{2k_2}.$$

Since this is an equation in $\langle x_2^2, y_2 \rangle \approx \mathbb{Z} \oplus \mathbb{Z}$, we can conclude that $4sk_1 = 0$, and so either $k_1 = 0$ or $s = 0$. If $k_1 = 0$, it follows that the curve L (which represents the elements $[a_1, b_1]$

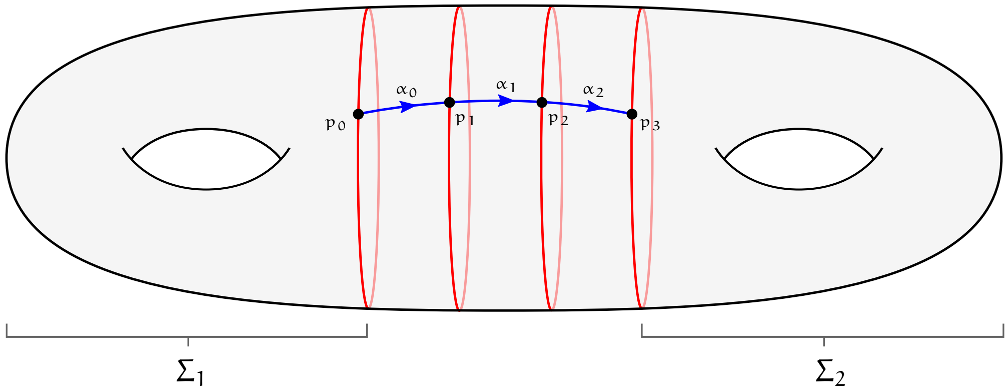


Figure 1.8: The arc α connecting the points p_i in the case $n = 3$.

and $[a_2, b_2]$ in $\pi_1\Sigma$) has image $y_1^{2k_1} = 1$, so L is a essential simple loop in the kernel of F_*. If $s = 0$, then by Theorem 1.4.2 it follows that M does not admit a geometric structure modeled on *Sol*.

CASE 2: $n > 0$. In this case, L has multiple components; we will show that F can be lifted to a torus semi-bundle cover of M in which the preimage of the middle torus is connected, thereby reducing to the case in which $n = 0$. Choose points $p_0, \ldots, p_n$ on the $n+1$ components of L, and let $\alpha \subset \Sigma$ be a simple arc with end points at p_0 and p_n whose intersection with L is the points p_i. For $i = 0, \ldots, n-1$ let α_i denote the segment of α between p_i and p_{i+1}, as shown in Figure 1.8. By adjusting F by a homotopy that preserves L, we may assume that $F(p_i) = q$ for some basepoint $q \in S \subset M$, and so $F(\alpha_i)$ is a loop in M based at q representing an element $w_i \in \pi_1 M$.

In the notation of the previous case, we assume that $F_*(a_1)$ and $F_*(b_1)$ lie in $\langle x_1, y_1 \rangle \leq \pi_1 M$, and by the definition of L we have that $w_i \in \langle x_{j_i}, y_{j_i} \rangle$ where $j_i = 1$ if i is odd and $j_i = 2$ if i even. We may also assume that $w_i \notin \langle x_{j_i}^2, y_{j_i} \rangle$, for if $w_i \in \langle x_{j_i}^2, y_{j_i} \rangle$ then α_i is a proper simple arc in a component $\Sigma \backslash\backslash L$ with image homotopic into S, and we can reduce the number of components of L, which contradicts the minimality assumption. If $w = w_0 \cdots w_{n-1}$, then we have

$$F_*(\pi_1\Sigma) \leq \left\langle x_1, y_1, w x_k w^{-1}, w y_k w^{-1} \right\rangle,$$

where $k = 1$ if n is odd and $k = 2$ if n is even.

If $D = \langle g_1, g_2 \mid g_1^2 = g_2^2 = 1 \rangle$ denotes the infinite dihedral group, then there is a homomorphism $f : \pi_1 M \to D$ given by $x_i \mapsto g_i$ and $y_i \mapsto 1$ for $i = 1, 2$. The cover of M corresponding to $\ker f$ is $T \times \mathbb{R}$ with deck group D, as described in Section 1.3.1. For each $i = 0, \ldots, n-1$, since $w_i \notin \langle x_{j_i}^2, y_{j_i} \rangle$ we have $f(w_i) = g_{j_i}$, and it follows that $f(w)$ is a reduced word in D of length n starting with g_2. The image of $\pi_1 \Sigma$ under the composition $f \circ F_*$ is the subgroup

$$H = \langle g_1, f(w) g_k f(w)^{-1} \rangle \leq D,$$

which itself is isomorphic to the infinite dihedral group. Let $\hat{M}$ be the quotient of $S \times \mathbb{R}$ by H, which is another torus semi-bundle that is the cover of M corresponding to the subgroup $f^{-1}(H)$. Then $\hat{M}$ contains $n + 1$ tori $S_0, \ldots, S_n$ that are lifts of S, and the result of splitting $\hat{M}$ along these tori is n products $T \times I$ (each of which double-covers N_1 or N_2) along with two twisted I-bundles over a Klein bottle (each of which projects to N_1 or N_2 by a homeomorphism). The S_i are parallel and one can show that $\hat{F}^{-1}(S_i)$ is connected for $i = 0, \ldots, n$, where $\hat{F} : \Sigma \to \hat{M}$ is the lift of F to $\hat{M}$. Hence we can take any of the S_i to be the "middle torus" of $\hat{M}$.

Therefore we may apply the argument of the first case of this proof to $\hat{F}$ to find either a essential simple loop in $\ker \hat{F}_*$ or that $\hat{M}$ is Seifert fibered. In the former case, an essential simple loop in $\ker \hat{F}_*$ is also an essential simple loop in $\ker F_*$. In the latter, if $\hat{M}$ is Seifert fibered then it carries a Euclidean or *Nil* structure, and therefore so does M. It follows that M is Seifert fibered as well. □

1.7 The Simple Loop Conjecture for Metabelian Groups

An *orientation character* on a group G is a homomorphism $\rho_G : G \to \mathbb{Z}/2$, and an *oriented group* is a pair (G, ρ_G) where ρ_G is an orientation on G. When G is the fundamental group of a manifold M, we take ρ_G to be the orientation character ρ_M defined in Section 1.3. Similarly, one can say what it means for a homomorphism between two oriented groups to be *2-sided*. It then seems natural to ask if the following generalization of the Simple Loop Conjecture holds for a fixed oriented group G.

Statement. *Let Σ be a closed surface and let (G, ρ_G) be an oriented group. If $f : \pi_1\Sigma \to G$ is a 2-sided homomorphism that is not injective, then there is an essential simple loop in Σ that represents an element of the kernel of f.*

When G is the fundamental group of an aspherical 3-manifold this is equivalent to the Simple Loop Conjecture for 3-manifolds. This statement is known to be false when $G = PSL(2, \mathbb{C})$ by work of Cooper-Manning [5] and when $G = PSL(2, \mathbb{R})$ by work of Mann [20]. (In both cases, G carries the trivial orientation character as it is identified with the groups of orientation-preserving isometries of hyperbolic 3- and 2-space, respectively.)

A group is called *metabelian* if it fits into a short exact sequence of the form

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1,$$

where A and B are abelian groups. For example, the fundamental groups of the torus bundles treated in Section 1.5 are metabelian with $A = \mathbb{Z} \oplus \mathbb{Z}$ and $B = \mathbb{Z}$. One might be led to ask if the group-theoretic version of the Simple Loop Conjecture holds for metabelian groups, and if a technique similar to that of Section 1.5 can be used to prove it. We provide the following result in this direction.

Theorem 1.7.1. *Let (G, ρ_G) be an oriented group that fits into an exact sequence of the form*

$$1 \longrightarrow A \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 1,$$

where A *is abelian, and suppose that* $A \leq \ker \rho_G$*. If* Σ *is a closed surface of genus at least two, then the group-theoretic version of the Simple Loop Conjecture holds for* Σ *and* G.

Proof. This will be a group-theoretic analogue to the proof of Theorem 1.5.2. Let $p : G \to \mathbb{Z}$ denote the projection map in the short exact sequence. For a surface Σ and a 2-sided homomorphism $f : \pi_1\Sigma \to G$, we may assume that f is surjective. For if not, then either $f(\pi_1\Sigma)$ lies in A and any separating simple loop in Σ represents an element of $\ker f$, or $p \circ f$ has nontrivial image and we replace G by $f(\pi_1\Sigma)$, ρ_G by $(\rho_G)|_{f(\pi_1\Sigma)}$, A by $A \cap f(\pi_1\Sigma)$, and $\mathbb{Z}$ by $(p \circ f)(\pi_1\Sigma) \approx \mathbb{Z}$.

There is a map $\Sigma \to S^1$ whose induced homomorphism on fundamental groups is $p \circ f$, and by applying Lemma 1.5.4 to this map we find a simple nonseparating loop $L \subset \Sigma$ such that every element of $\pi_1(\Sigma \backslash\backslash L) \leq \pi_1\Sigma$ is contained in $\ker(p \circ f)$. By exactness, $f(\pi_1(\Sigma\backslash\backslash L))$ is contained in A, and the assumptions that f is 2-sided and that $A \leq \ker \rho_G$ imply that $\Sigma\backslash\backslash L$ must be orientable.

As shown in the proof of Theorem 1.5.2 there are essential simple loops β, γ, and δ in Σ representing elements of $\ker(p \circ f)$ and with $[\beta]$ equal to the commutator of $[\gamma]$ and $[\delta]$. By exactness, $f[\beta]$, $f[\gamma]$, and $f[\delta]$ are contained in A, and since A is abelian we have that $f[\gamma]$ is trivial. $\square$

We conclude by showing that, despite the previous result, the group-theoretic Simple Loop Conjecture does not hold for *all* torsion-free metabelian groups. This is a torsion-free version of a finite example due to Casson [19, Section 2].

Example 1.7.2. Let Σ be a surface of genus $g \geq 2$. We will give a topological construction of the quotient of $\pi_1\Sigma$ by its second derived subgroup, which is sometimes called the *metabelianization* of $\pi_1\Sigma$. From the construction we will see that the kernel of $\pi_1\Sigma \to G$ does not contain any elements represented by simple loops in Σ.

First, let $B = H_1(\Sigma)$ (with $\mathbb{Z}$ coefficients understood), let $f_1 : \pi_1\Sigma \to B$ be the abelianization map, and let $K_1 = \ker f_1$. Let $P : \hat{\Sigma} \to \Sigma$ be the cover of Σ corresponding to K_1. Next, let $f_2 : \pi_1\hat{\Sigma} \to H_1(\hat{\Sigma})$ be the analogous natural map for $\hat{\Sigma}$, and let $K_2 = \ker f_2$. We have $K_2 \leq \pi_1\hat{\Sigma} \approx K_1 \leq \pi_1\Sigma$, and so we identify K_2 with its image under P_* and consider it a subgroup of $\pi_1\Sigma$.

Observe that K_1 does not contain any element of $\pi_1\Sigma$ represented by a nonseparating simple loop in Σ, but does contain every element represented by a separating simple loop in Σ. Hence every separating simple loop in Σ lifts to $\hat{\Sigma}$; we now show that every such loop lifts to a *nonseparating* simple loop in $\hat{\Sigma}$.

We first observe that $B \approx \mathbb{Z}^{2g}$ is a one-ended group. Since B acts properly on $\hat{\Sigma}$ with compact quotient Σ, it follows that $\hat{\Sigma}$ is a one-ended space. Any inessential separating simple loop in $\hat{\Sigma}$ must therefore separate $\hat{\Sigma}$ into a compact piece and a noncompact piece. Hence if β is a simple separating loop in Σ for which some (and hence any) lift $\hat{\beta}$ of β separates $\hat{\Sigma}$, then $\hat{\beta}$ cuts off a compact subsurface $\hat{\Sigma}_{\hat{\beta}} \subset \hat{\Sigma}$. If $\hat{\beta}'$ is another lift of β, then $\hat{\beta}$ and $\hat{\beta}'$ are disjoint, and the regularity of the cover $\hat{\Sigma} \to \Sigma$ implies that there is a deck transformation of $\hat{\Sigma}$ that takes $\hat{\beta}'$ to $\hat{\beta}$. This deck transformation must take $\hat{\Sigma}_{\hat{\beta}'}$ homeomorphically onto $\hat{\Sigma}_{\hat{\beta}}$. If one of these subsurfaces is contained in the other (say $\hat{\Sigma}_{\hat{\beta}'} \subset \hat{\Sigma}_{\hat{\beta}}$) then $\hat{\beta}$ and $\hat{\beta}'$ must be parallel. However, this is impossible: for by choosing hyperbolic metrics on Σ and $\hat{\Sigma}$ so that the covering action is by isometries, and choosing β, $\hat{\beta}$, and $\hat{\beta}'$ to be the unique geodesics in their homotopy classes, we see that if $\hat{\beta}$ and $\hat{\beta}'$ are parallel then they are not distinct lifts of β.

It follows that the subsurfaces $\hat{\Sigma}_{\hat{\beta}}$ (as $\hat{\beta}$ ranges over the lifts of β) must be disjoint. In particular, each such subsurface does not contain any lifts of β in its interior. Thus the covering map $\hat{\Sigma} \to \Sigma$ restricts to a cover of a component of $\Sigma \setminus \beta$ by $\hat{\Sigma}_{\hat{\beta}}$, and since $\hat{\beta}$ projects to β via a homeomorphism, the restricted cover is a homeomorphism. However, this is impossible, as $\hat{\Sigma}_{\hat{\beta}}$ is not a disk and so must contain a nonseparating simple loop, and this nonseparating loop is a lift of its image under the covering projection. We have already observed that such loops do not lift from Σ to $\hat{\Sigma}$, and so from this contradiction we conclude that $\hat{\beta}$ (and hence every lift of β to $\hat{\Sigma}$) must be nonseparating.

It follows that K_2 does not contain *any* elements represented by simple loops of Σ, since the nonseparating simple loops in Σ are homologically nontrivial, and the separating simple loops of Σ lift to homologically nontrivial loops in $\hat{\Sigma}$. Hence if we let $G = \pi_1\Sigma/K_2$ and let $f : \pi_1\Sigma \to G$ be the quotient map, then f is a noninjective map with no elements represented by essential simple loops in its kernel. If $A = \pi_1\hat{\Sigma}/K_2 \approx H_1(\hat{\Sigma})$, then A is abelian and we have

$$G/A = (\pi_1\Sigma/K_2)/(\pi_1\hat{\Sigma}/K_2) \approx \pi_1\Sigma/\pi_1\hat{\Sigma} \approx \pi_1\Sigma/K_1 \approx H_1(\Sigma),$$

which is also abelian. Thus we see that G is metabelian, for it fits into the short exact sequence

$$1 \longrightarrow H_1(\hat{\Sigma}) \longrightarrow G \longrightarrow H_1(\Sigma) \longrightarrow 1,$$

and so we have constructed the desired group G and map $f : \pi_1\Sigma \to G$.

CHAPTER 2

SURFACES IN 4-MANIFOLDS: TRISECTIONS

The notion of a *trisection* of a smooth 4-manifold was introduced by Gay and Kirby in 2012 [9] as a four-dimensional analogy of Heegaard splittings of 3-manifolds. Whereas a Heegaard splitting is a decomposition of a 3-manifold into two three-dimensional handlebodies that meet along a surface, a trisection of a 4-manifold is a decomposition into three four-dimensional handlebodies whose mutual intersection is a surface.

In this chapter we will introduce Heegaard splittings and trisections and explore ways in which the central surfaces in the latter reflects the topology of the ambient 4-manifold. We will further examine ways in which the data of a trisection of a 4-manifold can be expressed in terms of the mapping class group of the central surface.

2.1 Preliminaries and Heegaard Splittings

For the rest of the chapter, we will assume that every manifold comes with a smooth structure. This is automatic for manifolds of dimension two and three [29, 26, 27], but is a nontrivial assumption in dimension four.

We will use the word *glue* in this chapter to refer to the process of building a manifold from two manifolds with boundary by identifying their boundaries via a homeomorphism. Explictly, let M and N be manifolds with boundary and let $G : \partial M \to \partial N$ be a homeomorphism. The result of *gluing* M and N together using G is the quotient manifold

$$(M \sqcup N)\big/(x \sim G(x) \text{ for all } x \in \partial M).$$

It is well-known that if M and N are smooth manifolds and G is a diffeomorphism then the glued manifold admits a natural smooth structure, and furthermore that gluing M and N together via isotopic diffeomorphisms $\partial M \to \partial N$ produces diffeomorphic results.

Connect Sums and Boundary Connect Sums

If M and N are two oriented connected n-dimensional manifolds, the *connect sum* of M and N refers to the n-manifold constructed by deleting open n-balls from the interiors of M and N and gluing them together by an orientation-reversing diffeomorphism of the resulting $(n-1)$-sphere boundaries. The result, denoted $M\#N$, does not depend on the choice of n-ball in either summand, and this process can be adjusted so to give $M\#N$ a smooth structure if M and N have smooth structures as well.

If M and N are two oriented connected n-manifolds each with a single boundary component, the *boundary connect sum* of M and N, denoted $M\natural N$, is obtained by deleting $(n-1)$-balls from the interiors of ∂M and ∂N and gluing M and N together via an orientation reversing diffeomorphism of the resulting $(n-2)$-spheres. Similar to the connect sum, $M\#N$ is independent of the choices made in the construction and can be given a smooth structure when M and N are smooth.

For $g \geq 0$, the phrase *(three-dimensional) genus-*g *handlebody* will be used to refer to the result of attaching g copies of $D^2 \times I$ ("handles") to the boundary of a 3-ball D^3 via embeddings of $D^2 \sqcup D^2 \subset \partial(D^2 \times I)$ in $\partial D^3 = S^2$. The boundary of a genus-g handlebody is a genus-g surface.

The definition of a trisection will require the four-dimensional version of a handlebody, which we will simply call a *four-dimensional handlebody*. This is obtained by attaching four-dimensional handles $D^3 \times I$ to a 4-ball D^4 via embeddings of $D^3 \sqcup D^3$ in $\partial D^4 = S^3$. We will occasionally use the notation $\natural^k(S^1 \times D^3)$ to refer to this object; the *genus* $k \geq 0$ is the number of handles attached. We note that the boundary of a genus-k four-dimensional handlebody is diffeomorphic to $\#^k(S^1 \times S^2)$.

2.1.1 Disk Systems

The notion of a disk system will let us define Heegaard and trisection diagrams, which will be important tools for visualizing these structures.

Definition 2.1.1. If Σ is a genus-g surface, a *disk system* is a set of g pairwise-disjoint simple loops $\alpha = \{\alpha_1, \dots, \alpha_g\}$ on Σ such that $\overline{\Sigma \setminus \alpha}$ is diffeomorphic to a sphere with $2g$ open disks removed.

We think of a disk system on Σ as giving instructions for building a genus-g handlebody with boundary Σ. The procedure is as follows: start with the thickened surface $\Sigma \times [0,1]$, and for each i attach a thickened disk $D^2 \times I$ to $\Sigma \times \{1\}$ via a map that sends $S^1 \times I \subset \partial(D^2 \times I)$ to $N(\alpha_i) \times \{1\} \subset \Sigma \times [0,1]$, where $N(\alpha_i)$ is a regular neighborhood of α_i in Σ. After attaching all g of the disks, the result is a 3-manifold with two boundary components, one of which is diffeomorphic to Σ and the other of which is diffeomorphic to a 2-sphere. The construction is completed by filling in the 2-sphere boundary component with a 3-ball. To see that the result is a handlebody, consider the pieces in the opposite order in which they were added: there is the 3-ball that we added last, the disks that were attached to $\Sigma \times \{1\}$ look like handles attached to the 3-ball, and the $\Sigma \times [0,1]$ that we started with is a collar of the boundary of the handlebody.

Conversely, if Σ is identified as the boundary of a handlebody H, we obtain a disk system on Σ by choosing g pairwise-disjoint embedded disks in H whose union does not separate H and considering their boundary curves on Σ. By the definition of a handlebody, such a collection always exists, but if $g > 1$ then there are infinitely many such collections. The following notion is useful in dealing with the resulting ambiguities.

Definition 2.1.2. Let $\alpha = \{\alpha_1, \dots, \alpha_g\}$ be a disk system on Σ, and let c be an arc with endpoints on α_i and α_j for some $i \neq j$ whose interior is disjoint from α. Let α_i' be the simple loop obtained as the boundary of a regular neighborhood of $\alpha_i \cup c \cup \alpha_j$. Then α_i' is called a *handleslide of* α_i *over* α_j.

Observe that, by construction, $\alpha' = (\alpha \backslash \{\alpha_i\}) \cup \{\alpha_i'\}$ is also a disk system on Σ. For convenience we will also refer to the process of replacing α by α' as a *handleslide*.

Lemma 2.1.3. *If α and α' are two disk systems on Σ, then α and α' correspond to the same handlebody with boundary Σ if and only if there is a sequence of handeslides that transforms α into α'.*

This can be proved using Morse theory or via direct topological arguments, as in [15, Section 2].

2.1.2 Heegaard Splittings of 3-Manifolds

Definition 2.1.4. If M is a closed orientable 3-manifold, a *genus-*g *Heegaard splitting* is a decomposition

$$M = H_1 \cup H_2$$

such that

1. H_1 and H_2 are handlebodies of genus-g, and
2. $F = H_1 \cap H_2$ is a surface of genus$-g$ that is the boundary of both H_1 and H_2.

The surface F is called the *splitting surface* of the Heegaard splitting. We say two Heegaard splittings $M = H_1 \cup H_2$ and $M' = H_1' \cup H'$ are *equivalent* if there is a diffeomorphism $T : M \to M'$ such that $T(H_1) = H_1'$ and $T(H_2) = H_2'$.

Every closed, orientable 3-manifold admits a Heegaard splitting. Heegaard splittings have been studied since the early twentieth century and form part of the basis of classical 3-manifold topology. See, for instance, [13] for a more complete introduction.

Connect Sums of Heegaard Splittings

If $M = H_1 \cup H_2$ and $M' = H_1' \cup H_2'$ are two 3-manifolds with Heegaard splittings of genera g and g' and with splitting surfaces F and F', we obtain a genus-$(g + g')$ Heegaard splitting of the connect sum $M\#M'$ by carrying out the connect sum operation using open 3-balls $B \subset M$ and $B' \subset M'$ such that $B \cap F$ and $B' \cap F'$ are open disks, and choosing a diffeomorphism $\partial(M\backslash B) \to \partial(M'\backslash B')$ that restricts to a diffeomorphism $\partial(F\backslash B) \to \partial(F'\backslash B')$. The resulting Heegaard splitting is

$$M\#M' = (H_1 \natural H_1') \cup (H_2 \natural H_2)$$

with splitting surface $F\#F'$.

Definition 2.1.5. A Heegaard splitting is *reducible* if it is equivalent to a connect sum of two Heegaard splittings of positive genus. A Heegaard splitting that is not reducible is *irreducible*.

Examples of Heegaard Splittings

The decomposition of S^3 into the union of two 3-balls is the only genus-0 Heegaard splitting. The standard decomposition of S^3 into the union of two tori is a genus-1 splitting. More generally, if $H_1 \subset S^3$ is an *unknotted* embedded genus-g handlebody, then $H_2 = \overline{S^3 \backslash H_1}$ is also a handlebody, and the decomposition $S^3 = H_1 \cup H_2$ is a genus-g splitting of S^3.

If $S^2 = D_1 \cup D_2$ is a decomposition of the 2-sphere into two disks that meet along their boundaries, then

$$S^1 \times S^2 = (S^1 \times D_1) \cup (S^1 \times D_2)$$

is a genus-1 Heegaard splitting of $S^1 \times S^2$.

If $M = H_1 \cup H_2$ is a 3-manifold with genus-g Heegaard splitting, then we can obtain a genus-$(g+1)$ Heegaard splitting of $M \cong M\#S^3$ by connect summing $M = H_1 \cup H_2$ with the genus-1 splitting of S^3 mentioned above. The result is called the *stabilization* of the original Heegaard splitting of M and is unique up to equivalence. The Reidemeister-Singer theorem states that any two Heegaard splittings of a single 3-manifold have a common stabilization.

Heegaard Diagrams

We can express the information contained in a Heegaard splitting as a diagram on the splitting surface as follows.

Definition 2.1.6. If $M = H_1 \cup H_2$ is a Heegaard splitting with splitting surface F, let α and β be disk systems on F for H_1 and H_2, respectively. Then the tuple $(F; \alpha, \beta)$ is called a *Heegaard diagram* for the splitting $M = H_1 \cup H_2$.

Examples of Heegaard diagrams are shown in Figure 2.1. (It is worth spending some time thinking about how the diagrams shown correspond to the splittings described in the previous section.) Following the procedure described in Section 2.1.1, from a Heegaard diagram $(F; \alpha, \beta)$ we can reconstruct the 3-manifold with Heegaard splitting by attaching a handlebody to each boundary component of $F \times I$ using the two disk systems in the diagram. It follows from Lemma 2.1.3 that two diagrams $(F; \alpha, \beta)$ and $(F'; \alpha', \beta')$ correspond to equivalent Heegaard splittings if and only if there is a diffeomorphism $P : F \to F'$ and sequences of handleslides that transform $P(\alpha)$ into α' and $P(\beta)$ into β'.

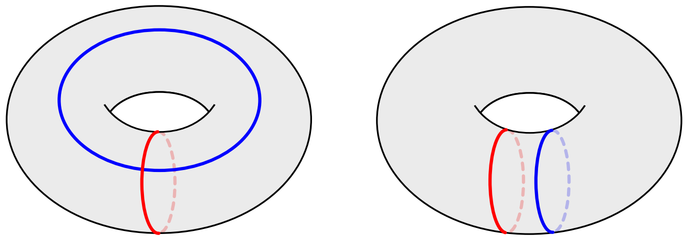


Figure 2.1: Genus-1 Heegaard diagrams for S^3 (left) and $S^1 \times S^2$ (right).

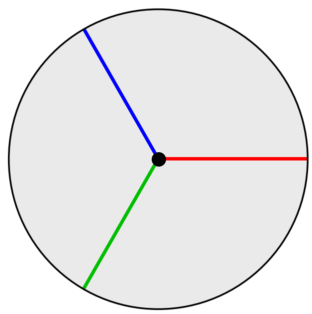


Figure 2.2: The schematic picture of a trisection.

2.2 Trisections of 4-Manifolds

We now define trisections of 4-manifolds and the associated concepts that are analogues of the concepts introduced for Heegaard splittings.

Figure 2.2 shows the schematic picture of a trisected 4-manifold. In the figure, the two-dimensional wedges correspond to four-dimensional handlebodies, the radial arcs between the wedges correspond to three-dimensional handlebodies, and the point in the middle corresponds to the central surface of the trisection. The schematic picture is often more useful than the formal definition when thinking about trisections, as it illustrates how the various pieces fit together. (Note that the boundary of the disk in the schematic picture does *not* correspond to the boundary of any of the pieces; it is just an artifact of the representation.)

Definition 2.2.1. For a closed orientable smooth 4-manifold X, a $(g; k_1, k_2, k_3)$-*trisection* of X is a decomposition

$$X = X_1 \cup X_2 \cup X_3$$

such that

1. for each $i = 1, 2, 3$, $X_i \approx \natural^{k_i}(S^1 \times D^3)$ is a four-dimensional handlebody[1] of genus k_i,
2. for each $i = 1, 2, 3$ (and with indices taken modulo 3), $H_i = X_{i-1} \cap X_i \approx \natural^g(S^1 \times D^2)$ is a three-dimensional handlebody of genus g, and
3. $F = X_1 \cap X_2 \cap X_3$ is a genus-g surface that is the boundary of H_i for each i.

Two trisections $X = X_1 \cup X_2 \cup X_3$ and $X' = X_1' \cup X_2' \cup X_3'$ are *equivalent* if there exists a diffeomorphism $T : X \to X'$ such that $T(X_i) = X_i'$ for each $i = 1, 2, 3$.

Remark 2.2.2. It is important to note that, in the description above, $\partial X_i = H_i \cup H_{i+1}$ is a genus-g Heegaard splitting of $\partial X_i \approx \#^{k_i}(S^1 \times S^2)$ with splitting surface F.

In [9], Gay and Kirby prove that every closed, oriented, smooth 4-manifold admits a trisection and that there is a notion of *stabilization* for trisections such that any two trisections of a 4-manifold X become equivalent after sufficiently many stabilizations.

2.2.1 Connect Sums of Trisections

If $X = X_1 \cup X_2 \cup X_3$ and $X' = X_1' \cup X_2' \cup X_3'$ are $(g; k_1, k_2, k_3)$- and $(g'; k_1', k_2', k_3')$-trisected 4-manifolds with central surfaces F and F', we obtain a $(g + g'; k_1 + k_1', k_2 + k_2', k_3 + k_3')$-trisection of $X \# X'$ analogously to how we take connect sums of Heegaard splittings: carry out the connect sum operation using open 4-balls that are sufficiently-small neighborhoods of points on the central surfaces of each trisection. The resulting trisection has

[1] We use the convention that $\natural^0(S^1 \times D^3) = D^4$ and $\#^0(S^1 \times S^2) = S^3$.

four-dimensional pieces $X_i \natural X_i'$ and central surface $F \# F'$. Similarly to the Heegaard splitting case, we have the following definition.

Definition 2.2.3. A trisection is *reducible* if it is equivalent to a connect sum of trisections of positive genus, and is *irreducible* otherwise.

2.2.2 Trisection Diagrams

From a trisection of X as described above we obtain a diagram on F by choosing disk systems α, β, and γ on F for H_1, H_2, and H_3, respectively. An example is shown in Figure 2.3.

Definition 2.2.4. If α, β, and γ are disk systems on a genus-g surface F, then the tuple $(F; \alpha, \beta, \gamma)$ is called a *trisection diagram* if $(F; \alpha, \beta)$, $(F; \beta, \gamma)$, and $(F; \gamma, \alpha)$ are each Heegaard diagrams for $\#^{k_i}(S^1 \times S^2)$ for some integers $0 \leq k_i \leq g$.

Conversely, we can build a trisected 4-manifold by attaching thickened disks to $F \times D^2$. The procedure for this is similar to a construction that will take place in the proof of Lemma 2.3.13 later; see [9, Section 2] for more specific details. As with Heegaard splittings, two trisection diagrams correspond to equivalent trisections if and only if one diagram can be transformed into the other by a sequence of handleslides (of α's over α's, β's over β's, and γ's over γ's) followed by a diffeomorphism of F.

We note that stabilizing a trisection amounts to connect summing a diagram for it with the diagram for one of the three unbalanced genus-1 trisections of S^4 shown in Figure 2.4.

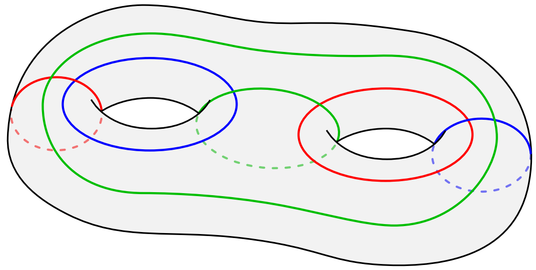


Figure 2.3: A trisection diagram for the $(2,0)$-trisection of $S^2 \times S^2$.

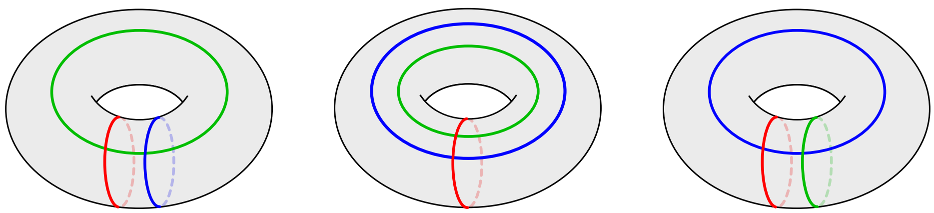


Figure 2.4: Diagrams for the $(1;1,0,0)$-, $(1;0,1,0)$-, and $(1;0,0,1)$-trisections of S^4.

2.3 Trisections and the Mapping Class Group

In this section, we denote by $\mathcal{M}(\Sigma)$ the *mapping class group* of a surface Σ, which is the group of isotopy classes of orientation preserving homeomorphisms of Σ. If H is a genus-g handlebody with boundary Σ, then we denote by $\mathcal{H}(H,\Sigma)$ the *handlebody group* of H, which is the subgroup of $\mathcal{M}(\Sigma)$ consisting of isotopy classes of homeomorphisms of Σ that extend to homeomorphisms of H.

Since we will mostly deal with homeomorphisms and mapping classes on the boundary Σ of a single handlebody H, there will be no harm in using the symbols $\mathcal{M}$ and $\mathcal{H}$ in place of $\mathcal{M}(\Sigma)$ and $\mathcal{H}(H,\Sigma)$, respectively.

2.3.1 Gluing Maps for Heegaard Splittings

Informally, a Heegaard splitting gluing map is a diffeomorphism used to glue two handlebodies together to obtain a 3-manifold with Heegaard splittings. This notion has been studied in various contexts: for instance, Birman [1] uses the image of a Heegaard splitting gluing map in $\mathrm{Aut}(H_1(\Sigma;\mathbb{Z}))$ to define an invariant of 3-manifolds, and Namazi and Souto show in [28] that a gluing map obtained by iterating a pseudo-Anosov diffeomorphism of a surface gives rise to a 3-manifold with a metric that is almost hyperbolic (i.e., it admits a metric of curvature arbitrarily close to -1).

Before we handle trisections, we will define and carefully study gluing maps for Heegaard splittings. This will both serve as a warm-up for the trisection gluing maps as well as provide some important notions that we will use later.

Gluing Map Pairs

Let $M = H_1 \cup H_2$ be a genus-g Heegaard splitting of a 3-manifold M with splitting surface F. To maintain formality it will be necessary to transfer information contained in the Heegaard splitting to the boundary of an external "model" handlebody. To that end, let H be a fixed genus-g handlebody with boundary Σ, choose diffeomorphisms $\Phi_1, \Phi_2 : H \to M$ such that $\Phi_i(H) = H_i$, and choose a diffeomorphism $\Psi : \Sigma \to M$ such that $\Psi(\Sigma) = F$. Then for $i = 1, 2$ there is a diffeomorphism $\Sigma \to \Sigma$ given by

$$\varphi_i = \Psi^{-1} \circ \Phi_i|_\Sigma.$$

Definition 2.3.1. If H is a genus-g handlebody and φ_1 and φ_2 are two diffeomorphisms of $\Sigma = \partial H$, the pair (φ_1, φ_2) is called a *genus-g gluing map pair* (on $\Sigma = \partial H$).

We think of a gluing map pair as a set of instructions for constructing a 3-manifold by gluing two copies of H to a single surface. Since there are some choices to be made in this process (similar to how we chose identifications Φ_1, Φ_2, and Ψ above), there are natural questions about the uniqueness of the resulting 3-manifold. The following lemma addresses these questions.

Lemma 2.3.2. *A genus-*g *gluing map pair* (φ_1, φ_2) *on a handlebody* H *with boundary* Σ *determines a 3-manifold with Heegaard splitting which is unique up to equivalence. Moreover, gluing map pairs* (φ_1, φ_2) *and* (ψ_1, ψ_2) *determine equivalent Heegaard splittings if and only if there are diffeomorphisms* h_1*,* h_2*, and* f *of* Σ *such that*

1. h_1 *and* h_2 *extend to diffeomorphisms of* H*; and*
2. $f \circ \varphi_i \circ h_i$ *is isotopic to* ψ_i *for* $i = 1, 2$.

Proof. We first give the method for constructing a 3-manifold with Heegaard splitting from a gluing map pair (φ_1, φ_2) on $\Sigma = \partial H$. We proceed in reverse order of how we obtained a gluing map pair from a Heegaard splitting at the beginning of the section: let H_1 and H_2 be diffeomorphic copies of H and choose identifications $\Phi_i : H \to H_i$, and let F be a diffeomorphic copy of Σ and choose an identification $\Psi : \Sigma \to F$. We construct a closed smooth 3-manifold M by gluing H_1 and H_2 to $F \times [1, 2]$ via the maps

$$G_i = \Psi_i \circ \varphi_i \circ (\Phi_i|_\Sigma)^{-1} : \partial H_i \to F \times \{i\},$$

where, for $i = 1, 2$, $\Psi_i : \Sigma \to F \times \{i\}$ is given by $\Psi_i(x) = (\Psi(x), i)$. Notice that M has a natural Heegaard splitting $M = \hat{H}_1 \cup \hat{H}_2$, where $\hat{H}_1 = H_1 \cup \left(F \times [1, \frac{3}{2}]\right)$ and $\hat{H}_2 = H_2 \cup \left(F \times [\frac{3}{2}, 2]\right)$.

CLAIM 1: The 3-manifold M and its Heegaard splitting are unique up to Heegaard splitting equivalence.

We show that carrying out the above construction using different choices of identifications yields an equivalent Heegaard splitting. Suppose that H_1' and H_2' are another pair of diffeomorphic copies of H with identifications $\Phi_i' : H \to H_i'$, that F is another copy of Σ with identification $\Psi' : \Sigma \to F'$, and that we construct a 3-manifold with Heegaard splitting $M' = \hat{H}_1' \cup \hat{H}_2'$ by gluing the H_i' to $F' \times [1,2]$ via

$$G_i' = \Psi_i' \circ \varphi_i \circ (\Phi_i'|_\Sigma)^{-1} : \partial H_i' \to F' \times \{i\}.$$

where Ψ_i' is defined analogously to Ψ_i. We construct a diffeomorphism $T : M \to M'$ by defining $T = \Phi_i' \circ \Phi_i^{-1}$ on H_i and $T = (\Psi' \circ \Psi^{-1}) \times \mathbb{1}$ on $F \times [1,2]$. To see that T is well-defined, it suffices to observe that

$$T \circ G_i = G_i' \circ T$$

on ∂H_i, so T respects the gluings of the two manifolds M and M'. Since $T(\hat{H}_i) = \hat{H}_i'$, the Heegaard splittings of M and M' are equivalent.

CLAIM 2: If there are diffeomorphisms h_1, h_2, and f of Σ that satisfy the conditions in the statement of the lemma, then (φ_1, φ_2) and (ψ_1, ψ_2) determine equivalent Heegaard splittings.

Notice from the definition of the gluing diffeomorphism G_i in the construction above that precomposing the φ_i by (potentially different) handlebody diffeomorphisms or post-composing both φ_1 and φ_2 by a surface diffeomorphism is functionally the same as choosing different identifications of H and Σ with the H_i and F. Thus the proof of this claim proceeds identically to that of Claim 1 (since we have already noted that isotopic gluing diffeomorphisms produce diffeomorphic results).

CLAIM 3: If (φ_1, φ_2) and (ψ_1, ψ_2) determine equivalent Heegaard splittings, then there are diffeomorphisms h_1, h_2, and f of Σ that satisfy the conditions in the statement of the lemma.

Suppose we have constructed a 3-manifold with Heegaard splitting M from H_1, H_2, and $F \times [1,2]$ as above using identifications $\Phi_i : H \to H_i$ and $\Psi : \Sigma \to F$ and gluing diffeomorphisms

$$G_i = \Psi_i \circ \varphi_i \circ (\Phi_i|_\Sigma)^{-1} : \partial H_i \to F \times \{i\},$$

and that we have similarly constructed M' from H_1', H_2', and $F' \times [1,2]$ using $\Phi_i' : H \to H_i'$, $\Psi' : \Sigma \to F'$, and

$$G_i' = \Psi_i' \circ \psi_i \circ (\Phi_i'|_\Sigma)^{-1} : \partial H_i' \to F' \times \{i\}.$$

(Recall that $\Psi_i : \Sigma \to F \times \{i\}$ is defined by $\Psi_i(x) = (\Psi(x), i)$, and Ψ_i' is defined similarly.) Suppose further that the Heegaard splittings $M = \hat{H}_1 \cup \hat{H}_2$ and $M' = \hat{H}_1' \cup \hat{H}_2'$ are equivalent, so there is a diffeomorphism T such that

$$T\left(H_1 \cup \left(F \times [1, \tfrac{3}{2}]\right)\right) = H_1' \cup \left(F' \times [1, \tfrac{3}{2}]\right) \quad \text{and}$$
$$T\left(H_2 \cup \left(F \times [\tfrac{3}{2}, 2]\right)\right) = H_2 \cup \left(F' \times [\tfrac{3}{2}, 2]\right).$$

Since the thickened surface $F \times [1, \frac{3}{2}]$ can be thought of as a collar on the boundary of H_1 (and similarly for the three other handlebodies H_2, H_1', and H_2'), we may modify by T by an isotopy so that it sends $F \times \{t\}$ diffeomorphically to $F' \times \{t\}$ for each $t \in [1,2]$ and that T sends each H_i diffeomorphically to H_i. After this adjustment, T satisfies the condition

$$T \circ G_i = G_i' \circ T$$

on each H_i.

For $t \in [1,2]$, let $\Psi_t : \Sigma \to F \times [1,2]$ be given by $\Psi_t(x) = (\Psi(x), t)$, and define Ψ_t' similarly. Define diffeomorphisms $h_1, h_2, f_t : \Sigma \to \Sigma$ by

$$h_i = \left(\Phi_i^{-1} \circ T^{-1} \circ \Phi_i'\right)\Big|_\Sigma,$$
$$f_t = (\Psi_t')^{-1} \circ T \circ \Psi_t.$$

The h_i extend to diffeomorphisms of $\mathcal{H}$ by definition. The maps f_t provide an isotopy between f_1 and f_2. Putting those definitions together with the definitions of G_i and G_i' and the fact that $T \circ G_i = G_i' \circ T$, we find that

$$f_i \circ \varphi_i \circ h_i = \psi_i \quad \text{for } i = 1, 2.$$

Finally, let $f = f_1$, so that $f \circ \varphi_1 \circ h_1 = \psi_1$ and $f \circ \varphi_2 \circ h_2$ is isotopic to ψ_2, as desired. The concludes the proof of Lemma 2.3.2. □

Heegaard Splitting Gluing Maps

The reader may notice that we have done things slightly more tediously than is necessary. Most notably, perhaps, is the fact that in constructing a gluing map pair on $\Sigma = \partial H$ from a Heegaard splitting $M = H_1 \cup H_2$, we have chosen identifications Φ_i of the model handlebody with the H_i and then a separate identification Ψ of the model surface with the boundary surface F of H_i. We chose to proceed in this way because it is the most "symmetric" approach: we are favoring neither H_1 nor H_2 in our description of the Heegaard splitting via a gluing map pair. This will make it easy to generalize to gluing map triples for trisections in Section 2.3.3.

We are also interested in approaching gluing maps in a non-symmetric fashion. We start off as we did at the beginning of this section: starting from $M = H_1 \cup H_2$ with model handlebody H and surface $\Sigma = \partial H$, choose identifications $\Phi_i : H \to H_i$ for $i = 1, 2$. This time we will take $\Psi = \Phi_1|_\Sigma$, so that the identification of Σ with the splitting surface F is obtained from identifications that we have already chosen. We form a gluing map pair as before:

$$\varphi_1 = \Psi^{-1} \circ \Phi_1|_\Sigma = \mathbb{1},$$
$$\varphi_2 = \Psi^{-1} \circ \Phi_2|_\Sigma = (\Phi_1|_\Sigma)^{-1} \circ \Phi_2|_\Sigma.$$

Thus we obtain a gluing map pair of the form $(\mathbb{1}, \varphi)$, where $\varphi = (\Phi_1|_\Sigma)^{-1} \circ \Phi_2|_\Sigma$ is a diffeomorphism of Σ. This leads to the following definition.

Definition 2.3.3. If H is a genus-g handlebody and φ is an orientation-preserving diffeomorphism of $\Sigma = \partial H$, then φ is called a *genus-g (Heegaard splitting) gluing map*.

Notice that, as with gluing map pairs, we include the handlebody as part of the definition.

Remark 2.3.4. The reader may notice that nothing about the construction guarantees that a gluing map will be an *orientation-preserving* diffeomorphism of Σ. However, this is not difficult to accomplish: fix an orientation on the 3-manifold M (and hence on H_1 and H_2) and on the handlebody H. If we choose the identification maps so that Φ_1 is orientation-preserving and Φ_2 is orientation-reversing, then $\varphi = (\Phi_1|_\Sigma)^{-1} \circ \Phi_2|_\Sigma$ will be orientation-preserving.

We obtain the following as a corollary of Lemma 2.3.2.

Lemma 2.3.5. *A genus-g Heegaard splitting gluing map φ on a handlebody H with boundary Σ determines a 3-manifold with Heegaard splitting which is unique up to equivalence. Moreover, gluing maps φ and ψ determine equivalent Heegaard splittings if and only if there are diffeomorphisms h_1 and h_2 of Σ such that*

1. *h_1 and h_2 extend to diffeomorphisms of H; and*
2. *$h_1^{-1} \circ \varphi \circ h_2$ is isotopic to ψ.*

Proof. Apply Lemma 2.3.2 to gluing map pairs $(\varphi_1, \varphi_2) = (\mathbb{1}, \varphi)$ and $(\psi_1, \psi_2) = (\mathbb{1}, \psi)$. Notice that, since $\varphi_1 = \psi_1 = \mathbb{1}$, condition (2) in the statement of Lemma 2.3.2 implies that $f = h_1^{-1}$. □

In light of the above lemmas, we will not distinguish between a homeomorphism φ and its isotopy class $[\varphi] \in \mathcal{M}$ for the rest of this chapter; the symbol φ will be used for both objects. It will be clear from context which of the two is being referred to.

Remark 2.3.6. Before we continue, it is worth observing the relationship between gluing maps and Heegaard diagrams. Let φ be a Heegaard splitting gluing map on $\Sigma = \partial H$, and let α be a disk system for H on Σ. If we take $\beta = \varphi(\alpha)$, then $(\Sigma; \alpha, \beta)$ is a Heegaard diagram for the Heegaard splitting corresponding to φ.

Conversely, given a Heegaard diagram $(\Sigma; \alpha, \beta)$ on a surface Σ, identify Σ with the boundary of a handlebody H so that the α-curves form a disk system for H. If φ is any diffeomorphism of Σ so that $\beta = \varphi(\alpha)$, then φ is a gluing map for the Heegaard splitting corresponding to the given diagram. (Notice that there are choices to be made in this construction, but the ambiguities that arise are equivalent to potentially pre- and post-composing φ with handlebody group elements, as in Lemma 2.3.5.)

From now on we will be considering the gluing map representation of Heegaard splitting data in the setting of the mapping class group $\mathcal{M} = \mathcal{M}(\Sigma)$. We will stop writing $f \circ g$ for the composition of two surface diffeomorphisms, and instead write fg to indicate group multiplication.

Definition 2.3.7. If G is a group, A and B are subgroups of G, and $g \in G$, then the set

$$AgB = \{agb : a \in A, b \in B\}$$

is called an (A, B)-*double coset*.

As with (single) cosets, the collection of (A, B)-double cosets in G forms a partition of G. We can rephrase Lemma 2.3.5 as the following.

Lemma 2.3.8. *Mapping classes $\varphi, \psi \in \mathcal{M}$ are gluing maps for equivalent Heegaard splittings if and only if φ and ψ lie in the same $(\mathcal{H}, \mathcal{H})$-double coset, that is, if and only if*

$$\mathcal{H}\varphi\mathcal{H} = \mathcal{H}\psi\mathcal{H}.$$

Thus there is a correspondence between the set of equivalence classes of genus$-g$ Heegaard splittings and the set of $(\mathcal{H}, \mathcal{H})$-double cosets in $\mathcal{M}$.

Gluing Maps for S^3 and $\#^k(S^1 \times S^2)$

Since our goal is to study trisections, the primary interest in gluing maps for Heegaard splittings will be for those of S^3 and connect sums of $S^1 \times S^2$'s. We will start with the

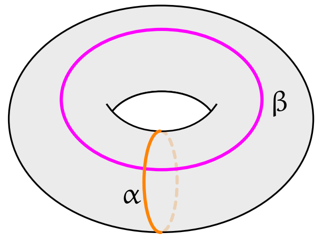


Figure 2.5: Dehn twists around these two curves give rise to generators of the mapping class group of the torus.

genus-1 case and then generalize to higher genus Heegaard splittings. Diagrams for the genus-1 splittings of S^3 and $S^1 \times S^2$ are shown in Figure 2.1.

Let Σ_1 be a torus, and let α and β be the curves on Σ_1 shown in Figure 2.5. Let a and b denote the isotopy classes of the *right Dehn twists*[2] about α and β, respectively. It is well-known that a and b generate $\mathcal{M}(\Sigma_1)$ and satisfy the braid relation $aba = bab$.

One can verify the following facts by computing Dehn twists by hand. For an oriented curve γ, we write $-\gamma$ to denote γ with the opposite orientation.

Lemma 2.3.9. *Let α and β be the curves in Σ_1 and a and b the mapping classes as specified above. If α and β are given orientations, then*

1. $aba(\alpha) = \beta$,
2. $aba(\beta) = -\alpha$, *and*
3. $(aba)^2 = -\mathbb{1}$,

where $-\mathbb{1}$ denotes the mapping class of Σ_1 that acts as multiplication by -1 on $H_1(\Sigma_1)$.

[2]The Dehn twist of a surface Σ around an embedded curve ℓ entails cutting Σ open along ℓ, twisting a neighborhood of one of the new boundary components by 2π, and then regluing. See [6, Chapter 3] for more details.

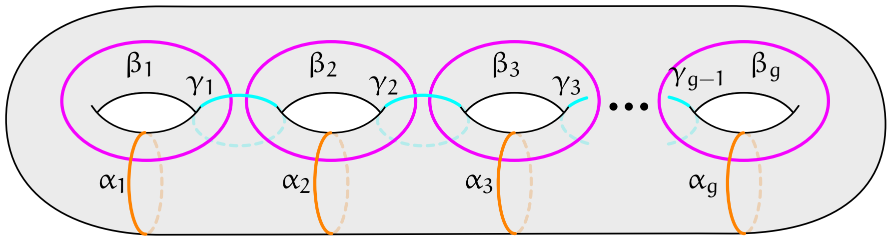


Figure 2.6: Dehn twists around these $3g - 1$ curves generate the mapping class group of the genus-g surface.

We now identify Σ_1 with the boundary of a solid torus H so that the curve α bounds a disk in H. Comparing the previous lemma to the diagrams in Figure 2.1 (and referring back to Remark 2.3.6), we find that $q_0 = aba$ and $q_1 = \mathbb{1}$ are gluing maps for the genus-1 Heegaard splittings of S^3 and $S^1 \times S^2$, respectively.

We now generalize to generalize to genus-g Heegaard splittings of $\#^k(S^1 \times S^2)$, where $g > 1$ and $0 \leq k \leq g$. Let $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$, and $\gamma_1, \ldots, \gamma_{g-1}$ be the curves on a genus-g surface Σ_g shown in Figure 2.6. Let a_i, b_j, and c_k denote the right Dehn twists about α_i, β_j, and γ_k, respectively. It is well-known that this collection of $3g - 1$ mapping classes generates $\mathcal{M}(\Sigma_g)$.

An analogous statement to Lemma 2.3.9 holds for these maps, where we replace a, b, α, and β in the statement of the lemma with a_i, b_i, α_i, and β_i. Thus we arrive at the main result of this section.

Lemma 2.3.10. *Let $g > 0$, let Σ_g be the surface depicted in Figure 2.6, and identify Σ_g with the boundary of a genus-g handlebody H so that the α-curves shown in the figure form a disk system for H. Let a_i and b_j be the mapping classes on the surface Σ_g defined above, and for $0 \leq k \leq g$ define a mapping class*

$$q_k = \prod_{i=k+1}^{g} a_i b_i a_i.$$

Then q_k is a gluing map for the genus-g Heegaard splitting of $\#^k(S^1 \times S^2)$.

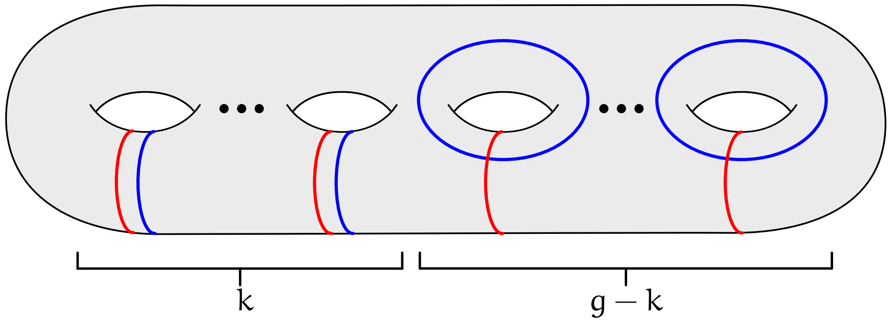


Figure 2.7: A Heegaard diagram for the genus-g splitting of $\#^k(S^1 \times S^2)$.

Proof. Observe that q_k sends the α curves in the diagram in Figure 2.7 to the β curves. □

We should also justify the use of the word "the" in the phrase "*the* genus-g Heegaard splitting of $\#^k(S^1 \times S^2)$" in the statement above. Waldhausen proved in [39] that for each $g \geq 0$ and $0 \leq k \leq g$ there is precisely one genus-g Heegaard splitting of $\#^k(S^1 \times S^2)$ up to equivalence.[3] Thus we will refer to the q_k as the *standard gluing maps* for the Heegaard splittings of $\#^k(S^1 \times S^2)$.

We can further use Waldhausen's theorem along with Lemma 2.3.8 to make the following observation about the q_k maps; we will make frequent reference to this when discussing trisection gluing maps.

Lemma 2.3.11. *If φ is a gluing map on $\Sigma_g = \partial H$ for the genus-g Heegaard splitting of $\#^k(S^1 \times S^2)$, then*

$$\varphi \in \mathcal{H} q_k \mathcal{H}.$$

Moreover, we have $q_k^2 \in \mathcal{H}$.

Proof. The first statement is a corollary of Waldhausen's theorem and Lemma 2.3.8. For the second claim, it suffices to notice that, for each i, the mapping class $(a_i b_i a_i)^2$ sends α_i

[3]See [33] for an English translation and elaboration on Waldhausen's proof. In particular, Section 6 of [33] notes that Waldhausen claimed but did not adequately prove the result for Heegaard splittings of $\#^k(S^1 \times S^2)$ for $k > 0$, but that other authors have since filled in the gaps in Waldhausen's argument.

to itself with the reverse orientation. It follows that q_k^2 preserves a disk system for H, so it extends to a diffeomorphism of H. $\square$

2.3.2 Gluing Maps for Trisections

In this section we will carry out much of the same development and analysis from the previous section in the context of trisections of 4-manifolds. Our goal is to express the data of a trisection in terms of a triple of mapping classes of a surface, and then later to compress that data into a single mapping class.

Gluing Map Triples

Let $X = X_1 \cup X_2 \cup X_3$ be a $(g; k_1, k_2, k_3)$-trisection of X with three-dimensional handlebodies H_i and central surface F as defined in Section 2.2. Proceeding similarly to how we did for Heegaard splittings, let H be a fixed genus-g handlebody with boundary Σ. For each $i = 1, 2, 3$ choose diffeomorphisms $\Phi_i : H \to X$ such that $\Phi_i(H) = H_i$, and choose a diffeomorphism $\Psi : \Sigma \to X$ such that $\Psi(\Sigma) = F$. Then for each i we obtain a diffeomorphism $\Sigma \to \Sigma$ given by

$$\varphi_i = \Psi^{-1} \circ \Phi_i|_\Sigma.$$

Notice that for each i (with indices taken modulo 3), the pair $(\varphi_i, \varphi_{i+1})$ is a gluing map pair for the Heegaard splitting $\partial X_i = H_i \cup H_{i+1}$ induced by the trisection. Thus $\varphi_i^{-1} \circ \varphi_{i+1}$ is a gluing map for a genus-g Heegaard splitting of $\#^{k_i}(S^1 \times S^2)$, and so by Lemma 2.3.11 we have

$$\varphi_i^{-1} \circ \varphi_{i+1} \in \mathcal{H} q_{k_i} \mathcal{H}.$$

This leads to the following definition.

Definition 2.3.12. If H is a genus-g handlebody and φ_1, φ_2, and φ_3 are diffeomorphisms of $\Sigma = \partial H$, then the triple $(\varphi_1, \varphi_2, \varphi_3)$ is called a *gluing map triple* (on $\Sigma = \partial H$) if there are integers $0 \leq k_1, k_2, k_3 \leq g$ such that $\varphi_i^{-1} \circ \varphi_{i+1} \in \mathcal{H} q_{k_i} \mathcal{H}$ for each i. The tuple $(g; k_1, k_2, k_3)$ is called the *type* of $(\varphi_1, \varphi_2, \varphi_3)$.

For convenience we will often abbreviate the notation $(\varphi_1, \varphi_2, \varphi_3)$ and $(g; k_1, k_2, k_3)$ by simply writing (φ_i) and $(g; k_i)$, respectively.

We've shown that a $(g; k_1, k_2, k_3)$-trisected 4-manifold determines a gluing map triple of type $(g; k_1, k_2, k_3)$. The following result shows how to contruct a trisected 4-manifold from a gluing map triple and additionally addresses the ambiguities that arise in the construction.

Theorem 2.3.13. *A gluing map triple (φ_i) of type $(g; k_i)$ on a genus-g handlebody H with boundary Σ determines a $(g; k_i)$-trisected 4-manifold which is unique up to equivalence. Moreover, gluing map triples (φ_i) and (ψ_i) determine equivalent trisections if and only if there are diffeomorphisms h_1, h_2, h_3, and f of Σ such that, for $i = 1, 2, 3$,*

1. *h_i extends to a diffeomorphism of H; and*
2. *$f \circ \varphi_i \circ h_i$ is isotopic to ψ_i.*

Proof. The proof proceeds similarly to that of Lemma 2.3.2. For the remainder of the argument, let $\varepsilon > 0$ be a small real number and let $I = [-\varepsilon, \varepsilon]$. Let D denote the unit disk in $\mathbb{C}$, let $p_1 = 1$, $p_2 = e^{2\pi i/3}$ and $p_3 = e^{4\pi i/3}$ be points on ∂D, and let $D = D_1 \cup D_2 \cup D_3$ be the decompostion of D into three wedges as shown in Figure 2.8. We will frequently abuse notation and identify portions of ∂D with subsets of $\mathbb{R}$; for example, we use $[p_1, p_2]$ to refer to $\{e^{i\theta} : 0 \leq \theta \leq 2\pi/3\}$ and $[p_3 - \varepsilon, p_3 + \varepsilon]$ to refer to $\{e^{i\theta} : 4\pi i/3 - \varepsilon \leq \theta \leq 4\pi i/3 + \varepsilon\}$.

We first show how to construct a trisected 4-manifold from a gluing map triple (φ_i) on $\Sigma = \partial H$. Let H_1, H_2, and H_3 be diffeomorphic copies of H with identifications $\Phi_i : H \to H_i$

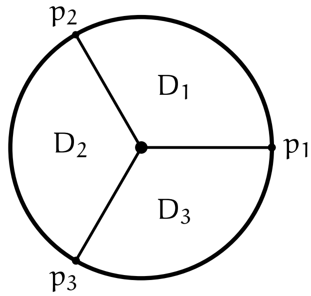


Figure 2.8: The unit disk D with points p_i and regions D_i.

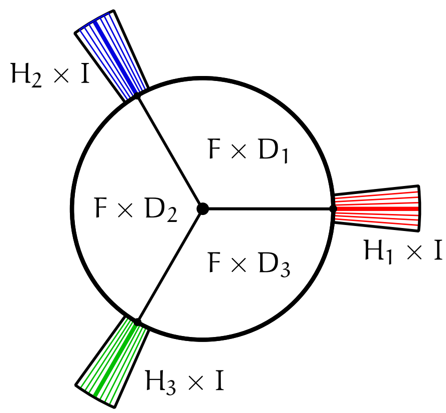


Figure 2.9: The spine of the trisection constructed from a gluing map triple.

and let F be a diffeomorphic copy of Σ with identification $\Psi : \Sigma \to F$. We attach thickened handlebodies $H_i \times I$ to the thickened surface $F \times D$ via the maps $G_i : \partial H_i \times I \to F \times [p_i - \varepsilon, p_i + \varepsilon]$ given by

$$G_i(x, t) = \left(\left(\Psi \circ \varphi_i \circ (\Phi_i|_\Sigma)^{-1}\right)(x), p_i + t\right).$$

Figure 2.9 shows a schematic of the resulting 4-manifold $\tilde{X}$. We refer to this as the *spine* of a trisection; it turns out that the spine of a trisection contains all of the information necessary to construct a closed trisected 4-manifold. To see this, notice that $\tilde{X}$ has three boundary components

$$M_i = (H_i \times \{p_i + \varepsilon\}) \cup (F \times [p_i + \varepsilon, p_{i+1} - \varepsilon]) \cup (H_{i+1} \times \{p_{i+1} - \varepsilon\})$$

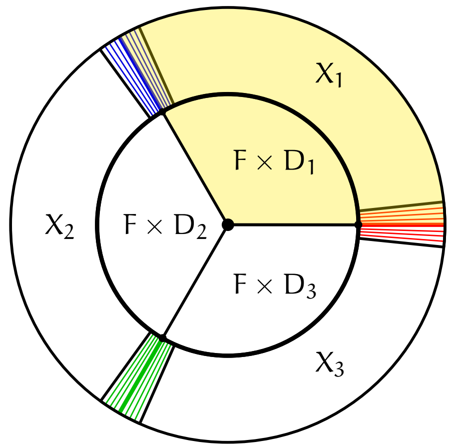


Figure 2.10: The trisected 4-manifold constructed from a gluing map triple. One of the pieces of the triseciton, $\hat{X}_1$, is highlighted for illustration purposes.

for $i = 1, 2, 3$. By the definition of a gluing map triple, each M_i is diffeomorphic to $\#^{k_i}(S^1 \times S^2)$, and so we can construct a closed 4-manifold X by gluing four-dimensional handlebodies $X_i = \natural^{k_i}(S^1 \times D^3)$ to $\tilde{X}$ via diffeomorphisms $\partial X_i \to M_i$. By the Laudenbach-Poenaru theorem [18], the resulting 4-manifold does not depend on the choice of gluing diffeomorphism. Thus we obtain a trisected 4-manifold $X = \hat{X}_1 \cup \hat{X}_2 \cup \hat{X}_3$, where

$$\hat{X}_i = (F \times D_i) \cup (H_i \times [0, \varepsilon]) \cup (H_{i+1} \times [-\varepsilon, 0]) \cup X_i.$$

See Figure 2.10.

Claim 1: The trisected 4-manifold is unique up to trisection equivalence.

Suppose that we construct another trisected 4-manifold in the same manner as above, but we use a different set of three-dimensional handlebodies H_i', surface F', and identifications $\Psi_i : H \to H_i$ and $\Phi : \Sigma \to F$. We attach $H_i' \times I$ to $F' \times D$ via the maps $G_i : \partial H_i' \times I \to F' \times [p_i - \varepsilon, p_i + \varepsilon]$ given by

$$G_i'(x, t) = \left(\left(\Psi' \circ \varphi_i \circ (\Phi_i'|_\Sigma)^{-1}\right)(x), p_i + t\right),$$

to obtain a spine $\tilde{X}'$ and then complete the trisected 4-manifold $X' = \hat{X}_1' \cup \hat{X}_2' \cup \hat{X}_3'$ by attaching four-dimensional handlebodies X_i' to $\tilde{X}'$. We construct a diffeomorphism $T : M \to M'$ such that $T(\hat{X}_i) = \hat{X}_i'$ by defining $T = \left(\Phi_i' \circ \Phi_i^{-1}\right) \times \mathbb{1}$ on $H_i \times I$ and $T = \left(\Psi' \circ \Psi^{-1}\right) \times \mathbb{1}$ on $F \times D$. As in the proof of Lemma 2.3.2, we have $T \circ G_i = G_i' \circ T$, and so we obtain a well-defined diffeomorphism $\tilde{X} \to \tilde{X}'$ between the spines. The Laudenbach-Poenaru theorem lets us extend T over the four-dimensional handlebodies X_i, and thus we obtain a trisection equivalence $T : X \to X'$.

CLAIM 2: If there are diffeomorphisms h_1, h_2, h_3, and f of Σ that satisfy the conditions in the statement of the theorem, then (φ_i) and (ψ_i) determine equivalent trisections.

As in the proof of Lemma 2.3.2, the proof of this statement proceeds identically to that of Claim 1.

CLAIM 3: If (φ_i) and (ψ_i) determine equivalent trisected 4-manifolds, then there are diffeomorphisms h_i and f that satisfy the conditions in the statment of the theorem.

Suppose we have constructed a trisected 4-manifold X from $H_i \times I$ and $F \times D$ as described above using identifications $\Phi_i : H \to H_i$ and $\Psi : \Sigma \to F$ and gluing diffeomorphisms $G_i : \partial H_i \times I \to F \times [p_i - \varepsilon, p_i + \varepsilon]$ given by

$$G_i(x,t) = \left(\left(\Psi \circ \varphi_i \circ (\Phi_i|_\Sigma)^{-1}\right)(x), p_i + t\right),$$

and that we have constructed an equivalent trisected 4-manifold X' from $H_i' \times I$, $F' \times D$, $\Phi_i' : H \to H_i'$, $\Psi' : \Sigma \to F'$, and $G_i' : \partial H_i' \times I \to F' \times [p_i - \varepsilon, p_i + \varepsilon]$ given by

$$G_i'(x,t) = \left(\left(\Psi' \circ \psi_i \circ (\Phi_i'|_\Sigma)^{-1}\right)(x), p_i + t\right).$$

Then there is a diffeomorphism $T : X \to X'$ such that $T(\hat{X}_i) = \hat{X}_i'$ for $i = 1, 2, 3$. If $0 \in D$ denotes the center of the disk, then by assumption we have

$$T(F \times \{0\}) = T(\hat{X}_1 \cap \hat{X}_2 \cap \hat{X}_3) = \hat{X}_1' \cap \hat{X}_2' \cap \hat{X}_3' = F' \times \{0\},$$

and by similar reasoning $T(H_i \times \{0\}) = H_i' \times \{0\}$ for each i (and where $0 \in I$ is the center of the interval). We may therefore assume that T (after an isotopy) sends $F \times \{p\}$ diffeomorphically to $F' \times \{p\}$ for each $p \in D$ and sends $H_i \times t$ diffeomorphically to $H_i' \times t$ for each $t \in I$ and $i = 1, 2, 3$. Thus we obtain the relation $T \circ G_i = G_i' \circ T$ on each $\partial H_i \times I$.

For $\theta \in \partial D$ let $\Psi_\theta : \Sigma \to F \times D$ be given by $\Psi_\theta(x) = (\Psi(x), \theta)$, and define $\Phi_{i,0} : \mathcal{H}_i \to H_i \times \{0\}$ by $\Phi_{i,0}(x) = (\Phi(x), 0)$. Define Ψ_θ' and $\Phi_{i,0}'$ similarly. We can now define diffeomorphisms $h_1, h_2, h_3, f_\theta : \Sigma \to \Sigma$ by

$$h_i = \left(\Phi_{i,0}^{-1} \circ T^{-1} \circ \Phi_{i,0}'\right)\Big|_\Sigma,$$
$$f_\theta = (\Psi_\theta')^{-1} \circ T \circ \Psi_\theta.$$

Combining the above defitions with those of the G_i and G_i' and the relation $T \circ G_i = G_i' \circ T$, it follows that

$$f_{p_i} \circ \varphi_i \circ h_i = \psi_i \quad \text{for } i = 1, 2, 3.$$

The h_i satisfy condition (1) of the statement of the theorem, and f_θ provides isotopies between the f_{p_i}. Taking $f = f_1$ completes the proof. □

2.3.3 Trisection Gluing Maps

We want to compress the data of trisected 4-manifold a single mapping class of a surface.

Let (φ_i) be a gluing map triple of type $(g; k_i)$ on a surface $\Sigma = \partial H$. By definition, we have $\varphi_i^{-1} \circ \varphi_{i+1} \in \mathcal{H} q_{k_i} \mathcal{H}$ for each i (with indices taken modulo 3), and so there are $h, h' \in \mathcal{H}$ so that

$$\varphi_1^{-1} \varphi_2 = h q_{k_1} h'.$$

(Once again we will starting writing compositions of diffeomorphisms of Σ multiplicatively from now on.) Define a new gluing map triple $(\varphi_1', \varphi_2', \varphi_3')$ as follows:

$$\varphi_1' = (\varphi_1 h)^{-1}\varphi_1 h = \mathbb{1},$$
$$\varphi_2' = (\varphi_1 h)^{-1}\varphi_2 (h')^{-1} = q_{k_1},$$
$$\varphi_3' = (\varphi_1 h)^{-1}\varphi_3.$$

By Theorem 2.3.13, (φ_i) and (φ_i') correspond to equivalent trisected 4-manifolds. This leads to the following definition.

Definition 2.3.14. A gluing map triple on Σ is called *standard* if it is of the form $(\mathbb{1}, q_k, \varphi)$ for some k and for some mapping class φ of Σ.

Definition 2.3.15. If H is a genus-g handlebody and φ is an orientation-preserving diffeomorphism of $\Sigma = \partial H$ such that $(\mathbb{1}, q_k, \varphi)$ is a gluing map triple of type $(g; k, k_2, k_3)$ for some k_2 and k_3, then φ is called a *(trisection) gluing map* (on $\Sigma = \partial H$) of type $(g; k)$.

As with Heegaard splitting gluing maps, we are not *a priori* guaranteed that a gluing map arising from the procedure above will be orientation-preserving. If φ is orientation-reversing, we replace it by $\varphi' = \varphi \circ (R|_\Sigma)$ where $R : H \to H$ is an orientation-reversing diffeomorphism. It follows from Theorem 2.3.13 that $(1, q_k, \varphi')$ corresponds to the same trisected 4-manifold as $(1, q_k, \varphi)$.

We additionally note that, given a gluing map φ of type $(g; k)$, the two integers k_2 and k_3 are the values so that

$$q_k^{-1}\varphi \in \mathcal{H} q_{k_2} \mathcal{H} \quad \text{and} \quad \varphi^{-1}\mathbb{1} \in \mathcal{H} q_{k_3} \mathcal{H},$$

the existence of which is guaranteed by $(\mathbb{1}, q_k, \varphi)$ being a valid gluing map triple.

Thus the above discussion shows that every trisected 4-manifold has a standard gluing map triple, and additionally it shows how to construct a standard triple from any given triple. Noticing that we made some choices in that construction (of $h, h' \in H$), it is natural to ask to what extent a gluing map uniquely determines and is determined by a trisected 4-manifold.

In the following, for $0 \leq k \leq g$ let $\mathcal{K}_k$ denote the subgroup of $\mathcal{M}$ given by

$$\mathcal{K}_k = q_k \mathcal{H} q_k^{-1} \cap \mathcal{H}.$$

It turns out that the groups $\mathcal{K}_k$ are interesting in their own right; we will discuss them further in Section 2.3.4.

Theorem 2.3.16. *A gluing map* φ *of type* $(g; k)$ *on a genus-*g *handlebody* H *with boundary* Σ *determines a* $(g; k, k_2, k_3)$*-trisected 4-manifold (for some* k_2, k_3*) which is unique up to equivalence. Moreover, gluing maps* φ *and* ψ *of type* $(g; k)$ *determine equivalent Heegaard splittings if and only if there are diffeomorphisms* $f \in \mathcal{K}_k$ *and* $h \in \mathcal{H}$ *so that*

$$f \varphi h = \psi.$$

Proof. Given a gluing map φ, we construct a trisected 4-manifold by applying Theorem 2.3.13 to the standard triple $(\mathbb{1}, q_k, \varphi)$.

Suppose that φ and ψ determine equivalent trisected 4-manifolds. Applying Theorem 2.3.13 to gluing map triples $(\mathbb{1}, q_k, \varphi)$ and $(\mathbb{1}, q_k, \psi)$, we obtain diffeomorphisms h_1, h_2, h_3, and f of Σ so that $h_i \in \mathcal{H}$ and the following equations hold.

$$f h_1 = \mathbb{1}$$

$$f q_k h_2 = q_k$$

$$f \varphi h_3 = \psi$$

The first equation implies that $f \in \mathcal{H}$ and the second implies that $f \in q_k \mathcal{H} q_k^{-1}$, so we have $f \in \mathcal{K}_k$. Taking $h = h_3$ proves one direction of the second claim of the theorem. The converse (that gluing maps φ and ψ that satisfy $f \varphi h = \psi$ determine equivalent trisected 4-manifolds) is a straightforward application of Theorem 2.3.13. □

Unlike Heegaard splittings, not every mapping class $\varphi \in \mathcal{M}$ is a gluing map for a trisection. It follows from the definition that φ is a gluing map of type $(g;k)$ if and only if there are integers k_2 and k_3 such that following three conditions are satisfied.

$$\mathbb{1}^{-1}q_k \in \mathcal{H}q_k\mathcal{H},$$
$$q_k^{-1}\varphi \in \mathcal{H}q_{k_2}\mathcal{H},$$
$$\varphi^{-1}\mathbb{1} \in \mathcal{H}q_{k_3}\mathcal{H}$$

The first condition always holds, and the second is equivalent to

$$\varphi \in q_k\mathcal{H}q_{k_2}\mathcal{H} = \{q_k h q_{k_2} h' : h, h' \in \mathcal{H}\}.$$

Recall that we chose the q_k gluing maps so that $q_k^2 \in \mathcal{H}$. It follows that $\mathcal{H}q_{k_3}^{-1}\mathcal{H} = \mathcal{H}q_{k_3}\mathcal{H}$, and hence the third condition is equivalent to

$$\varphi \in \mathcal{H}q_{k_3}\mathcal{H}.$$

Putting the two nontrivial conditions together, we have

$$\varphi \in q_k\mathcal{H}q_{k_2}\mathcal{H} \cap \mathcal{H}q_{k_3}\mathcal{H}.$$

Using this, we can express the collection of elements of $\mathcal{M}$ that are valid trisection gluing maps.

Definition 2.3.17. For $0 \leq k \leq g$, define $Q_k \subset \mathcal{M}$ by

$$Q_k = \bigcup_{0 \leq k_2, k_3 \leq g} q_k\mathcal{H}q_{k_2}\mathcal{H} \cap \mathcal{H}q_{k_3}\mathcal{H},$$

where the q_k are the standard Heegaard splitting gluing maps of $\#^k(S^1 \times S^2)$.

Notice that Q_k is defined as a *subset* of $\mathcal{M}$, and in general is not a subgroup of $\mathcal{M}$. However, Q_k does have notable interactions with the group theory of $\mathcal{M}$: it admits a left action by $\mathcal{K}_k$ and a right action by $\mathcal{H}$. Thus Q_k is a union of $(\mathcal{K}_k, \mathcal{H})$ double cosets, and combining this observation with Theorem 2.3.16 yields the main result of this section.

Theorem 2.3.18. *A mapping class $\varphi \in \mathcal{M}$ is a trisection gluing map if and only if $\varphi \in Q_k$ for some k. Moreover, gluing maps φ and ψ in Q_k determine equivalent trisections if and only if φ and ψ lie in the same $(\mathcal{K}_k, \mathcal{H})$-double coset, that is, if and only if*

$$\mathcal{K}_k \varphi \mathcal{H} = \mathcal{K}_k \psi \mathcal{H}.$$

Thus there is a bijective correspondence between the set of $(\mathcal{K}_k, \mathcal{H})$-double cosets in Q_k and the set of equivalence classes of $(g; k_1, k_2, k_3)$-trisections for which $k_1 = k$.

We now have a complete mapping class group formulation of the information contained in a trisection of a 4-manifold, and we can begin to ask questions about how gluing maps reflect properties of the 4-manifolds that they correspond to. For instance, if a valid gluing map is a sufficiently-high iterate of a pseudo-Anosov diffeomorphism of Σ, can we make any conclusions about the geometry of the corresponding 4-manifold? Alternatively, does the image of a trisection gluing map in $\mathrm{Aut}(H_1(\Sigma; \mathbb{Z}))$ give rise to any 4-manifold invariants in the sense of [1]? At the time of this writing, no such work in these directions is known to the author.

Trisection Gluing Map Examples

The correspondence between trisections gluing maps and trisection diagrams is similar to that for Heegaard diagrams (see Remark 2.3.6). Given a gluing map on $\Sigma = \partial H$ of type $(g; k)$, we build a diagram for the corresponding trisection by starting with a disk system α for H on Σ and setting $\beta = q_k(\alpha)$ and $\gamma = \varphi(\alpha)$. Given a trisection diagram $(\Sigma; \alpha, \beta, \gamma)$, we transform the diagram via handleslides and a diffeomorphism so that $(\Sigma; \alpha, \beta)$ looks like the standard diagram for $\#^k(S^1 \times S^2)$ shown in Figure 2.7 and then take φ to be any diffeomorphism of φ that takes the α-curves of the new diagram to the γ-curves.

As in Section 2.3.1, let Σ_g be the genus-g surface shown in Figure 2.6, identified as the

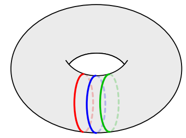
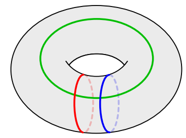
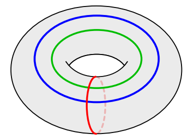
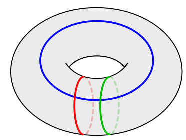
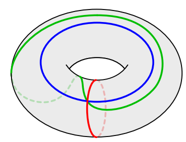
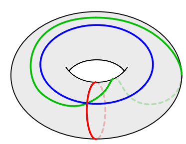
4-Manifold	**Type**	**Gluing map**	**Diagram**
$S^1 \times S^3$	$(1,1)$	$\mathbb{1}$	
S^4	$(1,1)$	aba	
S^4	$(1,0)$	aba	
S^4	$(1,0)$	$\mathbb{1}$	
$\mathbb{C}P^2$	$(1,0)$	b	
$\overline{\mathbb{C}P^2}$	$(1,0)$	b^{-1}	

Figure 2.11: Diagrams and gluing maps for genus-1 trisections.

boundary of handlebody H such that the α-curves in that figure form a disk system for H. Let a_i, b_j, and c_j be the mapping classes defined as Dehn twists about the curves in Figure 2.6.

In the genus-1 case we have two mapping class generators $a = a_1$ and $b = b_1$. In terms of these generators, the table in Figure 2.11 shows gluing maps for all six genus-1 trisections.

Figure 2.12 shows a trisection diagram for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ obtained as a connect sum of the two genus-1 diagrams for $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$. A gluing map of type $(2,0)$ for this genus-2

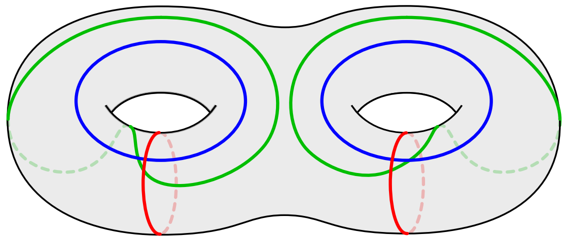


Figure 2.12: A diagram for the $(2;0,0,0)$-trisection of $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$.

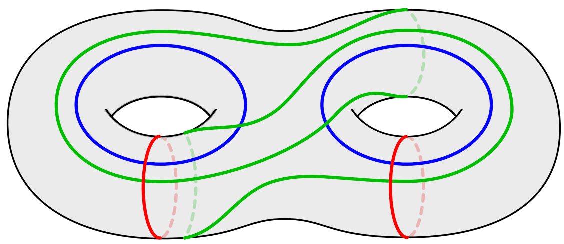


Figure 2.13: A diagram for the $(2;0,0,0)$-trisection of $S^2 \times S^2$.

trisection is $b_1 b_2^{-1}$.

Finally, Figure 2.13 shows a diagram of a genus-2 trisection that cannot be obtained by connect summing smaller genus diagrams: that of $S^2 \times S^2$. A gluing map of type $(2,0)$ for this trisection is $c_1 b_1 b_2$.

2.3.4 Mapping Class Groups of Heegaard Splittings and Trisections

We now explore a connection between Heegaard splitting and trisection gluing maps via the notions of *mapping class groups* of a Heegaard splitting or trisection. The three- and four-dimensional versions are defined similarly.

Definition 2.3.19. If $M = H_1 \cup H_2$ is a 3-manifold with Heegaard splitting, the *(Heegaard splitting) mapping class group* of M, denoted $\mathcal{M}(M)$, is the group of diffeomorphisms T :

$M \to M$ such that $T(H_i) = H_i$ for $i = 1, 2$, taken up to isotopies through diffeomorphisms of the same nature.

Definition 2.3.20. If $X = X_1 \cup X_2 \cup X_3$ is trisected 4-manifold, the *(trisection) mapping class group* of X, denoted $\mathcal{M}(X)$, is the group of diffeomorphisms $T : X \to X$ such that $T(X_i) = X_i$ for $i = 1, 2, 3$, taken up to isotopies through diffeomorphisms of the same nature.

(We will not be dealing with multiple Heegaard splittings or trisections of a single 3- or 4-manifold, so the notations $\mathcal{M}(M)$ and $\mathcal{M}(X)$ will not be ambiguous.) As with gluing maps, mapping class groups have been studied for Heegaard splittings (see, for instance, [16]) but not – to the knowledge of this author – for trisections.

Mapping Class Groups and Gluing Maps

We can relate mapping class groups to gluing maps (for both Heegaard splittings and gluing maps) as follows. Let $M = H_1 \cup H_2$ be a 3-manifold with genus-g Heegaard splitting and splitting surface F. As in Section 2.3.1, let H be a genus-g handlebody with boundary Σ, and let $\Phi_i : H \to M$ and $\Psi : \Sigma \to M$ (with $\Phi_i(H) = H_i$ and $\Psi(\Sigma) = F$) be identifications used to construct a gluing map pair (φ_1, φ_2), with $\varphi_i = \Psi^{-1} \circ \Phi_i|_\Sigma$. By definition, every element $T \in \mathcal{M}(M)$ preserves F setwise and hence restricts to a mapping class of F. Thus $\Psi^{-1} \circ T|_F \circ \Psi$ is an element $\mathcal{M}(\Sigma)$, and so we have an map $\mathcal{G} : \mathcal{M}(M) \to \mathcal{M}(\Sigma)$.

Theorem 2.3.21. *The map $\mathcal{G} : \mathcal{M}(M) \to \mathcal{M}(\Sigma)$ given by $\mathcal{G}(T) = \Psi^{-1} \circ T|_F \circ \Psi$ is an injective homomorphism with image*

$$\mathcal{G}(\mathcal{M}(M))) = \varphi_1 \mathcal{H} \varphi_1^{-1} \cap \varphi_2 \mathcal{H} \varphi_2^{-1}.$$

Thus $\mathcal{M}(M)$ is isomorphic to $\varphi_1 \mathcal{H} \varphi_1^{-1} \cap \varphi_2 \mathcal{H} \varphi_2^{-1}$.

Proof. It is straightforward to see that $\mathcal{G}$ is a homomorphism. We first establish that the image of $\mathcal{G}$ contained in $\varphi_1\mathcal{H}\varphi_1^{-1} \cap \varphi_2\mathcal{H}\varphi_2^{-1}$, and then show that $\mathcal{G}$ has a well-defined inverse on this subgroup.

If $T \in \mathcal{M}(M)$, then by definition $T(H_i) = H_i$ for $i = 1, 2$. It follows that $\Phi_i^{-1} \circ T|_{H_i} \circ \Phi_i$ is a diffeomorphism of H, and so

$$\begin{aligned}\varphi_i^{-1} \circ \mathcal{G}(T) \circ \varphi_i &= \left(\Psi^{-1} \circ \Phi_i|_\Sigma\right)^{-1} \circ \left(\Psi^{-1} \circ T|_F \circ \Psi\right) \circ \left(\Psi^{-1} \circ \Phi_i|_\Sigma\right) \\ &= (\Phi_i|_\Sigma)^{-1} \circ T|_F \circ \Phi_i|_\Sigma \\ &= \left(\Phi_i^{-1} \circ T|_{H_i} \circ \Phi_i\right)\Big|_\Sigma \in \mathcal{H}.\end{aligned}$$

We therefore have $\mathcal{G}(T) \in \varphi_i\mathcal{H}\varphi_i^{-1}$ for $i = 1, 2$, and so the image of $\mathcal{G}$ is contained in $\varphi_1\mathcal{H}\varphi_1^{-1} \cap \varphi_2\mathcal{H}\varphi_2^{-1}$.

To complete the proof, define $\tilde{T} : F \to F$ by $\tilde{T} = \Psi \circ f \circ \Psi^{-1}$ for $f \in \varphi_1\mathcal{H}\varphi_1^{-1} \cap \varphi_2\mathcal{H}\varphi_2^{-1}$. The fact that

$$(\Phi_i|_\Sigma)^{-1} \circ \tilde{T} \circ \Phi_i|_\Sigma = \varphi_i^{-1} \circ f \circ \varphi_i \in \mathcal{H}$$

guarantees that $\tilde{T}$ extends over each handlebody H_i, and so $\tilde{T}$ extends to a diffeomorphism $T : M \to M$. By construction, $T \in \mathcal{M}(M)$ and $\mathcal{G}(T) = f$. This completes the proof. □

Corollary 2.3.22. *If* M *is a 3-manifold with Heegaard splitting corresponding to a gluing map* φ *on* $\Sigma = \partial M$, *then*

$$\mathcal{M}(M) \approx \mathcal{H} \cap \varphi\mathcal{H}\varphi^{-1}.$$

Proof. Apply Theorem 2.3.21 to the gluing map pair $(\mathbb{1}, \varphi)$. □

The following theorem and its corollary are proved identically to Theorem 2.3.21 and Corollary 2.3.22, with the following caveat: when we extend $\tilde{T}$ to a diffeomorphism of the 4-manifold X, we must appeal to the Laudenbach-Poenaru theorem to extend over the four-dimensional handlebodies.

Theorem 2.3.23. *If* X *is a trisected 4-manifold corresponding to a gluing map triple* $(\varphi_1, \varphi_2, \varphi_3)$ *on* $\Sigma = \partial\mathcal{H}$*, then*

$$\mathcal{M}(X) \approx \varphi_1\mathcal{H}\varphi_1^{-1} \cap \varphi_2\mathcal{H}\varphi_2^{-1} \cap \varphi_3\mathcal{H}\varphi_3^{-1}.$$

Corollary 2.3.24. *If* M *is a 3-manifold with trisected 4-manifold corresponding to a gluing map* φ *on* $\Sigma = \partial M$ *of type* $(g; k)$*, then*

$$\mathcal{M}(X) \approx \mathcal{H} \cap q_k\mathcal{H}q_k^{-1} \cap \varphi\mathcal{H}\varphi^{-1}.$$

Thus we can express the mapping class group of a Heegaard splitting or trisection in terms of any gluing map for the splitting or trisection.

A Note on Goeritz Groups

We observe that Lemma 2.3.11 and Corollary 2.3.22 imply that, for $0 \leq k \leq g$, the mapping class group of a genus-g Heegaard splitting of $\#^k(S^1 \times S^2)$ is isomorphic to the group $\mathcal{K}_k$ from Section 2.3.3. These groups are often called *Goeritz groups* after the mathematician who proved that $\mathcal{K}_0$ is finitely generated in genus-2. In [31], Scharlemann proves that $\mathcal{K}_0$ is finitely presented in genus-2, and conjectures that the same might hold in higher genus. Cho and Koda show in [2] that $\mathcal{K}_1$ is finitely presented in genus-2, but the higher genus $\mathcal{K}_k$ have not been successfully analyzed as of this writing. A survey of known results on Goeritz groups can be found in [3].

It is interesting to consider to what extent the correpondence in Theorem 2.3.18 provides a connection between trisections and Goeritz groups. For instance, Meier and Zupan show in [24] that there are only finitely-many equivalence classes of genus-2 trisections; is this related to the finite presentation[4] of $\mathcal{K}_0$, $\mathcal{K}_1$, and $\mathcal{K}_2$ in genus-2? Such a link between Goeritz groups and trisections would be valuable in furthering the study of both topics.

[4]The group $\mathcal{K}_g = \mathcal{H}$ is known to be finitely-presented in all genus (see, for instance, [38]).

2.4 Trisections and the Curve Complex

In this section we generalize yet another means of studying Heegaard splittings of 3-manifolds into the realm of trisections of 4-manifolds. We briefly review a measure of complexity of 3-manifolds with Heegaard splittings, define some analogous complexities for trisected 4-manifolds, and then pose some questions about this new complexity.

Curve Complexes

Let $g \geq 2$ and let Σ be a closed genus-g surface. The *curve complex* of Σ, denoted $\mathcal{C}(\Sigma)$, is a simplicial complex whose vertices consist of isotopy classes of essential simple loops in Σ. The vertices corresponding to simple loops $\alpha_0, \ldots, \alpha_n$ in Σ span an n-simplex in $\mathcal{C}(\Sigma)$ if and only if the α_i have isotopy representatives that are pairwise-disjoint on Σ.

It is known that if we endow $\mathcal{C}(\Sigma)$ with a simplicial metric (that is, a metric that is Euclidean on each simplex such that each edge has distance one), then $\mathcal{C}(\Sigma)$ is an infinite-diameter δ-hyperbolic metric space [23]. We refer to [32] for a more thorough introduction to the geometry of curve complexes.

Handlebody Sets

If the surface Σ is identified with the boundary of a handlebody H, then we obtain a subcomplex $\Delta \subset \mathcal{C}(\Sigma)$ which is spanned by (isotopy classes of) essential simple loops in Σ that bound disks in H. This is the *handlebody set* of H. One can show that Δ is a connected subcomplex of $\mathcal{C}(\Sigma)$ of infinite diameter. Moreover, the geometry of Δ and the manner in which it is embedded in $\mathcal{C}(\Sigma)$ is well-understood: Δ is a quasi-convex subset of $\mathcal{C}(\Sigma)$ [21], and Δ admits a δ-hyperbolic metric but is not quasi-isometrically embedded in $\mathcal{C}(\Sigma)$ [22].

2.4.1 Heegaard Distance

Let $M = H_1 \cup H_2$ be a 3-manifold with Heegaard splitting of genus at least 2, and let F denote the splitting surface. We measure the complexity of the splitting as follows: let Δ_1 and Δ_2 be the handlebody sets in $\mathcal{C}(F)$ corresponding to H_1 and H_2, respectively. If d denotes the simplicial metric on $\mathcal{C}(F)$, then the *distance* of the Heegaard splitting of M is the number $d(\Delta_1, \Delta_2)$. This is the length of the shortest path in $\mathcal{C}(F)$ connecting a loop that bounds a disk in H_1 to a loop that bounds a disk in H_2.

An initial connection between the distance of a Heegaard splitting and the topology of the ambient 3-manifold is obtained by considering the Heegaard splittings of distance zero. Observe that if a Heegaard splitting $M = H_1 \cup H_2$ with splitting surface F has distance zero, then there is a simple loop $\alpha \subset F$ that bounds disks $D_1 \subset H_1$ and $D_2 \subset H_2$. We can assume that α is a separating loop in F, for if it is not we may replace α with the boundary of a regular neighborhood of $\alpha \cup \beta$, where β is any simple loop in F that intersects α exactly once. (Since the original α bounds a disk in each handlebody, so does this new curve.)

Gluing D_1 and D_2 together along α yields a 2-sphere $S \subset M$ which separates M. This means that if we cut M along S (and thus cut F along α), fill in the resulting S^2 boundary components with 3-balls, and cap off the boundary components of $F \setminus \alpha$ with disks, we obtain two closed 3-manifolds M_1 and M_2 with Heegaard splittings such that the original 3-manifold M and its Heegaard splitting can be obtained as a connect sum $M_1 \# M_2$ as in Section 2.1.2.

Definition 2.4.1. If $M = H_1 \cup H_2$ is a Heegaard splitting with splitting surface F, then a *reducing sphere* is an embedded 2-sphere $S \subset M$ such that $S \cap F$ is a single circle that is essential in F.

Observe that a Heegaard splitting is reducible (in the sense of Section 2.1.2) if and only if it admits a reducing sphere. Thus a Heegaard splitting of distance zero is reducible, and conversely we observe that a reducible Heegaard splitting has distance zero. A 3-manifold M is called *reducible* if there is a (smoothly) embedded 2-sphere in M that does not bound a ball in M. (If every embedded 2-sphere in M bounds a ball, then M is *irreducible*.) The following result from classical 3-manifold topology gives a connection between the two senses of "reducible" that we have introduced.

Lemma 2.4.2 ("Haken Lemma")**.** *If* M *is a reducible 3-manifold, then any Heegaard splitting of* M *is reducible.*

The relationship between the distance of a Heegaard splitting and the topology and geometry of the ambient 3-manifold goes deeper than we have discussed. The notion of distance was first introduced by Hempel in [14], where it was shown that any splitting of 3-manifold that either is Seifert fibered or contains an incompressible embedded torus must have distance at most two. Together with the proof of the Geometrization Conjecture that was completed a few years after Hempel's work, this implies that any 3-manifold with a splitting of distance greater than two must admit a hyperbolic structure.

2.4.2 Circumference of Trisections

We define analogues of Heegaard splitting distance for trisections as follows. Let $g > 1$ and let $X = X_1 \cup X_2 \cup X_3$ be a $(g; k_1, k_2, k_3)$-trisected 4-manifold with three-dimensional handlebodies $H_i = X_{i-1} \cap X_i$ and central surface F as in Section 2.2. Then we obtain three handlebody sets Δ_1, Δ_2, and Δ_3 in $\mathcal{C}(F)$ corresponding to H_1, H_2, and H_3, respectively. Recall that $H_i \cup H_{i+1}$ is a genus-g Heegaard splitting of the 3-manifold $\partial X_i \cong \#^{k_i}(S^1 \times S^2)$; since each such 3-manifold is reducible, it follows from Lemma 2.4.2 that the Heegaard splittings given by the pairs of three-dimensional handlebodies in the trisection are reducible. In other words, $d(\Delta_i, \Delta_{i+1}) = 0$ for each $i = 1, 2, 3$, and so $\Delta_i \cap \Delta_{i+1} \neq \emptyset$.

Definition 2.4.3. Let $g > 1$ and let $X = X_1 \cup X_2 \cup X_3$ be a $(g; k_1, k_2, k_3)$-trisection as above. A *handlebody circuit* for the trisection is a combinatorial loop $\ell \subset \mathcal{C}(F)$ with a decomposition $\ell = \ell_1 \cup \ell_2 \cup \ell_3$ such that each ℓ_i is a combinatorial arc contained in Δ_i.

In other words, a handlebody circuit is an edge loop in $\mathcal{C}(F)$ contained in $\Delta_1 \cup \Delta_2 \cup \Delta_3$ that passes through each Δ_i once. (We allow a segment ℓ_i to have length zero, meaning it consists of a single vertex of ℓ.) Since each handlebody set is connected and pairs of handlebody sets have nonempty intersection, every trisection admits a handlebody circuit in the curve complex of its central surface. This leads to the following notion of complexity for trisections.

Definition 2.4.4. The *circumference* of a trisection is the least combinatorial length of a handlebody circuit in the curve complex of the central surface of the trisection.

Remark 2.4.5. The reader may question if a difference is made by relaxing the restriction that a handlebody circuit stay in the union of the handlebody circuits, and instead define circumference as the length of the shortest loop in $\mathcal{C}(F)$ that meets each handlebody set at least once. However, an elementary argument shows that such a complexity is bounded above by a constant that depends only on the hyperbolicity constant of $\mathcal{C}(F)$ and the quasi-convexity constant of a handlebody set, each of which only depends on the genus of F. On the other hand, the measure we have defined above is not obviously bounded.

Remark 2.4.6. Since $\mathcal{C}(F)$ is simply-connected (see [10]), one may also consider a complexity of a trisection given by the least *area* of a combinatorial disk whose boundary is a handlebody circuit for the trisection. This was suggested as a possible direction of study in Gay and Kirby's original trisection paper [9]. However, it is unclear if the area complexity measure captures any information about a trisection that circumference does not.

Circumference Properties and Examples

Our first observation about trisection circumference is that it satisfies a property analogous to Heegaard distance.

Theorem 2.4.7. *A trisection is reducible if and only if it has circumference zero.*

Proof. Let $X = X_1 \cup X_2 \cup X_2$ be a trisected 4-manifold with three-dimensional handlebodies H_i and central surface F as usual. If the trisection of X is reducible, then there is an embedded 3-sphere $S \subset X$ (called a *reducing sphere*) that decomposes X into a connect sum of two other trisected 4-manifolds. It follows that S meets F in a single essential loop α and meets each handlebody H_i in a single disk with boundary α. Thus $\alpha \in \Delta_1 \cap \Delta_2 \cap \Delta_3$ and so the circumference of the trisection is zero.

Conversely, if the trisection has circumference zero, there is an essential simple loop $\alpha \subset F$ that bounds a disk D_i in each three-dimensional handlebody H_i. As in the Heegaard splitting case, we may assume that α is a separating loop. We construct a reducing sphere for the trisection by first building a *spine* $D = D_1 \cup D_2 \cup D_3$, similarly to how we constructed trisected 4-manifolds in the proof of Theorem 2.3.13. Notice that each $S_i = D_i \cup D_{i+1}$ is a separating 2-sphere in $\partial X_i = H_i \cup H_{i+1}$, so it bounds a properly embedded 3-ball B_i in X_i (see below). Filling in the spheres $S_i \subset D$ with B_i yields the desired reducing 3-sphere.

We point out that the fact that S_i bounds a 3-ball B_i is actually another consequence of the Laudenbach-Poenaru theorem [18]. For any separating 2-sphere in $S \subset \partial X_i$ there is a diffeomorphism $\partial X_i \to \#^{k_i}(S^1 \times S^2)$ taking S to one of the 2-spheres S' used to build the connect sum. This diffeomorphism extends to $X_i \to \natural(S^1 \times D^3)$, and the desired 3-ball bound by S is obtained as the pullback of the 3-ball in $\natural(S^1 \times D^3)$ bound by S'. □

Similar to 3-manifolds, smooth 4-manifold X is called *reducible* if there is a smoothly-embedded 3-sphere in X that does not bound a smooth 4-ball in X. We note that a trisection analogue to Lemma 2.4.2 is unproven as of this writing.

Conjecture 2.4.8 ("Haken Lemma for Trisections"). *If X is a reducible 4-manifold, then every trisection of X is reducible.*

We can compute the circumferences of the trisections of genus two using Theorem 2.4.7 the following observation.

Lemma 2.4.9. *If a $(2;0,0,0)$-trisected 4-manifold has nonzero circumference, then it has circumference at least six.*

Proof. We first make an observation about Heegaard splittings of the 3-sphere. If $S^3 = H_1 \cup H_2$ is a genus-g Heegaard splitting with splitting surface F and corresponding handlebody sets $\Delta_1, \Delta_2 \subset \mathcal{C}(F)$, then every vertex α in $\Delta_1 \cap \Delta_2$ gives rise to a 2-sphere $S \subset S^3$. Since every 2-sphere in S^3 is separating, it follows that α must be separating in F.

Now let X be a $(2;0,0,0)$-trisected 4-manifold with central surface F and handlebody sets $\Delta_i \subset \mathcal{C}(F)$, and let $\ell \subset \mathcal{C}(F)$ be a handlebody circuit of length at most five. By definition there is a decomposition $\ell = \ell_1 \cup \ell_2 \cup \ell_2$ into segments such that $\ell_i \subset \Delta_i$, and it follows from the length assumption that one of the segments (say ℓ_1) has length at most one. If ℓ_1 has length zero then it is a vertex of $\mathcal{C}(F)$ that lies in $\Delta_1 \cap \Delta_2 \cap \Delta_3$, so the trisection has length zero. If ℓ_1 has length one then it is an edge of $\mathcal{C}(F)$ whose endpoints lie in $\Delta_1 \cap \Delta_2$ and $\Delta_3 \cap \Delta_1$, so by the above paragraph the endpoints correspond to separating simple loops in F that are non-isotopic and disjoint. Since F is a genus-2 surface, this is impossible. This shows that a handlebody circuit for the trisection of X either has length zero or has length at least six. □

Theorem 2.4.10. *The $(2;0,0,0)$-trisection of $S^2 \times S^2$ has circumference six. Every other genus-2 trisection has circumference zero.*

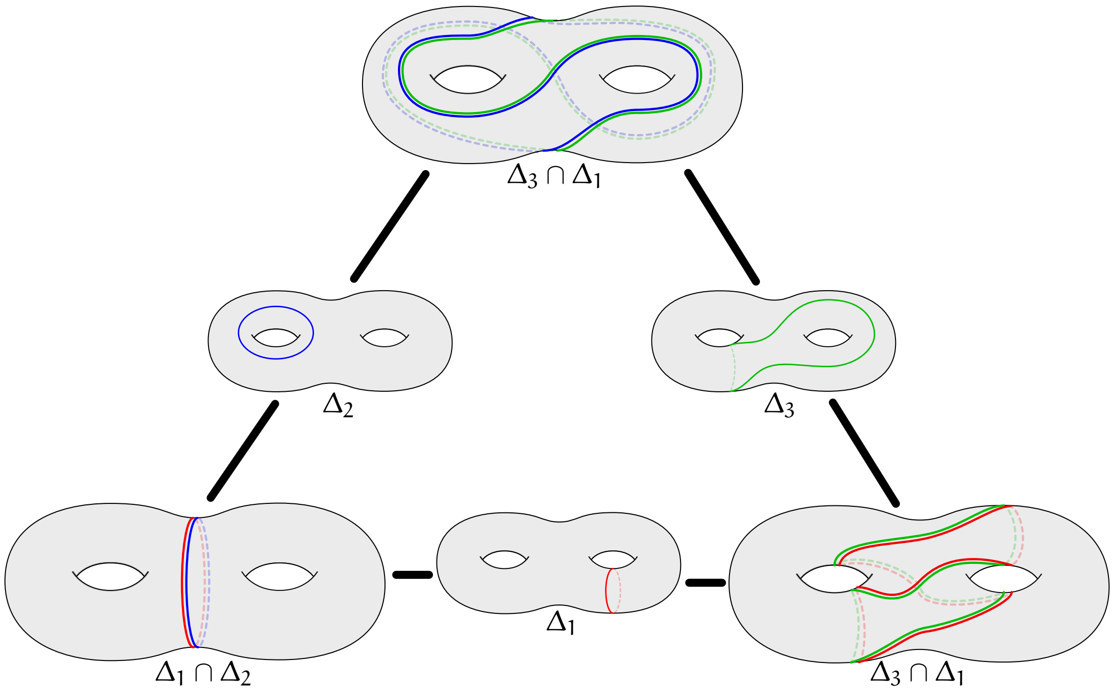


Figure 2.14: A handlebody circuit in the curve complex of the central surface of the $(2; 0, 0, 0)$-trisection of $S^2 \times S^2$. The three curves in the pairwise intersections of handlebody sets are each obtained as the boundary of a regular neighborhood of the union of two curves from the diagram in Figure 2.13.

Proof. Figure 2.13 shows a trisection diagram for the $(2; 0, 0, 0)$-trisection of $S^2 \times S^2$, and Figure 2.14 shows a handlebody circuit of length six for this trisection. Thus the trisection of $S^2 \times S^2$ has circumference at most six, and Lemma 2.4.9 shows that the circumference must be exactly six. (To see that the $(2; 0, 0, 0)$-trisection of $S^2 \times S^2$ is irreducible, notice that if it were reducible we could obtain it as the connect sum of two of the genus-1 trisections shown in Figure 2.11. One can show that this is impossible via intersection form arguments.) It is shown in [24] that every other genus-2 trisection is reducible, and so such trisections have circumference zero. □

Questions about Circumference

We conclude with some yet-unanswered questions about trisection circumference. These arose in the search for analogues to known facts about Heegaard distance.

Question A. Are there trisections with arbitrarily-large circumference?

Question B. What can be said about geometry or topology of a 4-manifold with a large circumference?

Question C. What can be said about irreducible trisections circumference three?

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