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1. Introduction.

In this paper we present a local convergence theorem for the method of nonlinear successive overrelaxation. The hypotheses of the theorem are weaker than those necessitated by earlier works on the subject. We also prove local convergence for a general class of related methods and then establish the convergence of some specific members of this class.

2. Description and Matrix Formulation of the Method.

Let $F = (f_1, \ldots, f_N)^T$ be defined on Ω, $\Omega \subset E^N$, N-dimensional Euclidean space, and let $F : \Omega \to E^N$. Let Ω_o be an open and bounded set in E^N with $\Omega_o \subset \Omega$ and let $f_i \in C^1(\Omega_o)$ for each i, $1 \leq i \leq N$. Since E^N is locally

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convex, we can assume without loss of generality that Ω_o is convex. We seek solutions of $F(x) = 0$ and consider the iterative method

$$(2.1)\quad \begin{aligned} x_1^{n+1} &= x_1^n - \omega \frac{f_1(x^n)}{f_{11}(x^n)} , \\ x_2^{n+1} &= x_2^n - \omega \frac{f_2(x_1^{n+1}, x_2^n, \ldots, x_N^n)}{f_{22}(x_1^{n+1}, x_2^n, \ldots, x_N^n)} , \\ &\quad \cdots \\ x_N^{n+1} &= x_N^n - \omega \frac{f_N(x_1^{n+1}, x_2^{n+1}, \ldots, x_{N-1}^{n+1}, x_N^n)}{f_{NN}(x_1^{n+1}, x_2^{n+1}, \ldots, x_{N-1}^{n+1}, x_N^n)} \end{aligned}$$

with x^o given and $n = 0,1,2,\ldots$. This method is known as <u>nonlinear successive overrelaxation</u> (NLSOR). The special case $\omega = 1$ is usually called the <u>nonlinear Gauss-Seidel</u> method (NLGS). These methods have been proposed and studied by several authors [1], [5], [7], [8], and [9].

Let $x^{n,i}$ denote the argument of the function f_i which appears in the <u>ith</u> equation of (2.1); i.e. $x^{n,i} \equiv (x_1^{n+1}, \ldots, x_{i-1}^{n+1}, x_i^n, \ldots, x_N^n)^T$. If, for each i , $x^{n,i}$ and x^n are in Ω_o , then by the mean value theorem we have

$$(2.2)\quad f_i(x^{n,i}) = f_i(x^n) + \nabla f_i(\xi^{n,i})^T (x^{n,i} - x^n) , \quad i = 1,\ldots,N$$

where $\xi^{n,i} = t_i x^{n,i} + (1 - t_i)x^n$ for some $t_i \in (o,1)$.

Comparing the vector $(x^{n,i} - x^n)$ with the vector $(x^{n+1} - x^n)$, we see that (2,2) can be rewritten as

$$(2.3)\qquad \begin{pmatrix} f_1(x^n) \\ f_2(x^{n,2}) \\ \cdots \\ f_N(x^{n,N}) \end{pmatrix} = F(x^n) + \begin{pmatrix} 0 & & \bigcirc \\ \vdots & \ddots & \\ \cdots\ f_{ij}(\xi^{n,i}) \cdots & & \\ \vdots & & 0 \end{pmatrix} (x^{n+1} - x^n) \ .$$

On substituting (2.3) into (2,1) and solving for x^{n+1}, we obtain

$$(2.4)\qquad x^{n+1} = x^n - G_\omega^{-1}(x^n)\, F(x^n) \ , \text{ with}$$

$$G_\omega^{-1}(x^n) \equiv \begin{pmatrix} 1 & & \bigcirc \\ \vdots & \ddots & \\ \cdots\ \dfrac{\omega f_{ij}(\xi^{n,i})}{f_{ii}(x^{n,i})} \cdots & & \\ \vdots & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \ddots & & \bigcirc \\ & \dfrac{\omega}{f_{ii}(x^{n,i})} & \\ \bigcirc & & \ddots \end{pmatrix} .$$

Let $x, x^* \in \Omega_0$ with $F(x^*) = 0$. We can apply the mean value theorem again to obtain

$$(2.5)\qquad F(x) = (\nabla f_i(\eta^i))^T (x - x^*)$$

where each η^i is a convex combination of x and x^*.

3. Local Convergence of Nonlinear Successive Overrelaxation.

THEOREM 1. (Local Convergence of NLSOR). Let $F \in C^1(\Omega_o)$ where Ω_o is an open set containing x^*, a zero of F. Let $\omega > 0$ and let

$$H^* = (L^* + \frac{1}{\omega} D^*)^{-1}[(1 - \frac{1}{\omega})D^* + U^*]$$

where L^*, U^* and D^* denote respectively the strictly lower triangular, strictly upper triangular and diagonal portions of $J(x^*)$, the Jacobian matrix of F at x^*. If H^* exists and has its largest eigenvalue less than one in modulus, then there is an open set Ω_1 such that NLSOR converges to x^* from any $x^o \in \Omega_1$.

Proof. Since H^* exists, $f_{ii}(x^*) \neq 0$ for every $i = 1,\ldots,N$; therefore, there is an open set containing x^* in which $f_{ii} \neq 0$. Without loss of generality we may take this set to be Ω_o itself.

Let $0 < \delta < 1 - \rho(H^*)$, and pick $||\cdot||'$ to be a norm such that $||H^*||' \leq \rho(H^*) + \delta < 1$. Here, as usual, $\rho(\ )$ denotes the spectral radius. Now there exist nonnegative constants c and C such that $c||x|| \leq ||x||' \leq C||x||$ for every $x \in E^N$ (see [4]). Choose $R > 0$ such that $\mathrm{cl}\, N(x^*,R) \subset \Omega_o$. (Here $N(x^*,R)$ denotes an open ball in E^N with center x^* and radius R.) Since F' is continuous on $\mathrm{cl}\ N(x^*,R)$, each f_i satisfies a Lipschitz condition there with constant, say L_i ; moreover, $|f_{ii}(x)| > 0$ for every x in the compact set $\mathrm{cl}\ N(x^*,R)$, so there is a $b > 0$ such that $|f_{ii}(x)| \geq b$ there. Pick

$$L \geq \max_i \frac{L_i}{b} .$$

<u>Claim</u>. If $x^o \in \mathrm{cl}\ N(x^*, \frac{R}{(1+\omega L)^N})$ then $x^1 \in \mathrm{cl}\ N(x^*,R)$.

<u>Proof of Claim</u>.

$$||x_2^o - x^*|| = ||x_1^o - x^*|| + \omega \left|\frac{f_1(x_1^o)-f_1(x^*)}{f_{11}(x_1^o)}\right|$$

$$\leq ||x_1^o - x^*|| + \omega L||x_1^o - x^*|| = (1+\omega L)||x_1^o - x^*||$$

$$\leq \frac{R}{(1+\omega L)^{N-1}} .$$

Suppose $||x_k^o - x^*|| \leq \frac{R}{(1+\omega L)^{N-k+1}}$, then

$$||x_{k+1}^o - x^*|| \leq ||x_k^o - x^*|| + \omega \left|\frac{f_k(x_k^o)-f_k(x^*)}{f_{kk}(x_k^o)}\right| \leq ||x_k^o - x^*||(1+\omega L)$$

$$\leq \frac{R}{(1+\omega L)^{N-k+1}}(1+\omega L) = \frac{R}{(1+\omega L)^{N-(k+1)+1}} .$$

Hence by induction $||x^1 - x^*|| \leq R$.

We now view the iteration (2.1) in the matrix form (2.4). Since the elements of $G_\omega(x)$ are composed of partials evaluated at points of $\mathrm{cl}\ N(x^*,R)$ when $||x^* - x|| \leq \frac{R}{(1+\omega L)^N}$, the mapping $x \to G_\omega(x)$ is continuous at x^* as is the mapping $x \to \hat{J}(x)$, where $\hat{J}$ denotes the Jacobian of F with each row evaluated at the points defined by 2.5. Hence $\eta(x) \equiv ||I - G_\omega^{-1}(x)\hat{J}(x)||'$ is continuous in $\mathrm{cl}\ N(x^*, \frac{R}{(1+\omega L)^N})$, $\eta(x^*) = ||E^*||'$, and $\eta(x^*) \leq \rho(E^*) + \delta < 1$. There is, then, a $\delta' < 1$ and $R' > 0$ such that $\eta(x) \leq \delta'$ on the set $\{x \in E^N : ||x - x^*||' \leq R'\}$. R' will be chosen so that $R' \leq \frac{cR}{(1+\omega L)^N}$, since this insures that $c||x - x^*|| \leq ||x - x^*||' \leq R' \leq \frac{cR}{(1+\omega L)^N}$, so that $||x - x^*|| \leq \frac{R}{(1+\omega L)^N}$ and $\eta(x)$ is defined. Now choose $\varepsilon_1 > 0$ such that $\mathrm{cl}\ N(x^*, \varepsilon_1(1+\omega L)^N) \subset \{x \in E^N : ||x - x^*||' \leq R'\}$ and $\varepsilon_2 > 0$ such that $\{x \in E^N : ||x - x^*||' \leq \varepsilon_2\} \subset \mathrm{cl}\ N(x^*, \varepsilon_1)$.

Now set $\Omega_1 \equiv \{x \in E^N : ||x - x^*||' < \varepsilon_2\}$. Let $x^o \in \Omega_1$. Then since $||x^o - x^*|| \leq \varepsilon_1$, x^1 is defined and $||x^1 - x^*|| \leq \varepsilon_1(1+\omega L)^N$; hence, $||x^1 - x^*||' \leq R'$ and (setting $e^i = x^i - x^*$) $e^1 = [I - G^{-1}(x^o)\hat{J}(x^o)]e^o$ so that $||e^1||' \leq \eta(x^o)||e^o||' \leq \delta'\varepsilon_2 < \varepsilon_2$. Hence $x^1 \in \Omega_1$. Suppose we have shown that $||e_k||' \leq (\delta')^k||e^o||'$. Then $x^k \in \Omega_1$

and by the same argument as above x^{k+1} is at least defined and $||e^{k+1}|| \leq \varepsilon_1(1+\omega L)^N$ so that $\eta(x^k) \leq \delta'$. Thus $||e^{k+1}||' \leq \delta'||e^k||' \leq (\delta')^{k+1}||e^o||'$. Hence by induction $\{x^n\}$ is defined and

$$||x^n - x^*||' \leq (\delta')^n||x^o - x^*||' \leq (\delta')^n\varepsilon_2 \to 0 \quad \text{as} \quad n \to \infty .$$

This completes the proof.

In the above proof we have used the fact that norm topologies are equivalent in E^N , so that convergence in one norm is equivalent to convergence in any other norm.

<u>COROLLARY</u>. If $F \in C^1(\Omega_o)$, $x^* \in \Omega_o$, $F(x^*) = 0$, $0 < \omega < 2$ and $J(x^*)$ is symmetric and positive definite, then there is an open set $\Omega_1 \subset \Omega_o$ such that NLSOR converges to x^* from any $x^o \in \Omega_1$.

<u>Proof</u>. If $J(x^*)$ is symmetric and positive definite, then $\rho(H(x^*)) < 1$ for each ω such that $0 < \omega < 2$. See [10].

4. <u>Convergence of Related Methods</u>.

Suppose that we consider a general Newton-like method of the form

$$(4.1) \qquad x^{n+1} = x^n - A(x^n)^{-1}F(x^n) .$$

The tedious part of the previous theorem was showing that for x sufficiently close to x^* , $A(x) = G_\omega(x)$ was defined.

If we are given an iteration function which can be rewritten in the form (4.1) in such a way that $A(x)$ is defined, then we can state the following lemma which extends some results in [2].

LEMMA 1. Let A be a function which assigns to any $x \in \Omega_o$ an $N \times N$ matrix $A(x)$. Furthermore, let $A(x)$ be continuous at x^* with $A(x^*)$ invertible and $\rho(I - A(x^*)^{-1}J(x^*)) < 1$. Then there is an open set $\Omega_1 \subset \Omega_o$ such that the sequence defined by (4.1) converges to x^* from any $x^o \in \Omega_1$.

Proof. Choose a norm, $||\cdot||'$, so that $||I - A(x^*)^{-1}J(x^*)||' < 1$ and choose $\delta > 0$ and $\Omega_1 \subset \Omega_o$ so that if x is in the open set Ω_1 which contains x^*, then $||I - A(x)^{-1}\hat{J}(x)||' \leq \delta < 1$. This follows from the continuity of $\hat{J}$, A, and matrix inversion. $\hat{J}$ is defined in the proof of Theorem 1.

Let $x^o \in \Omega_1$. Then, $A(x^o)$ is invertible and

$$\begin{aligned}||x^*-(x^o-A(x^o)^{-1}F(x^o))||' &= ||(I-A(x^o)^{-1}F)(x^*)-(I-A(x^o)^{-1}F)x^o||' \\ &\leq ||I - A(x^o)^{-1}\hat{J}(x^o)||'||x^* - x^o||' \\ &\leq \delta||x^* - x^o||' .\end{aligned}$$

Hence the iteration converges from x^o.

Let us now consider the method of nonlinear simultaneous displacements (NLSD)

$$(4.2) \qquad x^{n+1} = x^n - \begin{pmatrix} \ddots & & 0 \\ & f_{ii}(x^n) & \\ 0 & & \ddots \end{pmatrix}^{-1} F(x^n) \, .$$

The following local convergence theorem for NLGS and NLSD is easily obtained from the previous lemma and the usual linear theory [10].

<u>THEOREM 2</u>. Let $J(x^*)$ be irreducible and diagonally dominant. Then there exist open sets $\Omega_1, \Omega_2 \subset \Omega_0$ such that NLGS and NLSD converge to x^* from any $x^0 \in \Omega_1, \Omega_2$ respectively.

<u>Proof</u>. Since $J(x^*)$ is irreducible it cannot have any zero row or column. Since it is diagonally dominant it cannot have a zero diagonal element. Hence $|f_{ii}(x^*)| > 0$, $i = 1, \ldots, N$. Define $A(x)$ to be the diagonal of $J(x)$ for $x \in \Omega_0$. Clearly $A(x)$ is continuous at x^* and $A(x^*)^{-1}$ exists.

$$I - A(x^*)^{-1} J(x^*) = \begin{pmatrix} 0 & & \frac{f_{ij}(x^*)}{f_{ii}(x^*)} \\ & \ddots & \\ \frac{f_{ij}(x^*)}{f_{ii}(x^*)} & & 0 \end{pmatrix}$$

is also irreducible and (by applying Theorem 1.7 of Varga [10, p. 20] to its transpose) has spectral radius less than one. We know from Theorem 1 that $G_1(x)$ is defined and continuous for x sufficiently close to x^* and Theorem 3.4 of Varga [10, p. 73] ensures that $\rho(H^*) < 1$.

REMARK. The conclusions of the above theorem and the same technique of proof apply if we replace the hypothesis of "irreducible and diagonally dominant" with "strictly diagonally dominant."

The above techniques provide a clear indication as to how to formulate and prove local convergence theorems for those methods studied in [3], [6] and [7]. Roughly, these methods use a correction term whose numerator is successively displaced and whose denominator is simultaneously displaced.

REFERENCES

[1] P. Concus, Numerical solution of the minimal surface equation, Math. Comp. 21 (1967), pp. 340-350.

[2] J. W. Daniel, On Newton-like methods, Comp. Sci. Tech. Rept. 21, University of Wisconsin, 1968.

[3] D. Greenspan and S.V. Parter, Mildly nonlinear elliptic partial differential equations and their numerical solution, II, Numer. Math. 7 (1965), pp. 129-146.

[4] A. S. Householder, The Theory of Matrices in Numerical Analysis, Blaisdell, New York, 1964.

[5] H. M. Lieberstein, Overrelaxation for nonlinear elliptic partial differential equations, Math. Res. Cen. Tech. Rept. 80, University of Wisconsin, 1959.

[6] J. M. Ortega and W. C. Rheinboldt, Monotone iterations for nonlinear equations with applications to Gauss-Seidel methods, SIAM J. Numer. Anal. 4 (1967), pp. 171-190.

[7] J. M. Ortega and M. L. Rockoff, Nonlinear difference equations and Gauss-Seidel type iterative methods, SIAM J. Numer. Anal. 3 (1966), pp. 497-513.

[8] S. Schechter, Iteration methods for nonlinear problems, Trans. Amer. Math. Sco. 104 (1962), pp. 179-189.

[9] __________, Relaxation methods for convex problems, SIAM J. Numer. Anal. 5 (1968), pp. 601-612.

[10] R. S. Varga, Matrix Iterative Analysis, Prentice Hall, Englewood Cliffs, N. J., 1962.