SECTION VI.

DYNAMICS OF FLUID BODIES.

CHAPTER I.

THE GENERAL LAWS OF THE EFFLUX OF WATER FROM VESSELS.

§ 303. Efflux.—The doctrine of the efflux of fluids from vessels constitutes the first principal division of hydro-dynamics. We distinguish first between the efflux of air and the efflux of water, and then again between the efflux under uniform and that under variable pressure. We next treat of the efflux of water under constant pressure. The pressure of water may be assumed as constant when the same quantity of water is admitted on one side as flows out from the other, or when the quantity of water flowing out in a certain time is small compared with the size of the vessel. The chief problem, whose solution will be here treated, is that of determining the discharge through a given orifice, under a given pressure, in a definite time.

If the discharge in each second = \( Q \), we then have the expenditure, after the lapse of \( t \) seconds, under variable pressure: \( Q_t = Qt \). But to obtain the efflux per second, it is necessary to know the dimensions of the orifice, and the velocity of the particles of fluid issuing from it. For the sake of simplicity of investigation, we shall for the present assume that the particles of water flow out in straight and parallel lines, and on this account form a prismatic vein or stream of fluid. If, now, \( F \) be the transverse section of the vein and \( v \) the velocity of the water, or of each particle of water, the discharge per second will form a prism of the base \( F \) and height \( v \), and, therefore, \( Q \) will be = \( Fv \) units of volume and \( G = Fv\gamma \) units of weight, \( \gamma \) being the density of the water or the effluent liquid.

Examples.—1. If water flows through a sluice, of 1.7 square feet aperture, with a 14 feet velocity, the discharge will be \( Q = 14 \times 1.7 = 23.8 \) cubic feet, and hence the discharge per hour will be \( = 23.8 \times 3600'' = 85680 \) cubic feet.—2. If 264 cubic feet of water were to be discharged through an orifice of 3 square inches in 3 minutes and 10 seconds, the mean velocity of efflux would amount to

\[
v = \frac{Q}{Ft} = \frac{264}{5 \frac{5}{144} \times 190''} = \frac{264}{3 \times 140} = 40 \text{ feet.}
\]
§ 304. Velocity of Efflux.—Let us suppose a vessel $AC$, Fig. 397, filled with water, having a horizontal exit orifice, $F$, rounded from the inside, which forms but a small part of the transverse section or bottom surface $CD$. Let the head of water $FG$, supposed invariable during the efflux, = $h$, the velocity of efflux = $v$, and the discharge in each second = $Q$, and therefore its weight = $Qg$. The mechanical effect which this mass of water produces by sinking from the height $h$ is = $Qgh$, and the mechanical effect which the effluent mass $Qg$ accumulates in its transit from a state of rest into that of the velocity $v$, is $\frac{v^2}{2g}Qg$ (§ 71). If no loss of mechanical effect take place in its passage through the orifice, both mechanical effects will be equal, and therefore $h = \frac{v^2}{2g}Qg$; i. e., $h = \frac{Qgh}{2g}$, and, inversely, $v = \sqrt{2gh}$, or in feet, $h = 0.0155v^2$, and $v = 8.02\sqrt{h}$.

The velocity, therefore, of water issuing through an orifice is equivalent to the final velocity of a body falling freely from the height of the water.

The correctness of this law may be proved by the following experiments. If we apply an orifice directed upwards to the vessel $AC$, Fig. 398, the jet $FK$ will ascend vertically, and nearly attain the level $HR$ of the water in the vessel, and we may assume that it would exactly attain this height, were all resistances, such as those of the air, friction at the sides of the vessel, disturbances from the descending water, &c., entirely removed. But since a body ascending to a perpendicular height $h$, has the initial velocity $v = \sqrt{2gh}$ (§ 17), it accordingly follows that the velocity of efflux is $v = \sqrt{2gh}$.

For a different head of water $h_1$, the velocity is $v_1 = \sqrt{\frac{2g}{h_1}}$, hence we have $v : v_1 = \sqrt{h} : \sqrt{h_1}$; therefore, the velocities of efflux are to each other as the square roots of the heads of water.

Examples.—1. The discharge which takes place in each second through an orifice 10 inches square, under a pressure of 5 feet, is:

$Q = Fe = 10.12 \sqrt{2g}h = 120.802 \sqrt{5} = 962.4$. 2,236 = 2151.9 cubic inches.

2. That 252 cubic inches may be discharged through an orifice of 6 square inches in each second, the head of water required is:

$h = \frac{1}{2g} \left( \frac{Q}{F} \right)^2 = 0.0155 \left( \frac{252}{6} \right)^2 = 0.0155(42)^2 = 27.34$ inches.

§ 305. Velocity of Influx and Efflux.—If water flows in with a certain velocity $c$, the mechanical effect $\frac{c^2}{2g}$, corresponding to the ve-
Locality due to the height \( h_i = \frac{c^2}{2g} \), must be added to the mechanical effect \( h \). \( Q \gamma \); hence we have to put:

\[
(h + h_i) Q \gamma = \frac{v^2}{2g} Q \gamma, \text{ or } h + h_i = \frac{v^2}{2g},
\]

and therefore the velocity of efflux:

\[
v = \sqrt{\frac{2g}{h + h_i}} = \sqrt{\frac{2g}{h + \frac{c^2}{2g}}},
\]

Since the quantity of water flowing into a vessel kept constantly full is as great as that \( Q \) which flows out, we may put \( Gc = Fb \), where \( G \) represents the area of the transverse section \( HR \) (Fig. 397) of the water pouring in. Accordingly if we put \( c = \frac{F}{G} v \), we shall then obtain:

\[
h = \frac{v^2}{2g} - \left( \frac{F}{G} \right)^2 \frac{v^2}{2g} = \left[ 1 - \left( \frac{F}{G} \right)^2 \right] \frac{v^2}{2g};
\]

and hence:

\[
v = \frac{\sqrt{2g h}}{\sqrt{1 - \left( \frac{F}{G} \right)^2}}.
\]

According to this formula, the velocity increases, the greater the ratio of the sections \( \frac{F}{G} \) becomes; the velocity is least, viz., \( = \sqrt{2gh} \), if the transverse section \( F \) of the orifice of efflux is small compared with the transverse section \( G \) of the orifice of influx, and it approximates more and more to an infinitely great velocity, the smaller the difference is between these orifices. If \( F = G \), therefore \( \frac{F}{G} = 1 \), then \( v = \frac{\sqrt{2gh}}{0} = \infty \), and therefore also \( c = \infty \). This infinite value must be understood to express that the water must flow to and from a bottomless vessel \( \mathcal{A}C \), Fig. 399, with an infinite velocity, in order that the stream of fluid \( CF \) may entirely fill the orifice of discharge \( F \). If we put \( v = \frac{Gc}{F} \), we shall then obtain:

\[
h = \left[ \left( \frac{G}{F} \right)^2 - 1 \right] \frac{c^2}{2g}, \text{ hence } F = \frac{Ge}{\sqrt{1 + \frac{2g h}{c^2}}},
\]

which expression indicates that the transverse section \( F \) of the stream flowing out, for an infinite velocity of influx, is constantly less than the transverse section \( G \) of the stream flowing in, and hence, that the discharging orifice is not quite filled when it is greater than \( \frac{G}{\sqrt{1 + \frac{2g h}{c^2}}} \).
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Remark. The accuracy of this formula, given by Daniel Bernoulli,

\[ v = \sqrt{\frac{2gh}{1 - \left(\frac{F}{G}\right)^2}} \]

has of late been brought into doubt by many philosophers. I have endeavored to prove bow unfounded are the objections made, in an article "Efflux," in the "Allgemeinen Maschinenencyclopadie."

Example. If water runs from a circular orifice, 5 inches in width, in the bottom of a prismatic vessel of 60 square inches transverse section, under a pressure of 6 feet, the velocity is then:

\[ v = \frac{8.02 \sqrt{h}}{\sqrt{1 - \left(\frac{25 \pi}{4.60}\right)^2}} = \frac{8.02 \times 2.449}{\sqrt{1 - (0.327)^2}} = \frac{19.641}{\sqrt{0.8931}} = \frac{19.641}{0.945} = 20.78 \text{ feet.} \]

§ 306. Velocity of Efflux, Pressure, and Density.—The above formulæ are only true if the pressure of the air on the fluid surface is as great as its pressure against the orifice; but if these pressures are different from one another, we have then to extend these formulæ. If the upper surface HR, Fig. 400, is pressed by a piston K with a force \( P \), which case, for example, presents itself in that of the fire-engine, we may then imagine it to be replaced by the pressure of a column of water. If \( h \) be the height of this column, and \( \gamma \) the density of the liquid, we may therefore put \( P = Gh\gamma \). If we substitute for \( h \) the head of water augmented by \( h_1 = \frac{P_1}{G\gamma}, h + h_1 = h + \frac{P_1}{G\gamma} \), we then obtain for the velocity of efflux:

\[ v = \sqrt{\frac{2gh}{1 - \left(\frac{P_1}{G\gamma}\right)}} \]

when, moreover, we suppose \( \frac{F}{G} \) to be very small. If, further, we represent the pressure on each unit of surface of \( G \) by \( p_1 \), we have more simply \( \frac{P_1}{G} = p_1 \), and hence

\[ v = \sqrt{\frac{2gh}{1 - \left(\frac{p_1}{\gamma}\right)}} \]

Again, if we represent the pressure of water at the level of the orifice by \( p \), we may then put

\[ p = \left(h + \frac{P_1}{\gamma}\right) \gamma; \text{ therefore, } h + \frac{P_1}{\gamma} = \frac{p}{\gamma} \text{, whence } v = \sqrt{\frac{2gh}{1 - \left(\frac{p}{\gamma}\right)}} \]

The velocity of efflux, therefore, increases as the square root of the pressure on each unit of surface, and inversely as the square root of the density of the fluid. Under equal pressures, therefore, a fluid of a density represented by \( 4 \), runs out half as fast as one of density 1. Since the air is 770\(^*\) times lighter than water, it would,

* According to the experiments of Prout, 100 cubic inches of air at 60° temperature and 30 inches barometer, weigh 31,0117 grains troy, and consequently at 32° a cubic foot will weigh 31,0117 \( \times \frac{521}{483} \times 17.28 = 566.1 \) grains; and since a cubic foot of water weighs 62.5 lbs. avoirdupois = 62.5 \( \times \frac{14400}{20} = 437500 \) grains, the relative weights of
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If it were an inelastic body, flow out $\sqrt{770} = 27.5$ faster than water. If the water does not flow freely, but under water, in consequence of a counter-pressure, a diminution of the velocity of efflux then takes place. If the mouth of the vessel $AC$, Fig. 401, is the depth $FG = h$ below the surface of the upper water $HR$, and $FG_1 = h_1$ below the surface $H_1R_1$ of the lower water, we then have the pressure downwards $p = h\gamma$, and the counter pressure upwards $p_1 = h_1\gamma$, hence the force of efflux is:

$$ p - p_1 = (h - h_1)\gamma, $$

and the velocity of efflux

$$ v = \sqrt{2gh} = \sqrt{2gh_1} = \sqrt{2g(h - h_1)}. $$

For efflux under water the difference of level $h - h_1$ between the surfaces must be regarded as the head of water.

If the water on the side of the outer orifice be pressed by the force $p$, and on the side of the inner orifice or of the surface of water by the force $p_1$, we have then generally

$$ v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}. $$

Examples.—1. If the piston of a 12 inch cylinder, or that of a fire-engine, were pressed down with a force of 3000 lbs., and there were no obstacles in the tubes or pipes, the water would then issue through the mouth-piece of the tube and be directed vertically upwards with a velocity:

$$ v = \sqrt{2g\frac{p_1}{\gamma}} = \sqrt{2g\frac{3000}{62.5}} = 8.02 \sqrt{\frac{600}{12.5}} = 62,697 \text{ feet}, $$

and ascend to the height $h = 0.0155 \cdot 60.93 = 0.93$ feet.—2. If water rushes into a rarefied space; for example, into the condenser of a steam-engine, whilst it is pressed from above or on its exposed surface by the atmosphere, the last formula

$$ v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)} $$

for the velocity of efflux is then to be applied. If the head of water $h = 3$ feet, and the external barometer stand at 27 inches, and the internal at 4 Paris inches, we shall now have $p_1 - p = 27 - 4 = 23$ Paris inches $= \frac{23}{12}$ 1,035 Prussian feet $= 2,042$ English feet, or a column of water $= 135$, 2,042 $= 27.57$ feet, and the velocity of the water rushing into the vacuum $v = 8.02 \cdot \sqrt{3 + 27.57} = 44.34$ feet.—3. If the water in the feed-pipe of a steam-engine boiler stands 12 feet above the surface of the water in the boiler, and the pressure of steam be 20 lbs., and the pressure of the atmosphere only 15 lbs. on the square inch, the velocity with which the water enters into the boiler will be:

$$ v = 8.02 \sqrt{12 + \frac{(15 - 20) \cdot 144}{62.5}} = 8.02 \sqrt{12 - \frac{5.144}{62.5}} = 8.02 \sqrt{0.48} = 5.55 \text{ feet}. $$

Air and water at that temperature is $\frac{566.1}{437500} = \frac{1}{772}$. At 60° the relation will be 535.88.

$\frac{437500}{772} = 1.816$. And as $\sqrt{816} = 28.5$, the velocity of efflux will, under this condition, be $\frac{28.5}{27.75}$ part more rapid than in that supposed in the text.—Am. Ed.
§ 307. Hydraulic Pressure.—When water enclosed in a vessel is in motion, it then presses more feebly against the sides than when at rest. We must, therefore, distinguish the hydrodynamic or hydraulic pressure from the hydrostatic pressure of water. If \( p_1 \) be the pressure on each unit of surface \( H_1R_1 = G_1 \), Fig. 402, \( p \) the pressure without the orifice \( F \), and \( h \) the head of water \( FG_1 \), we then have for the velocity of efflux

\[
v = \sqrt{2g \left( h + \frac{p_1 - p}{\gamma} \right) + \frac{1 - \left( \frac{F}{G_1} \right)^2}{2g}},
\]

or

\[
h + \frac{p_1 - p}{\gamma} = \left[ 1 - \left( \frac{F}{G_1} \right)^2 \right] \frac{v^2}{2g}; \text{ if, further,}
\]

in another transverse section \( H_2R_2 = G_2 \), which lies at a height \( FG_2 = h_2 \) above the orifice, the pressure = \( p_2 \), we then have likewise:

\[
h_1 + \frac{p_2 - p}{\gamma} = \left[ 1 - \left( \frac{F}{G_2} \right)^2 \right] \frac{v^2}{2g}.
\]

If we subtract one expression from the other, it then follows that:

\[
h - h_1 + \frac{p_1 - p_2}{\gamma} = \left[ \left( \frac{F}{G_1} \right)^2 - \left( \frac{F}{G_2} \right)^2 \right] \frac{v^2}{2g},
\]

or, if the head of water \( G_1G_2 \) of the stratum \( H_2R_2 = G_2 \) be represented by \( h_2 \), the measure of the hydraulic pressure of water at \( H_2R_2 \) is:

\[
p_2 = h_2 + \frac{p_1 - p_2}{\gamma} = \left[ \left( \frac{F}{G_1} \right)^2 - \left( \frac{F}{G_2} \right)^2 \right] \frac{v^2}{2g}.
\]

But now \( F_0 \) is the velocity \( v_1 \) of the water at the surface \( G_1 \), and \( F_0 \)

\[
\frac{G}{G_2},
\]

the velocity \( v_2 \) of the water at the section \( G_2 \); hence more simply we may write \( p_2 = \frac{p_1}{\gamma} + h_2 - \left( \frac{v_2^2}{2g} - \frac{v_1^2}{2g} \right) \).

Therefore, from this it follows that the hydraulic head of water \( p_2 \) at any place in the vessel is equivalent to the hydrostatic head of water \( \frac{p_1}{\gamma} + h_2 \) diminished by the difference of the height due to the velocity at this point, and at the place of entrance. If the upper surface of the water \( G_1 \) is great, we may neglect the velocity of influx, and hence may put \( p_2 = \frac{p_1}{\gamma} + h_2 - \frac{v_2^2}{2g} \), and the hydraulic head of water is less by the height due to the velocity than the hydrostatic head of water. The faster, therefore, water flows in conduit pipes, the less it presses against the sides of the pipes. From this cause, pipes very often burst, or begin to leak, when its motion in them is checked, or when the pipes are stopped up, &c.
By means of the apparatus of efflux $ABC\text{D}$, Fig. 403, we may have ocular demonstration of the difference between hydraulic and hydrostatic pressure. If we carry upwards a small tube $ER$ from the transverse section $G_2$, it will become filled with water, which will ascend in it above the level of the fluid surface if $G_2$ is $> G_1$, therefore $v_2 < v_1$; for as the pressure $p_1$ on the fluid surface is counteracted by the pressure of the air against the mouth of the tube, we may put for the height which measures the pressure at $G_2$ viz. $x = \frac{p_2}{g} = h_2$

$$-\left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right),$$

and therefore $x$ is $> h_2$ if $\frac{v_2^2}{2g}$ is $< \frac{v_1^2}{2g}$. If, on the other hand, the transverse section $G_3$ be $< G_1$, and the water therefore flow through $G_3$ quicker than through $G_1$, we shall then have the height of the column of water in the small tube $E_1$ whose inner orifice is at $G_3$, $y = h_3 - \left(\frac{v_2^2}{2g} - \frac{v_1^2}{2g}\right)$ less than $h_3$, and hence it will not reach to the level $HR$ of $G_1$. Again, if $G_4$ be very small, and therefore the corresponding velocity $v_4$ very great, then $\frac{v_4^2}{2g} - \frac{v_2^2}{2g}$ may be $> h_4$, and hence the corresponding hydraulic head of water $z$ may be negative, i.e. the air may press more from without than the water from within. A column of water will therefore ascend in the tube $E_1K$, which is inserted below, and whose outer orifice is under water, which in conjunction with the pressure of the water, will balance that of the external atmosphere. If this small tube be short, the water, which may be colored for this purpose, will ascend from the vessel $K$ underneath, through the tube, enter the reservoir of efflux, and will arrive at $F$ and be discharged.

Remark. If the discharging vessel $ACE$, Fig. 404, consists of a wide reservoir $AC$ and of a narrow vertical tube $CE$, the hydraulic pressure at all places in this tube is then negative. If we disregard the pressure of the atmosphere $p_1$, the pressure of the water in the vicinity of the mouth $F$ may be put $= 0$, because the whole head of water here $GF = h$ will be expended in generating the velocity $v = \sqrt{2gh}$; on the other hand, at a place $D, E$, at the height $GG = h$, below the surface of water, the hydraulic pressure $= h = - (h-h_1)$ negative; if, therefore, a hole be bored in this tube, no water will run out, but air will rather be drawn in, which will arrive at $F$ and flow out. This negative pressure will be greatest directly below the water, in $G$ because $h_2$ is there least.

§ 308. By means of the formula $Q = F\sqrt{2gh}$, the discharge issuing in one second can only then be calculated directly when the orifice is horizontal, because here only the velocity throughout the whole
transverse section $F$ is the same; but if the transverse section of the orifice has an inclination to the horizon, for example, if it is at the side of the vessel, the particles of water at different depths will then flow out with different velocities, and the discharge $Q$ can no longer be considered as a prism, and hence, therefore, the formula $Q = Fv = F\sqrt{2gh}$ cannot be applied directly. The most simple case of this kind is presented in the efflux through a cut in the side of a vessel, or in what is called a weir, Fig. 405. This cut forms a rectangular aperture of efflux $EFGH$, whose breadth $EF = GH$ is represented by $b$, and height $EH = FG$ by $h$. If we divide this surface $bh$ by horizontal lines into a great number $n$ of equally broad laminae, we may suppose the velocity in each of these to be the same. Since the heads of water of these laminae from above downwards are $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}, \&c.$, we then have the corresponding velocities

$$\sqrt{2g\cdot\frac{h}{n}}, \sqrt{2g\cdot\frac{2h}{n}}, \sqrt{2g\cdot\frac{3h}{n}}, \&c.;$$

and since, further, the area of a lamina $= b\cdot\frac{h}{n} = \frac{bh}{n}$, we then have the discharge

$$Q = \frac{bh}{n} \left( \sqrt{2g\cdot\frac{h}{n}} + \sqrt{2g\cdot\frac{2h}{n}} + \sqrt{2g\cdot\frac{3h}{n}} + \ldots \right) = \frac{bh\sqrt{2gh}}{n\sqrt{n}} \left( \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n} \right).$$

But now:

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n},$$

or

$$1^{\frac{1}{3}} + 2^{\frac{1}{3}} + 3^{\frac{1}{3}} + \ldots + n^{\frac{1}{3}} = \frac{n^{1+\frac{1}{3}}}{1+\frac{1}{3}} = \frac{3}{2} n^{\frac{2}{3}} = \frac{3}{2} n^{\frac{2}{3}};$$

hence, the discharge required is:

$$Q = \frac{bh\sqrt{2gh}}{n\sqrt{n}} \cdot \frac{3}{2} n^{\frac{2}{3}} = \frac{3}{2} bh\sqrt{2gh};$$

If by the term "mean velocity ($v$)" be understood that velocity which must subsist at all places, that as much water, in consequence, does issue as with the variable velocities of efflux within the whole profile; we may then put: $Q = bh\cdot v$, and, consequently, $v = \frac{3}{2} \sqrt{2gh}$, i.e.
the mean velocity of water issuing through a rectangular cut in the side of a vessel is \( \frac{3}{2} \) of the velocity at the sill or lower edge of the cut.

If the rectangular aperture of efflux \( K G \), Fig. 406, with horizontal sill does not reach the surface of the water, we may find the discharge by regarding the aperture as the difference of the two cuts \( EFGH \) and \( EFLK \). Hence, if \( h \) is the depth \( HE \) of the lower, and \( KE = h_3 \) that of the upper edge, we then have the discharge from these apertures \( \frac{3}{4} b \sqrt{2g h_1^3} \) and \( \frac{3}{4} b \sqrt{2g h_2^3} \), and hence the quantity of water for the rectangular orifice \( GHKL \):

\[
Q = \frac{3}{4} b \sqrt{2g h_1^3} - \frac{3}{4} b \sqrt{2g h_2^3} = \frac{3}{4} b \sqrt{2g (h_1^3 - h_2^3)},
\]

and the mean velocity of efflux:

\[
v = \frac{Q}{b(h_1 - h_2)} = \frac{3}{4} \sqrt{2g \left( \frac{h_1^3 - h_2^3}{h_1 - h_2} \right)}.
\]

If \( h \) is the mean head of water \( \frac{h_1 + h_2}{2} \), or the depth of the centre of the orifice below the surface of water, and \( a \) the height of the orifice \( HK = h_1 - h_2 \), we may then put

\[
v = \frac{3}{4} \sqrt{2g \left( \frac{h + \frac{a}{2}}{h - \frac{a}{2}} \right)},
\]

or approximately:

\[
v = \frac{a}{\sqrt{2g h}}.
\]

**Example.** If a rectangular orifice is 3 feet wide and 1 1/2 feet high, and the lower edge lies 2 3/4 feet below the surface of water, the discharge is then:

\[
Q = \frac{3}{4} \times 8.023 \times (2.75^2 - 1.52^2) = 16.04 \times (4.560 - 1.837) = 16.04 \times 2.723 = 43.67 \text{ cubic feet.}
\]

From the formula of approximation the mean velocity of efflux is:

\[
v = \left[1 - \frac{1}{g/125} \left(\frac{1.25}{2}\right)^3\right] \approx 8.02 \sqrt{2,125} = 11.685 - 0.042 = 11,643 \text{ feet, and hence the discharge } Q = 3. \frac{3}{4} \times 11,643 = 43,65 \text{ cubic feet.}
\]

**Remark.**—If the cut in the side is inclined to the horizon at an angle \( \varphi \), we shall then have to substitute the height of the aperture \( \frac{h_1 - h_2}{\sin \varphi} \) for its vertical projection, whence we must put

\[
Q = \frac{3}{4} b \sqrt{2g} \left[ \sqrt{h_1^3} - \sqrt{h_2^3} \right].
\]

If the transverse section of the reservoir parallel to the aperture be not considerably greater than the section of the aperture, we shall then have to take into account the velocity \( v = \frac{F}{G} \) with which the water flows to it, and for this reason put:

\[
Q = \frac{3}{4} b \sqrt{2g \left[ \left( h_1 + \frac{v^2}{2g} \right)^{\frac{3}{2}} - \left( h_2 + \frac{v^2}{2g} \right)^{\frac{3}{2}} \right]}.
\]
§ 309. Triangular Lateral Orifice. — Besides rectangular lateral orifices, we have in practice triangular and circular. Let us first consider the efflux through a triangular orifice $EFG$, Fig. 407, with horizontal base, whose vertex $E$ lies in the surface of the water. Let the base $FG = b$, and the height $EF = h$, let us divide the last into $n$ equal parts, and carry through the points of division lines parallel to the base, we then resolve the entire surface into small elements of the areas:

$$
\frac{b}{n}, \frac{h}{n}, \frac{2b}{n}, \frac{3b}{n}, \frac{h}{n}, \ldots, \frac{h}{n},
$$

and the heads of water:

$$
\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}, \frac{h}{n}, \frac{h}{n}, \ldots, \frac{h}{n}, \frac{h}{n},
$$

The discharges for these are:

$$
\frac{bh}{n^2} \sqrt{2g \frac{h}{n}} \frac{2bh}{n^2} \sqrt{2g \frac{h}{n}} \frac{3bh}{n^2} \sqrt{2g \frac{h}{n}}, \ldots, \frac{h}{n},
$$

and we obtain the discharge for the whole orifice:

$$
Q = \frac{bh}{n^2} \sqrt{2g \frac{h}{n}} (1 + 2 \sqrt{2} + 3 \sqrt{3} + \ldots + n \sqrt{n}),
$$

or since the series in the parenthesis

$$
\frac{n^2}{n^2} \frac{n^2}{n^2} = \frac{3}{2} + 1 = \frac{3}{2} n^2,
$$

$$
Q = \frac{2}{3} bh \sqrt{2g h} = \frac{2}{3} bh \sqrt{2gh}.
$$

If the base of the orifice $EGK$ lies in the surface and the vertex lower by $h$, we then have the discharge $\frac{2}{3} bh \sqrt{2g h}$ flowing through the rectangle $EFGK$.

Through the trapezium $ABCD$, Fig. 408, whose upper base $AB = b_1$, lies in the surface of the water, and whose lower base is $CD = b_2$, and height $DE = h$, we may find the discharge by regarding the orifice as composed of a rectangle and two triangles, viz.,

$$
Q = \frac{2}{3} b_1 h \sqrt{2gh} + \frac{2}{5} (b_1 - b_2) h \sqrt{2gh} = \frac{2}{5} (2 b_1 + 3 b_2) h \sqrt{2gh}.
$$

Further, the discharge for a triangle $CDE$, Fig. 409, of the base $DE = b_1$, and of the height $h_1$, and whose vertex $C$ is distant $h$ from
the surface of water: $Q = \text{discharge through } ABC \text{ less the discharge through } AE$

$$Q = \frac{4}{3} bh \sqrt{2gh} - \frac{2}{3} (2b + 3b) h_i \sqrt{2gh_i}$$

As the breadth $AB = b$ may be determined by the proportion $b : b_1 = h : (h - h_i)$, it follows that

$$Q = \frac{2 \sqrt{2g} \cdot b_i}{15} \left( \frac{2 h (h^\frac{3}{2} - h_i^\frac{3}{2}) - 3 h_i^\frac{3}{2}}{h - h_i} \right)$$

Lastly, for a triangle $ACD$, Fig. 410, whose vertex lies above the base, the quantity discharged is

$$Q = \frac{3}{5} \sqrt{2g} \cdot b_i (h^\frac{3}{2} - h_i^\frac{3}{2}) - \frac{2 \sqrt{2g} \cdot b_i}{15} \left( \frac{2 h^\frac{5}{2} - 5 h h_i^\frac{3}{2} + 3 h_i^\frac{5}{2}}{h - h_i} \right)$$

Example. What quantity of water flows through the square $ABCD$, Fig. 411, whose vertical diagonal $AC = 1$ foot, if the angular point $A$ reaches the surface of the water?

The upper half of this square gives the expenditure:

$$Q = \frac{3}{5} \sqrt{2g} = \frac{3}{5} \sqrt{2 \times 9.81 \times 0.44} = 0.44 \times 8.02 \times 0.533 = 0.21 \times 0.533 = 0.1131 \text{ cubic feet,}$$

but the lower water expenditure:

$$Q = \frac{2 \sqrt{2g} \cdot b_i}{15} \left( \frac{2 h^\frac{5}{2} - 5 h h_i^\frac{3}{2} + 3 h_i^\frac{5}{2}}{h - h_i} \right)$$

$$= \frac{2 \sqrt{2g} \cdot b_i}{15} \left( \frac{2 \times 9.81 \times 0.533}{1 - \frac{1}{2}} \right)$$

$$= 32.08 \left( \frac{2 - 1.7678 + 0.5303}{15} \right) = 1.6307 \text{ cubic feet; the discharge through the entire orifice is } Q = 0.1131 + 1.6307 = 2.7438 \text{ cubic feet.}$$

§ 310. Circular Lateral Orifices.—The discharge through a circular orifice $AB$, Fig. 412, may be determined by an approximate formula in the following manner. Let us decompose the orifice by concentric circles into equally small annuli, and each annulus into very small elements, which may be regarded as parallelograms. If now $r$ is the radius of such an annulus, $b$ its breadth and $n$ the number of its elements, we have the magnitude of an element
CIRCULAR LATERAL ORIFICES.

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\[ K = \frac{2\pi rb}{n} \].  

If \( h \) is the depth \( CG \) of the centre \( C \) below the surface of water \( HR \), and \( \phi \) the angle \( ACK \), by which an element \( K \) is distant from the highest point \( A \) of the annulus, we have then the head of water of this element:

\[ KF = CG - CL = h - r \cos \phi, \]

and hence the discharge of this element

\[ = \frac{2\pi rb}{n} \sqrt{2g(h - r \cos \phi)}. \]

Now \( \sqrt{h - r \cos \phi} \)

\[ = \sqrt{h} \left[ 1 - \frac{1}{2} \frac{r}{h} \cos \phi - \frac{1}{8} \left( \frac{r}{h} \right)^2 \cos \phi^2 + \ldots \right] \]

\[ = \sqrt{h} \left[ 1 - \frac{1}{2} \frac{r}{h} \cos \phi - \frac{1}{8} \left( \frac{r}{h} \right)^2 (1 + \cos 2\phi) + \ldots \right]; \]

hence the discharge of an element:

\[ = \frac{2\pi rb}{n} \sqrt{\frac{2gh}{h}} \left[ 1 - \frac{1}{2} \frac{r}{h} \cos \phi - \frac{1}{8} \left( \frac{r}{h} \right)^2 (1 + \cos 2\phi) - \ldots \right]. \]

The discharge of an entire annulus is now known, if we put in the parenthesis for \( 1, n = 1 = n, \) for \( \cos \phi \) the sum of all the cosines of \( \phi \) from \( \phi = 0 \) to \( \phi = 2\pi, \) and for the cosine of \( 2\phi, \) the sum of all the cosines of \( 2\phi \) from \( 2\phi = 0 \) to \( 2\phi = 4\pi. \) But as the sum of all the cosines of a complete circle is \( 0, \) these cosines vanish, and the discharge for the annulus:

\[ = \frac{2\pi rb}{n} \sqrt{2gh} \left[ n - \frac{1}{8} \left( \frac{r}{h} \right)^2 \cdot n - \ldots \right] \]

If now for \( b \) we substitute \( \frac{r}{m}, \) and for \( r, \frac{r}{m}, \frac{2r}{m}, \frac{3r}{m}, \ldots, \frac{mr}{m}, \) we then obtain the discharge of all the annuli which make up the circular surface, and lastly, the quantity of efflux of the whole circle

\[ Q = \pi r^2 \sqrt{2gh} \left[ \frac{r}{m^3} (1 + 2 + 3 + \ldots + m) \right. \]

\[ - \frac{1}{8} \left( \frac{r^3}{m^4 h^2} (1^3 + 2^3 + 3^3 + \ldots + m^3) \right] \]

\[ = \pi r^2 \sqrt{\frac{2gh}{h}} \left[ 1 - \frac{1}{8} \left( \frac{r}{h} \right)^2 \right. - \ldots \left. \right], \]

or more accurately:

\[ Q = \pi r^2 \sqrt{\frac{2gh}{h}} \left[ 1 - \frac{1}{8} \left( \frac{r}{h} \right)^2 - \frac{n^4}{1024} \left( \frac{r}{h} \right)^4 - \ldots \right]. \]

If the circle reaches the surface of the water, then

\[ Q = 1024 \pi r^2 \sqrt{\frac{2gh}{h}} = 0.964 F \sqrt{\frac{2gh}{h}} \]

if \( F \) represents the area of the circle.

It is besides easy to conceive that in all cases where the head of
water at the centre is equal to or greater than the diameter, we may put the whole series $= 1$, and take \( Q = F \sqrt{2 \cdot gh} \). This rule may also be applied to other orifices, and, therefore, in all cases where the centre of gravity of an orifice lies at least as deep below the fluid surface as the figure is high, the depth \( h \) of this point may be regarded as the head of water, and \( Q \) put $= F \sqrt{2 \cdot gh}$.

If we consider that the mean of all the cosines of the first quadrant $= \frac{\pi}{4}$, and that all the cosines of the second $= -\frac{\pi}{4}$ and likewise that the mean of the first and of the second vanishes, we may then, by the method adopted above, find the discharge of the upper semicircle:

\[
Q_1 = \frac{\pi r^2}{2} \sqrt{2 \cdot gh} \left[ 1 - \frac{\pi}{12} \left( \frac{r}{h} \right) - \frac{1}{12} \left( \frac{r}{h} \right)^2 \right],
\]

and that of the lower:

\[
Q_2 = \frac{\pi r^2}{2} \sqrt{2 \cdot gh} \left[ 1 + \frac{\pi}{12} \left( \frac{r}{h} \right) - \frac{1}{12} \left( \frac{r}{h} \right)^2 \right].
\]

**Example.** What quantity of water flows hourly through a circular orifice 1 inch in diameter, above which the fluid surface stands 1 inch high?

\[
\frac{r}{h} = 0.4, \text{hence } \left( \frac{r}{h} \right)^4 = 0.735; \text{ further, } 1 - \frac{1}{12} \left( \frac{h}{r} \right)^2 = 1 - 0.023 = 0.977,
\]

and consequently the discharge per second:

\[
Q = \pi \cdot \frac{1}{4} \cdot 12 \cdot 8.02 \cdot \frac{7}{144} \cdot 0.977 = \pi \cdot \frac{1}{4} \cdot 8.02 \cdot 0.977 \sqrt{7} = 16.26 \text{ cubic inches},
\]

which, per minute, is 973, and per hour 33.78 cubic feet.

§ 311. Discharging Vessels in Motion.—The velocity of efflux varies if a vessel previously at rest or in uniform motion changes its condition of motion, because in this case every particle acts by its own weight, as well as by its inertia against the surrounding medium.

If we move the vessel \( AC \), Fig. 413, upwards with a vertical accelerating force, whilst the water flows through the bottom by the hole \( F \), an increase takes place, and if it be moved downwards vertically by an accelerating force, a diminution of the velocity of efflux ensues. If \( p \) is the accelerating force, each element of water \( M \) presses not only by its own weight \( Mg \), but also by its inertia \( Mp \); consequently the force of each element in the one case, must be put \((g + p) M \), and in the other \((g-p) M \), therefore instead of \( g \), \( g + p \). From this it follows then that \( \frac{v}{2} = (g + p) h \), and hence for the velocity:

\[
v = \sqrt{2 \cdot (g + p) \cdot h}.
\]

If the vessel ascends with the accelerating force \( g \), then is
v = \sqrt{2 \cdot 2 \cdot gh} = 2 \sqrt{gh}, therefore the velocity of efflux 1,414 times that of a vessel at rest. If the vessel falls by its own weight, therefore, with the accelerated motion g, there is \( v = \sqrt{0} = 0 \), no water therefore flows out. If the vessel moves uniformly up or down, there remains \( v = \sqrt{2 \cdot gh} \), but if it ascends with a retarded motion, then will \( v = \sqrt{2(g-p)h} \), and if it descends with the same retardation, then \( v = \sqrt{2(g+p)h} \).

If the vessel moves horizontally, or at an acute angle to the horizon (§ 274), the fluid surface will be inclined to the horizon, and a change in the velocity of flow will take place.

By the rotation of a vessel \( AC \), Fig. 414, about its vertical axis \( XX \), the concave surface forms a parabolic funnel \( AOB \), hence there will be over the middle \( F \) of the bottom a lesser head of water than at the edges, and hence the water will flow through the orifice \( F \), in the axis, more slowly than through any other orifice \( K \) at the bottom. If \( h \) represent the head of water in the middle, then the velocity of efflux at the middle will be \( = \sqrt{2 \cdot gh} \), if \( y \) be the distance \( FK = ME \) of any other orifice \( K \) from the axis, and \( \omega \) the angular velocity, we shall then have the corresponding elevation of the water above the middle:

\[
OM = \frac{1}{2} \quad TM = \frac{1}{2} \quad ME \cot \theta \quad T = \frac{1}{2} \quad y \quad \frac{\omega^2 y^2}{g} = \frac{\omega^2 y^2}{g} = \frac{w^2}{2g},
\]

if \( w \) be the velocity of rotation of the orifice \( K \). Hence then the velocity of efflux for this is

\[
v = \sqrt{2g(h + \frac{w^2}{2g})} = \sqrt{2gh + \frac{w^2}{2g}}.
\]

This formula is true for every arbitrarily shaped vessel, and also for one closed above, as \( AC \), Fig. 415, so that the funnel cannot be formed. Its application to wheels of reaction and to turbines will be found in the sequel.

Examples. — 1. If a vessel full of water \( AC \), Fig. 413, weighs 350 lbs., and by means of a rope passing over a roller \( K \) is drawn by a weight \( G \) of 450 lbs., it will ascend with an accelerating force \( p = \frac{450 - 350}{350} \cdot g = \frac{100}{800} \cdot g = \frac{1}{8} \cdot g \), and hence the velocity of efflux will be \( v = \sqrt{2 \cdot \left( g + \frac{1}{8} \cdot g \right)} = \sqrt{2 \cdot \frac{9}{8} \cdot gh} \). Were the head of water \( h = 4 \) feet, the velocity of efflux would be \( v = 1 \cdot \sqrt{9 \cdot g} = 3 \cdot \sqrt{32.2} = 16.01 \) feet. — 2. If a vessel \( AC \), Fig. 415, full of water revolves so that it makes 100 revolutions per minute, if the depth of the orifice \( F \) below the surface of water in the middle amounts to 2 feet, and the distance from the axis \( XX \) 3 feet, then the velocity of efflux is

\[
v = \sqrt{2 \cdot g \cdot h + \frac{w^2}{2g}} = \sqrt{64.4 \cdot 2 + \left( \frac{3 \cdot \omega \cdot 100}{30} \right)} = \sqrt{128.4 + 100 \cdot \omega^2} = \sqrt{238.4} = 33.4 \text{ feet.}
\]

If the vessel be at rest the velocity will be \( v = \sqrt{128.8} = 11.34 \text{ feet.} \)
CHAPTER II.

ON THE CONTRACTION OF THE FLUID VEIN BY THE EFFLUX OF WATER
THROUGH ORIFICES IN A THIN PLATE.

§ 312. Co-efficient of Velocity.—The laws of efflux developed in
the preceding chapter accord almost entirely with experiment, so long
as the head of water is not small compared with the width of the
orifice, and as long as the orifice gradually widens inwards without
forming corners or edges, and is close at the bottom or sides of the
vessel. The experiments made by Michelotti, by Eyetelwein, and
by the author on this subject, with smoothly polished metallic mouth-
pieces, have shown that the effective discharge, or that which actually
flows out, amounts to from 96 to 98 per cent. of the theoretical
quantity.

The mouth-piece $AD$, Fig. 416, represented in half its natural
size, gave for a head of water of 10 feet, 97,5
per cent., for 5 ft. 96,9 per cent., and for 1
ft. 96,8 per cent. Since for this efflux the
fluid vein has the same transverse section as
the orifice, we must then assume that this
diminution of discharge is accompanied
with a loss of velocity, which is caused by the friction
or adhesion of the water to the inner circum-
ference of the orifice, and by the viscosity of
the water. In what follows, we shall call the
ratio of the effective velocity of efflux to that of the theoretical
$= v \sqrt{2 gh}$, the co-efficient of velocity, and represent it by $\phi$. From
this, therefore, the effective velocity of efflux in the most simple case
is $v_1 = \phi v = \phi \sqrt{2 gh}$, and the discharge:

$$Q = Fv_1 = \phi Fv = \phi F \sqrt{2 gh}.$$ 

If we substitute for $\phi$ the mean value 0,97, we then obtain for the
quantity in feet

$$Q = 0,97 \cdot F \sqrt{2 gh} = 0,97 \cdot 8,02 F \sqrt{h} = 7,779 F \sqrt{h}.$$ 

A vis viva $Q_\gamma \cdot \frac{v^2}{g}$, is inherent in a discharge $Q$ issuing with the velo-
city $v$, by virtue of which it is capable of producing the mechanical
effect $Q_\gamma \cdot \frac{v^2}{2g}$. But since by its descent from the height $h = \frac{v^2}{2g}$, the
weight $Q_\gamma$ produces the mechanical effect $Q_\gamma \cdot \frac{v^2}{2g}$, it follows that
by the efflux of the water, this suffers a loss
\[ Q\gamma \left( \frac{v^2}{2g} - \frac{v_1^2}{2g} \right) = Q\gamma \cdot \frac{v^2}{2g} (1 - \phi^2) = (1 - 0.97^2) Q\gamma \cdot \frac{v^2}{2g} \]

\[ = 0.059 \ Q\gamma \cdot \frac{v^2}{2g}, \text{ or } 5.9 \text{ per cent.} \]

Therefore, the effluent water produces by its *vis viva* 5.9 per cent. less mechanical effect, than does its weight by falling from the height \( h \).

§ 313. Co-efficient of Contraction.—If water flows through an orifice in a thin plate, a considerable diminution of the discharge under otherwise similar circumstances takes place, whilst the particles of fluid rushing through the orifice move in convergent directions, and in this way give rise to a contraction of the fluid vein. The measurements of the vein made by many, and especially of late by the author, have shown that the vein at a distance which is about equal to one-half of the width of the orifice, has the greatest contraction, and a thickness equal to 0.8 that of the diameter of the orifice. If \( F_1 \) is the transverse section of the contracted vein, as also \( F \) the transverse section of the orifice, we then have from this \( F_1 = (0.8)^2 F = 0.64 \ F \).

The ratio \( F_1 \) of these transverse sections is called the *co-efficient of contraction*, and is represented by \( \alpha \), and accordingly, the mean value for the efflux of water through orifices in a thin plate may be put:

\[ \alpha = 0.64. \]

As long as we possess no more accurate knowledge on the contraction of the fluid vein, we may assume that the stream flowing through a circular orifice \( AB \), Fig. 417, forms a body of rotation \( ABEF \), whose envelope is generated by the revolution of a circular arc \( AF \) about the axis \( CD \) of the stream. Let the diameter \( AB \) of the orifice \( = d \), and the distance \( CD \) of the contracted section \( EF \), \( = \frac{1}{2} d \), we then obtain the radius

\[ \alpha = MF = r \]

of the generating arc \( AF \) by means of the equation:

\[ \frac{\beta \Delta s}{2} = FN \left( 2 \frac{MF}{2} - FN \right), \text{ or} \]

\[ \frac{d^2}{4} = \frac{d}{10} \left( 2r - \frac{d}{10} \right), \quad r = 1.3d. \]

Orifices made after this figure of the contracted vein give pretty nearly the velocity of discharge \( v_1 = 0.97 \cdot v \).

The contraction of the fluid vein is caused by the water which lies directly above the orifice flowing out together with that which comes to it from the sides. There takes place, therefore, in the interior of the vessels the convergence of the filaments of water, similar to that represented in the figure, and the contraction of the fluid vein con-
sists in a mere propagation of this convergence. We may convince ourselves of this motion of the water in the vicinity of the orifice by means of a glass apparatus of efflux; if we drop into the fluid minute substances which are either heavier or lighter than water, for example, such as oak saw-dust, bits of sealing wax, &c., and allow them to pass out with it from the orifice.

§ 314. Contraction of the Fluid Vein.—If water flows through triangular or quadrilateral orifices, and in a thin plate, the stream then assumes particular figures. The inversion of the jet, or the altered position of its transverse section with respect to that of the orifice, is very striking to the eye, in consequence of which a corner of this section comes to coincide with the middle of one side of the orifice.

Hence, from a triangular orifice \( \textit{ABC} \), Fig. 418, the section of the stream at a certain distance from the orifice forms a treble star-like vein \( \textit{DEF} \), from a quadrilateral orifice \( \textit{ABCD} \), Fig. 419, a star of four veins \( \textit{EFGH} \), from a five-sided orifice \( \textit{ABCDE} \), Fig. 420, a star \( \textit{FGHKL} \), consisting of five veins. These sections vary at different distances from the orifice: at a certain distance they diminish, and at a successive one again increase; hence the vein consists of plates or ribs of variable breadth, and thereby forms, when the efflux is observed under great pressure, bulges and nodes, similar to what is seen in the cactus. If the orifice \( \textit{ABCD} \), Fig. 421, is rectangular; at a lesser distance from the orifice, the section will then form a cross or star; and at a greater one, it will again assume the form of a rectangle \( \textit{EF} \).

Observations on various kinds of orifices have been made by Bidone, and accurate measurements of the vein from square apertures also by Poncelet and Lesbros. The last measurements have led to a small co-efficient of contraction 0,563. The measurements of water issuing through lesser orifices, give us, however, greater co-efficients of contraction; they show, moreover, that these are greater for elong-
gated rectangles than for rectangles which approximate more to the square.

§ 315. Co-efficient of Efflux.—If in the flow of water through orifices in thin plates, the effective velocity were equal to the theoretical \( v = \sqrt{2gh} \), we should have the effective discharge:

\[
Q = a \cdot F \cdot v = a \cdot F \cdot \sqrt{2gh},
\]

because \( a \cdot F \) represents the transverse section of the vein at the place of greatest contraction, where the particles of water move in parallel directions. But this is by no means the case: it is shown rather by experience that \( Q \) is smaller than \( a \cdot F \cdot \sqrt{2gh} \), that we must therefore multiply the theoretical discharge \( F \cdot \sqrt{2gh} \) by a co-efficient which is less than the co-efficient of contraction, in order to obtain the effective discharge. We must hence assume that for efflux from an orifice in a thin plate, a certain loss of velocity takes place, and therefore introduce a co-efficient of velocity \( \phi \), and hence put the effective velocity of efflux \( v_1 = \phi v = \phi \sqrt{2gh} \). From this then we have the effective discharge \( Q_1 = F_1 \cdot v_1 = a \cdot F \cdot \phi v = a \cdot \phi F \cdot \sqrt{2gh} \). Again, if we call the ratio of the effective discharge to the theoretical or hypothetical quantity, the co-efficient of efflux, and represent it in what follows by \( \mu \), we then have:

\[
Q_1 = \mu Q = \mu a \cdot F \cdot \sqrt{2gh},
\]

hence \( \mu = a \cdot \phi \), i.e. the co-efficient of efflux is the product of the co-efficients of contraction and of velocity.

Multiplied observations, but chiefly the measurements of the author, have led to this, that the co-efficient of efflux for orifices in thin plates is not constant; that for small orifices and for small velocities, it is greater than for large orifices and for great velocities: and that it is considerably greater for elongated and small orifices than for orifices which have a regular form, or which approximate to the circle.

For square orifices of from 1 to 9 square inches area, with from 7 to 21 feet head of water, according to the experiments of Bossut and Michelotti, the mean co-efficient of efflux is \( \mu = 0,610 \); for circular ones of from ½ to 6 inches diameter, with from 4 to 21 feet head of water, \( \mu = 0,615 \), or about \( \frac{18}{25} \). The single values observed by Bossut and Michelotti vary considerably from one another, but we cannot discover in them any dependence between the dimensions of the orifice and the magnitude of the head of water. From the author’s experiments at a pressure of 24 inches, the co-efficient for an orifice of

<table>
<thead>
<tr>
<th>Diameter (inches)</th>
<th>Co-efficient of Efflux ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,3937</td>
<td>0,628</td>
</tr>
<tr>
<td>0,7874</td>
<td>0,621</td>
</tr>
<tr>
<td>1,1811</td>
<td>0,614</td>
</tr>
<tr>
<td>1,5748</td>
<td>0,607</td>
</tr>
</tbody>
</table>

On the other hand, at a pressure of 10 inches for the round orifice of

<table>
<thead>
<tr>
<th>Diameter (centimetres)</th>
<th>Co-efficient of Efflux ( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0,637</td>
</tr>
<tr>
<td>2</td>
<td>0,629</td>
</tr>
<tr>
<td>3</td>
<td>0,622</td>
</tr>
<tr>
<td>4</td>
<td>0,614</td>
</tr>
</tbody>
</table>
From these it is manifest that the co-efficient of efflux increases when the dimensions of the orifice and the head of water decrease.

If \( \mu \) we take the mean value = 0.615, and for \( a = 0.64 \), we obtain the co-efficient of velocity for the efflux through orifices in a thin plate, \( \varphi = \frac{\mu}{a} = 0.96 \), therefore, nearly as great as for efflux through rounded or conoidal orifices.

**Remark 1.** Buff's experiments (see Poggendorf's Ann. Band 46), show that the co-efficient of efflux for small orifices and for small heads of water or velocities is considerably greater than for large or mean orifices and velocities. An orifice of 2.084 lines diameter, gave for \( \frac{1}{4} \) inch pressure, \( \mu = 0.692 \); for 35 inches \( \mu = 0.644 \); on the other hand, an orifice of 4.848 lines for \( \frac{1}{4} \) inches pressure, \( \mu = 0.682 \), and for 29 inches, \( \mu = 0.653 \).

**Remark 2.** According to the author's experiments, the co-efficients for efflux under water are about \( \frac{1}{2} \) per cent. less than for efflux in air.

§ 316. Rectangular Lateral Orifices.—The most accurate experiments on efflux through large rectangular lateral apertures are those made at Metz by Poncelet and Lesbros. The widths of these orifices were two decimetres, (nearly 8 inches); the depths, however, varied from one centimetre to two decimetres. In order to produce perfect contraction, a brass plate of four millimetres, = 0.1575 inches, thickness was used for these orifices. From the results of their experiments, these experimenters have calculated by interpolation the tables at the end of this paragraph for the co-efficients which may be used for the measurement or calculation of the discharge.

If \( b \) be the breadth of the orifice, and if \( h_1 \) and \( h_2 \) are the heads of water above the lowest, and above the uppermost horizontal edge of the orifice, we then have, from § 308, the discharge: \( Q = \frac{2}{3} b \sqrt{2gh} (h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}) \). But if we substitute the height of the aperture \( a \), and the mean head of water \( h = \frac{h_1 + h_2}{2} \), we then have approximately \( Q = (1 - \frac{a^2}{96h^2}) ab \sqrt{2gh} \), and hence the effective discharge \( Q_1 = \mu Q = \left(1 - \frac{a^2}{96h^2}\right) \mu ab \sqrt{2gh} \). If, further, we put \( \left(1 - \frac{a^2}{96h^2}\right) \mu = \mu_1 \), we have then simply \( Q_1 = \mu_1 ab \sqrt{2gh} \), and in order to allow of our calculating by this simple or general formula of efflux, not only the values of \( \mu \), but also those of \( \mu_1 \) are given in the following tables.

Since the water in the vicinity of the orifice is in motion, it stands lower directly before the aperture than at a greater distance from the plate in which the aperture is made; on this account two tables have been compiled, the one for heads of water measured at a greater distance from the orifice, and the other for those measured immediately at the side in which the orifice lies. It may be seen, moreover, from both tables, although with certain variations, that the co-efficients of efflux increase the lower the orifice is, and the less the head of water.
If the orifices have different breadths, we are compelled, so long as we have no further experiments, still to use the co-efficients of these tables in like manner for the calculation of the discharge. If, further, the orifices are not rectangular, we must determine their mean breadth and mean depth, and introduce into the calculation the co-efficients corresponding to these dimensions. Lastly, it is always preferable to measure the head of water at a certain distance from the side in which the orifice lies, and to use the first table, than directly at the orifice where the surface of water is curved and less tranquil, than a little above it.

Example.

1.—What quantity of water flows through a rectangular aperture, 2 decimetres broad and 1 decimetre deep, if the surface of water is 1 ½ metre above the upper edge? Here \( b = 0.2; a = 0.1; h = \frac{h_1 + h_2}{2} = 1.6 + 1.5 = 1.55 \); hence the theoretical discharge \( Q = 0.1 \times 0.2 \sqrt{2g} \times \sqrt{1.55} = 0.02 \times 4.429 \times 1.245 = 0.1103 \) cubic metre. But now Table I. gives for \( a = 0.1 \) and \( h = 1.5 \), \( \mu_1 = 0.611 \), hence the effective discharge \( Q = 0.611 \times 0.1103 = 0.0674 \) cubic metre. 2. What discharge corresponds to a rectangular orifice in a thin plate of 8 inches breadth, 2 inches depth, with a 15 inches head of water above the upper edge?* The theoretical discharge is \( Q = \frac{1}{2} \times 7.906 \sqrt{\frac{1}{2}} = 0.8784 \times 1.1547 = 1.014 \) cubic feet. But now 2 inches is about 0.05 metre, and 15 inches about 0.4 metre; hence, according to the table \( a = 0.05 \) and \( h = 0.4 \), the corresponding co-efficient \( \mu_1 = 0.628 \) is to be taken, and the quantity of water sought is \( Q = 0.028 \times 1.014 = 0.0287 \) cubic feet. 3. If the breadth = 0.25, the depth = 0.15, and the head of water \( h = 0.045 \) metre, then \( Q = 0.25 \times 0.15 \times 4.429 \times \sqrt{0.12} = 0.166 \times 0.3464 = 0.0575 \) cubic metre. To the height 0.15 corresponds for \( h = 0.04 \), the mean value: \( \mu_1 = 0.582 \times 0.0603 = 0.5925 \), and \( h = 0.05, \mu_1 = 0.582 \times 0.0605 = 0.595 \); but since \( h \) is given = 0.045, we must then substitute the new mean \( h = 0.594 \times 0.0575 = 0.03415 \) cubic metre.

* In using the following tables, the English measures will be furnished with the proper co-efficients by employing the first, or left-hand column, in which to find the height \( h \), and the column under the number of inches answering to the height of orifice \( a \).—A.M. E.B.
TABLE I.

The co-efficients for the efflux through rectangular orifices in a thin vertical plate, from Poncelet and Lesbros. The heads of water are measured at a certain distance back from the orifice, or at a point where the water may be considered as still.

<table>
<thead>
<tr>
<th>Head of water, or distance of the surface of water from the upper side of the orifice.</th>
<th>WEIGHT OF ORIFICE.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eng. in.</td>
<td>Metres.</td>
</tr>
<tr>
<td>0,00</td>
<td>0,000</td>
</tr>
<tr>
<td>0,19</td>
<td>0,003</td>
</tr>
<tr>
<td>0,39</td>
<td>0,010</td>
</tr>
<tr>
<td>0,57</td>
<td>0,015</td>
</tr>
<tr>
<td>0,78</td>
<td>0,020</td>
</tr>
<tr>
<td>1,18</td>
<td>0,030</td>
</tr>
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<tr>
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<tr>
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<tr>
<td>4,72</td>
<td>0,120</td>
</tr>
<tr>
<td>5,51</td>
<td>0,140</td>
</tr>
<tr>
<td>6,29</td>
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</tr>
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<td>7,08</td>
<td>0,180</td>
</tr>
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</tr>
<tr>
<td>118,11</td>
<td>3,000</td>
</tr>
</tbody>
</table>
Co-efficients of efflux through rectangular orifices in a vertical plate, from Poncelet and Lesbros. The heads of water are measured directly at the orifice.

<table>
<thead>
<tr>
<th>Head of water, or distance of the surface of water from the upper side of the orifice.</th>
<th>( \text{Eng. in.} )</th>
<th>( \text{Metres.} )</th>
<th>( 0.20 \text{ in. or 8 in.} )</th>
<th>( 0.10 \text{ in. or 4 in.} )</th>
<th>( 0.05 \text{ in. or 2 in.} )</th>
<th>( 0.03 \text{ in. or 1.18 in.} )</th>
<th>( 0.02 \text{ in. or 0.8 in.} )</th>
<th>( 0.01 \text{ in. or 0.4 in.} )</th>
</tr>
</thead>
<tbody>
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<td>0.00</td>
<td>0.000</td>
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<td>0.651</td>
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</tr>
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</tr>
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<td>0.618</td>
<td>0.617</td>
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</tr>
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<td>0.615</td>
<td>0.612</td>
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</tr>
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</tr>
<tr>
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<td>0.607</td>
<td>0.606</td>
<td>0.608</td>
<td>0.609</td>
<td></td>
</tr>
</tbody>
</table>
§ 317. Wiers.—If water flows through wiers, or through notches in a thin plate, as, for example, $FB$, Fig. 422, the fluid vein then suffers a contraction on three sides, by which a diminution of the discharge is effected, since the quantity discharged from these orifices is $Q_i = \frac{3}{4} \mu b h \sqrt{2gh}$. But here the head of water $EH = h$, or the head of water above the sill, of the wier must not be measured immediately at the sill, but at least two feet before the plate in which the orifice lies, because the fluid surface before the opening suffers a depression, which becomes greater and greater the nearer it is to the orifice, and in the plane of the orifice amounts to a quantity $GR$ of from 0.1 to 0.25 the head of water $FR$, so that the thickness $FG$ of the stream in this plane is only 0.9 to 0.75 of the head of water. Experiments instituted by many philosophers on the flow of water through notches in thin plates, have afforded a multiplicity of results, but not always of the desired accordance. The following short table contains the results of the experiments of Poncelet and Lesbros on wiers of two decimetres, or about 8 inches breadth.

**TABLE OF THE CO-EFFICIENTS OF EFFLUX FOR WIERS OF 2 DECIMETRES,**

$= 7.87$ INCHES BREADTH, ACCORDING TO PONCELET AND LESBROS.

<table>
<thead>
<tr>
<th>Head of water $h$</th>
<th>0.01 m</th>
<th>0.02 m</th>
<th>0.03 m</th>
<th>0.04 m</th>
<th>0.05 m</th>
<th>0.06 m</th>
<th>0.08 m</th>
<th>0.10 m</th>
<th>0.15 m</th>
<th>0.20 m</th>
<th>0.22 m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Co-efficient of efflux $\mu_1$</td>
<td>0.424</td>
<td>0.417</td>
<td>0.412</td>
<td>0.407</td>
<td>0.401</td>
<td>0.397</td>
<td>0.395</td>
<td>0.393</td>
<td>0.390</td>
<td>0.385</td>
<td></td>
</tr>
</tbody>
</table>

From the average of determinations, we may here put $\mu_1 = 0.4$. Experiments on wiers of greater breadth gave Eytelwein the mean $\mu = \frac{3}{4} \mu = 0.42$, and Bidone $\mu_1 = \frac{3}{4}.0.62 = 0.41$, &c. The most extensive experiments are those of d'Aubuisson and Castel. From these, d'Aubuisson asserts that for wiers whose breadth is no more than the third part of the breadth of the canal or side in which the wier lies, the mean of $\mu$ is $= 0.60$, therefore, we may put $\frac{3}{4} \mu = 0.40$: but, on the other hand, for wiers which extend over the whole side, or have the same breadth as the water-course: $\mu = 0.665$, therefore, $\mu_1 = 0.444$; lastly, for other relations between the breadth of the wier and that of the canal, the co-efficients of efflux are very different, and lie between 0.58 and 0.66. Experiments made by the author, reduce the variability of these co-efficients to certain laws (§ 322).

(During his investigations in the summer of 1845, to determine the relative value of the several sources for supplying water to the city of Boston, the editor had an opportunity of making extensive series of
experiments on the passage of water through wiers of 1, 2, and 3,01 feet in breadth, and from 0,066 foot to 2,087 feet in depth above the bottom of the notch. The water was measured in a cubical box, 6 feet on a side, to which was attached, on the exterior, a glass gauge tube with a scale extending to the top of the receptacle. In like manner, a gauge tube was inserted in the dam which contained the notch, and several feet distant from it, with the 0 of its scale accurately adjusted to the level of the bottom of the notch. A scale sliding vertically was placed immediately over the centre of the wier by which the depth over the edge of the notch-board could be ascertained. The reservoir from which the water was drawn was at least 6 times as wide as the opening of the notch. The following are some of the co-efficients \( \mu_1 = \frac{2}{3} \mu \) for the several breadths of wier:

1.—Wier 3,01 feet in breadth, cut in 2 inch planks——

<table>
<thead>
<tr>
<th>Full depth ( h ) over bottom of notch.</th>
<th>Co-efficients of discharge ( \mu ) at the notch.</th>
<th>Depression of surface at the notch.</th>
</tr>
</thead>
<tbody>
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<td>0,075 feet</td>
<td>0,3667</td>
<td>0,021 feet</td>
</tr>
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<td>0,3794</td>
<td>0,040 &quot;</td>
</tr>
<tr>
<td>0,280 &quot;</td>
<td>0,3973</td>
<td>0,070 &quot;</td>
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<tr>
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<td>0,4294</td>
<td>0,155 &quot;</td>
</tr>
<tr>
<td>0,801 &quot;</td>
<td>0,4208</td>
<td>0,158 &quot;</td>
</tr>
<tr>
<td>1,023 &quot;</td>
<td>0,4129</td>
<td>0,167 &quot;</td>
</tr>
</tbody>
</table>

2. Wier 2 feet wide, in a 1 inch board——

<table>
<thead>
<tr>
<th>Full depth ( h ) over bottom of notch.</th>
<th>Co-efficients of discharge ( \mu ) at the notch.</th>
<th>Depression of surface at the notch.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,199 feet</td>
<td>0,4195</td>
<td>unc. &quot;</td>
</tr>
<tr>
<td>1,020 &quot;</td>
<td>0,4344</td>
<td>0,196 &quot;</td>
</tr>
<tr>
<td>1,062 &quot;</td>
<td>0,4408</td>
<td>0,206 &quot;</td>
</tr>
<tr>
<td>1,232 &quot;</td>
<td>0,4477</td>
<td>0,228 &quot;</td>
</tr>
<tr>
<td>1,280 &quot;</td>
<td>0,4460</td>
<td>0,230 &quot;</td>
</tr>
</tbody>
</table>

3. Wier 1 foot wide, in 1 inch board——

<table>
<thead>
<tr>
<th>Full depth ( h ) over bottom of notch.</th>
<th>Co-efficients of discharge ( \mu ) at the notch.</th>
<th>Depression of surface at the notch.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,329 feet</td>
<td>0,4144</td>
<td>unc. &quot;</td>
</tr>
<tr>
<td>0,333 &quot;</td>
<td>0,4166</td>
<td>unc. &quot;</td>
</tr>
<tr>
<td>0,339 &quot;</td>
<td>0,4191</td>
<td>unc. &quot;</td>
</tr>
<tr>
<td>0,352 &quot;</td>
<td>0,4265</td>
<td>0,068 &quot;</td>
</tr>
<tr>
<td>0,360 &quot;</td>
<td>0,4265</td>
<td>0,070 &quot;</td>
</tr>
<tr>
<td>2,060 &quot;</td>
<td>0,4149</td>
<td>0,118 &quot;</td>
</tr>
<tr>
<td>2,087 &quot;</td>
<td>0,4130</td>
<td>0,125 &quot;</td>
</tr>
</tbody>
</table>

The "full depth over the notch" here signifies, of course, that of the general level of the reservoir, above the edge of the wier. In each case it will be observed that the above co-efficients increase with the increase of depth up to a certain point, and then diminish gradually as far as the observations were extended. As these experiments were made with a view to determine an important practical and economical question, they were conducted with great care, and are believed
to be worthy of reliance as the basis of computation for works on an extended scale.]

Examples.—1. A weir, 0.25 metre broad and 0.15 head of water, gives per second the discharge \( Q = 0.393 \cdot b h \sqrt{2gh} = 0.393 \cdot 4.129 \cdot 0.25 \cdot (0.15)^{\frac{3}{2}} = 0.435, 0.0581 \) = 0.02527 cubic metres. 2. What breadth must be given to a weir which, with a head of water of 8 inches, will allow 6 cubic feet per second to pass through? It is

\[
b = \frac{Q}{\mu h} = \frac{6}{0.4 \cdot 8.02 \sqrt{\frac{3}{2}}} = 3,208 \cdot 0.5443 = 3,436 \text{ feet.}
\]

If, according to Eytelwein, we take \( \mu = 0.42 \), it follows that:

\[
b = 2.368 \cdot 0.5443 = 3.27 \text{ feet.}
\]

§ 318. Maximum and Minimum of Contraction.—In the flow of water through orifices in a plane side, the axis of the stream is perpendicular to the surface of the side, and therefore the amount of the contraction is a mean, but if the axis of the orifice or of the fluid stream forms an acute angle with the portion of the side containing the orifice, the contraction will be less; and if the angle between this axis, and the inner surfaces of the edges of the aperture, be obtuse, the contraction will be still greater. The one case is represented in Fig. 423, and the other in Fig. 424. Without doubt this difference of contraction is caused by the particles of water, which flow towards the orifice from the sides, deviating less from their direction in the one than in the other case, when they pass through the orifice and unite to form a fluid stream.

Fig. 423.  
Fig. 424.

The contraction is a minimum, i.e. nothing, if by the gradual convergence of the side which embraces the orifice, the lateral flow is entirely prevented, and a maximum if the side has a direction opposite to that of the fluid stream, so that certain particles of water must revolve 180° before arriving at the orifice. Both cases are represented

Fig. 425.  
Fig. 426.
in Figs. 425 and 426. In the first case, the co-efficient of efflux is about 1, viz. 0.96 to 0.97; and in the second from the measurements of Borda, Bidone and the author, a mean of 0.53. Changes in the co-efficients of efflux through convergent sides very often present themselves in practice; they occur in dams, which are inclined to the horizon, as in Fig. 427. Poncelet found for a similar opening the co-efficient of efflux $\mu = 0.80$, when the board was inclined 45°, and on the other hand, $\mu = 0.74$ only for an inclination of 63½°, that is, for a slope of $\frac{1}{5}$. For similar wiers, Fig. 428, where, as in the

Poncelet sluice-board, contraction takes place at one side only, the author found $\mu = 0.70$, therefore, $\mu_1 = \frac{3}{4} \mu = 0.467$ for an inclination of 45°, and $\mu = 0.67$, therefore, $\mu_1 = 0.447$ for an inclination of 63½°.

Example. If a sluice-board, inclined at an angle of 50°, which goes across a channel 2$\frac{1}{4}$ feet broad, is drawn up $\frac{1}{2}$ foot high, and the surface of water stands 4 feet above the bottom of the channel, the height of the aperture may be put $a = \frac{1}{2} \sin 50^\circ = 0.3830$ feet, the head of water $h = 4 - \frac{1}{2} = 3.8085$ feet, and the co-efficient of efflux $\mu = 0.78$; hence, the discharge $Q = \mu \frac{h^2}{2g} \frac{a}{2}$, therefore, $Q = 0.78 \cdot 2.25 \cdot 0.3830 \cdot 8.02 \sqrt{3.8085} = 10.49$ cubic feet (English).

§ 319. Partial Contraction.—We have only hitherto considered those cases where the water flows from all sides towards the aperture, and forms a contracted vein around, and we must now investigate others, where the water flows from one or more sides to the aperture, and therefore produces a stream only partially contracted. To distinguish the circumstances of contraction, we will call the case, where the vein is contracted on all sides, general; and the case, where it is only contracted in one part of its circumference, partial, or imperfect contraction. Partial contraction is induced when an orifice in a plane thin plate is confined by other plates in the direction of the fluid stream at one or more sides.

In Fig. 429, are represented four orifices of equal size $a, b, c, d$, in the bottom $ABC$ of a vessel. The contraction by efflux through the orifice $a$ in the middle of the bottom is general, because the water can flow to it from all sides; the contraction from the efflux through $b, c, d$, is partial, because the water can only flow to them from one, two, or three sides. Likewise, if a rectangular lateral aperture goes to the bottom of the vessel, the contraction is then partial, because it falls away at the bottom side, if, further, the aperture of the
dam reaches the bottom in the lateral walls of the channel, there is then only a contraction on one side.

Partial contraction is remarkable in two respects; first, by giving an oblique direction to the stream; and, secondly, by increasing the quantity of discharge.

If the lateral aperture $F$, Fig. 430, reaches a second side $CD$, so that no contraction takes place there, the axis $FK$ of the fluid stream becomes deflected by an angle $KFG$ of about $9^\circ$ from the normal $FG$ to the plane of the orifice. The obliquity of the stream is much greater if two adjacent sides of the orifice have projecting borders. If the orifice has borders in two oppositely situated sides, and contraction at these prevented, such a deviation of course will not take place, but at the other side, the vein at some distance from the orifice, will spread out more than if the border were not there. Although a greater discharge is obtained by a partial contraction, we must, as a rule, endeavor to avoid this, because the fluid stream, in consequence, suffers a deviation in its direction and a greater extension in its breadth.

Experiments on the efflux of water with partial contraction have been made by Bidone and by the author. They allow us to assume that the co-efficients of efflux increase simultaneously with the ratio of the contracted part to the whole perimeter, though it is easy to perceive that this relation is different, if the perimeter is almost or entirely restricted, and the contraction almost or entirely suppressed. Let us put the ratio of this restriction to the entire perimeter $= n$, and let us represent by $x$, any number deduced from experiment, we may then, although only approximately, put the ratio of the corresponding co-efficient of efflux $\mu_n$ of partial contraction to the co-efficient of efflux of perfect contraction:

$$\frac{\mu_n}{\mu_0} = 1 + x \cdot n,$$

and consequently $\mu_n = (1 + x \cdot n) \cdot \mu_0$.

Bidone's experiments give for circular orifices $x = 0,128$, and for rectangular $x = 0,152$; the author's, however, give for the last, $x = 0,134$. Rectangular orifices with borders, are those which are most frequently met with in practice; we will assume for them the mean value $x = 0,143$, and hence put $\mu_n = (1 + 0,143 \cdot n) \cdot \mu_0$.

For a rectangular lateral orifice of the depth $a$ and breadth $b$, $n = \frac{b}{2 (a + b)}$, if the contraction on one side $b$ is suppressed; if, for instance, this side lies in the plane of the bottom; again, $n = \frac{a}{2}$, if a side $a$ and a side $b$ are bordered, and $n = \frac{2 a + b}{2 (a + b)}$, if on one side $b$, and both sides $a$, the contraction is prevented; if, for example, the
orifice takes up the whole breadth of the reservoir, and reaches the plane of the bottom.

**Example.** What quantity of water flows through a 3 feet broad and 10 inch deep vertical aperture of a dam at a pressure of 1½ feet above the upper side of the aperture, if the lower one coincides with the bottom of the channel, and hence there is no contraction at the bottom? The theoretical discharge is:

\[ Q = \frac{1}{9} \cdot 3 \cdot 8.02 \sqrt{1.5 + \frac{8.02}{1.9166}} = 8.02 \sqrt{1.9166} \approx 28.11 \text{ cubic feet}. \]

According to Poncelet's table for general contraction \( \mu = 0.604 \), we have, therefore,

\[ n = \frac{2(3 + \frac{1}{12})}{9} = \frac{18 + 5}{9} = \frac{23}{9}; \]

hence, for the above cases of partial contraction \( \mu_n \approx 1 + 0.143 \cdot \frac{28.11}{28.11} = 1.056 \cdot 0.604 = 0.638 \), and the effective discharge is

\[ Q_e = 0.638 \cdot Q = 0.638 \cdot 28.11 \approx 18.14 \text{ cubic feet}. \]

§ 320. **Imperfect Contraction.**—The contraction of the fluid vein depends, further, upon whether the water before the orifice is tolerably still, or whether it arrives before it with a certain velocity. The quicker the water flows to the orifice, the less contracted does the vein become, and the greater is the discharge. The relations of contraction and efflux above given and investigated, have reference only to the case where the orifice lies in a large side, and it can only be assumed that the water flows to it with a small velocity; hence we must know the relations of contraction and efflux, when the transverse section of the orifice is not much less than that of the affluent water, and when, consequently, the water arrives at the orifice with a considerable velocity. In order to distinguish these two cases from one another, we shall call the contraction, where the superincumbent water is still, **perfect**; and that where it is in motion, **imperfect contraction**. The contraction, for example, is imperfect in the efflux from a vessel \( \triangle ABC \), Fig. 431, because the transverse section \( F \) of the orifice is not much smaller than that of \( G \) of the arriving water, or the area of the side \( CD \), in which this orifice lies. If, on the other hand, the vessel had the form \( \triangle D_1 \), and, therefore, the area of the bottom surface \( C \) of the orifice, the efflux would then go on with perfect contraction. The imperfectly contracted vein is besides distinguishable, not merely by its greater thickness from the perfectly contracted fluid vein, but also by its not having so transparent and crystalline an appearance.

If the ratio of the area of the orifice \( F \), and the side containing the orifice \( G \), therefore, \( \frac{F}{G} = n \), the co-efficient of efflux for perfect contraction = \( \mu_p \), and that for imperfect = \( \mu_n \), we may with greater accuracy, according to the experiments and calculations made by the author, put:

1. For circular orifices:

\[ \mu_n = \mu_p [1 + 0.04564 (14.821 n - 1)], \]

and

\[ 32^\circ \]
CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX.

2. For rectangular orifices:
\[ \mu' = \mu_0 \left[ 1 + 0.0760 \left( 9^n - 1 \right) \right]. \]

To render the calculation easier in cases of application, the corrections \( \frac{\mu_n - \mu_0}{\mu_0} \) of the co-efficient of efflux on account of imperfect contraction are compiled in the following short tables.

**TABLE I.**

**CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR CIRCULAR ORIFICES.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>0.35</th>
<th>0.40</th>
<th>0.45</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu_n - \mu_0}{\mu_0} )</td>
<td>0.007</td>
<td>0.014</td>
<td>0.023</td>
<td>0.034</td>
<td>0.045</td>
<td>0.059</td>
<td>0.075</td>
<td>0.092</td>
<td>0.112</td>
<td>0.134</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>0.55</th>
<th>0.60</th>
<th>0.65</th>
<th>0.70</th>
<th>0.75</th>
<th>0.80</th>
<th>0.85</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu_n - \mu_0}{\mu_0} )</td>
<td>0.161</td>
<td>0.189</td>
<td>0.223</td>
<td>0.260</td>
<td>0.303</td>
<td>0.351</td>
<td>0.408</td>
<td>0.471</td>
<td>0.546</td>
<td>0.613</td>
</tr>
</tbody>
</table>

**TABLE II.**

**CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR RECTANGULAR ORIFICES.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>0.35</th>
<th>0.40</th>
<th>0.45</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu_n - \mu_0}{\mu_0} )</td>
<td>0.009</td>
<td>0.019</td>
<td>0.030</td>
<td>0.042</td>
<td>0.056</td>
<td>0.071</td>
<td>0.088</td>
<td>0.107</td>
<td>0.128</td>
<td>0.152</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>0.55</th>
<th>0.60</th>
<th>0.65</th>
<th>0.70</th>
<th>0.75</th>
<th>0.80</th>
<th>0.85</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu_n - \mu_0}{\mu_0} )</td>
<td>0.178</td>
<td>0.208</td>
<td>0.241</td>
<td>0.278</td>
<td>0.319</td>
<td>0.365</td>
<td>0.416</td>
<td>0.473</td>
<td>0.537</td>
<td>0.608</td>
</tr>
</tbody>
</table>

The different values of the ratio of the transverse sections \( n = \frac{F}{G} \) stands above in these tables, and immediately below additions to the

co-efficients of efflux, on account of imperfect contraction; for example, for the ratio of the transverse sections \( n = 0.35 \), i.e., for the case where the area of an orifice is 35 hundredths of the area of the whole side of the orifice, we have for circular orifices \( \frac{\mu_n}{\mu_0} = 0.075 \), and for rectangular orifices \( = 0.088 \); therefore, the co-efficient of efflux for perfect contraction in the first case is to be made about 75 thousandths, and in the second about 88 thousandths greater to obtain the corresponding co-efficients of efflux for imperfect contraction. Were the co-efficient of efflux \( \mu_0 = 0.615 \), we should have in the first case \( \mu_{0.35} = 1.075 \cdot 0.615 = 0.661 \), and in the second, \( \mu_{0.35} = 1.088 \cdot 0.615 = 0.669 \).

Example.—What discharge does a rectangular lateral aperture \( F \), \( 1\frac{1}{4} \) feet broad and \( \frac{1}{4} \) foot deep, give if it be cut in a rectangular wall \( CD \), Fig. 432, 2 feet broad and 1 foot deep, and the head of water \( EH = 2 \) feet? The theoretical discharge is \( Q = 1.25 \cdot 0.5 \cdot 8.02 \sqrt{2} = 5.012 \cdot 1.414 = 7.086 \) cubic feet, and the co-efficient of efflux for perfect contraction is, according to Poncelet, \( \mu_0 = 0.610 \); but now the ratio of the transverse sections \( n = \frac{2}{1.25} = 0.8 = 0.312 \), and for \( n = 0.312 \), from Table II.,
\[
\frac{\mu_n - \mu_0}{\mu_0} = 0.071 + \frac{4}{15} (0.088 - 0.071) = 0.071 + 0.004 = 0.075; \text{ hence it follows, }
\]
that the co-efficient of efflux for the present case is \( \mu_{0.312} = 1.075 \cdot 0.610 = 0.6557 \), and the discharge \( Q = 0.6557 \cdot 8.02 = 5.081 \) cubic feet.

\[ \text{§ 321. Efflux of Water in Motion.—We have hitherto assumed} \]
that the head of water has been measured in still water; we must now, therefore, consider the case when only the head of water in motion, and flowing with a certain velocity towards the orifice, can be measured. Let us suppose the case of a rectangular lateral orifice, and represent its breadth by \( b \), and the heads of water with respect to both horizontal edges \( h_1 \) and \( h_2 \), the height due to the velocity \( c \) of the affluent water by \( k \), we shall then have the theoretical discharge:
\[
Q = \frac{3}{2} b \sqrt{2g} \left[ h_1 + k \right]^{\frac{3}{2}} - \left( h_2 + k \right)^{\frac{3}{2}}.
\]
This formula is not directly applicable to the determination of the discharge, because the height due to the velocity:
\[
k = \frac{c^2}{2g} = \frac{1}{2g} \left( \frac{Q}{G} \right)^{\frac{1}{2}}
\]
is again dependent on \( Q \), and further transformation leads to a complicated equation of a higher order, hence it is far simpler to put the effective discharge \( Q_1 = \mu_1 ab \sqrt{2gh} \), and understand by \( \mu_1 \), not the mere co-efficient of efflux, but one especially dependent on the ratios of the transverse sections. Most frequently, this case presents itself when the object is to measure water flowing...
in canals and courses, because it is seldom possible in this case to dam up the water so high by a transverse section \( BC \), Fig. 433, containing the orifice of discharge, that the orifice \( EF \) becomes only a small part, compared with the transverse section of the stream of water flowing to it; and, hence, the velocity of the last very small compared with the mean velocity.

From experiments made by the author on this subject with Poncelet orifices, where the head of water is measured one metre above the plane of the orifice, the expression \( n \),

\[
\frac{\mu_{n} \mu_{0}}{\mu_{0}} = 0.641 \left( \frac{F}{G} \right)^{2} = 0.641 \cdot n^{2},
\]

may be taken as tolerably accurate, when \( n = \frac{F}{G} \) is the ratio of the transverse section, which, however, should not much exceed \( \frac{1}{2} \); further, \( \mu_{0} \) represents the co-efficient for general contraction, taken from Poncelet's table corresponding to the present case. If \( b \) be the breadth, \( a \) the depth of the orifice, \( B \) the breadth and \( A \) the depth of the fluid stream, and \( h \) the depth of the upper side of the orifice below the surface of water, we have, accordingly, the effective discharge:

\[
Q = \left( 1 + 0.641 \left( \frac{ab}{AB} \right)^{2} \right) \mu_{0a} \cdot ab \sqrt{2g \left( h + \frac{a}{2} \right)}.
\]

The following table serves for shortening the calculation in cases of application.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>0.35</th>
<th>0.40</th>
<th>0.45</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu_{n} \mu_{0}}{\mu_{0}} )</td>
<td>0.002</td>
<td>0.006</td>
<td>0.014</td>
<td>0.026</td>
<td>0.040</td>
<td>0.058</td>
<td>0.079</td>
<td>0.103</td>
<td>0.130</td>
<td>0.160</td>
</tr>
</tbody>
</table>

Example. To find the quantity of water conducted through a course 3 feet broad, a board is placed across, with a 2 feet wide and 1 foot deep rectangular orifice, and the water in this way is so dammed up that it at last attains a height of 2\( \frac{1}{2} \) feet above the bottom, and \( \frac{1}{2} \) above the lower edge of the orifice. The theoretical discharge is \( Q = ab \sqrt{2gh} = 1 \cdot 2 \cdot 8.02 \cdot \sqrt{1.25} = 16.04 \cdot 1.118 = 17.93 \) cubic feet; the co-efficient of efflux for perfect contraction may be put \( 0.689 \), and the ratio of the transverse sections \( a = \frac{F}{G} = \frac{ab}{AB} = 1 \cdot \frac{1}{2} = 0.296 \); hence, it follows, that the co-efficient of efflux for the present ratio of discharge:

\[
(1 + 0.641 \cdot 0.296) \mu_{0a} = 1.056 \cdot 0.602 = 0.6357,
\]

and the effective quantity discharged

\[
= 17.93 \cdot 0.6357 = 11.31 \) cubic feet.

§ 322. Imperfect contraction very often occurs in the efflux through wiers, as in Fig. 422. Wiers may take up a part only of the breadth of the reservoir or canal, or the whole breadth. In the latter case, contraction at the sides of the aperture does not take place, and for this reason more water flows through them than through wiers of the first kind. The author has made experiments also on these circum-
stances of efflux, and deduced from the results formulas by which the corresponding co-efficients may be estimated with tolerable certainty with the assistance of the ratio of the sections \( n = \frac{G}{V} = \frac{h}{A} \). If we retain the denominations of the former paragraph, we then have for the Poncelet wiers:

\[
\frac{\mu_n - \mu_0}{\mu_0} = 1,718 \left(\frac{F}{G}\right)^4 = 1,718 e^{n^4},
\]

and for wiers occupying the entire breadth of the canal:

\[
\frac{\mu_n - \mu_0}{\mu_0} = 0,041 + 0,3693 n^2,
\]

hence, in the first case, the discharge is:

\[
Q_1 = \frac{3}{2} \left[ 1 + 1,718 \left(\frac{h}{A}B\right)^4 \right] \mu_0 \cdot b \sqrt{2gh^3}.
\]

And in the second:

\[
Q_1 = \frac{3}{2} \left[ 1,041 + 0,3693 \left(\frac{h}{A}B\right)^3 \right] \mu_0 \cdot b \sqrt{2gh^3},
\]

where \( h \) represents the head of water \( EH \) above its sill \( F \), measured at about 3 feet 6 inches back from the wier.

In the following tables, the corrections \( \frac{\mu_n - \mu_0}{\mu_0} \), for the most simple values of \( n \) are put down.

### TABLE I.

**CORRECTIONS FOR THE PONCELET WIERS.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>0,05</th>
<th>0,10</th>
<th>0,15</th>
<th>0,20</th>
<th>0,25</th>
<th>0,30</th>
<th>0,35</th>
<th>0,40</th>
<th>0,45</th>
<th>0,50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu_n - \mu_0}{\mu_0} )</td>
<td>0,000</td>
<td>0,000</td>
<td>0,001</td>
<td>0,003</td>
<td>0,007</td>
<td>0,014</td>
<td>0,026</td>
<td>0,044</td>
<td>0,070</td>
<td>0,107</td>
</tr>
</tbody>
</table>

### TABLE II.

**CORRECTIONS FOR WIERS OVER THE ENTIRE SIDE, OR WITHOUT ANY LATERAL CONTRACTION.**

<table>
<thead>
<tr>
<th>( n )</th>
<th>0,00</th>
<th>0,05</th>
<th>0,10</th>
<th>0,15</th>
<th>0,20</th>
<th>0,25</th>
<th>0,30</th>
<th>0,35</th>
<th>0,40</th>
<th>0,45</th>
<th>0,50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\mu_n - \mu_0}{\mu_0} )</td>
<td>0,041</td>
<td>0,042</td>
<td>0,045</td>
<td>0,049</td>
<td>0,0560</td>
<td>0,064</td>
<td>0,074</td>
<td>0,086</td>
<td>0,100</td>
<td>0,116</td>
<td>0,133</td>
</tr>
</tbody>
</table>

**Example.** To determine the quantity of water carried off by a canal 5 feet broad, a waste board is applied, with an outward sloping edge, over which the water is allowed to flow after it has ceased to rise; the head of water above the bottom of the canal is 3½ feet, and above the edge 1½ feet, hence the theoretical discharge is \( Q = \frac{3}{2} \cdot 5 \cdot 8,02 \).
\[ \left( \frac{3}{2} \right)^{\frac{3}{2}} = 48.12 \text{ cubic foot}. \]

The coefficient of efflux is, since \( \frac{h}{d} = \frac{1.5}{3.5} = \frac{3}{7} \) and \( \mu_0 = 0.577 \),

\[ \mu_\theta = [1.041 + 0.3693 \cdot (\frac{3}{7})^3]. \]

\[ 0.577 = 1.11. \]

\[ 0.577 = 0.64 \text{, hence the effective discharge} \]

\[ Q_r = 0.64 \cdot Q = 0.64 \times 48.12 = 30.79 \text{ cubic feet}. \]

---

### Chapter III.

**ON THE EFFLUX OF WATER THROUGH TUBES.**

§ 323. **Short Tubes, or Mouth-pieces.**—If water is allowed to flow through short tubes, or mouth-pieces, other relations take place than when it flows through orifices in a thin plate, or through outwardly sloping orifices in a thick plate. When the tube is prismatic, and its length \( 2 \frac{1}{2} \) to \( 3 \) times that of its width, it then gives an uncontracted and opaque stream, which has a small distance of projection, and hence, also, a smaller velocity than that of a jet flowing, under otherwise similar circumstances, through an orifice in a thick plate. If, therefore, the tube KL has the same transverse section as the orifice F, Fig. 434; and if also the head of water of both is one and the same, we then obtain in LR a troubled and uncontracted, and, therefore, a thicker jet, and in FH a clear and contracted, and, therefore, thinner one; and, it may be observed, that the distance of the projection ER is less than that of DH. This ratio of efflux only takes place when the tube is of a given length; if it is shorter, or scarcely as long as it is broad, then the jet KL, Fig. 435, will not touch the sides of the tube, the tube will have no influence on the efflux, and the jet will be the same as through orifices in a thin plate.

Sometimes in tubes of greater length, the fluid stream does not entirely fill the tube, namely: when the water is not allowed to come into contact with the sides of the tube, but if in this case we close the outer orifice by the hand or by a board for a few moments, a stream will then be formed which will entirely fill the tube, and the so-called full flow will then take place. Contraction of the fluid vein takes place also in the flow through tubes, but the place of contrac-
tion is here in the interior of the tube. We may be convinced of this, if we avail ourselves of glass tubes, such as $KL$, Fig. 436, and color the water, for in this case we shall remark, that there is progressive motion only in the middle of the transverse section $G$ close behind the place of entrance $K$, but not at the outside of it, and that it is a sort of eddying motion which takes place. But it is the capillarity, or the adhesion of the water to the sides of the tube, which causes the fluid entirely to fill the end $FL$ of the tube. The water flowing from the tube has only a pressure equal to that of the atmosphere, but the contracted section $G$ is only $a$ times the size of the section $F$ of the tube, and for this reason the velocity in it is \( \frac{1}{a} \) times as great as the velocity of efflux $v$; hence the pressure of the water in the vicinity of $G$ is

\[
\left( \frac{1}{a} v \right)^2 \frac{v^2}{2g} = \left[ \left( \frac{1}{a} \right)^2 - 1 \right] \frac{v^2}{2g} \quad (\S \, 307)
\]

less than at its exit, or than the atmospheric pressure. If we bore a narrow hole in the tube at $G$, no discharge will pass through it, but there will be an absorption of air rather; the full discharge and the action of the tube will at last entirely cease if the hole be made wider, or more holes bored.

\( \S \, 323. \) Cylindrical Tubes.—Numerous experiments have been made on the flow of water through cylindrical additional tubes; but the results vary considerably from one another. The co-efficients of Bossut are those, which from their smallness ($0.785$) have been found to vary most from others. From the experiments of Michelotti, with tubes from $\frac{1}{2}$ to 3 inches width, and with a head of water of from 3 to 20 feet, the mean of this co-efficient is $\mu = 0.813$. The experiments of Bidone, Eytelwein and d’Aubuisson vary very little from this. The mean, however, which may be adopted, and which corresponds particularly with the author’s experiments on the discharge through short mouth-pieces $= 0.815$. As we have found this for orifices in a thin plate $0.615$, it follows that, under otherwise similar circumstances and relations, $\frac{815}{615} = 1.325$ times as much water flows through cylindrical additional tubes, as through round orifices in a thin plate. These co-efficients, moreover, increase as the width of tubes becomes less, and but slightly with the increase of the head of water or velocity of efflux. According to the author’s experiments, under a pressure of from 9 to 24 inches for tubes three times as long as broad:

\* A considerable series of experiments on the flow of water through additional was, some years since, performed by a committee of the Franklin Institute, which yet await a proper reduction to render the results available.—Am. Ed.
According to this table, therefore, the co-efficients increase considerably as the width of the tubes decreases. Buff found for tubes 2,79 lines wide, and 4,3 lines long, the co-efficients of efflux gradually to increase from 0,826 to 0,855, when the head of water sank from 33 to 1½ inches successively.

The author found a co-efficient of efflux of 0,819 for the flow of water through rectangular additional tubes.

If the additional tubes $KL$, Fig. 437, are on the inside partially confined; if, for instance, one side is contiguous to the bottom, and if a partial contraction is produced thereby, then the co-efficient of efflux, from the author's experiments, does not perceptibly increase, but the water flows away at different parts of the section, with different velocities, and, of course, from the side $BC$ faster than from the side opposite to it. If the inner anterior surface of a tube does not coincide with the side surface, but projects, as $a$, $b$, $c$, Fig. 438, then this tube is called an internal additional tube. If the anterior surface of this tube is at least $\frac{1}{3}$ as broad as the tube is wide, as for example $a$, then the co-efficient of efflux will remain the same as if this surface were in the plane of the side, but if the anterior surface be less, as $b$, $c$, the co-efficient will then be less. For a very small and almost vanishing anterior surface, according to the experiments of Bidone and the author, this amounts to 0,71 if the vein fills the tube, and 0,53 (compare § 318) if it does not quite fill the inner sides of the tube.

In the first case ($b$) the fluid stream is broken and divergent, like a brush; and in the second ($c$) strongly contracted and quite crystalline.

§ 324. Co-efficient of Resistance.—As water flows without contraction from prismatic additional tubes, it follows that, in its efflux through these mouth-pieces, the co-efficient of contraction = unity, and the co-efficient of velocity $\phi = \text{the co-efficient of efflux } \mu$. A discharge $Q$ with the velocity $v$, possesses a vis viva $\frac{Qv^2}{2}$, and is ca-
pable of producing the mechanical effect $\frac{v^2}{2g} Q \gamma$ (§ 71). But now the theoretical velocity of efflux $= \frac{v}{\sqrt{\phi}}$, hence the mechanical effect $\frac{1}{\sqrt{\phi}} \cdot \frac{v^3}{2g}$. $Q \gamma$ corresponds to the mass of water flowing out, and the discharge $Q$ accordingly loses by efflux the mechanical effect

$$\left(\frac{1}{\sqrt{\phi}} \cdot \frac{v^3}{2g} - 2g\right) Q \gamma = \left(\frac{1}{\sqrt{\phi}} - 1\right) \frac{v^3}{2g} Q \gamma.$$

For efflux through orifices in a thin plate, the mean of $\phi = 0.97$, hence the loss of effect here amounts to

$$\left[\left(\frac{1}{0.97}\right)^2 - 1\right] \frac{v^3}{2g} Q \gamma = 0.063 \frac{v^3}{2g} Q \gamma;$$

for efflux through short cylindrical tubes $\phi = 0.815$, and the corresponding loss of effect

$$\left[\left(\frac{1}{0.815}\right)^2 - 1\right] \frac{v^3}{2g} Q \gamma = 0.505 \frac{v^3}{2g} Q \gamma,$$

i.e. eight times as great as for efflux through orifices in a thin plate. In rendering available the vis viva of flowing water, it is consequently better to let the fluid flow through orifices in a thin plate, than through prismatic tubes. But if the inner edges in which the tube meets the sides of the cistern are rounded, and by this a gradual passage from the cisterns into the tube effected, the co-efficient of efflux will then rise to 0.96, and the loss of mechanical effect will be brought down to 8½ per cent. In shorter adjutages, accurately rounded, having the form of the contracted fluid vein $\mu = \phi = 0.97$, and hence the loss of mechanical effect as for orifices in a thin plate $= 6$ per cent.

A head of water $\left(\frac{1}{\sqrt{\phi}} - 1\right) \frac{v^3}{2g} Q \gamma$ is due to the loss of mechanical effect $\left(\frac{1}{\sqrt{\phi}} - 1\right) \frac{v^3}{2g}$; hence we may also suppose that from the obstacles to the efflux, the head of water suffers the loss $\left(\frac{1}{\sqrt{\phi}} - 1\right) \frac{v^3}{2g}$, and assume after deduction of this loss, that the residuary part of the head of water is expended in generating the velocity. We may call this loss $\left(\frac{1}{\sqrt{\phi}} - 1\right) \frac{v^3}{2g}$, which increases with the square of the velocity of efflux, the height due to the resistance, and the co-efficient $\frac{1}{\sqrt{\phi}} - 1$, with which the height due to the velocity is to be multiplied to obtain the height due to the resistance, the co-efficient of resistance. We shall represent in what follows, the co-efficient, expressing the ratio of the height of resistance to the head of water, by the letter $\xi$; therefore, the height due to the resistance itself may be expressed by $\xi \cdot \frac{v^3}{2g}$. By the formula $\xi = \frac{1}{\sqrt{\phi}} - 1$ and $\phi = \frac{1}{\sqrt{1+\xi}}$. 
the co-efficient of resistance may be calculated from the co-efficient of velocity, and vice versa.

**Example.**—1. What discharge will flow through a 2 inch wide tube, under a head of water of 3 feet, which corresponds to a co-efficient of resistance \( \zeta = 0.4 \)?

\[
\phi = \frac{1}{\sqrt{1.4}} = 0.845; \quad \text{hence, } v = 0.845 \cdot 8.02 \sqrt{3} = 12.05 \text{ feet; further, } F = \left( \frac{1}{1.4} \right)^{10}.
\]

\( w = 0.02182 \) square feet; consequently, the quantity of water sought is \( Q = 0.263 \) cubic feet.—2. If a tube of 2 inches width, under a pressure of 2 feet, deliver in a minute 10 cubic feet of water, its co-efficient of efflux, or of velocity, is then \( v = \frac{Q}{F \sqrt{2gh}} \).

\[
= \frac{10}{60 \cdot 0.02182 \cdot 8.02 \sqrt{2}} = \frac{1}{1.05 \sqrt{2}} = 0.674, \text{ the co-efficient of resistance } = \frac{1}{\left( \frac{0.674}{1} \right)^{10} - 1} = 1.201; \text{ and lastly, the loss in head of water produced by the resistances of the tube:}
\]

\[
= \frac{1.201}{2g} \cdot \frac{1.201 \cdot 0.0155 \left( \frac{Q}{F} \right)^{10}}{0.130} = 1.085 \text{ feet.}
\]

§ 325. Oblique Additional Tubes.—Obliquely attached or obliquely cut tubes give a smaller quantity of water than rectangularly attached, or rectangularly cut additional tubes, because the direction of the water in them becomes changed. Experiments conducted upon an extensive scale, have led the author to the following. If \( \theta \) be the angle which the axis of the tube \( KL \), Fig. 439, makes with the normal \( KN \) to the plane \( AB \) of the inner orifice of the tube; and if \( \zeta \) be the co-efficient of resistance for rectangularly cut tubes, we shall then have the co-efficient of resistance for the inclined tube:

\[
\zeta' = \zeta + 0.303 \sin \theta + 0.226 \sin \theta^2.
\]

Let us take for \( \zeta \) the mean value 0.505, and we shall obtain:

<table>
<thead>
<tr>
<th>for ( \theta^0 = )</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>The co-efficient of resistance ( \zeta ) =</td>
<td>0.505</td>
<td>0.565</td>
<td>0.635</td>
<td>0.713</td>
<td>0.794</td>
<td>0.870</td>
<td>0.937</td>
</tr>
<tr>
<td>The co-efficient of efflux ( \mu ) =</td>
<td>0.815</td>
<td>0.799</td>
<td>0.782</td>
<td>0.764</td>
<td>0.747</td>
<td>0.731</td>
<td>0.719</td>
</tr>
</tbody>
</table>

From this, for example, the co-efficient of resistance of an additional tube deviating by \( 20^\circ \) from the axis is \( \zeta = 0.635 \), and the co-efficient of efflux \( \frac{1}{1.635} = 0.782 \), and for \( 35^\circ \) deviation, the first = 0.753, and the last = 0.755.

In general, these inclined and additional tubes are larger than we have hitherto assumed, and they should be longer too, because the water would not otherwise perfectly fill the tube. The preceding formula represents only that part of the resistance which is due to
that portion of tube at the inner orifice, which is three times as long as the tube is wide. The resistance which the remaining portion of tube opposes to the motion of the water, will be given subsequently.

Example. If the plane of the inner orifice $AB$ of a horizontally lying pond sluice $KL$, Fig. 440, as likewise the interior surface of the pond dam, is inclined $40^\circ$ to the horizon, then the axis of the pipe makes, with the normal to this plane, an angle of $50^\circ$, and hence the co-efficient of resistance for efflux through the portion of the interior orifice of this tube is $0,870$; and if now the co-efficient of resistance $0,650$ were due to the remaining and longer portion, the co-efficient of resistance of the entire tube would then be $0,870 + 0,650 = 1,520$, and hence the co-efficient of efflux $= \frac{1}{\sqrt{1 + 1,520}} = \frac{1}{\sqrt{2,520}} = 0,630$. For a 10 feet head of water and 1 foot width of tube, the following discharge would be given:

$$Q = 0,63 \times \frac{\pi}{4} \times 8,02 \sqrt{10} = 12,55 \text{ cubic feet}. $$

§ 326. Imperfect Contraction.—When a cylindrical additional tube $KL$, Fig. 441, is inserted into a plane wall $AB$, whose area $G$ does not much exceed the transverse section $F$ of the tube, the water then comes to the place of insertion with a velocity which must not be disregarded, and it then issues into the tube with imperfect contraction only, on which account the velocity of efflux is again greater than when the water before entrance into the tube is to be assumed as still. Again if $\frac{F}{G} = n$ is the ratio of the section of the tube to that of the area of the side, and further, $\mu_o$ be the co-efficient of efflux for perfect contraction, where $\frac{F}{G}$ may be equated to 0, we shall have, according to the experiments of the author, to put the co-efficient of efflux for imperfect contraction, or for the ratio of the sections $n$:

$$\mu_n = \mu_0 \left(1 + 0,102 n + 0,067 n^2 + 0,046 n^3\right)$$

$\mu_n$ being put $= 0,815$, $\mu_0 = 0,815$. 1,019 $= 0,830$.

The following useful and convenient table gives somewhat more accurately the values for correction $\frac{\mu_n}{\mu_0}$.
CORRECTION FOR IMPERFECT CONTRACTION.

TABLES
OF CORRECTION FOR IMPERFECT CONTRACTION, BY EFFLUX THROUGH SHORT CYLINDRICAL TUBES.

<table>
<thead>
<tr>
<th>n</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>0.35</th>
<th>0.40</th>
<th>0.45</th>
<th>0.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_n - \mu_0)</td>
<td>0.006</td>
<td>0.013</td>
<td>0.020</td>
<td>0.027</td>
<td>0.035</td>
<td>0.043</td>
<td>0.052</td>
<td>0.060</td>
<td>0.070</td>
<td>0.080</td>
</tr>
<tr>
<td>(\mu_0)</td>
<td>0.050</td>
<td>0.060</td>
<td>0.070</td>
<td>0.080</td>
<td>0.090</td>
<td>0.100</td>
<td>0.110</td>
<td>0.120</td>
<td>0.130</td>
<td>0.140</td>
</tr>
</tbody>
</table>

By efflux through short parallelopipedical tubes, these corrections are nearly the same.

These co-efficients are especially applicable to the efflux of water through compound tubes, for example, in the case represented in Fig. 442, where the orifice of the short tube KL enters the wider short tube CK, whose orifice again lies in the cistern AC. Imperfect contraction takes place at the entrance of the water from the wider into the narrower tube, and hence the co-efficient of efflux must be determined by the last rule. If we put the co-efficient of resistance corresponding to the co-efficient of efflux = \(\zeta_i\), the co-efficient of resistance for the entrance into the wider tube from the cistern = \(\zeta\), the head of water = \(h\), the velocity of efflux = \(v\), and the ratio \(\frac{F}{G}\) of the section of the tubes = \(n\), therefore, the velocity of the water in the wider tube = \(nv\), then the formula gives:

\[
h = \frac{v^2}{2g} + \zeta \cdot \frac{(nv)^2}{2g} + \zeta_1 \cdot \frac{v^2}{2g}, \text{ i.e. } h = (1 + n^2 \zeta + \zeta_1) \frac{v^2}{2g}.
\]

And hence:

\[
v = \frac{\sqrt{2gh} e}{\sqrt{1 + n^2 \zeta + \zeta_1}}.
\]

Example. What discharge will the apparatus delineated in Fig. 442 deliver, if the head of water \(h = 4\) feet, the width of the narrower tube 2 inches, and that of the wider one 3 inches? \(n = (\frac{4}{2})^2 = \frac{4}{2}, \text{ hence } \mu_\frac{4}{2} = 1.009 \cdot 0.815 = 0.817, \text{ and the corresponding co-efficient of resistance } \zeta_i = (\frac{1}{0.817})^2 - 1 = 0.318; \text{ but now we have } \zeta = 0.505 \text{ and } n^2 \cdot \zeta = \frac{1}{8} \cdot 0.505 = 0.063; \text{ hence it follows, that } 1 + n^2 \zeta + \zeta_1 = 1 + \frac{1}{8} + 0.318 + 0.063 = 1.491.\]
CONICAL TUBES.

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\[\begin{align*}
389 = 1 + 0.099 + 0.318 = 1.417, \text{ and the velocity of efflux } v &= \frac{8.02 \cdot \sqrt{4}}{\sqrt{1.417}} = 16.04 \\
= 14.34 \text{ feet}. \text{ Again, since the transverse section of the tube is 0.02182 square feet, the discharge will be } Q = 14.34n 0.02182 = 0.313 \text{ cubic feet.}
\end{align*}\]

\[\begin{align*}
\text{§ 327. Conical Tubes.—Additional conical tubes give a discharge different from that of prismatic or cylindrical tubes; they are either conically convergent, or conically divergent; in the first case the outer orifice is smaller, and in the second case larger than the inner orifice. The co-efficients of efflux for the first tubes are greater, and for the last, less than for cylindrical tubes. One and the same conical tube gives more water when the wider orifice is made the exit orifice, as } K \text{ in Fig. 443, than when it is turned inwards, as } L \text{ in the same figure, except that it does not give a greater quantity in proportion as the wider orifice exceeds the narrower. If many, as Venturi and Eytelwein, give for conically divergent tubes, a greater co-effi-}
\end{align*}\]

\[\begin{align*}
\text{cients of efflux, than for conically convergent tubes, it must always be borne in mind, that they take the narrower transverse section for the orifice. The following experiments instituted at pressures of from 0,25 to 3,3 metres, with a tube } AD \text{ 9 centimetres long, Fig. 444, bring before us the effect of conicalness in tubes. The width of these tubes at one extremity amounted to } DE = 2.468, \text{ at the other } AB = 3.228 \text{ centimetres, and the angle of convergence, i.e. the angle } AOB, \text{ which the oppositely situated sides } AE \text{ and } BD \text{ of a section in the direction of the longer axis include } = 4^\circ, 50' \text{. By efflux through the narrower orifice, the co-efficient was } = 0.920 \text{, but by efflux through the wider it was } = 0.553, \text{ and if in the calculation, we take the narrower entrance orifice for the transverse section, it will give } = 0.946. \text{ In the first case, when the tube was applied as a conically convergent adjutage, the fluid vein was little contracted, thick and smooth; but, in the second case, when the tube served as a conically divergent adjutage, it was strongly divergent, broken, and spouting. Venturi and Eytelwein have experimented further on efflux through conically divergent tubes. Both philosophers have applied these conical tubes to cylindrical and conoidal adjutages, made after the form of the contracted fluid vein. By such a connection as is represented in Fig. 445, where the divergent portion } KL \text{ of the outer orifice is between 12 and } 21\frac{1}{2} \text{ lines wide, and } 8\frac{1}{8} \text{ of an inch long, and the angle}
\end{align*}\]
of convergence estimated at 5°, 9'. Eytelwein found $\mu = 1.5526$ when he took the narrow end for the orifice, and, on the other hand, $\mu = 0.483$ for the wider end, in which he was right. Through this combined adjutage there certainly flows $\frac{1.5526}{0.615} = 2.5$ times as much as through a simple orifice in a thin plate, and $\frac{1.5526}{0.815} = 1.9$ times as much as through a short cylindrical tube. With small velocities and greater divergence, it is scarcely possible, even by previously closing the tubes, to bring about a full flow.

§ 328. The most ample experiments have been made by d'Aubuisson and Castel on efflux through conically convergent additional tubes. The tubes for this purpose were of great variety, of different lengths, widths, and angles of convergence. The most extensive experiments were those made with tubes of 1.55 centimetres width at the discharging orifice, and of from 2.6 times greater, i.e., of 4 centimetres in length, for which reason we will here communicate the results in the following table. The head of water was, throughout, 3 metres. The discharges were measured by a special gauge-cistern; but in order to obtain besides the coefficients of efflux, those of the velocity and contraction, the amplitude of the jet, due to given heights, were measured, and from these the velocity of efflux (see § 38, Ex. 2) calculated. The ratio $\frac{v}{\sqrt{\frac{2gh}{F\sqrt{2gh}}}}$ of the effective velocity $v$ to the theoretical $\sqrt{\frac{2gh}{F\sqrt{2gh}}}$ gave the coefficient of velocity $\mu$, as also the ratio $\frac{Q}{F\sqrt{2gh}}$ of the effective discharge $Q$ to the theoretical $F\sqrt{2gh}$ gave the coefficient of efflux $\mu$, and, lastly, the ratio of both coefficients, i.e., $\frac{\mu}{\mu}$, determined the coefficient of contraction $a$.

**Table of the Co-efficients of Efflux and Velocity for Efflux through Conically Convergent Tubes.**

<table>
<thead>
<tr>
<th>Angle of convergence</th>
<th>Co-efficient of efflux</th>
<th>Co-efficient of velocity</th>
<th>Angle of convergence</th>
<th>Co-efficient of efflux</th>
<th>Co-efficient of velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0° 0'</td>
<td>0.829</td>
<td>0.829</td>
<td>13° 24'</td>
<td>0.946</td>
<td>0.963</td>
</tr>
<tr>
<td>1° 36'</td>
<td>0.866</td>
<td>0.867</td>
<td>14° 28'</td>
<td>0.941</td>
<td>0.966</td>
</tr>
<tr>
<td>3° 10'</td>
<td>0.895</td>
<td>0.894</td>
<td>16° 36'</td>
<td>0.938</td>
<td>0.971</td>
</tr>
<tr>
<td>4° 10'</td>
<td>0.912</td>
<td>0.910</td>
<td>19° 28'</td>
<td>0.924</td>
<td>0.970</td>
</tr>
<tr>
<td>5° 26'</td>
<td>0.924</td>
<td>0.919</td>
<td>21° 0'</td>
<td>0.919</td>
<td>0.972</td>
</tr>
<tr>
<td>7° 52'</td>
<td>0.930</td>
<td>0.932</td>
<td>23° 0'</td>
<td>0.914</td>
<td>0.974</td>
</tr>
<tr>
<td>8° 58'</td>
<td>0.934</td>
<td>0.942</td>
<td>29° 58'</td>
<td>0.895</td>
<td>0.975</td>
</tr>
<tr>
<td>10° 20'</td>
<td>0.938</td>
<td>0.951</td>
<td>40° 20'</td>
<td>0.870</td>
<td>0.980</td>
</tr>
<tr>
<td>12° 4'</td>
<td>0.942</td>
<td>0.955</td>
<td>48° 00'</td>
<td>0.847</td>
<td>0.984</td>
</tr>
</tbody>
</table>
From this table it is seen that the co-efficients of efflux attain their maximum 0.946 for a tube of 13½° lateral convergence; that, on the other hand, the co-efficients of velocity come out always greater and greater, the greater is the angle of convergence. The following example will show how this table may be used in those cases which present themselves in practice.

Example.—What discharge will a short conoidal tube of 1½ inches width at the outer orifice, and of 10° convergence, deliver under a pressure of 16 feet? According to the experiments of the author, a cylindrical tube of this width gives μ = 0.810; d'Aubuisson's tube, however, gave μ = 0.829, therefore about 0.829 — 0.810 = 0.019 more. Now from the table for a tube of 10° convergence, μ is = 0.937; hence, for the given tube we may put μ = 0.937 — 0.019 = 0.918, whence the discharge

\[
Q = 0.918t \cdot \frac{v^3}{4 \cdot 8^4} \cdot \frac{8.02 \cdot 0.918}{64} = 0.3614 \text{ cubic feet.}
\]

§ 329. Resistance of Friction.—The longer prismatic or cylindrical tubes are, the more they retard the efflux; hence we must assume that the sides of the tubes offer resistance to the motion of the water by the friction, adhesion, or viscosity of the fluid acting against them. From reason and from numerous observations and measurements, we may assume that this resistance of friction is independent of the pressure, that it increases directly as the length l, and inversely as the width d, and, therefore, proportional to the ratio \( \frac{l}{d} \). Moreover, it appears that this resistance is greater for great and less for less velocities, and that it very nearly increases with the square of the velocity \( v \) of the water. If we measure this resistance by the height of a column of water, which must be deducted from the entire head \( h \), in order to obtain the height requisite for the generation of the velocity, we may then put for this height, which we shall term the height due to the resistance, \( h_1 = \zeta_1 \cdot \frac{l}{d} \cdot \frac{v^2}{2g} \), where \( \zeta_1 \) represents a number, from experiment, which we may call the height due to the resistance of friction. There is a greater loss, therefore, of pressure or head of water from the friction of the water in the tube, the greater the ratio \( \frac{l}{d} \) of the length to the width is, and the greater the height due to the velocity \( \frac{v^2}{2g} \). From the discharge \( Q \) and the transverse section of the tube \( F = \frac{\pi d^2}{4} \) there follows the velocity \( v = \frac{4Q}{\pi d^2} \), and hence the height due to the friction:

\[
h_1 = \zeta_1 \cdot \frac{l}{d} \cdot \frac{1}{2g} \left( \frac{4Q}{\pi d^2} \right)^2 = \zeta_1 \cdot \frac{l}{2d} \cdot \left( \frac{4}{\pi} \right)^2 \cdot \frac{Q^2}{d^4}.
\]

In order to obtain the least possible loss of head of water, or fall, in leading a certain quantity of water \( Q \) the pipe must be made as wide as possible, and not unnecessarily long. A double width requires, for instance, only \( (\frac{1}{2})^4 = \frac{1}{16} \) of the fall that the single width does.
If the transverse section of a tube be rectangular, and of the depth \(a\) and the breadth \(b\), we must substitute for

\[
\frac{1}{d} = \frac{1}{4} \cdot \frac{\pi d}{\pi d^2} = \frac{1}{4} \cdot \frac{\text{circumference}}{\text{area}} = \frac{1}{4} \cdot \frac{2(a + b)}{ab} = \frac{a + b}{2ab},
\]

whence it follows: \(h = \frac{\xi}{2ab} \cdot \frac{(a + b)}{2g} \cdot v^2\).

By means of this formula for the resistance of friction in pipes, we may also find the velocity and the quantity of efflux which a pipe, of given length and width, will conduct under a given pressure. It is quite the same, whether the tube \(KL\), Fig. 446, is horizontal, inclines, or ascends, if only by the head of water is understood the depth \(RL\) of the middle point \(L\) of the orifice of the tube below the surface of water \(HO\) of the efflux reservoir. If \(h\) is the head of water, \(h_1\), the height due to the resistance for the orifice of entrance, and \(h_2\), that for the remaining portion of the tube, we then have:

\(h - (h_1 + h_2) = \frac{v^2}{2g}\), or \(h = h_1 + h_2 + \frac{v^2}{2g}\).

If \(\xi\) represents the co-efficient of resistance for the portion of tube next the cistern, and \(\xi\), the co-efficient of the resistance of friction of the rest of the tube, we then have

\[h = \frac{\xi}{2g} \cdot \frac{v^2}{2g} + \frac{\xi_1}{d} \cdot \frac{l}{d} \cdot \frac{v^2}{2g} + \frac{v^2}{2g},\]

or, 1. \(h = \left(1 + \frac{\xi + \xi_1}{d}\right) \cdot \frac{v^2}{2g},\)

and 2. \(v = \sqrt{\frac{2gh}{1 + \xi + \xi_1 \cdot \frac{l}{d}}}.

From the last formula the discharge \(Q = Fv\) is given.

For very long tubes \(1 + \xi\) is small compared with \(\xi_1 \cdot \frac{l}{d}\), whence, simply, \(h = \frac{\xi_1}{d} \cdot \frac{v^2}{2g}\), and inversely,

\[v = \sqrt{\frac{1}{\xi_1} \cdot \frac{d}{l} \cdot 2gh}.

\(\xi 330.\) The co-efficient of friction, like the co-efficient of efflux, is not quite constant; it is greater for small, and less for great velocities; i.e. the resistance of the friction of water in tubes does not increase exactly with the square of the velocity, but with some other power of it. Prony and Eytelwein have assumed, that the head of water lost by the resistance due to friction ought to increase as the
simple velocity and as its square, and have given for it the expression
\[ h = (a \cdot v + \beta \cdot v^2) \frac{l}{d}, \]
where \( a \) and \( \beta \) are co-efficients deduced from experiment. To determine these co-efficients, 51 experiments, which, at various times, were made by Couplet, Bossut, and Du Buat, on the motion of water through long tubes, were made use of by these hydraulicians.

Prony found from this, that
\[ h = (0,0000693 v + 0,0013932 v^2) \frac{l}{d} \]
Eytelwein
\[ h = (0,0000894 v + 0,0011213 v^2) \frac{l}{d} \]
d’Aubuisson assumes
\[ h = (0,0000753 v + 0,001370 v^2) \frac{l}{d} \]
metres.

A formula, discovered by the author, agrees more accurately with observation. It has the form
\[ h = \left( a + \frac{\beta}{\sqrt{v}} \right) \frac{l \cdot v^2}{d \cdot 2g} \]
and is based on the hypothesis, that the resistance of friction increases simultaneously as the square, and as the square root of the cube of the velocity. From this we have the co-efficient of resistance
\[ \zeta = a + \frac{\beta}{\sqrt{v}}, \]
and the height due to the resistance of friction
\[ h = \zeta \cdot \frac{l \cdot v^2}{d \cdot 2g}. \]

For the measurement of the co-efficient of resistance \( \zeta \), or of the auxiliary constants \( a \) and \( \beta \), not only the determinations of Prony and Eytelwein from the 51 experiments of Couplet, Bossut, and Du Buat were used by the author, but also 11 experiments made by him, and 1 experiment by Gueymard in Grenoble. The older experiments extend only to velocities of from 0.043 to 1.930 metres; in the experiments of the author, however, the extreme limit of velocity reached to 4,648 metres. The widths of the tubes, in the older experiments, were 27 mm. = 1.06 in.; 36 mm. = 1.42 in.; 54 mm. = 2.12 in.; 135 mm. = 5.31 in.; and 490 mm. = 19.29 in. ; later experiments were conducted with tubes of 33 mm. = 1.29 in.; 71 mm. = 2.79 in.; and 275 mm. = 5.31 in. By means of the method of least squares, it has been found from the 63 experiments laid down:

\[ \zeta = 0,01439 + \frac{0,0094711}{\sqrt{v}}; \]
\[ h = (0,01439 + \frac{0,0094711}{\sqrt{v}}) \frac{l \cdot v^2}{d \cdot 2g} \text{ metre}; \]

for Prussian measure:
\[ h = (0,01439 + \frac{0,016921}{\sqrt{v}}) \frac{l \cdot v^2}{d \cdot 2g} \text{ feet}, \]
or for English measure:
\[ h = (0,01439 + \frac{0,017963}{\sqrt{v}}) \frac{l \cdot v^2}{d \cdot 2g} \text{ feet}. \]

§ 331. For facilitating the calculation, the following table of the co-efficients of resistance has been compiled. We see from this, that
the variability of these co-efficients is not inconsiderable, as for 0.1 metre velocity it is $0.0443$, for 1 metre $0.0239$, and for 5 metres $0.0186$.

### TABLE OF THE CO-EFFICIENTS OF FRICTION.

<table>
<thead>
<tr>
<th>$v$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>ft. in.</td>
<td>0</td>
<td>0.0443</td>
<td>0.0356</td>
<td>0.0317</td>
<td>0.0294</td>
<td>0.0278</td>
<td>0.0266</td>
<td>0.0257</td>
<td>0.0250</td>
<td>0.0244</td>
</tr>
<tr>
<td>3.4</td>
<td>0.039</td>
<td>0.0363</td>
<td>0.0322</td>
<td>0.0291</td>
<td>0.0268</td>
<td>0.0250</td>
<td>0.0228</td>
<td>0.0219</td>
<td>0.0211</td>
<td>0.0205</td>
</tr>
<tr>
<td>6.7</td>
<td>0.0211</td>
<td>0.0209</td>
<td>0.0208</td>
<td>0.0206</td>
<td>0.0205</td>
<td>0.0204</td>
<td>0.0203</td>
<td>0.0202</td>
<td>0.0201</td>
<td>0.0200</td>
</tr>
<tr>
<td>9.10</td>
<td>0.0199</td>
<td>0.0198</td>
<td>0.0197</td>
<td>0.0196</td>
<td>0.0195</td>
<td>0.0194</td>
<td>0.0193</td>
<td>0.0192</td>
<td>0.0191</td>
<td>0.0190</td>
</tr>
<tr>
<td>13.0</td>
<td>0.0191</td>
<td>0.0191</td>
<td>0.0190</td>
<td>0.0190</td>
<td>0.0189</td>
<td>0.0188</td>
<td>0.0188</td>
<td>0.0187</td>
<td>0.0187</td>
<td>0.0187</td>
</tr>
</tbody>
</table>

We find in this table the co-efficients of resistance due to a certain velocity, when we look for the whole metre in the vertical, and the tenths in the first horizontal column, then proceed from the first number horizontally, and from the last vertically to the place where both motions meet; for example, for $v = 1.3$ metre, $\zeta = 0.0227$, for $v = 2.8$, $\zeta = 0.0201$.

For the Prussian measure we may put:

<table>
<thead>
<tr>
<th>$v$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9 ft.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>0.0522</td>
<td>0.0453</td>
<td>0.0411</td>
<td>0.0383</td>
<td>0.0382</td>
<td>0.0346</td>
<td>0.0333</td>
<td>0.0322</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v$</th>
<th>1</th>
<th>1$\frac{1}{2}$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>12</th>
<th>20 ft.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>0.0313</td>
<td>0.0296</td>
<td>0.0282</td>
<td>0.0263</td>
<td>0.0242</td>
<td>0.0229</td>
<td>0.0213</td>
<td>0.0204</td>
<td>0.0192</td>
<td>0.0182</td>
<td></td>
</tr>
</tbody>
</table>

§ 332. Long Tubes.—With respect to the motion of water in long tubes or conducting pipes, the three following fundamental problems present themselves for solution.

1. The length $l$ and the width $d$ of the tube and the quantity of water $Q$ to be conducted are given, and the head of water is required. We have first to calculate the velocity $v = \frac{Q}{F} = \frac{4Q}{\pi d^2} = 1.2732\frac{e}{d^2}$, then to look for the co-efficient of friction $\zeta$, corresponding to this value, in one of the last tables, and, lastly, to substitute the values of $\zeta$.
we then have $2h = \left(1 + \zeta + \frac{1}{d} l\right) \frac{v^2}{2g}$.

2. The length and width of the tube, as well as the head of water or fall, are given to determine the discharge. We must here find the velocity by the formula

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \frac{1}{d} l}};$$

but since the coefficient of resistance is not quite constant, but varies somewhat with $v$, we must know $v$ approximately beforehand, in order to be able to find out $\zeta$.

From $v$ it then follows that $Q = \frac{\pi d^2}{4} v = 0.7854 d^2 v$.

3. The discharge, the head of water, and the length of the tube are given, to determine the requisite width of the tube.

As $v = \frac{4Q}{\pi d^2}$ therefore $v^3 = \left(\frac{4Q}{\pi}\right)^3 \cdot \frac{1}{d^3}$, we then have $2gh = \left(1 + \zeta + \frac{1}{d} l\right) \left(\frac{4Q}{\pi}\right)^2 \cdot \frac{1}{d^3}$, or

$$2gh \cdot \left(\frac{\pi}{4Q}\right)^2 = (1 + \zeta) \frac{1}{d^3} + \zeta \frac{l}{d^3},$$

$$2gh \cdot \left(\frac{\pi}{4Q}\right)^2 d^3 = (1 + \zeta) d + \zeta_1 l,$$

hence the width of the tube

$$d = \sqrt[3]{\frac{(1 + \zeta) d + \zeta_1 l}{2gh}} \cdot \left(\frac{4Q}{\pi}\right)^2.$$

But now $\left(\frac{4Q}{\pi}\right)^2 = 1.6212$, $1 + \zeta$ as a mean $= 1.505$ and $\frac{1}{2g} = 0.0155$, hence we may put:

$$d = 0.4787 \sqrt[3]{(1.505a d + \zeta_1 l) \frac{Q^2}{h}} \text{ feet.}$$

This formula can only be used as a formula of approximation, because the unknown quantity $d$, and also the coefficient $\zeta_1$, dependent on $v = \frac{4Q}{\pi d^2}$ appear in it.

Examples.—1. What head of water does a conducting pipe, of 150 feet length and 5 inches width, require, if it is to carry off 25 cubic feet of water per minute? Here $v = 1.2732 \cdot \frac{25 \cdot 12^2}{60 \cdot 5^2} = 3.056$ feet, hence we may put $\zeta = 0.0242$, and the head of water or the entire fall of the pipe will be:

$$h = \left(1.505 + 0.0242 \cdot \frac{1.505 \cdot 12}{5}\right) \cdot 0.0155 \cdot 3.056 = (1.505 + 8.712) \cdot 0.0155 \cdot 9.339 = 1.479 \text{ feet, (English.)}$$

2. What discharge will a conducting pipe, 48 feet long and 2 inches wide, with a 5 feet head of water, deliver? It will be:
If we previously take \( \zeta_1 = 0.020 \), we shall obtain
\[
v = \frac{\frac{\sqrt{1.505 + 48.12}}{2}}{\sqrt{1.505 + 288 . \zeta_2}} = 17.88\text{,}
\]
If we previously take \( \zeta_1 = 0.020 \), we shall obtain
\[
v = \frac{\frac{\sqrt{1.505 + 48.12}}{2}}{\sqrt{1.505 + 288 . \zeta_2}} = 17.88\text{,}
\]
gives more correctly \( \zeta = 0.0211 \), hence we shall have more correctly:
\[
v = \frac{\frac{\sqrt{1.505 + 48.12}}{2}}{\sqrt{1.505 + 288 . \zeta_2}} = 17.88\text{,}
\]
and the quantity of water
\[
Q = 0,7854a \left( \frac{b}{12} \right)^2 \times 0,50 = 0,137 \text{ cubic feet = 236,7 cubic inches.}
\]

3. What width must be given to a conducting pipe, 100 feet in length, which at a head of water of 5 feet, delivers half a cubic foot of water per second? Here
\[
d = 0.4787 \sqrt{1.505 d + 100 \zeta_2} \text{, or approximately}
\]
\[
d = 0.4787 \sqrt{1.505 d + 100} \approx 0.4787 \sqrt{1.505 + 100.1} \approx 0.4787 \sqrt{1.505 + 0.100} \text{, or approximately}
\]
\[
d = 0.4787 \sqrt{0.100} = 0.30 \text{, therefore more correctly,}
\]
\[
d = 0.4787 \sqrt{0.1280} = 0.3173 \text{ feet.}
\]
Remark. Experiments with 2½ and 4½ inch wide common wooden pipes have given the author the coefficient of resistance 1.75 times as great as for metallic pipes, to which refer the values given in the table of the former §. Whilst, therefore, for example, for a velocity of 3 feet, \( \zeta_2 = 0.0242 \) for metallic pipes, we must put it for wooden pipes
\[
d = 0.4787 \sqrt{0.1280} = 0.3173 \text{ feet.}
\]
\[
d = 0.4787 \sqrt{0.1280} = 0.3173 \text{ feet.}
\]
§ 339. Bent Tubes.—Particular resistances are opposed to the motion of water in pipes when they are bent or knee-shaped. Experiments have been made by the author on both kinds of resistances, on which account it is necessary here to communicate the results.

If a pipe \( \mathcal{ACB} \), Fig. 447, forms a knee, or if, as it is termed, it be angled, a loss of pressure ensues, which increases uniformly with the height \( \frac{v^3}{2g} \) due to the velocity, and, further, increases with half the angle of deflection \( \angle \mathcal{ACF} = \angle \mathcal{BCE} = \delta \). The height of water lost through the knee, or the height due to the resistance corresponding to its transit through the knee, may be given by the expression
\[
\mathcal{h} = \zeta_2 \frac{v^3}{2g},
\]
where \( \zeta_2 \) expresses the coefficient of the knee resistance, dependent on the magnitude of the angle of deviation of the tube. Experiments made on different knees have led to the expression
\[
\zeta_2 = 0.9457 \sin. \beta^2 + 2.047 \sin. \delta^2,
\]
and from this the following table has been calculated.
\[ h = \left( 1 + \zeta_1 \frac{l}{d} + \zeta_2 \right) \frac{v^2}{2g} \]

\[ h = (1,505 + 8,712 + 2 \cdot 0,956) \frac{v^2}{2g} = 12,129 \times 0,0155 \text{ ft.} 9,339 = 12,129 \text{ ft.} 0,14475 = 1,753 \text{ ft.} \]

§ 334. Curved Tubes.—Curved tubes \( \mathcal{A} \mathcal{B} \), Fig. 449, offer, under otherwise similar circumstances, far less resistance than unrounded knee tubes. The height due to the resistance which measures this obstacle increases likewise as the square of the velocity, but at the same time also as the simple angle of deflexion or curvature \( \mathcal{A} \mathcal{C} \mathcal{B} = \mathcal{B} \mathcal{D} \mathcal{E} = \beta \), and may be expressed, therefore, by the formula:

\[ h = \zeta \cdot \frac{\beta^0}{180^0} \cdot \frac{v^2}{2g} = \zeta \cdot \frac{\beta}{\pi} \cdot \frac{v^2}{2g} \]

The corresponding co-efficient of resistance is by no means constant, it depends much more on the ratio of the width of the tube to the radius of curvature of its axis, and is the less, the less is this ratio. An extensive series of experiments made by the author, and the well known experiments of Du Buat, have, by their combinations, led to the expression \( \zeta = 0,131 + 1,847 \left( \frac{r}{R} \right)^2 \), for tubes with circular transverse sections, and for tubes with quadrangular or rectangular
transverse sections $\zeta = 0,124 + 3,104 \left( \frac{r}{R} \right)^{\frac{3}{2}}$, where $r$ represents half the width of the tube, and $R$ the radius of curvature of the axis.

The two following tables have been calculated accordingly.

**TABLE I.**

CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH CIRCULAR TRANSVERSE SECTIONS.

<table>
<thead>
<tr>
<th>$\frac{r}{R}$</th>
<th>0,1</th>
<th>0,2</th>
<th>0,3</th>
<th>0,4</th>
<th>0,5</th>
<th>0,6</th>
<th>0,7</th>
<th>0,8</th>
<th>0,9</th>
<th>1,0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>0,131</td>
<td>0,138</td>
<td>0,158</td>
<td>0,206</td>
<td>0,294</td>
<td>0,440</td>
<td>0,661</td>
<td>0,977</td>
<td>1,408</td>
<td>1,978</td>
</tr>
</tbody>
</table>

**TABLE II.**

CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH RECTANGULAR TRANSVERSE SECTIONS.

<table>
<thead>
<tr>
<th>$\frac{r}{R}$</th>
<th>0,1</th>
<th>0,2</th>
<th>0,3</th>
<th>0,4</th>
<th>0,5</th>
<th>0,6</th>
<th>0,7</th>
<th>0,8</th>
<th>0,9</th>
<th>1,0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>0,124</td>
<td>0,135</td>
<td>0,180</td>
<td>0,250</td>
<td>0,398</td>
<td>0,643</td>
<td>1,015</td>
<td>1,546</td>
<td>2,271</td>
<td>3,228</td>
</tr>
</tbody>
</table>

It is from hence seen, that for a round tube whose radius of curvature is twice as great as the radius of the tube, the co-efficient of resistance is $= 0,294$, and for a tube whose radius of curvature is at least ten times as great as the radius of the transverse section, this co-efficient $= 0,131$.

**Example.**—1. If the conducting pipe in the second example of § 332 be long small curves of $90^\circ$ curvature, and of the ratio $\frac{r}{R} = \frac{1}{3}$ (Fig. 452), what quantity of water will it deliver? The height due to the resistance of the one curve

$$v = \frac{90^9 \cdot 90^2}{188^3} = 0,147 \cdot \frac{v^2}{2g};$$

hence, for all five curvatures, it

$$5 \cdot 0,147 \cdot \frac{v^2}{2g} = 0,735 \cdot \frac{v^2}{2g},$$

and, accordingly, the velocity sought:

$$v = \sqrt{\frac{7,584 + 0,735}{17,88}} = 8,318 \text{ feet}.$$  

$$Q = 0,7854 \times 8,318 = 6,153 \text{ cubic feet} = 231 \text{ cubic inches}.$$  

2. If the curved buckets of a turbine form channels 12 inches long, 2 inches broad, and 2 inches deep, as $ABC$, Fig. 450, and if the water flows through them with a velocity of 50 feet, and the mean radius of curvature $R$ of this axis of the channels amounts to 8 inches, then $\frac{r}{R} = \frac{1}{4}$, the co-efficient of the resistance of curvature $= 0,297$; further, $\frac{r}{\pi} = \frac{12}{\pi} = \frac{10}{3} = 0,4774$; and lastly, the height due to the resistance corresponding to the curvature of the scoop.
§ 335. Jets d’Eau.—A conducting tube either discharges into the air or under water. The discharge under water is applied when the tube at its outer orifice is so wide that the entrance of air may be feared. Here of course the head of water \( RC \), Fig. 451, must be taken from the surface \( H \) of the upper water to that of \( C \) of the lower water. If the tube, for example, \( KLM \), Fig. 452, discharges into the open air, it will give a stream of water \( OR \), which, when allowed to ascend, is called a jet d’eau. We shall here consider what is most required for these jets. That a jet may ascend to the utmost possible height, it is necessary that the water should flow from the adju­tage with great velocity; hence such adju­tages must be applied which offer the fewest obstacles to the water in its passage, to which, therefore, the greatest co-efficients of velocity are due. Orifices in a thin plate, short tubes fashioned like the contracted fluid vein, and long and conically convergent ones, are those which give the greatest velocities of efflux. Orifices in a thin plate are little suitable, because a jet formed by them presents nodes and bulgings, and, therefore, is sooner scattered by the external air than the prismatic jet. The same takes place in a certain degree with short mouth-pieces, shaped like the contracted vein. Hence, for fountains and fire-engines, mostly long and slightly conically convergent, similar to those which d’Aubuisson used for his experiments, are very properly made use of. Sometimes entirely cylindrical jets are used. Where these mouth-pieces, as, for example, \( KL \), Fig. 453, are screwed on to the con­ducting tube \( AB \), they should gradually widen, that no contraction may occur in passing into them. If these mouth-pieces or discharging tubes are very long, like those of fire-engines, the friction of the water in them will then cause a considerable loss of pressure, because the water has here
a great velocity. For great velocities we may properly put the co-efficient of resistance \( \zeta = 0,016 \), and, therefore, the loss of head of water \( \frac{1}{d} \cdot \frac{v^2}{2g} \). If now the length of a hose is twenty times as great as the mean width, we shall then obtain the height of the resistance due to friction

\[
= 0,016 \cdot 20 \cdot \frac{v^2}{2g} = 0,32 \cdot \frac{v^2}{2g},
\]

thus, from this cause, above 32 per cent. of the height of ascent is lost. These tubes are generally much longer, hence this loss is greater.

The velocity with which water passes out of a mouth-piece or hose, and of which the jet or the height of ascent principally depends, may be estimated by means of the above principles. If we put this velocity of efflux \( v \), the width of the mouth-piece at the exit orifice \( d \), and the mean width of the conducting tube \( d_1 \), we shall then obtain the velocity of the water in it \( v_1 = \frac{d^3}{d_1^2} v \). If \( \zeta \) represent the co-efficient of resistance at the inner orifice of the tube, \( \zeta_1 \) that of the resistance of friction in the pipe, and \( \zeta_2 \) the co-efficient for the knees or curvature of the pipe, the height due to the resistance for the motion of water in the conduit pipes will be:

\[
h = (\zeta + \zeta_1 \frac{d}{d_1} + \zeta_2) \frac{v_1^2}{2g} = (\zeta + \zeta_1 \frac{d}{d_1} + \zeta_2) \frac{d_1^4}{d^4} \cdot \frac{v^2}{2g},
\]

It is seen from this, that the resistance to the water is less, the wider the conduit pipe is. It is hence an important rule, to employ as wide pipes and hoses as possible, for leading water to jets d'eau and for fire-engines.

If, further, we represent the co-efficient of resistance for the mouth-piece by \( \zeta_3 \), we have then the height due to the resistance for this is \( \zeta_3 \frac{v^2}{2g} \), and the sum of all the heights due to the resistance is then:

\[
h = \left[ (\zeta + \zeta_1 \frac{d}{d_1} + \zeta_2) \frac{d_1^4}{d^4} + \zeta_3 \right] \frac{v^2}{2g}.
\]

If, lastly, the height of pressure, i.e. the depth \( RO \), Fig. 452, of the outer orifice \( O \) below the surface of water \( H \) in the reservoir = \( h \), the formula

\[
h = \left[ 1 + \zeta_3 + (\zeta + \zeta_1 \frac{d}{d_1} + \zeta_2) \frac{d_1^4}{d^4} \right] \frac{v^2}{2g},
\]

holds true, and hence the velocity of efflux is:

\[
v = \frac{\sqrt{2gh}}{\sqrt{1 + (\zeta + \zeta_1 \frac{d}{d_1} + \zeta_2) \frac{d_1^4}{d^4} + \zeta_3}}.
\]

If the jet were to rise perpendicularly and in vacuo, the height of ascent would be:
but because the air and the descending water offer impediments to the ascent, and to the direction of the jet, as is the case in fire-engines, the effective height of ascent is somewhat less. According to d'Aubuisson's conclusions from the experiments undertaken upon this subject by Mariotte and Bossut, the effective height of ascent is \( s = s - 0.01 \cdot s^2 = s (1 - 0.01 \cdot s) \) metres, or for English measure = \( s (1 - 0.00305 \cdot s) \) feet.

We see from this that in great ascents proportionately more height is lost than in small velocities. Thick jets ascend somewhat higher than thin ones. In order to diminish the resistance of the descending water, the jet must be directed a little inclined. As to the height and amplitude of oblique jets, see § 38.

Example. If the conduit pipe for a fountain be 250 feet long, and 2 inches diameter, if the coefficient of resistance corresponding to the mouth-piece = 0.32, if the entrance orifice at the reservoir be sufficiently rounded, and the bends that occur have sufficient radii of curvature to allow of our neglecting the corresponding coefficients of resistance, to what height will a jet \( \frac{3}{4} \) inch thick, under a head of water of 30 feet, rise? If we take the coefficient of friction = 0.025, we shall then obtain the entire height due to the resistance:

\[
h = (1 + 0.025 \cdot \frac{250}{\frac{3}{4}} \cdot \frac{\frac{3}{4}}{\frac{3}{4}} + 0.32) \frac{v^2}{2g} = 1.47 \frac{v^2}{2g},
\]

hence, the height due to the velocity \( \frac{h}{1.47} = \frac{30}{1.47} = 20.41 \) feet, and the effective height of ascent \( s = 20.41 (1 - 0.00305 \cdot 20.41) = 20.41 - 1.27 = 19.14 \) feet.

**Chapter IV.**

**On the Resistances of Water in Passing Through Contraction.**

§ 337. Abrupt Widening.—Changes in the transverse section of a tube, or of any other reservoir of efflux, produce changes in the velocity of the water. The velocity is inversely proportional to the transverse section of the stream. The wider the vessel is, the less is the velocity, and the narrower the vessel, the greater the velocity of the water flowing through it. If the transverse section of a vessel be suddenly altered, as, for example, in the tube \( \text{ACE} \), Fig. 454, there then ensues a sudden alteration of the velocity, and this is accompanied by a loss of *vis viva*, or connected with a corresponding diminution of pressure. This loss may be as accurately measured as the mechanical effect in the impact of inelastic bodies (§ 258).

Every particle of water which passes...
from the narrower tube $BD$ into the wider tube $DG$, strikes against the slowly moving mass of water in this tube, and, after impulse, joins itself to and proceeds onwards with it. It is exactly the same with the collision of solid and inelastic bodies; these bodies go on likewise after impact with a common velocity. Since we have already found that the loss of mechanical effect by the impact of these bodies is

$$L = \frac{(v_1 - v_2)^2}{2g} \frac{G_1 G_2}{G_1 + G_2},$$

so we may here, as the impinging particle of water $G_1$ is indefinitely small compared with the impinged mass of water $G_2$, put:

$$L = \frac{(v_1 - v_2)^2}{2g} G_1,$$

and, consequently, the corresponding loss of head $h = \frac{(v_1 - v_2)^2}{2g}$.

There arises, therefore, from a sudden change of velocity a loss of head, which is measured by the height due to the velocity corresponding to this change.

If now the transverse section of the one tube $AC = F_1$, and that of the other $CE = F$, the velocity of the water in the first tube $= v_1$, and that in the other $= v$, we then have $v_1 = \frac{Fv}{F_1}$, hence the loss in head of water in the passage from one tube to the other is

$$h = \left(\frac{F}{F_1} - 1\right)^2 \frac{v^2}{2g},$$

and the corresponding co-efficient of resistance $\zeta$

$$= \left(\frac{F}{F_1} - 1\right)^2.$$

The experiments undertaken by the author on this subject accord well with theory. That the tube $DG$ may be filled with water, it is requisite that it be not very short, nor much wider than the tube $AC$. This loss vanishes, when, as represented in Fig. 455, a gradual passage from one tube into the other is accomplished by the rounding of the edges.

**Remark.** The head of water found $h_1 = \left(\frac{F}{F_1} - 1\right)^2 \frac{v^2}{2g}$ cannot, of course, be utterly lost; we must rather assume that the mechanical effect produced by it is expended on the separation of the previous continuity of the particles of water.

**Example.** If the diameter of a tube, of the construction in Fig. 454, is as great again as that of another tube, then $\frac{F}{F_1} = \left(\frac{2}{1}\right)^2 = 4$, hence the co-efficient of resistance $\zeta = (4-1)^2 = 9$, and the corresponding height due to the resistance on passing from the narrow into the wide tube $= 9 \cdot \frac{v^2}{2g}$. If the velocity of the water in the latter tube $= 10$ feet, the height due to the resistance is then $9 \cdot 0.0155e 10^8 = 13.95$ feet.

§ 338. Abrupt Contraction.—A sudden change of velocity also
occurs when water passes from a cistern $AB$, Fig. 456, into a narrow tube $DG$, especially when there is a diaphragm at the place of entrance $CD$ whose orifice is less than the transverse section of the tube $DG$. If the area of the contraction is $= F_1$, and $a$ the co-efficient of contraction, we have then the transverse section $F_2$ of the contracted fluid vein $= a F_1$; and if, on the other hand, $F$ is the transverse section of the tube and $v$ the velocity of efflux, we then find that the velocity at the contracted section $F_2$ is, $v_2 = \frac{F}{a F_1} v$, and hence the loss of head in passing from $F_2$ into $F$, or from $v_2$ into $v$: \[ h = \left( \frac{v-v}{2g} \right)^2 = \left( \frac{F}{a F_1} - 1 \right)^2 \frac{v^2}{2g} \], and the corresponding co-efficient due to the resistance $\xi = \left( \frac{F}{a F_1} - 1 \right)^2$.

Without the diaphragm, we have a mere short tube, Fig. 457; hence, $F = F_1$ and $\xi = \left( \frac{1}{a} - 1 \right)^2$.

If we assume $a = 0,64$, we then obtain:
\[ \xi = \left( \frac{1}{a} - 0,64 \right)^2 = \left( \frac{0,36}{0,64} \right)^2 = 0,316. \]

But the co-efficient due to the resistance for the transit through an orifice in a thin plate is about 0,07; hence, here, where the water flows out \( \frac{1}{a} \) times as fast as from the contracted transverse section, the corresponding height due to the resistance
\[ = 0,07 \cdot \left( \frac{v}{a} \right)^2 \cdot \frac{1}{2g} = 0,07 \cdot \frac{1}{a^2} \cdot \frac{v^2}{2g} = 0,07 \cdot \frac{v^2}{2g} = 0,07 \cdot \frac{v^2}{2g} = 0,41 \cdot \frac{v^2}{2g}. \]

By combining these two resistances, we obtain the entire height due to the resistance for efflux through a short tube:
\[ = 0,316 \frac{v^2}{2g} + 0,171 \frac{v^2}{2g} = 0,49 \frac{v^2}{2g}, \]
whilst we before found it $= 0,50 \frac{v^2}{2g}$.

Experiments on the efflux of water through an additional tube, with a narrow inner orifice, as in Fig. 456, have led the author to the following results. The co-efficient of resistance for transit through a diaphragm, and for a contraction at the wider tube, may be expressed by the formula $\xi = \left( \frac{F}{a F_1} - 1 \right)^2$, but there must be put:
EFFECT OF IMPERFECT CONTRACTION.

For $\frac{F_1}{F}$: 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

| $\alpha$  | 0.616 0.614 0.612 0.610 0.607 0.605 0.603 0.601 0.598 0.596 |

And it follows that:

| $\zeta$  | 231.7 50.99 19.78 9.612e 5,256 3,077e 1,876 1,169 0.734 0.480 |

From this, for example, the co-efficient of resistance in the case where the narrow transverse section is half as great as that of the tube, is $\zeta = 5,256$, i.e., for transit through this contraction, a head of water is required which is $5\frac{1}{4}$ times as great as the height due to the velocity.

Example. What discharge will the apparatus delineated in Fig. 458 give, if the head of water is $1\frac{1}{2}$ feet, the width of the circular contraction $1\frac{1}{2}$, and that of tube 2 inches?

We have here $\frac{F_1}{F} = \left(\frac{1\frac{1}{2}}{2}\right)^2 = \frac{9}{16}$, hence $\alpha = 0,066, and \zeta = \left(\frac{16}{9} - 0,066 - 1\right)^2 = \left(\frac{16 - 5,454}{5,454}\right)^2 = \left(\frac{10,546}{5,454}\right)^2 = 3,74$. If now we put $h = (1 + \zeta)\frac{v^2}{2g}$, we shall then obtain the velocity of efflux:

$v = \sqrt{2gh} = 8,02\sqrt{1,5} = 4,56$ feet, and consequently the quantity discharged:

$Q = \text{area} \cdot v = \frac{\pi \cdot 4 \cdot 12 \cdot 4,56 = 54,72}{\text{cubic inches}} = 172$ cubic inches.

§ 339. Effect of Imperfect Contraction.—In the case considered in the last paragraph, the water issues from a large cistern, hence the contraction may be regarded as perfect; but if the transverse section of the cistern, or of the stream of fluid arriving at the narrow part, is not great with respect to the transverse section $F_1$, Fig. 458, of this contracted part; the contraction is then imperfect, and hence also the corresponding co-efficient of resistance is less than in the case above investigated. If we retain the former denominations, we have then also here the height due to the resistance, or the head of water expended by the transit through $F_1$, $h = \left(\frac{F}{\alpha F_1} - 1\right)^2 \frac{v^2}{2g}$, only, for $\alpha$, we must substitute variable numbers which are greater the greater the ratio $\frac{G}{F}$.

Fig. 458.

Fig. 459.
From experiments undertaken by the author, we must put into the formula \( \zeta = \left( \frac{F}{a F_1} - 1 \right)^2 \) for the co-efficient of resistance,

<table>
<thead>
<tr>
<th>( \frac{F}{F_1} )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0.624</td>
<td>0.632</td>
<td>0.643</td>
<td>0.650</td>
<td>0.681</td>
<td>0.712</td>
<td>0.755</td>
<td>0.813</td>
<td>0.892</td>
<td>1.000</td>
</tr>
</tbody>
</table>

And it follows

\[
\zeta \quad 225.9 \quad 47.77 \quad 17.50 \quad 7.801 \quad 3.753 \quad 1.796 \quad 0.797 \quad 0.290 \quad 0.060 \quad 0.000
\]

These losses are diminished when, by rounding off the edges, the contraction is diminished or counteracted, and they may be entirely neglected, if, as is represented in Fig. 460, a gradually widening tube \( MN' \) is put on.

**Example.** What head of water is requisite that the apparatus in Fig. 461 may deliver 8 cubic feet of water per minute? If the width of the diaphragm \( F_1 = \frac{1}{2} \), the width of the efflux tube \( DG = 2 \) inches, and the lower width of the efflux tube \( AC = 3 \) inches, we shall then have \( F = \left( \frac{1}{2} \right)^{\frac{2}{3}} = \frac{1}{4} \), hence \( a = 0,637 \); further, \( \frac{F}{F_1} = \left( \frac{2}{1} \right)^{\frac{2}{3}} = (\frac{2}{1})^{\frac{2}{3}} = (\frac{2}{1})^{\frac{2}{3}} = (\frac{2}{1})^{\frac{2}{3}} = \frac{1}{2} \), and the co-efficient of resistance:

\[
\zeta = \left( 16 \cdot 9 \cdot 0.637 \right)^{-1} = \left( 10.267 \right)^{-1} = 3.207. \quad \text{The velocity of efflux is now:}\n\]

\[
v = \frac{4 Q}{\pi d^2} = \frac{4 \cdot 8}{60. \pi (\frac{2}{3})} = 19.2 \quad \text{feet; hence, the head of water in question is}\n\]

\[
h = \left( 1 + \frac{\zeta}{2g} \right) \frac{v^2}{2g} = 4.207n \cdot 0.0155 \cdot 6.112 = 2.43 \text{ feet.}\n\]

§ 340. **Slides, cocks, valves.**—For regulating the flow of water from pipes and cisterns, slides, cocks, valves, &c., are used, by which contractions are produced which offer obstacles to the passage of the water, and these may be determined in a manner similar to the loss estimated in the last paragraph. But since the water here undergoes further efficiencies \( a \) and \( \zeta \) cannot be determined directly, but special experiments are necessary for this purpose. Such experiments have been also made, *and their principal results are communicated in the following tables.*

---

* Experiments on the efflux of water through valves, slides, &c., undertaken and calculated by Julius Weisbach, under the title "Untersuchungen im Gebiete der Mechanik und Hydraulik," &c. Leipzig, 1842.
TABLE I.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDING VALVES IN RECTANGULAR TUBES.

<table>
<thead>
<tr>
<th>Ratios of transverse section $F_r / F$</th>
<th>1.0</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
<th>0.6</th>
<th>0.5</th>
<th>0.4</th>
<th>0.3</th>
<th>0.2</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Co-efficient of resistance $\zeta$</td>
<td>0.00</td>
<td>0.09</td>
<td>0.39</td>
<td>0.95</td>
<td>2.08</td>
<td>4.02</td>
<td>8.12</td>
<td>17.8</td>
<td>44.5</td>
<td>193</td>
</tr>
</tbody>
</table>

TABLE II.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDES IN CYLINDRICAL TUBES.

<table>
<thead>
<tr>
<th>Height of place $e$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of transverse section.</td>
<td>1,000</td>
<td>0,948</td>
<td>0,856</td>
<td>0,740</td>
<td>0,609</td>
<td>0,466</td>
<td>0,315</td>
<td>0,159</td>
</tr>
<tr>
<td>Co-efficient of resistance $\zeta$.</td>
<td>0.00</td>
<td>0.07</td>
<td>0.26</td>
<td>0.81</td>
<td>2.06</td>
<td>5.52</td>
<td>17.0</td>
<td>97.8</td>
</tr>
</tbody>
</table>

TABLE III.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A RECTANGULAR TUBE.

<table>
<thead>
<tr>
<th>Angle of position.</th>
<th>5°</th>
<th>10°</th>
<th>15°</th>
<th>20°</th>
<th>25°</th>
<th>30°</th>
<th>35°</th>
<th>40°</th>
<th>45°</th>
<th>50°</th>
<th>55°</th>
<th>60°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of transverse section.</td>
<td>0,926</td>
<td>0,849</td>
<td>0,769</td>
<td>0,687</td>
<td>0,604</td>
<td>0,520</td>
<td>0,436</td>
<td>0,352</td>
<td>0,269</td>
<td>0,188</td>
<td>0,110</td>
<td>0</td>
</tr>
<tr>
<td>Co-efficient of resistance.</td>
<td>0.05</td>
<td>0.31</td>
<td>0.88</td>
<td>1.84</td>
<td>3.45</td>
<td>6.15</td>
<td>11.2</td>
<td>20.7</td>
<td>41.0</td>
<td>95.3</td>
<td>275</td>
<td>∞</td>
</tr>
</tbody>
</table>

TABLE IV.

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A CYLINDRICAL TUBE.

<table>
<thead>
<tr>
<th>Angle of position.</th>
<th>5°</th>
<th>10°</th>
<th>15°</th>
<th>20°</th>
<th>25°</th>
<th>30°</th>
<th>35°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of transverse section.</td>
<td>0.926</td>
<td>0.850</td>
<td>0.772</td>
<td>0.692</td>
<td>0.613</td>
<td>0.535</td>
<td>0.458</td>
</tr>
<tr>
<td>Co-efficient of resistance.</td>
<td>0.05</td>
<td>0.29</td>
<td>0.75</td>
<td>1.56</td>
<td>3.10</td>
<td>5.47</td>
<td>9.68</td>
</tr>
<tr>
<td>Angle of position.</td>
<td>$40^\circ$</td>
<td>$45^\circ$</td>
<td>$50^\circ$</td>
<td>$55^\circ$</td>
<td>$60^\circ$</td>
<td>$65^\circ$</td>
<td>$82\frac{1}{2}^\circ$</td>
</tr>
<tr>
<td>-------------------</td>
<td>-----------</td>
<td>-----------</td>
<td>-----------</td>
<td>-----------</td>
<td>-----------</td>
<td>-----------</td>
<td>----------------</td>
</tr>
<tr>
<td>Ratio of transverse section.</td>
<td>$0,385$</td>
<td>$0,315$</td>
<td>$0,250$</td>
<td>$0,190$</td>
<td>$0,137$</td>
<td>$0,091$</td>
<td>$0$</td>
</tr>
<tr>
<td>Co-efficient of resistance.</td>
<td>$17,3$</td>
<td>$31,2$</td>
<td>$52,6$</td>
<td>$106$</td>
<td>$206$</td>
<td>$486$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

**TABLE V.**

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN RECTANGULAR TUBES.

<table>
<thead>
<tr>
<th>Angle of position.</th>
<th>$5^\circ$</th>
<th>$10^\circ$</th>
<th>$15^\circ$</th>
<th>$20^\circ$</th>
<th>$25^\circ$</th>
<th>$30^\circ$</th>
<th>$35^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of transverse section.</td>
<td>$0,913$</td>
<td>$0,826$</td>
<td>$0,741$</td>
<td>$0,658$</td>
<td>$0,577$</td>
<td>$0,500$</td>
<td>$0,426$</td>
</tr>
<tr>
<td>Co-efficient of resistance.</td>
<td>$0,28$</td>
<td>$0,45$</td>
<td>$0,77$</td>
<td>$1,34$</td>
<td>$2,16$</td>
<td>$3,54$</td>
<td>$5,72$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Angle of position.</th>
<th>$40^\circ$</th>
<th>$45^\circ$</th>
<th>$50^\circ$</th>
<th>$55^\circ$</th>
<th>$60^\circ$</th>
<th>$65^\circ$</th>
<th>$70^\circ$</th>
<th>$90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of transverse section.</td>
<td>$0,357$</td>
<td>$0,293$</td>
<td>$0,234$</td>
<td>$0,181$</td>
<td>$0,134$</td>
<td>$0,094$</td>
<td>$0,060$</td>
<td>$0$</td>
</tr>
<tr>
<td>Co-efficient of resistance.</td>
<td>$9,27$</td>
<td>$15,07$</td>
<td>$24,9$</td>
<td>$42,7$</td>
<td>$77,4$</td>
<td>$158$</td>
<td>$368$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

**TABLE VI.**

THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN CYLINDRICAL TUBES.

<table>
<thead>
<tr>
<th>Angle of position.</th>
<th>$5^\circ$</th>
<th>$10^\circ$</th>
<th>$15^\circ$</th>
<th>$20^\circ$</th>
<th>$25^\circ$</th>
<th>$30^\circ$</th>
<th>$35^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of transverse section.</td>
<td>$0,913$</td>
<td>$0,826$</td>
<td>$0,741$</td>
<td>$0,658$</td>
<td>$0,577$</td>
<td>$0,500$</td>
<td>$0,426$</td>
</tr>
<tr>
<td>Co-efficient of resistance.</td>
<td>$0,24$</td>
<td>$0,52$</td>
<td>$0,90$</td>
<td>$1,54$</td>
<td>$2,51$</td>
<td>$3,91$</td>
<td>$6,22$</td>
</tr>
</tbody>
</table>
§ 341. By means of the co-efficients derived from the above tables, we may not only assign the loss of pressure corresponding to a certain slide, cock, or position of a valve, but also deduce what position is to be given to this apparatus that the velocity of efflux or the resistance may be of a certain amount. Such a determination is, of course, the more to be relied on, the more the regulating arrangements are like to those used in the experiments. The numerical values given in the tables are only true for the case where the water, after its transit through the contractions produced by means of this apparatus, again fills this tube. That this full flow may take place for small contractions, the tube should have a considerable length. The transverse sections of the rectangular tubes were 5 centimetres (1.97 inch) broad and 2½ (0.98 inch) deep. The transverse sections of the cylindrical tubes had, however, a width of 4 centimetres (1.57 inch). By the slide, Fig. 462, there is a simple contraction, whose transverse section forms in the one tube a mere rectangular $F_1$, Fig. 463; in the second, however, a lune, $F_p$, Fig. 464. In the case of cocks, two contractions present themselves, and also two changes of direction; on this account the resistances are also very considerable. The transverse sections of the greatest contractions have very peculiar figures. The stream in throttle valves, Fig. 466, divides itself into
two portions, each of which passes through a contraction. The transverse sections of these contractions are, in the throttle valve of rectangular tubes, rectangular, and in cylindrical ones lunar-shaped. The following examples will suffice to show the application of the above tables.

Examples.—1. If a sliding valve is applied to a cylindrical conducting pipe 500 feet long and 3 inches wide, and this be drawn up to \( \frac{1}{2} \) of its entire height, and therefore close \( \frac{1}{2} \) of the pipe, what discharge will it deliver under a pressure of 4 feet? The coefficient of resistance \( \zeta \) for entrance into the pipe may be put from the above at 0,025, and that of the resistance of the slide from Table ii. \( \zeta = 5,52 \), hence the velocity of the flow \( \sqrt{\frac{8,02 \cdot \sqrt{4}}{8,02 \cdot 2}} = 16,04 \).

If we put the coefficient of friction \( \zeta = 0,025 \), we shall then obtain \( \sqrt{\frac{57,025}{59,025}} = 2,12 \) feet. But the velocity \( v = 2,12 \) feet, gives more correctly \( \zeta = 0,026 \); hence, more accurately, \( v = \frac{16,04}{\sqrt{57,025}} = 2,08 \) feet, and the discharge per second is \( \frac{\pi \cdot 9 \cdot 12 \cdot 2,08}{60 \cdot \pi (\frac{3}{1})^2} = 1,91 \) feet, and after putting on the valve \( \frac{3}{1} \cdot 1,91 = 1,528 \) feet. The coefficient of efflux is \( \sqrt{\frac{8,02 \cdot \sqrt{5}}{9,016}} = 1,91 \), hence the coefficient of resistance \( \frac{1}{0,106} = 88 \); the coefficient of efflux for the second case is \( \frac{1}{0,106} = 0,0848 \); hence the coefficient of resistance \( \frac{1}{0,0848^2} = 1 = 138,0 \), and consequently to produce the coefficient of resistance of the throttle-valve; \( \xi = 138 - 88 = 50 \). But now, from Table vi., the angle of position \( a = 50^\circ \), \( \xi = 92,6 \), and the angle of position \( a = 55^\circ \), \( \xi = 98,8 \); hence we may be allowed to assume, for a position of \( 30^\circ + \frac{15^\circ}{26,2} \cdot 30^\circ = 53^\circ \), the desired quantity of discharge may be obtained. If we consider, further, for a change of velocity of 1,91 feet to 1,528 feet, the coefficient of resistance passes from 0,0268 into 0,0281; then, more correctly, \( \xi = 138,0 - 88 \cdot 281 = 138,0 - 92,96 = 45,04 \), and, accordingly, the angle of position \( 50^\circ + \frac{10,9}{26,2} \cdot 5^\circ = 52^\circ \).

§ 342. Valves.—The knowledge of the resistance produced by valves is of great importance. Experiments have been made by the author on this subject. The conical and slack, or flap-valves, Figs. 467 and 468, are those which most frequently are met with in practice. In both, the water passes through the aperture formed by a
ring $RG$; the conical valve $KL$ has a guide rod, by which it is fixed in guides, and admits of an outward push only in the direction of the axis; the clack valve $KL$ opens by turning like a door. It is easily seen in both apparatuses that a resistance is opposed to the water, not only by the valve ring, but also by the valve plate.

For the conical valve with which the experiments were undertaken, the ratio of the aperture in the valve ring, to the transverse section of the whole tube was 0.356, and, on the other hand, the ratio of the surface of the ring for the opened valve to the transverse section of the tube 0.406; hence, for the mean, we may put $\frac{F_1}{F} = 0.381$. Whilst the efflux in different positions of the valve was observed, it was found that the co-efficient of resistance diminished when the valve slide was greater, and that this diminution was almost insignificant when it exceeded half the width of the aperture. Its amount was in this case = 11, therefore, the height due to the resistance or the loss of head of water = $11 \cdot \frac{v^2}{2g}$, $v$ being the velocity of the water in the full tube. This number may be also used for determining the co-efficients of resistance corresponding to the other ratios of the transverse sections. Let generally $\zeta = \left(\frac{F}{F_1} - 1\right)^3$, we then obtain for the observed case $\frac{F_1}{F} = 0.381$, $\zeta = 11$, and $11 = \left(\frac{1}{0.381} - 1\right)^3$, hence

$$a = \frac{1}{0.381 \left(1 + \sqrt[3]{11}\right)} = \frac{1}{4.317 \cdot 0.381} = 0.608,$$

and, lastly, in general:

$$\zeta = \left(\frac{F}{0.608 F_1} - 1\right)^3 = (1,645 \cdot \frac{F}{F_1} - 1)^3.$$

If, for example, the transverse section of the aperture is one half that of the tube, the co-efficient of resistance will accordingly = $(1,645 \cdot 2 - 1)^3 = 2.29^3 = 5.24$.

For the clack, or trap valve, the ratio of the transverse section of the aperture to the tube was $\frac{F_1}{F} = 0.535$; but the following table shows in what degree the co-efficient of resistance diminishes with the size of the aperture.

**Table**

<table>
<thead>
<tr>
<th>Angle of aperture</th>
<th>15°</th>
<th>20°</th>
<th>25°</th>
<th>30°</th>
<th>35°</th>
<th>40°</th>
<th>45°</th>
<th>50°</th>
<th>55°</th>
<th>60°</th>
<th>65°</th>
<th>70°</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coefficient of resistance</td>
<td>90</td>
<td>62</td>
<td>42</td>
<td>30</td>
<td>20</td>
<td>14</td>
<td>9.5</td>
<td>6.8</td>
<td>4.6</td>
<td>3.2</td>
<td>2.3</td>
<td>1.7</td>
</tr>
</tbody>
</table>
The co-efficients of resistance of these valves may be calculated approximately with the help of this table, even when the ratio of the transverse section is any other. The same method must be adopted as that followed for conical valves.

Example. A forcing-pump delivers by each descent of the piston, 5 cubic feet of water in 4 seconds, the width of the tube of ascent, in which lies the valve opening upwards, is 6 inches, the aperture of the valve-ring 3½ inches, and the greatest diameter of the valve 4½ inches; what resistance has the water in its passage through the valve to overcome? The ratio of the transverse section for the aperture is
\[
\left(\frac{\frac{3.5}{6}}{\frac{4.5}{6}}\right)^2 = (\frac{7}{12})^2 = 0.34,
\]
and that of the annular contraction to the transverse section of the tube is
\[
\frac{2}{(\frac{1.645}{6} - 1)^2} = 3.22^2 = 10.4.
\]
The velocity of the water in
\[
v = \frac{\sqrt{2 gh}}{\sqrt{1 + \zeta}} = \frac{20}{5} = 6.37 \text{ feet},
\]
the height due to the velocity = 0.629 feet; and, consequently, the resistance due to the height = 10.4 x 0.629 = 6.54 feet. The quantity forced up in one second weighs \( \frac{4}{5} \) :: 62.5 = 77.6 lbs.; hence the mechanical effect which by the transit of the water through the valve is consumed in this time = 77.6 x 6.54 = 507.5 ft. lbs.

§ 343. Compound Vessels.—The principles already laid down on the resistance of water in its passage through contractions, find their application in the efflux of water through compound vessels. The apparatus \( \mathcal{AD} \) represented in Fig. 469, is divided by two partition walls containing the orifices \( F_1 \) and \( F_2 \), and on this account forms three vessels of communication.

There were no partition walls, and the edges, where one vessel passes into the other, rounded off, we should then have as for a single vessel the velocity of flow through \( F \): \( v = \frac{\sqrt{2 gh}}{\sqrt{1 + \zeta}} \), representing the depth \( FH \) below the surface of water, and \( \zeta \) the co-efficient of resistance for the passage through the orifice \( F \). But since obstacles are to be overcome on passing through \( F_1 \) and \( F_2 \), we then have to put
\[
v = \sqrt{\frac{2 gh}{\sqrt{1 + \zeta} + \zeta_1 + \zeta_2}},
\]
and to substitute for \( \zeta_1 \) and \( \zeta_2 \), the co-efficients of resistance corresponding to the contractions \( F_1 \) and \( F_2 \). If we represent the transverse sections of the vessels \( CD \), \( BC \) and \( AB \), by \( G_1 \), \( G_1 \), and \( G_2 \), we may further put (§ 338):
\[
\zeta_1 = \left(\frac{G}{\alpha F_1} - 1\right)^2, \text{ and } \zeta_2 = \left(\frac{G_1}{\alpha F_2} - 1\right)^2,
\]
hence also:
\[
v = \frac{\sqrt{2 gh}}{\sqrt{1 + \zeta + \left(\frac{G}{\alpha F_1} - 1\right)^2 + \left(\frac{G_1}{\alpha F_2} - 1\right)^2}}.
\]
Exactly the same relations take place in the compound apparatus of discharge represented in Fig. 470, except only that the friction of the water in the tube of communication $CE$ has perhaps to be taken into account. If $l$ is the length, and $d$ the width of this tube, but $\xi$, the co-efficient of friction, and $v$, the velocity of the water in the tube of communication, we then have the height which the water loses in passing from $AC$ to $GL$:

$$h_1 = \left[1 + \left(\frac{1}{a} - 1\right)v + \xi, \frac{l}{d}\right] \frac{v^2}{2g},$$

or, since the velocity is to be put:

$$v_1 = \frac{aF}{F}v, h_1 = \left[1 + \left(\frac{1}{a} - 1\right)v + \xi, \frac{l}{d}\right] \left(\frac{aF}{F_1}\right)^2 \frac{v^2}{2g}.$$

If this height be deducted from the whole head of water $h$, there will remain the head of water in the second vessel $h_2 = h - h_1$, and hence the velocity of efflux:

$$v = \sqrt{\frac{2gh}{1 + \xi}} = \sqrt{\frac{2gh}{1 + \xi + \left[1 + \left(\frac{1}{a} - 1\right)v + \xi, \frac{l}{d}\right] \left(\frac{aF}{F_1}\right)^2}}.$$

This determination is very simple with the apparatus represented in Fig. 471, because the transverse sections $G, G_1, G_2$ of the cisterns may be made indefinitely great with respect to the transverse sections of the orifices $F, F_1, F_2$. Hence the first difference of level $OH_1$, or height due to the resistance in passing through:

$$F_1 \text{ is } h_1 = \frac{1}{2g} \left(\frac{v_1}{a_1}\right)^2 = \left(\frac{aF}{a_1F_1}\right)^2 \frac{v^2}{2g},$$

and likewise the second difference of level $O_1 H_1$, or the height due to the resistance in passing through

$$F_2 \text{ is } h_2 = \left(\frac{aF}{a_2F_2}\right)^2 \frac{v^2}{2g},$$

where $a, a_1, a_2$, represent the co-efficients of contraction for the orifices $F, F_1$ and $F_2$. It accordingly follows that:

$$v = \sqrt{\frac{2gh}{1 + \left(\frac{aF}{a_1F_1}\right)^2 + \left(\frac{aF}{a_2F_2}\right)^2}},$$

and the quantity of discharge:
of water; if, further, is the co-efficient of friction for entrance from the co-efficient of friction of the reservoir into the tube, and PIEZOMETERS.

\[ Q = \frac{aF \sqrt{2gh}}{\sqrt{1 + \left(\frac{aF}{a_1 F_1}\right)^2 + \left(\frac{aF}{a_2 F_2}\right)^2}} \]

\[ = \frac{\sqrt{\left(\frac{1}{aF}\right)^2 + \left(\frac{1}{a_1 F_1}\right)^2 + \left(\frac{1}{a_2 F_2}\right)^2}}{\sqrt{2gh}}. \]

It is easy to perceive that compound reservoirs of efflux deliver less water, under otherwise similar circumstances, than simple ones.

Example. If in the apparatus, Fig. 470, the whole head of water or depth of the centre of the orifice \( F \) below the surface of water of the first cistern is 6 feet, the orifice 8 inches broad and 4 inches deep, the tube connecting both reservoirs 10 feet long, 12 inches broad, and 6 inches deep, what discharge will this reservoir give? The mean width of the tube \( = \frac{1}{2} \cdot 1.5 = \frac{3}{2} \) ft., hence \( \frac{l}{d} = \frac{3}{10} = 0.3 \); let us now put the co-efficient of friction \( \zeta_1 = 0.025 \), and it follows that \( \zeta_1 \cdot \frac{l}{d} = 0.025 \cdot 15 = 0.375 \); if the co-efficient of resistance for entrance into prismatic tubes be here put 0.505, we obtain

\[ 1 + \left(\frac{1}{a_1} - 1\right)^2 + \zeta_1 \frac{l}{d} = 1 + 0.505 + 0.375 = 1.88. \]

As \( \frac{aF}{F_1} = \frac{0.64 \cdot 8.4}{12.6} = 0.2845 \), the co-efficient of resistance for the entire connecting tube \( = 1.88 \cdot 0.2845^2 = 0.152 \), and the co-efficient of resistance for the transit through \( F \) \( = 0.07 \), we then obtain the velocity of efflux \( v = \frac{8.02 \sqrt{6}}{\sqrt{1.07 + 0.152}} = \frac{8.02 \sqrt{6}}{1.222} = 17.77 \) feet. The contracted section is \( 0.64 \cdot 1 \cdot \frac{3}{2} = 0.32 \) square feet, hence the discharge \( = 0.32 \cdot 17.77 = 5.68 \) cubic feet.

§ 344. Piezometers.—The loss of pressure which water suffers in conduit pipes from contractions, friction, &c., may be measured by columns of water, which are sustained in vertically placed tubes, which, when used for this purpose, are called piezometers.

If \( v \) is the velocity of the water at a place \( B \), Fig. 472, where a piezometer is applied, \( l \) the length, \( d \) the width of the portion of tube \( AB \), \( h \) the head of water or the depth of the point \( B \) below the surface of water; if, further, \( \zeta \) is the co-efficient of resistance for entrance from the reservoir into the tube, and \( \zeta \), the co-efficient of friction, we then have for the height of the piezometer measuring the pressure at \( B \),

\[ z = h - \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g}. \]

If the length of a portion of the tube \( BC = l_1 \), and its fall \( = h_1 \), we then have the height of the piezometer at \( C \):

\[ 35^* \]
\[ z_1 = h + h_1 - \left(1 + z + \frac{\zeta_1 l}{d} + \frac{\zeta_1 l^2}{2g} \right) \frac{v^2}{2g}, \]

and hence the difference of these two heights:

\[ z_1 - z = h_1 - \frac{\zeta_1 l}{d} \cdot \frac{v^2}{2g}, \]

hence, inversely, the height of the portion of tube \( BC \), due to the resistance:

\[ \frac{\zeta_1 l}{d} \cdot \frac{v^2}{2g} = h_1 + z - z_1 = \text{the fall of the portion of tube} \]

plus the difference of the heights of the piezometers.

From this it is seen that piezometers are applicable to the measurement of the resistance which the water has to overcome in conduit pipes. If a particular impediment is found in the tubes; if, for instance, some small body is found fixed there, this will immediately be shown by the falling of the piezometer, and the amount of the resistance produced, expressed. The resistances which are caused by regulating apparatus, such as cocks, slides, &c., may be likewise expressed by the height of the piezometer. The piezometer, for example, stands lower at \( D \) than at \( C \), not only in consequence of the friction of the water in the portion of water \( CD \), but also in consequence of the contraction which the slide \( S \) produces in this tube. If for a perfectly opened slide the difference \( NO \) of the height of the piezometer = \( h_1 \), and for the slide partly closed = \( h_2 \), the new difference or depression \( h_3 \) = \( h_1 \), gives the height due to the resistance which corresponds to the passage of the water through the slide. Lastly, the velocity of efflux may be also estimated by the height of the piezometer. If the height of the piezometer \( PQ = z \), the length of the last portion of tube \( DE = l \), and its width = \( d \), we then have:

\[ z = \frac{\zeta_1 l}{d} \cdot \frac{v^2}{2g}, \text{ and hence } v = \sqrt{\frac{2g z}{e \cdot \frac{l}{\zeta_1 d}}} = \sqrt{\frac{d}{l} \cdot \frac{2gz}{\zeta_1}}, \]

**Example.** If the height of the piezometer \( PQ = z \), Fig. 471, \( \frac{1}{4} \) foot, and the length of the tube \( DE \), measured from the piezometer to the discharging orifice, \( = 150 \) feet, the width of the tube \( 3\frac{1}{4} \) inches, the velocity of efflux then follows:

\[ v = 8,02 \cdot \sqrt{\frac{3.5}{150} \cdot 0.75} = 8,02 \cdot 0.2415 = 1,937 \text{ feet, and the discharge} \]

\[ Q = \frac{\pi}{4} \cdot \left(\frac{3.5}{12}\right)^{\frac{1}{2}} \cdot 1,937 = 0.474 \text{ cubic feet.} \]