CHAPTER III.

ON COMBINATIONS FOR PRODUCING AGGREGATE PATHS.

432. I HAVE already stated in the beginning of this work (Art. 39), that pieces in a train may be required to describe elliptical, epicycloidal, or sinuous lines, and that such motions are produced by combining circular and rectilinear motions by aggregation. The process being, in fact, derived from the well-known geometrical principle by which motion in any curve is resolved into two simultaneous motions in co-ordinate lines or circles.

If the curve in which the piece or point is required to move be referred to rectangular co-ordinates, let the piece be mounted upon a slide attached to a second piece, and let this second piece be again mounted upon a slide attached to the frame of the machine at right angles to the first slide. Then if we assume the direction of one slide for the axis of abscissæ, the direction of the other will be parallel to the ordinates of the required curve. And if we communicate simultaneously such motions to the two sliding pieces as will cause them to describe spaces respectively equal to the corresponding abscissæ and ordinates, the point or piece which is mounted upon the first slide will always be found in the required curve.

This first slide, being itself carried by a transverse slide, falls under the cases described in the first Chapter of this Part, and the motion may be given to it by any contrivance for maintaining the communication of motion between pieces the position of whose paths is variable, as, for example, by a rack attached to the slide and driven by a long pinion. For the purpose of communicating the velocities to the two slides, any appropriate contrivance from the first part of the work may be chosen.

433. If the curve in which the point is to move be referred to polar co-ordinates, these may be as easily translated into mechanism, by mounting the point upon a slide and causing this slide to revolve round a center, which will be the pole. Then connecting these pieces by mechanism, so that while the slide revolves round its pole the point shall travel along the slide with the proper velocity, this point will always be found in the given curve.

434. Fig. 223 is a very simple arrangement, by which a short curve may be described upon the above principles.

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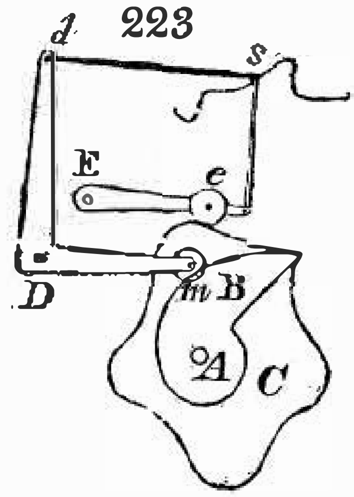

E is the center of motion of an arm *Ee* which is connected by a link with the describing point *s*; *D* is the center of motion of a second arm *Dd* which is connected by a link *ds*, with the same describing point *s*. If now *Ee* be made to move through a small arc, it will communicate to *s* a motion round *d* which will be nearly vertical, and if *Dd* be made to move through a small arc, it will communicate to *s* a motion round *e*, which will be nearly horizontal; and as the motion of the describing point *s* is solely governed by its connexion with these two links, these motions may be separately or simultaneously communicated to it. *A* is an axis, upon which are fixed two cam-plates, the lower of which, *C*, is in contact with a roller *e* at the end of the arm *Ee*, and the upper, *B*, in contact with a roller *m* at the end of an arm *Dm*, fixed at right angles to the arm *Dd*.

When the axis A revolves the cams communicate simultaneously motions to the two arms, which motions are given to the describing point, one in a direction nearly perpendicular to the other, the point will thus describe a curve of which the horizontal co-ordinates are determined by the cam B, and the vertical by the cam C.

In practice the shape of the cams may be obtained by trial: the machine must be previously constructed, and plain disks of a sufficient diameter substituted for the cams, then if the required path of s be traced upon paper, and it be placed in succession upon a sufficient number of positions upon this path, the cam-axis being also shifted, the corresponding positions of the rollers e and m may be marked upon the disks, and the shape of the cams thus ascertained.

435. If the object of the machine be merely to trace a few curves upon paper or other material, the principle of relative motion* will enable us to dispense with the difficulties that are introduced by the necessity of maintaining motion with a piece whose path itself travels. For since every complex path is resolvable into two simple paths, let the describing point move in one component path, and the surface upon which it traces the curve move in the other component path with the proper relative velocity, then will the curve be described by the relative motion of the point and surface.

Thus to describe polar curves, the surface upon which the curve is to be described may be made to revolve while the describing point travels with the proper velocity along a fixed slide, in a path the direction of which passes through the axis of motion of the surface. And as in this arrangement the axis of motion of the surface and the path of the describing point are both fixed in position, the simultaneous

* Already employed in Arts. 256, 404, 405.

motions may be communicated to them by any of the contrivances in our first Part, without having recourse to the principle of Aggregate Motion. And thus, in general, a firmer and simpler machine will be obtained.

Also the tracing of curves upon a surface is sometimes accomplished under the Aggregate principle by causing the *surface* to move with the double motion, while the describing point is at rest*.

436. Screw-cutting and boring machines are reducible to this head. For the cutting of a screw is in fact the tracing of a spiral upon the surface of a cylinder, and the motion of boring is also the tracing of a spiral upon the surface of a hollow cylinder; the tool being in both cases the describing point, and the plain cylinder the surface. Now as the tracing of this spiral is resolvable into two simultaneous motions, one of revolution with respect to the axis of the cylinder, and the other of transition parallel to that axis, we have in the construction of machines for boring and screw cutting the choice of four arrangements.

(1) The cylinder may be fixed and the tool revolve and travel. This is the case in all simple instruments for boring and tapping screws, in machines for boring the cylinders of steam engines, and in engineers' boring machines.

(2) The tool may be fixed and the cylinder revolve and travel. Screws are cut upon this principle, in small lathes with a traversing mandrel, as it is called.

(3) The tool may revolve and the cylinder travel. The boring of the cylinders of pumps is often effected upon this principle.

(4) The cylinder may revolve and the tool travel. Guns are thus bored, and engineers' screws cut in the lathe.

437. But motion in curves may be often more simply obtained by means of some geometrical property that may

* The motion which must be communicated to a plane to enable it to receive a given curve from a fixed describing point, is not the same as that which would cause a point, carried by the moving plane, to trace the same curve upon a fixed plane. Vide Clairaut, Mem. de l'Acad. des Sciences, 1740.

admit of being employed in mechanism, as the ellipse is described by the trammel fig. 224. This consists of a fixed cross $abcd$, in which are formed two straight grooves meeting in C, and perpendicular to each other; a bar PGH

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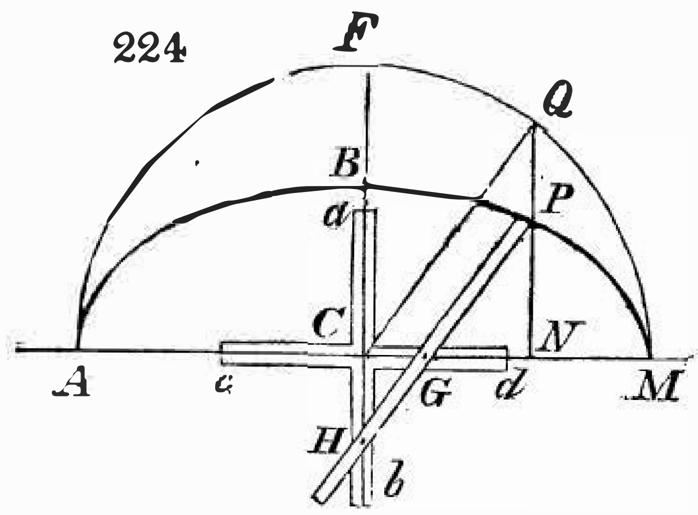

has pins attached to it at G and H, which fit and slide in these grooves, and a describing point is fixed at P. When the bar moves it receives simultaneously the rectilinear motion of the pin H in the groove ab, and that of the pin G in the groove cd, by which the describing point P traces a curve MPB, which can be shewn as follows to be the ellipse.

When HP coincides with ab, G comes to C, and therefore $GP = BC$, and when HP coincides with Cd, H comes to C and therefore $HP = CM$.

With center C and radius CQ equal to HP, describe a semicircle AFM, and through P draw QPN perpendicular to cd produced, join CQ, then QP is parallel to CH, also $HP = CM = CQ$, $\therefore$ $CHPQ$ is a parallelogram.

$$\therefore \frac{CQ}{GP} = \frac{QN}{PN}.$$

But $CQ = CF$ and $GP = BC$,

$$\therefore \frac{QN}{PN} = \frac{CF}{BC},$$

and the curve is an ellipse.

438. Thus also epicycloids or hypocycloids are described mechanically in Suardi's pen*, by fixing the describing point

* Adams' Geometrical and Graphical Essays.

at the end of a proper arm upon the extreme axis B, fig. 211, of an epicyclic train in the manner already explained in the first Chapter (Art. 386.) And in this instance we may also avail ourselves of the principles of Art. 435, and describe these curves by causing the plane and the arm which carries the describing point to revolve simultaneously with the proper angular velocity ratio, round parallel axes fixed in position.

439. But the most extensively useful contrivance of this class is that which is termed a *parallel motion*, by which a point is made to describe a right line by the joint action of two circular motions, and as this is a contrivance of great practical importance, it is necessary to examine it in detail.

ON PARALLEL MOTIONS.

440. A parallel motion is a term somewhat aukwardly applied to a combination of jointed rods, the purpose of which is to cause a point to describe a straight line by communicating to it simultaneously two or more motions in circular arcs, the deviations of these motions from rectilinearity being made as nearly as possible to counteract each other.

The rectilinear motion so produced is not strictly accurate, but by properly proportioning the parts of the contrivance, the errors are rendered so slight that they may be neglected.

441. Let Aa, Bb, fig. 225, be rods capable of moving round fixed centers A and B, and let them be connected by a third rod or link ab jointed to the extremities of the first rods respectively, as in Art. 326. The rods Aa, Bb are termed radius rods. This system may be moved in succession through a series of positions, the principal ones of

which are indicated by the figures 1, 1, 2, 2, 3, 3, 4, 4, *a*, *b*,

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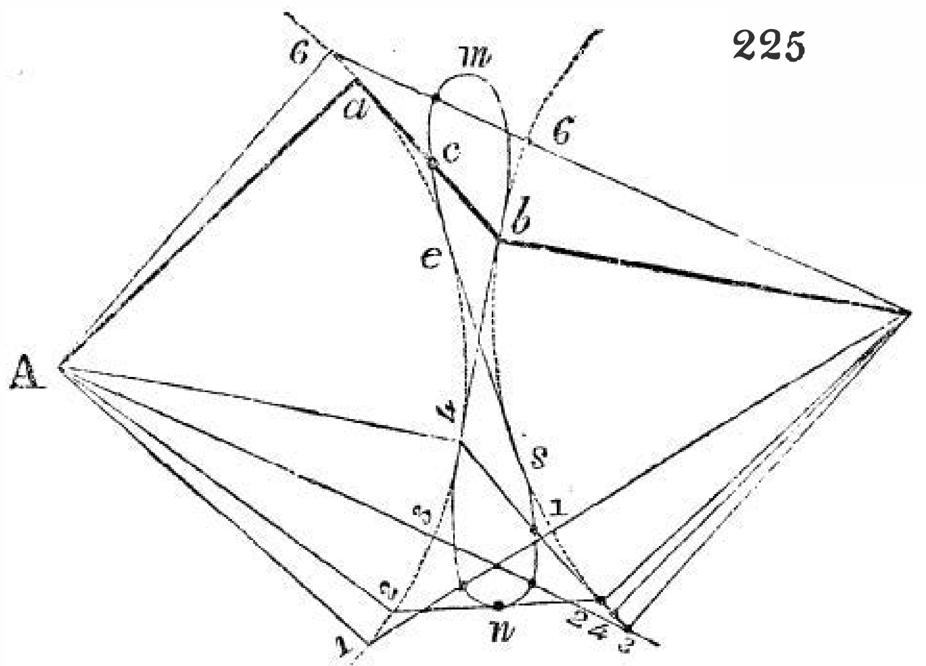

6, 6, 1, 1, and so on repeatedly. If a tracing point *c* be attached to some part of the link near its center, it will describe a curve ***mceεn4bm***, somewhat resembling the figure 8. If the position of the tracing point be properly assumed, a very considerable length of the intersecting portion of this curve will be found to approximate so nearly to a right line, that it may, for all practical purposes, be considered and employed as such.

442. For example, let *Ee*, fig. 226, be a crank or excentric, which, by its revolution is intended to communicate a reciprocating motion to the piston *P* through a link *ec*, jointed to the top of the piston rod *Pc*. In the common mode the upper end *c* of the piston rod would be guided in a vertical line, either by sliding through a collar or in a groove. If, however, the end *c* be jointed to the center of a link *ab* connecting two equal radius rods *Aa*, *Bb*, whose centers of motion *B*, *A* are attached to the frame of the machine; then

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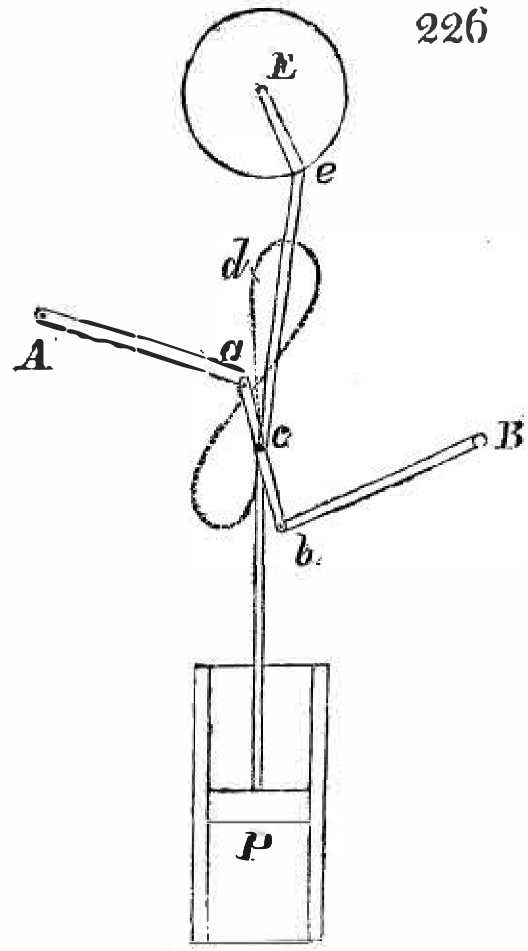

the path of *c* will be a certain segment *cd* of the curve described in Art. 441 ; and if the motion of *c* be not too great with respect to the length of the radius rods, this curve will vary so slightly from a right line that it may be safely employed instead of a sliding guide. An algebraical equation may be found for the entire curve*, but it is exceedingly involved and complex, and of no use in obtaining the required practical results, which are readily deduced by simple approximate methods, as follows.

443. Let *A*, *C*, fig. 227, be the centers of motion, *AB*, *CD* the radius rods, *BD* the link, and let the link be perpendicular to the two radius rods in the mean position of the system *ABDC*.

Let *AB* be moved into the position *Ab*, and *Cc*, *bc* be the corresponding positions of the other rod and the link. Draw *bf* parallel to *BD*. Now in the first position the link *BD* is perpendicular, and in the second position this link is thrown into the oblique position *bc*, by which the upper end is carried to the left, and the lower to the right of the vertical line *BMd*, through spaces *be*, *dc*, which are respectively equal to the versed sines of the angles described by

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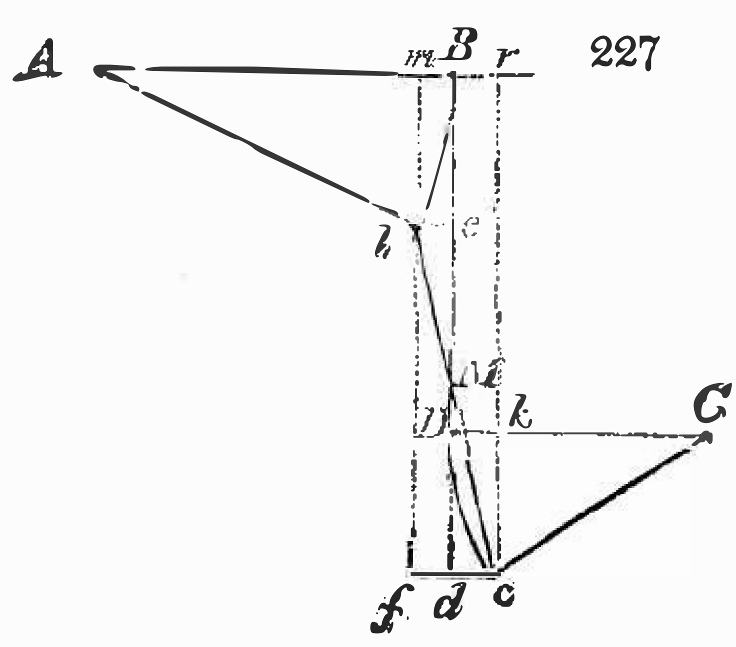


the radius rods *AB*, *DC* in moving to their second positions *Ab*, *Cc*. But as the ends of the link move different ways,

* This is completely worked out by Prony, Architecture Hydraulique, Art. 1478.

there will be one point M between them that will be found in the vertical line BMd, and its place is determined by the proportion (Art. 395).

$$bM : Mc :: be : dc.$$

Let $AB = R$, $CD = r$, $BD = l$,

$BAb = \theta$, $DCc = \phi$, and $bM = x$;

$$\therefore \frac{x}{l-x} = \frac{R \text{ versin } \theta}{r . \text{versin } \phi} = \frac{R . \sin^2 \frac{\theta}{2}}{r . \sin^2 \frac{\phi}{2}} = \frac{r}{R} \times \frac{R^2 . \sin^2 \frac{\theta}{2}}{r^2 . \sin^2 \frac{\phi}{2}}.$$

Now as the angle BAb never exceeds about $20°$ in practice the inclination cbf of the link is small, and $Bb (= R\theta)$ very nearly equal to $Dc (= r\phi)$; and as these angles are small we may assume without sensible error $R \sin \frac{\theta}{2} = r \sin \frac{\phi}{2}$;

$$\therefore \frac{x}{l-x} = \frac{r}{R}, \text{ and } x = \frac{lr}{R+r},$$

which is the usual practical rule.

This rule may be simply stated in words, by saying that the segments of the link are inversely proportional to their nearest radius rods.

Ex. Let $R = 7$ feet, $r = 4$ feet, $l = 2$ feet.

$$\therefore x = \frac{2 \times 4}{7+4} = \frac{8}{11} = .727 \text{ feet} = 8.72 \text{ inches}.$$

444. The deviation of the point M from the line BD may be measured with sufficient accuracy as follows, and it is necessary to know it in order to ascertain how great a value of the angle θ may be safely employed. For simplicity I shall confine myself to the case in which the radius rods AB, CD are equal in length, and taking their length equal to unity, let the link $BD = l$, draw bf, rc parallel to BD, and let the inclination fbc of the link to the vertical $= \gamma$;

$$\therefore\ \gamma = \frac{fc}{l},\ \text{since } fbc \text{ is small,}$$

$$= \frac{\text{versin}\,\theta + \text{versin}\,\phi}{l} = \frac{2\ \text{versin}\,\theta}{l},\ \text{very nearly.}$$

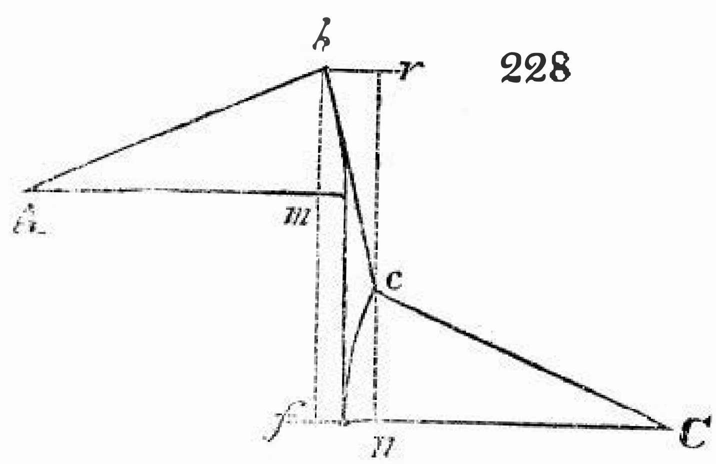
Now $mf = rc$, that is, $\sin\theta + l\cos\gamma = l + \sin\phi$;

$$\therefore\ \sin\phi = \sin\theta - l\ \text{versin}\,\gamma = \sin\theta - \frac{l\gamma^2}{2}$$

$$= \sin\theta - \frac{2}{l} \times (\text{versin}\,\theta)^2.$$

From this expression the value of ϕ corresponding to any given value of θ may be calculated.

When the radius rods are inclined upwards, we have (fig. 228) retaining the same notation,



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$$bm + mf = rc + cn,\ \text{ that is, }\ \sin\theta + l = l \cdot \cos\gamma + \sin\phi;$$

$$\therefore\ \sin\phi = \sin\theta + \frac{2}{l} \times (\text{versin}\,\theta)^2.$$

Let the link be half of the radius rods; therefore

$$\sin\phi = \sin\theta \pm 4\,(\text{versin}\,\theta)^2,$$

where the upper sign is taken when the radius rods are above the horizontal line, and the lower sign when below.

Also the deviation of the central point M of the link from a vertical right line is equal to

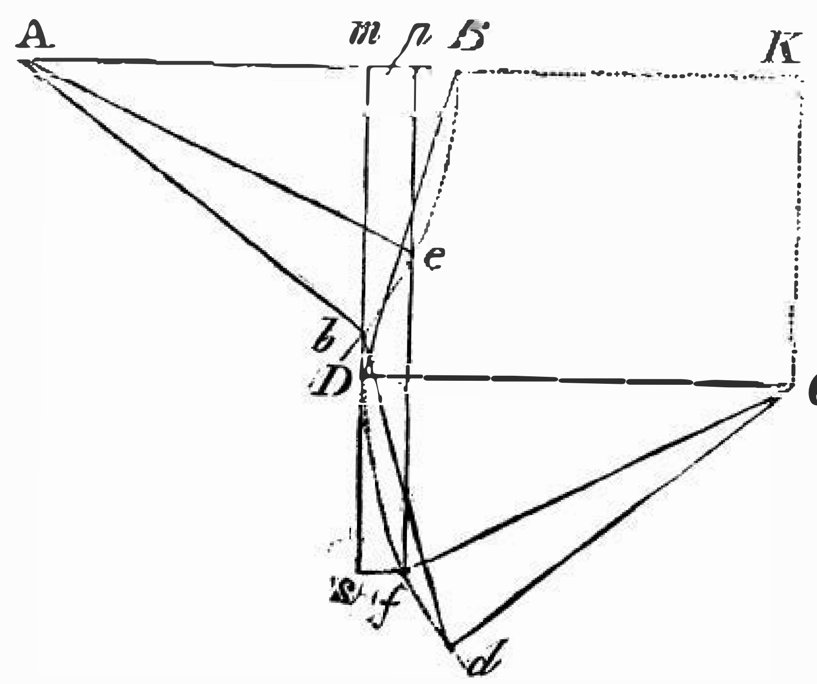
$$\frac{\text{versin}\,\theta}{2} - \frac{\text{versin}\,\phi}{2} = \frac{\cos\phi - \cos\theta}{2}.\quad \text{(Art. 395.)}$$

and the actual values of ϕ, and of the deviation which correspond to the principal values of θ, are given in the following table.

Values of θ.	ABOVE HORIZONTAL LINE.		BELOW HORIZONTAL LINE.	
	Values of ϕ.	Deviation.	Values of ϕ.	Deviation.
25^0	$27^0\ 15'$	.00864	$22^0\ 48'$	.00777
20^0	$20^0\ 54'$	.00274	$19^0\ 7'$	.00258
15^0	$15^0\ 17'$	.00064	$14^0\ 44'$	.00060
10^0	$10^0\ 3'$	.00007	$9^0\ 57'$	.00007

Thus if the radius rod or *beam* AB have 3 ft. radius, the deviation at 25^0 amounts to $.0086 \times 36$ inches $= .31$ inches, and at 20^0 to .097 inch.; generally the entire beam is made equal to three times the length of the stroke, and therefore describes an angle of about 19 degrees on each side of the horizontal line.

445. Even this error may be greatly reduced by a different mode of arranging the rods. Supposing the rods to be of equal length and equal to unity, let Ab, fig. 229, be the extreme angular position of the rod AB, let $BAb = \theta$, and let the horizontal distance AK of the centers of motion A, C, be made equal to $AB + CD -$ versin θ,



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instead of being equal to the sum of the radii AB, CD, as in the former case. In this arrangement the radii being supposed parallel in the first position AB, CD, it is clear from the mere inspection oftthe figure, that the link is inclined to the left in one position as far as it is inclined to the right in the other very nearlyt; and therefore the central point of the link in the lowest position *bd* will be in the vertical line which passes through the place of its central point in the position *BD*.

But as the link is continually changing its inclination in the intermediate positions between these two, there will be in these intermediate positions a deviation of the central point from this vertical line, which it is easy to see will be at a maximum when the link is vertical. Let this happen when the radius rod is at an angle $BAe = \theta_{,}$,

$$\text{and let } DCf = \phi_{,}, \text{ and } mDB = dbs = \gamma;$$

$$\text{then we have } mD + Ds = pe + ef,$$

$$\text{that is, } l \,.\cos\gamma + \sin\phi_{,} = \sin\theta_{,} + l;$$

$$\therefore\ \sin\phi_{,} = \sin\theta_{,} + l\,.\,\text{versin}\,\gamma = \sin\theta_{,} + \frac{l\,.\,\gamma^2}{2}.$$

$$\text{But } \gamma = \frac{Bm}{l} = \frac{\text{versin}\,\theta}{l};$$

$$\therefore\ \sin\phi_{,} = \sin\theta_{,} + \frac{(\text{versin}\,\theta)^2}{2l},$$

also the deviation of the middle point $= \dfrac{\cos\phi_{,} - \cos\theta_{,}}{2}$.

The following table exhibits the corresponding values of the angles and deviation, supposing as before that $l = \frac{1}{2}$; and also that versin $\theta = 2$ versin $\theta_{,}$ which is very nearly true.

θ	$\theta_{,}$	$\phi_{,}$	Deviation.
20^0	$14^0\ 7'$	$14^0\ 20'$	.00046
25^0	$17^0\ 36'$	$18^0\ 8'$	.00143
30^0	$21^{\bullet}\ 1'$	$22^0\ 6'$	.00347
35^0	$24^0\ 33'$	$26^0\ 38'$	.00785

In practice the angle θ never exceeds 20^0. Let the radius rods be 3 feet in length, then the deviation in inches is $36 \times .0005 = .018$ instead of $.097$, as in Art. 444.

446. If the radius rods AB, DC are arranged on the same side of the link CB, and the link be produced downwards, as in fig. 230, then the upper rod being made shorter

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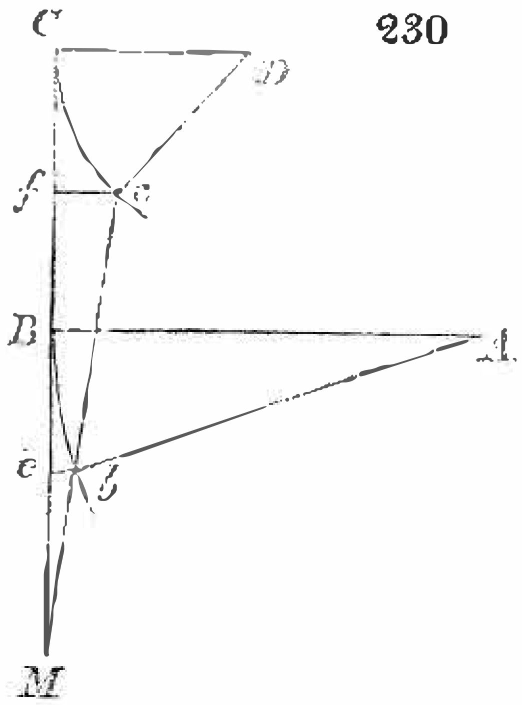

than the lower will move through a greater angle, and carry the upper end c of the link through a deviation cf greater than be, which is that produced by the longer rod. There will therefore be a point M in the link below the lower rod, which will remain in the line CB produced; and this point will be found by the proportion

$$\frac{cM}{bM} = \frac{fc}{be} = \frac{r \text{ versin } \phi}{R \text{ versin } \theta} = \frac{R}{r} \times \frac{r^2 \sin^2 \frac{\phi}{2}}{R^2 \sin^2 \frac{\theta}{2}},$$

when $AB = R$, $CD = r$, $BAb = \theta$, $CDc = \phi$.

But $r \sin \frac{\phi}{2} = R \sin \frac{\theta}{2}$ very nearly, whence $\frac{cM}{bM} = \frac{R}{r}$ gives the position of the point M.

447. The complete parallel motion which is most universally adopted in large steam-engines is shewn in fig. 231.

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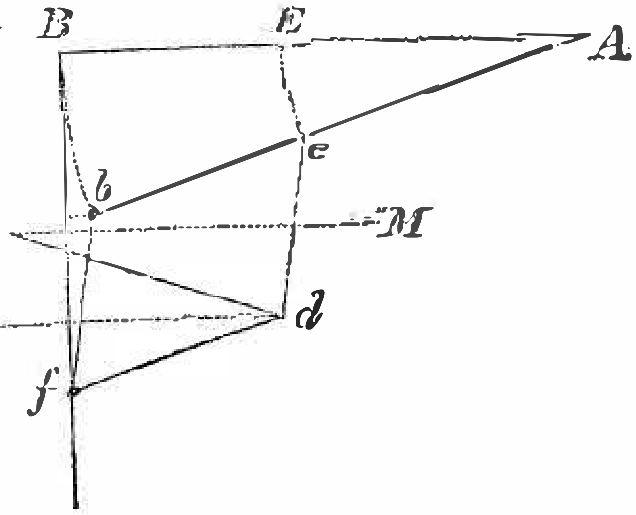

When so employed the beam of the engine becomes one of the radius rods of the system. Ab is half this beam whose center of motion is A. It has two equal links ed, bf jointed to it, of which bf is termed the *main link*, and ed the *back link*, and these are connected below by a third link df, termed the *parallel rod*, and equal to be. The radius rod or *bridle-rod* Cd is jointed to the extremity d of the back link ed, and its center C is fixed at a vertical distance below A equal to ed or bf. The length of the rods are so proportioned that f shall be the point to which the rectilinear motion is communicated, or *parallel point* as it is termed. To find the proportions let

$$Ae = R, \quad be\ (= fd) = R_{,}, \quad Cd = r,$$

draw Kd parallel to AB;

$$\therefore\ Kdf = BAb (= \theta), \text{ and let } MCd = \phi,$$

then, as before, the point d is carried towards K through a space equal to Cd versin $\phi = r$ versin ϕ, and the point f receives simultaneously this motion towards K, and a motion in the opposite direction arising from the inclination of the parallel rod df, which motion is equal to df versin fdK $= R_{,}$ versin θ. If these two motions be equal the point f will remain in the vertical line Bf, as required;

$$\therefore\ r \cdot \text{versin}\, \phi = R_{,}\ \text{versin}\, \theta, \text{ or } \frac{r}{R_{,}} = \frac{\sin^2 \frac{\theta}{2}}{\sin^2 \frac{\phi}{2}}.$$

But the rods Ae, Cd, connected by the link ed, form a system similar to that of Art. 443, and, as before, we may assume

$$Ae.\sin\frac{\theta}{2} = Cd\sin\frac{\phi}{2}, \quad \text{or } R\sin\frac{\theta}{2} = r.\sin\frac{\phi}{2} \text{ very nearly;}$$

$$\therefore \frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\phi}{2}} = \frac{r^2}{R^2} = \frac{r}{R_{,}}; \quad \therefore\ r = \frac{R^2}{R_{,}},$$

that is, Ae is a mean proportional between cd and df.

448. Since the joints Ae, ed, Cd, considered separately, form a system similar to the first simple arrangement, it follows that if the proper point be taken between d and e an additional parallel motion is obtained; so that this form combines two parallel motions in one, and is commonly so employed in steam engines, by suspending the great piston rod from f and the lesser air-pump rod from the link ed. The three parallel motions described (figs. 227, 230, and 231) are all due to Mr. Watt, and are to be found in his patent of 1784.

449. Let $AbfedC$, fig. 232, be an arrangement similar to the last; produce bf, and make $bp = ed\,\frac{Ab}{Ae}$.*

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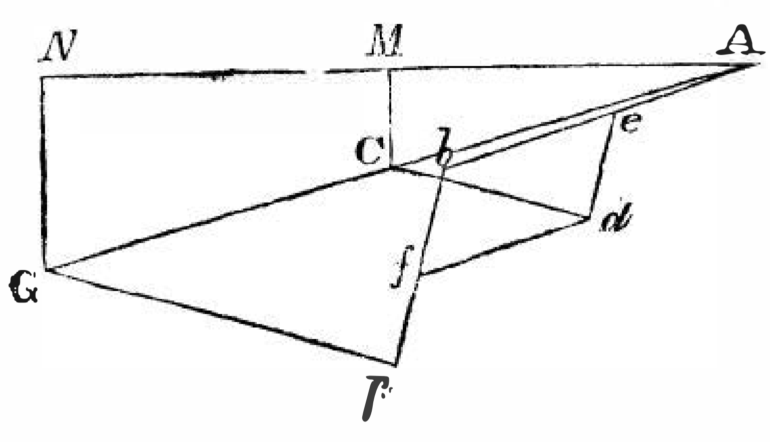

Join AC and produce it to G, making $AG = AC\,.\,\frac{Ab}{Ae}$, join Gp.

* From Prony, Arch. Hyd. Art. 1491.

Suppose Gp to be a new radius rod, moving round a fixed center G, it is clear in all positions of this arrangement that the lines Gp and Cd, bp and ed will remain parallel, on account of the fixed proportion of these lines respectively, therefore the point f would describe its straight line if fd were removed. But in that case the arrangement Ab, bp, pG considered separately forms a simple parallel motion of the first kind, and it appears that the more complex arrangement is equivalent to a simple one, occupying a greater space in the proportion of $AN : AM :: Ab : Ae$. Hence the convenience of the complex system.

450. There are various modifications of the latter arrangements, but the proportions of the rods may always be found in a similar manner to those already given. For example, in steam boats the beam is placed below the machinery, and the entire arrangement of the parallel motion inverted and otherwise altered to accommodate it to the necessity of compressing the entire machine into the smallest possible space.

Fig. 233 represents an arrangement of the parallel

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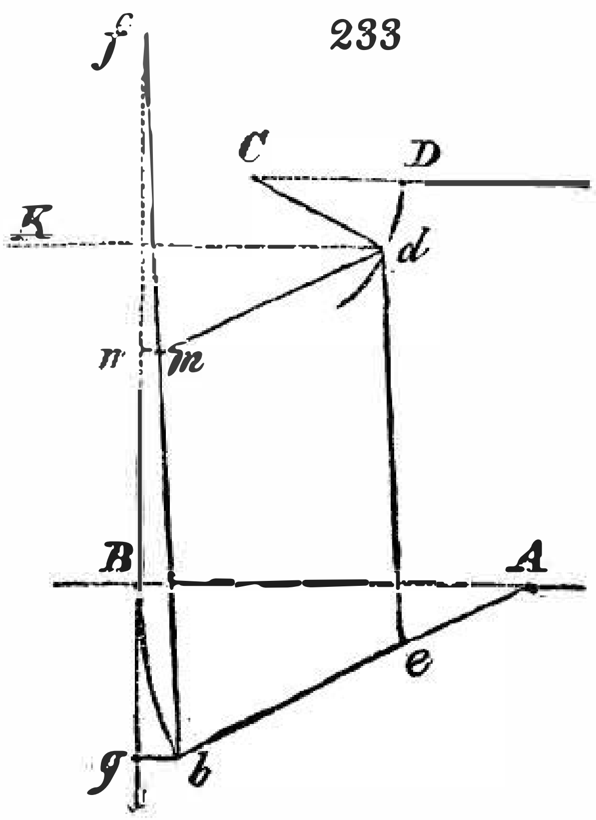

motion for steam boats, in which Ab is the beam, A its center of motion ; a short bridle rod, Cd, is employed, and the parallel rod dm is jointed to the main link bf below the parallel point f.

Let $Ae = R$, $eb = dm = R_,$, $Cd = r$, $DCd = \phi$, $BAb = \theta$. Draw AB, Kd horizontal, and fB vertical ; then the point d is carried towards fB through a horizontal space

$$= Cd \text{ versin } DCd = 2r \,.\, \sin^2 \frac{\phi}{2}.$$

And the point m is carried horizontally to the left by this movement of d, and at the same time to the right through a space $= dm \times \text{versin } Kdm = 2R_, \sin^2 \frac{\theta}{2}$, since $dm = eb$ and $Kdm = BAb$.

The horizontal deviation of m from the vertical fB is therefore equal to $mn = 2R_, \sin^2 \frac{\theta}{2} - 2r \sin^2 \frac{\phi}{2}$.

Also the deviation of b from the vertical fB, is equal to $bg = Ab \times \text{versin } BAb = 2 \,.\, \overline{R + R_,} \,.\, \sin^2 \frac{\theta}{2}$,

and since f is the parallel point, we have

$$\frac{fm}{fb} = \frac{nm}{bg} = \frac{R_, \sin^2 \frac{\theta}{2} - r \sin^2 \frac{\phi}{2}}{\overline{R + R_,} \,.\, \sin^2 \frac{\theta}{2}}.$$

But in the system Cd, de, eA, we may assume

$$r \sin \frac{\phi}{2} = R \sin \frac{\theta}{2}, \quad \therefore \sin^2 \frac{\phi}{2} = \frac{R^2}{r^2} \sin^2 \frac{\theta}{2};$$

$\therefore$ putting $fm = x$, and $mb = l$, and arranging the terms,

we have $\dfrac{x}{l + x} = \dfrac{R_, r - R^2}{(R + R_,)\, r}$, and $x = \dfrac{l}{R} \cdot \dfrac{R_, r - R^2}{R + r}$.

If $R_, r = R^2$, $x = 0$ and the parallel point coincides with m, as in Art. 447. If $R_, r < R^2$, x becomes negative and the parallel point will fall between m and b.

451. Let an isosceles triangle *GFE*, fig. 234, be suspended by two equal radius rods *CE*, *AF*, moving on fixed centers *A* and *C*, and jointed to the two extremities of the base *FE* respectively.

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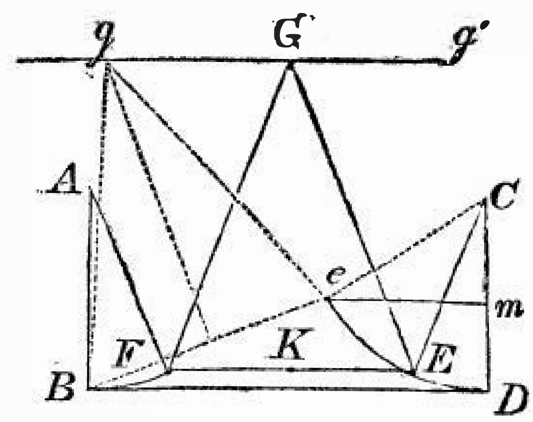

If now this triangle be swung from its central position, (that is, when the apex is equidistant from the points of suspension *A* and *C*), so as to carry its apex *G* to a little distance on either side, as for example to the position *g*, and to a similar one on the opposite side *g'*, then a describing point at *G* will draw a curve which will be found to vary very little from a right line whose direction is parallel to the base of the triangle when in its central position *GFE*, provided the proportions of the system be so arranged, that the three points *g G g'* are situated in a right line. This arrangement, which is the invention of Mr. Roberts of Manchester, furnishes a parallel motion which is in many cases more convenient than the former ones, especially if the path required be horizontal.

To investigate the proportions, draw the arcs *FB*, *DE*, make *AB*, *CD* perpendicular to *FE* and join *BD*. Let the extreme position be that in which the radius rod *AF* becomes perpendicular and coincident with *AB*, and the middle position that in which the base *FE* of the triangle is horizontal, and therefore parallel with *BD*. Then it remains to find such an altitude for the point *G*, that its vertical distance above *BD* may be the same in the middle and in the extreme position, in which case as the two extreme positions are symmetrical to the middle one, a right line parallel to *BD* will pass through the three positions of the apex *G*, as required.

Let $AB = CD = r$, $FE = b$, $BD = d$, $GK = h$,

$DCE = BAF = \theta$, $DCe = \phi$, $eBD = \psi$,

Then in the middle position, we have

$$2r \cdot \sin\theta + b = d, \qquad (1)$$

in the extreme position,

$$b \cos\psi + r \cdot \sin\phi = d, \qquad (2)$$

and also $b \cdot \sin\psi = r \cdot \text{versin}\,\phi$, (3).

Again, in the middle position, the altitude of G above BD is

$$h + r \cdot \text{versin}\,\theta,$$

and in the extreme position the altitude of g above BD is

$$h \cdot \cos\psi + \frac{b}{2}\sin\psi,$$

and these are equal by the conditions of the problem;

$$\therefore\ h + r \cdot \text{versin}\,\theta = h \cdot \cos\psi + \frac{b}{2}\sin\psi. \quad (4).$$

In these four equations we are at liberty to assume three of the quantities ϕ, ψ, θ, r, d, b, h, and the others may be determined, the most convenient is to assume values for r, d, and b. If $r = d = 1$, then the following table shews a few corresponding values of b and h.

b	h
$\frac{2}{3}$	3.95
.577	1.100
$\frac{1}{2}$	.943
.414	.654

But a convenient expression may be found by approximation, as follows: supposing that the angles of the system ϕ, θ and ψ are much smaller than those shewn in the figure;

for by (3) and (4) $\frac{h}{r} = \frac{\frac{1}{2}\text{versin}\,\phi - \text{versin}\,\theta}{\text{versin}\,\psi}$,

in which if we assume

$$\text{versin}\,\phi = \frac{em^2}{2r^2},\ \text{versin}\,\theta = \frac{\left(\frac{em}{2}\right)^2}{2r^2} = \frac{1}{4}\cdot\frac{em^2}{2r^2},$$

$$\text{versin}\,\psi = \frac{mD^2}{2b^2},\ \text{where } mD = \frac{em^2}{2r},$$

$$\text{and } em = d - b,$$

$$\text{we finally obtain } \frac{h}{r} = \frac{b^2}{(d-b)^2}.$$

452. Let the lever AB, fig. 235, be jointed at the extremity A to a rod or frame EA moving round a fixed center E, and so long that the small arc Aa, through which the extremity of the lever A moves, may be taken for a right line in the direction of the line AF. CD is a bridle rod whose fixed center of motion C is in the line AF.

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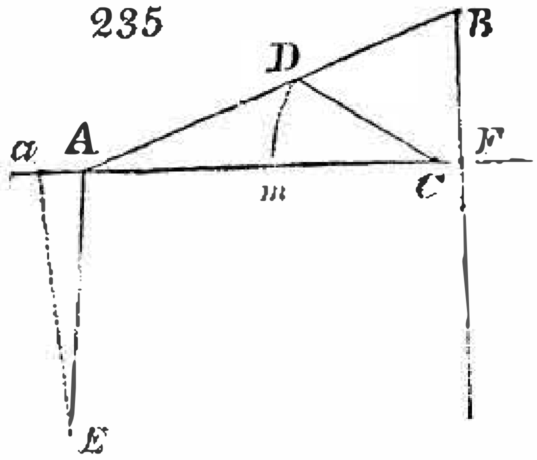

Let $CD = r$, $AD = R$, $DB = R_{,}$, $DCA = \phi$, $DAC = \theta$, then, supposing as before for convenience that the machine is in a vertical plane and the line AF horizontal, the point D is carried horizontally to the right through a space $= r$ versin ϕ, and the point B receives this motion, and is also carried to the left horizontally by means of its inclination through a space $= R_{,}$ versin θ, and if these be equal, the horizontal distance of B from A will be the same as when the rods coincided with the horizontal line AF; therefore we must have

$$R_{,}\ \text{versin}\,\theta = r\ \text{versin}\,\phi, \qquad (1)$$

$$\text{also } Dm = R\sin\theta = r\,.\,\sin\phi \qquad (2).$$

From these two equations the value of $R_{,}$ may be obtained for any given values of R, r and θ; also,

$$\text{since } R_{,} \,.\, \sin^2\frac{\theta}{2} = r\sin^2\frac{\phi}{2}, \quad \text{by (1)};$$

$$\text{and } R\sin\frac{\theta}{2} = r \,.\, \sin\frac{\phi}{2} \text{ very nearly, we obtain}$$

$$R_{,}r = R^2.$$

If the distances *AD*, *DC*, *DB* be equal, and the point *A* be made to travel in an exact straight line by sliding in a groove instead of the radial guide, then the parallel point will describe a true straight line perpendicular to *AF*, instead of the sinuous line which in all the other arrangements is substituted for it. For in this case the angle *DAF* is equal to *DCA* in all positions, and since $DB = DC$, a perpendicular from *B* upon *AC* will always pass through the same point *C*. In this respect this parallel motion has the advantage over all others.

If the friction of a sliding guide at *A* be considered objectionable, a small parallel motion of the first kind (Art. 443) may be substituted for it.

453. Toothed wheels are sometimes employed in parallel motions; their action is necessarily not so smooth as that of the link-work we have been considering, but on the other hand the rectilinear motion is strictly true, instead of being an approximation, as will appear by the two examples which follow.

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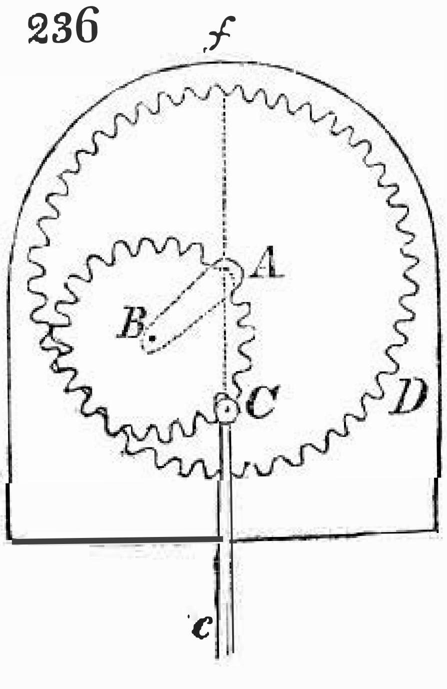

454. Ex. 1. In fig. 236 a fixed annular wheel *D* has an axis of motion *A* at the center of its pitch-line. An arm or crank *AB* revolves round this center of motion, and carries the center of a wheel *B*, whose pitch-line is exactly

of half the diameter of the annular wheel with whose teeth it geers. By the well known property of the hypocycloid any point C in the circumference of the pitch-line of B will describe a right line coinciding with a diameter of the annular pitch-circle. If then the extremity C of a rod Cc, be jointed to this wheel B by a pin exactly coinciding with the circumference of its pitch-circle, the rotation of the arm AB will cause C to describe an exact right line Cf, passing through the center A. This is termed White's parallel motion, from the name of its inventor*.

Since $AC = 2 . \cos BAC$, it is evident that the velocity ratio of C to BA is the same as in a common crank, and the motion produced in C equal to that which would be given by a crank with a radius equal to $2AB$, and an infinite link (Art. 328).

455. Ex. 2. Two equal toothed wheels, A and B, fig. 237, carry pins c and d at equal radial distances; and

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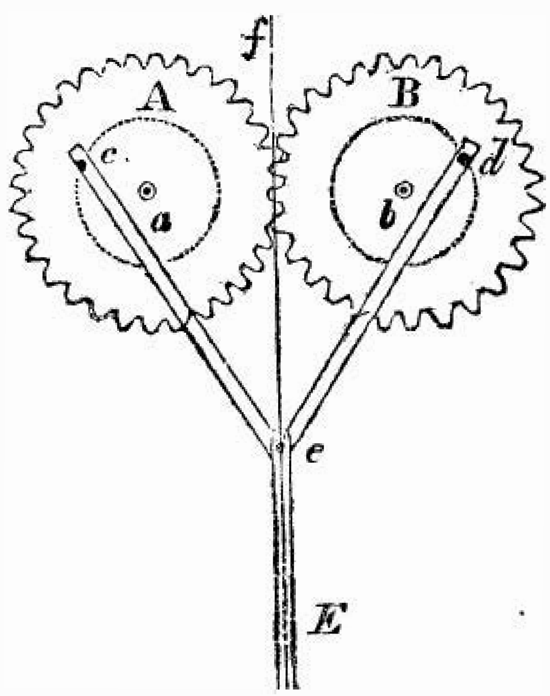

symmetrically placed with respect to the common tangent of the pitch-circles fe. If two equal links ce, de be jointed to these pins and to the extrémity of a rod eE, the point e will plainly always remain in the common tangent, by virtue of

* Vide White's Century of Inventions.

the similar triangles formed by the rods, the tangent *fe*, and the line *cd*.

The velocity ratio of *e* to the wheels is not however the same as that produced by the common crank and link of fig. 168, Art. 328, for the path of *e* does not pass through the center of motion of the crank.

If however r be the radius of the crank *ac* or *bd*, R the radius of the pitch-circles of the wheels, l the length of the link *ce* or *ed*, and the angle $cab = \frac{\pi}{2} + \theta$, then it can be easily shewn that the distance of *e* from the line of centers *ab* is equal to $\sqrt{l^2 - (R \pm r \sin \theta)^2} \pm r \cdot \cos \theta$.