## CHAPTER II.

## ON COMBINATIONS FOR PRODUCING AGGREGATE VELOCITY.

394. I SHALL in this Chapter proceed to shew the principal methods of obtaining the complex motion of a body in a given path by the simultaneous communication to it of two or more simple motions in that path; arranging the solutions under the same divisions as in the first part of this Work, but taking them in a somewhat different order, for the sake of convenience.

### BY LINK-WORK.

395. Let a bar $ABC$, fig. 203, be bisected in $B$, and let a small motion $Aa$ perpendicular to the bar be communi-

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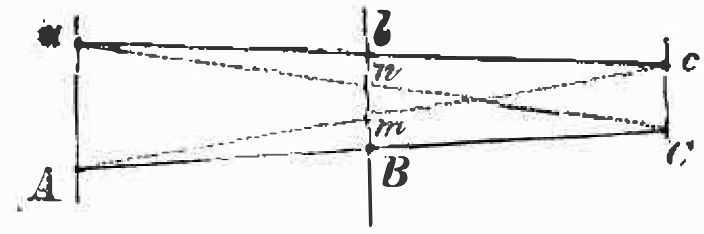


cated to the extremity $A$, $C$ remaining at rest; then will the central point $B$ move through a space $Bn = \frac{Aa}{2}$. On the other hand, had $A$ remained at rest, and a small transverse motion $Cc$ been given to the other extremity $C$, the central point $B$ would have moved through a space $Bm = \frac{Cc}{2}$. If these two motions are communicated either simultaneously or successively to the two extremities, the center $B$ will be carried through a space $Bb = \frac{Aa + Cc}{2}$. Or, if starting from the position $Ac$, the two motions had been communicated in the opposite directions, so as to carry the bar into the position $aC$, then the center of the bar would receive a motion $mn = \frac{Aa - Cc}{2}$. The length of the bar being always supposed so great, compared with the motions, that its inclination in the different positions may be neglected, and therefore the lines $Cc$, $Bb$, $Aa$, be all considered perpendicular to $AC$. Hence *two small independent motions being communicated to the extremities of a bar; its center receives half their sum or difference, according as the motions are in the same or in opposite directions.*

If the motions be communicated to $A$ and $B$, then $C$ will receive the whole motion of $A$ in the opposite direction, and twice the motion of $B$ in the same direction. The bar $AC$ has been divided in half at $B$ for simplicity only, for it is evident that by dividing it in any other ratio we can communicate the component motions in any desired proportions. But in general it is the law of motion which is to be communicated, and the quantity is of less consequence, especially if reduced for both motions in the same proportion.

396. Let $FG$, fig. 204, be a bar whose center is $E$, and to whose extremities are fixed pins $F$ and $G$, upon which the centers of other bars $AB$, $CD$, turn. Then if four independent motions be communicated to the points $A, B, C, D$, the motions of $A$ and $B$ will be concentrated upon $F$, and those of $C$ and $D$ upon $G$, and the motions of $F$ and $G$ being concentrated in like manner upon $E$, this point will receive the four motions. By jointing other levers to the extremities of these, and so

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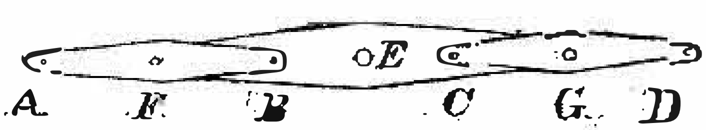

on, any number of independent motions may be concentrated upon the point $E$*.

## BY WRAPPING CONNECTORS.

397. If a bar $Bb$, fig. 205, be capable of sliding in the direction of its length and carry a pully $A$ round which is passed a cord $DE$, then it can be shewn in the same manner, that the bar will receive half the sum of independent motions communicated to the extremities $D$, $E$, the bar being supposed to be urged in the direction $bB$, by a weight or spring. This is a more compendious contrivance than the former, as the motions may be of considerable extent. If the component motions be communicated to one extremity of the string $D$ and to the bar, then will the other extremity $E$ receive the entire motion of $D$ in the reverse direction, and also twice the motion of $Bb$ in the same direction†.

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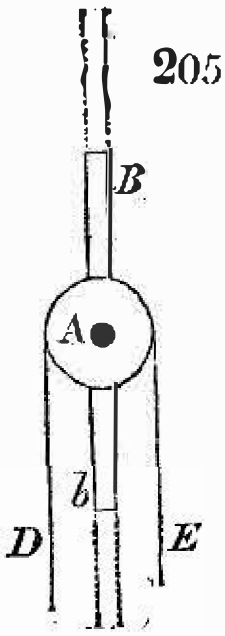



398. If a second similar combination be placed at the side of this, with its bar parallel to that of the first, and if a cord whose ends are tied to the upper extremities of each bar be passed over a third intermediate pully, the center of this latter pully will receive the aggregate motion of the cords of the two systems, as shewn for the lever in Art. 396.

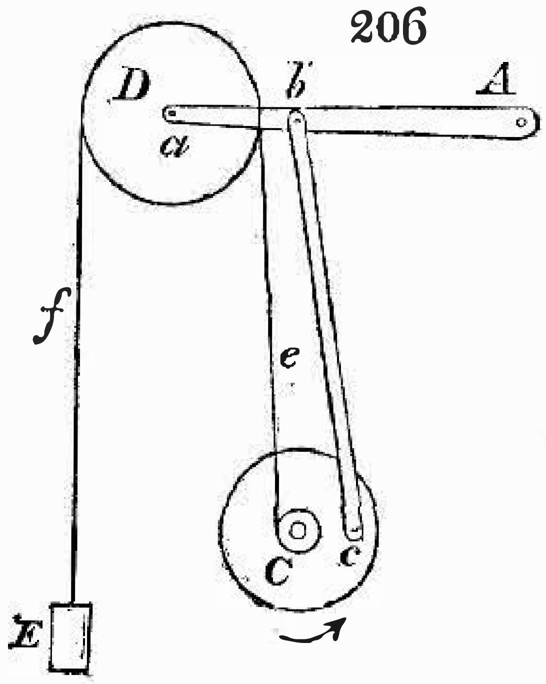
399. As an example of the employment of these combinations, let $C$, fig. 206, be an axis of motion upon which is fixed a small barrel round which the cord $e$ is rolled, and also a disk with an excentric pin $c$, which by means of a

* Another example of aggregate velocity by Link-work is the well-known reticulated frame termed *Lazy tongs*, which resembles a row of X's, thus ×××××. It is too weak from its numerous joints to be of much practical service.

† The first application of this principle appears to be the *Rouet de Lyon*, for winding silk. Vide Enc. Meth. Manufactures, t. II. p. 44.

link $cb$ communicates a reciprocating motion to an arm $Aa$, whose center of motion is $A$. The extremity of this arm carries a revolving pully $D$, and the cord which is coiled round the band is laid over this pully and fixed to a heavy piece $E$, which moves in the vertical path $Ef$. Now when $C$ revolves, the center $a$ of the pully $D$ moves up and down through a small arc which is nearly a right line parallel to $fE$, and by virtue of this motion the string $f$ and the body $E$ will receive a reciprocating motion of double its extent. But the string $e$ will be also slowly coiled upon the barrel by which it, as well as $E$, will receive a slow travelling motion in a constant direction upwards. By what has preceded, therefore, the body $E$ receiving these motions simultaneously, will, as in the example of Art. 387, move vertically with a reciprocating motion, of which the downward trip is shorter than the upward one.

206




400. Let $Aa$, fig. 207, be an axis to which are fixed two cylinders $B$ and $C$, nearly of the same diameter, and let a cord be coiled round $B$, passed over a pully $D$, and then brought back and coiled in the opposite direction round $C$. When $Aa$ revolves, one end of the cord will be coiled and the other uncoiled, and if $R$ be the radius of $B$, and $r$ of $C$, $A$ the angular velocity of the axis, the velocities of the two extremities of the cord will be $AR$ and $Ar$; and by Art. 397, the center of the pully $D$ will travel with a velocity equal to half the difference of these velocities, since they are in opposite

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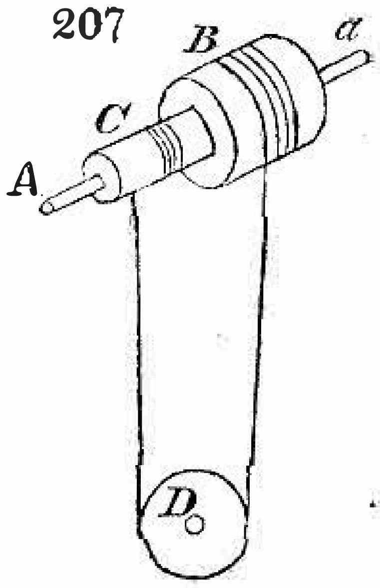


directions, or to $\frac{A(R-r)}{2}$. This velocity is the same as would be obtained if the center of the pully $D$ were suspended from the axis $Aa$ by a cord wrapped round a single barrel whose radius $= \frac{R-r}{2}$.

401. This combination belongs to a class which has received the name of *differential motions*, their object being to communicate a very slow motion to a body, or rather to produce by a single combination such a velocity ratio between two bodies that under the usual arrangement a considerable train of combinations would be required *practically* to reduce the velocity, for, *theoretically*, a simple combination will always answer the same purpose. Thus in the above machine, although theoretically a barrel with a radius $\frac{R-r}{2}$ would do as well as the double barrel, yet its diameter in practice would be so small as to make it useless from weakness. Whereas each barrel of the differential combination may be made as large and as strong as we please.

If a considerable extent of motion however be required, this contrivance becomes very troublesome, on account of the great quantity of rope which must be wound upon the barrels. For by one turn of the differential barrel the space through which the pully is raised $= \pi(R-r)$, but the quantity of rope employed is the sum of that which is coiled upon one barrel, and of that which is uncoiled from the other $= 2\pi(R+r)$. Now in the equivalent simple barrel the quantity of rope coiled is exactly equal to the space through which the body is moved, and therefore in this case $= \pi(R-r)$, so that for a given extent of motion

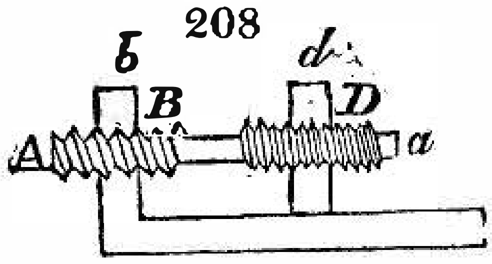
$$\frac{\text{rope for differential barrel}}{\text{rope for common barrel}} = 2\frac{R+r}{R-r},$$

when $R - r$ is by hypothesis very small. This inconvenience has been sufficient to banish the contrivance from practice, for although it is represented in all mechanical books under the name of the Chinese windlass, it is never actually employed.

## BY SLIDING CONTACT.

402. $Aa$, fig. 208, is an axis upon which are formed two screws $B$ and $D$, whose pitches are $C$ and $c$ respectively. $B$ passes through a nut $b$ fixed to the frame, and $D$ through a nut $d$, which is capable of sliding parallel to the axis of the screw*.

208




Now when a screw is turned round it travels with respect to its nut through a space equal to one pitch for each revolution, consequently one turn of $Aa$ will cause it to move with respect to $b$ through the space $C$. But the same motion will cause the nut $d$ to move with respect to its screw through a space $c$. The nut $d$, therefore, receives two simultaneous motions, for by the advance of the screw $Aa$ through the fixed nut $b$, the nut $d$ is carried forwards through the space $C$, but by the revolving action of the screw $Aa$ it will be at the same time carried backwards through the space $c$; its motion during one rotation of the screw $Aa$ is therefore equal to the difference of the two pitches $= C - c$. If $C$ be greater than $c$ this will be positive, and the nut will advance slowly when the screw $Aa$ advances; but if $c$ be greater than $C$, the nut will move slowly in the opposite direction to the endlong motion of the screw. If $C = c$ then $C - c = 0$, and the nut $d$ receives no motion, which is indeed obvious. All this supposes that the threads of the two screws are both right-handed or both left-handed. If

* This contrivance is claimed by White, (Century of Inventions, p. 84,) and also for M. Prony, by Lanz and Betancourt, (Essay, D. 3).

one be right-handed and the other left-handed, each revolution of the screw $Aa$ will cause the nut $d$ to advance through a space $= C + c$.

403. In fig. 209*, $Ff$ is a screw which passes through a nut $g$, this nut is mounted in a frame so as to be capable of revolving but not of travelling endlong in the direction of the axis of the screw. So that if the nut were turned round, and the screw itself prevented from revolving, this screw would receive an endlong motion in the usual manner, at the rate of one pitch for each revolution of the nut. A toothed wheel $E$ is fixed to the nut, and engaged with a pinion $C$, which is fixed to the axis $Aa$, parallel to the screw. To the screw is also fixed a toothed wheel $D$, which engages with a long pinion $B$ upon the same axis $Aa$ which carries the pinion $C$. When $Aa$ revolves therefore, it communicates rotation both to the screw and to the nut. If $B$ and $C$, $D$ and $E$ were respectively equal, it is plain that the nut and screw would revolve as one piece, and consequently no relative motion take place between them; but as these wheels are purposely made to differ, the nut and screw revolve with different velocities, and thus a motion arises between the nut and its screw, which causes the latter to travel in the direction of its length, with a velocity ratio that may be thus calculated.

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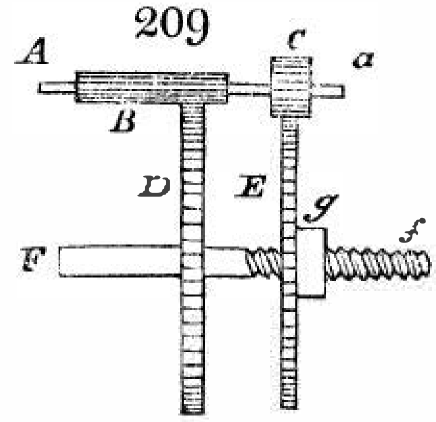


Let the letters $B\,C\,D\,E$ applied to the wheels, represent their respective numbers of teeth, and let $P$ be the pitch of the screw. Also, let the synchronal rotations of the axis $Aa$, the nut and the screw, be $L\,L_n$, and $L_s$ respectively,

$$\therefore\ L_n = \frac{LC}{E} \text{ and } L_s = \frac{LB}{D}.$$

* This combination occurs in White's Century of Inventions.

But the endlong motion of the screw depends upon the *relative* rotations of the screw and nut, and not upon their absolute rotations. Now it is obvious, that if the screw make $L_s$ rotations, and the nut $L_n$ rotations in the same direction, that the screw and nut will have made $L_s - L_n$ rotations with respect to each other, and therefore that the screw will have advanced endlong through a space

$$= (L_s - L_n).P = L.P\left(\frac{B}{D} - \frac{C}{E}\right),$$

which may be made very small with respect to $L$.

This combination is applied to machinery for boring, for the motion of a boring instrument consists of a quick rotation combined with a slow advance in the direction of its axis, which is precisely the motion given to the screw $Ff$. Nothing more is therefore required than to fix the boring tool to one end of this screw.

The long pinion $B$ (Art. 390) is employed for the obvious purpose of maintaining the action of $B$ upon $D$ during the endlong motion of the screw, and this endlong motion is in fact the difference of two motions that are simultaneously given to the screw. For $Aa$ revolving, if $B$ and $D$ were removed the rotation of the nut would cause the screw to travel endlong with one velocity, and if $C$ and $E$ were removed instead of $B$ and $D$, then the rotation of the screw in its fixed nut would cause it to travel endlong with another velocity; but these two causes operating simultaneously, the screw travels with the difference of these velocities.

404. A slow relative rotative motion of two concentric pieces may be produced, as in fig. 210, in which $Dd$ is a fixed stud, $B$ an endless screw-wheel revolving upon the stud, and $C$ a second endless screw-wheel revolving upon the tube which carries the preceding wheel $B$. $A$ is an

endless screw so placed as to act at once upon both wheels*. Now if these wheels had the same number of teeth they would move as one piece, but if one of them has one or two teeth more or less than the other, this will not disturb the pitch of the teeth sufficiently to interfere with the action of the endless screw. And as the revolutions of this screw will pass the same number of teeth in each wheel across the plane of centers, it follows that when one wheel has thus made a complete revolution, the other will have made more or less than a complete revolution by exactly the number of deficient or excessive teeth.

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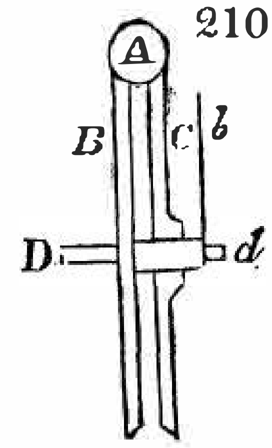



Let $B$ have $N$ teeth, and $C$, $\overline{N+m}$ teeth, then since the same number of teeth in each wheel will simultaneously pass the plane of centers, $N \times \overline{N+m}$ teeth of each will pass during $N$ rotations of $C$, and $\overline{N+m}$ of $B$, which are therefore their synchronal rotations, and their relative rotations in the same time are $\overline{N+m} - N = m$.

This contrivance is used in counting the revolutions of machinery, for by attaching an index to the tube which carries $B$, and graduating the face of $C$ into a proper dial-plate, $b$ revolves so slowly with respect to $C$, that it may be made to record a great number of rotations of $A$ before it returns again to the beginning of the course. Thus if $B$ have 100 teeth and $C$ 101, the hand will make one rotation round the dial during the passage of $100 \times 101$ teeth of either wheel across the plane of centers, that is, during 10100 rotations of the screw. Also the same hand $b$ may read off sub-divisions upon a small dial attached to the extremity of the fixed axis $d$.

405. This contrivance does not strictly belong to the problem we are at present considering, but it has a kind of

* From Wollaston's Odometer, for registering the number of turns made by a carriage-wheel.

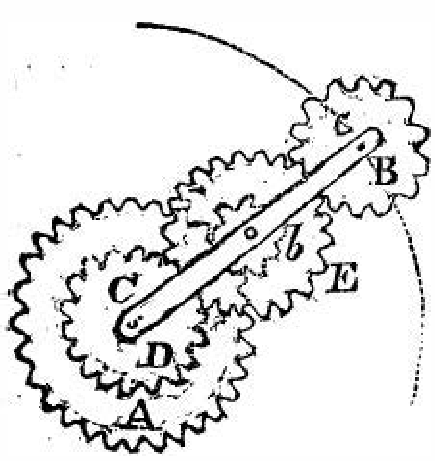
natural affinity with it that induced me to give it a place here. Similarly, a thick pinion upon an axis parallel to $Dd$, may be employed to drive the two wheels in lieu of an endless screw, but the relative motion will not be so slow*. But by employing two pinions of different numbers of teeth to drive the two wheels a very slow [relative] motion may be obtained; thus, if in fig. 209, the screw and nut be suppressed, and the wheel $E$ be the dial-plate, and the wheel $D$ carry the index, as in fig. 210, then we have found

$$\frac{L_s - L_n}{L} = \frac{C}{E} - \frac{B}{D},$$ which may be made very small.

## BY EPICYCLIC TRAINS.

406. A train of mechanism the axes of which are carried by an arm or frame which revolves round a center, as in figs. 211, 212, 213, is termed in this work an *Epicyclic train.*

211

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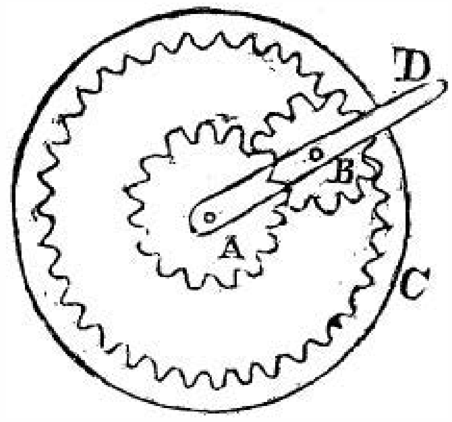


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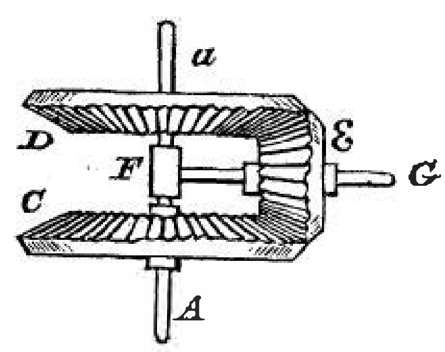



The two wheels which are at each end of such a train, or at least one of them, will be always concentric to the revolving frame.

Thus in fig. 211, $CB$ is the frame or *train-bearing arm*, a wheel $A$ concentric to this frame geers with a pinion $b$, upon whose axis is fixed a wheel $E$ that geers with a wheel $B$. And thus we have an epicyclic train $A$ (Art. 233)

$$\begin{array}{ll} A & \\ b \text{———} E & \\ & B, \end{array}$$

* This combination occurs in a clepsydra, by Marcolini, described in the notes to the ninth book of Vitruvius, by Dan. Barbaro, 1556. Vide also Art. 256.

of which if the first wheel *A* be fixed, and a motion be given to the arm, the train will then revolve round the fixed wheel, and the relative motion of the arm to the fixed wheel will communicate rotation through the train to the last wheel *B*; or the first wheel as well as the arm may be made to revolve with different velocities, in which case the last wheel *B* will revolve with a motion that will be presently calculated.

If the wheel *E*, instead of geering with *B*, be engaged with a wheel *D*, which, like the wheel *A*, is concentric to the arm, then we have an epicyclic train *A*

*b*——*E*

*D*,

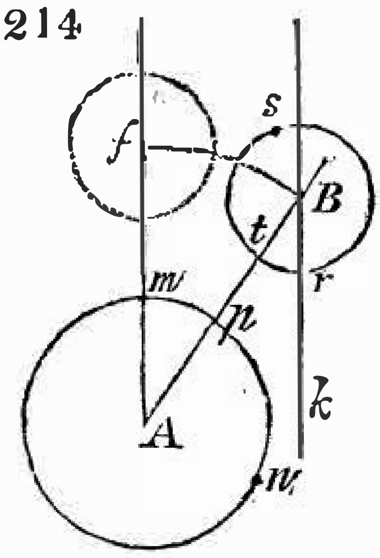
of which both the extremities are concentric to the arm. In such a train we may either communicate motion to the arm and one extreme wheel in order to produce an aggregate rotation in the other extreme wheel, or motion may be given to the two extreme wheels *A* and *B* of the train, with the view of communicating the aggregate motion to the arm.

Fig. 212 is a simple form of the epicyclic train, in which the arm *AD* carries a pinion *B*, which geers at once with a spur-wheel *A* and an annular wheel *C*, both concentric with the train-bearing arm.

Fig. 213 is another simple form in which *FG* is the arm, *Aa* the common axis; *D*, *C*, two bevil-wheels moving freely upon it, and *E* a pinion carried by the arm, and geering at once with the two bevil-wheels. These two arrangements contain the least number of wheels to which an epicyclic train can be reduced, if its two extreme wheels are to be concentric to the arm; and, as in fig. 211, motion may either be given to the two wheels in order to produce aggregate motion in the arm, or else to the arm and one wheel, in order to produce aggregate motion in the other. Or very commonly, one of the concentric wheels is fixed, and motion being then given to the arm, will be communicated to the other wheel, or *vice versa*, according to a law which we shall proceed to investigate. In these examples, toothed wheels only are employed, but the subsequent formulæ will apply as well to epicyclic trains in which any of the combinations of Class A are used.

407. *To find the velocity ratios of Epicyclic trains.* Let $AB$, fig. 214, be the train-bearing arm revolving round $A$, and carrying a train of which the first wheel $A$ is concentric to the arm, and the last wheel $B$ may either be concentric with $A$ or not. These two wheels are connected by a train of any number of axes carried by the arm or frame $AB$. Now the revolutions of the wheels of the train may be estimated in two ways; First, with respect to the *fixed* frame of the machine, that is, by measuring the angular distance of a given point on the wheel from the fixed line $Af$; or, if the wheel be excentric as $B$, from a line $Bk$ parallel to $Af$. Secondly, they may be measured with respect to the arm which carries them. The first may be termed the absolute revolutions, and the second the relative revolutions, or motions relative to the train-bearing arm.

214




Let the arm with its train move from the position $Af$ to $AB$, and during the same time let a point $m$ in the wheel $A$ move to $n$ from any external cause, and the point $r$ in the wheel $B$ move to $s$ by virtue of its connexion with the wheel $A$, all being supposed for simplicity to revolve in the same direction as the arm. Then $mAn$, $rBs$ are the *absolute* motions of the wheels $A$ and $B$, and $pAn$, $tBs$ their *relative* motions to the arm,

$$\text{but } mAn = mAp + pAn, \text{ and } rBs = rBt + tBS$$
$$= mAp + tBS;$$

where $mAp$ is the motion of the arm.

If, on the other hand, the wheels had moved in the opposite direction to the arm, then

$$mAn = pAn - mAp, \text{ and } rBs = tBs - mAp,$$

and these are true whatever be the magnitude of the angles described, and are therefore true for entire revolutions, for the angular velocity ratios in these trains are constant. Hence it appears that the absolute revolutions of the wheels of epicyclic trains are equal to the sum of their relative revolutions to the arm, and of the revolutions of the arm itself, when they take place in the same direction, and equal to the difference of these revolutions when in the opposite direction.

408. Let $a$, $m$, $n$, be the synchronal absolute revolutions of the train-bearing arm, of the first wheel of the train, and of the last wheel respectively; and let $\epsilon$ be the epicyclic train, that is, let it represent the quotient of the relative revolutions of the last wheel divided by those of the first; $\epsilon$ is therefore the quantity which is represented by $\frac{L_m}{L_1}$, or by $\frac{D}{F}$ in Chapter VII, the motions of the wheel-work being estimated with respect to the train-bearing arm alone. Also, the first and last wheel of the epicyclic train are included in the expression $\epsilon$, although one or both of them may be concentric to the arm.

Then the relative revolutions of the first wheel with respect to the arm $= m - a$, and of the last wheel $= n - a$, and as the motions of the train, considered with respect to the arm, will be the same as those of an ordinary train, we have $n - a = \epsilon \,.\, \overline{m - a}$.

$$\epsilon = \frac{n - a}{m - a};$$

$$\text{whence } a = \frac{m\epsilon - n}{\epsilon - 1}, \quad n = a + \overline{m - \alpha} \,.\, \epsilon,$$

$$\text{and } m = \alpha + \frac{n - \alpha}{\epsilon}.$$

If the first wheel of the train be fixed, which is a common case, its absolute revolutions = 0; $\therefore m = 0$, and we have

$$a = \frac{n}{1 - \epsilon}, \text{ and } n = \overline{1 - \epsilon} \,.\, a.$$

If the last wheel of the train be fixed, then $n = 0$, and

$$\text{we have } a = \frac{m\epsilon}{\epsilon - 1}, \text{ and } m = \left(1 - \frac{1}{\epsilon}\right) a.$$

But when these wheels are not fixed,

$$\alpha = \frac{m\epsilon - n}{\epsilon - 1} = \frac{m\epsilon}{\epsilon - 1} + \frac{n}{1 - \epsilon},$$

that is, the revolutions of the arm are equal to the sum of the separate revolutions which it would have received from the train, supposing its extreme wheels to have been fixed in turn.

In the formulæ of this Article the rotations of the first and last wheel and of the arm are all supposed to be in the same direction; if either of them revolve in the opposite, the sign of $m$, $n$, or $\alpha$ must be changed accordingly. With respect to the sign of $\epsilon$, see Art. 412.

409. But in trains of this kind it often happens that if neither the first nor last wheel of the epicyclic train be fixed, then either motion is communicated from some original driver to the two extreme wheels of the epicyclic train with a view to produce an aggregate motion of the arm, or else the original driver communicates motion to one of these extreme wheels and to the arm, for the purpose of producing the aggregate motion of the other extreme wheel.

Fig. 215 is an example of the first case. $mn$ is an axis to which is fixed the train-bearing arm $kl$, which carries the two wheels *d* and *e* united together and revolving upon the arm itself. The wheels *b* and *c* are united and revolve together upon the axis *mn*, but are not attached to it. Likewise the wheels *f* and *g* are fixed together, and revolve freely round the axis *mn*. The wheels *c*, *d*, *e*, and *f* constitute an epicyclic train, of which *c* is the first, and *f* the last wheel. An axis *A* is employed as a driver, and carries two wheels *a* and *h*, the first of which geers with the wheel *b*, and thus communicates motion to the first wheel *c* of the epicyclic train, and the wheel *h* drives the wheel *g*, which thus gives motion to the last wheel *f* of the epicyclic train. When the axis *A* is turned round it thus communicates motion to the two ends of the epicyclic train, through which the train-bearing arm *kl* receives an aggregate rotation, which we shall presently calculate.

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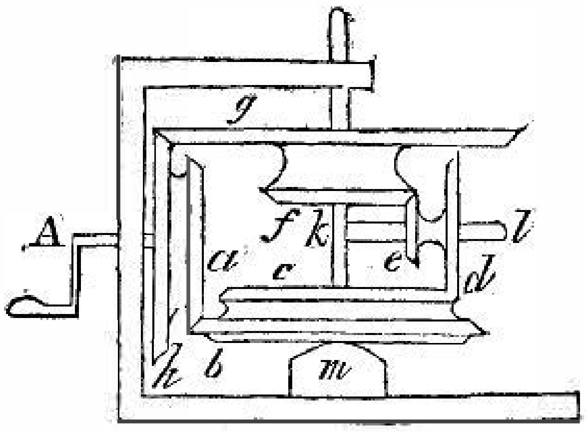



As an example of the second case, we must suppose the wheels *g* and *f* to be disunited, *g* being now *fixed* to the axis *mn*, and *f* only running loose upon it. The driving axis *A* will thus communicate, as before, rotation to the first wheel of the epicyclic train *c* by means of the wheels *a* and *b*, and will also by *h* cause the wheel *g*, the axis *mn*, and the train-bearing arm *kl* to revolve, by which the compound rotation will be given to the loose wheel *f*. In this second combination however, the last wheel *f* of the train is not necessarily concentric to the train-bearing arm, which it must be in the first case.

410. To obtain a formula adapted to this first case. Let the driving axis be connected with the first wheel of the train by a train $\mu$, and with the last wheel by a train $\nu$; and

let the synchronal rotations of this driver with these wheels be $p$;

$$\therefore \; m = \mu . p, \text{ and } n = \nu . p;$$

$$\therefore \; \frac{a}{p} = \frac{\mu\epsilon - \nu}{\epsilon - 1} = \frac{\mu}{1 - \frac{1}{\epsilon}} + \frac{\nu}{1 - \epsilon}.$$

The first part of which is due to the action of the train $\mu$, and the second to that of the train $\nu$.

For suppose the train $\mu$ removed, then would the firs wheel of the epicyclic train remain fixed, and $m = \mu p = 0$;

$$\therefore \; \frac{a}{p} = \frac{\nu}{1 - \epsilon},$$

and in like manner, if the train $\nu$ were removed,

$$\frac{a}{p} = \frac{\mu}{1 - \frac{1}{\epsilon}}.$$

The arm moves, therefore, with the sum or difference of the separate actions of the two trains from the original driving axis.

411. In the second case, let the driving axis be connected with the first wheel of the epicyclic train by a train $\mu$, and with the arm by a train $\alpha$,

$$\text{then } m = \mu p, \text{ and } a = \alpha p;$$

$$\therefore \; n = \alpha p \, \overline{1 - \epsilon} + \mu p \epsilon,$$

$$\frac{n}{p} = \alpha \, . \, \overline{1 - \epsilon} + \mu\epsilon.$$

The revolutions, therefore, of the last wheel of the epicyclic train are the aggregate of those due to the train $a$, which produces the motion of the arm, and of those due to the train $\mu$, which produces the motion of the first wheel of the epicyclic train.

412. The only difficulty in the application of these formulæ lies in the signs which must be given to the symbols of the trains. But these it must be remembered, are each of them the representatives of a fraction, whose numerator and denominator are respectively equal to the synchronal rotations of the last follower and first driver of the train.

One direction of rotation being assumed positive, the opposite one will be negative, and therefore if the extreme wheels revolve in the same direction, whether that be back or forwards, the symbol of the train will be positive; and if they revolve in the opposite direction it will be negative. The rotations of the train $\mu$, $\nu$ are *absolute;* and those of $\epsilon$ *relative to the arm.* To find the sign of $\epsilon$, we must suppose the arm to be for the moment fixed, and then analyse the train in the usual manner to find whether the motions of its extreme wheels are in the same or in opposite directions, and the directions of rotation must be estimated accordingly. In a similar way, the signs of $\mu$ and $\nu$ are easily determined by considering them separately, and observing whether their extreme wheels move in the same or in opposite directions. If in the same, then $\mu$ and $\nu$ have the same signs; and if in opposite, then different signs. In the formulæ the symbols are all supposed positive, and therefore in every particular case positive trains retain the signs which are already given to them in these formulæ, but negative trains take the opposite signs. And although the term epicyclic train strictly implies that all the axes of the train are carried excentrically round the centre of the arm, yet I must repeat that the first and last wheel must be included in it, although one or both may happen to be concentric with the arm.

413. Let, for example, these principles and formulæ be applied to the simple epicyclic trains in figs. 211, 212, 213,

and suppose the letters to represent the numbers of teeth. The epicyclic train formed by the wheels $A$, $B$, $C$, in fig. 212, is of such a nature that the extreme wheels $A$ and $C$ revolve in opposite directions, therefore $\epsilon$ is negative, and so also in the train $C, E, D$, in fig. 213, but in the train $\begin{array}{ll} A & \\ b\text{——} & E \\ & B \end{array}$ or $\begin{array}{ll} A & \\ b\text{——} & E \\ & D \end{array}$ of fig. 211, the extreme wheels revolve the same way, and therefore $\epsilon$ is positive. Also in fig. 211,

$$\epsilon = + \frac{AE}{bB}, \text{ in fig. } 212 \ \epsilon = - \frac{A}{C},$$

$$\text{and in fig. } 213 \ \epsilon = - \frac{C}{D} = -1.$$

Let the first wheels of these trains be fixed, then when the arm revolves we have

$$\text{for } 211. \quad n = \left(1 - \frac{AE}{bB}\right)a,$$

$$212. \quad n = \left(1 + \frac{A}{C}\right)a,$$

$$213. \quad n = 2a,$$

where $n$ and $a$ are the synchronal rotations of the last wheel of the train and of the arm respectively.

In fig. 213, therefore, it appears that when one wheel $C$ is fixed, the other revolves twice as fast as the arm in the same direction.

In fig. 215, in its first case $\epsilon = \frac{ce}{df}$, and if the arm were fixed, $c$ and $f$ would revolve opposite ways, therefore $\epsilon$ is negative; $\mu = \frac{a}{b}$ and $\nu = \frac{h}{g}$, also $g$ and $b$ revolve opposite ways, and therefore $\mu$ and $\nu$ must have different signs, and thus the formula becomes

$$\frac{a}{p} = \frac{\mu\epsilon - \nu}{1 + \epsilon} = \frac{\frac{ace}{bdf} - \frac{h}{g}}{1 + \frac{ce}{df}} = \frac{aceg - hbdf}{bg(df + ce)}.$$

But under the second case, $\epsilon$ is negative, as before;

$$\mu = \frac{a}{b} \quad a = \frac{h}{g},$$

and these have different signs;

$$\therefore \frac{n}{p} = a\,(1 + \epsilon) - \mu\epsilon = \frac{h}{g}\left(1 + \frac{ce}{df}\right) + \frac{ace}{bdf}.$$

414. Epicyclic trains are employed for several different purposes, each of which will be exemplified in turn.

(1.) For the representation of planetary motion, and for all machinery in which epicyclic motion is a part of the effect to be produced, as in the geometric pen and epicycloidal chuck, where real epicycloids are to be traced, or in the machinery for laying ropes. Some of these effects more properly belong to the next chapter.

In all these cases a frame containing mechanism is carried, by the action of machinery, round other fixed frames, and the motion can only be communicated to the machinery in this travelling frame upon the principle of epicyclic trains.

(2.) When a velocity ratio is required to be accurately established between two axes whose centers are fixed in position, and this ratio is composed of unmanageable terms when applied to the formation of a simple train, the epicyclic principle will generally effect the decomposition required, as we shall presently see.

(3.) For producing a small motion by what is termed the Differential principle, of which examples by other aggregate combinations have been already given.

(4.) To concentrate the effect of two or more different and independent trains upon one wheel or revolving piece, when one or both of them are variable in their action.

This was first applied to what are termed Equation clocks, in which the minute-hand points to true time, and its motion therefore consists of the equable motion of an ordinary minute-hand, plus or minus the equation, or difference between true and mean time.

The same principle has been applied with the greatest success to the bobbin and fly-frame.

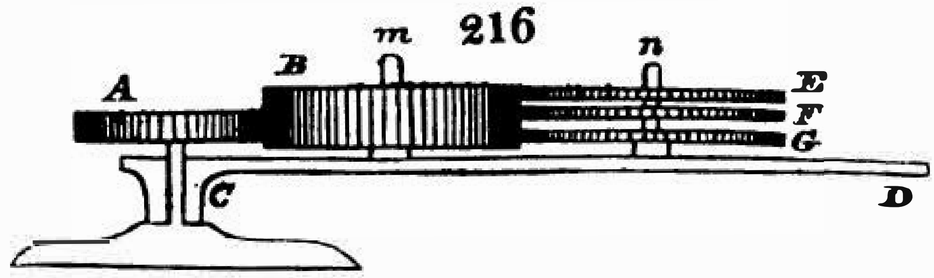
415. The train which is carried on the arm, and the arm itself, receive various forms; the train should be as light as possible, and consist of few wheels, especially when it revolves in a vertical plane; because being excentric its weight interferes with the equable rotation of the arm or wheel which carries it, unless it be balanced very carefully. When the excentric train is necessarily heavy, this difficulty is in some degree got over by making the train-bearing axis vertical, as in planetary machinery and in rope-laying machinery.

## EXAMPLES OF THE FIRST USE OF EPICYCLIC TRAINS.

416. Ex. 1. *Ferguson's Mechanical Paradox.*

This was contrived to shew the properties of a simple epicyclic train, of which the first wheel is fixed to the frame of the machine.

216




It consists of a wheel *A*, fig. 216, of 20 teeth, fixed to the top of a stud which is planted in a stand that serves to support the apparatus. An arm *CD* can be made to revolve round this stud, and has two pins *m* and *n* fixed into it, upon one of which is a thick idle wheel $B$ of any number of teeth, which wheel geers with $A$ and also with three loose wheels $E$, $F$, and $G$, which lie one on the other about the pin $n$.

When the arm $CD$ is turned round, motion is given to these three wheels which form respectively with the intermediate wheel $B$ and the wheel $A$ three epicyclic trains.

Now in this machine the extreme wheels of each epicyclic train revolve in the same direction, and therefore $\epsilon$ is positive, and the formula applicable to this case is $\frac{n}{a} = 1 - \epsilon$, where $n$ and $a$ are the absolute synchronal rotations of the last wheel and of the arm. But the object of this machine is only to shew the directions of rotation.

If $\epsilon = 1$ $\frac{n}{a} = 0$, and the last wheel of the train will have no absolute rotation. If $\epsilon$ be less than unity $\frac{n}{a}$ will be positive, and the last wheel will revolve absolutely in the same direction as the arm. But if $\epsilon$ be greater than unity $\frac{n}{a}$ will be negative, and the absolute rotations of the arm and wheel will be in opposite directions.

Let $E$, $F$, $G$ have respectively 21, 20, and 19 teeth, then

$$\text{in the upper train } \epsilon = \frac{A}{E} = \frac{20}{21}$$

is less than unity, and $E$ will revolve the same way as the arm.

$$\text{in the middle train } \epsilon = \frac{A}{F} = \frac{20}{20}$$

equals unity, $\frac{n}{a} = 0$ and $F$ will have no absolute revolution.

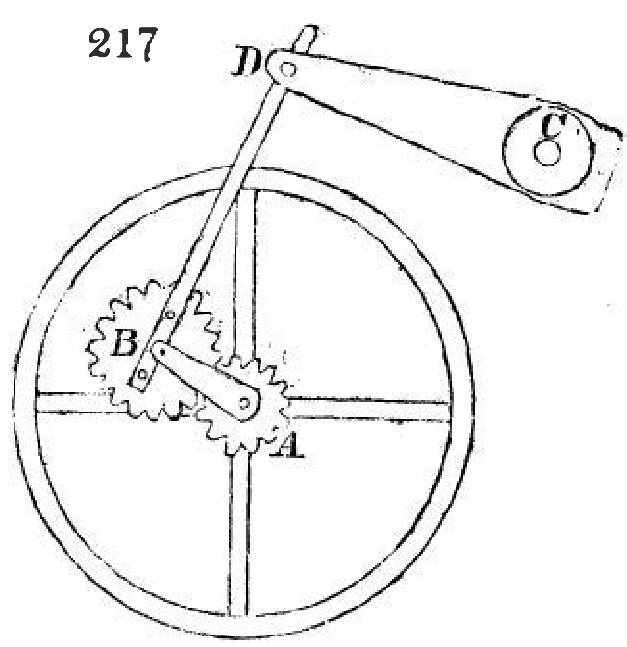
and in the lower train $\epsilon = \frac{A}{G} = \frac{20}{19}$

is greater than unity, and $G$ will revolve backwards.

It follows from this that when the arm is turned round, $E$ will revolve one way, $G$ the other, and $F$ will stand still, or rather continually point in the same direction. Which being an apparent paradox, gave rise to the name of the apparatus, which is well adapted to shew the more obvious properties of trains of this kind. But Ferguson was not the first who studied the motions of epicyclic trains; Graham's orrery in 1715, appears to be the original of this curious class of machinery, but for which no general formula appears to have been hitherto given*.

417. Ex. 2. The contrivance termed *sun and planet-wheels* was invented by Watt as a substitute for the common crank in converting the reciprocating motion of the beam of the steam engine into the circular motion of the fly-wheel. The rod $DB$, fig. 217, has a toothed wheel $B$ fixed to it, and the fly-wheel has a toothed wheel $A$ also attached

217




to it, a link $BA$ serves to keep these wheels in geer. Now when the beam is in action the link or arm $BA$ will be made to revolve round the center $A$, just as a common crank

* In Rees' Cyclopædia, Art. Planetary Numbers, are a few arithmetical rules for the calculation of planetary trains, given without demonstration.

would, but as the wheel $B$ is attached to the rod $DB$ so as to prevent it from revolving absolutely on its own center $B$, every part of its circumference is in turn presented to the wheel $A$, which thus receives a rotatory motion, the proportionate value of which is easily ascertained hy the formula already given.

The wheels $AB$ with the arm constitute an epicyclic train $\frac{A}{B} = \epsilon$, in which $\epsilon$ is negative, since the wheels revolve in opposite directions considered with respect to the arm, and in which the last wheel $B$ has no absolute rotation, being pinned to the arm $D$; the formula

$$m = a + \frac{n - a}{\epsilon}$$

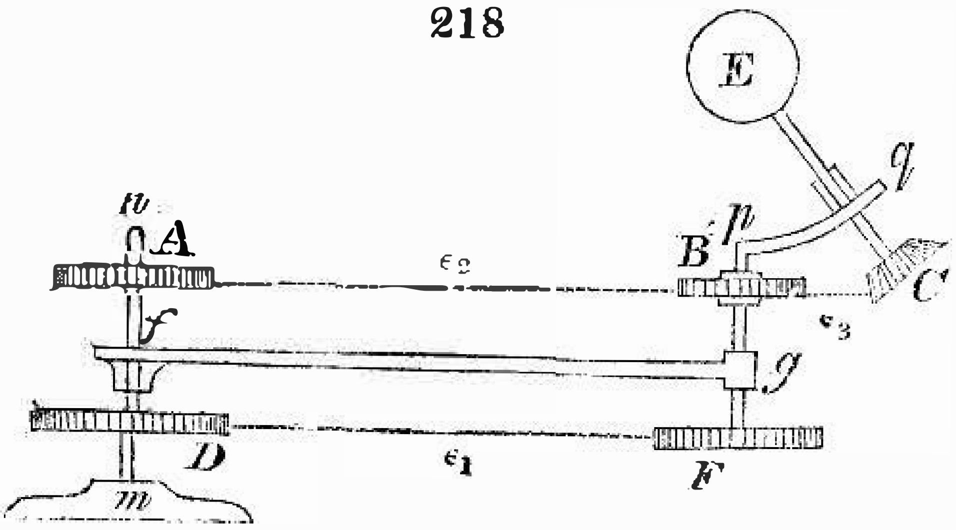
becomes

$$\left.\begin{array}{l}\text{making } n = 0 \\ \text{and } \epsilon = -\dfrac{A}{B}\end{array}\right\} \frac{m}{a} = 1 + \frac{B}{A}.$$

In Watt's Engine the wheels were equal and therefore $m = 2a$, and the fly-wheel revolved twice as fast as the crank-arm.

418. Ex. 3. *Planetary Mechanism.* $mn$ is a fixed central axis, upon which a train-bearing arm $fg$ turns, carrying two separate epicyclic trains $\epsilon_1$ and $\epsilon_2$.

218




One of these, $\epsilon_1$, has a first wheel $D$, and a last wheel $F$, connected by any train of wheel-work, and the axis of this last wheel passes through the end of the arm $fg$, and carries a second arm $pq$.

The other train $\epsilon_2$ has a first wheel $A$ connected to its last wheel $B$, by any train of wheel-work, but this last wheel is united to the first wheel of an epicyclic train $\epsilon_3$ borne by the arm $pq$, of which train the last wheel is $C$. The question is, to find the absolute rotations of this last axis. The arrangement is one that occurs in some shape or other in most orreries, for the purpose of representing the Diurnal rotation of the Earth's axis, in which case $fg$ is the annual bar, and $E$ a ball representing the Earth.

Let the absolute synchronal rotations of the bar $fg = a$, those of $D = m_1$; of $F$ (and therefore of the arm $pq$) $= n_1$; of $A = m_2$; of $B$ (and therefore of the first wheel of the train $\epsilon_3$) $= n_2$; and of $C$ (and therefore of the Earth) $= n_3$.

$$\text{Then } n_1 = a \,.\, \overline{1 - \epsilon_1} + m_1 \epsilon_1$$

$$n_2 = a \,.\, \overline{1 - \epsilon_2} + m_2 \epsilon_2$$

$$n_3 = n_1 \,.\, \overline{1 - \epsilon_3} + n_2 \epsilon_3.$$

In an orrery by Mr. Pearson for equated motions, described in Rees' Cyclopædia, the arm or annual bar $fg$, is carried round by hand, and the wheels $A$ and $D$ are fixed to the central axis. In this case $m_1$ and $m_2$ vanish, and we obtain the formula

$$\frac{n_3}{a} = 1 - \epsilon_1 + \epsilon_1 \epsilon_3 - \epsilon_2 \epsilon_3.$$

But the arm $pq$ which carries the Earth's axis must preserve its parallelism, and therefore having no absolute rotation $n_1 = 0$. The train $\epsilon_1$ will therefore $= +1$;

$$\text{(1.)} \qquad \text{and } \frac{n_3}{a} = \epsilon_3 - \epsilon_2 \epsilon_3 = \epsilon_3 \,.\, \overline{1 - \epsilon_2},$$

which must be positive, since the Earth performs its daily and annual revolutions in the same direction. The train $\epsilon_3$ in Mr. Pearson's orrery consists of three wheels of 40 each *en suite*; $\therefore\ \epsilon_3 = +1$,

$$\text{also his train } \epsilon_2 = \frac{269 \times 26 \times 94}{10 \times 10 \times 18},$$

in which the extreme wheels revolve in opposite directions, therefore $\epsilon_2$ is negative;

$$\therefore\ \frac{n_3}{a} = 1 + \frac{269 \times 26 \times 94}{10 \times 10 \times 18} = \frac{164809}{450}.$$

In making these calculations it must be remembered that the absolute period of $E$ is a sidereal day and its period relative to the arm $fg$ is a solar day, also the period of $fg$ is a year. Now from Art. 407 it appears that the absolute revolutions of any wheel or piece of an epicyclic train are equal to the sum of its relative revolutions and of the revolutions of the arm when they revolve in the same direction, and the same reasoning shews that the number of sidereal days in a year is equal to the number of solar days + 1.

Also $n_3$ and $a$ are the synchronal absolute rotations of the arm or annual bar $fg$, and Earth's axis $CE$; therefore $\frac{n_3}{a}$ = number of sidereal days in a year; but the fractions in Art. 247 represent the number of solar days in a year, and we may therefore employ them for $\frac{n_3}{a}$ by adding unity as above. We may thus obtain other and simpler trains than that already given. The train $\epsilon_3$ being carried by a small arm should be as simple and light as possible. But it may be reduced to only two wheels by making $\epsilon_3$ negative, and at the same time $\epsilon_2$ positive, since $\frac{n_3}{a}$ *must* be positive.

For example, employing the fraction $\frac{94963}{260}$ (vide p. 233) and remembering that the rotations $n_3$ are sidereal days, we have

$$\frac{n_3}{a} = 1 + \frac{94963}{260} = \frac{95223}{260} = \frac{3}{2} \times \left(\frac{7 \times 29 \times 157}{2 \times 5 \times 13} - 1\right),$$

which, compared with (1), gives

$$\epsilon_3 = -\frac{3}{2}, \text{ and } \epsilon_2 = \frac{7 \times 29 \times 157}{2 \times 5 \times 13} = \frac{203 \times 157}{10 \times 13}.$$

Otherwise,

$$\frac{10 \times 164809 - 27 \times 58965}{10 \times 450 - 27 \times 161}$$

$$= \frac{56035}{153} = \frac{5 \times 7 \times 1601}{3^2 \times 17}$$

$$= \frac{7}{3} \times \frac{8005}{51} = \frac{7}{3} \times \left(\frac{8056}{51} - 1\right)$$

$$= \frac{7}{3} \times \left(\frac{2^3 19 . 53}{3 \times 17} - 1\right)$$

with an error of $33''.9$ in defect.

$$\text{Again } \frac{7 \times 164809 - 18 \times 58965}{7 \times 450 - 18 \times 161} = \frac{92293}{252} = \frac{17 \times 61 \times 89}{2^2 \times 3^2 \times 7}$$

$$= \frac{61}{9} \times \left(\frac{23 \times 67}{4 \times 7} - 1\right)$$

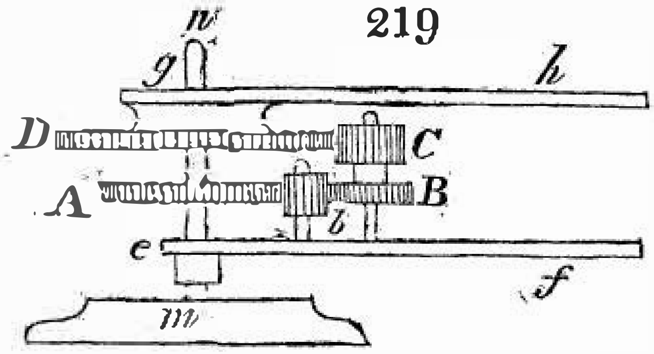
with an error of $13''.7$ in defect.

419. Ex. 4. In the ordinary construction of a planetarium, difficulty arises on account of the number of concentric tubes which are required to communicate the motion of the wheels to the arms which carry the planets. This is avoided in a planetarium by Mr. Pearson. By interposing an epicyclic train between each pair of planetary arms he makes them each derive their motion from the next one in the series, so that the tubes are entirely dispensed with. Referring to Rees' Cyclopædia, Art. Planetary Machines, for

an elaborate description and drawings of this machine, I shall quote one portion as an example of the use of our formulæ.

A fixed stud *mn*, fig. 219, carries the whole of the arms

219




in order, of which the arms of Mercury and of Venus are only shewn in this diagram, the others being disposed in the same manner. Between these arms a wheel *A* is fixed to the stud, and the arm of Venus carries an epicyclic train, of which *A* is the first wheel, and the last wheel *D* is fixed to the arm of Mercury. If, then, the period of Venus $a =$ ♀ and of Mercury $a =$ ☿, we have

$$\frac{n}{a} = 1 + \epsilon,$$

since $\epsilon$ by virtue of the intermediate idle wheel *b* is negative,

$$\text{where } \frac{n}{a} = \frac{♀}{☿} = \frac{1553}{608}, \text{ nearly;}$$

$$\therefore \quad \epsilon = \frac{AC}{BD} = \frac{945}{608} = \frac{63 \times 30}{16 \times 76},$$

which are Mr. Pearson's numbers.

If on the other hand *ef* were the Earth's arm, and *gh* that of Venus, we should have

$$\frac{⊕}{♀} = \frac{3277}{016} = 1 + \frac{AC}{BD}; \therefore \frac{AC}{BD} = \frac{⊕ - ♀}{♀} = \frac{1261}{2016} = \frac{13 \times 97}{2^5 . 3^2 . 7}.$$

To examine whether the idle wheel *b* cannot be dispensed with, it must be observed that it is introduced to make $\epsilon$ negative, and that if it were removed $\epsilon$ would be positive, and $\frac{n}{a} = 1 - \epsilon$. Now because the two arms must revolve in the same direction, $\frac{n}{a}$ is positive, therefore $\epsilon$ if positive must be less than unity, which makes $n$ less than $a$, and the train-bearing arm revolve quicker than the other. If, then, the arm of Mercury were to carry the train instead of the arm of Venus, the idle wheel would be got rid of.

Supposing, therefore, in the figure, that Mercury is changed for Venus, the whole being inverted, we have

$$\epsilon = + \frac{AC}{BD}, \quad \text{and} \quad \frac{\text{☿}}{\text{♀}} = 1 - \frac{AC}{BD} = \frac{608}{1553},$$

$$\text{whence } \frac{AC}{BD} = 1 - \frac{\text{☿}}{\text{♀}} = \frac{945}{1553} = \frac{2 \times 5 \times 53}{13 \times 67} \text{ nearly},$$

$$\text{or on the second supposition } \frac{\text{♀}}{\oplus} = \frac{2016}{3277} = 1 - \frac{AC}{BD};$$

$$\therefore \frac{AC}{BD} = \frac{\oplus - \text{♀}}{\oplus} = \frac{1261}{3277} = \frac{13 \times 97}{29 \times 113}.$$

## EXAMPLES OF THE SECOND USE OF EPICYCLIC TRAINS.

7

**420.** The second use which I have mentioned of epicyclic trains is for the establishment of an exact ratio of angular velocity between two axes when the terms of the ratio are unmanageable if applied to the arrangement of the ordinary trains of wheel-work, and when an approximation (Art. 243) is not admissible.

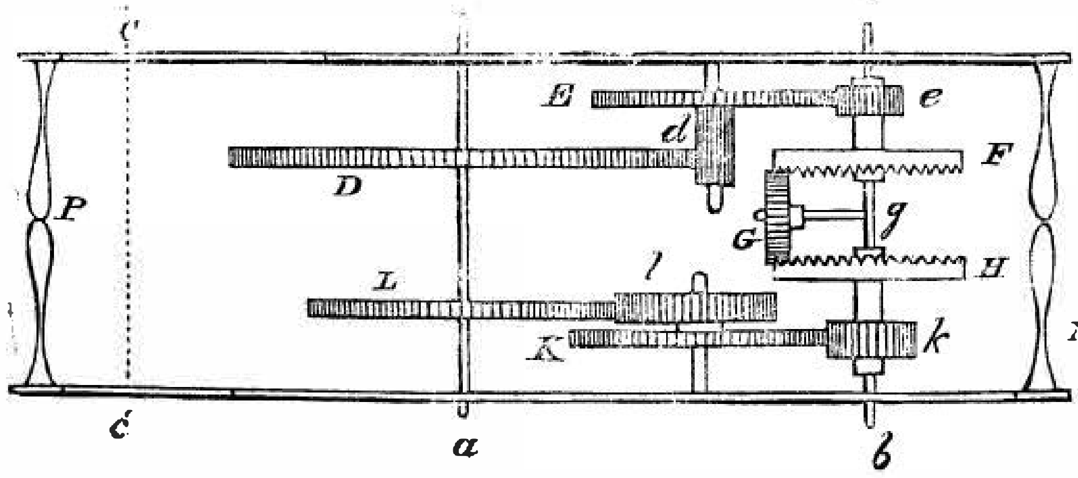
In Art. 410 we have shewn that if $\epsilon$ be an epicyclic train, and if a driving axis be connected with the first wheel of the train $\epsilon$ by a train $\mu$, and with the last wheel of the train $\epsilon$ by a train $\nu$, we have

$$\frac{a}{p} = \frac{\mu}{1 - \frac{1}{\epsilon}} + \frac{\nu}{1 - \epsilon},$$

when $a$ and $p$ are the synchronal rotations of the train-bearing arm and of the driving axis respectively.

As the epicyclic train is in this case employed merely to concentrate the effect of the two trains $\mu$ and $\nu$ upon the axis of the train-bearing arm, the epicyclic train itself may be employed in the simplest form, as in fig. 220, which shews one form of the mechanism which results.

220




*Bb* is the axis of the train-bearing arm *Gg*, this arm carries a wheel *G* which geers with two equal crown-wheels *F* and *H*, which are concentric to the axis *B b*, but are each fixed to tubes or cannons which run freely upon it.

The epicyclic train consists therefore of these three wheels, *F*, *G* and *H*, of which *F* may be considered to be the first wheel, and *H* the last wheel.

*Aa* is the driving axis, and this carries two wheels *D* and *L*; *D* serves to connect the axis with the first wheel *F* of the epicyclic train by means of the train of wheel-work *d*, *E* and *e*; and *L*, together with *l*, *K* and *k*, constitute a train of wheel-work which connects the axis *Aa* with the last wheel *H* of the epicyclic train. We have therefore

$$\mu = \frac{DE}{de}, \text{ and } \nu = \frac{LK}{lk}.$$

If the motion of the epicyclic train be considered with respect to the arm, it is clear that its extreme wheels *F*, *H* move in opposite directions, therefore $\epsilon$ is negative and equal to $-\dfrac{FG}{GH} = -1$;

$$\therefore \quad \frac{a}{p} = \frac{1}{2}(\mu + \nu) = \frac{1}{2}\left(\frac{DE}{de} + \frac{LK}{lk}\right).$$

If therefore a ratio of angular velocity $\frac{a}{p}$ be given, of which the numerator or denominator, or both, are not decomposable, we must endeavour to find two manageable fractions whose sum shall be equal to the proposed fraction, and employ them to form a train of wheel-work similar to that shewn in fig. 220.

This employment of epicyclic trains is given by Francœur*, from whom I have derived the calculations in the following articles. He attributes the mechanism to Messrs. Péqueur and Perrelet, about 1823, but the first idea of this method appears due to Mudge, who obtained an exact lunar train by epicyclic wheels before 1767†.

421. *First case.* Let $\frac{a}{p}$ be a fraction of which the denominator is decomposable into factors, but not the numerator.

Let the denominator $p = fgh$, therefore the fraction which represents the ratio of the velocities will be $\frac{a}{fgh}$. The denominator may often be susceptible of a division into three factors in various manners, each of which will furnish a distinct solution of the problem, subject to a condition which will presently appear. To decompose $\frac{a}{fgh}$ into two reducible fractions, assume

$$\frac{a}{fgh} = \frac{fx}{fgh} + \frac{gy}{fgh},$$

that is to say, $a = fx + gy$. It is easy to resolve this equation in prime numbers for $x$ and $y$, and obtain an infinity of values for $x$ and $y$ that will satisfy the problem, and give

* Dict. Technologique, t. XIV. p. 431.
† Vide Mudge on the Timekeeper, or Reid's Horology, p. 70.

$$\frac{a}{fgh} = \frac{x}{gh} + \frac{y}{fh};$$

$f$ and $g$ must however be prime to each other, since $a$ is prime, which is the condition already alluded to.

For example, let $\frac{271}{216}$ be the fraction proposed. Since

$216 = 4 \times 9 \times 6$ we may assume $271 = 9x + 4y, f = 9, g = 4$. The ordinary methods employed in equations of this kind will give $x = 31 - 4t$, $y = 9t - 2$, where $t$ is any whole positive or negative number, $gh = 24$, $fh = 54$. Hence we have

$$\begin{array}{lllllll} & x = 27, & 23, & 19 \ldots & 31, & 35, & 39, \\ & y = \phantom{2}7, & 16, & 25 \ldots & -\ 2, & -11, & -20, \\ \text{corresponding to} & t = \phantom{2}1, & 2, & 3 \ldots & -\ 0, & -\ 1, & -\ 2, \end{array}$$

The fraction $\frac{271}{216}$ is therefore equal to

$$\frac{27}{24} + \frac{7}{54}, \quad \frac{23}{24} + \frac{16}{54}, \quad \frac{19}{24} + \frac{25}{54},$$

$$\text{or to} \quad \frac{31}{24} - \frac{2}{54}, \quad \frac{35}{24} - \frac{11}{54}, \quad \frac{39}{24} - \frac{20}{54}, \quad \text{and so on.}$$

The first set referring to the case in which the crown-wheels turn in the same direction, the second to that in which they turn different ways.

But since 8 and 3 have no common factor, the denominator 216 might have been decomposed into $8 \times 3 \times 9$, whence assuming $271 = 8x + 3y$, we should have had

$$x = 3t - 1, \quad y = 93 - 8t, \quad \text{and}$$

$$\begin{array}{llllllll} x = & 2, & 5, & 8, \ldots\ldots & -1, & -4, & -7\ldots \\ y = & 85, & 77, & 69, \ldots\ldots & 93, & 101, & 109\ldots. \end{array}$$

whence the new decompositions

$$\frac{2}{27} + \frac{85}{72}, \quad \frac{5}{27} + \frac{77}{72}, \quad \frac{8}{27} + \frac{69}{72}, \quad \frac{93}{72} - \frac{1}{27},$$

and so on, all of which are solutions of the question.

Generally the proposed denominator must be resolved into prime factors under the form $m^{\alpha}.\, n^{\beta}.\, p^{\gamma} \ldots\ldots$ and any two of the divisors of this quantity may be assumed for $f$ and $g$, provided they be prime to each other. Thus if the equation $a = fx + gy$ be resolved in whole numbers, the component fractions will be $\frac{x}{gh} + \frac{y}{fh}$, where $h$ is the product of all the remaining factors of the denominator, after $f$ and $g$ have been removed.

422. Ex. 1. A mean lunation $= 29^{d}.\ 12^{h}.\ 44'.\ 3''$ $= 2551443''$, therefore the ratio of a lunation to twelve hours $= \frac{850481}{14400}$, of which the numerator is a prime. But this fraction may be by the above method resolved into two:

$$\text{thus } \frac{850481}{14400} = \frac{40 \times 50}{6 \times 6} + \frac{71 \times 79}{50 \times 32}.$$

And if these fractions be employed for the trains $\mu$ and $\nu$, the axes $Aa$, $Bb$ will revolve with the required ratio,

$$\text{for } \frac{a}{p} = \tfrac{1}{2}(\mu + \nu) = \tfrac{1}{2}\left(\frac{80 \times 50}{6 \times 6} + \frac{71 \times 79}{25 \times 32}\right) = \tfrac{1}{2}\left(\frac{DE}{de} + \frac{LK}{lk}\right).$$

And the periods are inversely as the synchronal rotations. If therefore a period of twelve hours be given by a clock to the axis $Bb$, $Aa$ will receive a period accurately equal to a lunation.

The mechanism may be thus represented in the notation already explained.

| Axes. | Trains. | Periods. |
|---|---|---|
| First Axis ............ | 79——80......a................ | Lunation. |
| Upper Stud ............ | . ....... 6—50 | |
| Upper Cannon ...... | . ............. 6—Crown Wheel F. | |
| Lower Stud............ | 32—71 | |
| Lower Cannon ...... | ......25———— Crown Wheel H | |
| Train-bearing Axis. | ————————Epicyclic Wheel G.. | 12 hours. |

If the fraction be resolved into a difference instead of a sum, as in the example $\frac{271}{216} = \frac{35}{24} - \frac{11}{54}$, this may be translated into mechanism, by making the trains $\mu$ and $\nu$ of different signs, that is, by making their extreme wheels revolve different ways.

423. Ex. 2. Mean time is to sidereal time nearly as 8424 : 8401.

$$\text{Now } \frac{8401}{8424} = \frac{31 \times 271}{39 \times 216} = \frac{31}{39} \times \left\{\frac{19}{24} + \frac{25}{54}\right\};$$

$$\therefore \frac{a}{p} = \tfrac{1}{2}(\mu + \nu) = \left(\frac{19}{24} + \frac{25}{54}\right); \quad \therefore \mu = \frac{19}{12} \quad \nu = \frac{25}{27},$$

and we obtain the following train, which differs from fig. 220 only in fixing the wheels $E$ and $K$ upon a single axis, which also carries a wheel of 39, geering with a wheel of 31 upon $Aa$, as appears in the following notation.

| Axes. | Trains. | Periods. |
|---|---|---|
| First Axis ............ | 31 ................................ | Sidereal Day. |
| Second Axis ......... | 39—19—25 | |
| Upper Cannon ...... | .......1...27—Crown Wheel F. | |
| Lower Cannon ...... | ......12———Crown Wheel H | |
| Train-bearing Axis. | ............Epicyclic Wheel G... | Solar Day. |

424. *Second case.* The fraction in the first case has been supposed to have a decomposable denominator. Let now both denominator and numerator be prime. Form two fractions $\frac{a}{A}$ and $\frac{a'}{A}$, in which $A$ is an arbitrary quantity and commodiously decomposable into factors, and proceed to obtain from each of these fractions the sums or differences of two decomposable fractions as before, which may be employed in wheel-work as follows.

Let an axis $Aa$, fig. 220, be connected to one axis $Bb$, by two trains and an epicyclic train, as in the figure, and also to another axis $Cc$ by a precisely similar arrangement. Then if the synchronal rotations of the axes $Aa$, $Bb$, $Cc$ be $A$, $a$ and $a_{,}$, $\mu$, $\nu$ the trains which connect $Aa$ with $Bb$, and $\mu_{,}\nu_{,}$ the trains that connect $Aa$ with $Cc$, we shall have

$$\frac{a}{A} = \frac{\mu + \nu}{2} \text{ and } \frac{a_{,}}{A} = \frac{\mu_{,} + \nu_{,}}{2}; \quad \therefore \frac{a}{a_{,}} = \frac{\mu + \nu}{\mu_{,} + \nu_{,}},$$

will be the ratio of the synchronal rotations of $Bb$ and $Cc$.

Suppose for example that it be required to make one axis perform 17321 turns, while another makes 11743; both being prime numbers, the fraction $\frac{17321}{11743}$ is irreducible, and indecomposable into factors.

Assume a divisor $5040 = 7 \times 8 \times 9 \times 10$, and form separately two trains whose velocities are represented by

$$\frac{17321}{5040} \text{ and } \frac{11743}{5040}.$$

For the first we have

$$\frac{17321}{5040} = \frac{1480}{630} + \frac{783}{720} = \frac{148}{63} + \frac{87}{80},$$

whence the trains $\frac{74}{63}$ and $\frac{87}{40}$, as in the first method. (Art. 421).

For the second train,

$$\frac{11743}{5040} = \frac{830}{633} + \frac{729}{720} = \frac{83}{63} + \frac{81}{80},$$

whence the trains $\frac{166}{63}$ and $\frac{81}{40}$.

If we represent the wheels which in the left-hand train correspond to $F$, $G$ and $H$, by $f$, $g$ and $h$, we have the following notation of the resulting machine.

| Axes. | Trains. | Synch. Rotations. |
|---|---|---|
| Axis $Aa$ | 87—74—9—6 ............................<br>9—83<br>21—Crown Wheel F.<br>8—36<br>20—Crown Wheel H | 5040 |
| Axis $Bb$ | Epicyclic Wheel G.<br>63—Crown Wheel $f$. | 11743 |
| Axis $Cc$ | 40—Crown Wheel $h$<br>Epicyclic Wheel $g$. | 17321 |

## EXAMPLES OF THE THIRD USE OF EPICYCLIC TRAINS.

425. The third employment of epicyclic trains, is to produce a very slow motion. In the formula $\frac{a}{p} = \frac{\mu\epsilon - \nu}{\epsilon - 1}$ Art. 410, all the trains are at present taken positive.

Let $\epsilon$ be made negative, and let $\mu$ and $\nu$ have different signs,

$$\therefore \frac{a}{p} = \frac{\mu\epsilon - \nu}{\epsilon + 1},$$

in which, by properly assuming the numbers of the trains, $a$ may be made very small with respect to $p$, and therefore the arm to revolve very slowly. This leads to such an arrangement as that of fig. 215, (Art. 409.)

$$\text{for } \frac{a}{p} = \frac{aceg - hbdf}{bg(ce + df)}, \quad \text{(Art. 413.)}$$

and in this expression the two terms of the numerator having no common divisor, may be so assumed as to differ by unity, by which an enormous ratio may be produced.

For example, put $a, c, e, g$ each equal 83,

$$b = 106,\ d = 84,\ f = 65,\ h = 82,$$

and we get

$$\frac{a}{p} = \frac{83^4 - 82 \times 106 \times 84 \times 65}{106 \times 83\,(83^2 + 84 \times 65)} = \frac{1}{108646502}.$$

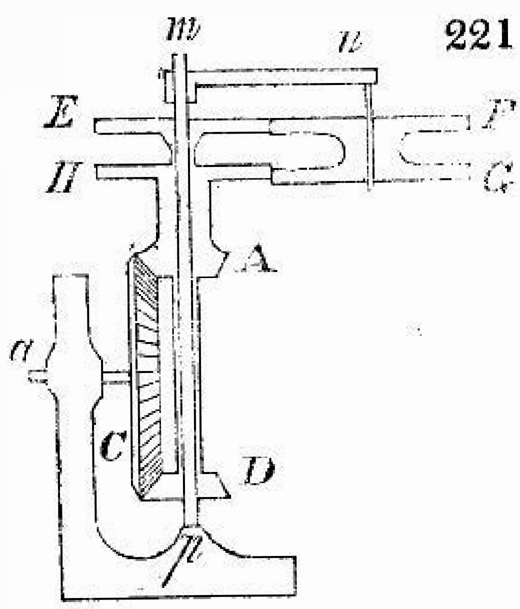
If in this machine we suppress the wheels $h$ and $e$ by making $a$ turn both $b$ and $g$, and $d$ turn both $f$ and $c$, we have*

$$\frac{a}{p} = \frac{a}{bg} \times \frac{cg - bf}{c + f} = \frac{20}{100 \times 99} \times \frac{101 \times 99 - 100^2}{101 + 100} = \frac{1}{99495}.$$

426. If on this contrary we wish to make the shaft, whose revolutions are $p$, revolve slowly with respect to the arm; then the numerator of the fraction $\frac{a}{p}$ must be a sum, and the denominator a difference; therefore $\epsilon$ must in the expression $\frac{a}{p} = \frac{\mu\epsilon - \nu}{\epsilon - 1}$ be positive, and nearly equal to unity, and $\mu$ and $\nu$ must have different signs.

* Putting $a = 20$, $b = 100$, $c = 101$, $g = 99$, and $f = 100$. This latter combination is given with these numbers by White (Century of Inventions).

Fig. 221 is a combination that will answer the present purpose: $mp$ is a fixed axis upon which turns a long tube, to the lower end of which is fixed a wheel $D$, and to the upper a wheel $E$; a shorter tube turns upon this, which carries at its extremities the wheels $A$ and $H$. A wheel $C$ is engaged both with $D$ and $A$, and a train-bearing arm $mn$, which revolves freely upon $mp$, carries upon a stud at $n$ the united wheels $F$ and $G$. The epicyclic train therefore is formed of the wheels $EFG$ and $H$, and is plainly positive, the extreme wheels $EH$ revolving in the same direction.




Let $H$ be the first wheel, $\therefore\ \epsilon = \dfrac{HF}{GE}$,

also $\mu = \dfrac{C}{A}$ and $\nu = \dfrac{C}{D}$ with different signs, since $A$ and $D$ revolve different ways;

$$\therefore\ \frac{a}{p} = \frac{\dfrac{C}{A}\cdot\dfrac{HF}{GE} + \dfrac{C}{D}}{\dfrac{HF}{GE} - 1},$$

put $A = 10$, $C = 100$, $D = 10$, $E = 61$, $F = 49$, $G = 41$, $H = 51$, and we shall obtain $\dfrac{a}{p} = 25000$, that is, 25000 rotations of the train-bearing arm $mn$ will produce one of the wheel $C$.

427. Generally, however, the first wheel of the epicyclic train is fixed, in which case the formula becomes $\dfrac{n}{a} = 1 - \epsilon$. If $\epsilon$ be positive and very near unity, this will be very small, or $n$ small with respect to $a$, that is, the motion of the last wheel of the train slow with respect to that of the arm. In

the simple forms of epicyclic trains, figs. 211, 212, and 213, the two latter are excluded, because $\epsilon$ is negative, but the former with the train $\begin{array}{c} A \\ b\text{———}E \\ D \end{array}$ is usually selected, $A$ being a fixed wheel, and $\frac{n}{a} = 1 - \frac{AE}{bD}$ is made as small as possible; which is effected by making $AE - bD = \pm 1$.

Thus if $\epsilon = \frac{101 \times 99}{100 \times 100}$ be the numbers of the wheels,

$$\text{we have } \frac{n}{a} = \frac{1}{10000},$$

but as these large numbers are inconvenient for the wheels that are carried upon the arm,

$$\text{let } \epsilon = \frac{111 \times 9}{100 \times 10}, \quad \therefore \frac{n}{a} = \frac{1}{1000},$$

$$\text{or let } \epsilon = \frac{31 \times 129}{32 \times 125}, \quad \therefore \frac{n}{a} = \frac{1}{4000}.$$

428. This combination is used for registering machinery for the same purpose as the contrivances in Arts. 404 and 405; and since the concentric wheels $A$ and $D$ (fig. 211) are very nearly of the same size, the pinions $b$ and $E$ carried by the arm may be made of the same number of teeth, or in other words, a thick pinion substituted for them which geers at once with the fixed wheel $A$ and the slow-moving wheel $D$*.

Let $M$, $M - 1$, and $K$ be the numbers of teeth of $D$, $A$, and the thick pinion respectively, then

$$\frac{n}{a} = 1 - \frac{K \times (M-1)}{K \times M} = \frac{1}{M},$$

where $M$ is the number of teeth of the slow-moving wheel.

* In Roberts' self-acting mule.

EXAMPLES OF THE FOURTH USE OF EPICYCLIC TRAINS.

429. The fourth employment of epicyclic trains consists in concentrating the effects of two or more different trains upon one revolving body when these trains move with respect to each other with a variable velocity ratio. I have already shewn how this may be effected when the extent of motion is small, as in Arts. 395, 397, but by epicyclic trains an indefinite number of rotations may be produced.

As an example of this application I shall take the equation clock, as it is the earliest problem of this class which presents itself for solution in the history of mechanism, and actually occupied the attention of mechanists for a long period*. The object of this machine is to cause the hand of a clock to point on the usual dial, not to mean solar time, but to *true solar time.* For this purpose we may resolve its motion as astronomers resolve the motion of the sun; namely, into two, one of which is the uniform motion which belongs to the mean time, and the other the difference between mean and true time or the equation. If, then, two trains of mechanism be provided, one of them an ordinary clock, and the other contrived so as to communicate a slow motion corresponding to the equation of time, and if we then concentrate the effects of these separate trains upon the hand of our equation clock by means of an epicyclic train, we shall obtain the desired result. There are three possible arrangements, as in Art. 406, (1) the equation may be communicated to one end of the train, and the mean motion to the other, the arm receiving the solar motion†; (2) the equation may be given to one end of the train, and the mean motion to the arm, the other end of the train will then receive the solar motion; (3) the equation may be communicated to the

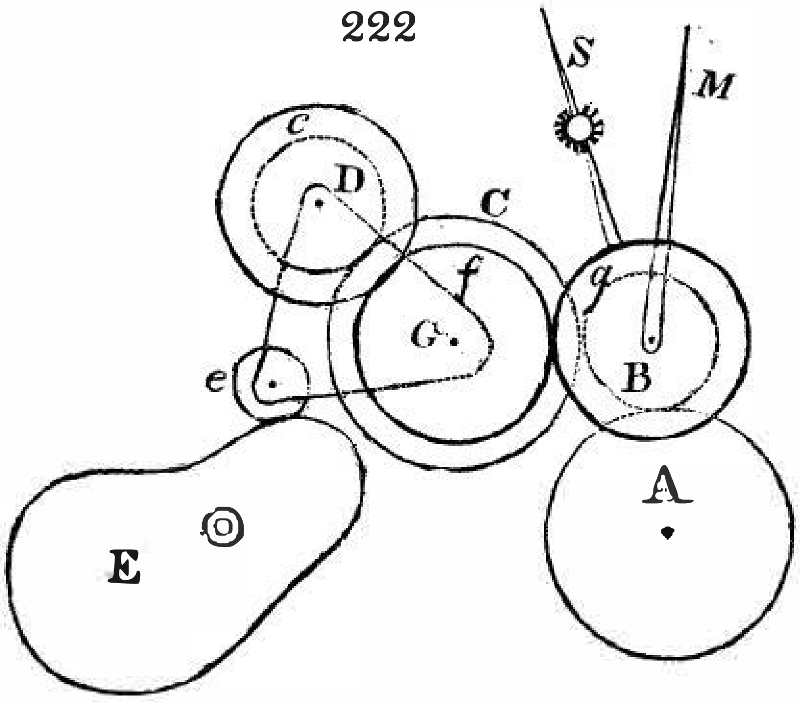
* Vide the Machines Approuvées of the Acad. des Sciences.

† Employed in the equation clock of Le Bon, 1722.

arm, and the mean time to one end of the train, when the other end of the train will receive the solar motion*. I shall describe the mechanism of the latter arrangement.

430. Fig. 222 is a diagram which will serve to shew the wheel-work of that part of an equation clock by which the motion is given to the hands. This wheel-work is commonly called the dial-work. *G* is the centre of motion of the epicyclic train, *GDe* the train-bearing arm. The wheels *f*

222




and *C* turn freely upon the axis *G*, and the axis *D* carried by the arm has two wheels *D* and *c* fixed to it, which geer with *f* and *C* respectively.

The epicyclic train consists, therefore, of the four wheels *C*, *c*, *D* and *f*, of which let *C* be the first wheel. In this arrangement the equation is to be communicated to the train-bearing arm, and the mean motion to the first wheel *C* of the epicyclic train. Now for this purpose *C* is driven by the wheel *B*, dotted in the figure, which derives its motion from a wheel *A* connected with an ordinary clock, and as the minute-hand *M* of the clock is fastened to the axis of *B*, this minute-hand will shew *mean time* upon the dial in the usual manner.

The equation is communicated to the train-bearing arm *GDe*, as follows. *E* is a cam-plate, which by its connexion

* In the clocks of Du Tertre, 1742, and Enderlin.

with the clock is made to revolve in a year (Art. 247). A friction roller $e$ upon the train-bearing arm rests upon the edge of the cam-plate, and is kept in contact with it by means of a spring or weight. The cam-plate is shaped so as to communicate the proper quantity of angular motion to the arm. We have seen how one end of the epicyclic train receives the mean motion, and $f$, which is the other extremity of the train, geers with a wheel $g$ concentric to the minute-wheel $B$, and turning freely upon it; the solar hand $S$ is fixed to the tube or cannon of $g$, and thus receiving the aggregate of the mean motion and the equation, will point upon the dial to the true time which corresponds to the mean time indicated by $M$.

The formula which belongs to this case is, (Art. 411)

$$n = a \cdot \overline{1 - \epsilon} + m\epsilon,$$

in which $\epsilon$ is positive and $= \frac{Cc}{Df}$. Now if the synchronal rotations of the minute-hand $M$ and of $C$ be $M$ and $m$ respectively, we have $m = M \cdot \frac{B}{C}$, and if those of $f$ and $g$ be $n$ and $s$, we have $n = s \cdot \frac{g}{f}$; substituting these values in the formula, we obtain

$$s = a \cdot \frac{Df - Cc}{Dg} + M \cdot \frac{Bc}{Dg},$$

of which the first part belongs to the equation, and the second to the mean motion.

Now the mean motion of $S$ must be the same as that of $M$; $\therefore \frac{Bc}{Dg} = 1$. And for that part of the motion of $S$ which is due to the equation, the expression $a \cdot \frac{Df - Cc}{Dg}$ shews the proportion between the angular motion of the train-bearing arm and of the hand $s$, synchronal rotations being directly proportional to angular velocity (Art. 20.). If the arm is to move with the same angular velocity as the hand,

$$\text{then } \frac{Df - Cc}{Dg} = 1,$$

and this is readily effected by making $f = c = g$ and $C = 2D$; also, since $Bc = Dg$ where $c = g$, we must have $B = D$, and these are the actual proportions employed by Enderlin. But if it be required that the arm move through a less angle than the hand, through half the angle, for example, then $C = 3D$, and so on.

431. In the treatises on Horology, and in the machines of the French Academy, may be found a great number of contrivances for equation clocks, which was a favourite subject with the mechanists of the last century. The machine itself is merely curious, and the desired purpose may be effected in a much more simple manner, if indeed it be worth doing at all, by placing concentrically to the common fixed dial a smaller moveable dial, and communicating to the latter the equation, by which the ordinary minute-hand of the clock will simultaneously shew mean time on the fixed, and true time on the moveable dial, without the intervention of the epicyclic train*.

Nevertheless, I have selected this machine as the best for the purpose of explanation, as being easily intelligible. The most successful machine of this class is undoubtedly the Bobbin and Fly-frame, in which, by means of an epicyclic train, the motions of the spindles are beautifully adjusted to the increasing diameter of the bobbins and consequent varying velocity of the bobbins and flyers. But this machine involves so many other considerations, that the complete explanation of it cannot be given in the present stage of our subject.

* This is done in the early equation clocks of Le Bon, 1714, Le Roy, &c.