CHAPTER VIII.

ELEMENTARY COMBINATIONS.

CLASS B. { DIRECTIONAL RELATION CONSTANT.
VELOCITY RATIO VARYING.

257. THE elementary combinations which are the subject of the preceding chapters, include those which are employed in all the largest and most important machines; for the parts of heavy machinery are always made to move with uniform velocity, if possible; and consequently with a constant velocity ratio and directional relation to each other. In the combinations which remain to be considered, either the velocity ratio, or directional relation, or both, vary; but as the arrangement of them is for the most part derived from some one or other of the previous contrivances, it will no longer be necessary to enter so much at large into the explanation of principles and of various forms, as a reference to the preceding chapters will for the most part suffice, at least for the less important machines. For this reason I have not thought it necessary to assign a separate chapter to each division of the classes *B* and *C*, as in class *A*, but shall include these classes each in a single chapter.

CLASS B. DIVISION A.

COMMUNICATION OF MOTION BY ROLLING CONTACT.

258. It has been already shewn, in Art. 35, that when a pair of curves revolving in the same plane in contact are of such a form as to roll together, the point of contact remains in the line of centers. The two radii of contact

coincide therefore with this line, and the tangents of the angles made by the common tangent of the curves at the point of contact with their radii respectively are the same.

259. Ex. 1. In the *logarithmic spiral* the tangent makes a constant angle with the radius vector. Let two equal logarithmic spirals be placed in reverse positions, and made to turn round their respective poles as centers of motion, and let these centers be fixed at any distance that will permit the curves to be in contact. Then in every position of contact the common tangent will make the same angle with the radius vector of one curve that it makes on the opposite side with the radius vector of the other. The two radii of contact will therefore be in one line, and coincide with the line of centers, and hence, *equal logarithmic spirals are rolling curves.*

Ex. 2. Let aPm, APM be two similar and equal ellipses of which s, h; S, H are the foci, and let them be

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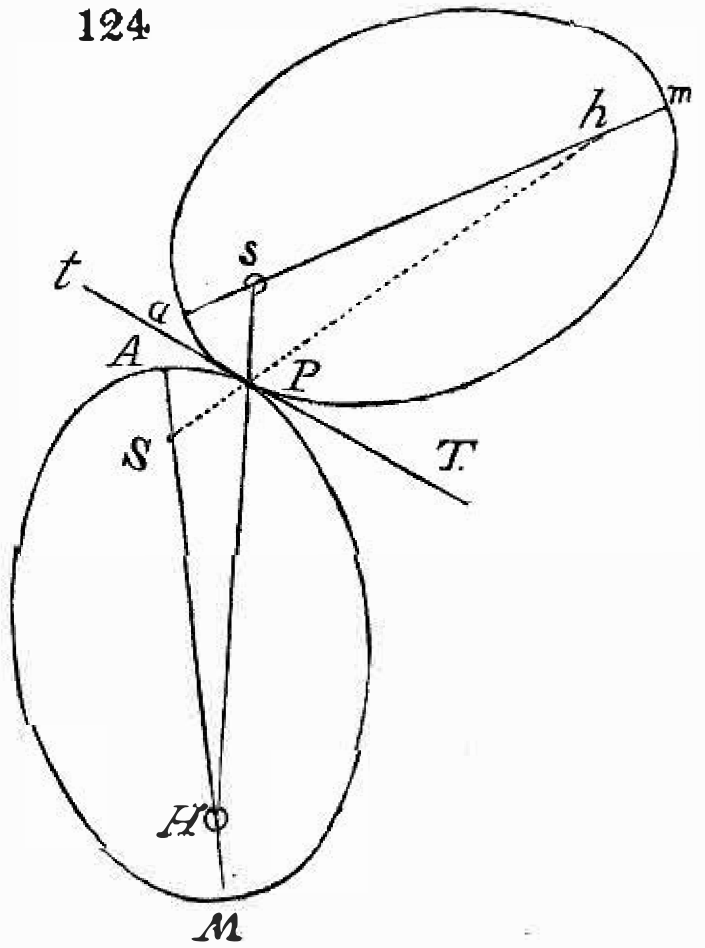


placed in contact at any point P situated at equal distances aP, AP from the extremities of their major axes, and draw tPT the common tangent at P.

Now by the property of the ellipse the tangent makes equal angles with the radii sP, Ph; and because $aP = AP$, and the ellipses are equal, the tangent makes the same angle with the radii $SP\ PH$; whence $tPs = TPH$, and sPH is a right line. Also $sP = SP$; $\therefore sP + PH = SP + PH = AM$ is a constant distance, whatever be the distance of the point of contact P from the extremity of the axes major. If, therefore, the foci s, H be made centers of motion, and their distance equal to the major axes of the ellipses, the curves will roll together.

The logarithmic spiral and ellipse round the focus appear to be the only two rolling curves that admit of simple independent demonstrations of their possessing this property.

260. Suppose fig. 124 to represent any pair of rolling curves, and let $r = sP$ be the distance of their point of contact P from the center of rotation s of the first curve, and $\theta = asP$ the angle made by r with a fixed radius sa, and let $r_{,} = PH, \theta_{,} = PHA$, be the corresponding quantities in the second curve, and c the distance sH of the centers; then since r and $r_{,}$ are in the same straight line,

$$r + r_{,} = c, \quad \therefore dr = -dr_{,};$$

also the lengths of those parts of the curves aP, AP, that have been in contact are equal;

$$\therefore \int\sqrt{dr^2 + r^2 d\theta^2} = \int\sqrt{dr_{,}^2 + r_{,}^2 d\theta_{,}^2},$$

and as $dr = -dr_{,}$, $\therefore rd\theta = r_{,}d\theta_{,} = (c - r)\, d\theta_{,}$.

Again, $\dfrac{rd\theta}{dr}$ is the tangent of the angle the first curve makes with r, and $\dfrac{r_{,}d\theta_{,}}{dr_{,}}$ is the tangent of the angle the second curve makes with $r_{,}$, and these angles are the same;

$$\therefore \frac{rd\theta}{dr} = -\frac{r_{,}d\theta_{,}}{dr_{,}}, \text{ whence } rd\theta = r_{,}d\theta_{,}, \text{ as before.}$$

Hence, if one curve be given by an equation between r and θ, the other is determined by the equations

$$r_{\prime} = c - r, \text{ and } \theta_{\prime} = \int \frac{r\,d\theta}{c - r}.$$

Ex. Let the first curve be the logarithmic spiral, (Art. 259) and let ϕ be the constant angle between the radius vector and the curve, $\therefore \theta = \phi \log \frac{r}{b}$ is its equation;

$$\therefore d\theta = \phi \frac{dr}{r}, \quad \theta_{\prime} = \int \frac{r\,d\theta}{c - r} = \phi \int \frac{dr}{c - r} = C - \phi \log r.$$

Now when $\theta_{\prime}$ vanishes, $r = c - b$; $\therefore 0 = C - \phi \log \overline{c - b}$;

$\therefore -\theta_{\prime} = \phi \log \frac{r}{c - b}$ is the equation to the second curve, which is the same logarithmic spiral in the reverse position.

The general equation of this article is given by Euler, in the fifth volume of the Acta Petropolitana, but it is not easy to obtain many convenient results in this manner. The properties of one class of rolling curves have been investigated in the most complete and able manner, in a paper in the Cambridge Philosophical Transactions, by the Rev. H. Holditch, to which I must refer those of my readers who are desirous of following out the subject I have substituted in the next article a simpler but more limited investigation, for which I am indebted to the author of the paper in question.

261. Let each rolling curve enter into itself, in which case it must have greater and less apsidal distances, a and b. And as in revolving the greater apsidal distance of one must come into contact with the less apsidal distance of the other,

$$a + b = c = r + r_{\prime}.$$

Let p be the perpendicular from the center upon the tangent to the point of contact; then since $\frac{p}{r}$ is the sine of the angle the tangent makes with the radius vector, and this angle must be the same from the very nature of rolling curves, when $a + b - r$ is substituted for $r_,$; it is plain that if we assume

$$\frac{r^n}{p^n} = \frac{A + Br + Cr^2 + \&c.}{A_, + B_, r + C_, r^2 + \&c.}$$

that this equals

$$\frac{A + B . \overline{a + b - r} . + C . \overline{a + b - r}|^2 + \&c.}{A_, + B_, . \overline{a + b - r} + C_, . \overline{a + b - r}|^2 + \&c.}$$

Let $r = c - x$, $2c = a + b$;

$$\therefore \frac{A + B . \overline{c - x} + C . \overline{c - x}|^2 + \&c.}{A_, + B_, . \overline{c - x} + C_, . \overline{c - x}|^2 + \&c.}$$

$$= \frac{A + B . \overline{c + x} + C . \overline{c + x}|^2 + \&c. \ldots}{A_, + B_, . \overline{c + x} + C_, . \overline{c + x}|^2 + \&c. \ldots}$$

And as the coefficients of the even powers of x are the same in both fractions, and the coefficients of the odd powers only differ in their signs; if O be the sum of the odd, and E of the even powers in the numerators, and $O_,$ of the odd and $E_,$ of the even in the denominators, then

$$\frac{E - O}{E_, - O_,} = \frac{E + O}{E_, + O_,}, \quad \therefore \frac{E}{E_,} = \frac{O}{O_,};$$

$$\therefore \frac{r^n}{p^n} = \frac{E - O}{E_, - O_,} = \frac{E}{E_,} = \frac{\alpha + \beta x^2 + \gamma x^4 + \ldots}{\alpha_, - \beta_, x^2 + \gamma_, x^4 + \ldots},$$

$$\text{or } \frac{r^n}{p^n} = \frac{A + A_2 \left(r - \frac{a + b}{2}\right)^2 + A_4 \left(r - \frac{a + b}{2}\right)^4 + \ldots}{B + B_2 \left(r - \frac{a + b}{2}\right)^2 + B_4 \left(r - \frac{a + b}{2}\right)^4 + \ldots},$$

which is a more convenient notation.

Now at the apses $r = a$ or b, and either of these values will give the equation

$$1 = \frac{A + A_2\left(\frac{a-b}{2}\right)^2 + A_4\left(\frac{a-b}{2}\right)^4 + \&c\ldots}{B + B_2\left(\frac{a-b}{2}\right)^2 + B_4\left(\frac{a-b}{2}\right)^4 + \&c\ldots},$$

which being substracted from the former, we have

$$\frac{r^n - p^n}{p^n} = \frac{A + A_2\left(r - \frac{a+b}{2}\right)^2 + \ldots}{B + B_2\left(r - \frac{a+b}{2}\right)^2 + \ldots} - \frac{A + A_2\left(\frac{a-b}{2}\right)^2 + \ldots}{B + B_2\left(\frac{a-b}{2}\right)^2 + \ldots}.$$

If these fractions be reduced to a common denominator, the general term in the numerator is

$$A_{2m} \cdot B_{2n}\left(r - \frac{a+b}{2}\right)^{2m}\left(\frac{a-b}{2}\right)^{2n}$$

$$- A_{2m} B_{2n}\left(r - \frac{a+b}{2}\right)^{2n}\left(\frac{a-b}{2}\right)^{2m}$$

$$= A_{2m} \cdot B_{2n}\left(r - \frac{a+b}{2}\right)^{2m}\left(\frac{a-b}{2}\right)^{2n}$$

$$\left\{\left(r - \frac{a+b}{2}\right)^{2m-2n} - \left(\frac{a-b}{2}\right)^{2m-2n}\right\},$$

which is divisible by

$$\left(r - \frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2, \text{ or by } (a-r)\cdot(r-b);$$

$$\therefore \frac{r^n - p^n}{p^n} = (a-r)\cdot(r-b)\,\frac{C_1 + C_2\left(r - \frac{a+b}{2}\right)^2 + \ldots}{D_1 + D_2\left(r - \frac{a+b}{2}\right)^2 + \ldots}, \quad (1)$$

in which equation $C_1, C_2 \ldots D_1, D_2 \ldots$ are arbitrary constants. If then, for the sake of simplicity, we limit the investigation to the first term by taking $C_2 \ldots D_2 \ldots = 0$, and make $n = 2$, we have*

$$\frac{p}{\sqrt{r^2 - p^2}} \;(= \tan\theta) = \frac{r\,d\theta}{dr} = \frac{k}{\sqrt{\overline{a-r}\,.\,\overline{r-b}}}:$$

to integrate this, let

$$\sqrt{\overline{a-r}\,.\,\overline{r-b}} = \overline{a-r}\,.\,x\,; \quad \therefore\ r - b = \overline{a-r}\,.\,x^2\,;$$

$$\therefore\ r = \frac{b + x^2 . a}{1 + x^2}, \quad dr = \frac{2\,\overline{a-b}\,.\,x\,dx}{\overline{1+x^2}|^2}, \quad a - r = \frac{a-b}{1+x^2};$$

$$\therefore\ d\theta = \frac{2k\,dx}{b + ax^2} = \frac{2k}{\sqrt{ab}}\;\frac{\sqrt{\frac{a}{b}}\,.\,dx}{1 + \frac{ax^2}{b}}:$$

$$\text{let } \frac{k}{\sqrt{ab}} = \frac{1}{n};$$

$$\therefore\ \frac{n\theta}{2} = \tan^{-1} x\sqrt{\frac{a}{b}} = \tan^{-1}\sqrt{\frac{a}{b}}\sqrt{\frac{r-b}{a-r}},$$

$$\therefore\ \frac{ar - ab}{ab - br} = \tan^2\frac{n\theta}{2},$$

$$\text{and } r = \frac{ab}{a\cos^2\frac{n\theta}{2} + b\sin^2\frac{n\theta}{2}} = \frac{2ab}{a + b + \overline{a-b}\,.\cos n\theta},$$

one equation amongst many others† that may be obtained

* Otherwise; the solution of the equation $\frac{r\,d\theta}{dr} = \frac{r_{,}\,d\theta_{,}}{dr}$ is $\frac{r\,d\theta}{dr} = u(r\,.\,\overline{c-r})$ where $u(r\,.\,\overline{c-r})$ is any symmetric function of r and $\overline{c-r}$.

Now $\overline{a-r}\,.\,\overline{r-b} = r\,.\,\overline{a+b} - ab - r^2 = r(a+b-r) - ab = r\,.\,\overline{c-r} - ab$ is a symmetric function of r and $\overline{c-r}$, therefore we may assume

$$\frac{r\,d\theta}{dr} = \frac{k}{\sqrt{a-r}\,\sqrt{r-b}}, \text{ as in the text.}$$

† For example, the equation $\frac{r\,d\theta}{dr} = \frac{k_{,} + k\,.\left(r - \frac{a+b}{2}\right)^2}{\sqrt{\overline{a-r}\,.\,\overline{r-b}}}$, which gives a much greater variety of rolling curves than the one in the text. This equation has been fully discussed in the paper already referred to.

from the expression (1). This however includes a variety of curves, according as different values are taken for ab and n.

262. Now if c be the distance of the centers of the two curves, it is evident from what has been said before, that if $\frac{r\,d\theta}{dr} = f\,(r)$ (any function of r) be the differential equation of one curve, then $\frac{r_{,}\,d\theta_{,}}{dr_{,}} = f\,(c - r_{,})$ (the same function of $\overline{c - r_{,}}$) will be the differential equation of the other.

Since therefore we have taken $\frac{r\,d\theta}{dr} = \frac{k}{\sqrt{\overline{a - r}\,.\,\overline{r - b}}}$ for the equation of the first curve, that of the second will be

$$\frac{r_{,}\,d\theta_{,}}{dr_{,}} = \frac{k}{\sqrt{\overline{a - c + r_{,}}\,.\,\overline{c - r_{,} - b}}},$$

$$= \frac{k}{\sqrt{\overline{a_{,} - r}\,.\,\overline{r_{,} - b_{,}}}}, \quad \text{(if } c - b = a_{,}, \text{ and } c - a = b_{,})$$

which is the same form as the differential equation of the first curve; and being solved in the same manner we have these equations to a pair of rolling curvese

$$r = \frac{2ab}{a + b + \overline{a - b}\,.\cos n\theta} \qquad (1),$$

$$r_{,} = \frac{2a_{,}b_{,}}{a_{,} + b_{,} + \overline{a_{,} - b_{,}}\,.\cos n_{,}\theta_{,}} \qquad (2).$$

263. Let $n\theta = 2m\pi$;

$$\therefore\ \theta = \frac{2m}{n}\,.\,\pi, \quad \text{and } r = b,$$

which shews that the minor distances recur at equal angles, the major in like manner correspond to

$$\theta = \frac{2m+1}{n}\pi,$$

and therefore bisect the angles between the minor apsidal distances. If the portion of curve between two minor distances, including as they do a major distance between them, be called a Lobe, then $\frac{2\pi}{n}$ is the angle which contains a lobe, and there are therefore n lobes in one revolution. In order that the curve may return to itself, and so be capable of successive revolutions, n must be an integer.

264. The constants in the pair of equations (1), (2), may be assumed at pleasure, subject to the conditions

$$\frac{ab}{n^2} = \frac{a_ib_i}{n_i^2}, \quad \text{since } k = \frac{n}{\sqrt{ab}} = \frac{n_i}{\sqrt{a_ib_i}},$$

$$\text{and } a - b = a_i - b_i.$$

For the greater apsidal distance of one curve corresponds to the less of the other,

$$\text{that is, } c - a = b_i, \text{ and } c - b = a_i.$$

A system of wheels or curves thus found will roll together in pairs or in any combinations.

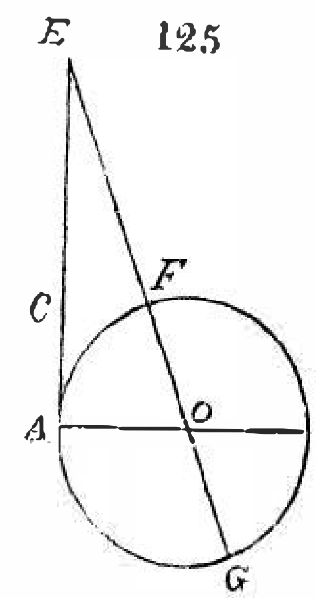
Let $a - b = l$, then since $\frac{ab}{n^2} = k^2$, $b^2 + bl = n^2k^2$;

$$\therefore\ b = \sqrt{n^2k^2 + \frac{l^2}{4}} - \frac{l}{2},$$

$$a = \sqrt{n^2k^2 + \frac{l^2}{4}} + \frac{l}{2};$$

and if n be taken successively equal to 1, 2, 3, &c. we have thus the major and minor distances of a system of wheels of one two three &c. lobes, which will work together, where k and l may have any assignable values, but must be the same for the same system.

265. Describe therefore a circle whose diameter is b and draw a tangent at any point A, (fig. 125,) in which take $AC = k$, and $AE = nk$, and draw EG through the center, then the apsidal distances for a wheel of n lobes are EG and EF;



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$$\text{for } EF = EO - FO = \sqrt{n^2k^2 + \frac{l^2}{4}} - \frac{l}{2} = b,$$

$$\text{and} EG = EO + OG = \sqrt{n^2k^2 + \frac{l^2}{4}} + \frac{l}{2} = a,$$

Ex. If $k^2 = \frac{2}{9}$, and $l = 1$, then if $n = .1, 3, 4...$

we have $b = .56, 1, 1.45...$

$a = 1.56, 2, 2.45...$

the figures will roll together or in any pairs, or two similar ones will roll together.

Substituting these values of a and b in (1) (2), the equation to a curve of n lobes will be

$$r = \frac{2n^2k^2}{2\sqrt{n^2k^2 + \frac{l^2}{4}} + l \cdot \cos n\theta} \quad (3).$$

266. In the equation (3) let $n = 1$;

$$\therefore r = \frac{2k^2}{2\sqrt{k^2 + \frac{l^2}{4}} + l \cdot \cos\theta},$$

the equation to an ellipse round the focus, of which the major axis

$$= 2\sqrt{k^2 + \frac{l^2}{4}} = a + b,$$

and $l = a - b =$ the distance between the foci.

The curve of one lobe in the system defined by the equation (3), is therefore always an ellipse round the focus, which has been already shewn to be capable of rolling with another equal and similar ellipse; and this equation will also give curves of any number of lobes capable of rolling with it.

267. These curves may be set out practically as follows. Having determined the values of a and b for a curve in a system of any given number of lobes, describe an ellipse whose axis major is $a + b$, and $a - b$ the distance between its foci.

Draw straight lines from one of the foci to the elliptic circumference, making equal angles with each other. Divide the base of each lobe into as many equal parts as there are equal angles round the focus, then the distances from the center to the several points of the lobe are easily shewn to be equal to the elliptic distances, and may therefore be set off from them.

268. Thus, let it be required to construct a set of three rolling curves of one, three, and four lobes respectively, in a system of which the constants l and k are given.

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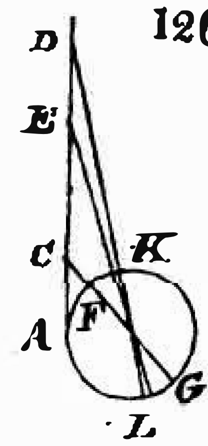

Describe the circle AKG, fig. 126, with a diameter $= l$, and upon the tangent AD set off $AC = k$, $AE = 3k$, and $AD = 4k$. Draw through the center CG, EL, and DL.

The curve of one lobe will be an ellipse round the focus M, fig. 127, whose apsidal distances are CF and CG, and major axis consequently $= CF + CG$.

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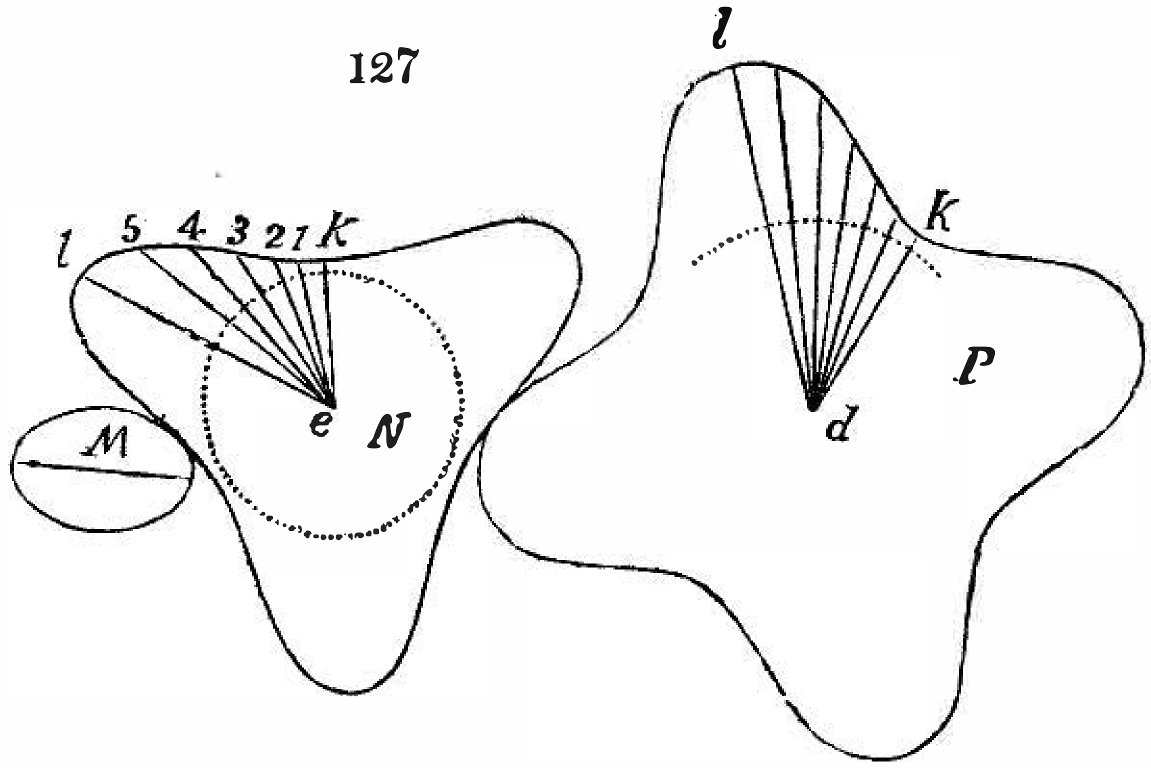

For the curve of three lobes describe a semi-ellipse Q, fig. 128, with apsidal distances ek, el respectively equal to EK, EL; and from e draw a sufficient number of radii $e1$, $e2$, $e3$, &c....at equal angular distances.

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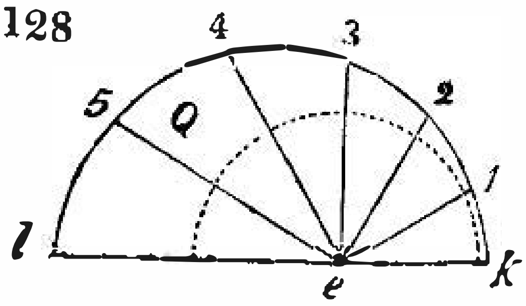

To construct the three-lobed curve N, describe a circle round its center e, which divide into six equal sectors, each one of which will contain half a lobe. Divide this into as many equal angles as those of the semi-ellipse Q, and draw radii, upon which set off in order distances equal to the radii of the semi-ellipse, as indicated by the corresponding letters and figures. Through the points thus obtained draw the curved edge of the semi-lobe, and this curve repeated to right and left alternately will complete the three-lobed curve.

To describe the four-lobed curve P, draw an ellipse whose apsidal distances are DK, DL, and major axis $DK + DL$, and proceed in a precisely similar manner to divide it and transfer its radial distances from the focus to the semi-lobe dkl of the four-lobed curve P.

Any two of these curves will roll together, or if two of them be made alike, the pair so obtained will roll together.

I cannot conclude these extracts without strongly recommending a perusal of the original paper, in which the forms and properties of a great variety of these curves are completely worked out.

269. If, however, a curve be given, and another be required to roll with it, which has been shewn (Art. 260) to be a problem that admits of solution, this in practice can only be solved by tentative methods, which will readily occur, but require some patience in application.

270. By Art. 35, the angular velocities in rolling contact are inversely as the segments into which the point of contact divides the line of centers.

In a pair of rolling ellipses, let A, $A_{\prime}$ be the angular velocities of the driver and follower respectively, r, $r_{\prime}$ their radii,

$$\text{then } \frac{A_{\prime}}{A} = \frac{r}{r_{\prime}} = \frac{r}{a+b-r}.$$

$$\text{This is at a maximum when } r = a; \therefore \frac{A_{\prime}}{A} = \frac{a}{b},$$

$$\text{and at a minimum when } r = b; \therefore \frac{A_{\prime}}{A} = \frac{b}{a}.$$

Let the ratio of the maximum to the minimum $= m$;

$$\therefore m = \frac{a^2}{b^2}.$$

$$\text{But, } r = \frac{2ab}{a + b + e(ae - b) . \cos\theta};$$

$$\therefore \frac{A_{\prime}}{A} = \frac{2ab}{a^2 + b^2 + (a^2 - b^2)\cos\theta} = \frac{2 . m^{\frac{1}{2}}}{m + 1 + (m-1) . \cos\theta};$$

which will also apply to a pair of equal rolling curves (as in Art. 262) of any number of lobes; but if they have different numbers of lobes, and a, $a_{\prime}$, b, $b_{\prime}$, be the respective apsidal distances, we should find $m = \frac{aa_{\prime}}{bb_{\prime}}$.

271. *To employ rolling curves in practice.* In fig. 124, let the upper curve be the driver, and let it revolve in the direction from T to t. Then since the radius of contact sP increases by this motion, and the corresponding radius PH decreases, the edge of the driver will press against that of the follower, and so communicate a motion to it of which the angular velocity ratio will be $\frac{PH}{sP}$. But when the point m has reached M, the radii of contact in the driver will begin to diminish, and its edge to retire from that of the follower, so that the communication of motion will cease.

To maintain it, it is necessary to provide the retreating edge with teeth, as in fig. 129, which will engage with similar teeth upon the corresponding edge of the follower, and thus maintain the communication of motion until the point a has reached A, when the advancing side of the driver will come into operation, and the teeth be no longer necessary.

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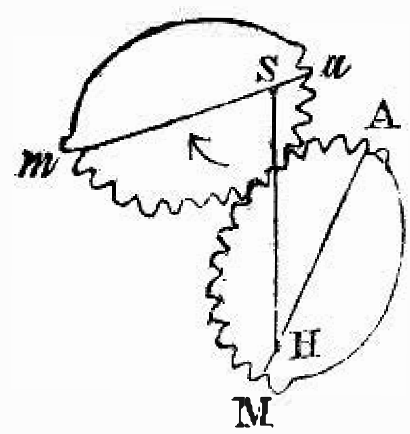

These teeth, however, necessarily destroy the advantage of no friction, and another practical difficulty is introduced. If the curves be not very accurately executed, it may happen that the first pair of teeth and spaces that ought to come together at M, m in each revolution, may not accurately meet, and that either the tooth may get into the wrong space, or become jammed against another tooth, by which the machinery may be broken.

272. To prevent this accident, a curved guide-plate n (fig. 130) may be fixed to one of the wheels, and a pin p to

the other. The edge of this plate must be made of such a form that the pin p may be certain of engaging with it, even if the wheels are not exactly in their proper relative position. When the pin has fairly entered the fork of the plate, it will press either on the right or left side, and so correct the position, and guide the first pair of teeth into contact. It is easy to see that the edge of this plate should be the epicycloid that would be described by p, if the lower plate were taken as a fixed base, and the upper made to roll upon it; but the outer edge of the plate must be sloped away from the true form, to ensure the entrance of the pin into the fork.

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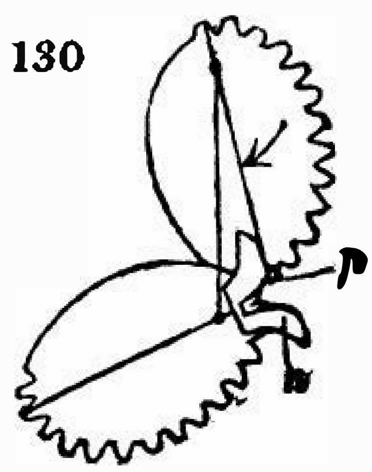

273. Another method is to carry the teeth all round the two plates, which effectually prevents them from getting entangled in the above manner, but at the same time entirely destroys the rolling action. This method, however, is the one always adopted in practice, as for example, in the Cometarium, and in the silk-mills, and is an excellent method of obtaining a varying velocity ratio. Fig. 131 represents a pair of such wheels that were employed by Messrs. Bacon and Donkin in a printing machine.

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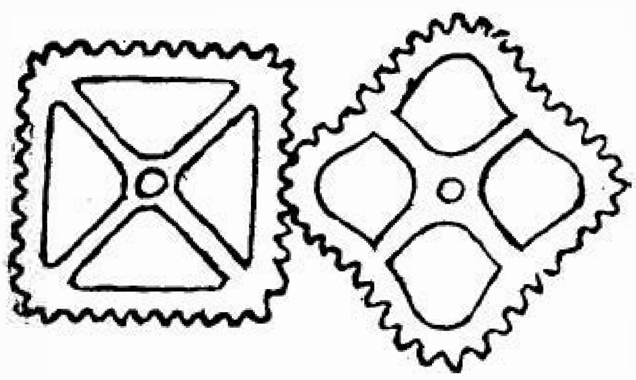

274. The forms of the teeth to be applied to these rolling curves may be obtained by a slight extension of the general solution in Art. 82. For calling the rolling curves pitch curves, it can be shewn for them, precisely in the same manner as it is there shewn for pitch circles, that if any given circle or curve be assumed as a describing curve, and if it be made to roll on the inside of one of these pitch curves, and on the outside of the corresponding portion of

the other pitch curve, that the motion communicated by the pressure and sliding contact of one of the curved teeth so traced upon the other, will be exactly the same as that effected by the rolling contact of the original pitch curves.

275. *The Cometarium* is a machine which has two parallel axes of motion carrying indices or clock-hands; one of which axes is the center of a circle, and the other the focus of an ellipse, which represents the orbit of a comet. The two axes must be connected by mechanism, so that when the first revolves uniformly, the second shall revolve with an angular velocity that will make it describe equal areas of its ellipse in equal times, and thus *represent the motion of a comet round the sun**, for which purpose the machine is constructed. Now, according to what is termed Seth Ward's hypothesis, if one radius vector HP of an ellipse (fig. 124), revolve uniformly round the focus H, the other SP will describe equal areas round the focus S. This, although a very coarse approximation, is considered sufficient for the mechanical representation of planetary or cometary motions in this instrument, and is accordingly obtained by connecting the two axes with a pair of rolling ellipses, as in fig. 124. For by Art. 259, it appears that $HP = hP$, and the angle $SHP = shP$. The motion therefore of HP and hP with respect to the axis major of their respective ellipses is the same, and the ratio of the angular velocities of sP and hP round their foci s and h is the same as those of SP and HP round S and H Also, since the corresponding radii sP, PH have been shewn to

* In any ellipse APM (fig. 124), we have

$$\frac{\text{Angular velocity of } SP \text{ round } S}{\text{Angular velocity of } HP \text{ round } H} = \frac{HP}{SP} = \frac{SP.HP}{SP^2} = \frac{CD^2}{SP^2},$$

where CD is the conjugate diameter of the ellipse. If the ellipse be nearly a circle, CD *may be supposed constant*, in which case if the angular velocity of HP be uniform, that of SP will vary as $\frac{1}{SP^2}$, which is the law of motion of the radius vector of a planet. This is termed Seth Ward's hypothesis, but is a very coarse approximation.

coincide with the fixed line of centers, it follows that the angular velocities of SH and sa round the centers H and s are respectively the same as those of HP and sP, that is, of HP and SP with respect to the major axes of the ellipses.

276. This machine was first introduced by Dr Desaguliers, and may be considered as the first attempt to employ rolling curves in machinery. He did not however furnish his ellipses with teeth, but connected them by means of an endless band of catgut, which embraced the circumference of each ellipse, lying in a groove in the circumference. The addition of teeth was a subsequent improvement.

277. When the required periodic variation in the ratio of angular velocity is not very great, a pair of equal common spur-wheels, with their centers of motion a little excentric, may be substituted for the equal ellipses revolving round their foci; but in this method the action of the teeth will become very irregular, unless the excentricity be very small.

278. The difficulty of forming a *pair* of rolling curves is sometimes evaded in the manner represented by fig. 132. *A* is a curved plate revolving round the center *B*, and having its edge cut into teeth. *C* a pinion with teeth of the same pitch. The center of this pinion is not fixed, but is carried by an arm or frame, which revolves on a center *D*. So that as *A* revolves, the frame rises and falls to enable the

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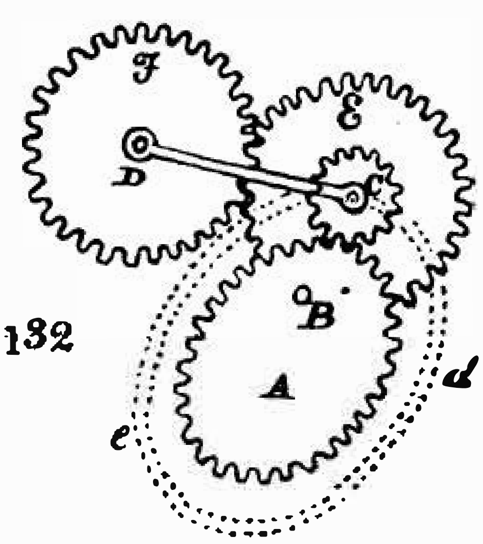


pinion to remain in geer with the curved plate, notwithstanding the variation of its radius of contact. To maintain the teeth at a proper distance for their action, the wheel *A* has a plate attached to it which extends beyond it, and is furnished with a groove *de*, the central line of which is at a

* Vide Rees' Cyclopædia, art. Cometarium; or Ferguson's Astronomy.

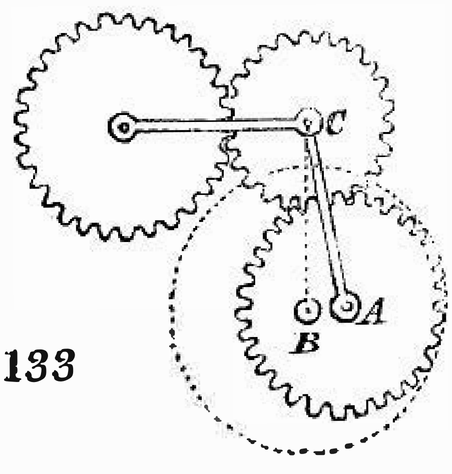
constant normal distance from the pitch line of the teeth equal to the pitch radius of the pinion. A pin or small roller attached to the swinging frame D and concentric with the pinion C rests in this groove. So that as the wheel A revolves, the groove and pin act together, and maintain the pitch lines of the wheel and pinion in contact, and at the same time prevent the teeth from getting entangled, or from escaping altogether.

Let R be the radius of C, r the radius of contact of A, ϕ the angle between R and r; then it can be easily shewn

$$\text{that } \frac{\text{ang. vel. of } A}{\text{ang. vel. of } C} = \frac{R}{r} \times \cos\phi.$$

But as the center of motion of C continually oscillates, and it is generally necessary to communicate the rotation of A to a wheel revolving on a fixed center of motion, a wheel E must be fixed to the pinion C, and this wheel must geer with a second wheel D concentric to the center of the swing-frame. When A revolves, the rotation of C will be communicated through E to F, but will also be compounded with the oscillation of the swing-frame, in a manner that will be explained under the head of Aggregate Motions, in the second Part of this work.

279. If for the curved wheel A an ordinary spur-wheel A, (fig. 133) moving on an *excentric center of motion* B, be substituted, a simple link AC connecting the center of the wheel A with that of its pinion C will maintain the proper pitching of the teeth, in a more simple manner than the groove and pin. The wheel A must be of course fixed to the extremity of its axis, to prevent the link from striking it in the course of its revolutions*. This



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* From a machine by Mr Holtzapfel.

combination being wholly formed of spur-wheels, is one of the simplest modes of effecting a varying angular velocity ratio.

280. *On Roëmer's wheels.* These wheels were proposed by the celebrated astronomer Olaus Roëmer*, to effect the varying motion of planetary machines. Aa, Bb are two parallel axes, of which the lower one is provided with a cone C, fluted into regular teeth like those of ordinary bevel-wheels, but occupying the surface of a much thicker frustum of the cone than usual. Opposite to this cone is fixed upon the axis Aa a smooth frustum D, whose apex d is in the reverse direction, and this latter cone is so formed as just to clear the tops of the teeth of C. Upon the surface of D are planted a series of teeth or pins, so arranged as to fall in succession between the teeth of C. By placing these pins at different distances from the apex d, we can obtain any velocity ratio we please between the extremes; for if R, r be the greatest and least radii of D, and $R_{,}$, $r_{,}$ of C; then the angular velocity ratio of C to D will vary between the limits of $\frac{R}{r_{,}}$ and $\frac{r}{R_{,}}$; the first being obtained by placing the pins close to the large end of D, and the second by fixing them at the small end; and when the pins are fixed in any intermediate position, an intermediate velocity ratio will be obtained.

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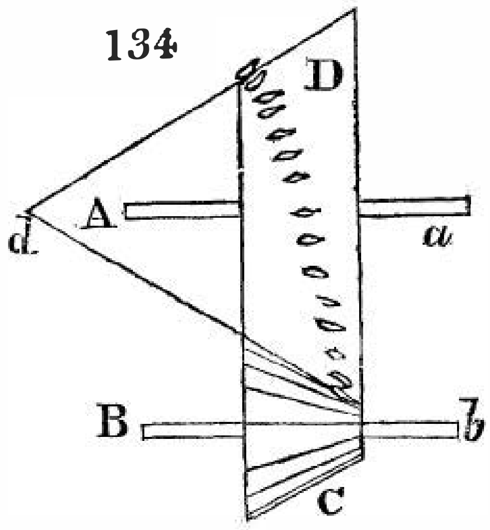

281. If the axes be not parallel, a varying ratio of angular velocity may be obtained by the excentric crown-wheel.

This was invented by Huyghens, for the purpose of representing the motion of the planets in his Planetarium.†

* Machines Approuvées, t. I. † Descriptio Automati Planetarii.

AB is an axis, to the extremity of which is fixed a crown-wheel F, exactly similar to that represented in fig. 26, page 50, only that its center of motion B is excentric to its circumference. This wheel is driven by a long cylindrical pinion CD, whose axis meets that of AB in direction, and is at right angles to it. Now since the radius of contact of the pinion is constant, while the radius of contact of the teeth of the hoop varies at different points of the circumference by virtue of its excentricity, it follows that the angular velocity ratio of the axes will vary.

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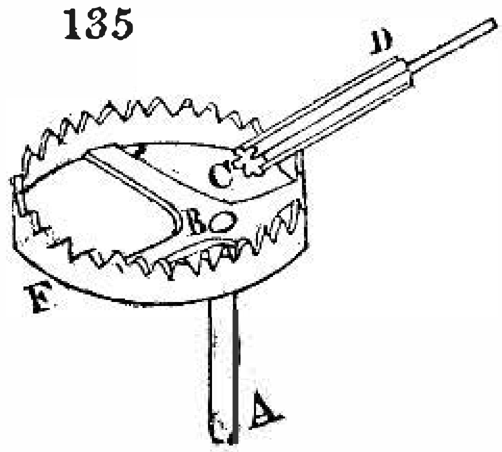

In Huyghen's machine the pinion is the driver, and is supposed to revolve uniformly, but if the contrivance be adopted in other machines, the wheel or pinion may be made the driver, according to the law of velocity required. Also, by making the circumference of the crown-wheel of any other curve than a circle, different laws of velocity may be obtained at pleasure. The action of the teeth however will be irregular, if the excentricity of the hoop be too much increased.

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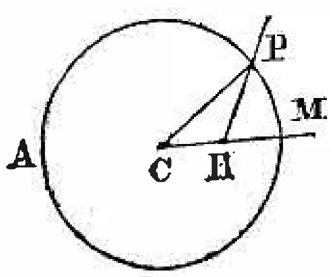

282. Let H, fig. 136, be the center of motion of the crown-wheel, C the center of its circumference,

$CP = R$, $HP = r$, $MHP = \theta$, and $HC = E$.

Then, since the axis of the pinion is directed to H in the line of the excentric radius HP, the perimetral velocity of the pinion will be communicated to this radius in a direction perpendicular to it; and if ρ be the radius of the pinion, we have

$$\frac{\text{angular velocity of pinion}}{\text{angular velocity of crown-wheel}} = \frac{r}{\rho}.$$

$$\text{But } R^2 = r^2 + E^2 \mp 2rE\cos\theta,$$

$$\text{whence } r = \pm E\cos\theta + R \,.\, \sqrt{1 - \frac{E^2}{R^2} \,.\, \sin^2\theta}.$$

Now in planetary machines E is small with respect to R;

$$\therefore\ r = \pm E\cos\theta + R.$$

And since the pinion revolves uniformly, angular velocity of crown-wheel

$$\propto \frac{1}{r} \propto \frac{1}{R \pm E\cos\theta} \propto R \mp E\cos\theta \text{ nearly.}$$

But if MP were the elliptic orbit of a planet, of which C the center, H the focus, HP the radius vector, and AM $(= 2R)$ the axis major, we should have angular velocity of HP

$$\propto \frac{1}{HP^2} \propto (R \mp E\cos\theta)^2 \propto R \mp 2E\cos\theta \text{ nearly.}$$

By making therefore the *excentric distance CH* of the crown-wheel equal to the *distance of the foci* of the elliptic orbit, the radius vector HP will revolve with an approximate representation of planetary motion, when the driving pinion revolves uniformly*.

283. Huyghens also proposed another method of obtaining the varying velocity; namely, by varying the pitch of the teeth. If in a pair of ordinary spur-wheels the pitch of one wheel be constant as usual, but in the other it vary so that a given arc of the circumference shall contain N teeth in one part, and an equal arc n teeth in another part of the circumference, and so on; then as every tooth of the first wheel causes one tooth of the other wheel to cross the line of centers, and the driver is supposed to move uni-

* In the article Equation Mechanism, in Rees' Cylopædia, will be found a minute and popular account of the various contrivances employed to represent planetary motion. Those that I have introduced into the text are applicable to machinery in general, and on this account, as well as from the celebrity of their authors, deserve to be studied.

formly, it follows that these equal arcs of the follower will pass the line in times that will be directly as their numbers of teeth *N* and *n*, and thus an unequal velocity will be obtained for the follower. But it is evident that this contrivance is but a make-shift, since teeth of unequal pitch will never work well together, although, if the variations from the mean pitch be small, they may be made to act so as to pass tooth for tooth across the line, with a kind of hobbling motion.

Nevertheless, a pair of wheels very similar to these admit of having their teeth formed upon correct geometrical principles; but the difficulty of executing them would be so much greater than those of the rolling curves (Art. 274), that I do not think it worth while to occupy space by developing their theory, which may be easily deduced from the preceding pages.

284. It may happen that the variation of angular velocity in the follower may consist in a sudden change from motion to rest, and *vice versa;* that is, that the follower may be required to move by short trips with intervals of complete rest between, or with an *intermittent* motion.

This may readily be effected with a pair of common spur-wheels, by cutting away the teeth of the driver, as in fig. 137, where the follower *B* is an ordinary spur-wheel, and

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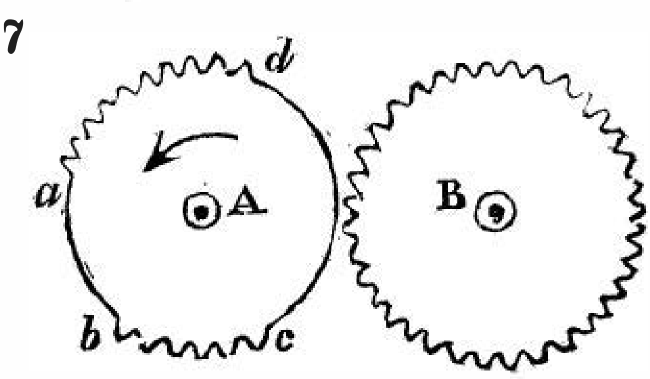

the driver *A* is a wheel of the same pitch whose teeth have been cut away between *a* and *b*, *c* and *d*; consequently, when *A* revolves it will cease to turn *B* while the plain parts of its circumference are passing the line of centers, but will turn it in the usual manner when the teeth come into action. By properly proportioning the plain arcs to those which contain the teeth, we can obtain any desired ratio of rest and motion that can be included within one revolution of the driver.

285. These intermitted teeth are liable to the same objection as those in Art. 271, namely, the chance of the first pair of teeth in each row getting jammed together, and a similar remedy may be employed—a guide-plate and pin. Thus in fig. 138, the wheel A will revolve in the direction of

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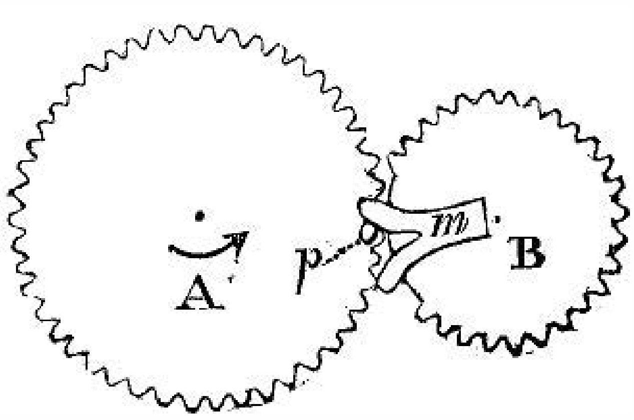
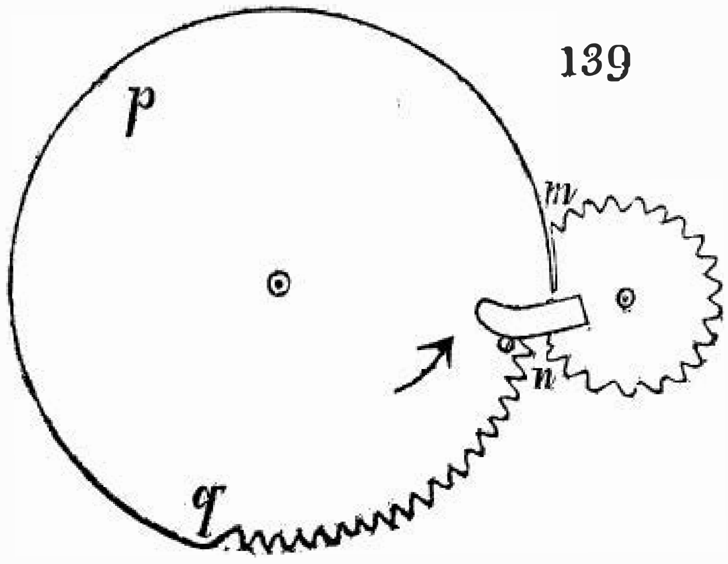


the arrow without communicating any motion to B, until the pin p enters the fork of the guide-plate m, and thus communicates to it a motion which brings the teeth of B into geer with those of A; and A will then continue to turn B until the plate m again reaches the position of the figure, when B will rest until the pin p returns.

In this combination B must make a complete revolution, (unless there be more guide-plates than one) and if R, r be the respective radii of driver and follower, it is easy to see that when A revolves uniformly, the time of B's rest is to the time of its motion as $R - r : r$. Also, several pins may be fixed to A if required, and the intermitted teeth may be given to A instead of to B, or to both.

286. As there is no contrivance in the above to protect B from being displaced during its period of rest, and thereby

preventing the guide-plate from receiving the pin, the action will be rendered more complete by the arrangement of fig. 139.



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Here the follower has its edge *mn* formed into an arc of a circle whose center is the center of motion of the driver, and the circumference of the driver is a plain disk *npq* of a greater diameter than the pitch circle of the toothed portion *qn*. This plain edge runs past *mn* without touching it, but effectually prevents the follower from being moved out of its position of rest, and therefore ensures the meeting of the pin and guide-plate.

287. Bevil or crown-wheels may be employed if necessary, and the combinations may be thrown into a great many other differentr forms. The pin and disk of fig. 139 have this advantage, that, when properly formed, they allow the intermittent wheel to begin and end its motion gradually, whereas in fig. 137 the motions begin with a jerk, and the follower is apt to continue its motion through a small space, after the teeth of the driver have quitted it.

288. In many machines a lever is required to move another by the mere contact of their extremities. As the angular motion required is always small, these extremities may be formed into rolling curves, by which the friction will be entirely got rid of, and the small variation in the angular velocity ratio will generally be of little or no con-

sequence. Arcs of the logarithmic spiral or ellipse round the focus will be the most easily described; but since the motion is small, arcs of circles may be substituted as an approximation for the rolling curves, and these may be described as follows.

Let A, B, fig. 140, be the centers of motion of the levers, AB the line of centers divided in T in the proportion of the

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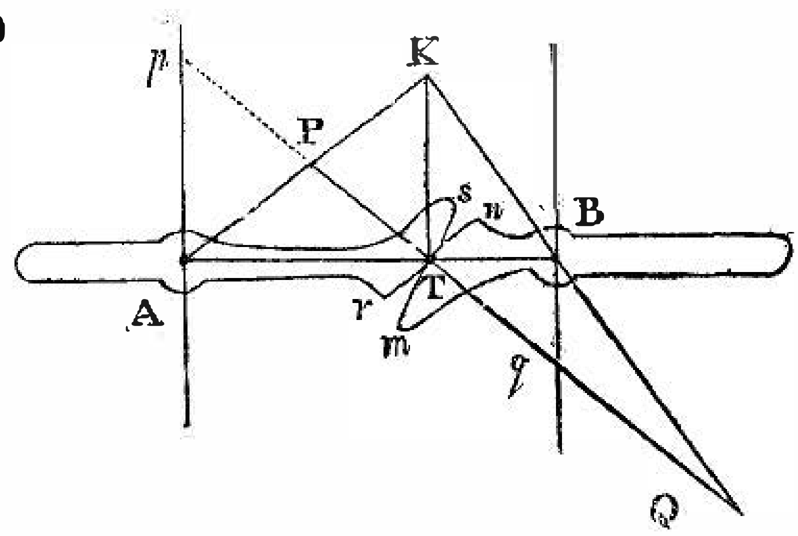

radii in their mean position. Draw KT perpendicular to AT, and through T draw PTQ inclined to AT at any angle less than a right angle. Assume a point K in KT. Join AK intersecting PTQ in P, and join KB, producing it to meet PTQ in Q. With center P and radius PT describe an arc rTs, and with center Q and radius QT describe an arc mTn. These arcs will roll together in the mean position of the figure.

For by Art. 33, it appears that the action of these arcs is equivalent to that of a pair of rods AP, BQ, connected by a link PQ. Now during the motion of this system the link may be considered as revolving round a momentary center, which center is always changing its position. But as the extremity P of the link begins to move in a direction perpendicular to AP, this ceuter must be somewhere in the line AP produced; and in like manner, as the extremity Q begins to move perpendicularly to BQ, the center must be somewhere in BQ produced; it must therefore be in K, the intersection of AP and BQ. But since K is the momentary center of motion of the link, and KT is perpendicular to AB, it follows that the point of contact T of the arcs rs, mn, will begin to move in the line of centers, and therefore the contact will be *rolling contact*.

289. Since the distance of K from T is arbitrary, let it be supposed infinite, in which case AK, QK become parallel to each other, and perpendicular to the line of centers, as at Ap and Bq, and p, q are now the centers of the arcs. This is a simpler construction.

In practice the angle PTA must be made much greater than in the figure, to avoid oblique action.

Class B. Division B. COMMUNICATION OF MOTION BY SLIDING CONTACT.

290. The simplest mode of obtaining a varying angular velocity ratio, when the rotations are to be continued indefinitely in the same direction, is by the pin and slit, fig. 141, where Aa, Bb are axes parallel in direction, but placed with their ends opposite to each other. Aa is provided with an arm carrying a pin d, which enters and slides freely in a long straight slit formed in a similar arm, which is fixed to the extremity of Bb. If one of these axes revolve, it will communicate a rotation to the other with a varying velocity ratio; for the pin in revolving is continually changing its distance from the axis of Bb.

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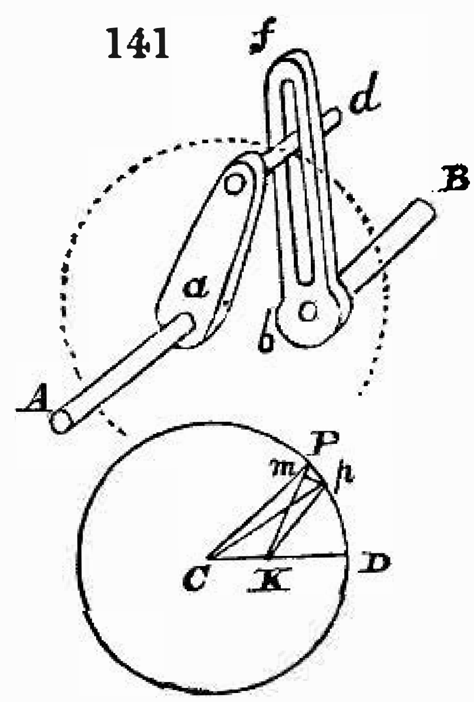

Let C be the center of motion of the pin-arm, K the center of motion of the slit-arm, P the pin, R the constant radius of the pin from C, r the radial distance from K, and let P move to p through a small angle; draw pm perpen-

-cular to CP, then angular velocity of pin : angular velocity of slit

$$:: \frac{Pp}{PC} : \frac{pm}{PK} :: \frac{1}{R} : \frac{\cos CPK}{r}.$$

If CP revolve uniformly, the angular velocity of KP will vary as $\frac{\cos CPK}{r}$, or if CK be small, as $\frac{1}{r}$; therefore when the centers of motion are near, this contrivance produces the same law of motion as that of Art. 282.

If $PCD = \theta$, $PKD = \beta$, $CK = E$, we have

$$R \sin\theta = (R\cos\theta - E)\tan\beta;$$

$$\therefore \tan\beta = \frac{R \,.\, \sin\theta}{R\cos\theta - E},$$

will give the position of KP corresponding to any given position of CP.

By altering the direction of the slit, or by making it curvilinear, other laws of motion may be obtained.

291. In the endless screw and wheel (Art. 170), the thread of the screw is inclined to the axis of the cylinder at a constant angle ϕ, and the angular velocity ratio of screw and wheel is constant. If, however, the inclination ϕ of the thread be made to vary at different points of the circumference, as shewn in fig. 142, the angular velocity ratio will vary accordingly. For example, if the threads through half the circumference lie in planes perpendicular to the axis of the screw, the wheel will revolve with an intermittent motion, remaining at rest during the alternate half rotations of the screw. If A, a be the respective angular velocities of the screw and wheel, R, r their pitch-radii, it appears, from Art. 166, that $\frac{A}{a} = \frac{r}{R}\tan\phi$.

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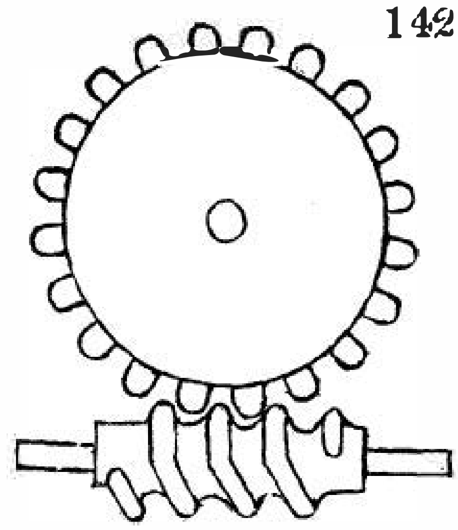

But as the inclination ϕ changes, the teeth of the wheel must be made in the form of solids of revolution, having their axes radiating from the center of the wheel.

292. A simple intermittent motion is effected by a pinion of one tooth *A*, fig. 143. This tooth will in each revolution pass a single tooth of the wheel *B* across the line of centers; but during the greatest portion of its rotation will leave the wheel undisturbed. To prevent the wheel *B* from continuing this motion by inertia through a greater space than this one tooth, a *detent* *C* may be employed. This turns freely upon its center, and may be pressed by a weight or spring against the teeth. It will be raised as the inclined side of the tooth passes under it by the action of *A*, and will fall over into the next space, but when *A* quits the wheel, the detent pressing upon the inclined side of the tooth will move it through a short space backwards, until the point *m* rests at the bottom of the nook, as shewn. The detent thus retains the wheel in its position during the absence of the tooth *A*. These detents receive other forms, for which I shall refer to the section on Link-work, in Chap. IX.

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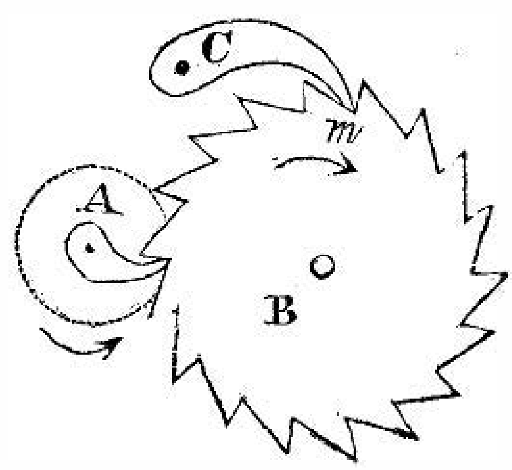

293. A better intermittent motion is produced by a contrivance (fig. 144) which may be termed the *Geneva stop*, as it is introduced into the mechanism of the Geneva watches.

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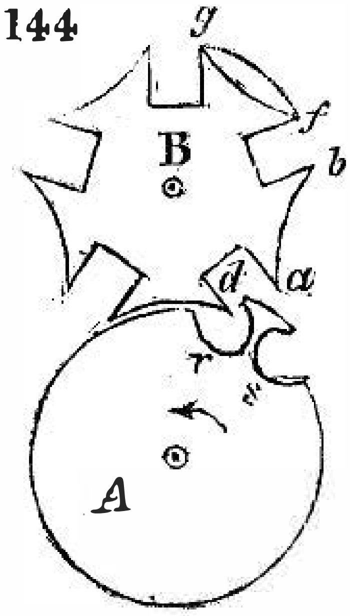


A is the driver which revolves continually in the same direction, *B* the follower, which is to receive from it an intermittent motion, with long intervals of rest. For this purpose its circumference is notched alternately into arcs of circles as ab, concentric to the center of A when placed opposite to it, and into square recesses, as shown in the figure.

The circumference of A is a plain circular disk, very nearly of the same radius as the concave tooth which is opposed to it; this disk is provided with a projecting hatchet-shaped tooth, flanked by two hollows r and s. When it revolves (suppose in the direction of the arrow), no motion will be given to B so long as the plain edge is passing the line of centers, but at the same time the concave form of the tooth of B will prevent it from being moved (as in fig. 139).

But when the hatchet-shaped tooth has reachedt the square recess of B, its point will strike against the side of the recess at d, and carry B through the space of one tooth, so as to bring the next concave arc ab opposite to the plain edge of the disk, which will retain it until another revolution has brought the hatchet into contact with the side of the next recess bf.

The hollow recess at r is necessary to make room for the point d, which during the motion is necessarily thrown nearer to the center of A than the circumference of the plain edge of the latter. The hatchet-tooth being symmetrical will act in either direction.

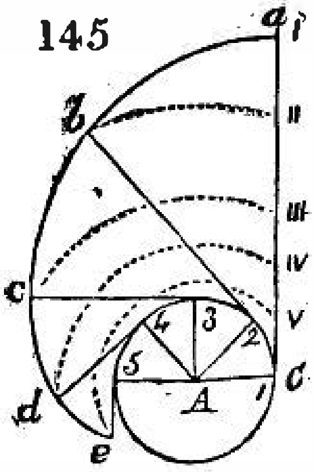
294. The office of this contrivance in a Geneva watch is to prevent it from being over-wound, whence it is termed *a stop*; and for this purpose one of the teeth is made convex, as shown in dotted lines at fg. If A be turned round, the hatchet-tooth will pass four notches in order, but after passing the fourth across the line of centers, the convex edge gf will prevent further rotation, so that in this state the combination becomes a contrivance to prevent an axis from being turned more than a certain number of times in the same direction.

For the wheel A is attached to the axis which is turned by the key in winding, and the wheel B thus prevents this axis from being turned too far, so as to overstrain the spring. As the watch goes during the day the axis of A revolves slowly in the opposite direction, carrying the stop-wheel with it by a similar intermitting motion.

The late Mr. Oldham applied this kind of mechanism to intermittent motions*, and his arrangement is in some respects superior to that of fig. 144. Instead of the hatchet-tooth he employed a pin carried by a plate fixed to the back of the driver, by which means he was enabled to reduce the size of the square notches of the follower.

295. Any required variation in the ratio of angular velocities may be produced by a cam-plate; but if the directional relation is constant the motion will necessarily be limited, as in fig. 71, (page 153). In this contrivance, by altering the form of the curve we may obtain different velocity ratios at every point of its action; as, for example, if a portion of the edge of the cam-plate be concentric to its axis, the pin or bar which it drives will receive no motion while that part of the edge is sliding past it.

296. The curve for a cam of this kind is generally described by points. The methods of doing this will readily occur in each particular case, but one example may serve to shew the nature of the process. In the combination of fig. 72, page 153, let the angular velocity ratio vary so that when a series of points 1, 2, 3, 4, 5, fig. 145, in the circumference of the circle C 3, 5 shall have reached in order the point C, the pin in the sliding bar shall be moved into the corresponding positions I, II, III, IV, V. To each of the position points in



* In the machinery of the Banks of England and Ireland.

the circumference of the circle draw tangents, and with center A draw circular arcs in order, each intersecting one of the position points I, II, III, &c., and the corresponding tangent, as at a, b, c, d, e; thus is obtained a series of points through which, if a curve be drawn, it will be the cam required; for it is manifest, that if any point (as 3) of the circle be brought to C, the corresponding point c of the curve will be moved to III, and thus the pin will be placed in its required position; and so for every other pair of positions.

The curve for a pin of sensible diameter must be obtained from this by the usual method (Art. 88).

CLASS B. DIVISION C. COMMUNICATION OF MOTION BY WRAPPING CONNECTORS.

297. If an indefinite number of rotations be required to be communicated from one revolving axis to another, an endless band may be employed, as in fig. 146. A is a driving pully, whose edge is shaped to the curve required, and is also grooved or otherwise adapted for the reception of an endless band, (Art. 180). The follower C is a cylindrical pully of the usual form. A stretching pully D (Art. 188) will be required for one side of the band, and if Ap be a perpendicular upon the direction of the other side, and Cq be the radius of the follower pully, we have by Art. 37 and 38,

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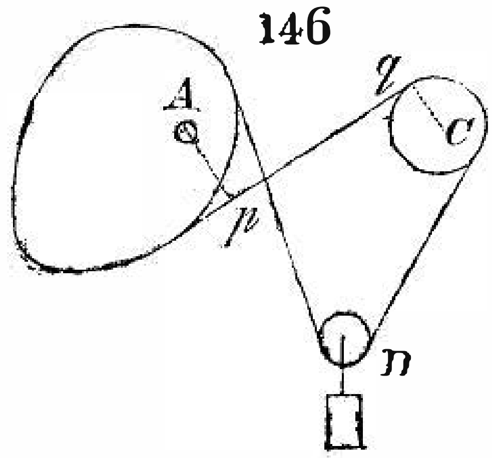

$$\frac{\text{ang. vel. of } A}{\text{ang. vel. of } C} = \frac{Cq}{Ap}.$$

298. If the motion be limited to a small arc the combination assumes the form of fig. 3, (p. 21), but if the limited motion extend to more than a complete revolution, a spiral groove is employed, as in the *fusee* of a watch.

Aa, *Bb*, fig. 147, are parallel axes, one of which carries

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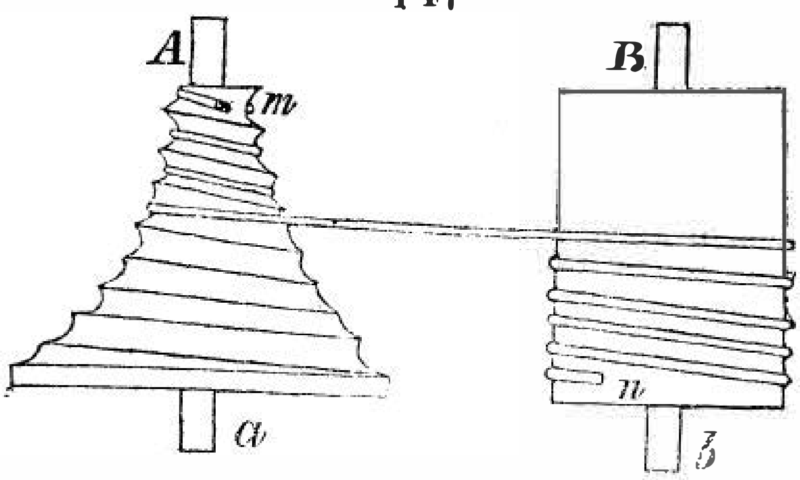

a solid pully, or fusee, as it is termed, upon whose surface is formed a spiral groove, extending in many convolutions from one end to the other. The axis *Bb* carries a plain cylinder; a band, a cord, or chain, is fastened as at *m* to one end of the fusee, and coiled round it, following the course of the spiral; the other end of the cord is fixed to the barrel at *n*. If the cord be kept tight by the action of a weight or spring upon one of the axes, the rotation of the other axis will communicate by means of the cord a rotation to the first axis, the velocity ratio of which will vary inversely as the perpendiculars from the axes upon the direction of the cord. And the motion may be continued through as many revolutions as there are convolutions in the spiral.

In like manner a pair of fusees may be employed instead of a fusee and cylinder.

299. If the fusee be required to communicate motion in both directions without the use of the re-acting weight or spring, a double cord will answer the purpose. Thus let it be required to employ the fusee in the manner of the barrel *A*, fig. 104, (p. 181), to give motion to a carriage *B*. The fusee will enable us to obtain a varying velocity ratio between *A* and *B*. In fig. 148, *Aa* is the axis of the fusee, which in this example is made to diminish at both ends. One cord is fas-

tened at *m*, and being coiled round the fusee is carried away at *n*, and attached to the carriage, as at *c*, fig. 104. The other cord is fixed at *p* to the fusee, and being coiled in the opposite direction, leaves the fusee at the same point at which the first cord is carried off. But this cord is taken in the opposite direction, as at *q*, and fixed to the end *d* (fig. 104) of the carriage, (or, which is better, both cords are carried over pullies and brought back to the carriage.)

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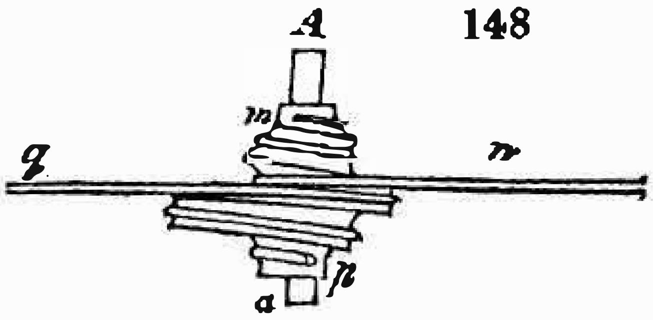

When the axis *Aa* revolves, one cord will unwrap itself from the fusee, while the other wraps upon it, and *vice versa*. But they will always leave its surface in opposite directions at the same point.

Since the fusee (fig. 148) is small at each end and large in the middle, it will, if turned with a uniform angular velocity, have the effect of gradually accelerating the motion of the carriage, till it has reached the middle of its path, and then of gradually retarding it to the end. It is employed in this manner in the self-acting mule of Mr. Roberts, of Manchester.

Class B. Division D. COMMUNICATION OF MOTION BY LINK-WORK.

300. Let AB, CD be two axes parallel in direction, but not opposite to each other, and let the arms AP, CQ be fixed to their extremities and connected by a short link PQ, jointed to the opposite faces of their arms; then if AP and CQ be each greater than AC, the perpendicular distance of the axes, a continual rotation of one axis will communicate a continual

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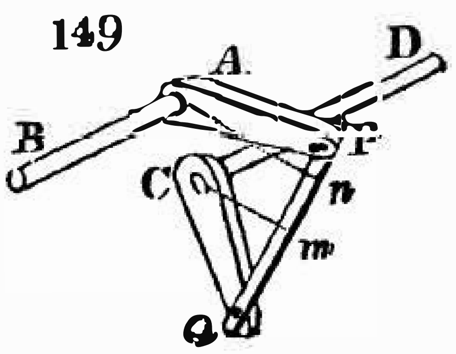

rotation to the other, but with a varying angular velocity ratio; for, if An, Cm be perpendiculars from the centers of motion upon the link, we have

$$\frac{\text{ang. vel. of } AP}{\text{ang. vel. of } CQ} = \frac{Cm}{An},$$

by Art. 32, Cor. 1; which perpendiculars continually change during the motion of the system.

But the properties of this kind of link-work will be more conveniently discussed in the corresponding division of the next Chapter.

301. The combination which is termed Hooke's Joint, however, properly belongs to this division of the subject. This is a method of connecting by link-work two axes whose directions meet in a point, so that the rotation of one shall communicate rotation to the other with a varying angular velocity. It has another use, as an universal joint of flexure, which will be afterwards considered.

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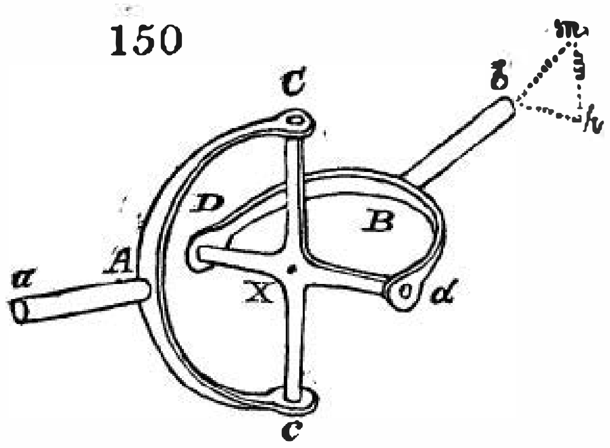

302. This contrivance was invented by Dr. Robert Hooke, and fully described by him in his Cutlerian Lectures*, as well as its properties and the uses to which he intended to apply it, of which however no demonstrations are given. To use his own words, somewhat abridged, "The Universal Joynt consisteth of five several parts. The two first parts are

* No. 2, Animadversions on the Mach. Cælestis, 1674, p. 73. No. 3, Description of Helioscopes, p. 13...1676.

the axes *Aa* and *Bb*, on which the semicircular arms are fastened which are to be joyned together so, as that the motion of one may communicate a motion to the other, according to a proportion which, for distinction's sake, I call elliptical or oblique. The two next parts are the two semicircular arms *CAc* and *DBd*, which are fastned to the ends of those rods, which serve to take hold of the four points of the *ball*, *circle*, *medium*, or *cross* in the middle, X; each of these pair of arms has two center holes, into which the sharp ends of the medium are put, and by which the elliptical or oblique proportion of motion is steadily, exactly, and most easily communicated from the one rod or axis to the other. These center holes I call the *hands*. The fifth and last thing is the ball, round plate, cross, or medium X in the middle, taken hold of by the hands both of one and the other pair of semicircular arms, which, for distinction's sake, I henceforth call the *medium*; and the two points *C*, *c*, taken hold of by the hands of the (driving) axis I call the *points*; and the other two points *D*, *d*, taken hold of by the second pair of arms I call the *pivots*.

"Great care must be had that the pivots and points lie exactly in the same plane, and that each two opposite ones be equally distant from the center, that the middle lines of them cut each other at right angles, and that the axes of the two rods may always cut each other in the center of the medium cross or plate, whatever change may be made in their inclination.

"The shape of this medium may be either a cross, whose four ends hath each of them a cylinder, which is the weakest way; or secondly, it may be made of a thick plate of brass, upon the edge of which are fixed four pivots, which serve for the hands of the arms to take hold of. This is much better than the former, but hath not that strength and steadiness that a large ball hath, which is the way I most approve of, as being strong, steady, and handsome."

To this may be remarked, that at present a stout ring or hoop is generally employed for the medium.

303. *To find the angular velocity ratio of axes connected by a Hooke's joint.* Let C be the intersection of the

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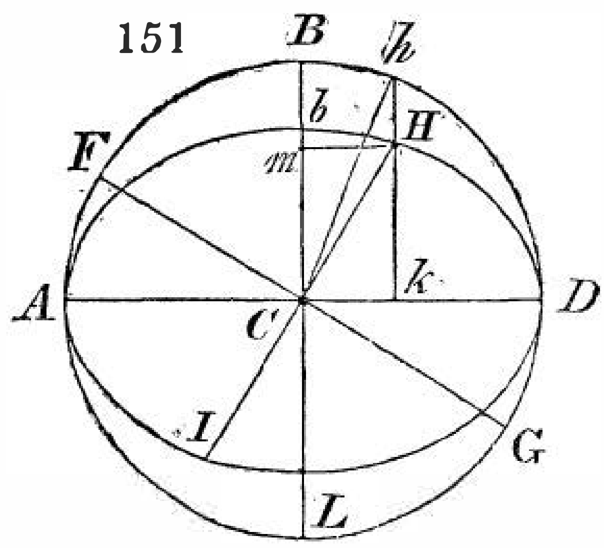


axes, the circle $ABDL$ that described by the extremities of the driver's arms, the plane of the paper being supposed perpendicular to the driving axis. Let the plane which contains the two axes intersect the paper in BCL, and let the ellipse AbD be the projection of the circle described by the extremities of the follower's arms. If θ be the inclination of one axis to the direction of the other produced, we have

$$bC = BC \cdot \cos\theta.$$

Let FCG be that branch of the medium cross which is jointed to the driver; then as this branch is always in the plane of the circle ABD, the projection of the other arm will be perpendicular to it. Draw HCI at right angles to FCG, and passing through the center C, then will this be the projection of that branch of the cross which is jointed to the follower, and H the position of its extremity.

Now in the projected circle AbD all lines parallel to the major axis ACD are unaltered in length by the projection; $\therefore$ Hm, perpendicular upon BC, is the sine of the angle through which the axis of the follower has been moved, if we reckon the motion from bC; and drawing hHk perpendicular to ACD, hCB is that angle. But the corresponding angle, through which the driver has moved, is $ACF = HCB$. Let α be the angle through which the driver has moved, β the corresponding angle of the follower, both being measured as above from that position of the machine in which the arms of the follower lie in the plane which contains the two axes (1),

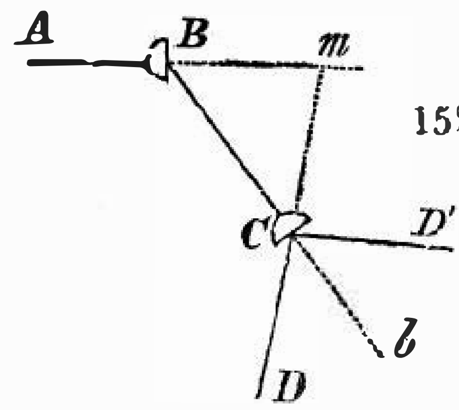
$$\text{then } \frac{\tan\alpha}{\tan\beta} = \frac{Hk}{hk} = \frac{bC}{BC} = \cos\theta;$$

$$\therefore \tan\beta = \frac{\tan\alpha}{\cos\theta}, \text{ or } \frac{\tan\beta}{\tan\alpha} = \textit{constant}.$$

The relative positions of the driver and follower are exhibited graphically by means of the ellipse and circle, (fig. 151), where if HCB be the angular distance of any given radius HC of the driver from its position at the beginning of the motion at B, reckoned as above at (1), then will hCB be the corresponding angular distance of the radius hC of the follower, which coincided with it at starting from B.

If we follow these radii round the circle, it appears that they coincide at four points B, D, L, and A; that at starting from B the follower moves slower than the driver at first, and falls behind it, and then accelerates, until it overtakes it at D, beyond which it takes the lead through the next quadrant DL, first moving quicker than the driver and then retarding; so that the driver overtakes it at L, and passes it. The motion through LA is similar to that through BD; and that from A to B the same as that from D to L. The amount of retardation and acceleration depends upon the value of θ; and therefore if a single joint be employed, the axes must be inclined to each other sufficiently to produce the desired variation of velocity.

304. By means of two joints, however, the axes may be parallel or inclined at any angle, and a greater variety of motion be procured.



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Thus let AB, fig. 152, be the driving axis, and let it be connected to the first following axis BC by a Hooke's joint at B, and let this be similarly jointed to a second axis CD at C. The plane of ABC may be different from that of BCD.

First, let the angular motion of the second joint at C be reckoned like that of the first, from the position in which the fork of the follower lies in the plane of the two axes. Then for the motion of the joint B we have, as before,

$$\tan\beta = \frac{\tan\alpha}{\cos\theta};$$

and if γ be the corresponding angles of the axis CD, and $\theta_{,}$ its inclination to BCb,

$$\tan\gamma = \frac{\tan\beta}{\cos\theta_{,}} = \frac{\tan\alpha}{\cos\theta\,.\,\cos\theta_{,}}.$$

If there be a series of similar axes, whose successive mutual inclinations are θ, $\theta_{,}$, $\theta_{,,}\ldots\theta_{n}$, δ the angular distance of a radius of the last, corresponding to α,

$$\text{then,}\ \tan\delta = \frac{\tan\alpha}{\cos\theta\,.\,\cos\theta_{,}\,\cos\theta_{,,}\ldots\cos\theta_{n}}.$$

In a system of this kind any desired amount of variation may be obtained, and the last follower may be set at any given angle to the first driver, or even in its own direction produced, by three Hooke joints only.

In the system just described the shafts may lie in different planes, but it is supposed that the joints are all so adjusted that when the following arms of the first joint B lie in the plane ABC of its two axes, that the following arms of every other joint also lie in the plane of their two axes.

Let there be a system of three axes with two joints, as fig. 152, but let the *driving* arms of the second lie in the plane BCD, when the *following* arms of the first lie in the plane ABC. The angles of the second are therefore now reckoned from a fixed radius distant one quadrant from those of the first.

If $\tan\beta = \dfrac{\tan\alpha}{\cos\theta}$ be the equation to the first,

$$\tan\gamma = \frac{\tan\left(\frac{\pi}{2}+\beta\right)}{\cos\theta_{,}} \text{ is the equation to the second.}$$

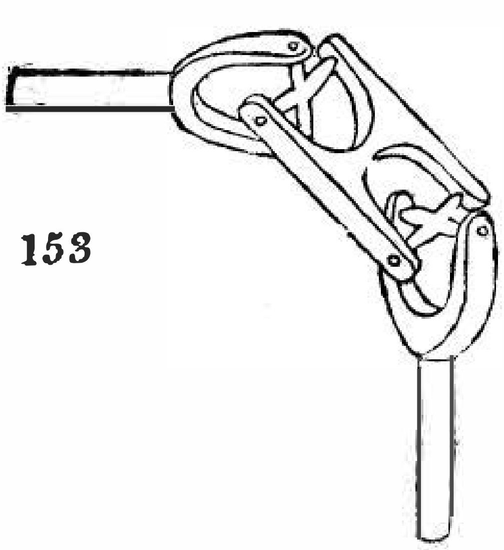
$$\text{But } \tan\left(\frac{\pi}{2}+\beta\right) = \frac{1}{\tan\beta};$$

$$\therefore \tan\gamma = \frac{\cos\theta}{\tan\alpha \,.\, \cos\theta_{,}}.$$

$$\text{Let } \theta = \theta_{,}; \quad \therefore \tan\gamma = \frac{1}{\tan\alpha};$$

which shews that if the forks be set as above, and if the angles of inclination of the axes be equal, then the variations of motion will counteract each other, and the angular velocity ratio of the extreme axes AB, CD, remain constant.

305. When the double Hooke's joint is thus employed it is commonly for this purpose of correcting the varying ratio of angular velocity, and the intermediate piece may therefore be made short, as in fig. 153. Care must be taken however that the extreme axes are so placed as to meet in a point, and that the angles they each make with the intermediate piece are the same.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In this form the Hooke's joint properly belongs to the Link-work of the previous ~~Chapter.~~ Class. It may be used either to communicate equal rotation between two axes inclined at an angle AmD, fig. 152, or between parallel axes, as AB, CD'.

306. Considering only the elementary form of this contrivance, it is evident that two branches to each axis are necessary only to give greater steadiness to the motion, and that a single pair of arms AC, BD (fig. 150), connected by a straight link from C to D, would produce the same motion; so that in this way we are brought to a form very similar to that of fig. 149. Also, Oldham's joint, in Art. 176, becomes a Hooke's joint, if the axes, instead of being parallel, are set so as to intersect in the center of the cross.

307. More complex relations of angular velocity may be obtained by making the two arms of each axis unequal, as AC greater than Ac; or setting the arms of the cross at different angles. For this purpose in Hooke's figure the arms are provided with the power of adjustment, in length. He proposed to employ this contrivance in the resolution of spherical questions and various other purposes, which however have been long forgotten; but may be found in the original Essay already referred to.

308. One of the applications which Hooke made of this device was to the construction of Sun-dials.

Two axes being connected by this joint, let one of them be fixed parallel to the axis of the Earth, and the other perpendicular to the plane of the required dial; and let the first be furnished with an index, travelling round a common equally-divided hour-circle, and let the other carry a similar index which travels round the circle of the dial. The two indexes being so adjusted that when the first points to noon the other shall coincide with the twelve-o'clock line; then by turning round the upper axis "till you see its index to point at those hours, halves, quarters, or minutes, you have a mind to take notice of in your dial, by the second index you are directed to the true corresponding point in the plane of the dial itself *;" which may be shown as follows.

Let AOE, fig. 154, be a dial-plate inclined at any given angle to the horizon, C its center, PC the style, PO an arc of the circle of declination perpendicular to the plate, therefore CO is the substyle. Let PE be an arc of the meridian, therefore CE is the twelve-o'clock line; and if the sun decline through any angle EPA from the plane of the meridian, we have in the spherical triangle POA, right-angled at O,

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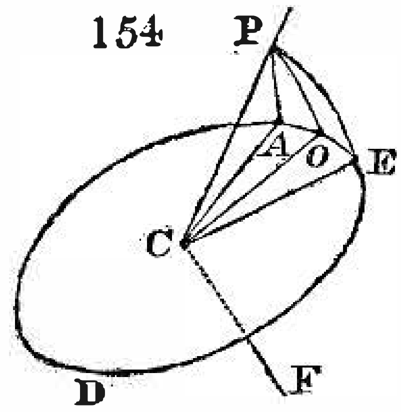

$$\sin PO = \frac{\tan AO}{\tan OPA}.$$

If then PC be the position of one axis of the Hooke's joint, and the other axis CF be perpendicular to the dial-plate, $\sin PCO = \cos PCF$. And the expression becomes the same as that we have already found for the synchronal motions of the two axes; AO measuring the motion of the shadow of the style over the plate, and OPA the corresponding rotation of the solar hours.

The axis PC having been fixed in its due situation parallel to the axis of the earth, the meridian line or twelve-o'clock point E may be found by a plummet. And in employing the instrument the actual dial-plate will of course be fixed at the lower extremity of the axis CF, parallel to the circle AOE; but this does not affect the demonstration.

* Hooke, *Desc. of Helioscopes*, p. 14.

309. With respect to the use of Hooke's joint as a universal joint of flexure, let Aa, fig. 150, be a fixed rod, and let it be required to move the extremity b of Bb in any direction bm concentric to X the intersection of the rods. Draw bh, mh meeting in h, and also concentric to X, and in planes respectively perpendicular to the axes of rotation or joints Cc, Dd. Then b can move in the direction bh, by virtue of the joint Cc, and in the direction hm by virtue of the joint Dd; and if it be made to revolve simultaneously round these joints with velocities respectively proportional to bh and hm, it will describe the path bm.

The motion bm is performed round an axis passing through X, which being perpendicular to the plane bmX is also situated in the plane of the joints Cc, Dd; and whatever be the angle between the joints Cc and Dd, that is, between bh and hm, the triangle bhm can be constructed upon bm. Bb is here supposed perpendicular to the plane of the joints, hut the same will be true for any piece or rod rigidly connected with Bb, and therefore for a rod making any angle with the plane of the joints. It follows therefore that if two rods or pieces are connected by two joints of flexure whose axes meet at any angle, these piecesaare at liberty to turn round any axis of flexure situated in the plane of these two axes, and passing through their point of intersection.

310. Now let the second rod be required to bend with respect to the first round any axis passing through the intersection X.

Let AB, BC, fig. 155, be the rods, B their point of intersection, FBD the plane containing the two axes of flexure, as before, and not necessarily perpendicular to BC, and let BC be required to move about an axis BK passing through B, but not in the plane of FBD. In this plane draw BD, which suppose to be rigidly connected with BC, and let Dn be the direction of motion of D in revolving round the axis BK. Now by virtue of the two axes in the plane FBD, the point D is at liberty to move in the direction Dm perpendicular to that plane, but in no other. But let a third axis of flexure be introduced into the system passing through B in any direction not in the plane FBD, and let mn be the line of motion round this new axis, then the triangle Dmn can always be constructed upon Dn, and thus, as before, Dn be described by the simultaneous motions Dm and mn.

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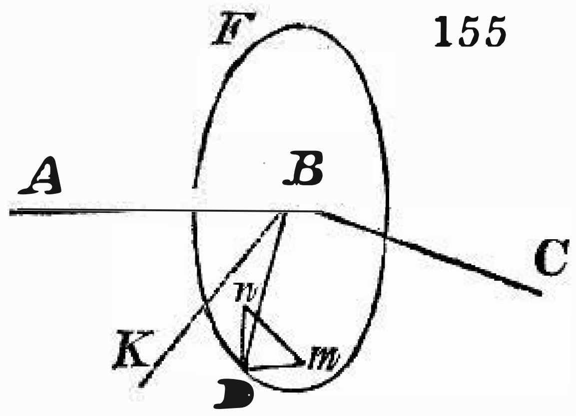

Hence it appears, that if two pieces are connected by three joints of flexure whose axes intersect in a point, and make any angles with each other, but are not in the same plane, these pieces are at liberty to turn about the point of intersection round any axis of flexure whatever which passes through that point.

If the axes of the joints pass each other without meeting in a point, it can be similarly shewn that the moving piece has still the unlimited choice of an axis in direction, and that this axis will lie somewhere between the component axes.

311. The joints by which the members of crustaceous animals and insects are united, furnish many beautiful examples of these principles.

Every separate joint in these animals is a hinge-joint very curiously constructed, but of course possessing but a single axis of flexure; these axes however are grouped so as to produce compound joints having two or three axes of flexure, and therefore forming universal joints, or swivel-joints, in the manner explained in the previous Article.

As an example of this we may take the front claw of the common crab, represented in fig. 156. This consists in fact of five separate pieces, *A*, *B*, *C*, *D*, *E*, not including the moveable jaw *F* of the actual claw; each piece is jointed to the next by a hinge-joint. But upon our principles the entire limb may be considered to consist of two principal members *C* and *E*; of which the first is jointed to the body of the animal by a universal joint of three axes of flexure, and the second to the first by a joint of two axes of flexure, or Hooke's joint.

For the piece *C* is united to the claw *E* by means of an

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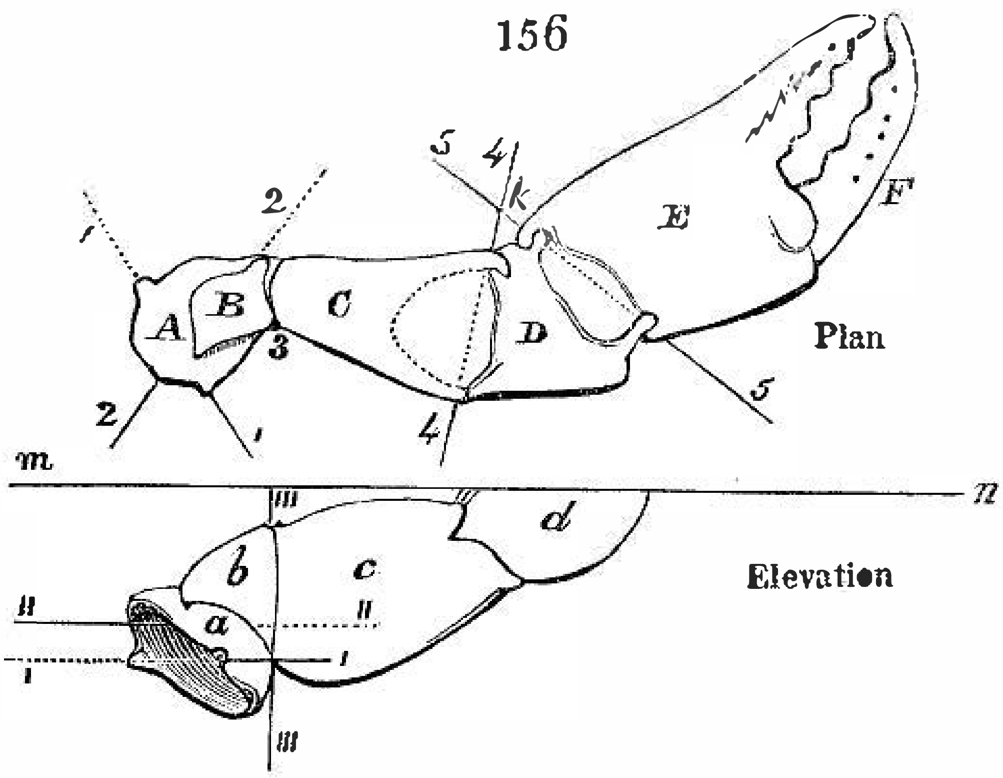

intermediate piece *D*, and the axes of the joints which connect them are shewn by the line 5,5 between *E* and *D*, and 4,4 between *D* and *C*. These axes meet in a point *k*, and therefore by what has preceded, it appears that *E* moves with respect to *C* about the point *k*, and that it is at liberty to turn round any axis of flexure passing through that point and in the plane 5, *k*, 4. So that this is in fact a natural Hooke's joint. The compound joint which connects the piece *C* with the body of the animal is more complex; and to exhibit its arrangement, two projections are given, one upon a plane perpendicular to the other, and intersecting it in the line *mn*.

We may suppose the claw to be laid down on the table in the upper figure, in which case this becomes the Plan and and the lower the Elevation, although the figures are drawn without any relation to the position of the claw with respect to the body of the animal, but only so as best to exhibit the joints, as will appear presently.

A ring *A* or *a* is jointed to the body of the animal by a joint whose axis is 1, 1, in the Plan, and I, I, in the Elevation. This is jointed to a second ring *B*, or *b*, by an axis 2, 2, or II, II; and *B* is jointed to *C* by a third axis vertical in the Plan, whose projection is therefore a point 3. It is shewn at III, III, in the Elevation. *C* is therefore connected to the body of the animal by a compound joint of three axes, whose directions nearly meet, but of which no two are parallel, neither are they in three parallel planes, and therefore, by Art. 310, *C* is at liberty to move about an axis situated at any angle with respect to the body. The compound joint, in fact, corresponds to the ball and socket joint employed for the shoulder of vertebrate animals. Its motions in different directions are of course limited by the extent of angular motion of which each separate hinge is capable.

The diagram is reduced from a very careful drawing. I found that the axis 2,2 was as nearly as possible in a plane perpendicular to 3, and that when the ring *A* was placed in its mean position, the axis 1, 1 was also in a plane perpendicular to 3. This determined the choice of the position of the planes of projection.

That of the Plan is parallel to the joints 1,1, 2,2, and therefore perpendicular to the joint 3, which thus becomes a point. The plane of the Elevation is parallel to the point 3.

As to the joints 4,4, 5,5, the joint 4,4 is in the drawing a little overstrained to allow 5,5 to come into parallelism with the plane of the paper; and 4,4, is also not in reality exactly perpendicular to 3. However, it must be understood that my object here is not to shew the relation of the limb to the body of the animal, but merely the principle of arrangement of the joints.

The claw E is shewn in its extreme outward position with respect to C; in its mean position it would be at right angles to the paper; and in the extreme inward position E and C would come into contact, to allow of which the shape of the intermediate piece and position of the hinges are beautifully adapted.

(See Errata)

Class B. Division ~~B~~ E. COMMUNICATION OF MOTION BY REDUPLICATION.

312. In the examples of Reduplication contained in the corresponding division of Class A, the strings and the motion of the follower are all parallel, and the velocity ratio constant. If the strings and the paths make angles with each other, a varying velocity ratio will ensue; as in the following example. Let the string be fixed at A, fig. 157, and passing over a pin B, let it be attached to a point C; let Bb be the path of the pin, Cc that of the extremity of the string, and when C is moved to c, very near to its first position, let B be carried to b; draw perpendiculars bm, bn, Cp, upon the two directions of the string in its new position.

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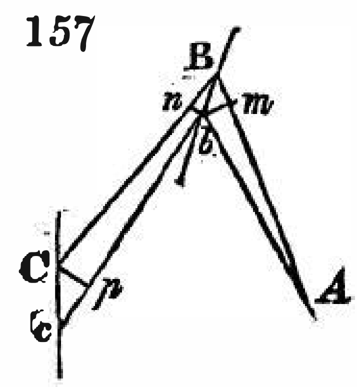

Then since the length of the string is the same in both positions, we have $AB + BC = Ab + bc$, that is,

$$Am + mB + Bn + nC = Ab + bp + pc,$$

But ultimately,

$$Ab = Am, \text{ and } bp = nC; \quad \therefore mB + Bn = pc,$$

$$\text{or} \quad Bb(\cos bBA + \cos bBC) = Cc \cdot \cos cCB;$$

$$\therefore \frac{Bb}{Cc} = \frac{\cos cCB}{\cos bBA + \cos bBC};$$

where the angles are those made by the direction of the string with the respective paths of the pin B and of the extremity C. But by the motion of the system these angles alter, and thus the velocity ratio varies.

If the strings and the path of B become parallel, the cosines become unity, and $\frac{Bb}{cC} = \frac{1}{2}$, as before (Art. 30).