CHAPTER VII.

TRAINS OF ELEMENTARY COMBINATIONS.

CLASS A. $\begin{cases}\text{DIRECTIONAL RELATION CONSTANT.}\\ \text{VELOCITY RATIO CONSTANT.}\end{cases}$

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217. THE elementary combinations which have been the subject of the preceding chapters consist, for the most part, of two principal pieces only, a driver and a follower; and we have shewn how to connect these so as to produce any required constant velocity ratio, or constant directional relation, whatever may be the relative position of the axes of rotation. There are many cases however in which, although *theoretically* possible, it may be *practically* inconvenient, or even impossible, to effect the required communication of motion by a single combination; in which case a series or train of such combinations must be employed, in which the follower of the first combination of the train is carried by the same axis or sliding piece to which the driver of the second is attached; the follower of the second is similarly connected to the driver of the third, and so on.

218. In all the combinations hitherto considered the principal pieces either revolve or travel in right lines. In a train of revolving pieces, the first follower and second driver being fixed to the same axis, revolve with the same angular velocity; and this is true for the second follower and third driver, and generally for the m^{th} follower and $\overline{m+1}|^{\text{th}}$ driver, which will also, if the piece which carries them travel in a right line, move with the same linear velocity.

But, for simplicity, let us consider all the pieces in the train to revolve (Art. 39), and let the synchronal rotations of the axes of the train in order be

$$L_1, L_2, L_3, L_4, \&c. \ldots\ldots L_m,$$

m being the number of axes;

$$\therefore \frac{L_1}{L_2} \times \frac{L_2}{L_3} \times \frac{L_3}{L_4} \ldots\ldots\ldots \frac{L_{m-1}}{L_m} = \frac{L_1}{L_m};$$

that is; *the ratio of the synchronal rotations of the extreme axes of the train is found by multiplying together the separate synchronal ratios of the successive pairs of axes.* Also, if $A_1 A_2 \ldots A_m$ be the angular velocities of the axes, we have

$$\frac{A_1}{A_2} \times \frac{A_2}{A_3} \ldots\ldots \frac{A_{m-1}}{A_m} = \frac{A_1}{A_m} = \frac{L_1}{L_m} \text{ (Art. 20).}$$

219. And since the values of any one of these separate ratios will be unaffected by the substitution of any pair of numbers that are in the same proportion, we may substitute indifferently in any one the numbers of teeth (N), the diameters (D), or radii (R), of rolling wheels, pitch-circles, or pullies, the periods (P) in uniform motion; or express the value of the ratio in any other equivalents that may be most easily obtained from the given machine or train whose motions we wish to calculate, recollecting that

$$\frac{L}{l} = \frac{A}{a} = \frac{n}{N} = \frac{r}{R} = \frac{p}{P}, \text{ (Art. 69).}$$

220. Ex. 1. In a train of wheel-work let the first axis carry a wheel of N_1 teeth driving a wheel of n_2 teeth on the second axis; let the second axis carry also a wheel of N_2 teeth driving a wheel of n_3 teeth on the third axis, and so on.

$$\frac{A_m}{A_1} \text{ or } \frac{L_m}{L_1} = \frac{N_1}{n_2} \times \frac{N_2}{n_3} \times \ldots\ldots\ldots \frac{N_{m-1}}{n_m},$$

that is, *to find the ratio of the synchronal rotations, or angular velocity of the last axis in a given train of wheel-work to those of the first, multiply the numbers of all the drivers for a numerator, and of all the followers for a denominator.*

It is scarcely necessary to remark that the number of drivers and of followers in a train of this kind is less by one than the number of axes.

221. Ex. 2. The ratios may each be expressed in a different manner: thus in a train of five axes, let the first revolve once while the second revolves three times;

$$\therefore \frac{L_1}{L_2} = \frac{1}{3}.$$

Let the second carry a wheel of 60 teeth driving a pinion of 20 on the third;

$$\therefore \frac{N_2}{n_3} = \frac{60}{20}.$$

Let the third axis drive the fourth by a belt and pair of pullies of 18 and 6 inches diameter respectively;

$$\therefore \frac{D_3}{d_4} = \frac{18}{6}.$$

And let the fourth perform a revolution in ten seconds, and the last in two, when the machinery revolves uniformly;

$$\therefore \frac{P_4}{p_5} = \frac{10}{2};$$

therefore we have,

$$\frac{L_1}{L_5} = \frac{1}{3} \times \frac{20}{60} \times \frac{6}{18} \times \frac{2}{10} = \frac{1}{135};$$

that is to say, that the first axis will perform one revolution while the last revolves 135 times.

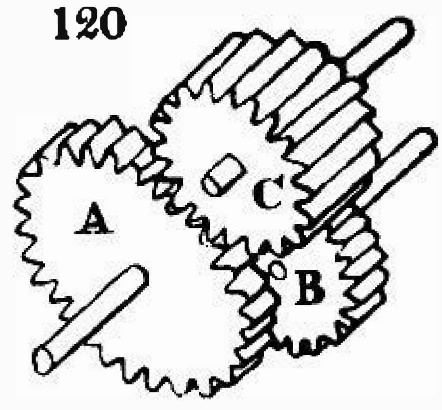
222. In this manner the synchronal rotations of the extreme axes in any given machine may be calculated; their directional relation may also be found, by examining in order the connexion of the axes, and by help of the few remarks which follow.

In a train of wheel-work consisting solely of spur-wheels or pinions with parallel axes, the direction of rotation will he alternately to right and left. If therefore the train consist of an even number of axes, the extreme axes will revolve in opposite directions, but if of an odd number of axes, then in the same direction. If an annular wheel be employed, its axis revolves the same way as that of the pinion (Art. 58).

223. If a wheel *A* (fig. 17, page 42) be placed between two other wheels *C* and *B*, it will not affect the velocity ratio of these wheels, which is the same as if the teeth of *B* were immediately engaged with those of *C*, but it does affect the directional relation; for if *B* and *C* were in contact, they would revolve in opposite directions, but in consequence of the introduction of the intermediate axis of *A*, *B* and *C* will revolve in the same direction. Such an intermediate wheel is termed an *idle wheel*.

224. When the shafts of two wheels *A* and *B*, fig. 120, lie so close together that the wheels cannot be placed in the same plane without making them inconveniently small, they may be fixed as here shewn, so as to lie one behind the other, and be connected by an idle wheel *C*, of rather more than double the thickness of the wheels it connects. Such a thick idle wheel is termed a *Marlborough wheel*, in

120

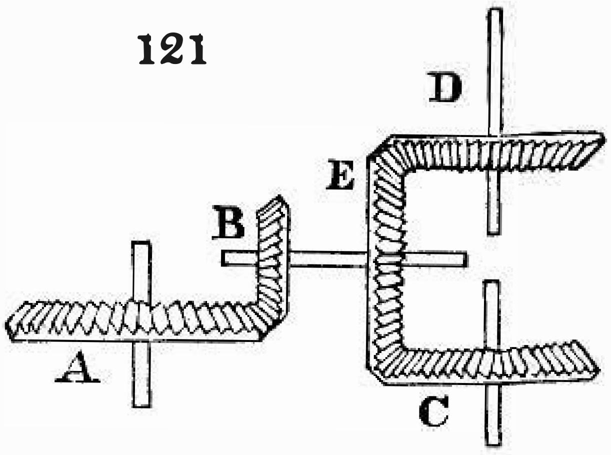


some districts. It is employed in the roller frames of spinning machinery.

225. When the axes in a train are not parallel, the directional relation of the extreme axes can only be ascertained by tracing the separate directional relations of each contiguous pair of axes in order.

By intermediate bevil-wheels parallel axes may be made to revolve either in the same or opposite directions according to the relative positions of the wheels; for example, in fig. 121 the wheel *A* drives *B*, upon whose shaft is fixed the wheel *E*. Now if the wheel *C* be fixed on the same side of the intermediate axis as *A*, the parallel axes of *A* and *C* will revolve in opposite directions; but if the wheel be fixed as at *D*, on the opposite side of the intermediate axis, then the axes of *A* and *D* will revolve in the same direction, the same number of wheels being employed in both cases.

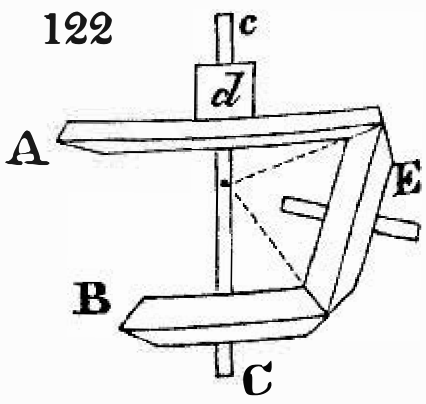
121



Endless screws may be represented in calculation by a pinion of one or more leaves, according to the number of their threads, (Art. 175), but their effect upon the directional relation of rotation will be different, according as they are right-handed or left-handed screws. (Art. 169).

226. Two separate wheels or pieces in a train may revolve concentrically about the same axes, as for example, the hands of a clock. Also, in fig. 122, the wheel *B* is fixed to an axis *Cc*, and the wheel *A* to a tube *d* or *cannon*, which turns freely upon *Cc*. If these wheels may revolve in op-

122



posite directions, a single bevil-wheel E will serve to connect them, if the three cones have a common apex as in the figure; and since E is an idle wheel (Art. 223), the velocity ratio of B to A will depend solely upon the radii of their own frusta.

But if the wheels B, A are to revolve in the same direction, they must be made in the form of spur-wheels, and connected by means of two other spur-wheels fixed to an axis parallel to Cc.

227. Millwrights imagine that in a given pair of toothed wheels it is desirable that the individual teeth of one wheel should come into contact with the same teeth of the other wheel as seldom as possible, on the ground that the irregularities of their figure are more likely to be ground down and removed by continually bringing different pairs of teeth into action.

This is a very old idea, and is stated nearly in the above words by De la Hire. It has also been acted upon up to the present time. Thus Oliver Evans tells us, that "great care should be taken in matching or coupling the wheels of a mill, that their number of cogs be not such that the same cogs will often meet; because if two soft ones meet often, they will both wear away faster than the rest, and destroy the regularity of the pitch; whereas if they are continually changing they will wear regular, even if they be at first a little irregular*."

The clockmakers on the other hand, think that the wearing down of irregularities will be the best effected by bringing the same pair of teeth into contact as often as possible†.

* O. Evans, Young Millwright's Guide, Philadelphia, 1834, p. 193. Vide also Buchanan's Essays, by Rennie, p. 117.

† Francœur, Mécanique Elémentaire, p. 143.

Let a wheel of M teeth drive a wheel of N teeth, and let $\frac{M}{N} = \frac{m}{n}$ when m and n are the least numbers in that ratio;

$$\therefore \; nM = mN,$$

and n is the least whole number of circumferences of the wheel M that are equal to a whole number of circumferences of the wheel N.

If, therefore, we begin to reckon the circumferences of each wheel that pass the line of centers, after a given pair of teeth are in contact, it is clear that after n revolutions of M, and m of N, the same two teeth will be again in contact. Neither can they have met before; for as the entire circumference of one wheel applies itself to the entire circumference of the other tooth by tooth, and as the numbers m and n are the least multiples of the respective circumferences that are equal, it follows that it is only after these respective lengths of circumferences have rolled past each other that the beginnings of each can again meet.

If we act on the watchmaker's principle, by which the contacts of the same pair are to take place very often, the numbers of the wheels M and N must be so adjusted that m and n may be the smallest possible, without materially altering the ratio $\frac{M}{N}$; and this will be effected by making the least of the two numbers m, n equal to unity, and therefore M a multiple of N.

But if the millwright's principle be adopted, m and n must be as large as possible, that is, equal to M and N, or in other words, M and N must be prime to each other. The millwrights employ a *hunting cog* for this purpose. Suppose, for example, that a shaft is required to revolve about three times as fast as its driving shaft, 72 and 24 are a pair of numbers for teeth that would produce this effect and would suit a watchmaker, one being a multiple of the other; but the millwright would add one tooth to the wheel (the hunting cog), and thus obtain 73 and 24, which are prime to each other, and very nearly in the desired ratio*.

228. Sometimes also the nature of the mechanism requires that the wheels shall come as seldom as possible into the same relative positions, and in that case the principle may be applied to a train of several axes. For example, in a train of three axes, in which the drivers have each 22 teeth, and the followers 25 and 35 teeth, we have

$$\frac{L_1}{L_3} = \frac{25 \times 35}{22 \times 22} = \frac{484}{875};$$

which numbers are prime to each other, and therefore the extreme wheels of the train will not return to the same relative position, until one has made 484, and the other 875 revolutions. These are the numbers of the old Piemont silk-reel (1·724), which is an excellent example of this principle†.

229. We are now able to calculate the relative motions of the parts in a given machine in which the velocity ratios are constant. The inverse problem is one of considerable importance in the contrivance of mechanism; namely, *Given the velocity ratio of the extreme axes or pieces of a train, to determine the number of intermediate axes, and the proportions of the wheels, or numbers of their teeth.* For simplicity we may suppose the train to consist of toothed wheels only; for a mixed train, consisting of wheels, pullies, link-work, and sliding pieces, can be calculated upon the same principles. Let the synchronal rotations of the first and last axes of the train be

* In a pair of wheels whose numbers are so obtained, any two teeth which meet in the first revolution are distant by one in the second, by two in the third, and so on; so that one tooth may be said to *hunt* the other, whence the phrase, a hunting cog.

† Encycl. Méthodique, Manufactures et Arts, tome II. p. 20.

L_1 and L_m respectively, and let $N_1\, N_2 \ldots$ &c. be the numbers of teeth in the drivers, and $n_1\, n_2 \ldots$ in the followers: then by Art. 220,

$$\frac{L_m}{L_1} = \frac{N_1 . N_2 . N_3 \ldots}{n_1 . n_2 . n_3 \ldots}$$

and by hypothesis the value of $\frac{L_m}{L_1}$ is given, and we have to find an equal fraction whose numerator and denominator shall admit of being divided into the same number of factors of a convenient magnitude for the number of teeth of a wheel. Also to find the value of m.

Synchronal rotations are preferred to angular velocities in stating the question, because it is generally in this form that the data are supplied.

230. In any given train of wheel-work the drivers may be placed in any order upon the axes as well as the followers; for the value of the fraction $\frac{N_1 . N_2 . N_3 \ldots}{n_1 . n_2 . n_3 \ldots}$ will be unaffected by any change of order in the factors, and therefore N_1 may be placed either upon the first, second, or third axes; and similarly for the others.

231. Let w be the greatest number of teeth that can be conveniently assigned to a wheel, and p the least that can be given to a pinion. The train may be either required for the purpose of reducing or increasing velocity. In the first case, L_m will be less than L_1, and the pinions the drivers; but in the second case, L_m will be greater than L_1, and the wheels the drivers.

Let $\therefore\ \frac{L_1}{L_m}$ or $\frac{L_m}{L_1} = \left(\frac{w}{p}\right)^k$ where k may be a whole number, or a fraction. Take m equal to $k+1$ (Art. 220) if a whole number, or to the next greatest whole number to $k+1$ if a fraction. This will plainly be the least value that can be given to m.

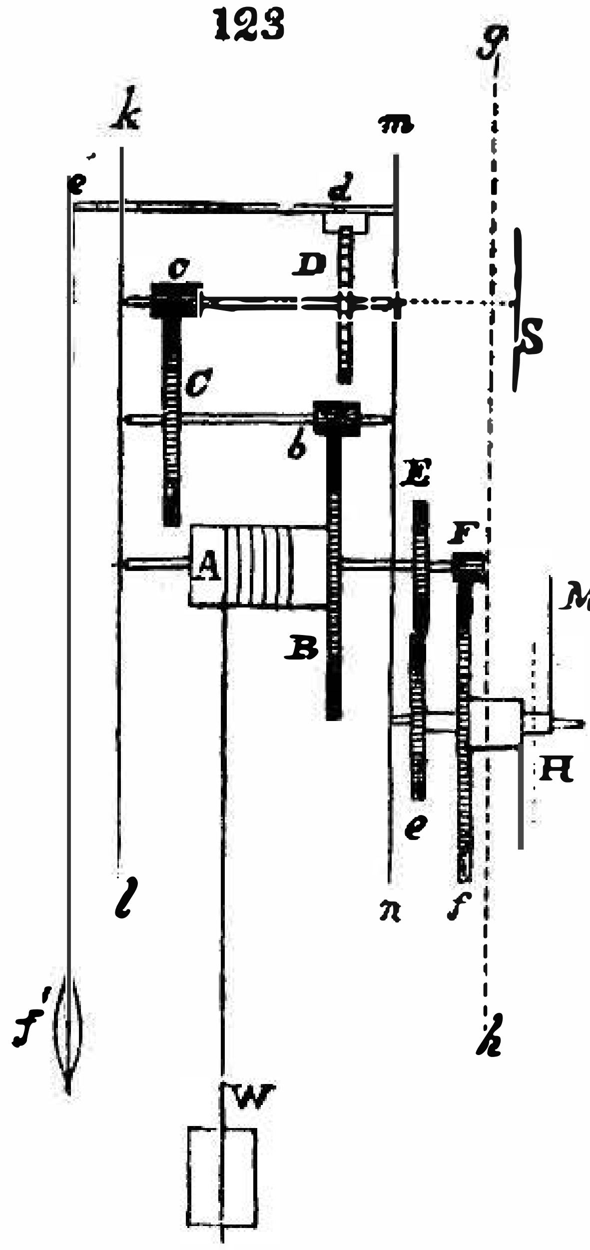
For m must be a whole number, and if it be taken less than $k+1$ then the values of $\frac{w}{p}$ will be greater; that is, either w will become a greater number than can be assigned to a wheel, or p a less than can be given to a pinion, which is absurd.

No general rule can be given for determining the values of w and p, which are governed by considerations that vary according to the nature of the proposed machine; also, it will rarely happen that the fraction will admit of being divided into factors so nearly equal as to limit the number of axes to the smallest value so assigned.

The discussion of a few examples will best explain the mode of proceeding in particular cases.

232. Fig. 123 is a diagram to represent the arrangement of the wheel-work of a clock of the simplest kind, for the purpose of illustrating what follows upon trains of wheel-work in general.

123



The weight W is attached to the end of a cord, which is coiled round the barrel A. Upon the same axis or *arbor** as the barrel is fixed a toothed wheel B, and this wheel drives a pinion b, which is fixed to the second arbor Cb of the train, which also carries a wheel C. This wheel drives a pinion c upon the third arbor, and upon this arbor is also fixed a toothed wheel D of a peculiar

* *Arbor* is the watchmakers' term for an axis; vide Note p. 44.

construction, termed an escapement wheel or swing-wheel. Above this wheel is an arbor *ed* termed the verge, which is connected with the pendulum *ef* of the clock, and vibrates together with it through a small arc. The verge also carries a pair of teeth which are termed pallets, and are engaged with the teeth of the swing-wheel D in such a manner, that every vibration of the pendulum and verge allows one tooth of the wheel to *escape* and pass through a space equal to half the pitch. With the nature of this connexion we have at present nothing to do; for, as the motion of the clock-work is our only object, it is sufficient to know that one tooth of the swing-wheel passes the line of centers for every two vibrations of the pendulum.

Let the time of a vibration of the pendulum be t seconds, where t is a whole number or a fraction, and let the swing-wheel have e teeth, then the period or time of a complete rotation of this wheel is $2te$. To take a simple case, let the pendulum be a seconds' pendulum; $\therefore\ t = 1$, and if $e = 30$, the swing-wheel will revolve in a minute; and if B have 48 teeth, and C 45, and the pinions 6 leaves each, we have for the train

$$\frac{L_3}{L_1} = \frac{48 \times 45}{6 \times 6} = 60\,;$$

therefore A will revolve in an hour; and supposing the cord to be coiled about sixteen times round the barrel, the weight in its descent will uncoil it and turn the barrel round, communicating motion to the entire train until the cord is completely uncoiled, which it will be after sixteen hours.

This train of wheel-work is solely destined to the purposes of communicating the action of the weight to the pendulum in such a manner as to supply the loss of motion from friction and the resistance of the air. But besides this, the clock is required to indicate the hours and minutes by the rotation of two separate hands, and accordingly two other trains of wheel-work are employed for this purpose.

The train just described is generally contained in a frame consisting of two plates, shewn edgewise at *kl*, *mn*, which are kept parallel and at the proper distance by means of three or four pillars, not shewn in the diagram. Opposite holes are drilled in these plates, which receive the pivots of the axes or arbors already described. But the axis which carries *A* and *B* projects through the plate, and other wheels *E* and *F* are fixed to it.

Below this axis and parallel to it a stout pin or *stud* is fixed to the plate, and a tube revolves upon this stud, to one end of which is fixed the minute-hand *M*, and to the other a wheel *e* engaged with *E*. In our present clock *E* revolves in an hour, consequently the wheels *E* and *e* must be equal.

A second and shorter tube is fitted upon the tube of the minute-hand so as to revolve freely, and this carries at one end the hour-hand *H*, and at the other a wheel *f*, which is driven by the pinion *F*; and because *f* must revolve in twelve hours, it must have twelve times as many teeth as *F*.

233. To exhibit the ramifications of motion in a machine, and the order and nature of the several parts of which the trains are composed, it is convenient to employ a *notation*. This notation should be of such a form as not only to exhibit these particulars, but also to admit of the addition, if necessary, of dimensions and nomenclature, as well as to allow of the necessary calculations by which the velocity ratios may be deduced. To exhibit in this way the actual arrangement of the parts is out of the question; this can only be done by drawings, and the very object of a notation is to unravel the apparent confusion into which the trains of

motion are thrown by the *packing* of the parts into the frame of the machine, and to place them in the order of their successive action.

Clock and watchmakers have long employed a system which consists simply in representing the wheels by the numbers of their teeth, and writing these numbers in successive lines, placing the wheels which are fixed on the same arbor on the same horizontal line, with the sign – interposed, and writing the numbers of the wheels that are in geer vertically over each other. The first driver in the train is always placed at the top of the series.

Thus in the principal train of the clock, fig. 123, if the letters represent the wheels we should write down the train thus:

B
b——*C*
    c——*D*;

or, employing the numbers already selected,

48
6——45
    6——30,

and adding the names, which is sometimes done,

Great wheel 48,
  Pinion 6——45 second-wheel,
    Pinion 6——30 swing-wheel*.

* Farey in Rees' Cyclopædia, art. *Clockwork*, calls this the ordinary mechanical method of writing down the numbers. Oughtred in his Opuscula, 1677, proposes another method in which the wheels which are on the same axis are written vertically over one another, and those which are in geer are placed in the same line with the character) between; thus, (the first driver being at the bottom, and all the drivers to the right of the followers):

30
6)45
  6)48

He employs, however, letters in lieu of figures, and introduces other artifices which are scarcely worth dwelling upon. Derham (Artificial Clockmaker, 1696)

234. This method requires very little addition to make it a very convenient system for mechanism in general. Thus the entire movement of the clock, fig. 123,

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Barrel 48 ———————— 25 ———————— 4
        6 — 45       :          :
            6 — 30 swing-wheel  :
                     25 — minute-hand
                                48 — hour-hand
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may be thus represented, and by which is shown very clearly the three trains of mechanism from the barrel to the swing-wheel, the minute-hand and the hour-hand; as well as the distinction of the pieces into drivers and followers, and the nature of their connexion; namely, whether they be permanently united by being fixed upon the same axis, or connected by geering. If however other connexions are introduced, as by wrapping-bands, or links, this must be written in the diagram, or expressed by a proper sign. I shall have occasion to return to this subject in a future page*.

235. In the explanation of the clock, fig. 123, I have assumed the numbers of the wheel-work and of the axes; let us now examine whether these are the best for the purpose, or generally how such numbers would be determined.

If the arbor of the swing-wheel revolve in a minute,

follows this method, and also uses another which consists in writing all the numbers in one line, thus, 48) 6 — 45) 6 — 30, where the character) implies that the wheels between which it lies geer together, and — that they are fixed on the same axis. Allexandre, Traité général des Horloges, 1735, writes the numbers thus, 48.6 — 45.6 — 30; and Derham also gives the "usual way of watchmakers in writing down their numbers," thus,

48
45 — 6
30 — 6

which, to use his own words, "though very inconvenient in calculation, representeth a piece of work handsomely enough, and somewhat naturally."

* Mr. Babbage is the only one who has endeavoured to extend Notation to Mechanism in general. His elaborate and complete system is fully explained in his paper on "A method of expressing by signs the action of Machinery," in the Philosophical Transactions, 1826, vide below, Chap. ~~ix.~~ x (See Errata)

and that of the barrel in an hour, we have $\frac{L_m}{L_1} = 60$; or if D be the product of all the drivers, and F of the followers, $D = 60nF$, an indeterminate equation, for the solution of which any numbers may be employed that are proper for the teeth of wheels. Now in common clocks six is the least number of leaves that is ever employed in a pinion, and 60 teeth the greatest number that can be given to a wheel;

$$\therefore \frac{w}{p} = \frac{60}{6} = 10.$$

Now $\frac{L_m}{L_1} = 10^{1.8}$, therefore by Art. 231, 3 is the least number of axes; and there will be two pinions of six each, $\therefore D = 60 \times 6^2 = 2160$, which is the product of two wheels.

We are at liberty to divide this into any two suitable factors. The best mode of doing it is to begin by dividing the number into its prime factors, writing it in this form:

$$2160 = 2^4 \times 3^3 \times 5.$$

For this enables us to see clearly the composition of the number; and it is easy to distribute these factors into two groups; as for example,

$$2^4 . 3 \times 3^2 . 5 = 48 \times 45, \text{ or } 2^3 . 5 \times 2 . 3^3 = 40 \times 54,$$

$$\text{or } 2^2 . 3^2 \times 2^2 . 3 . 5 = 36 \times 60.$$

The nearest to equality is the first, 48 and 45; and these will probably be selected for the train, which will stand thus:

$$\frac{D}{F} = \frac{48 \times 45}{6 \times 6}.$$

This is the best form in which to exhibit the numbers for a train when they have been merely divided into proper factors for teeth. If the distribution of the wheels and pinions upon the several axes is also settled, the train may then be written in the form 48

6——45

6.

236. Six is however too small a number of leaves to ensure perfect action in a pinion, for it appears in the Table (p. 93) that a pinion of 6 will only work with a wheel of 20 when the receding arc of action is equal to $\frac{2}{3}$ × pitch, and that if this arc be greater, the pinion becomes impossible. A pinion of 8 will be better, but 10 or 12 should be employed if a very perfect action is required. If 8 be selected, we have $F = 8^2 = 64$, and $D = 64 \times 60$, which will form a good train.

But in well-made clocks we may allow more than 60 teeth to the wheel: 100 or even 120 is very admissible. If we begin, then, with the wheels, and assume that three arbors are to be employed,

$$\text{let } \frac{D}{F} = \frac{(100)^2}{p^2} = 60\,; \quad \therefore \; p = 13, \text{ nearly.}$$

$$\text{Assume, therefore, } F = 12 \times 14\,; \quad \therefore \; D = 60 \times 12 \times 14$$
$$= 96 \times 105\,;$$

which gives the train 105

14——96

12

237. In a train of $\overline{k+1}$. axes of which every wheel has w teeth, and every pinion p leaves, we have

$$\frac{L_m}{L_1} = \left(\frac{w}{p}\right)^k = x^k \text{ if } \frac{w}{p} = x.$$

Now xp $(= w)$ is the number of teeth in each wheel, and $k\,(p + xp)$ is the entire number of teeth in the train.

$$\text{Let } \left(\frac{w}{p}\right)^k \text{ or } x^k = \text{constante} = C\,;$$

$$\therefore \; k = \frac{1\,C}{1\,x},$$

$$\text{and number of teeth} = \frac{1\,C}{1\,x} \cdot p \cdot (1 + x)$$
$$= \text{a minimum.}$$

Differentiating we obtain in the usual manner,

$$1\,x = \frac{1 + x}{x}\ ; \quad \text{whence } x = 3.59.$$

If therefore a given angular velocity ratio is to be obtained with the least number of teeth, we must make $\frac{w}{p} = 3.59$. This theorem is due to Dr. Young*.

As a practical rule this is not of much value, for it proceeds on the assumption that simplicity is best consulted by reducing the number of teeth *only* as much as possible; but, in fact, it is necessary in doing this to avoid also increasing the number of axes in a train. For example, in our clock $\frac{L_m}{L_1} = 60$, which being greater than the cube of 3.59 would require for the least number of teeth at least three wheels; and, in fact, if we compute the number of teeth required in the case of one, two, three, and four wheels, assuming the number of leaves in the pinions to be six, we find, putting D for the denominator, and dividing it into convenient factorsn

	Wheels.	Total Number of Teeth.
one wheel	$D = 6 \times 60 = 360$	$360 + 6 = 366$
two wheels	$D = 6^2 \times 60 = 45 \times 48$	$45 + 48 + 2 \times 6 = 105$
three wheels	$D = 6^3 \times 60 = 20 \times 27 \times 24$	$20 + 27 + 24 + 3 \times 6 = 89$
four wheels	$D = 6^4 \times 60 = 15 \times 16 \times 18 \times 18$	$15 + 16 + 18 + 18 + 4 \times 6 = 91$
five wheels	$D = 6^5 \times 60 = 12^3 \times 15 \times 18$	$3 \times 12 + 15 + 18 + 5 \times 6 = 99$

So that, as the theorem has already taught us, the least number of teeth, 89, is required when three wheels are employed. But the universal practice is to employ two wheels and pinions only in the train between the hour-arbor and swing-wheel arbor, for, in fact, the increase in the number of

* Young's Nat. Philosophy, vol. II. p. 56.

teeth does not occasion so great a loss of simplicity as the additional arbor with its wheel and pinion would do. Some mechanicians have fallen into the opposite error of supposing that the simplicity of the clock would be still more improved by reducing the train to a single wheel and pinion, and hence increasing inordinately the number of teeth in the wheel. Of this nature are Ferguson's and Franklin's clocks*.

238. If a clock has no seconds' hand there is no necessity for the arbor of the swing-wheel to perform its revolution in a minute, which when the pendulum is short, would become impracticable, from the great number of teeth required. Now from Art. 232, if t be the time of vibration of the pendulum in seconds, and e the number of teeth of the swing-wheel, $2te$ is time of rotation of the swing-wheel.

But the vibrations of small pendulums are commonly expressed by stating the number of them in a minute. Let p be this number, $\therefore \frac{2e}{p}$ is the time of one rotation of the swing-wheel in minutes, and the hour-arbor revolves in 60 minutes; the train between them is represented by $\frac{D}{F} = \frac{30p}{e}$.

Ex. The pendulum of a clock makes 170 vibrations in a minute, and there are 25 teeth in the swing-wheel, and eight leaves are to be given to the pinions; to find the wheels:

$$\frac{D}{64} = \frac{30 \times 170}{25};$$

$$\text{whence } D = 13056 = 128 \times 102.$$

239. In a watch the vibrations of the balance are much more rapid than in any pendulum-clock, varying in different constructions from 270 to 360 in a minute. Also, from the

* Vide Ferguson's Mechanical Exercises, or any Encyclopædia.

small size of the machinery, it becomes impossible to put so many teeth into the wheels. The escapement-wheel, termed in a watch the balance-wheel, has from 13 to 16 teeth, instead of having, as in a clock, from 20 to 40, and the numbers of teeth in the wheels vary from 40 to 80, or in chronometers and larger work are sometimes carried as high as 96, whereas in large clocks, 130 may even be employed. Now as the number of leaves in the pinions do not admit of reduction, the consequence is, that an additional arbor must be employed in watches, and the train of wheel-work between the hour-arbor and the arbor of the balance-wheel consists of 3 wheels and 3 pinions, instead of the two pair employed in a clock.

Ex. The balance of a watch makes 360 vibrations in a minute, and there are 15 teeth in the balance-wheel, and eight leaves in the pinions; to find the wheels:

Here $F = 8 \times 8 \times 8$,

$$\text{and } D = 8^3 \frac{30 \times 360}{15} = 368640 = 80 \times 72 \times 64.$$

240. The examples of clock-trains already given, refer merely to the connexion between the hour-arbor and the swing-wheel, and it has been assumed throughout that the barrel for the weight is carried by the hour-arbor; but in this case the clock will not go for more than sixteen hours, and must therefore be wound up every night and morning. If it be required to go longer the barrel must be fixed to a separate axis, and this connected by wheel-work with the hour-arbor, so that the barrel may revolve much more slowly, and consequently allow the weight to occupy a longer time in its descent.

Now the cord, as we have seen, is wound spirally round the barrel, and by making the barrel of the requisite length, we could of course make it hold as many coils as we please.

But in practice it is found that if more than about sixteen coils are placed on it, it becomes inconveniently long. So that if the clock be required to go for eight days without fresh winding up, each turn of the barrel will occupy twelve hours. As the arbor of the hour-hand revolves in one hour, any pair of wheels whose ratio is 12 will answer the purpose of connecting them; 96 and 8 are the numbers usually employed, which will produce this train:—

Train for Eight-day Clock.	Periods.
96	12^h
8 — 105	1^h
8 — 96	
8 — 30....	$1'$

241. If the clock be required to go a month, or 32 days, without winding, then supposing the barrel, as before, to have sixteen turns, each turn of the barrel will occupy 48 hours, and the train from the barrel to the hour-arbor $= \frac{D}{F} = 48$, which is too great a number for a single pair, but will do very well for two. If pinions of nine are employed,

$$D = 9 \times 9 \times 48 = 72 \times 54;$$

which numbers being small we are at liberty to employ larger pinions; for example, if we take twelve and sixteen,

$$D = 12 \times 16 \times 48 = 96 \times 96;$$

whence the following train:—

Train for Month-Clock.	Periods.
96	48^h
16 — 96	...
12 — 105..............	1^h
8 — 96......	...
8 — 30	$1'$

242. Now in the clock (fig. 123), the arbor of A is made to revolve in an hour, because the wheels E and e are equal. By making these wheels of different numbers, we get rid of the necessity of providing an arbor in the principal train that shall revolve in an hour, and may by that means, in an eight-day clock, or month-clock, distribute the wheels more equally. For example, in an eight-day clock let the swing-wheel revolve in a minute; and let the train from the barrel-arbor to this minute-arbor be $\frac{108 \times 108 \times 100}{12 \times 12 \times 10} = 810$, in which the barrel will revolve in 810 minutes or thirteen hours and a half, and consequently fourteen or fifteen coils of the cord will be sufficient.

The second wheel in this train, which in fig. 123 corresponds to B, will revolve in $\frac{12}{108} \times 810$ minutes, or an hour and a half, and on its arbor must be fixed, as in the figure, the two wheels E and F for the minute and hour-hands; consequently, the ratio of

$$\frac{F}{f} = \frac{1}{8}, \text{ and } \frac{E}{e} = \frac{3}{2}.$$

It is convenient that the size or pitch of the teeth in these two pairs should be about the same. To effect this, let x be the multiplier of the first ratio, and y of the second;

so that x and $8x$ are the numbers of teeth in the first pair, and $3y$, $2y$ in the second. Then, if the teeth of the two pairs be of the same pitch, we have

$$x + 8x = 3y + 2y, \text{ or } 9x = 5y; \therefore\ x = \frac{5y}{9}.$$

$$\text{Let } y = 9z;\ \therefore\ x = 5z;$$

and if $z = 1$, $y = 9$, $x = 5$, numbers are $\frac{5}{40}$ and $\frac{27}{18}$

$z = 2$, $y = 18$, $x = 10$, ... $\frac{10}{80}$ and $\frac{54}{36}$;

either of which may be adopted.

Train of Eight-day Clock.	Periods.
108	810′
12 — 108 —————— 54 —————— 10	90′
12 — 110	...
10 — 30	1′
36 — minute-hand	60′
80 — hour-hand	720′

I have confined the above examples to clock-work, because its action is more generally intelligible than that of other machines; but the principles and methods are universally applicable, or at least require very slight modifications to adapt them to particular cases.

TO OBTAIN APPROXIMATE NUMBERS FOR TRAINS.

243. If $\frac{L_m}{L_1} = a$ when a is a prime number, or one whose prime factors are too large to be conveniently employed in wheel-work, an approximation may be resorted to. For example, assume $\frac{L_m}{L_1} = a \pm E$. This will introduce an error of $\pm E$ revolutions of the last axis, during one of the first, and the nature of the machinery in question can alone determine whether this is too great a liberty.

But we may obtain a better approximation than this, without unnecessarily increasing the number of axes in the train; for determine in the manner already explained the least number m of axes that would be necessary if a were decomposable, and the number of leaves that the nature of the machine makes it expedient to bestow on the pinions, and let F be the product of the pinions so determined;

$$\therefore \frac{L_m}{L_1} \text{ or } \frac{D}{F} = \frac{Fa}{F}, \text{ supposing the wheels to drive.}$$

$$\text{Assume } \frac{D}{F} = \frac{Fa \pm E}{F};$$

where E must be taken as small as possible, but so as to obtain for $Fa \pm E$ a numerical value decomposable into factors. There will be in this case an error of $\pm E$ rotations in the last axis during F of the first, or of $\dfrac{\pm E}{F}$ rotations during one of the first.

If the pinions be the drivers, then in the same manner assume $\dfrac{L_1}{L_m} = \dfrac{Da \pm E}{D}$; and there will be an error of $\dfrac{\pm E}{D}$ rotations in the first axis during one of the last.

244. Ex. Let it be required to make $\dfrac{L_m}{L_1} = 269$ nearly. Now if the nearest whole number 270 be taken, a train may be formed, but with an error of one revolution in 270. But suppose that from the nature of the machine, a ratio of $\frac{1}{8}$ is the greatest that can be allowed between wheel and pinion,

then since 269 lies between 8^2 and 8^3, it appears that three pair of wheels and pinions are necessary.

If pinions of 10 are employed, $\dfrac{D}{F} = \dfrac{269000}{1000}$,

and $\dfrac{269001}{1000} = \dfrac{3^8 \times 41}{10^3}$, will make a very good train,

with an error of $\dfrac{1}{1000}$ of a revolution only in 269.

245. Ex. 2. Let it be required to find a train that shall connect the twelve hour-wheel of a clock with a wheel revolving in a lunation, $= 29^d. \ 12^h. \ 44'$ nearly, for the purpose of shewing the Moon's age upon a dial. Reducing the periods to minutes, we have

$$\frac{L\mathcal{S}_1}{L\mathcal{S}_m} = \frac{42524}{720},$$

of which the ~~denominator~~ $(= 2^2 \times 10631)$ contains a large prime, but

$$\frac{42524 + 1}{720} = \frac{945}{16} = \frac{3^3 . 5 . 7}{2^4},$$

is well adapted to form a train of wheel-work, with an error of one minute in a lunation.

246. This method is sufficient for ordinary purposes, but if greater accuracy be required, or if the terms of the fraction, although divisible into proper factors, should require so many wheels and pinions, as to make it necessary to find a fraction which shall approximate to the value in smaller terms, then *continued fractions* must be resorted to.

(See Errata)

$\dfrac{L\mathcal{S}_1}{L\mathcal{S}_m}$ being given in the form of a fraction with large terms, must be treated in the usual manner* to obtain the

* Vide Euler's Algebra, Barlow on Numbers, or Bonnycastle's Algebra, &c.

series of principal and intermediate fractions, which must be separately examined until one is found that will admit of a convenient division into factors, and at the same time approximate with sufficient accuracy.

247. Ex. *To find an annual train.*

Let it be required to find a train of wheel-work for a clock, by means of which a wheel may be made to revolve in an exact year, that is, in 365 days, 5 hours, 48 minutes, 48 seconds*.

If the hours, minutes, and seconds, be reduced to decimals of a day, the period becomes $365.24\dot{2}$ days; and supposing the pinion from which the motion is to be derived to revolve in one day, the required ratio becomes $\frac{365.24\dot{2}}{1.000}$, which by the common rule for circulating decimals is equal to

$$\frac{365242 - 36524}{900} = \frac{328718}{900} = \frac{164359}{450},$$

when in its lowest terms.

Now as the nearest whole number to this is 365, it appears that three axes, at least, would be required to produce this variation of motion, and therefore the fraction itself would not be in terms too great, provided it were manageable. Now

$$\frac{164359}{450} = \frac{269 \times 47 \times 13}{10 \times 9 \times 5};$$

which has an inconveniently large number, 269, but has been actually employed to form a train, in Mr. Pearson's Orrery for Equated Motions†, in this form,

$$\frac{269 \times 26 \times 94}{10 \times 10 \times 18}.$$

* The length of the year determined by different astronomers varies in the number of seconds from 47″.95 to 51″.6; the mean of five results is 49″.77.

† Rees' Cyclopædia, art. Orrery.

If the ratio be treated by the method of continued fractions, we obtain in the usual manner,

Quotients.		365	4	7	1	3	1	2	
Principal Fractions.	$\frac{0}{1}$	$\frac{1}{0}$	$\frac{365}{1}$	$\frac{1461}{4}$	$\frac{10592}{29}$	$\frac{12053}{33}$	$\frac{46751}{128}$	(B) $\frac{58804}{161}$	(A) $\frac{164359}{450}$
Intermediate Fractions.							$\frac{34698}{95}$ $\frac{22645}{62}$		(C) $\frac{105555}{289}$

The whole of these fractions will be found unmanageable, from containing large primes, with the exception of those marked *A*, *B* and *C*, of which *A* is the original fraction.

$$(B) = \frac{241 \times 61 \times 4}{7 \times 23} = \frac{241 \times 61 \times 52}{23 \times 13 \times 7}$$

corresponds to a period of $365^d.\ 5^h.\ 48'.\ 49''.19218$; this has been employed by Janvier*.

$$(C) = \frac{105555}{289} = \frac{227 \times 31 \times 15}{17 \times 17}$$

is equivalent to a period of $365^d.\ 5^h.\ 48'.\ 47''.3$, and is rather more accurate than the last; but as they each include a large wheel, it appears that the original fraction is quite as convenient.

248. If, as in the example just cited, the series of fractions obtained will not give a sufficiently convenient result, the more general method which follows may be employed, which however requires the calculation of the continued fractions, at least of the principal fractions, as they are called, and which, therefore, will not supersede the

* Rees' Cyclopædia, art. Planetary Machines.

method just explained, but may be used after it, should it be found to fail.

To find a fraction $\frac{x}{y}$ very near to $\frac{a}{b}$, we have their difference $= \frac{a}{b} - \frac{x}{y} = \frac{ay - bx}{by} = \frac{k}{by}$, suppose: k will be by the supposition a very small integer, compared with by, and either positive or negative; to find k, we have the indeterminate equation $ay - bx = k$. Let the fraction $\frac{a}{b}$ be converted into a series of principal converging fractions, and let $\frac{p}{q}$ be the last but one, then it can be shewn* that the following expressions will include all the solutions of this equation that are possible in integer numberse $x = pk + ma$, $y = qk + mb$,

$$\text{and } \frac{x}{y} = \frac{pk + ma}{qk + mb}$$

will be the approximate fraction required, in which m may be any whole number, positive or negative, as well as k, but k must be small with respect to by or ax. Thus a multitude of values of $\frac{x}{y}$ may be obtained, from whence the oneemay be chosen that best admits of decomposition into factors. The only part of this process which is left to choice is the selection of values for k and m. The numbers obtained from them for x and y must necessarily be small, for we are seeking numbers less than a and b, and therefore k and m must have different signs, but even with this limit there is an infinite latitude given to the choice.

Assume $k = 0, -1, +1, -2$, and so on; and in each case take such values of m as will make the values of x and y,

* Euler's Algebra, p. 530. Barlow on Numbers, p. 317. Francœur, Cours de Mathematiques, Art. 565. Par. 1819.

not too great for the purpose, trying always whether the pair of results are decomposable into factors, and if they be, then proceeding to calculate the consequent error. In this way a pair of numbers will at last be found, that will give sufficient exactness without employing too much wheel-work*. Tables of factors will greatly assist in these operations†.

249. For example, to find a fraction $\frac{x}{y}$ very near to $\frac{45}{14}$, (Art. 251.) the last fraction but one of the series of principal converging fractions, is $\frac{16}{5}$, and putting these numbers in the expression for $\frac{x}{y}$, we have

$$\frac{16k + m\,45}{5k + m\,14}.$$

$$\text{Let } m = 1 \;\; k = -1, \quad \therefore \frac{x}{y} = \frac{29}{9}.$$

$$m = 1 \;\; k = -2 \qquad \frac{x}{y} = \frac{13}{4}.$$

$$m = 2 \;\; k = -3 \qquad \frac{x}{y} = \frac{42}{13}.$$

Two of these have already been obtained from the series of converging fractions, but the third $\frac{42}{13}$ is a new one. In fact, since the expression $\frac{ma + pk}{mb + qk}$ includes the whole of the principal and secondary converging fractions, as well as many other approximate values of the original fraction, it must be expected that some assumed values of m and k will reproduce these already calculated approximations.

* Francœur, Dict. Technologique, tom. XIV. p. 423, and Traité de Mécanique, p. 146.

† Such as Barlow's New Mathematical Tables, 1814. Chernac. Cribrum Arithmeticum, Davent. 1811. Burckhardt, Table des Diviseurs. Par. 1817. Barlow's Table extends only to 10000, Chernac's to 1019999, and Burckhardt's to 3035999.

But the coexisting values of m and p that belong to the converging fractions, may be obtained at once, to save this useless trouble. For this purpose, write the quotients obtained from the original fraction in a *reverse order*, and proceed to deduce converging fractions from them in the usual manner, both principal and intermediate. Then will the numerator and denominator of each fraction of this new set be the coexisting values of m and k, that belong to a corresponding fraction in the first set, supposing it to be represented by the formula $\frac{ma - pk}{mb - qk}$, the principal fractions in one set corresponding reversely to those of the other set, and likewise the intermediates to the intermediates. It is useless therefore to try a pair of values of m and k so obtained, but any other pair will give new fractions.

250. For in the series of converging fractions,

$$\frac{A}{A_1}, \quad \frac{B}{B_1}, \quad \frac{C}{C_1}, \quad \frac{D}{D_1}, \quad \frac{E}{E_1},$$

in which $\alpha, \beta, \gamma, \delta, \epsilon$ are the quotients, it is known that

$$A = \alpha,$$

$$B = \beta A + 1 = \alpha\beta + 1,$$

$$C = \gamma B + A = (\alpha\beta + 1)\gamma + \alpha,$$

$$D = \delta C + B = \{(\alpha\beta + 1)\gamma + \alpha\}\delta + \alpha\beta + 1,$$

$$E = \epsilon D + C = \&c. \ldots$$

$$A_1 = 1,$$

$$B_1 = \beta,$$

$$C_1 = \beta\gamma + 1,$$

$$D_1 = (\beta\gamma + 1)\delta + \beta, \quad \&c.$$

(Euler's Algebra, p. 476.)

whence we obtain

$$C = E - \epsilon D,$$
$$-B = \delta E - (\delta\epsilon + 1) . D,$$
$$A = (\gamma\delta + 1) E - \{(\delta\epsilon + 1)\gamma + \epsilon)\} D.$$

In which the coexisting values of the coefficients of E and D, the last and last but one of the series of numerators, are 1 and ϵ, δ and $\delta\epsilon + 1$, $\gamma\delta + 1$ and $(\delta\epsilon + 1)\gamma + \epsilon$, and so on, which manifestly follow the same law as the corresponding values of A_1 and A, B_1 and B, &c., if we substitute $\epsilon\delta\gamma\beta\alpha$ for $\alpha\beta\gamma\delta\epsilon$ respectively. Also the same may be similarly shewn for the denominators A_1, B_1, C_1,... &c., as well as for the intermediate fractions. The coefficients of E and D will therefore be obtained from these quotients, if we treat them in this reverse order in the same manner as when we obtain from them the values of the successive converging fractions. And since E and D correspond to a and p, their coefficients are the values of m and k in the formula $\frac{ma - pk}{mb - qk}$, which belong to the continued fractions.

251. To shew this more clearly take this example, $\frac{45}{14}$, which treated in the usual manner gives the following set of quotients and converging fractions.

Quotients.		3	4	1	2	
Principal Fractions.	(a) $\frac{0}{1}$	(b) $\frac{1}{0}$	(c) $\frac{3}{1}$	(d) $\frac{13}{4}$	(e) $\frac{16}{5}$	(f) $\frac{45}{14}$
Intermediate Fractions.			(b′) $\frac{2}{1}$	(c′) $\frac{10}{3}$		(e′) $\frac{29}{9}$
			(b″) $\frac{1}{1}$	(c″) $\frac{7}{2}$		
				(c‴) $\frac{4}{1}$		

Writing the quotients in the reverse order and proceeding as before, we obtain the following set.

Quotients.		2	1	4	3	
Principal Fractions.	$(f)\ \frac{0}{1}$	$(e)\ \frac{1}{0}$	$(d)\ \frac{2}{1}$	$(c)\ \frac{3}{1}$	$(b)\ \frac{14}{5}$	$(a)\ \frac{45}{16}$
Intermediate Fractions.			$(e')\ \frac{1}{1}$		$(c''')\ \frac{11}{4}$	$(b'')\ \frac{31}{11}$
					$(c'')\ \frac{8}{3}$	$(b')\ \frac{17}{6}$
					$(c')\ \frac{5}{2}$	

Now every one of the fractions in the last set consist of the value of $\frac{m}{k}$ that belongs to one of the fractions of the first set, as shown by the corresponding letters of reference; the fractions of the first set being supposed to be represented by the formula

$$\frac{m \times 45 - k \times 16}{m \times 14 - k \times 5}.$$

This is shown in the following table:

	$\frac{m}{k}$	Principal Fractions.	$\frac{m}{k}$	Intermediate Fractions.	
f	$\frac{0}{1}$	$\frac{1 \times 45 - 0 \times 16}{1 \times 14 - 0 \times 5} = \frac{45}{14}$	$\frac{1}{1}$	$\frac{1 \times 45 - 1 \times 16}{1 \times 14 - 1 \times 5} = \frac{29}{9}$	e'
e	$\frac{1}{0}$	$\frac{0 \times 45 - 1 \times 16}{0 \times 14 - 1 \times 5} = \frac{16}{5}$	$\frac{11}{4}$	$\frac{4 \times 45 - 11 \times 16}{4 \times 14 - 11 \times 5} = \frac{4}{1}$	c'''
d	$\frac{2}{1}$	$\frac{1 \times 45 - 2 \times 16}{1 \times 14 - 2 \times 5} = \frac{13}{4}$	$\frac{8}{3}$	$\frac{3 \times 45 - 8 \times 16}{3 \times 14 - 8 \times 5} = \frac{7}{2}$	c''
c	$\frac{3}{1}$	$\frac{1 \times 45 - 3 \times 16}{1 \times 14 - 3 \times 5} = \frac{3}{1}$	$\frac{5}{2}$	$\frac{2 \times 45 - 5 \times 16}{2 \times 14 - 5 \times 5} = \frac{10}{3}$	c'
b	$\frac{14}{5}$	$\frac{5 \times 45 - 14 \times 16}{5 \times 14 - 14 \times 5} = \frac{1}{0}$	$\frac{31}{11}$	$\frac{11 \times 45 - 31 \times 16}{11 \times 14 - 31 \times 5} = \frac{1}{1}$	b''
a	$\frac{45}{16}$	$\frac{16 \times 45 - 45 \times 16}{16 \times 14 - 45 \times 5} = \frac{0}{1}$	$\frac{17}{6}$	$\frac{6 \times 45 - 17 \times 16}{6 \times 14 - 17 \times 5} = \frac{2}{1}$	b'

Any other integrals substituted for m and k will give new approximate fractions; as for example,

$$\frac{2 \times 45 - 3 \times 16}{2 \times 14 - 3 \times 5} = \frac{42}{13} = 3.230,$$

$$\frac{3 \times 45 - 7 \times 16}{3 \times 14 - 7 \times 5} = \frac{23}{7} = 3.285,$$

the decimals serve to show the closeness of the approximation for the original fraction, $\frac{45}{14} = 3.\dot{2}1\dot{4}$.

252. If we apply this method to the example (Art. 247) of an annual movement, the approximate fraction becomes

$$\frac{164359 \times k - m \times 58804}{450 \times k - m \times 161},$$

in which k and m may have any values; for example,

$$\frac{7 \times 164359 - 22 \times 58804}{7 \times 450 - 22 \times 161} = \frac{143175}{392} = \frac{25 \times 69 \times 83}{8 \times 7 \times 7},$$

corresponding to a period of $365^d. 5^h. 48'. 58''.6944$. (error $10''.69$). This is the annual train which has been calculated by a different method by P. Allexandre, in 1734, and afterwards by Camus and Ferguson.

However, the expression

$$\frac{3 \times 164359 - 10 \times 58804}{3 \times 450 - 10 \times 161} = \frac{94963}{260} = \frac{11 \times 89 \times 97}{2^2 \times 5 \times 13},$$

which corresponds to a period of $365^d. 5^h. 48'. 55''.38$, is quite as convenient, and rather more accurate.

In a train of this kind one or more endless screws may be introduced, by way of saving teeth; for example, in the fraction last cited the numerator does not admit of being divided into less than three wheels; but the denominator may be distributed between two pinions and an endless screw, (remembering that the latter is equivalent to a pinion of one leaf) thus, $1 \times 20 \times 13$, or $1 \times 10 \times 26$. If the endless screw be not convenient, then the terms of the fraction must be multiplied by 4, to make the numbers of the denominator large enough for three pinions, and the train will stand thus,

$$\frac{44 \times 89 \times 97}{8 \times 10 \times 13}.$$

253. Ex. *To find a Lunar train that shall derive its motion from the twelve-hour arbor of a clock.*

The mean synodic period of the Moon is $29^d.\ 12^h.\ 44'.\ 2''.8032$, which is exactly equal to $29^d.530588$, or nearly $29^d.5306$, and since twelve hours is equal to $0^d.5$, the ratio will be $\frac{295306}{5000}$, or, dividing each term by 2, $\frac{147653}{2500}$; from which the following quotients and fractions may be obtained.

Quotients.	59	16	2	1	16	3	
Principal Fractions.		$\frac{59}{1}$	(A) $\frac{945}{16}$	$\frac{1949}{33}$	$\frac{2894}{49}$	$\frac{48253}{817}$	$\frac{147653}{2500}$
Secondary Fractions.					(C) $\frac{4843}{82}$	(D) $\frac{99400}{1683}$	
					(B) $\frac{19313}{327}$		

Now as the whole number nearest to the original fraction is 59, which is less than 8^2, it is clear that two pair of wheels should suffice. The whole of the secondary fractions which would not admit of reduction, are omitted. The principal fractions are refractory, with the exception of (A), $\frac{945}{16} = \frac{3^2 . 5 . 7}{4^2}$, which has been employed by Ferguson and by Mr. Pearson; it corresponds to a period of $29^d.\ 12^h.\ 45'$ exactly, and has an error in excess of $57''.2$; as it is a multiple of seven, it may be introduced into a clock which has a weekly arbor.

This fraction has been already obtained by a coarser method in (Art. 245.).

$(B) = \dfrac{19313}{327} = \dfrac{7 \times 31 \times 89}{3 \times 109}$ has an error in defect of $0''.6$ in each lunation.

$(C) = \dfrac{4843}{82} = \dfrac{29 \times 167}{2 \times 41}$ has an error of $-\ 8''.6$.

$(D) = \dfrac{99400}{1683} = \dfrac{2^3 \times 5^2 \times 7 \times 71}{3^2 \times 11 \times 17}$ has an error of $+\ 1''.03$.

Other results may be obtained from the expression,

$\dfrac{147653 \times k - m \times 48253}{2500 \times k - m \times 817}$, as in the following Table.

	Values of k	Values of m	$\dfrac{D}{F}$	$\dfrac{D}{F}$ in Factors.	Error in a Lunation.
a	12	59	$\dfrac{41520}{703}$	$\dfrac{5 \times 48 \times 173}{19 \times 37}$	$-\ 0''.4$
	31	97	$\dfrac{103298}{1749}$	$\dfrac{2 \times 13 \times 29 \times 137}{3 \times 11 \times 53}$	$+\ 0''.08$
b	29	89	$\dfrac{12580}{213}$	$\dfrac{2^2 \times 5 \times 17 \times 37}{3 \times 71}$	$-\ 6''.18$
c	76	233	$\dfrac{21321}{361}$	$\dfrac{103 \times 23 \times 9}{19 \times 19}$	$-\ 9''.84$
d	29	92	$\dfrac{157339}{2664}$	$\dfrac{7^2 \times 13^2 \times 19}{2^3 \times 3^2 \times 37}$	$+\ 0''.44$
e	11	35	$\dfrac{64672}{1095}$	$\dfrac{2^5 \times 43 \times 47}{3 \times 5 \times 73}$	$+\ 0''.48$
	1633	5000	$\dfrac{147651}{2500}$	$\dfrac{3 \times 7 \times 79 \times 89}{2^2 \times 5^4}$	$-\ 33''.5$

Of these *a* is a train given by Francœur, *b* and *c* by Allexandre, *d* by Camus, *e* by Mr Pearson; each of these writers having arrived at his result by a method of his own*.

* Vide Francœur, Mécanique Elementaire, p. 146. Allexandre, Traité Général des Horloges, p. 188. Camus on the Teeth of Wheels. Rees' Cyclopædia, art. Planetary Numbers.

254. The early mechanists were content with much more humble approximations, and employed a great number of unnecessary wheels. In the annual movement of the planetary clock, by Orontius Finæus (about 1700), the following annual train is employed, from a wheel which revolves in three days*.

$$\begin{array}{llllll}
12 & \text{——} 48 & & & & \\
 & 36 & \text{——} 180 & & & \\
 & & 48 & \text{——} 48 & & \\
 & & & 24 & \text{——} 1460 = \dfrac{365}{1}. &
\end{array}$$

A train of half the number of wheels would do as well,

$$\text{thus } \frac{600 \times 730}{6 \times 6}, \quad \text{or } \frac{146 \times 1800}{12 \times 18}.$$

Again Oughtred†, in 1677, is satisfied to represent the synodic period of the Moon by $29\frac{1}{2}$ days, and employs the train $\frac{40 \times 59}{10 \times 4}$. Huyghens employed for the first time continued fractions in the calculation of this kind of wheelwork‡.

255. Let it be required to connect an arbor with the hour arbor of an ordinary clock, in such a manner that it may revolve in a sidereal day; so as to indicate sidereal time upon a dial, while the ordinary hands of the clock shew mean time upon their own dial.

Twenty-four hours of sidereal time are equivalent to $23^h.\ 56'.\ 4''.0906$ of mean solar. Neglecting the decimals and reducing to seconds, we obtain $86400''$ of sidereal time equivalent to $86164''$ of mean time, and therefore one wheel must make 86400 turns while the other makes 86164, or dividing by the common factor 4, we get

$$\frac{S_1}{S_m} = \frac{21600}{21541}, \text{ an unmanageable fraction.}$$

* Allexandre, p. 167. † Oughtred, Opuscula. ‡ Hugenii Op. posth. 1703.

Approximating as before, we obtain the expression

$$\frac{3651\,k + 21541.m}{3661\,k + 21600.m},$$

in which $k = -4$, $m = 7$, gives

$$\frac{1096}{1099} = \frac{8 \times 137}{7 \times 157},$$

with a daily sidereal error of $0''.0586$, or $21''\frac{1}{2}$ in the year*.

256. Another mode of indicating sidereal and solar time in the same clock, consists in placing behind the ordinary hour hand a moveable dial concentric with and smaller than the fixed dial†. Both dials must in this case be divided into twenty-four hours. The hand of the clock performs a revolution in twenty-four solar hours, and therefore indicates mean solar time upon the fixed dial as usual, but a slow retrograde motion is given to the moveable dial, so that the same hand shall point upon the latter to the sidereal time, which corresponds to the solar time shewn upon the fixed dial. For this purpose it is evident that during each revolution of the hour hand, the moving dial must retrograde through an angle corresponding to the quantity which sidereal time has gained upon solar time in twenty-four hours; which is $3'.56''.555 = 236''.555$, and as the entire circumference of the dial contains $86400''$, we have

$$\frac{\text{Ang. vel. of hour hand}}{\text{Ang. vel. of dial}} = \frac{86400000}{236555} = 60 \times \frac{288000}{47311}.$$

From this fraction approximate numbers may be obtained, by which the proper wheel-work for the motion of the dial can be set out.

* This is Francœur's result.

† This method is due to Mr Margett, the details of his mechanism may be found in Rees' Cyclopædia, art. Dialwork.

The fraction $\frac{288000}{47311}$ reduced to continued fractions gives

Quotients.	6	11	2	3	1	152	
Fractions.		$\frac{6}{1}$	$\frac{67}{11}$	$\frac{140}{33}$	$\frac{487}{80}$ (A)	$\frac{627}{103}$ (B)	&c.

(A) contains a large prime 487, but is employed by Mr Margett. (B) $= \frac{3 \times 11 \times 19}{103}$ contains a smaller number, and is a better approximation.