# CHAPTER II.

## ELEMENTARY COMBINATIONS.

CLASS A. { DIRECTIONAL RELATION CONSTANT. VELOCITY RATIO CONSTANT.

### DIVISION A. COMMUNICATION OF MOTION BY ROLLING CONTACT.

41. IN rolling contact it has been shewn that the point of contact is always in the line of centers; and the angular velocities are inversely as the segments into which the point of contact divides that line. Therefore if the velocity ratio is constant, the segments must be constant, and the curves become circles, revolving round their centers, and whose radii are the segments, and no other curves will answer the purpose.

Let $R$ be the radius of the driving circle, and $r$ that of the following circle; $L$ and $l$ their synchronal rotations; then as they are (by § 20) in the ratio of the angular velocities:

$$\therefore \frac{L}{l} = \frac{r}{R}.$$

This ratio will be preserved, whatever be the absolute velocity of the driver, but when this is uniform, which is generally the case, let $P$ and $p$ be the respective periods of the driver and follower;

$$\therefore \text{(by § 20)} \ \frac{P}{p} = \frac{l}{L} = \frac{R}{r}.$$

The motions being supposed hitherto to be in the same plane, the axis of rotation of the circles will be parallel.

42. *When the axes are parallel.* Let $Aa$, $Bb$ be two parallel axes, mounted in any kind of framework that will allow them to revolve freely, but retain them parallel to each other at a constant distance, and prevent endlong motion, and let two cylinders or rollers, $E, F$, be fixed opposite to each other, one on each axis, and concentric to it; the sum of their radii being equal to the distance of the axes. The cylinders will therefore be in contact in all positions, and if one of these axes, and consequently its attached cylinder, be made to revolve, its superficial motion will be communicated to the surface of the other cylinder by the adhesion of the parts which are brought successively into contact; and thus the second cylinder will be *driven* by the first by rolling contact, and their perimetral velocities will be equal.

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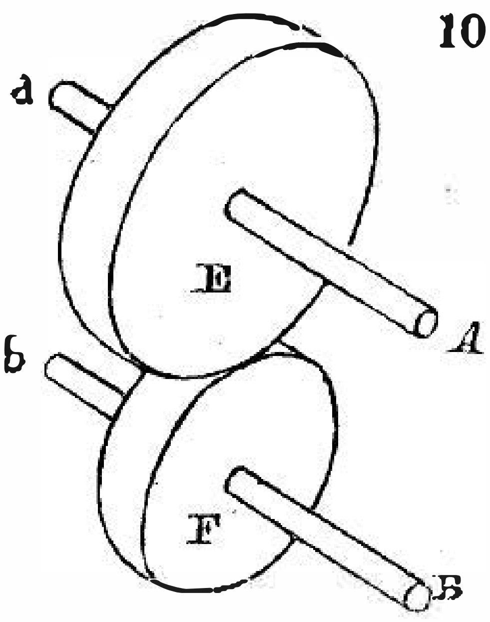


Let $R$ be the radius of the driver, and $r$ of the follower; then a section of the cylinders, made by a plane passing through them at any point at right angles to the axis, will present a pair of circles in contact, whose radii are $R$ and $r$;

and therefore, as before, $\dfrac{P}{p} = \dfrac{l}{L} = \dfrac{R}{r}$;

which is indeed manifest, for since the same length of circumference of the driver and of the follower passes the line of centers* in the same time, let $M$. circumferences of the driver, equal $m$. circumferences of the follower;

$$\therefore\ 2\pi RM = 2\pi rm, \text{ and } \frac{M}{m} = \frac{r}{R}.$$

* The *line of centers* is the right line which joins the centers of motion, as already stated, and, in the case of rolling circles, passes through their point of contact. The *plane of centers* is the plane which contains the two axes, whether they be parallel or intersect. These two phrases are of continual use.

But the number of circumferences that pass a given point measure the number of revolutions of the wheel;

$$\therefore \frac{M}{m} = \frac{L}{l} = \frac{r}{R}, \text{ as before.}$$

43. If the axes of rotation be not parallel, they may either meet in direction or not, and these cases must be considered separately.

*Axes meeting.* Let $AB$, $AC$ be two axes of rotation intersecting in $A$,

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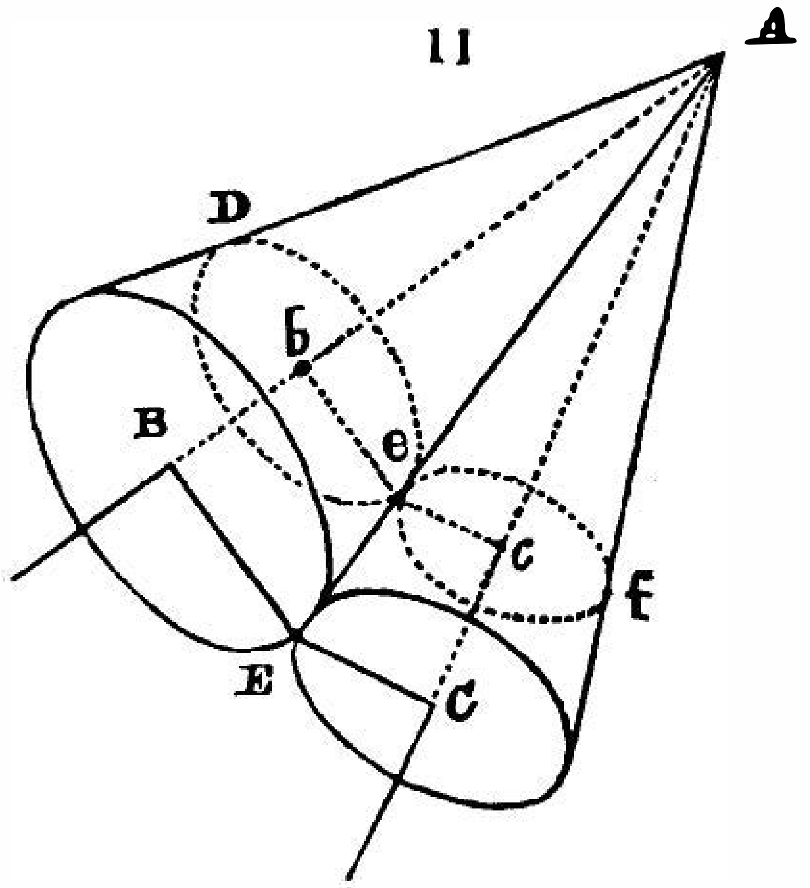


to which are attached cones $ABE$, $AEC$, whose apices coincide with $A$, and which have angles at their vertices of such a magnitude that their surfaces are in contact. Let $AE$ be the line of contact, and $Dbe$, $ecf$ sections of the cones at any point $e$ of the line and respectively perpendicular to their axes, which sections are necessarily circles touching at $e$, whose radii are $be$, $ce$. If angular velocities $A$, $a$ be given to the cones $ABE$, $AEC$, the perimetral velocities of these sections will be $A.be$ and $a.ce$, and if these are equal,

$$\frac{A}{a} = \frac{ce}{be} = \frac{CE}{BE}$$

a constant ratio. If then the perimetral velocities of any pair of corresponding sections be equal, those of every other such pair will be equal; therefore the cones will roll together as in the former case, and the ratio of the angular velocities be inversely as the radii of the bases of the cones.

44. In practice, a thin frustum only of each cone is employed. Let the position of the axes be given, and also the ratio of the angular velocities, it is required to describe the cones, or rather the frusta.

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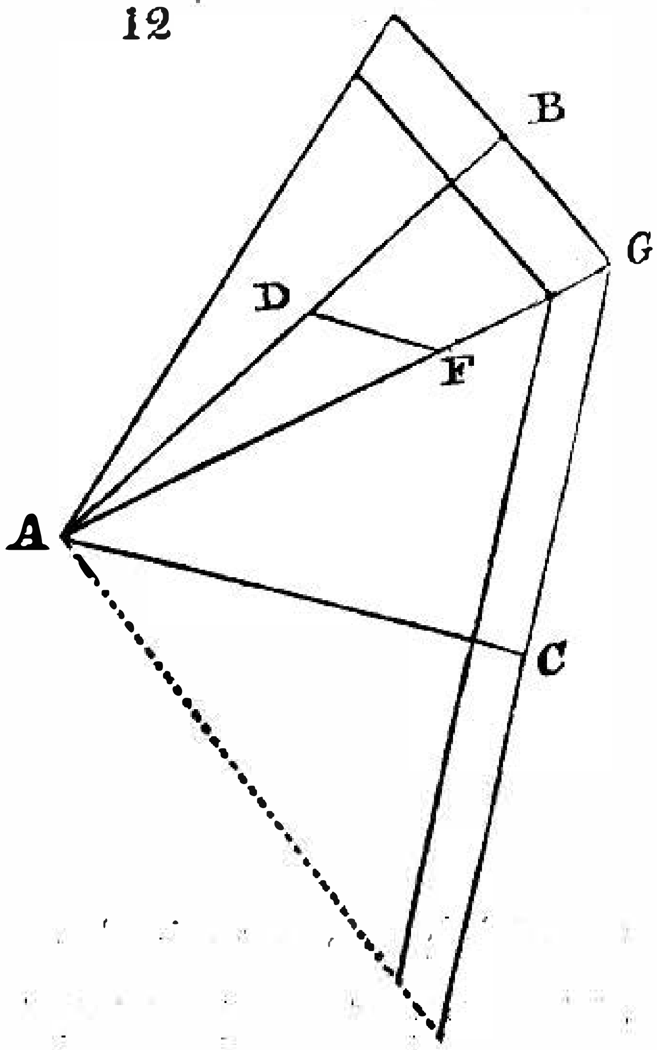


Let $AB$, $AC$ be the axes intersecting in $A$. Through any point $D$ in $AB$ draw $DF$ parallel to $AC$, and make $DF : AD$ in the ratio of the angular velocity of $AB$ to that of $AC$. Join $AF$, then will $AF$ be the line of contact of the two cones, by means of which the required frusta may be described at any convenient distance from $A$,

$$\text{for } \frac{DF}{AD} = \frac{\sin DAF}{\sin AFD}$$

$$= \frac{\sin DAF}{\sin FAC} = \frac{BG}{CG};$$

that is, the angular velocities are in the ratio required by last Article.

Cor. The angles at the vertices of the cones may be readily found thus:

Let $\theta$ be the angle $BAC$, $\kappa$ the semiangle of the vertex of the cone of $AB$, $\frac{m}{n}$ the given ratio of the angular velocities;

$$\therefore \frac{\sin \kappa}{\sin \theta - \kappa} = \frac{m}{n}; \quad (m \text{ being the less})$$

$$\text{whence } \tan \kappa = \frac{\sin \theta}{\frac{n}{m} + \cos \theta};$$

which may be adapted to logarithms by taking a subsidiary angle $\phi$, so that $\cos \phi = \frac{m}{n} \cos \theta$;

$$\text{whence } \tan \kappa = \frac{m \sin \theta}{2n \cos^2 \frac{\phi}{2}}.$$

If $\theta$ be a right angle, which is generally the case, then

$$\tan \kappa = \frac{m}{n}.$$

45. *Axes neither parallel nor meeting.* The hyperboloid of revolution is a well known surface, generated by the revolution of a straight line about an axis to which it is not parallel and which it does not meet*. If two such hyperboloids $EF$ be placed so that their generating lines

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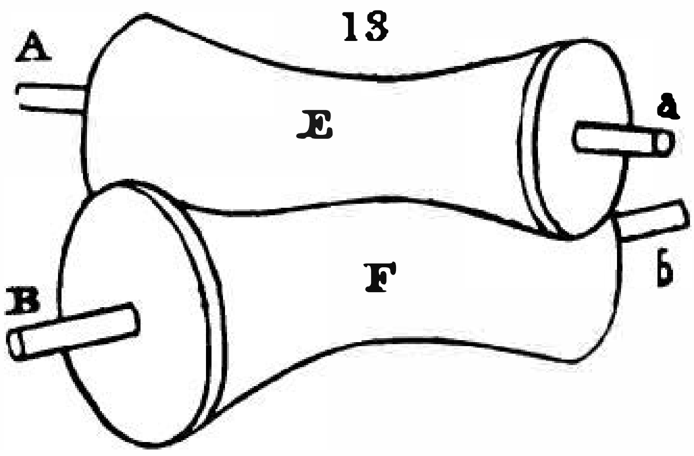


coincide, the solids will touch along this line, and their axes $Aa$, $Bb$ will neither be parallel nor meeting in direction.

If the solids revolve about their axes, the contact will plainly continue throughout this line; and that they may be so proportioned as to roll together, can be shewn as follows.

Let $AB$, $CD$ be the axes of rotation, $ab$, $cd$ their respective projections upon a plane to which they are both

* Vide Newton's Universal Arithmetic, Prob. 33. Hymers' Analytical Geometry, p. 142; or any Treatise on Analytical Geometry.

parallel; $gk$ their common perpendicular* produced to

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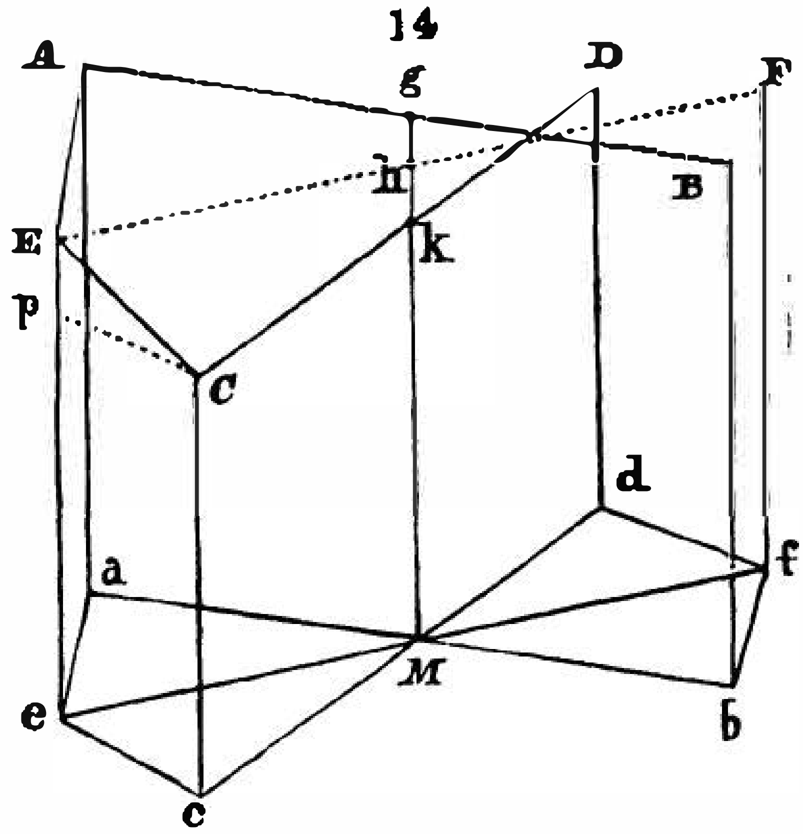



meet the plane in $M$; $EF$ a line intersecting $gk$ in $h$, and also parallel to the plane and projected in $ef$.

Let $EF$ revolve round $AB$ to generate one hyperboloid, and round $CD$ to generate the other; then $EF$ will be their line of contact. From any point $E$ let fall $EA$, $EC$, perpendiculars upon $AB$ and $CD$.

Now as the hyperboloids are solids of revolution, these lines $AE$, $CE$ will be the radii of a pair of circles in contact at $E$.

Draw $Cp$ parallel to $ce$, therefore $Ep = hk$ and $Cp = ce$; whence $EC^2 = hk^2 + ce^2$; and similarly $AE^2 = gh^2 + ae^2$.

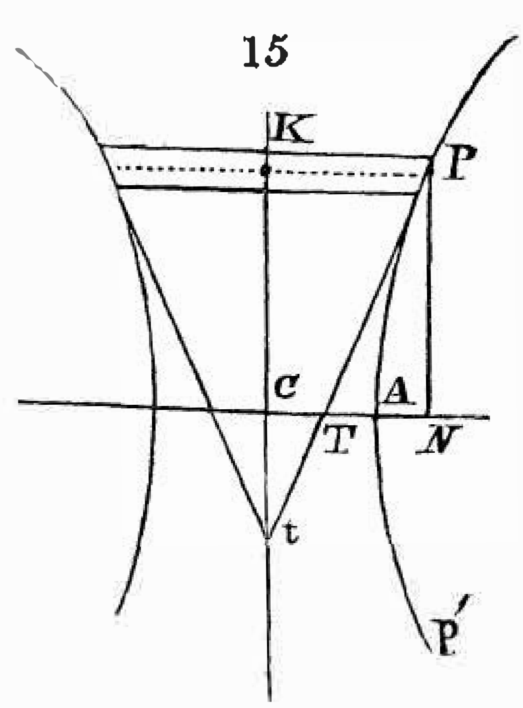
If therefore $\dfrac{ce}{ae} = \dfrac{hk}{gh}$, we have $\dfrac{EC}{AE} = \dfrac{hk}{gh}$, a constant ratio for every corresponding pair of circles; whence it follows, that if the superficial velocities be equal at any point of contact, they will be equal at every other; and as in the cones, the angular velocity ratio $\dfrac{A}{a} = \dfrac{r}{R}$, where $r$ and $R$ are the corresponding radii of the hyperboloids.

* Vide Playfair's Geometry, Sup. B. II. Prop. xix.

46. But in rolling cones, the perimeters of every corresponding pair of circles move in the same direction at the point of contact, although the circles are not in the same plane; for the radii of contact are perpendicular to the line of intersection of the two planes which contain the circles, and consequently the tangents at the point of contact coincide with that line and with each other. On the contrary, this is not the case in rolling hyperboloids.

For the circles whose radii are $CE$, $AE$, lie in planes whose intersection is the line $Ee$; and the tangents to these circles at the common point $E$, plainly cannot coincide either with that line or with each other; so that the motion of the surfaces is not in the same direction, and the rolling action is imperfect, and the more so the greater the angle $ECp$, that is, the greater the distance $gk$ between the axes; for as this distance diminishes, the hyperboloids approach to a pair of cones whose common apex is $h$.

47. In practice, as in the case of cones, a thin frustum only is required of each hyperboloid, and these frusta include so small a portion of the curve surface, that a frustum of the tangent cone at the mean point of contact may be substituted without sensible error†; and this may be found as follows:




Let $CK$ be the axis of the given hyperboloid; $CK = NP = y$ the mean distance of the required frustum from the center $C$, $KP = CN = k$ its radius; $CA = a$ the least radius of the hyperboloid and semi-axis-major of its generating hyperbola $pAP$; all which may be found when the position of the axes and ratio of the angular velocities are given in any particular case, by the last proposition; observing that in Fig. 14. $AE$, $EC$ are the mean radii of the frusta; $g$, $k$, the centers of the generating hyperbolas, and $gh$, $hk$ their semiaxes-major.

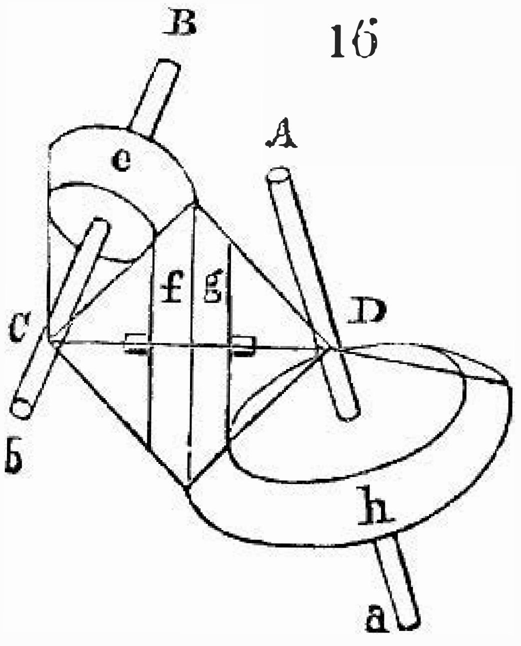
Let $PTt$ be the tangent at $P$; then, by the known properties of the hyperbola,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2), \text{ and } Ct = \frac{b^2}{y};$$

$$\therefore \; Ct = \frac{a^2 y}{x^2 - a^2} \text{ gives the apex } t \text{ of the required cone.}$$

Or, take $CT = \dfrac{a^2}{y}$; join $PT$, and produce it to $t$.

48. This third case of axes, neither parallel nor meeting, admits of solution by means of the cones of the second case; thus*:



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Let $Aa$, $Bb$ be the two axes, take a third line intersecting the axes at any convenient points $C$ and $D$ respectively; and let a short axis be mounted so as to revolve in the direction of this third line between the other two axes.

Now a pair of rolling cones, $e$, $f$, with a common apex at $c$, will communicate motion from the axis $Bb$ to the intermediate axis; and another pair $g$, $h$, with a common apex at $D$, will communicate motion from the intermediate axis to $Aa$; and thus the rotation of $Bb$ is communicated to $Aa$ by pure rolling contact.

Let $A$, $A_{\prime}$, $a$, be the respective angular velocities of the axes $Bb$, $CD$, $Aa$; and $R$, $R_{\prime}$, $r$ the radii of the bases of their cones, those of the cones, $f$, $g$, being the same:

* From Poncelet, Mécanique Industrielle, p. 300.

$$\therefore \frac{A}{A_{\prime}} = \frac{R_{\prime}}{R}, \text{ and } \frac{A_{\prime}}{a} = \frac{r}{R_{\prime}}, \text{ whence } \frac{A}{a} = \frac{r}{R},$$

exactly as if the cones $e$, $h$, were in immediate contact.

## *To apply these Solutions to Practice.*

49. Theoretically we have now the complete solution of the problem in all the three cases; having shewn how to find a pair of cylinders in the first case, and of conical frusta in the other cases, by which a given angular velocity ratio will be effected. If these solids could be formed with mathematical precision, then, the axes having been once adjusted in distance so that the surfaces should touch in one position, they would touch in every other positiont; but in practice this is impossible, and various artifices are employed to maintain the adhesion upon which the communication of motion depends.

The surface of one or both rollers may be covered with thick leather, which by giving elasticity to the surface enables it to maintain adhesional contact, notwithstanding any small errors of form.

One of the axes may be either made to run in slits at its extremities instead of round holes, or else it may be mounted in a swing frame. Both methods allowing of a little variation of distance between the two axes, the contact of the rollers will in this way also be maintained, notwithstanding small errors of form.

If the weight of the uppermost roller is not sufficient to produce the required adhesion, or if the rollers lie with their axes in the same horizontal plane, then weights or springs may be employed to press the axes together. The practical details of these methods belong rather to the department of Constructive Mechanism than to the plan of the present work.

50. But the most certain method of maintaining the action of the surfaces is to provide them with teeth. The plain cylindrical or conical surfaces of contact are exchanged for a series of projecting ridges with hollow spaces between. These ridges or *teeth* are distributed at equal distances from each other on the two surfaces, and generally in the direction of planes passing through the axis, so that when the driving wheel is turned, its teeth enter in succession the spaces between those of the follower. They are so adjusted that before one tooth has quitted its corresponding space the next in succession will have entered the next space, and so on continually; consequently, the surfaces cannot escape from each other, and there can be no slipping, notwithstanding slight errors of form.

The action of this contrivance falls partly under the head of rolling contact, and partly under that of sliding contact; for the teeth considered separately act against each other by sliding contact, and the forms of their acting surfaces must be determined, as we shall see, upon that principle.

On the other hand, the total action of a pair of toothed wheels upon each other is analogous to that of rolling contact. Equal lengths of the two circumferences contain equal numbers of teeth, and therefore equal lengths will pass the line of centers in the same time, if measured by the unit of the space occupied by one tooth and a hollow between. In fact, the adhesion which enables the surface of one plain roller to communicate motion to another arises from the roughness of the surfaces, the irregular projections of one indenting themselves between those of the other, or pressing against similar projections; and the contrivance of teeth is merely a more complete developement of this mode of action, by giving to these projections a regular form and arrangement. I shall proceed therefore to explain in this Section all that relates to the general action, arrangement, and construction of toothed wheels; leaving the exact form of the individual teeth to the next Section, and observing, that this arrangement corresponds to the ordinary practical view of the subject; for all that belongs to the complete action or construction of a pair of toothed wheels is always referred to a pair of corresponding plain rollers, or rolling circles, which are termed the *pitch circles*, or *geometrical circles*.

51. *Geering* is a general term applied to trains of toothed wheels. Two toothed wheels are said to be *in geer* when they have their teeth engaged together, and to be *out of geer* when they are separated so as to be put out of action; and generally speaking, a driver and follower, whatever be the nature of their connexion, are said to be in geer when the connexion is completely adjusted for action, and out of geer when the connexion is interrupted.

52. Toothed wheels with few teeth are termed *pinions*. This phrase is merely to be considered as the diminutive of *toothed wheel*; and there is no impropriety or ambiguity in calling a pinion a toothed wheel, if more convenient.

53. The teeth of wheels may be either made in one piece with the body or rim of the wheel, or they may be each made of a separate piece and framed into the rim of the wheel.

The first method is employed in cast-iron wheels of all sizes, from the largest to the smallest; also for brass or other metal wheels in smaller machinery, which are formed out of plain disks by cutting out a series of equidistant notches round the circumference, and thus leaving the teeth standing.

Figure 17, *A* and *C*, represents the form of the modern cast-iron wheels, in which, for the sake of uniting lightness and stiffness, a thin web or fin runs along the inner edge of the rim and on each side of the arms, so that the transverse section of the arm is a cross.

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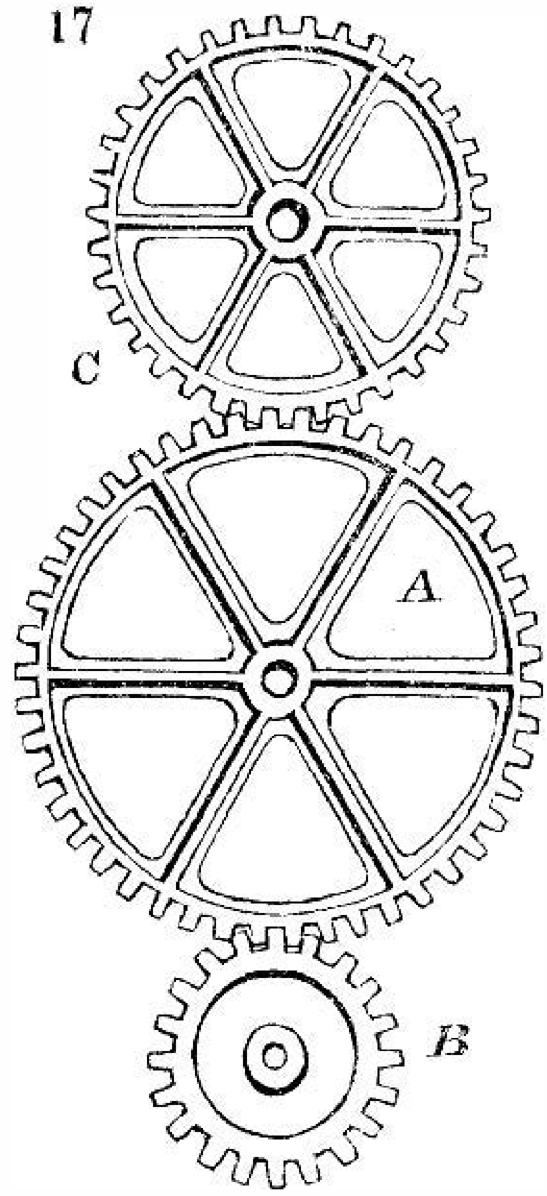


In smaller wheels the arms are omitted, as at *B*, and the rim of teeth united to the central boss by a thin continuous plate. These wheels are *plate* wheels, and when arms are employed, wheels are said to be *crossed out*; but this phrase rather belongs to clock-work. Wooden wheels in one piece with their teeth are too weak to be trusted beyond the construction of models, or wheel-work which transmit little pressure. The wheels of Dutch clocks of the coarser kind are constructed in this manner.

54. Figure 18 exemplifies the construction of mill-work, and larger machinery, previous to the introduction of cast-iron wheels by Messrs. Smeaton and Rennie, at the latter end of the last century*. The wheel *A* is framed of wood, not like carriage-wheels with radial spokes, but with two pair of parallel bars set at right angles, so as

* Mr. Smeaton was the first who began to use cast-iron in mill-work at the Carron Iron-works, in 1769. It was first employed for the large axes of water-wheels, and soon afterwards for large cog-wheels; but the complete introduction of it is due to Mr. Rennie.—Vide Farey on the Steam Engine, p. 443.

to leave a square opening in the midst for the reception of the shaft, which is also of wood, and square, and the opening being purposely left larger than the section of the shaft, the wheel is secured upon it by driving wedges in the intermediate space. This frame carries the rim of the wheel, which is made truly cylindrical on the outer surface, and annular in front. Equidistant mortises are pierced through the rim in number equal to those of the teeth or *cogs*, as they are called when made in separate pieces.

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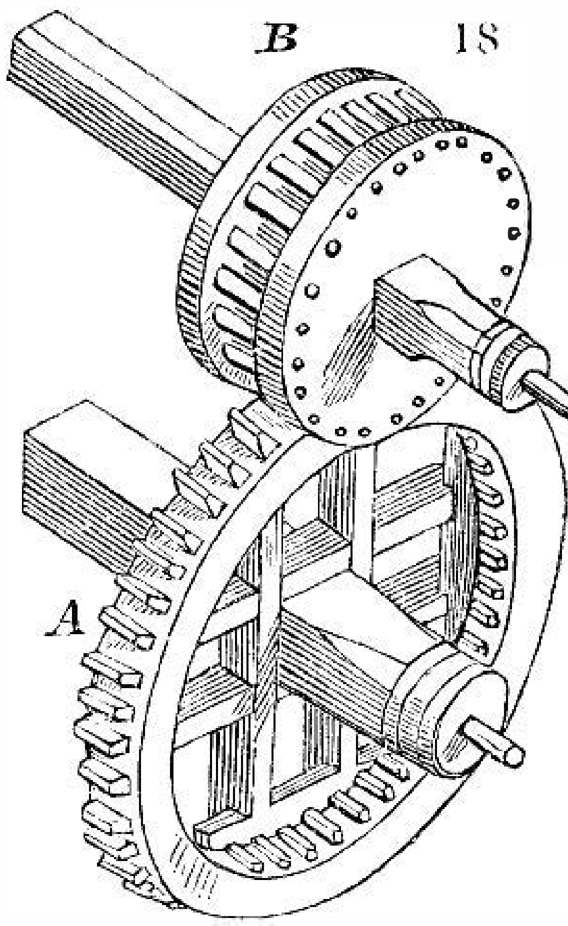


The cogs are made of well-seasoned hard wood, such as mountain-beech, hornbeam, or hickory; the grain is laid in the direction of the length, which being the radial direction, gives them the greatest transverse strength. A cog consists of a head *a*, and a shank *b*, of which the head is the acting part or actual tooth which projects beyond the rim, and the shank or tenon is made to fit its mortise exceedingly tight, and is left long enough to project on the inside of the rim. When the cog is driven into its mortise up to its shoulders a pin *c* is inserted in a hole bored close under the rim of the wheel, by which it is secured in its place.

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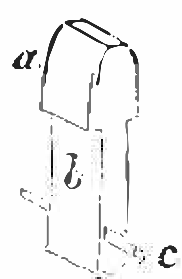


55. This construction of a toothed wheel has been partly imitated in modern mill-work, for it is found that if in a pair of wheels the teeth of one be of cast-iron, and in the other of wood, that the pair work together with much less vibration and consequent noise, and that the teeth abrade each other less than if both wheels of the pair had iron

teeth. Hence in the best engines one wheel of every large sized pair has wooden cogs fitted to it exactly in the manner just described; only that instead of employing a wooden framed wheel to receive them, a cast-iron wheel with mortises in its circumference is employed. Such a wheel is termed a *Mortise wheel.*

Wheels of the kind hitherto described, in which the teeth are placed radially on the circumference, whether the teeth be in one piece with the wheel, or separate, are termed *spur-wheels;* and when the term *pinion* is applied to a wheel its teeth are usually called *leaves*.

56. The pinions in large wooden machinery are commonly formed by inserting the extremities of wooden cylinders into equidistant holes, in two parallel disks attached to the axis or shaft*, as at *B*, (fig. 18.) thus forming a kind of cage, which is termed a *lantern*, *trundle*, or *wallower*; the cylindrical teeth being named its *staves*, *spindles*, or *rounds*. This construction is very strong, and the circular section of its teeth or *staves* gives it the advantage of a very smooth motion, when the *lantern* is *driven*, as will be shewn in its proper place. In Dutch clock-work this plan is imitated on a small scale, and small wire used for the staves.

57. A similar system to this is of great antiquity, for in early machinery the toothed wheels were often cut out of thin metal plates; and it would be obviously impossible to make a pair of such thin wheels work together, as in fig. 17; for the smallest deviation of one of the wheels from the plane of rotation of the pair, would cause the teeth to lose hold of each other sideways. For this reason one of the wheels of a pair were always made either in the lantern form as just

* *Axis* is the general and scientific word, *shaft* the millwright's general term, and *spindle* his term for smaller shafts; *axle* is the wheelwright's word, and *arbor* the watchmaker's.

described, or with pins inserted at one end only into a disk, as at *A* fig. 20, or else the teeth of one of the wheels were cut out of a hoop, as at *C*, forming what is termed a crown wheel, or contrate wheel.

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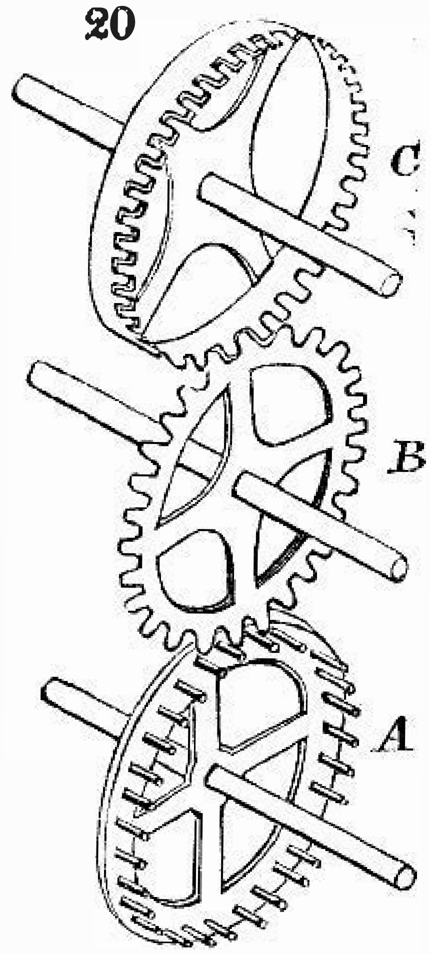


In this figure it is evident that the thin wheel *B* would retain hold of the pins of *A*, or of the teeth of *C*, notwithstanding a little deviation from the plane of rotation, or a little end-play in the axes.

58. *Annular wheels* have their teeth cut on the inside edge of an annulus, so that the pinion which works with them shall lie within the pitch circle. Hence *the two axes revolve in the same direction.* The arms of an annular wheel necessarily lie behind the annulus, in order to make room for the pinion, and the latter must be fixed at the extremity of its axis, otherwise this will stop the wheel by passing between the arms. Annular wheels are more difficult to execute than common spur-wheels, but it will be shewn that the action of their teeth is smoother. A pin-wheel like *A*, fig. 20, may be employed as an annular wheel, and is much easier to construct.

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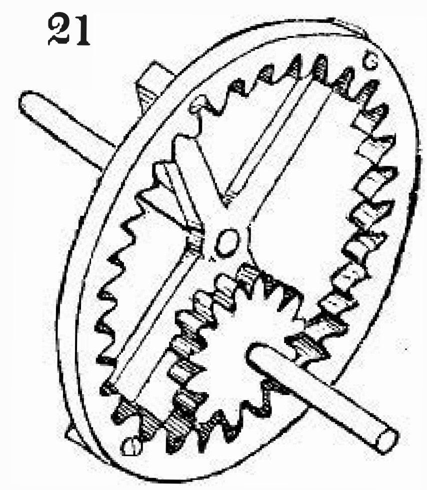

59. When the path of one of the pieces is rectilinear, or, in other words, if it be a sliding piece, then the teeth are cut on the edge of a bar attached to this piece, so that the teeth may work with those of the wheel or pinion, which is to drive or

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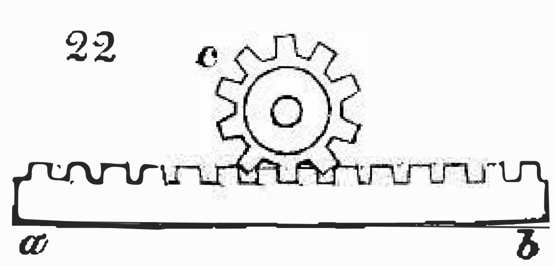

follow it, as in this figure, where the bar *ab* is supposed to be confined by proper guides, so as to move only in the direction of its length, and the pinion *c* to geer with it either as a driver or a follower.

Such a toothed bar is termed a *rack*. The teeth admit of all the different forms and arrangements of which the teeth of wheels in general are susceptiblet; the rack being merely a toothed wheel whose radius is infinite. Similarly, an annular wheel may be considered as a toothed wheel whose radius is negative.

60. If the space through which the bar moves is less than the circumference of the wheel, the latter may assume the form of a sector, as in this figure.

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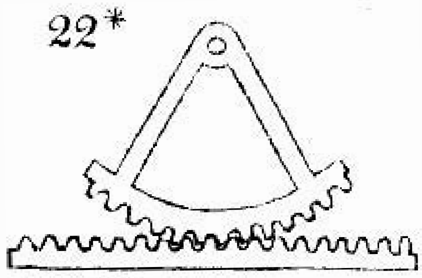

61. All these examples belong to the first case of position in the axes, that is, when they are parallel; but the second case, in which their directions meet, presents itself also very early in the history of mechanism.

A water-wheel, for example, has its axis necessarily horizontal, and near the surface of the water. The axis of a mill-stone, on the other hand, is vertical, and it is convenient to place the latter in an upper floor of the building. This is the disposition of the water-mill of Vitruvius, and is in fact universal.

But the exact method of deriving the form of the toothed wheels from a pair of rolling cones, was not introduced until the middle of the last century, when its mathematical principles were completely laid down by Camus, in 1766*.

* Camus, Cours de Mathématique, Par. 1766. The part relating to toothed wheels has been printed separately in England, and is well known. The principle of rolling cones was first published in England by Imison. In his treatise of the Mechanical Powers, 1787, he uses the term *bevel geer*, and speaks of such wheels as well known.

Previously to this it was thought sufficient to dispose the teeth of the wheels, as in this figure, upon the face

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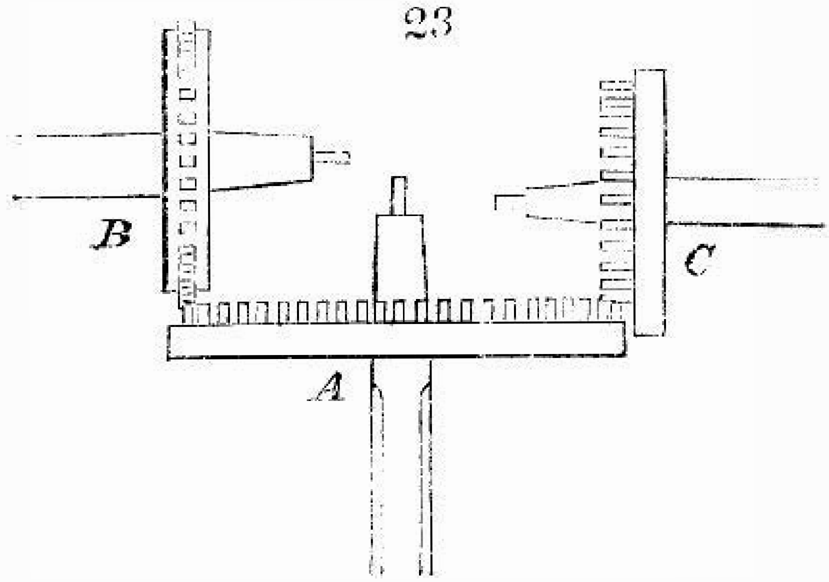


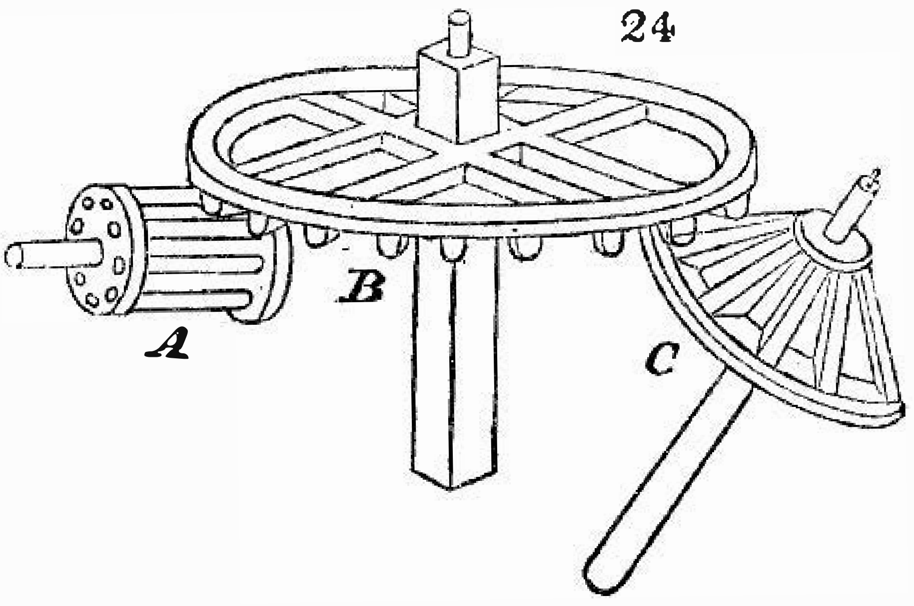

of one of the wheels as *A*, so as to catch those of an ordinary spur-wheel *B* with teeth on the circumference; or else to place the teeth of both wheels on the face, as in those of *A* and *C*. Sometimes the teeth of both wheels were placed on the circumference, as in the ordinary spur-wheels; with this difference, that the teeth require to be much longer, to enable them to lay hold of each other in this relative position. For the forms of the individual teeth no certain principles were followed, and for the arrangements in question the only principle appears to have been to place the teeth so that on passing the line or rather plane of centers*, the teeth should present themselves in the same relative position as if they belonged to a pair of wheels with parallel axes.

A similar principle is, indeed, clearly stated by De la Hire, in the extract which follows the next paragraph.

62. When the axes intersected each other at right angles, and one of them revolved much quicker than the other, a cylindrical lantern was universally given to the latter, and the teeth of the former placed on its face,

* Vide Note, p. 32.

as in this figure, at *A* and *B*. This form and arrange-




ment is found in mills of all kinds, from the earliest known printed figures to the wooden mill-work of the last century.

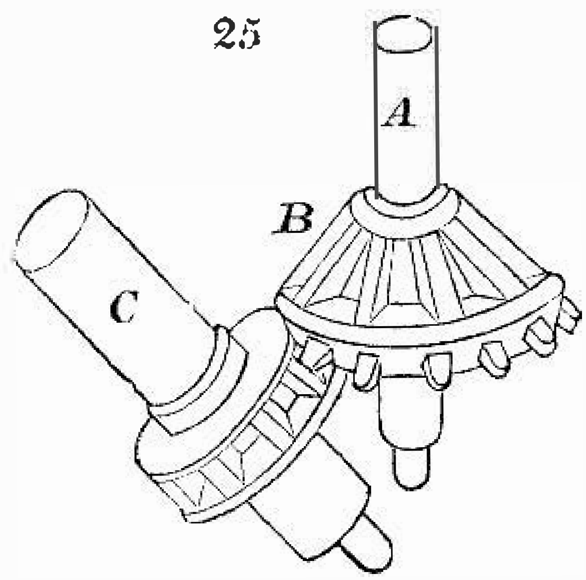
The wheel *B* is termed a face-wheel; it generally revolved in a vertical plane. This figure is copied from one in De la Hire's Mechanics*, in a chapter where he proposes to shew how the direction of motion may be changed by toothed wheels; and after giving the cylindrical lantern *A* for the case of axes at right angles, he proceeds to axes inclined at any other angle, thus:—"If a lantern *C* be constructed having staves inclined to the axis at any given angle, then will the horizontal motion of the power be changed into a motion inclined to it at any angle we please, provided only that the staves of this lantern *C* must be so arranged that they come successively into the horizontal position at the moment of meeting the teeth of the wheel *B*, in order that they may apply themselves to the teeth in the same manner as if this lantern was like the other *B*. These changes of direction in motions may be of great use in machinery."

* De la Hire's Treatise on Mechanics, Par. 1695. Prop. LXVI. This was early translated into English, in part, by Mandey, in his Mechanical Powers, 1709, p. 304.

It is rather singular, that upon the authority of this conical lantern the invention of *bevilgeer* has been attributed to De la Hire, when it is plain that the principle of rolling cones, which is essential to them, has nothing whatever to do with this arrangement; which is solely founded upon the notion of presenting the teeth to each other at the plane of centers, in the same relative position as in spur or face-wheels. The apex of the cone is turned in the wrong direction for bevil-wheels, and the cylindrical lantern is employed for the axes at right angles.

63. But the necessity of changing the direction of motion through other angles than right angles had arisen long before the time of De la Hire; suggested, as I believe, by the use of the Archimedean screw for raising water, which appears to have been a great favourite with the early mechanists. Figure 25, for example, is part of a complex piece of mill-work extracted from one of the early printed collections of machinery*. The object of the mechanism in question is to enable a water-wheel to give motion to a series of three Archimedean screws placed one above the other. A face-wheel, carried by the axis of the water-wheel, geers with a trundle (Art. 56) at the lower extremity of a vertical axis, which extends to the top of the building, and of which *A* is a portion.

25




Three conical wheels, similar to *B*, are placed one opposite to the lower end of each screw, as *C*, which it turns by geering with a square-staved trundle, as shewn in the figure.

* Le Diverse et Artificiose Machine del Capitano A. Ramelli. Par. 1588, ch. XLVIII.

These conical wheels are derived from the common spur-wheel, by the same principle of placing the teeth so that they shall, in crossing the line of centers, lie in the same relative position as if the axis of the wheel had been parallel to that of the trundle; which principle it was, in this case, oddly enough, thought necessary to extend also to the spokes or arms of the wheel.

64. The common *crown-wheel* and pinion, Fig. 26, which is used in clock and watch-work, in cases where axes meet at right angles, is another example of the same principle. The axis $A$, which carries the pinion, is at right angles to $B$, which carries the crown-wheel.

The teeth are cut on the edge of a hoop, and the action

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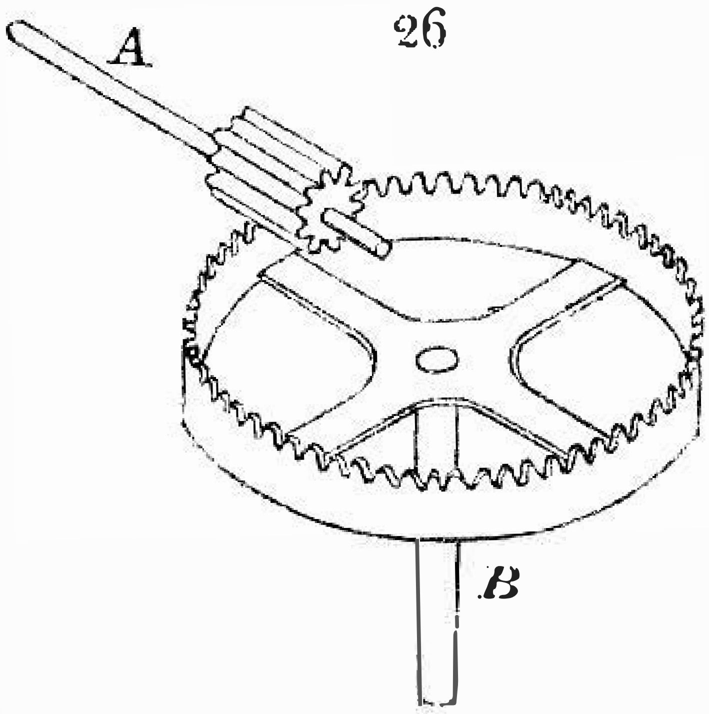


of the pinion upon them is nearly the same as if it worked with a rack; the combination being made on the presumption, that the curvature of that portion of the hoop whose teeth are engaged is so small, that it may be neglected; in which case, the hoop coincides with a rack which is tangent to it, along its line of intersection with the plane of centers, and which travels in a direction perpendicular to that plane.

The *crown-wheel* is often termed a *contrate wheel*.

65. *To form a pair of bevil-wheels*, a pair of conical frusta having been described (by Art. 44) to suit the required

angular positions of the axes and the given velocity ratio,

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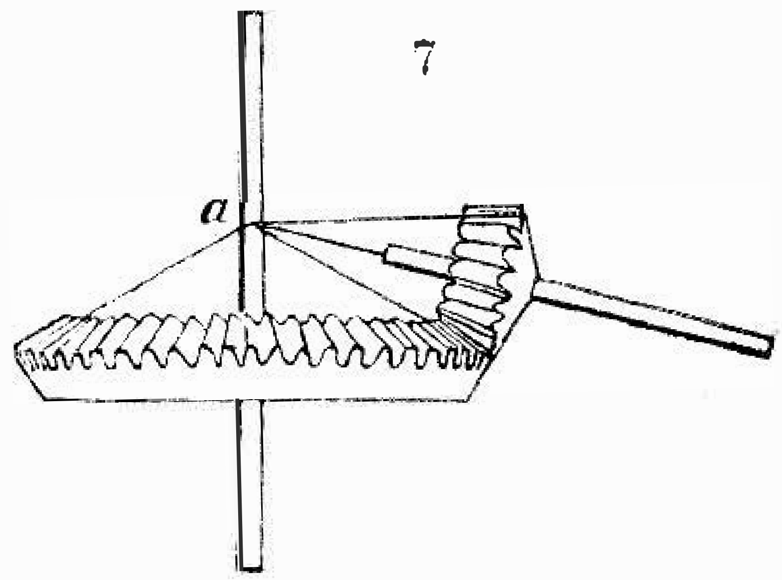
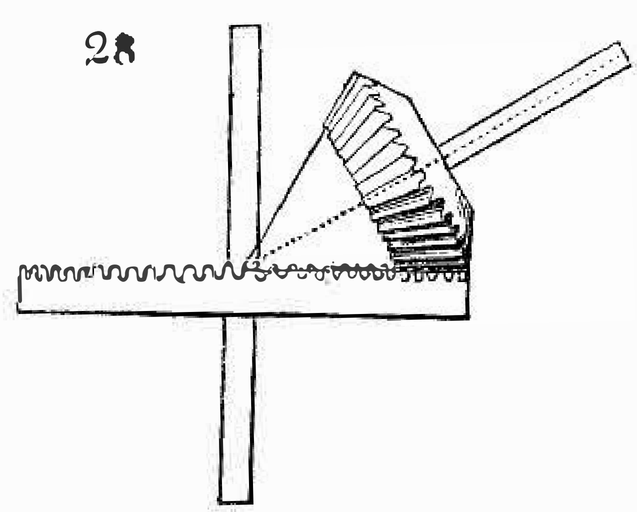



the smooth surface of these cones must be exchanged for a regular series of equidistant teeth, projecting nearly as much beyond the surface as the intermediate hollows lie below it, and directed to the apex of the cone, so that a line passing through this apex shall, if brought into contact with any part of the side of a tooth, touch it along its whole length. Thus the contact of one tooth with another will also take place along the line; whereas in face geering the contact of the teeth is between two convex surfaces at a point only.

28



66. It may happen that the common apex of the two cones shall lie so that one of them becomes a plane surface, as in fig. 28; in which case the teeth become radial. Also one of the cones may even be hollow, as in fig. 29.

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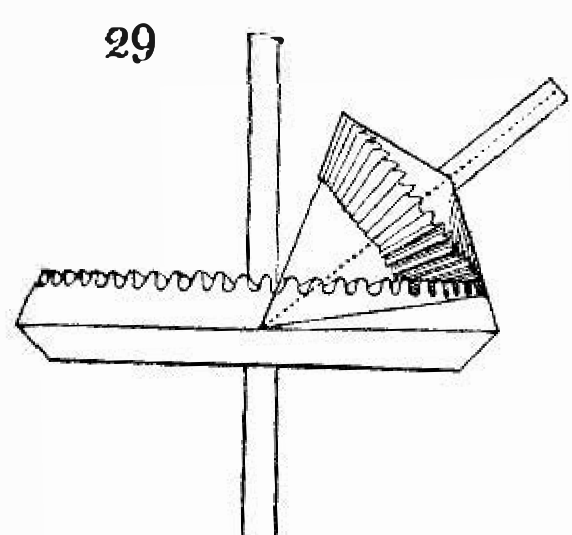
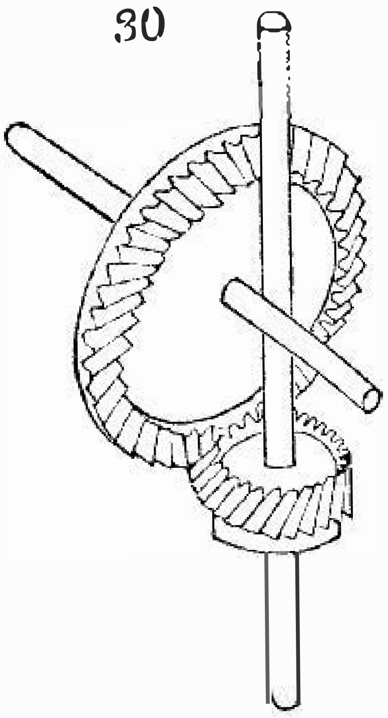


For every given position of the axes, however, we have a choice of two positions for the wheel which belongs to that shaft whose direction is carried past the other. In these last figures this wheel is placed below, but if it had been above, a different and smaller pair of cones would have been obtained for the given velocity ratio, in which these peculiarities of form would have been avoided.

67. When the axes are inclined to each other without meeting in direction, an intermediate double bevil-wheel may be employed, arranged as in Art. 48, or else frusta are employed, which are derived from the tangent cones of a pair of hyperboloids. (Art 47.)

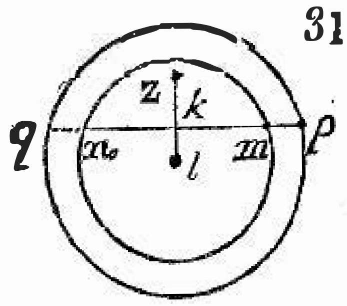
30



The direction of their teeth or flutes must be inclined to the base of the frustum, to enable them to come into contact; and the oblique position thus given to teeth has procured for wheels of this kind the name of *Skew Bevils.* If the teeth be cut in the direction of the generating line of each hyperboloid, they will obviously meet, since this line is the line of contact of the two surfaces.

To find this line upon a given frustum of the tangent cone, let fig. 31 be the plan of this frustum, $l$ the center; set off $lz$ equal to the shortest distance of the axes, (their common perpendicular) and divide it in $k$, so that $lk$ is to $kz$ as the mean radius of the frustum to the mean radius of that with which it is to work, draw $km$ perpendicular to $lz$, and meeting the circumference of the conical surface at $m$. Perform a similar operation on the base of the frustum, by drawing a line parallel to $km$, and at the same distance $lk$ from the center, meeting the circumference in $p$; join $mp$, which is plainly the line of direction of the teeth, (vide Art. 45).

31




We are also at liberty to employ the equally inclined line $qn$ in the opposite direction, but care must be taken that in the two wheels that pair of directions be taken of which the inclinations correspond.

But this question may also be satisfied upon the principle of face-wheel geering, and was so disposed of by the older mechanists, the teeth being merely arranged on the principle already explained, so that they should pass at the instant of contact, in the same relative positions as if the axes had been parallel, or meeting in direction.

68. It has been already shewn that there is no rubbing friction when the point of contact of two edges is on the line of centers. Of this Dr Hooke was certainly aware, as appears from his remarkable contrivance to get rid of the friction of wheel-work. This, to use his own words, "I called the perfection of Wheel-work; an invention which I made and produced before the Royal Society in 1666."

"It is, in short, first, to make a piece of wheel-work so that both the wheel and pinion, though of never so small a size, shall have as great a number of teeth as shall be desired, and yet neither weaken the work, nor make the teeth so small as not to be practicable by any ordinary workman. Next, that the motion shall be so equally communicated from the wheel to the pinion, that the work being well made, there can be no inequality of force or motion communicated. Thirdly, that the point of touching and bearing shall be always in the line that joins the two centers together. Fourthly, that it shall have no manner of rubbing, nor be more difficult to be made than the common way of wheel-work, save only that workmen have not been accustomed to make it*."

This fourth condition of no rubbing is, however, as we have seen (Art. 35), necessarily included in the third.

* Vide Cutlerian Lectures, by R. Hooke, No. 2, entitled Animadversions on the first part of the Machina Cœlestis, 1674, p. 70.

First, then, if there be a certain large number of teeth required to be made in a small wheel, then must the wheel and pinion consist of several plates or wheels lying one beside the other, as in this figure *A*, where eight plates

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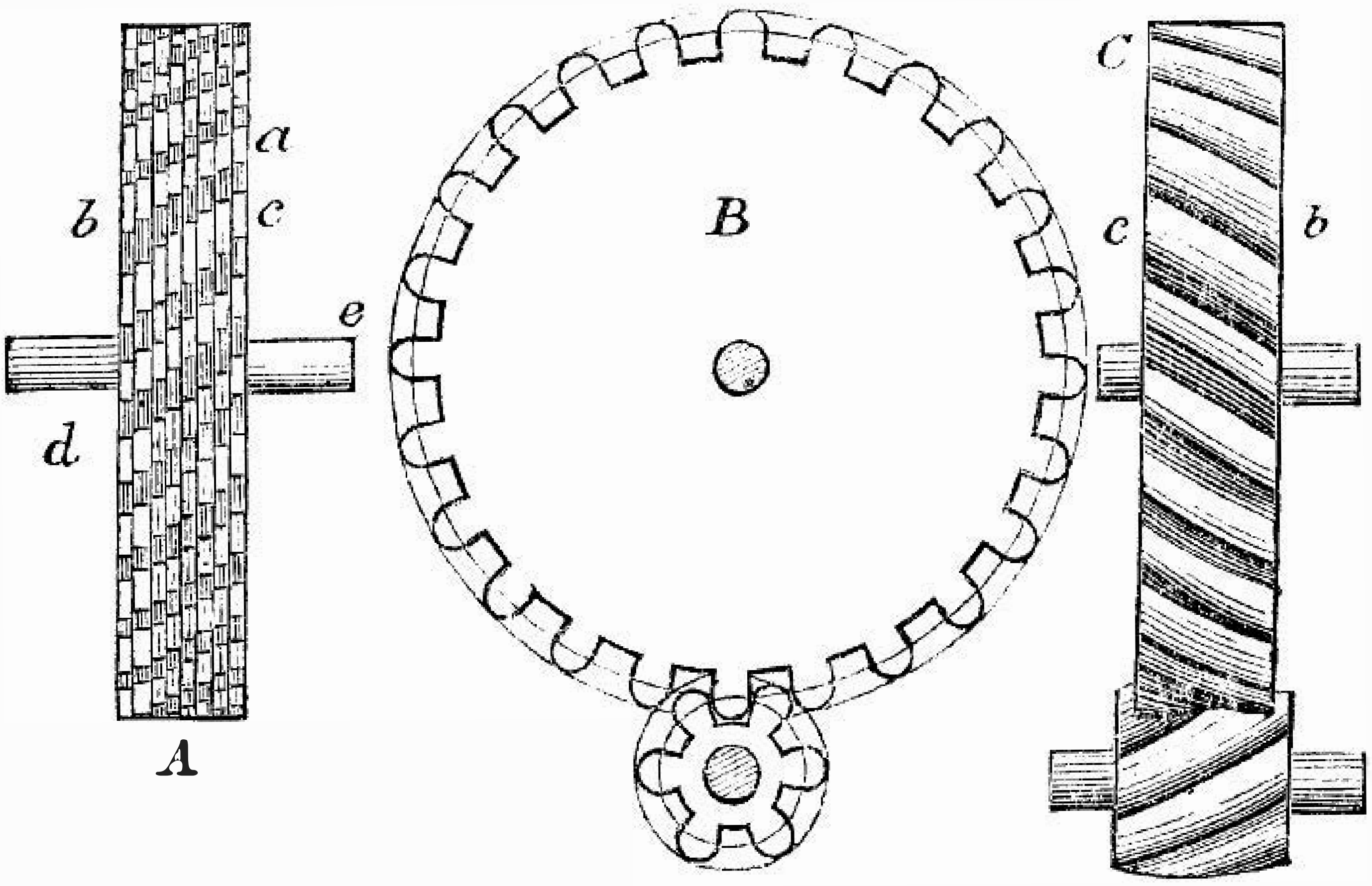


of equal thickness and size, are each cut into a wheel of twenty-five teeth, as shewn in front elevation at *B*; the wheels are fitted close together upon one arbor *de*, and fixed in such order that the teeth of the successive plates follow each other with such steps that the last tooth of each group may within one step answer to the first tooth of the next group. Thus, reckoning from *a* to *b*, the teeth follow each other in equidistant steps of such a magnitude that *b* is distant one such step from *c*, the first tooth of the next group.

The pinion being constructed upon a similar principle, and of the same number of plates, it is clear that the inequalities in the touching, bearing, or rubbing of such wheel-work, would be no more than what would be between the two next teeth of one of the sets, that is, about the same as in a wheel of 200 teeth, and yet the teeth are as large as those of a wheel of 25 teeth.

Secondly, if it be desired that the wheel and pinion should have infinite teeth, all the ends of the teeth must, by a diagonal slope, be filed off and reduced to a straight or rather a spiral edge, as in *C*, which may indeed be best made by one plate of a convenient thickness, which thickness must be more or less according to the bigness of the sloped tooth. And this is to be always observed in the cutting thereof, that the end of one slope tooth on the one side be full as forward as the beginning of the next tooth on the other; that is, that the end *b* of one tooth on the right side be full as low as *c*, the beginning of the next tooth on the left side.

Thus far I have employed nearly the words of Hooke, who has, however, said nothing respecting the *form* of the teeth, which must evidently, in the second system, be so shaped as to begin and end contact upon the very line of centers; the mode of effecting which will appear in the next chapter. The contact of the teeth will be at every instant at a single point, which point will, as the wheel revolves, travel from one side of the wheel to the other; a fresh contact always beginning on the first side, just before the last contact has quitted the other side. And as the point of contact is always on the line, or rather plane, of centers, it is strictly rolling, and there will be no sliding or friction between the teeth.

Hooke's system has been several times re-invented, for example, by Mr White, of Manchester, who patented it before 1808*; and endeavoured, in vain, to introduce it into the machinery of that place. The motion of such wheel-work is remarkably smooth and free from vibratory action, but it

* Vide White's Century of Inventions, 1822, Memoirs of Lit. and Phil. Soc. of Manchester, also Sheldrake, Theory of Inclined Plane Wheels, 1811. It has besides been reproduced as new in America, and lately in London, under the name of a Helix Lever.

has the defect of introducing an endlong pressure upon the axes, occasioned by the obliquity of the surfaces of contact to the planes of rotation. But there are many cases in which this property, when understood and provided for, would not be injurious. The first form of Hooke's geering, in which it appears as separate concentric wheels, as at *A*, has been employed successfully in cases where smooth action is necessary*; and is free from the oblique pressure, but loses the advantage of the perfect rolling action.

## ON PITCH.

69. Let $N$ and $n$ be the numbers of teeth of the driver and follower respectively, then as the teeth are equally spaced upon the circumference of the two wheels, these numbers are proportional to the circumferences and radii of their respective wheels; hence

$$\frac{N}{n} = \frac{R}{r} = \frac{P}{p} = \frac{l}{L}. \quad \text{(Vide Art. 42.)}$$

70. The *pitch circle* of a toothed wheel is the circle whose diameter is equal to that of a cylinder, the rolling action of which would be equivalent to that of the toothed wheel (Art. 50); therefore in the above equation $R$ and $r$ are the radii of the pitch circles of the driver and follower respectively; these rolling cylinders being the limit to which the toothed wheels approach, as their teeth are indefinitely diminished in size and increased in number, the distance of the axes remaining the same.

This circle is variously termed the pitch circle of the wheel, the primitive circle, or the geometrical circle. I

* I have seen it in a planing engine by Mr. Collier, of Manchester.

prefer the term pitch, as less liable to ambiguity, and as, I believe, the one most usually employed. In conical wheels the pitch circle will be the base of the frustum.

71. Let the circumference of the pitch circle be divided into equal parts, in number the same as that of the teeth to be given to the wheel; the length of one of these parts is termed the *pitch* of the teeth, or of the wheel, and evidently contains within itself the exact distance occupied by one complete *tooth and space.* The word *space* is employed here in its technical meaning, as denoting the hollow or gap that separates each tooth from the neighbouring one.

Let $C$ be the pitch, $D$ the diameter of the pitch circle, both expressed in inches and parts; and let $N$ be the number of teeth, then $NC = \pi D$*; from which expression if any two of the quantities $C$, $D$, $N$ be given, the third may be found. The arithmetical rules which are immediately deducible from this equation are in constant requisition amongst millwrights.

72. In English practice it has been found convenient to employ only a given number of standard values for the pitch, instead of using an indefinite number. The values most commonly chosen are 1in., $1\frac{1}{8}$in., $1\frac{1}{4}$in., $1\frac{1}{2}$in., 2in., $2\frac{1}{2}$in., 3in. And it very rarely happens that any intermediate values are necessary. Below inch pitch the values $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, and $\frac{3}{4}$, are perhaps sufficient.

These remarks apply to cast-iron wheels principally, as the great utility of this system of definite values for the pitch resides in its limiting the number of founders' patterns. Cast-iron teeth of less than $\frac{1}{4}$in. pitch are seldom employed; and, for machinery of a less size than this, the wheels would

* Where $\pi = 3.1415$. The millwrights commonly use $\frac{22}{7}$ for $\pi$.

be cut out of disks of metal in a cutting engine. Nevertheless the same system of sizes might be introduced with advantage into wheels of this latter kind.

73. Since the values of $C$ are few and definite, the use of the expression $NC = \pi D$ may be facilitated by calculating beforehand the values of $\frac{C}{\pi}$ and $\frac{\pi}{C}$ that belong to these cases.

For $N = \frac{\pi}{C} \cdot D$, and $D = \frac{C}{\pi} \cdot N$; and the following Table furnishes the factor corresponding to each of the established values of the pitch, by the use of which the number of teeth may be readily found for any given diameter, or *vice versa*.

| Pitch in inches. | $\frac{\pi}{C}$ | $\frac{C}{\pi}$ |
|---|---|---|
| 3 | 1.0472 | .9548 |
| $2\frac{1}{2}$ | 1.2566 | .7958 |
| 2 | 1.5708 | .6366 |
| $1\frac{1}{2}$ | 2.0944 | .4774 |
| $1\frac{1}{4}$ | 2.5132 | .3978 |
| $1\frac{1}{8}$ | 2.7924 | .3580 |
| 1 | 3.1416 | .3182 |
| $\frac{3}{4}$ | 4.1888 | .2386 |
| $\frac{5}{8}$ | 5.0265 | .1988 |
| $\frac{1}{2}$ | 6.2832 | .1590 |
| $\frac{3}{8}$ | 8.3776 | .1194 |
| $\frac{1}{4}$ | 12.5664 | .0796 |

## EXAMPLES.

Given, a wheel of 42 teeth, 2 inch pitch, to find the diameter of the pitch circle. Here the factor corresponding to the pitch is .6366 which multiplied by 42 gives 26.7 inches for the diameter required.

Given, a wheel of four feet diameter, $2\frac{1}{2}$ pitch, to find the number of teeth; the factor is 1.257 which multiplied by 48, the diameter in inches, gives 60 for the number of teeth.

Given, a wheel of $30\frac{1}{2}$ inches diameter, and 96 teeth, to find the pitch. Here $\frac{D}{N} = \frac{30.5}{96} = .317 = \frac{C}{\pi}$; which value of $\frac{C}{\pi}$ corresponds in the Table to inch pitch.

Questions of this kind are continually occurring in the execution of machinery; and simple as the calculation may appear to a mathematician, they require more multiplication and division than is always at the command of a workman. By way of simplifying the expression of the relations between the size of the teeth, their number, and the diameter of the pitch circle, a different mode of sizing the teeth in small machinery has been adopted in Manchester, which may be thus explained.

74. Suppose the diameter of the pitch circle to be divided into as many equal parts as the wheel has teeth; and let one of these parts be taken for a modulus instead of the pitch hitherto employed; and accordingly, let the few necessary values be assigned to it in simple fractions of the inch. Call this new modulus the *diametral pitch* of a wheel, to distinguish it from the common pitch, which may be named the *circular pitch*, and let $M$ be the diametral pitch;

$\therefore \frac{D}{N} = M$, and, as $M$ is a simple fraction of the inch, let

$M = \frac{1}{m}$; $\therefore mD = N$, in which $N$ and $m$ are always whole numbers.

The values of $m$, commonly employed, are 20, 16, 14, 12, 10, 9, 8, 7, 6, 5, 4, 3; and all wheels being made to correspond to one of the classes indicated by these numbers, the diameter or number of teeth of any required wheel is ascertained with much less calculation than in the common system of circular pitch.

This Table* shews the value of the circular pitch $C$, corresponding to the selected values of $m$ already given.

| $m$ | $C$, in decimals of inch. | $C$, in inches to nearest $\frac{1}{16}$. |
|---|---|---|
| 3 | 1.047 | 1 |
| 4 | .785 | $\frac{3}{4}$ |
| 5 | .628 | $\frac{5}{8}$ |
| 6 | .524 | $\frac{1}{2}$ |
| 7 | .449 | $\frac{7}{16}$ |
| 8 | .393 | $\frac{3}{8}$ |
| 9 | .349 | |
| 10 | .314 | $\frac{5}{16}$ |
| 12 | .262 | $\frac{1}{4}$ |
| 14 | .224 | |
| 16 | .196 | $\frac{3}{16}$ |
| 20 | .157 | $\frac{1}{8}$ |

* This table is founded on the practice of the well-known factory of Sharp, Roberts, and Co., at Manchester, and may therefore be relied on as exhibiting the present most perfect methods employed in the smaller class of mill-work, or cast-iron mechanism. In this system, a wheel in which $m = 10$ would be called a ten-pitch wheel, and so on.

Since $\frac{D}{N} = M$, we have $M = \frac{C}{\pi}$; therefore the diametral pitch is the quantity which has been calculated in the second column of the Table in page 58. In fact, it is easy to see that this scheme differs from the first, merely in expressing in small whole numbers the quantity $\frac{\pi}{C}$ instead of $C$.

In small machinery, of the kind that would be classed as clock or watch-work, and in which the wheels are cut out of plain disks by means of a cutting engine, the size of the teeth is often denoted by stating the number of them contained in an inch of the circumference, which may vary from about four to twenty-five. The word pitch is unknown to clockmakers, and their pitch circle is termed the geometrical circle; but, for the sake of uniformity, I shall apply the term pitch indifferently to all kinds of wheel-work. In cut wheels it is necessary to calculate the pitch for the purpose of obtaining the size of the cutter, which, as it operates by cutting out the spaces between the teeth, ought of course to be exactly of the same form and breadth as those spaces. When the number of teeth and geometrical diameter of a wheel are given, the pitch of these small teeth may be determined, in decimals of the inch, from the general expressions already given for the teeth of mill-work; and after the forms of the teeth have been described according to the methods contained in the next chapter, the shape and size of the cutter will be obtained.