CHAPTER III.

PAIRS OF ELEMENTS.

§ 14.

Different Forms of Pairs of Elements.

WE have already found, in the general solution of the machine problem (p. 35 *et seq.*), that the elementary—or what may be called the elementary—parts of a machine are not single, but occur always in pairs,—so that the machine, from a kinematic point of view, must be divided rather into pairs of elements than into single elements. It is the geometrical form of these pairs with which we must first of all make ourselves acquainted.

We shall in the first instance limit our investigations to rigid bodies,—that is, to such as possess approximately complete rigidity;—the problem before us in the construction of pairs of elements will then be the determination of a given or required motion by means of two such bodies or elements only. As we have found in the last chapter, these elements must satisfy the following conditions:—

1. That one element be fixed relatively to the surrounding portion of space,—itself assumed to be stationary;

2. That this element be so formed as to carry upon or within itself the envelope of the second and moving element, which

3. must be so arranged as to prevent every motion of the second element except the one which is required.

The stationary element then holds the moving one as it were imprisoned,—preventing every motion except a single one,—forcing every point in it, when it has begun to move, to travel in a determinate path, on which account such a pair of bodies may be called constrained. We referred in the last article to the immense number of forms which the relative motion of two bodies might take; remembering this, it is easily seen that pairs of constrained bodies may have very many geometrical forms. All pairs of such forms, however, which conform to the two last of the above conditions, have this in common, that they are envelopes, and indeed reciprocal envelopes, for the given motion,—which can be represented through their axoids. Hence they may, like their axoids, be more or less simple. We can imagine a case fulfilling both the conditions, and in which at the same time the one element not merely forms an envelope for the other, but encloses it,—in which, that is to say, the forms of the elements are geometrically identical, the one being solid or full and the other hollow or open. Such a pair of bodies may be called a closed pair.

It is evident that, in their simplicity, closed pairs differ notably from pairs in which the elements are not identical in form. We shall on this account consider them separately and in the first instance.

§ 15.

The Determination of Closed Pairs.

The geometrical properties of the bodies from which closed pairs can be constructed are so well defined that we do not require to look first for these pairs in existing machinery, but may attempt to discover them by *a priori* reasoning.

Two bodies forming a closed pair cover each other with their surfaces; on these we may imagine any number of pairs of coincident curves, and among these some may be supposed to be such that the single motion possible for the time being occurs along them,—such in other words, as slide on one another. If a pair of these curves,—(one belonging to the one and the other to the other element),— be isolated, they can therefore be caused to slide upon one another without destroying their coincidence. Thus if in each of two points A and A' of both curves (Fig. 37), an

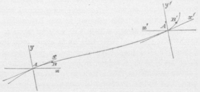


FIG. 37.

osculating plane be laid, and if further by its means the homologous systems of coordinates $x\,y\,z$, $x'\,y'\,z'$ be drawn, then if $A\,B$ and $A'\,B'$ be portions of equal length, and A be brought to A' and x to x' B must come to B', and the whole line $A\,B$ must coincide with $A'\,B'$. Among the lines which fulfil this condition we have first, where extension alone is concerned, the straight line;—among plane curves, that is curves of two dimensions, we have the circle only, and among general curves of three dimensions, the cylindric helix only. The two first may, however, be considered as special cases of the last, so that we may say that the only curve fulfilling the required condition is the cylindric helix. The common screw and nut therefore form a closed pair (Fig. 38).

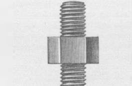


FIG. 38.

The form of the screw is not however in this case completely indifferent, on account of the third condition that only a single motion shall be possible, which here must be along the helix itself. Screw and nut must therefore be so shaped that each and every motion normal to the screw line must be impossible. With this object various forms are given in the common screw to the section perpendicular to the direction of sliding,—all being formed so that the sectional profile is double-sided, as in the various forms shown in the annexed figure.

Fig. [illegible]

In a closed pair having a profile such as the foregoing every motion oblique to the helix is rendered impossible,—so that only motion along the helix itself can take place. Everyone is familiar with the use of these profiles in the ordinary screws employed in construction. If a straight line parallel to the axis and longer than the pitch of the screw be used to generate a screw-surface, the surface obtained coincides with that of a circular cylinder, so that in a pair of bodies formed of screw-surfaces of this kind only relative radial motion would be prevented,—they would not be constrained so far as motion in any other direction is concerned.

We may now proceed to examine more closely the effect of altering the two essential dimensions of the closed pair which we have found, a screw and nut having a thread of suitable profile. Any such screw owes its special properties to two quantities, its radius or circumference and its pitch-angle, (or angle which has for its tangent the ratio pitch : circumference).

The alteration of the radius alone, (the pitch-angle remaining constant), gives us no new form,—all the properties of the original screw remain unchanged.

With the angle of pitch it is otherwise. If this be gradually diminished,—the diameter remaining constant,—the pitch becomes less and less, until at last if the angle be reduced to zero the pitch disappears altogether; the profile has a motion of rotation only, and describes simply a solid of revolution. The profile-section, however, in a plane at right angles to the direction of sliding remains unaltered, and has become the form by which the solid is generated, and we have as the result a closed pair consisting of two solids of revolution, of which the profiles are such as to prevent all motion in the direction of the axis. Fig. 40 is an illustration of such a pair,—in which the nut has become an eye (shown in section); the only motion which is possible for it being rotation.

If we now suppose the pitch-angle to increase instead of diminishing, the screw becomes steeper and steeper. If we make the angle = 90°, the screw lines become parallel to the axis,—

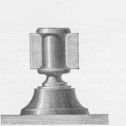


Fig. 40.

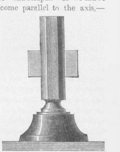


Fig. 41.

the screw becomes a prism, and the nut a corresponding hollow prism. The normal profile of the screw thread is now at right angles to the axis, and so is transformed into the normal section of a prism, which always retains a profile preventing cross-motions,—that is, a non-circular profile. As a result we have a closed pair consisting of two prisms having such profiles as to prevent any turning about their axes (Fig 41). The single possible motion of the open prism is here sliding in the direction of the edges of the full one.

Further alteration of the pitch-angle gives no new result—if we make it > 90° the screw passes from right to left-handed; it remains, however, always a screw; and we have found it unnecessary to make any distinction between a right and left-handed thread. The problem of the closed pair has thus been exhausted by our investigation. It will be well to consider the three forms which we have found as distinct from each other,—although they might all be considered as modifications of the screw,—and we have thus three closed pairs to distinguish. These are,—to recapitulate in a few words :—

1. The common screw and nut (twisting pair);
2. The solid and hollow solid of revolution, which forthe sake of brevity we shall call full and open revolutes respectively (turningpair);
3. The full and open prism, (sliding pair);

all three having normal profiles which prevent what we have calledn cross-motion; theyn aren adaptedn forn then productionn of

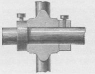


Fig. 42.

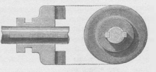


Fig. 43.

three kinds of constrained motion,—viz. motion (*a*) in helical paths; (*b*) inncirenlar paths; and(*c*) in rectilinear paths.

All three are well known in machinenconstruction,—the screw-pair both in fastenings andin movingpieces; the pairof revolutes in journals, bearings, &c. and the prism-pair in guides of all sorts. The use of such normal profiles as prevent any motion of a pair except in the one required direction, is also verynfamiliar. Therings,collars,or flanges of the bearings of shafts and spindles, for instance, carry thesen profiles. If it ben desired to usen a cylindrical shaft, which for convenience has been made without collars, as one element of a turning-pair,—the well-known "loose ring" (Fig. 42) is used to close the pair. The ease with which cylinders can be formed in the lathe causesnthem often tonbe employedn inn placesn wheren theyn haven ton becomen elements in sliding pairs. To make a cylinder into a prism for this purpose is the object of the common arrangement of feather and groove shown in Fig. 43. The same process of converting a cylinder into a prism is often used in cases where two bodies have to be so connected that they may resist all forces tending to move one upon the other;—here keys, cutters, and so on are employed. In short, the machine-maker is accustomed to fulfil the above described condition in most numerous ways in his practical work.

§ 16.

Motion in Closed Pairs.

We have found in the last section that there are three pairs of elements which fulfil the conditions necessary for the complete and continuous enclosure of the bodies of which they consist. It is specially notable that there are only three,—in itself a remarkable result of the investigation,—for judging from the immense variety of cases which occur in machinery we might have been inclined beforehand to assume the existence of a very much larger number. These three single cases are, however, still further characteristic, on account of the nature of the constrained motions which can be carried out by their means.

In the screw-pair all points in the nut describe helices,—and equal helices if the describing points lie at equal distances from the axis. These motions are compounded of a rotation about an axis and a sliding along it, and this axis is always that of the screw-spindle itself. The axoid of the screw-spindle (see § 13) is hence a straight line coinciding with the axis of the screw. We can find the axoid of the nut at once, by supposing it fixed and causing the spindle to move: all points of the spindle then describe helices relatively to the nut, and equal helices if the points are equally distant from the axis,—exactly the same motion, that is, as that of the screw. The axoid of the nut is therefore likewise a straight line coinciding with its geometrical axis. This axoid slides endlong upon the first and at the same time revolves about it, the angular motion bearing always a constant relation to the sliding. We have then before us in this pair of elements,—the screw and nut,—the most general case of the twisting of axoids, reduced at the same time to the most simple imaginable form, where both axoids are concentrated in the twisting axes themselves.

With the turning-pair we can observe something very similar to this. Here all points in the moving eye, the open element, describe circles about points in the geometrical axis of the stationary cylinder, these circles being equal for points at equal distances from the axis. The axoid of the fixed body is thus again a straight line coinciding with its geometrical axis, and we find the axoid of the eye to be the same, if we fix it and cause the cylinder to move. Thus for the axoids of this pair of elements,—the full and open revolute,—we have again two coincident axes turning about each other, forming the simplest case of cylindric rolling which we can conceive, one in which both the cylinders of instantaneous axes have become merely straight lines.

With the prism-pair all rotation ceases; the twisting of the instantaneous axes becomes a simple sliding of them one along the other. The geometrical axes of both prisms may be considered to be their axoids, but the notion of the geometrical axis is not so determinate in the prism as in the cylinder or screw; and we can consider any given pair of coincident edges or parallels to edges to be axoids.e

Here therefore the other extreme of the most general case of twisting is realized—that in which the sliding alone remains.

We may now advance a small but important step. We have above laid down as the first condition for the attainment of a given motion by one pair of bodies that one of the elements must be rigidly connected with the portion of space which we have considered as stationary. We may now release ourselves from this condition. For if two elements which have been rightly paired be both set in motion, there still continues between the one element and its partner the former absolute motion, (or motion which we have agreed to consider absolute,) but it has now become the relative motion of the element to its partner. We may therefore arrange the pairs of elements which we have found in kinematic chains, where then the relative motion of the paired elementse becomes that also of the links which they connect.

From all this we see that the three closed pairs represent the three typical cases of the most general form of relative motion described in §§ 12 and 13; viz. (reversing their order), simple sliding or translation, simple rotation, and simple sliding combined with simple rotation proportional to the sliding.

This is one of the characteristics of the closed pairs. Another very notable one we have already noticed, but without enlarging

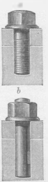


Fig. 44.

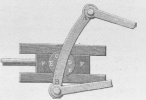
Fig. 45.

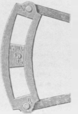
Fig. 46.

upon it. It is this, that the exchange of the fixed element with the movable one causes no alteration in the resulting absolute motion. The equality of the axoids proves this generally. It is however extremely important, and for the purposes of machine construction extraordinarily valuable. The exchange of one element of a pair with the other,—or, as we can say, the exchange, in respect to its fixedness or movability, of an element with its partner, —we shall call the inversion of the pair:—the inversion of closed pairs causes no alteration in the motion belonging to them.

Continual use is made of this principle in machine construction. Where, for example, a screw with a head (Fig. 44 *a*) is employed instead of one with a nut (Fig. 44 *b*), it is simply an inversion of the closed pair screw and nut. In the common waggon wheel the axle is fixed to the waggon body, the wheel with the open revolute moving upon it; in a railway carriage the latter is attached to the frame, the solid body or shaft being connected with the wheel, and movable. For a guide for rectilinear motions either of the arrangements shown in Figs. 45 and 46 is used, as may be more convenient; in the first of these a solid prism *A* slides in a prismatic slot, or open prism, *B*, while in the other an open prism, *A*, is moved backwards and forwards upon a straight bar *B*.

The familiar and easy realization of this invertibility of the elements of closed pairs is in many cases of the greatest value





to the constructor, in using such pairs he at once recognises the possibility of employing either the one or the other element as the hollow body or making the contact of each with its partner partly internal and partly external. The recognition of this principle sometimes removes differences between constructions which in their external appearance differ more or less widely, or at least gives a simple expression to what was before an indistinct sense of relationship between them. There may be mentioned, for instance, the exchange, which has of late been frequently seen, of the cylinder with the piston, which (*e.g.*) distinguishes Condie's Steam-hammer from Naismyth's. What the constructor has here carried out is the inversion of a prism-pair, whilst in other matters the functions of the different mechanisms remain unaltered; the changed arrangement of the ports presented some difficulties, the slide-valve also required to be somewhat differently arranged for convenience' sake, but in essence they have remained as before.

As another example I may mention the reversing link of Humphry and Tennant, Fig. 47 (or more rightly of Naismyth),* as compared with the older and more common one of Stephenson, Fig. 48. Here inversions of two pairs have taken place. First, Humphry's bar-link $A\,B$ is an inversion of the slotted link $A_1\,B_1$ used by Stephenson, and hence the element paired with the link in the latter case, the slide $C_1\,D_1$, becomes in Fig. 47 a hollow block $C\,D$, in which $A\,B$ can slide. Naismyth has also changed the cross pin F_1 into a body $F\,F$ having a cylindric hole, and the piece E_1, which in Stephenson's has such a hole, into a solid cylinder $E\,E$, of such a size as is necessary to allow the link to pass through it in the way shown. Kinematically the pieces $C\,D\,E$ and $C_1\,D_1\,E_1$ are completely identical,—both having for their element-forms a curved sector having a prismatic cross-section, and a cylinder normal to it.

These inversions frequently afford great advantages in construction, and on this account they are matters of considerable importance in machine-design. In kinematic science they are examples of the application of a simple general law, which as we have seen affects the simplest element-pairs generally and *a priori.*

§ 17.

The necessary and sufficient Restraint of Elements.

While in the course of our examination of closed pairs we considered the forms screw, revolute, and prism, and examined the relation between the corresponding solid and hollow pieces, we took no notice of the fact that the mutually enveloping geometrical forms were not always equally large or equally extended in the cases we used as illustrations. We found, and find almost always in practice, the nut to be much shorter than its screw,—

* Cf. *Practical Mech. Journ.*, 1862–3, p. 232.

the prismatic sliding block than its guide; in plummer-blocks spaces are left between the brasses, and in these themselves oil-grooves are frequently made,—all of which cases are equivalent to removing the corresponding parts of the closed figure.

This procedure is so useful and is taken so much for granted in machine construction, where the portions of surface left are always of sufficient extent, that the question how far it can be carried scarcely presents itself to the designer. Where the working of a machine is accompanied by the transmission of forces of considerable magnitude, its designer furnishes it with large bearing-surfaces to prevent wear; but the question here is as to the absolute dimensions of the surfaces, and not as to

FIG. 29.

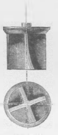


FIG. 30.

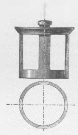


FIG. 31.

their distribution among the different bodies. If the forces are small, consideration as to wear scarcely enters into the question; the extent to which the surfaces can be diminished is here limited by the consideration that what is left must be sufficient to ensure that the paired elements shall always occupy the intended positions relatively to each other. The common cone-faced valve furnishes one among many illustrations of this. In the above three figures a simple valve, such as might be used to check the passage of water in a pipe, is shown in forms more and more removed from that of a solid cylinder. Such portions of the solid cylinder are always left to form the valve that in neither of the three cases can it be so moved that its axis shall not coincide with that of the hollow cylinder which forms its seat. In the first case (Fig. 49) three small segments of the cylinder are cuthaway; in the second (Fig. 50) four screw-formed webs only are left, and in the third (Fig. 51) four thin strips of the cylinder parallel to its axis, and connected below by a ring forming part of its cross section. There must evidently be some general principle underlying the arrangement of these small strips or other portions of the cylindrical surface which have to be retained in order that the bodies may keep their required mutual positions, or, as we may say, in order that they may mutually restrain each other. A definite number of such points is necessary, but is at the same time sufficient, in order to ensure this mutual restraint. This minimum of points of restraint we shall now endeavour to find. It is not an investigation to which hitherto any special value has been attached, but it is unquestionably one which should be kept in view, not only because no property of machine-elements can be unimportant in a scientific examination of the nature of machines, but also because of the important results which are directly connected with it.

§ 18.

Restraint against Sliding.

We shall first consider the case of a plane figure moving in a plane,—or, if it be preferred, of a plane section of a cylinder prevented in any way from leaving the plane in which it lies. By the expression point of restraint of the figure we shall mean a point in its circumference towards which the figure is prevented from sliding along or parallel to a normal to the tangent at that point. Sliding of the figure implies here an equal and similar motion of all points in it.

Single Point of Restraint.—Let the given figure *A* be prevented from moving freely in its own plane by contact in one point with a second and con-plane figure *B*;—we shall examine to what extent its motion is limited. The definition of a point of restraint just given renders it unnecessary that we should concern ourselves about the shape of the restraining figure B (Fig. 52);—we require only to draw a tangent TT' to the restrained figure A at a, and erect upon this through a the normal NN';—the direction from A towards a and N' is then that in which the point of restraint renders sliding impossible (Fig. 53).

No sliding therefore which has a component in this direction can occur. The only motions, however, which have not such components are those whose directions are included in the angle $T N T'$, as indicated by the arrows. This straight-angle may be called the field of sliding for a figure restrained only at a, and having the normal $N'a$ to the tangent $T T'$ as the direction of restraint. All the directions in which motion is prevented by the restraining point fall within the second straight-angle $T N' T'$,—

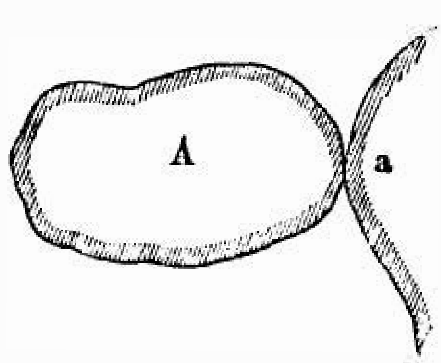


FIG. 52.

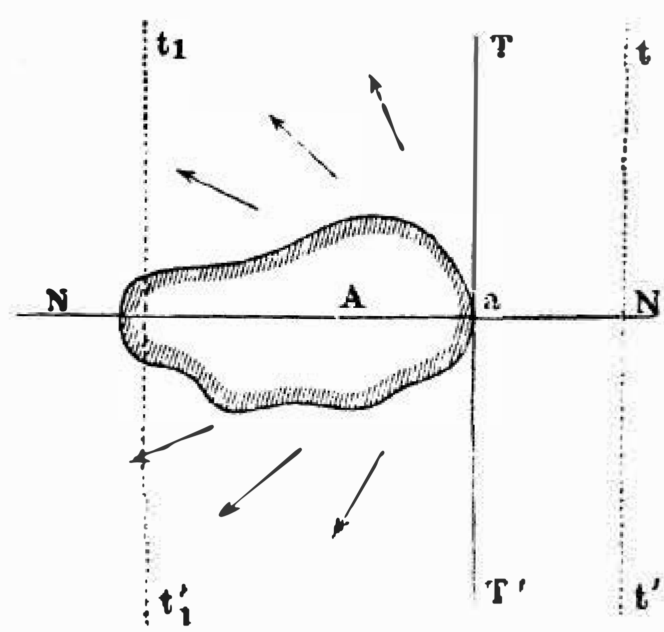


FIG. 53.

this we may therefore call the field of restraint for the point a. The fields of sliding and restraint for any point of restraint contain together four right angles. They are separated at the point of restraint by the tangent $T T'$; but as the essential difference between them is a question only of angle or direction, this line of separation may take different positions, such as $t\,t'$ or $t_1\,t'_1$, so long as it remains parallel to $T T'$. In general we may therefore say that any normal to the direction of restraint is a division line between the fields of sliding and restraint.

Two Points of Restraint.—If a figure have two restraining points, a and b, Fig. 54, these limit the possible directions of sliding to the angle enclosed between the two tangents $a\,T$ and bT, because all directions falling outside this angle, as those marked

3 or 4, must have a component parallel to one or other of the directions of restraint 1 and 2. If the division line between the fields of sliding and of restraint for each of the points a and b be drawn through the intersection O of their normals, the shaded angle $P\,O\,Q$ enclosed between them is the field of sliding, and the exterior angle $Q\,O\,P$ the field of restraint, for the case before us. Both points of support would equally prevent sliding in any direction falling within the opposite angle to $P\,O\,Q$

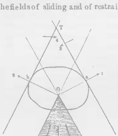


Fig. 54.

By reducing the angle $a\,T\,b$ between the tangents the field of sliding is made smaller and smaller. When they become parallel, as in Fig. 55, the angle covered by the field becomes infinitely small. But just as sliding could take place before along the lines of separation $O\,P$ and $O\,Q$, it can still occur along those lines now that they have become coincident,—that is, it can take place in a direction parallel to the two tangents. Motion is possible, therefore, not only along $O\,P$, but also along $O\,R$, the arms of the now infinitely small opposite angle. In other words, the field of sliding is now reduced simply to a line parallel to the tangents, and along this sliding can take place in both directions.

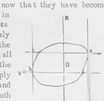


Fig. 55.

If the parallel directions of restraint 1 and 2 had not been opposite, as here shown, but in the same direction, the resulting restraint, so far as sliding is concerned, would not have differed from that given by a single point, so that it is not a case which need be further considered.

Three Points of Restraint.—The result of adding a third point of restraint, as c Fig. 56, to the two already examined, is easily found. We draw the tangent $c\,U$ and the normal through c, and also the line $O\,R$ separating the fields of sliding and restraint, placing the latter so as to pass through the intersection O of the two first normals. It is then evident from the figure that sliding is no longer possible through the whole angle $P\,O\,Q$, but that the field of sliding is diminished to the angle $P\,O\,R$. Here we see at once that we have the means of entirely preventing the sliding of a figure by the use of three points of restraint. For as the field of restraint of each single point covers 180°, nothing more is necessary than to place the third so that its field of restraint covers the sliding field of the other two. Fig. 57 represents this case. The third point c is so placed that its field of restraint,—extending to $R\,O$,—entirely covers the field of sliding $P\,O\,Q$ (shown by dotted lines) left by the other points a and b. The condition for the attainment of this end is that the three points of restraint be so placed that the angle between two consecutive normals should always be less than 180°. Figs. 58 and 59 represent separately the relative directions of the normals at the points of restraint in Figs. 56 and 57 respectively, and we see from them that in the first case the angles between the normals 1 & 2 and 2 & 3 are each less than 180°, but that between 3 and 1 is greater; while in the second case each of the three corresponding angles is less than two right angles.

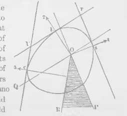


Fig. 56.

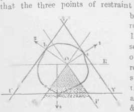


Fig. 57.

In the case in which the two first directions of restraint are parallel and opposite, Fig. 60, the third point c is not sufficient to prevent all sliding;—a fourth point d must still be added. For the directions $O c$ and OR, parallel to the tangents, in which sliding can take place, are themselves 180° apart. The addition therefore of two points of restraint, one between V and W, and another between T and U, is required in order that the angle between every pair of consecutive normals may be less than 180°. The

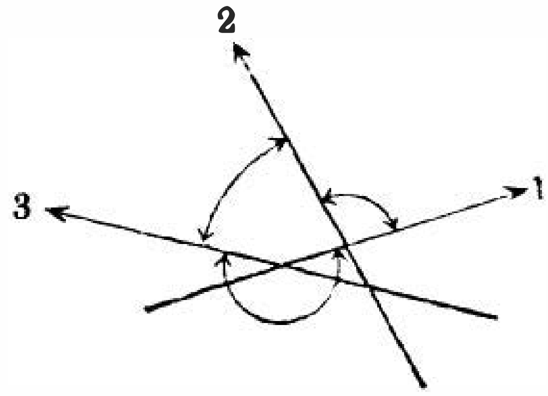


FIG. 58.

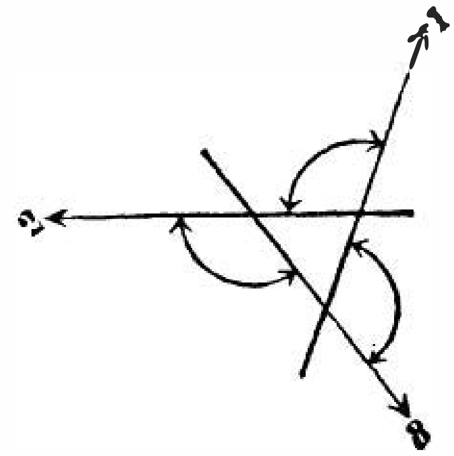


FIG. 59.

directions of restraint at c and d may in these circumstances again be 180° apart,—their tangents thus becoming parallel.

The minimum number of points of restraint which can completely prevent the sliding of a plane figure is therefore three, or if two of the directions of restraint enclose between them 180°, four. In the illustrations given in § 17 the sections of the

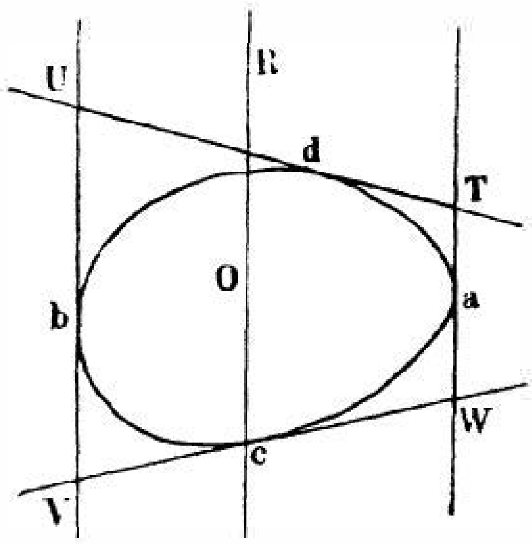


FIG. 60.

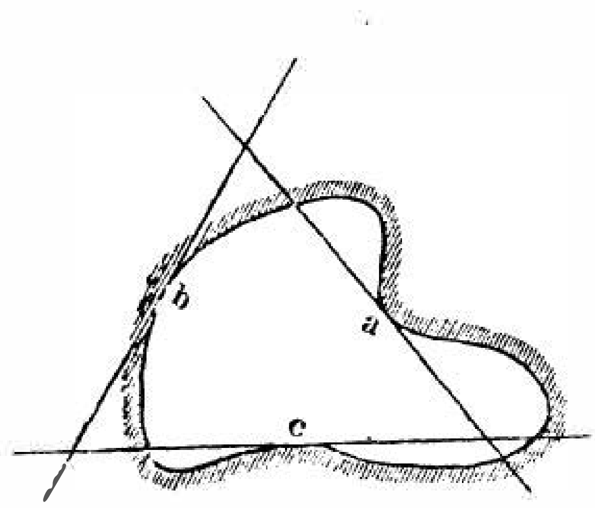


FIG. 61.

valve show that an arrangement is used which is in one case equivalent to three and in the others to four points of support. Every plane figure which has an outline returning upon itself can be completely restrained from sliding. It is entirely indifferent for this purpose whether the figure be restrained by external or internal contact, as Fig. 61, in which all the conditions of complete restraint are fulfilled, shows. Indeed, the investigation already made points this out,—although in another way,—for the second figure *B*, Fig. 52, which carries the point of restraint for the first, *A*, must be open or hollow if the first have an external profile (such as we have shown), while the action of the figures as to restraint is reciprocal, just as we have found to be the case with the elements from which pairs are formed.

§ 19.

Restraint against Turning.

Here also we shall consider first the case of plane motions of a plane figure, and shall understand by "turning" such a motion of the figure that some one point connected with it retains continuously or for an instant its position relatively to the plane. Two kinds of turning must be distinguished,—that which takes place in the same direction as the hands of a watch we call right-handed (*R. H.*), and that occurring in the opposite direction left-handed (*L. H.*), turning.

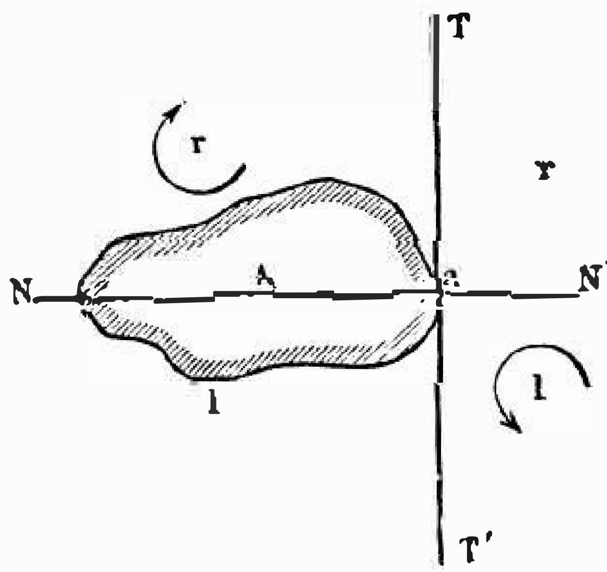


FIG 62.

Single Point of Restraint.—If the figure *A*, Fig. 62, have a single point of restraint *a*, at which *T T'* and *N N'* are again tangent and normal,—a right-handed turning may be given to it about any point in the quadrants *N a T* and *T a N'*, while left-handed turning cannot occur about any of these points, because the motion of the point *a* in these cases would always have a component in the direction of restraint. *L. H.* turning is possible, and *R. H.* impossible, about every point in the remaining quadrants *N' a T'* and *T' a N*. The turnings possible from points upon both sides of the normal *N N'* are indicated in the figure by the letters *r* and *l*. The whole field *N T N' T'* is thus a field of turning, and is divided by the normal *N N'* into fields of right- and left- handed turning. The normal itself belongs to both fields, so that turning in either direction is possible from points in it. About points in aN both turnings are possible to any extent, but about points in aN' they can occur only through infinitely small angles. For so soon as turning has commenced about any of the last-mentioned points, the normal passes to one or the other side of the centre of motion, which is thus thrown into the field either of right- or of left-handed turning,— and a glance at the figure shows that it must necessarily pass into that field which does not permit the continuance of the turning commenced.

Two Points of Restraint.—If a figure have two points of restraint, as a and b Fig. 63, and their fields of turning be separated by drawing the normals $a\,a'$ and $b\,b'$, we find at once that the field of $R.\,H.$ turning of a is covered throughout the angle $a\,O\,b$ between the normals by the field of $L.\,H.$ turning of b, and in this way both turnings are rendered impossible. Similarly, turning cannot occur about points in the angle $a'\,O\,b'$, — where again fields of right- and left-handed turning cover each other. In the angle $b\,O\,a'$ two fields of $R.H$ and in $a\,O\,b'$ two fields of $L.\,H.$ turning coincide. About points in these areas therefore (which are shaded in the figure) right-and left-handed turning are respectively possible. Thus of the two pairs of angles at the intersection of the normals,—one pair (that facing the intersecting point T of the tangents) forms a field of restraint for both turnings, while the remaining pair is the field of turning, one half of it being the field of right-handed and the other half of left-handed turning. The point O being common to both fields, turning in both directions may take place about it.

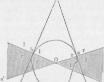


Fig. 63.

If the normals to the given points of restraint be parallel and opposite in direction (*i.e.* at an angle of 180° to each other), the angular field becomes a strip between the normals, about points in which either right- (Fig. 64) or left-handed turning (Fig. 65) is possible, according to the relative position of the normals. If the latter coincide (Fig. 66), the strip becomes a straight line, about points in which, as it forms the boundary line of two fields of turning in opposite directions, both right- and left-handed turning can take place.

If the parallel normals have the same direction (Fig. 67), the strip between them becomes a field of restraint, the space beyond

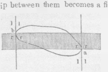


Fig. 66.

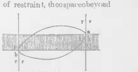


Fig. 67.

it being on the one side a field of $R. H.$ and on the other of $L. H.$ turning.

Three Points of Restraint.—If a third point of restraint c (Fig. 68) be added to any two others a and b, the tangents at which enclose any angle less than 180°, its influence upon the turning depends entirely upon its relative position. If, for instance, c be taken upon that part of the figure lying within

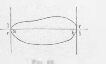


Fig. 68.

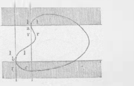


Fig. 69.

the tangent angle $a T b$, its normal cuts the other two in the points P and Q; its field of $L. H$ turning covers the similar field already existing, $a O b'$, so that this remains a field of $L. H$ turning. The $R. H.$ field of c covers the part $b P Q a'$ of the similar field already existing, which therefore also remains as before. Of the original field of turning only the small triangle $P O Q$ is covered by a pair of dissimilar fields,—so that turning about points in it only has been rendered impossible by the addition of the third restraining pointo

If c be sooplacedothat itsonormal passes through both halves of the field of turning (Fig. 69), the triangle POQ falls in the field of restraint of a and b instead of in their field of turning. The parts bOP and $c'Qb'$ areoeachocoveredoby a pair of dissimilar fields, and only the parts cPa' and $c'QOa$ remain as fields of right- andoleft-handed turningo respectively.

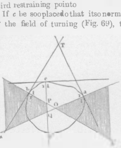


FIG. 68.

If theothirdo pointoofo restrainto be osooplacedothatoits normal makes a smalleroangle than 180° with those next it, as in Fig. 70, the result is widely different. If the normal to c then pass (as shown in the figure), through the original field of right-handed turning, its $R. H.$ field entirely covers the $L. H.$ one aOb', and its left-handed

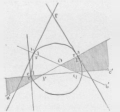


FIG. 69.

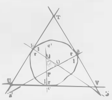


FIG. 70.

one the portion $a'PQb$ of the right-handed field $a'Ob$, so that turning about points in bothothese areas isprevented. The triangle POQ only is covered by a pair of similar fields, so thatoin it alone turning,andoright-handed turningonly, is possibleo If c had been so placed that its normal passed through the left-handed field $a\,O\,b'$,—the case would have been reversed, and a triangle would have been left about points in which $L.H.$ turning only could have occurred.

It will now be easily seen how the turning can be still further limited. For this purpose it is only necessary to diminish the size of the triangle $P\,O\,Q$. This becomes a minimum,—that is, a single point,—if c be so placed (Fig. 71) that its normal passes through the intersection O of the two first normals. The turning is then reduced as far as it can be,—but it still remains possible about one point.

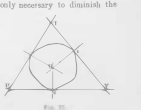


FIG. 71.

If the two first points of restraint have parallel normals we obtain other and entirely different results.

If the parallel normals to a and b have opposite directions (Fig 72), the normal to any point c between them divides the field of turning into two parts, of which one remains a field of turning, being covered by a pair of similar fields, while the other,

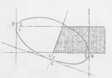


FIG. 72.

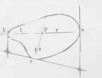


FIG. 73.

which is covered by dissimilar fields, becomes a field of restraint. If the field of turning of the first two points be a line only, (the normals coinciding, Fig. 73), about points in which, as we saw above by Fig. 66, both right- and left-handed turning may take place, the normal to c divides the line at P into two parts, about points in one of which, Pa, right-handed turning, and in the other, Pb, left-handed turning, is possible.

If the normals to the first two points of restraint be parallel and have similar directions, the addition of a third point may, as Fig. 74 shows, convert a part of each of the fields of turning into a field of restraint;—or it can, if its normal be parallel and opposite to the first two, entirely neutralize one of the fields of turning and reduce the other to a strip of limited breadth (Fig. 75);—or, lastly, if its normal be placed between the other two, and opposite to them in direction, the whole of the original fields of turning will be covered by fields of dissimilar name, so that turning will be entirely prevented.

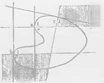


Fig. 74.

Four and Five Points of Restraint.—In no cases where turning cannot be prevented by three points of restraint, and we have seen that this is the rule, the object can be attained by the addition of a fourth point, if it be so placed that its field of turning covers those of the other points dissimilarly (that is to say; a right-over a left- and a left- over a right-handed field.) If for instance a fourth point d be added to the three shown in Fig. 70, in such a way that

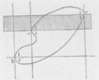


Fig. 77.

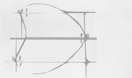
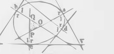


Fig. 76.

its normal passes to one side of the field of turning OPQ, so as to cover it with a field of dissimilar name (Fig. 77), no turning can take place. In a case like Fig. 72 a fourth point of restraint d, covering with its $L.\ H.$ field the remaining field of $R.\ H.$ turning converts it, as shown in Fig. 78, into a field of restraint. In the cases shown in Figs. 74 and 75 this can also be done

In the case shown in Fig. 78, however, the end cannot be reached in this way. For if the fourth point d (Fig. 79) pass to one side of P, and has its *R. H.* field to the left, and its *L. H* field to the right of it, the portions $Q\,b$ and $P\,a$ of the line are covered by unlike fields, so that no turning can take place about their points, but the piece $Q\,P$ is covered with a pair of like fields, and remains a line of centres for left-handed turning. To make $Q\,P$ disappear, the normal to d must pass through P itself, in which case turning can still take place about that point alone,—which is the point of intersection of all the normals, and is therefore common to the boundary lines of all the fields. In order to make turning impossible in such a case, a fifth point must be added to those shown in Fig. 79, so as to cover dissimilarly (as shown in Fig. 80) the still remaining field of turning PQ. If the fourth point of restraint be so placed that P and Q coincide, two more points are necessary to prevent turning, one for left- and the other for right-handed turning.



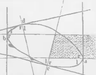
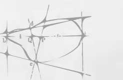


FIG. 78.



The same thing holds good for the cases shown in Fig. 71, where the normals to three points of restraint intersect at one point; for here also one additional point of restraint is required to prevent *R. H* and another to prevent *L. H.* turning. This and the foregoing case may be stated generally in the proposition: If the normals to the three points of restraint of any figure cut none another in one point, at least five points of restraint are required to render turning of the figure impossible.

It will be seen that it is a much more difficult problem so to restrain a figure that it shall not turn than to prevent its sliding. As a rule, it requires at least four points of restraint (three suffice in an exceptional case, Fig. 76, only), while very frequently five are necessary. The form of the figure, also, cannot vary within such wide limits as when sliding only is to be prevented,—its profile must have a curvature varying in such a way that it may be possible to find normals occupying the required relative positions. Hence the rotation of a circle, as of course can be recognised *a priori*, cannot be prevented by any number of points of restraint. When this form occurs in machine construction,—(as on account of the ease with which it can be made it so often does),—as the cross section of a body which it is desired to restrain from turning, it is necessary in some way to convert its form into one which can be so restrained. We have already (§ 15) looked at this fact from another point of view, and can now examine it in the light of the foregoing investigation.

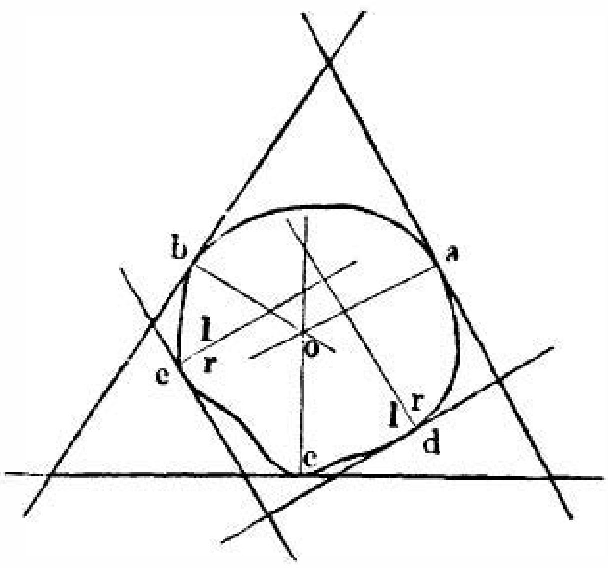


FIG. 80.

The fastening of a wheel or pulley upon a cylindrical shaft furnishes us with a very familiar example (Fig. 81). Here, if the original form of the shaft were retained, all the normals would cut at the centre O. A rectangular groove is therefore made in it, against the sides of which the key which holds shaft and wheel together can press at e or f. The one normal $e\,e'$ covers O with its field of left-, and the other $f\,f'$ with its field of right-handed turning, exactly as we found above (Fig. 80) to be necessary. In similar cases the turning is often restrained simply by flattening a portion of the shaft. The key then exerts pressure at such points as e and f (Fig. 82), so that the normals

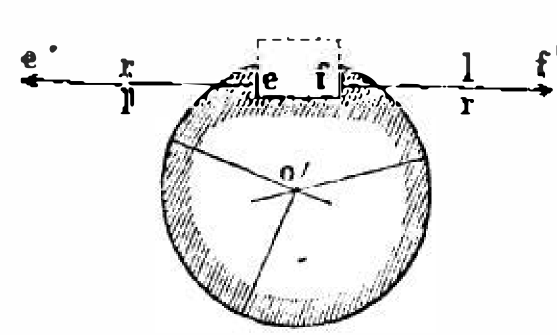


FIG. 81.

$e\,e'$ and $f\,f'$ pass left and right of O, covering it with unlike fields as before. The moment of the pressure in the direction of the normal has here, however, a much smaller arm than in the former case, on which account it is employed only where the effort tending to cause turning is small.

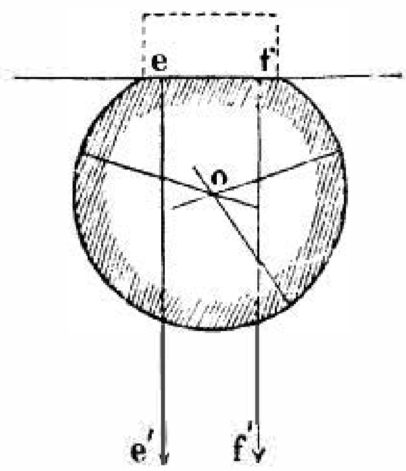


Fig. 82.

Large heavy water-wheels are frequently secured upon the shaft with three, or more often with four keys, a space being left between the solid and hollow cylindric surfaces, which are therefore altogether dispensed with for restraining purposes (Figs. 83 and 84). Such fastenings can offer obviously but small resistance to turning forces, as the normals to the faces of the key pass so very near to the centre O. On this account such modes of keying the shaft are employed rather as methods of centering, that is, as restraints against the cross sliding of the shaft in the wheel boss in the manner indicated in Figs. 57 and 60, and for this purpose they are correctly designed. Where, however, there is any considerable torsion to be resisted, as in the driving pinions of rolling mills,

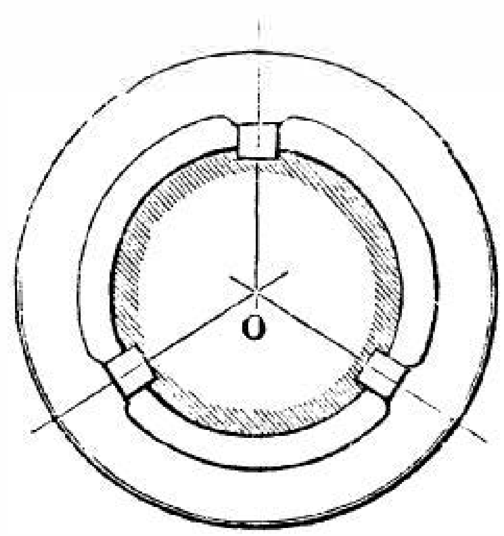


Fig. 83.

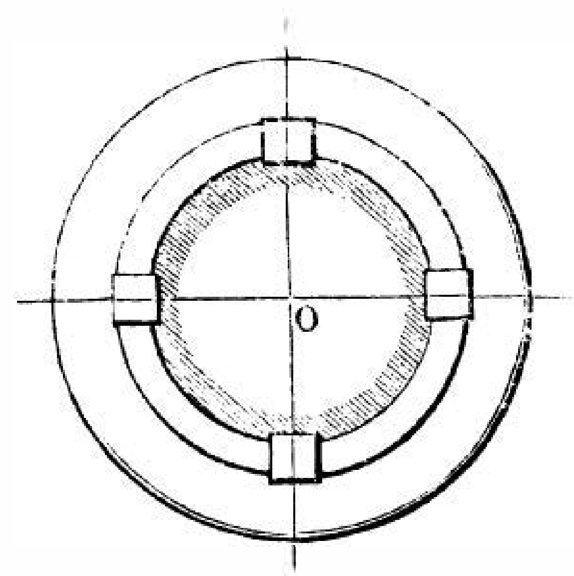


Fig. 84.

such an arrangement as shown in Fig. 85 is used which serves as a restraint both against sliding and turning.

Theoretically, complete restraint might be effected by four of these keys only (as f, g, b and c),—or the arrangement of Fig. 86 might be used, where a, b, and c are arranged as in Fig. 76, while d and e are added as in Fig. 60. But the arrangement shown in Fig. 85 is very much better, for the torsion is resisted at four points instead of at two, while the arm of the moment of resistance is doubled, so that the resistance of each single key is only one-fourth as much as in the former case; the placing of the keys near the corners of each side of the shaft is much better also than placing them in the centre.

A comparison of Figures 53 and 62 shows at once how it is in

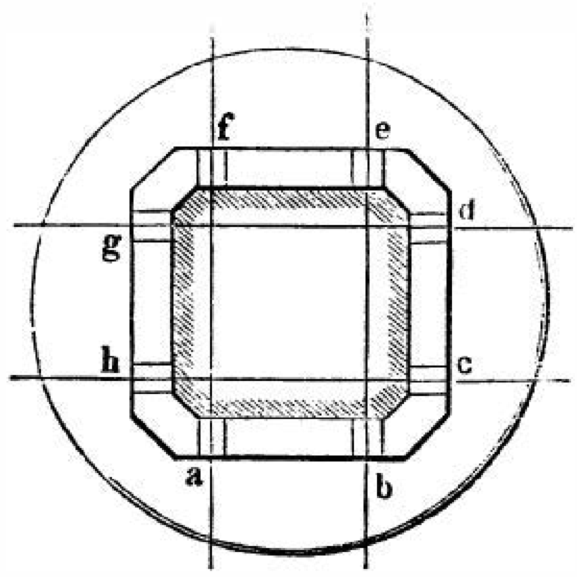


FIG. 85.

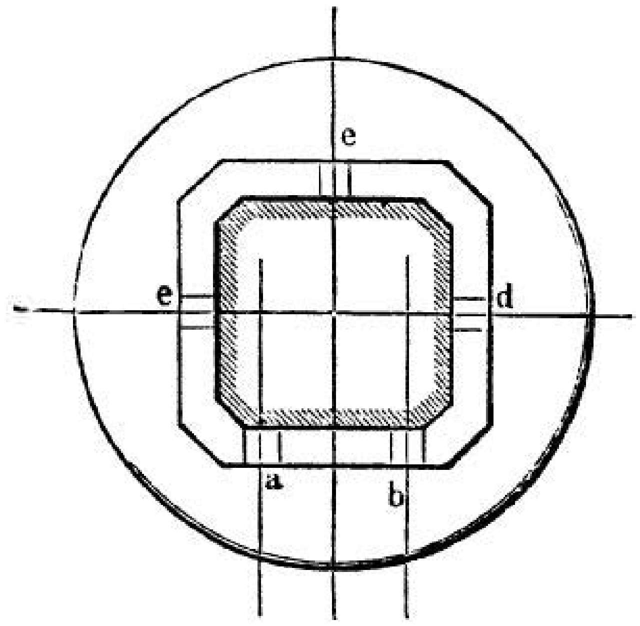


FIG. 86.

general so much more difficult to restrain turning than sliding,—namely, that while one point of restraint can prevent sliding throughout a field of 180°, it can do nothing more to prevent turning than to divide the whole field into two parts,—from points in one of which right-handed and in the other left-handed turning can take place.

§ 20.

Simultaneous Restraint of Sliding and Turning.

In proceeding now to apply the results found above to cases where sliding and turning take place simultaneously, we may first state the following propositions relating to plane figures:—

(1) Neither the sliding nor the turning of a plane figure in a plane can be prevented by two points of restraint.

(2) By three suitably placed points of restraint (*a*) sliding can be prevented, but not turning at the same time; and (*b*) turning can be prevented, but not sliding at the same time.

(3) Only by four suitably placed points of restraint, and with certain profiles only by five, can turning and sliding be prevented simultaneously.

We shall apply these propositions to the closed pairs.

Restraint in the Sliding-pair.—Let it be required with a minimum number of points so to restrain a solid prism that it shall be prevented from having any other relative motion than that which it would have if it formed one element, along with an open prism, of a closed pair. This can be done if, taking any two plane sections of the prism, perpendicular to its axis, we prevent either the sliding or the turning of their profiles. This requires four suitably placed points of restraint on each section (Fig. 87), so that no turning, and sliding parallel to the axis only, can take place,—which is the characteristic of the sliding pair. This gives eight points of restraint—*a*, *b*, *c*, *d*, *e*, *f*, *g*,

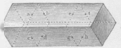


Fig. 87.

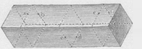


Fig. 88.

and *h*,—the four in each section being placed as in Fig 87. If, however, a third plane section, parallel to and between the two first be taken, two pairs of similarly-situated points, as *c* and *g*, *d* and *h*, may be placed together upon it. Two out of the eight points are thus dispensed with, and the required restraint is obtained with six—*a*, *b*, *c*, *d*, *e*, and *f* (Fig. 88).

Restraint in the Turning-pair.—To examine what minimum of points is necessary in order that a solid circular cylinder with flat ends (Fig. 89) may be restrained as completely as if it formed one element of a closed turning pair, we may in the same way take two plane sections of it normal to its axis, and restrain these from cross sliding,—which requires three points in each, *a*, *b*, *c*, and *d*, *e*, *f*,—and at the same time restrain a longitudinal section from sliding in the direction of the axis,—which can be done by taking one point of restraint upon each end, as *g* and *h*. Here, therefore, eight points of restraint are again required. If, however, the first six points be taken upon the edges of the end surfaces (Fig. 90), and suitable inclinations be given to the restraining surfaces, these sixopoints will also restrain the endlongosliding of the cylinder, and so will suffice for the whole requiredrestrainto

FIG. 89.

Restraint in the Twisting pair.—In order to restrain a screw spindle (Fig. 91) with the minimum number of points, we must restrain twoolongitudinalosections of it,—passing through its axis and at right angles to each other,—against both sliding ando turning. Ifo this beo done theo spindle will be as completely

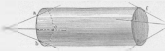


FIG. 90.

restrained as if it were enclosed by a nut. For this purpose each section (see Fig. 87) requires four points of restraint, so that here again eight points of restraint in all are required. Two pairs of these, however, can be reduced to one by a method similar to that shown in Fig. 88, so that the actual minimum number of points ~~of restraint is~~ once more six.

FIG. 91

We find therefore that for all three closed pairs, eight points of restraint are sufficient,—and that by the use of double points six may suffice,—to hold the moving element in the same position as that which it would have were it restrained by the infinite number of points of its partner element in a closed pair.

§ 21.

The Higher Pairs of Elements.

From the foregoing examination into the restraint of plane figures, we see that pairs of figures may be constructed in which the sliding of the one figure relatively to the other is prevented, while their relative rotation remains possible,—and further, that if the normals of restraint, their number not being less than three, intersect in one point, the rotation which remains possible will be about this point only. Such a rotation is a definite motion, excluding the possibility of all others; and this is just what we have recognised as the distinguishing characteristic of a pair of elements. If, therefore, a pair of figures be so conditioned that after the completion of any indefinitely small turning about a centre O, they have again three points of restraint with their normals cutting in a new point,—and that this occurs continually for every new mutual position of the figures —such figures may evidently be used as the foundation of a pair of elements. To construct the elements we require, *e.g.*, only to erect cylinders upon them, and provide these with end surfaces so as to prevent axial sliding.

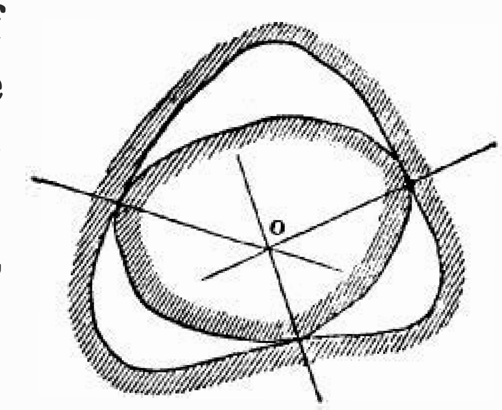


FIG. 92.

If there be two figures of such form that in all their relative positions the sliding of the one relatively to the other be impossible, then their only relative motion at each instant must be turning. If the normals of restraint have always a common point of intersection, then this turning can take place at each instant about this point only,—but if this be not the case, then turning (if it remain possible) must occur about a point outside at least one of the normals. But such a motion would cause a separation of the figures at the corresponding point of restraint, and is therefore inconsistent with the assumption of continued restraint against sliding. Where, therefore, such continued restraint is required, the normals must always have a common point of intersection, and the single motion possible at each instant will be turning about this point. We therefore have the important proposition :—If it can be shown for any two figures that in all consecutive mutual positions relative sliding is impossible, it follows that the normals to their points of restraint intersect always in one point.

The series of consecutive centres of rotation or points of intersection of the normals for the two figures form the centroids, and the cylinders erected upon these the axoids of the two paired bodies.

Pairs of elements formed in this way are not closed, like the pairs we have before examined, but possess the more general and higher characteristic of mutual envelopment (§ 3). We shall therefore distinguish them as higher pairs of elements from the closed pairs,—which, on account of the smaller variety of their characteristics, we shall term the lower pairs. In order to understand the higher pairs we shall examine in detail one example.

§ 22.

Higher Pairs.—Duangle and Triangle.

If from the ends of any straight line, PQ, with a radius equal to the length of the line, we describe intersecting arcs of a pair of circles, these will enclose a plane figure $PRQS$ (Fig. 93), which we may call a duangle. This will be touched in three points, Q, R, and S, by an equilateral triangle, ABC, of a height equal to $2\,PQ$, if Q be placed in the centre of one side of the triangle. For AB is $\perp$ to QR, ($\angle PRA$ being $= \angle BAQ = 30°$, and $\angle QRP = 60°$), and also Q, R, A and S are all points in a circle described about P with a radius PQ. The normals to the points of restraint Q, R, and S cut each other in Q, the angle between each pair being 120°. Sliding, therefore, is entirely restrained, and rotation can take place about one point only. The same holds good also for any other position of the duangle,—such, for instance, as that dotted,—as will be seen from what follows.

If we consider in the first instance continuous contact between the duangle and two sides only, AB and BC, of the triangle, it will be seen that if *L. H.* rotation take place the point P must move along a straight line $T\,U$ parallel to $B\,C$, (for P being the centre of the curve $R\,Q\,S$ is equidistant from all points in it), and similarly the point Q moves in a straight line $Q\,T$ parallel to $A\,B$. The two paths intersect at T at an angle of 60°—the same angle, namely, as that at which $P\,S$ and $Q\,S$ intersect in S. The as yet unknown path of the point S relatively to the figure $A\,B\,C$ is therefore simply that of the vertex of a triangle $P\,S\,Q$, of which the ends P and Q of the base slide upon the arms of an angle equal to the vertex angle of the triangle itself. Let $P\,Q\,S$,

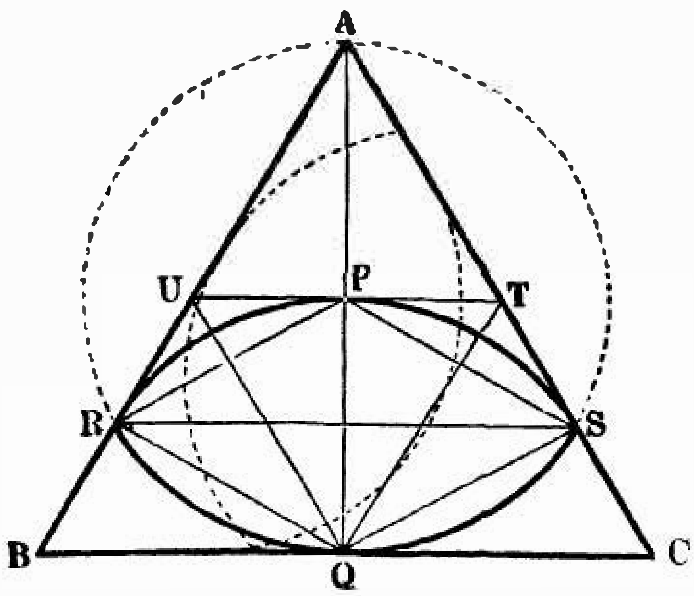


FIG. 93.

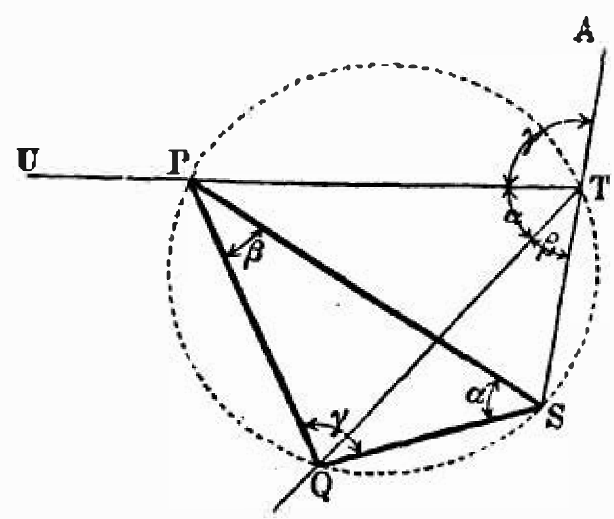


FIG. 94.

Fig. 94, be such a triangle, α, β and γ the angles at its vertex and base respectively. $U\,T\,Q = \alpha$, is the angle upon the arms of which the points P and Q slide. The points S and T are, however, points in a circle passing through P and Q,—α being the angle at the circumference upon the chord $P\,Q$. If, then, we join S and T, we have $\angle\ Q\,T\,S$ equal to the angle β at P, the circumferential angle upon the chord $Q\,S$, and therefore constant for all positions of the triangle. If, further, $S\,T$ be produced through T to A, the angle $A\,T\,P = 180° - (\alpha + \beta)$, that is, = the base angle γ at Q. The point S therefore moves in a straight line, which makes with the arms of the given angle angles respectively equal to those at the base of the given triangle. This straight line is in Fig. 93 above the third side $A\,C$ of the triangle, which makes at T with $U\,T$ and $Q\,T$ the angle 60°—the base angle of the equiangular (and also equilateral) triangle $P\,S\,Q$; all three sides of the triangle therefore touch the duangle continuously.

The contact of the side $C A$ of the triangle with the vertex S is continuous, along with that of the other two sides with the arcs which form the sides of the duangle; the angle between each pair of consecutive normals is therefore always 120°, because they remain always perpendicular to the sides of the triangle. Thus the conditions necessary to the continuous restraint of sliding are fulfilled by this pair of figures, and consequently (by § 21) the normals must always intersect in one point, so that the figures will serve as the basis of one of the higher pairs of elements. We have now to find the corresponding centroids.

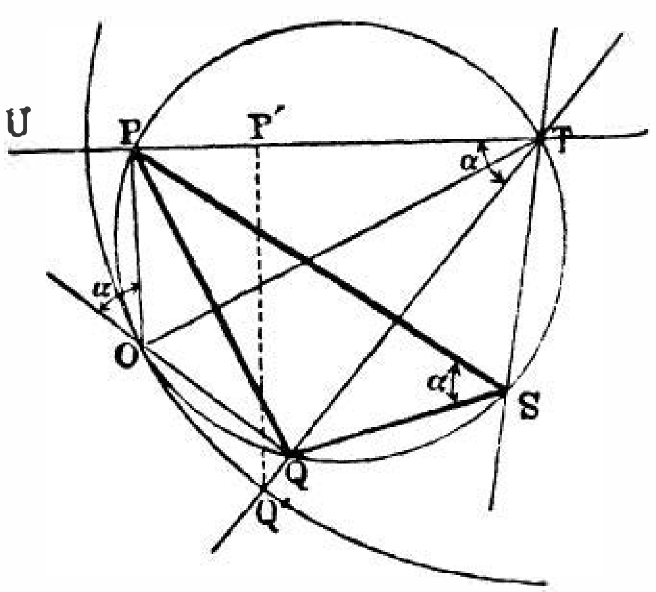


FIG. 95.

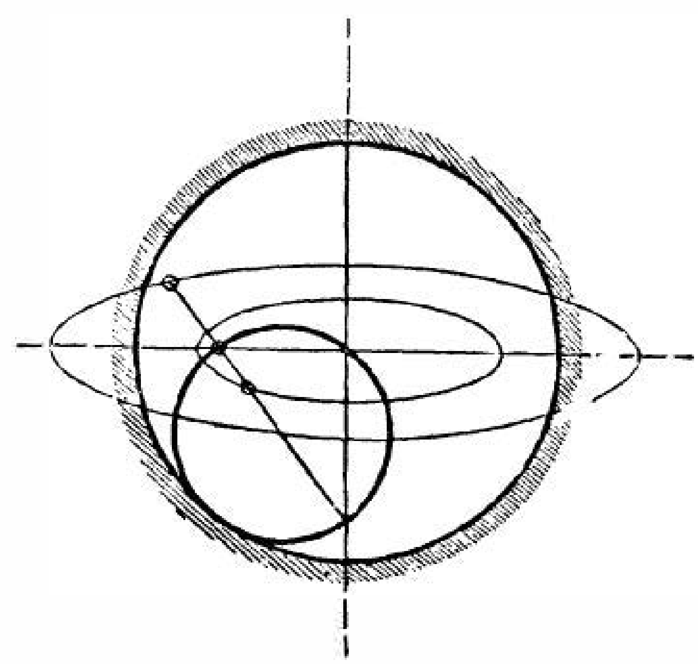


FIG. 96.

(*a.*) Centroid of the Triangle.—In order to make our investigation as general as possible we shall take the problem in the form used above,—of angle and triangle,—considering the line $P Q$ as a plane figure (see § 5), the motion of which relatively to the figure for which the angle $U T Q$ stands is the same as that of the duangle to the triangle. We require to know,—as was shown in § 8,—the paths of at least two points in the moving figure. We do know, however, the rectilinear paths $P T$ and $Q T$ of the points P and Q. The normals to these paths cut each other in O, Fig. 95, and this point must lie in the circle already found, for the angle $P O Q$ enclosed by the normals is obviously equal to $180° - \alpha$. Further, both $O P T$ and $O Q T$ being right angles, the line joining O and T must be the diameter of the circle $P T S Q$. The chord $P Q$ and the angle α being constant, the size of this circle is fixed, the distance $T O$ of the instantaneous centre O from the point T is constant;—the centroid is therefore a circle described about T

with the radius $T\,O$. To find the magnitude of this radius in terms of quantities already known, suppose $P\,Q$ to slide until it stands perpendicular to either of the arms $T\,P$ or $T\,Q$, as, for instance, at $P'Q'$. Then one of the normals coincides with $P\,Q$ itself, and the other has become zero, and it will be seen at once that

$T\,Q' = T\,O = \frac{P'\,Q'}{sin\ a} = \frac{P\,Q}{sin\ a}$, or if we denote $O\,T$ by R and $P\,Q$ by a, $R = \frac{a}{sin\ a}$.

(*b*.) Centroid of the Duangle.—If now, in order to find the second centroid, we suppose the line $P\,Q$ stationary, and set the angle $P\,T\,Q$ in motion, the points passing through P and Q must move always in the direction of the arms $T\,P$ and $T\,Q$ themselves. The normals cut in O as before. The locus of this point is now, however, that of the vertex of a triangle having a base $P\,Q$ and a vertex angle $180° - a$,—which is evidently the circle $Q\,O\,P\,T\,S$ having a diameter $T\,O$, and circumscribed about the given triangle $P\,Q\,S$. If we denote the radius of this circle by r, we have

$$r = \frac{T\,O}{2} = \frac{a}{2\ sin\ a} = \frac{R}{2}.$$

The centroids of our supposed pair of figures, angle and triangle, are therefore, if completely constructed, two circles, having the relative magnitude 1 : 2, of which the smaller rolls in the larger. The relative paths themselves are therefore trochoidal, the hypotrochoids for the rolling of r in R, becoming ellipses (Fig. 96), of which the one described by any point in the circumference of r has a semi-axis major equal to R, and a semi-axis minor equal to zero, and therefore conicides with the diameter of R. For the rolling of R upon r the point-paths are peri-trochoids, of which the common form is the cardioid. The common, curtate and prolate forms of these curves are shown in Figs. 96 and 97.[13] The former of these cycloid problems was first treated—(although by no means completely)—so far as my knowledge goes, by the celebrated mathematician Cardano, in the sixteenth century.[14] As I shall frequently have to refer again to this pair of circles I shall, for the sake of shortness, call them Cardanic circles. In the figures actually before us, the duangle and triangle, the relative motions correspond to certain parts only of that motion of the angle and triangle which would give us these centroids complete. The actual sequence of the motions of the pair is as follows:

So long as the point P, Fig. 98, moves towards U, T is the vertex of the angle on the arms of which $P\,Q$ moves, the diameter $R = \frac{P\,Q}{sin\ a} = \frac{P\,Q}{sin\ 60°}$ is the line $T\,Q$, and its half, $V\,Q$, the radius r,—so that the arc $Q\,U$ belongs to the larger, and $Q\,W\,P$ to the smaller Cardanic circle. Further, as $\angle\ U\,T\,Q = 60°$ and $\angle\ P\,V\,Q = 120°$, the arc $U\,Q$ is equal to the arc $Q\,W\,P$. From U onwards P moves upon the chord $U\,Q$ to W, and Q along the half chord $V\,T$,—this time Q has become the vertex of the angle along the arms of which

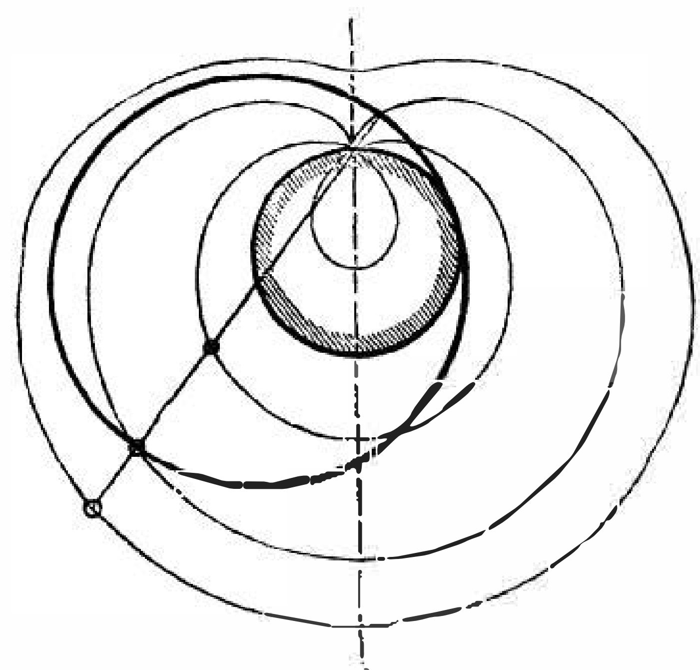


FIG. 97.

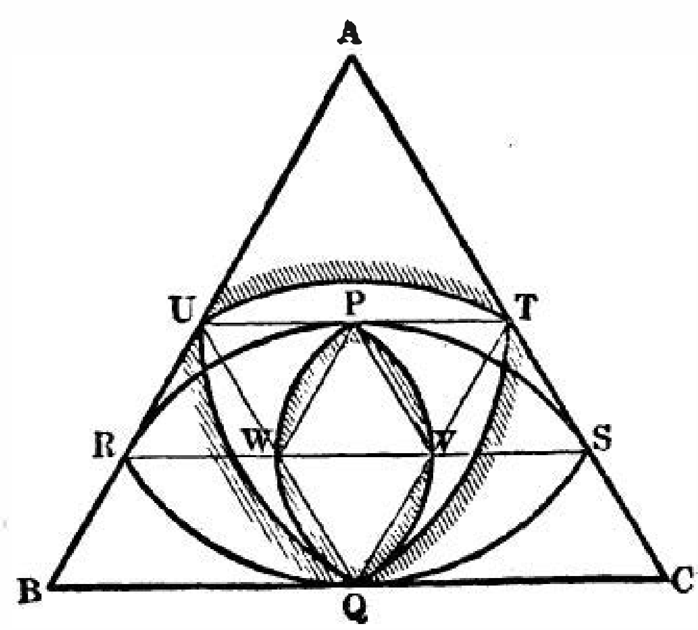


FIG. 98.

$P\,Q$ slides;—$Q\,U$ is the radius R and $W\,Q$ the radius r, by which we obtain the arcs $U\,T$ and $Q\,V\,P$. Proceeding from W, P moves next along the half chord $W\,Q$, while Q moves from T to P; U is now the vertex-angle, from which (with radius R) the arc $Q\,T$ is described, on which again the curve $Q\,W\,P$ rolls. After these motions P has reached the position Q, and *vice versâ*, and the duangle has turned through an angle of 180°. With its further rotation through two right angles the curve-triangle $Q\,U\,T$ makes another complete revolution, and the duangle $P\,V\,Q\,W$ one and a half revolutions. Thus, when the duangle has returned to its original position, the instantaneous centre has twice traversed the three sides of $Q\,U\,T$, and three times the two sides of $P\,V\,Q\,W$,—

continuous rolling having occurred between the two centroids.[1] These centroids we have now found completely. They are (*a*) for the equilateral triangle an equilateral curve-triangle inscribed within it,—(*b*) for the duangle a similar duangle, which has the minor axis of the first for its major axis, and which rolls in the centroid of the triangle.

§ 23.

Point-paths of the Duangle relatively to the Equilateral Triangle.

(Plates I and II.)

The paths described by points in the duangle relatively to the triangle can now be completely determined; for we know the centroids of both figures, and can fix that of the triangle and set that of the duangle in motion upon it. As these paths are formed by the rolling of one centroid upon another, they all belong to the class of curves known as roulettes. We have already determined the paths of two important points of the duangle, the points P and Q. These points always belong to the smaller Cardanic circle, and so describe always parts of hypocycloids coinciding with portions of the diameter of the larger circle. These portions form, as has been already noticed, two coincident equilateral triangles, UTQ, Fig. 1 Plate I. All other points of the duangle describe necessarily arcs of prolate or curtate hypocycloids, which are, as we have mentioned, ellipses. All these prolate and curtate curves are known by the common name of trochoids.[13] We may therefore say that all the remaining points in the duangle have for their paths hypotrochoids, of which the equilateral triangle UTQ is the foundation. As this triangle consists of six portions of hypocycloids,—so all the other point-paths must consist of six hypotrochoidal arcs. The figures thus built up take very various forms with different positions of the describing point. Fig. 1 shows three of them external to the triangle. The describing points themselves lie upon the production of the minor axis QP of the duangle, that is, of the major axis of the centroid Qm_1, Pm_2, and are numbered 1, 2, 3, commencing with the outermost point 4, which coincides with P. The figures are all three-cornered, and approach more and more nearly the triangular form, which is that actually described by the point 4. In Fig. 2 the paths of three more points 5, 6, and 7 are shown on a larger scale; the last of these is the centre M of the duangle. The path of 5 contains three loops; in the case of point 6, which is so chosen as to coincide with the centre M_1 of the triangle $A\ B\ C$, the loops have a common point of intersection. For any describing point between 6 and 7, the curves which intersect at M_1 in the former case open out, enclosing between them a triangular space; and lastly, point 7 gives the three loops fallen together into a continuous curve, which is the smallest curve which can be described by any point of the duangle. This curve is two-fold,—in a whole period, that is, the describing point passes twice through M; this can be seen by an examination of the curve 6, the tangent to which twice turns through four right angles.

If the describing point be taken further from P than 7 we simply obtain repetitions, in reverse order, of the curves already described.

By choosing describing points upon the major axis of the duangle we get a further series of curves, of which some examples are shown in Plate II. Point 1 again gives us an elliptic triangle; point 2, coinciding with the end S of the axis, gives a three-cornered figure, bounded partly by straight lines and partly by elliptic arcs; point 3 gives an elliptic triangle with concave sides, which is shown on a larger scale in Fig. 2. The point 4 coincides with the end m_2 of the short axis of the smaller centroid. It describes the remarkable figure No. 4 shown in Fig. 2, consisting of three circular arcs (described by m_2 as centre of the arc Pm_1Q), and three (twofold) rectilinear continuations of them (described by m_2 as a point in the circumference of the arc Pm_2Q). The point 5 gives a curve with three loops, intersecting in the point M_1; point 6 gives a three-looped curve with an inner open triangular space, and point M as the centre gives again the curve shown in Fig. 2 Plate I., and there marked 7. It is to be noted that the trochoidal triangles which form the paths of the points in the major axis are turned through an angle of 60° relatively to the point-paths of the minor axis.

The paths of all points between these two axes lie between

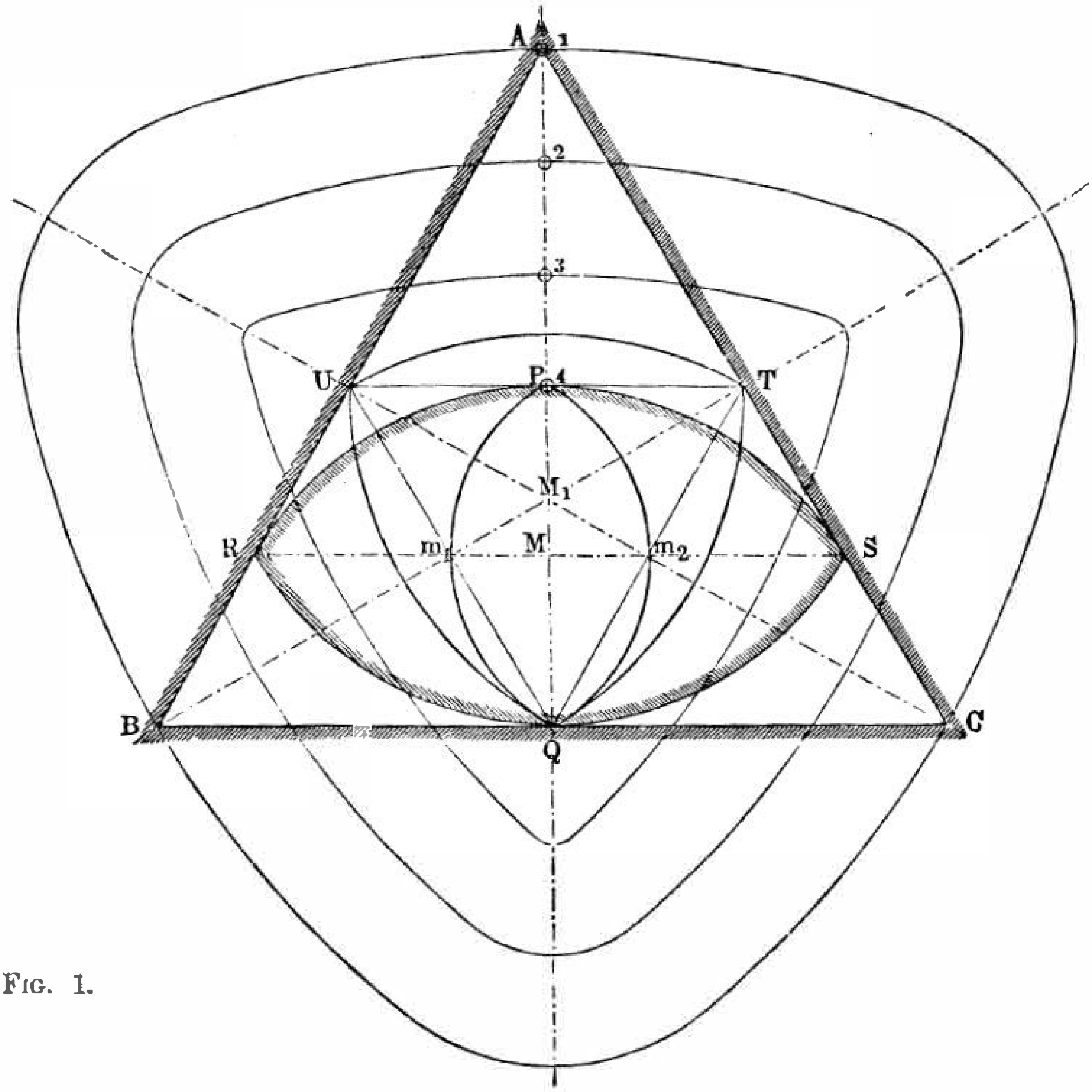


FIG. 1.

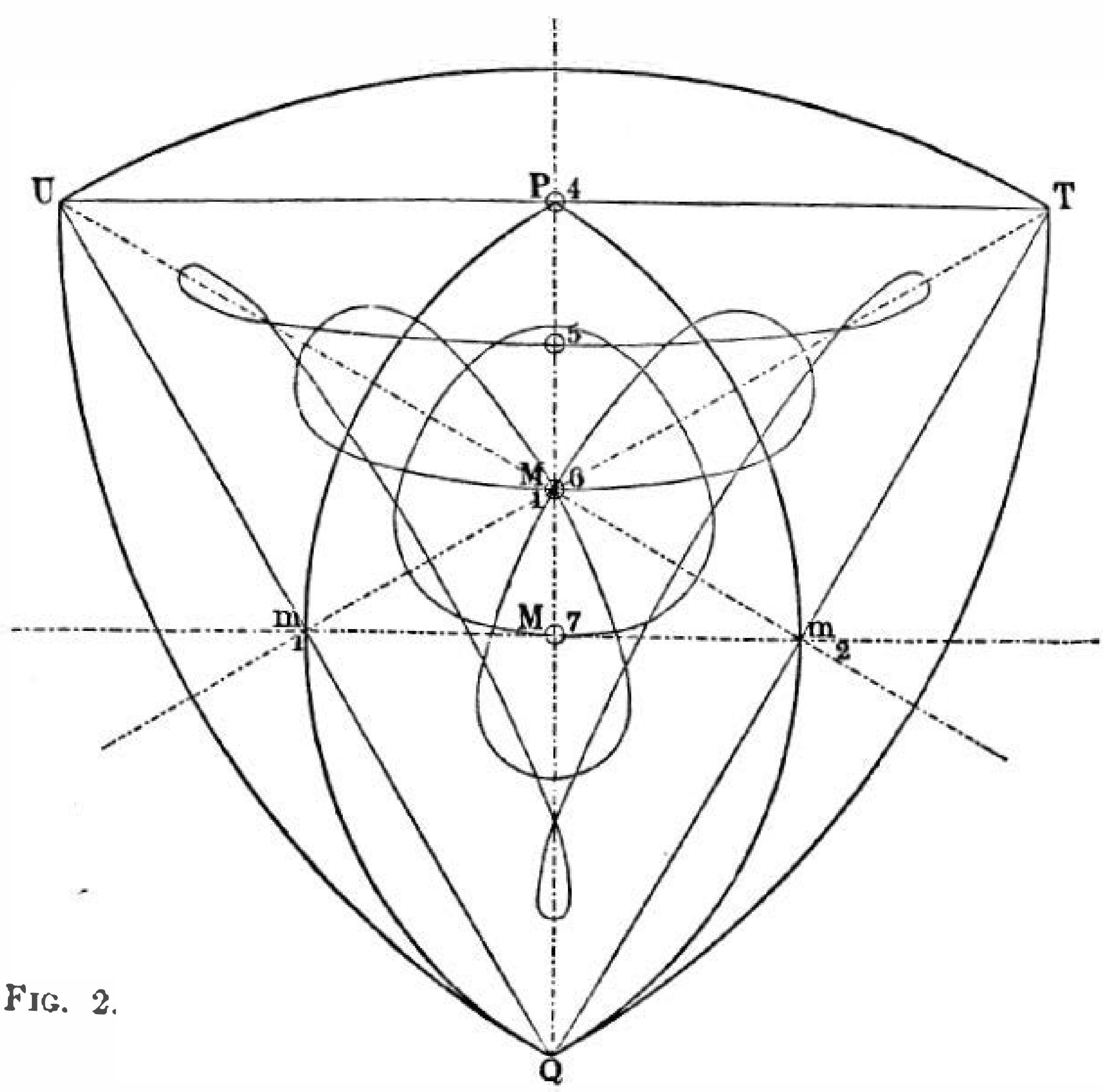


FIG. 2.

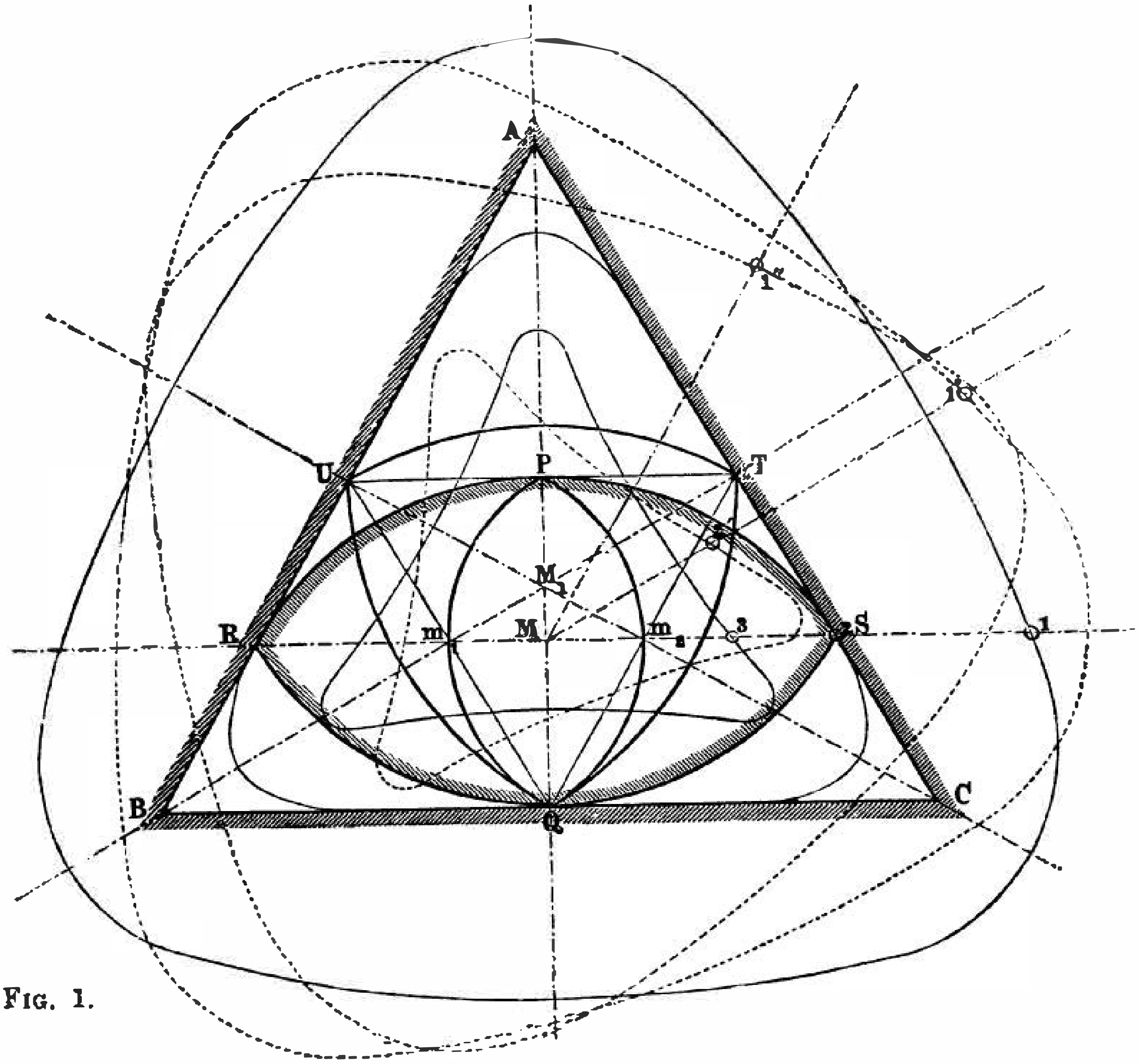


FIG. 1.

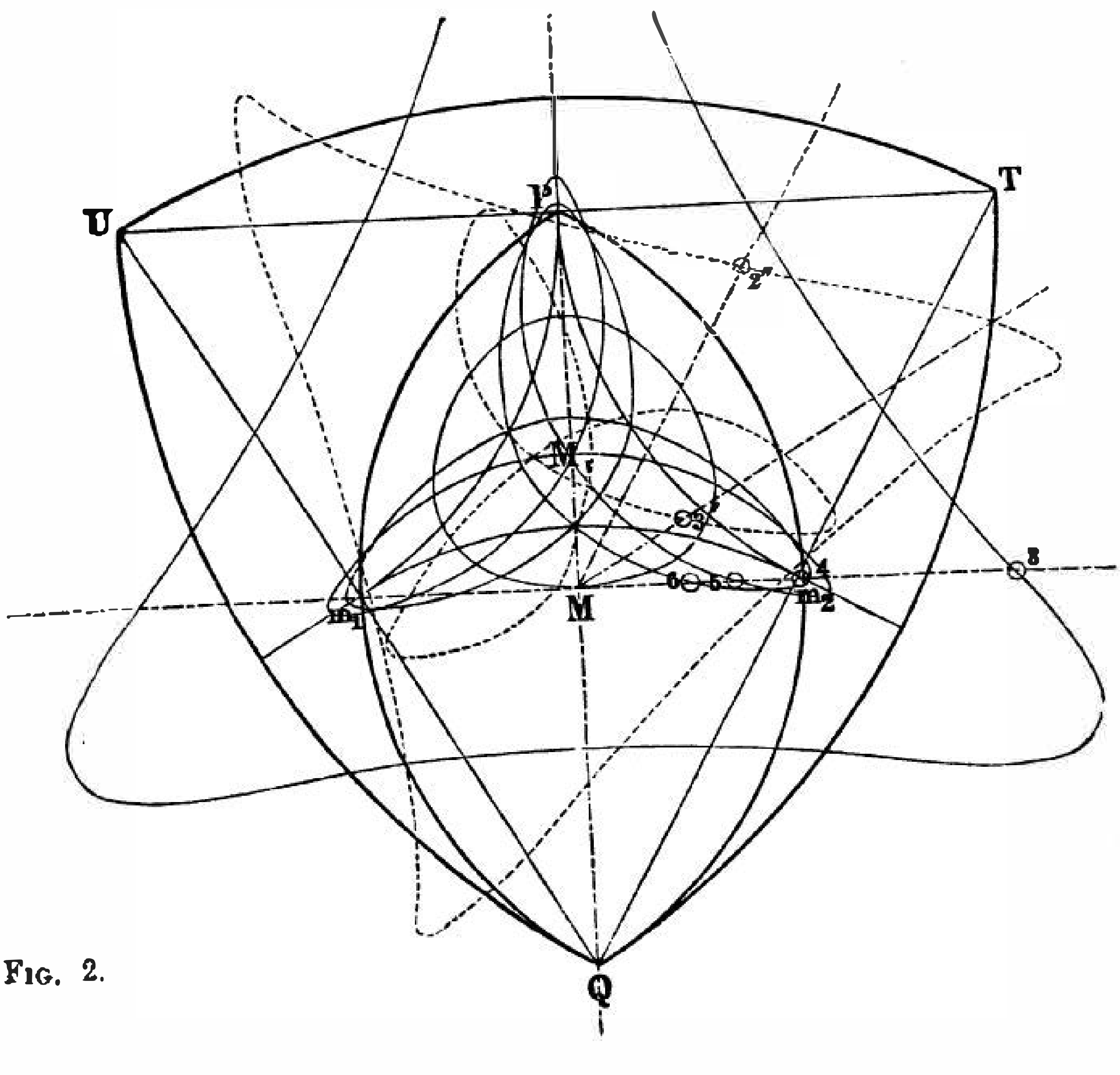


FIG. 2.

these two positions,—they are in general of an elliptical three-cornered form. Four of them are shown dotted in Plate II. and indicated by the numbers 1′, 2′, 1″, and 2″. These figures are no longer symmetrical about their three axes, as the former were;—this can be specially seen from No. 2″.[16]

§ 24.

Point-paths of the Triangle relatively to the Duangle.

(Plates III. and IV.)

To determine the point-paths of the triangle relatively to the duangle the latter must be fixed and the former set in motion The centroid, $U\,T\,Q$, Plate III. 1, then rolls upon the centroid, Pm_1 Qm_2. The figures described are formed of arcs of peri-trochoids. All describing points lying upon the rolling centroid here describe arcs of cardioids, as we have seen in connection with Fig. 97.

It is at once noticeable how greatly these figures differ from the former ones. This forms an illustration which exactly meets a mistake made by many former writers on this subject, that the inversion of such a pair of figures, although it produces the greatest alteration, in the manner of turning, does not alter the form of the point-paths.[17] This circumstance has given me occasion to construct these pairs of elements, to which, otherwise, I cannot ascribe any particular use.

The figures in Plate III., show the paths of points in the axis $M\,A$ of the triangle. The point 1 describes a rounded oval, consisting, like all the other figures, of six peri-trochoidal arcs. Point 2, coinciding with the vertex A of the triangle, gives an oval with concave sides, as does also point 3; the path of point 4 consists of two simple cardioids joined in the points m_1 and m_2 of the stationary centroids. The path of 4 is repeated in Fig. 2 upon a larger scale. The paths of 5 and 6 each have two loops, which in 7 fall together into one oval curve. Point 7 itself coincides with the centre point M_1 of the triangle $A\,B\,C$, and it must be noted that its path is really three-fold; that is, is traversed thrice in each period by the point M_1. This can be recognised from the looped paths 5 and 6, the tangents to which turn three times through four right angles. The path 1 is also remarkable, for the three homologous points 1, 1′, and 1″ lie continually in it, so that complete restraint occurs, as in Fig. 59.

Plate IV. shows the further point-paths obtained by choosing points in the line $M_1\,Q$, or (what is the same thing) in the lines $M_1\,T$ or $M_1\,U$. It is seen at once that the principal axis of the figures is now turned through 90°, and also that the loops form themselves about an axis perpendicular to the original one. The curve $T\,S$ (Fig. 1) and its symmetrical repetitions are characteristic, —the former is the circular arc described by the centre T of the rolling curve $U\,Q$.

If describing points be taken upon radii lying between $A\,M_1$ and $T\,M_1$, paths are obtained which are not, as before, symmetrical about two axes. It has not been thought necessary to give examples of these; their nature will be made sufficiently clear by the analogy of the paths 1′, 2′, &c. in Plate II.

We have found that the point-paths of the pairs of elements which we have considered possess extraordinary variety of form,—they can, however, be somewhat systematised by the use of a method and nomenclature similar to that employed for trochoidal curves. Our curves form themselves into two series, corresponding to the fixing of one or the other element, and each series divides itself into groups according to the position of the line on which its describing points are taken. The paths of points in the centroids themselves, —as, *e.g.*, the triangle $U\,T\,Q$, Plate I. 1, the three-cornered paths of the point m_2 in Plate II. 2, &c.—are specially characteristic. These paths may be called the common form of the roulettes concerned,— as in the case of the cycloid. By the same analogy we may call all paths of points which lie beyond or within the rolling centroids, curtate or prolate point-paths respectively. Among the last, one is specially characteristic, and common to all groups of point-paths,—the path of the centre point of the moving centroid, M in Plates I. and II., M_1 in Plates III. and IV. This roulette is always the smallest of its series; the point-paths concentrate themselves upon it as the path of the point relatively nearest to the centroidal curve itself, just as a circle concentrates itself upon its centre

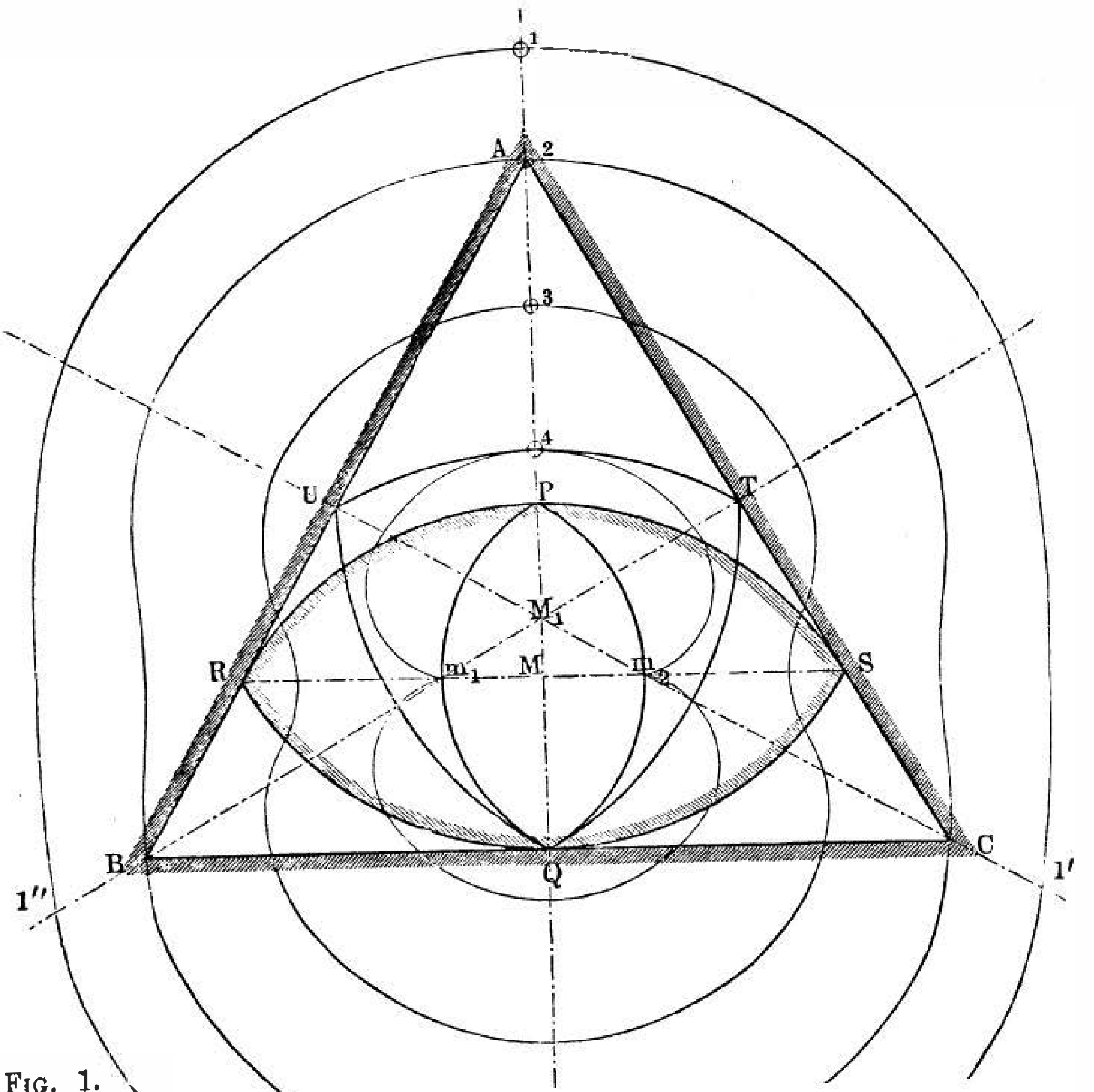


FIG. 1.

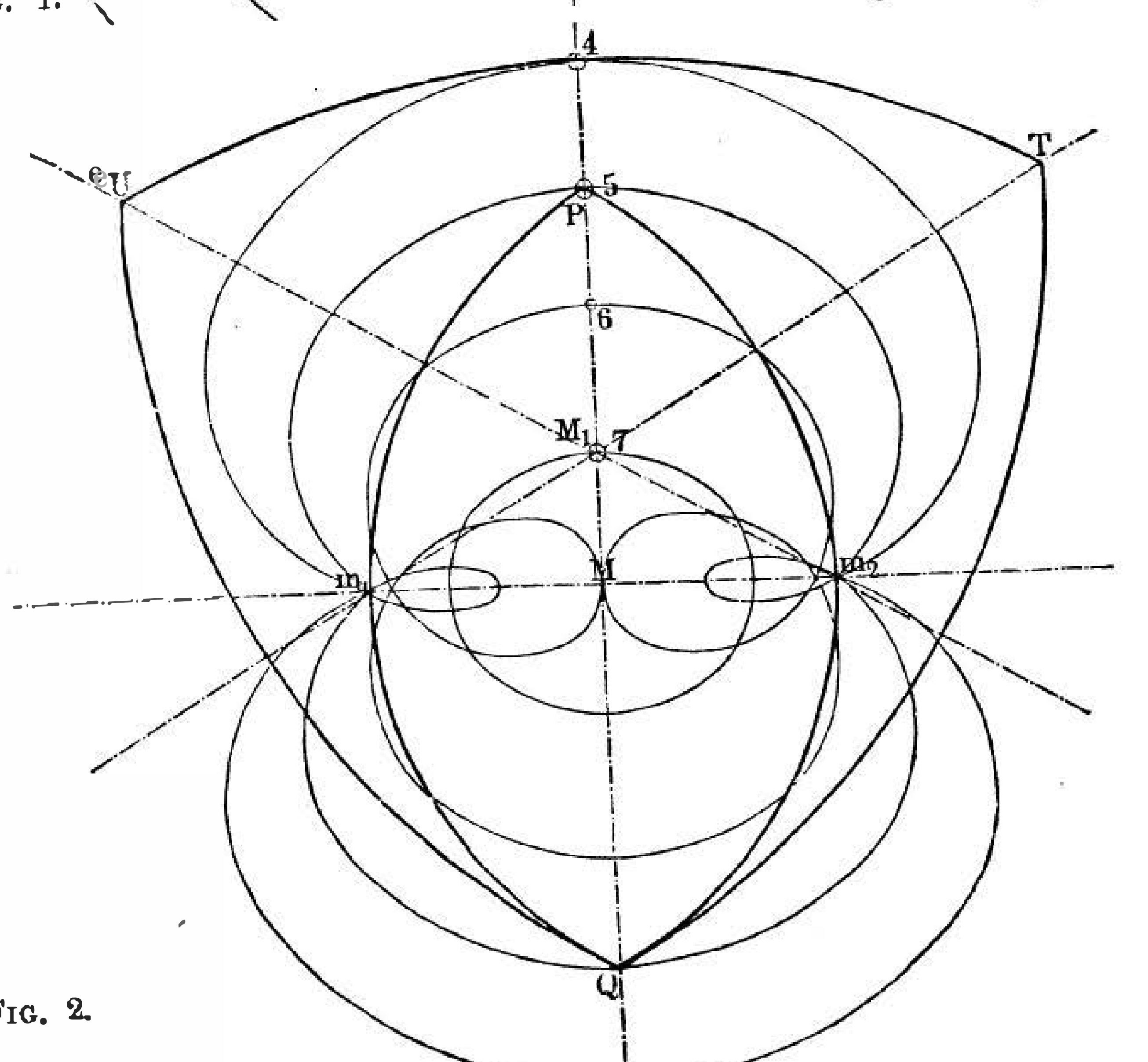


FIG. 2.

PLATE IV.

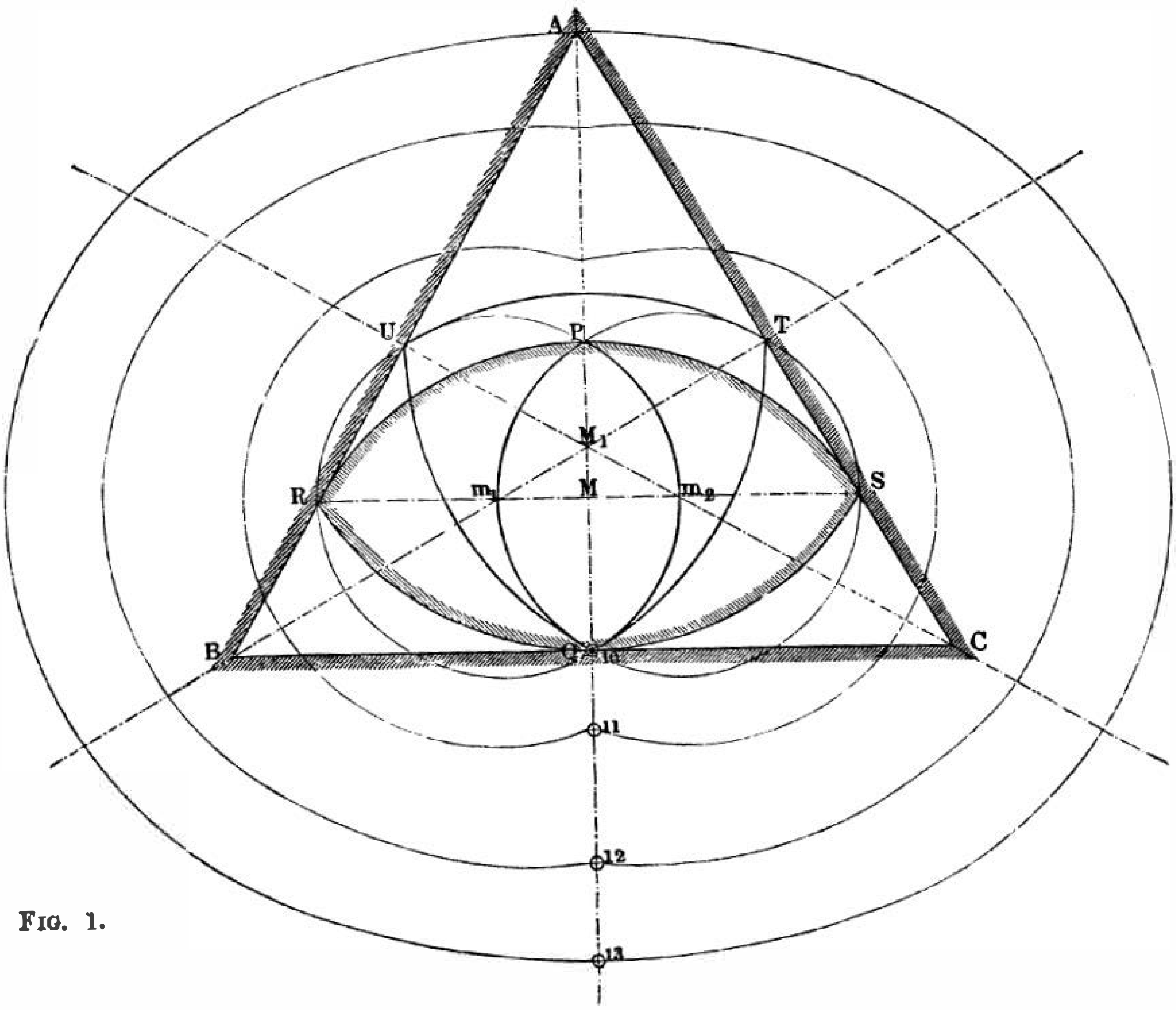


FIG. 1.

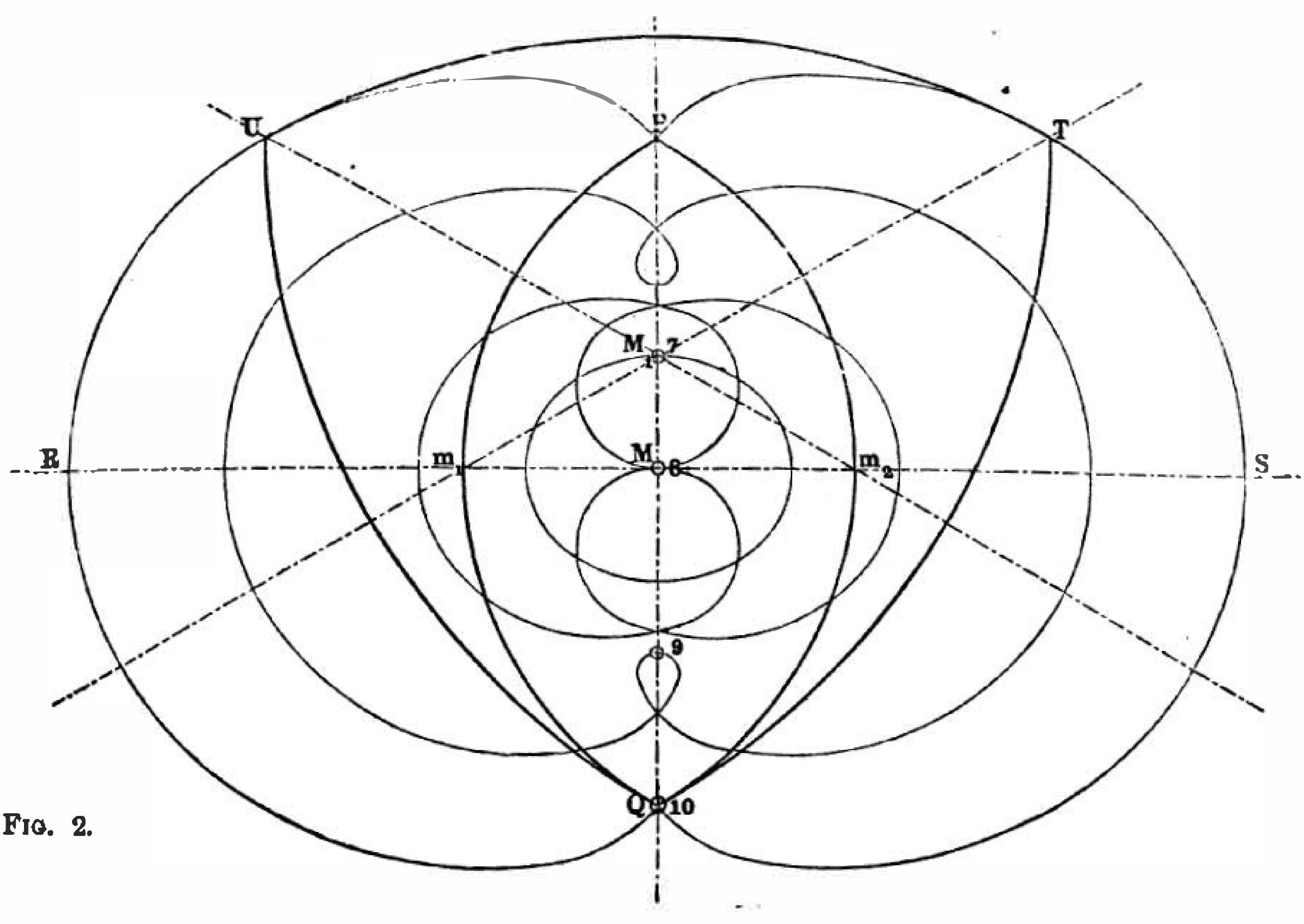


FIG. 2.

point when its radius is diminished to zero;—this form we may therefore call the concentral form of the common point-path.

Further, those roulettes also are noticeable which pass through the middle point of the whole series of curves, as No. 6 in Plate I., 2. These roulettes we shall call homocentral. In our example they take a series of forms of which some are shown in the second figures of Plates I. to IV. It must be noticed that homocentral point-paths can only be described by those points which, as the moving centroid revolves, pass through the centre of the stationary centroid; or conversely by those points through which the centre of the stationary centroid might pass if the pair were inverted. Such points are, however, only those of the concentral point-paths. In other words —the points of concentral point-paths describe homocentral point-paths if the pair be inverted. Thus the homocentral curves 6, Plate I., 2, and 5, Plate II., 2, are described by points in the concentral curve M_7 of Plates III., 2 and IV., 2. The points of the concentral curves which have been used can easily be seen from the figures.

This way of looking at the curves may also be extended to the examination of trochoids,—which, indeed, we should actually obtain if the centroids of the pair of elements were circles. Here the concentral roulettes would be the circles described by the centres of the rolling circles, and the homocentral paths those star-shaped figures which are described by points in the circumference of a circle concentric with the rolling circle, and having a radius equal to the difference between the radii of the two centroids.[18]

§ 25.

Figures of Constant Breadth.

The conclusions of § 21 lead us synthetically to a series of other pairs of elements, of which we may examine a few here. If upon any plane figure two parallel tangents be laid, as *A B* and *C D*, Fig. 99, the distance between them, *c*, measures the extension of the figure in the direction of the normals of restraint. This extension may be called the breadth of the figure,—in general, it is not constant for the same figure. There are figures, however, in which the breadth *is* constant; in which, that is, all pairs of parallel tangents on opposite sides are at the same distance apart. The circle gives us a familiar example of this. If on any figure having this property we place two pairs of the restraining tangents supposed, they touch it in four points and completely restrain it—as was shown in § 18—from sliding. This restraint, however, does not prevent the turning of the figure, and this turning may be so arranged that it can take place about one point only. That this may be the case the normals to the four points of restraint must intersect in that point—the opposite normals, that is to say, must coincide, as in Fig. 100, where the normal on *a* passes through *c*, and that on *b* through *d*.

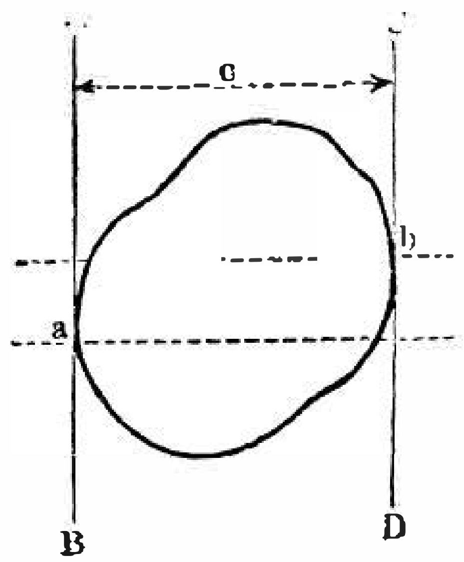


FIG. 99.

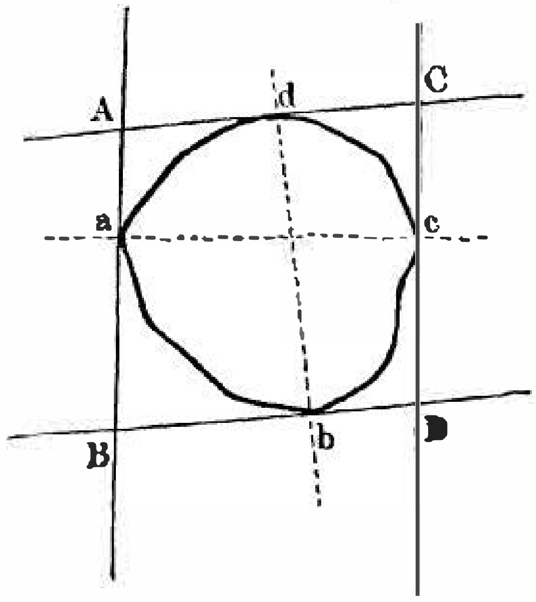


FIG. 100.

Then, the breadth of the figure being constant, the restraint is uninterrupted or continuous for all alterations of its position within the four tangents, from which it follows (see § 21) that the normals always intersect in one point. This shows that figures of constant breadth have the property that on any radius there lies not only the centre of curvature of the element of the circumference to which that radius belongs, but also the centre of curvature for the opposite element. The four tangents enclose a square, or more generally a rhombus, as *A B C D*. The foregoing shows also that every figure of constant breadth can be constrained in such a rhombus, so that from it and the rhombus a pair of elements may be formed.

§ 26.

Higher Pairs of Elements.—Equilateral Curve. Triangle and Rhombus.

Figures of constant breadth can easily be constructed of circular arcs. If from the corners of an equilateral triangle *P Q R*, Fig. 101, arcs be drawn with radii equal to the length of one of the sides, we obtain a figure which we may call an equilateral curve-triangle. This has everywhere a breadth equal to the side *P Q*, so that it can be constrained in a square or rhombus *A B C D*, the distance between whose opposite sides is *P Q*. In the square the normals intersect at right angles, in the rhombus obliquely. If we

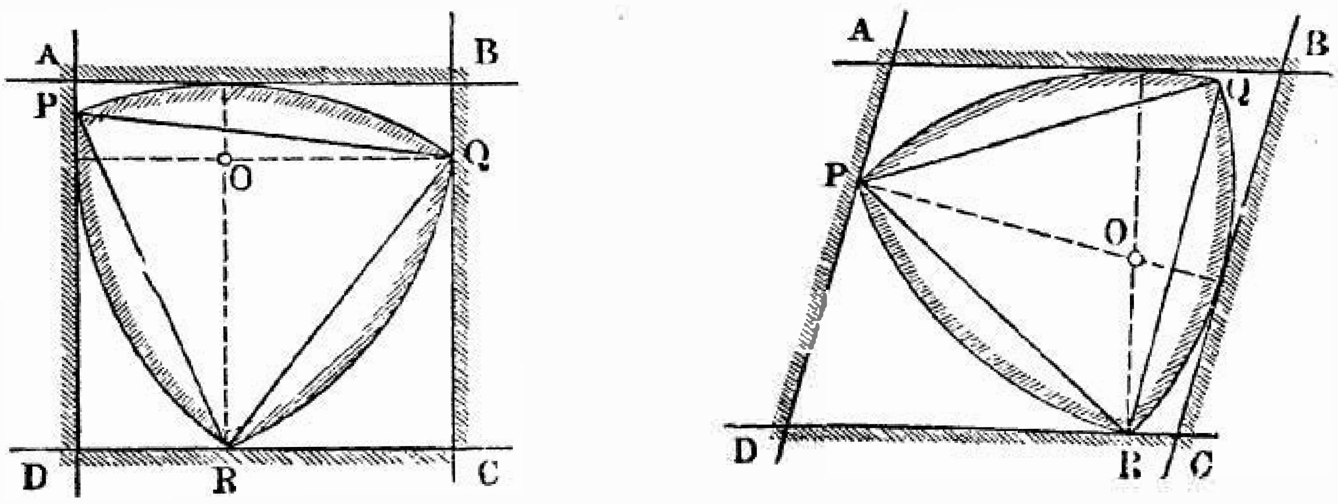


FIG. 101.

take these figures as cross sections for cylinders, and give to the latter such profiles as will prevent end-long sliding, we have constructed a higher pair of elements.

We may examine the centroids of these figures,—taking first that of the square. In Fig. 102 the instantaneous centre of motion is the point *O*, which in the position shown in the figure falls upon the vertical bisector *R O*, of the square, and is also the centre of one side of the triangle. Let the curve-triangle make *L. H.* rotation about this point. Its corner *P* then slides downwards along the side *A D*, while *R* moves to the right along *D C*. The normals from *P* and *R* always intersect at right angles, so that the locus of the instantaneous centre is that of the vertex of a right-angled triangle, of which the ends of the hypothenuse slide upon the arms (*D A* and *D C*) of a right angle. *O* is thus always the corner of a rectangle *P D R O*, of which the diagonal ($O\,D$) is constant and is equal to $P\,R$. The locus of O, or centroid, is therefore a circular arc having D for its centre and $P\,R = P\,Q = A\,B =$ the length of the side of the square, for its radius. The centre continues in this curve until R has arrived at the same distance from C at which P is shown from A in the figure —*i.e.*, up to the point 2. The chord $P\,Q$ then slides in the same way on $A\,B$ and $A\,D$, giving the arc 2, 3, similar to the former one, as the continuation of the centroid, in the same way the arc 3, 4 is obtained, and lastly the arc 4, 1. The centroid for the square is therefore a curve-square having for its sides four circular arcs drawn from the four corners of the square with radii equal to its sides.

To find the centroid of the triangle we invert the pair,—that is, imagine the triangle stationary and the square moving upon it. The centroid is then the locus of the vertex O of a right angled triangle having its hypothenuse $= P\,R$, *i.e.* a circle with the diameter $P\,R$ described from its middle point 3′. The arc of this circle which forms part of the centroid ends at 2′, the middle point of the side $Q\,R$. Then follows the similar curve 2′ 3′, and lastly,—returning to O,—the similar curve 3′ 1. Hence the centroid of the curve-triangle is itself a curve-triangle, and is equilateral, its sides being arcs described from the centres of the sides $P\,Q$, $Q\,R$, and $R\,P$, and having radii equal to half their length.

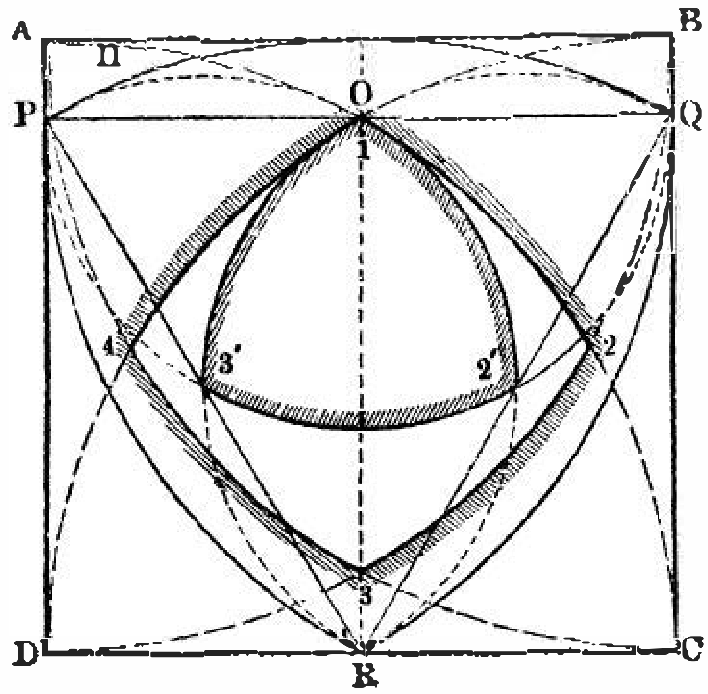


FIG. 102.

As the one figure rolls relatively to the other, the curve 1.2′ rolls on 1.2, then 2′.3′ on 2.3,—3′.1 on 3.4, and so on. In order again to reach its initial position the instantaneous centre must traverse equal distances on both centroids,—must therefore traverse the three sides of the curve-triangle four times, and the four sides of the curve square three times. For each completed rolling of the former on its three sides the element to which it belongs turns through an angle of 90° relatively to the square, so that after the first revolution its corner 1 comes to the point 4, after the second to 3, after the third to 2, and after the fourth back again to 1.

§ 27.

Paths of Points of the Curve-Triangle relatively to the Square.

(Plates V. and VI.)

The point-paths of this pair have, as follows from the nature of their centroids, a close relationship to those of the pair shown in Plates I. to IV. All paths of points of the curve-triangle relatively to the square consist of arcs of hypocycloids or hypotrochoids, which are in this case ellipses,—while all point-paths of the square relatively to the triangle consist of arcs of peri-trochoids (including the special case of cardioids, as on p. 123). Let us first suppose the square fixed and the triangle in motion.

Plate V. shows a series of point-paths for which the describing points lie upon a line drawn from the centre M of the triangle perpendicular to the chord $P\,Q$. Point 1 gives a four-sided figure composed of elliptic arcs, its corners being elliptically rounded off by a pair of similar arcs having a common tangent; at 1 for instance one of these is given by the rolling of m_1m_2 on O_4O_1, and the other by the rolling of m_1m_3 on O_4O_3. When m_2 reaches O_1, m_2m_3 begins to roll on O_1O_2. The point 1 is however so chosen that $m_1 1$ is equal to the radius of the curves of the centroid $m_1m_2m_3$, and being therefore upon the circumference of the smaller Cardanic circle, it describes a straight line. Thus the portions of the point-paths passing through A and B,—the centres of the greater Cardanic circles O_2O_3 and O_2O_1;—are straight lines,—or more strictly, are elliptic arcs which have become straight lines. The continuation of the curve can easily be understood. It is completed when each side of the centroid $m_1m_2m_3$ has rolled on each side of the centroid $O_1O_2O_3O_4$, and consists therefore of twelve (four times three) separate arcs.

The point 2 describes a four-cornered figure with slightly concave sides, and the point 3 a similar figure in which the concavity is more distinct; in both cases all the curves are elliptic. The end point m_1, fourth in the series of describing points, describes immediately right and left from O_4 straight lines directed towards the centres C and D of the base circles O_4O_1 and O_4O_3, like the straight lines O_1m_2 and O_3m_3 from the homologous points m_2 and m_3; joining each pair of straight lines is a circular arc described by the centre of the circle m_2m_3 rolling in O_1O_2, m_3m_1 in O_2O_3, etc. This point-path is the common form for this series of curves.

The fifth point describes a four-cornered figure of elliptic arcs, in which loops make their appearance. This figure is shown on a doubled scale in Plate V., 2. In the point-path of 6 the loops have separated and intersect each other, while with point 7, which is the centre M itself, the loops run over each other in a dumpy figure which in each period, or whole revolution of the element, traverses M three times. (See § 23). The centre M_1 of the square is also the centre of this point-path, which is the smallest of those which can be obtained by the motion of the curve triangle, or what we have called the concentral form of its point-path.

Plate VI. shows curves described by points on the prolongation through M_1 of the line on which were the points just considered. Point 1 gives us a four-cornered figure consisting of elliptic arcs—point 2 a straight-sided quadrilateral, covering part of the square $ABCD$, but having elliptically rounded corners;—the points 3 and 4 elliptic quadrilaterals with concave sides. The last figure is shown to double the scale in Fig. 2; within it is the path for the point 5, which being a point upon the centroid gives us again a common form for this series of curves;—it consists of four elliptic arcs with tangential prolongations at each cusp. The path 6 is described by the point which in Plate VI., 1, coincides with the centre point M of the square;—it is therefore the homocentral form of this series of curves; the point M, lastly, again gives us the concentral curve 7.

The point-paths 1′, 2′, and 3′, shown in dotted lines, are examples of those described by points which do not lie in either of the three principal axes of the centroid.

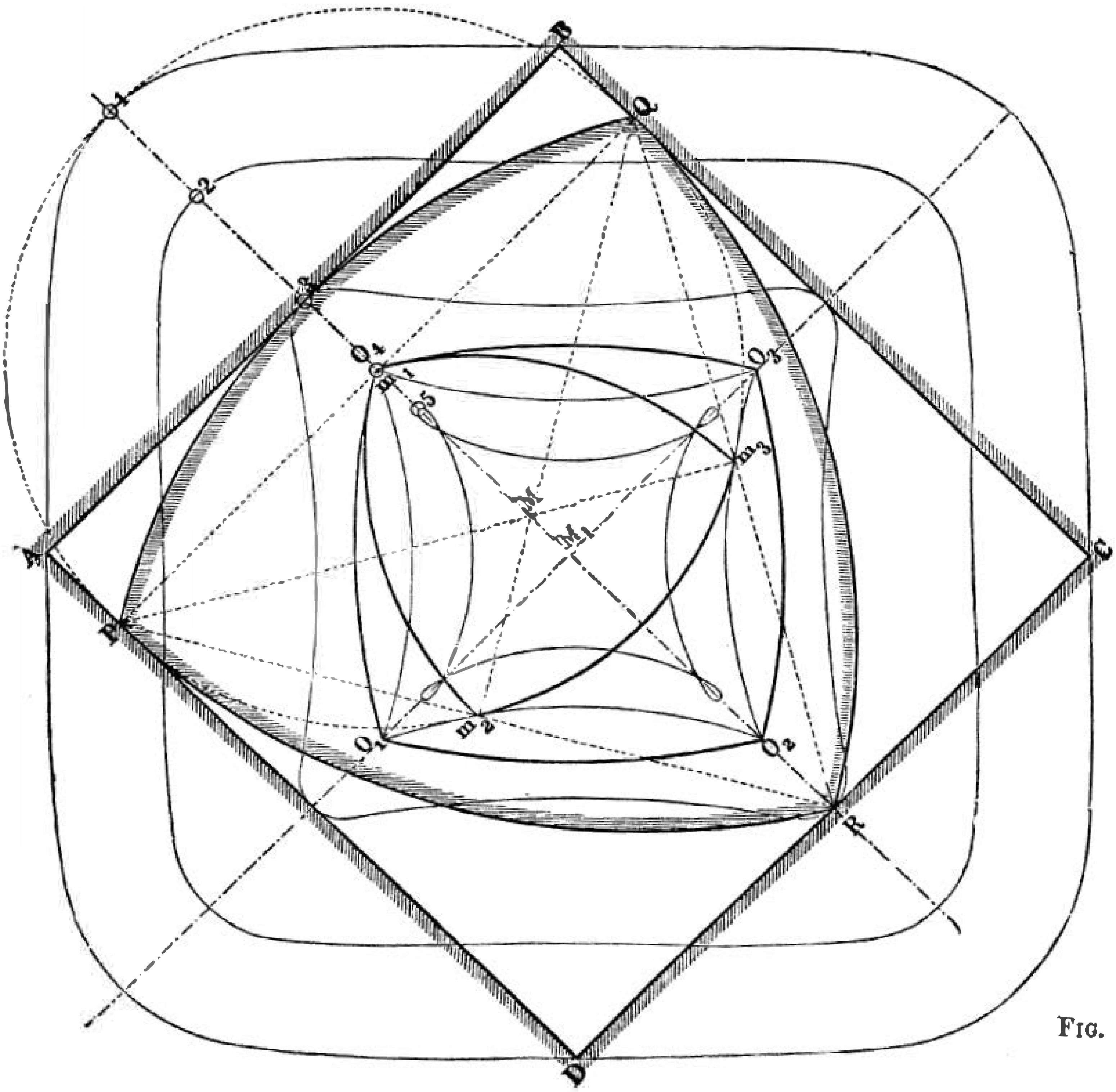


FIG. 1.

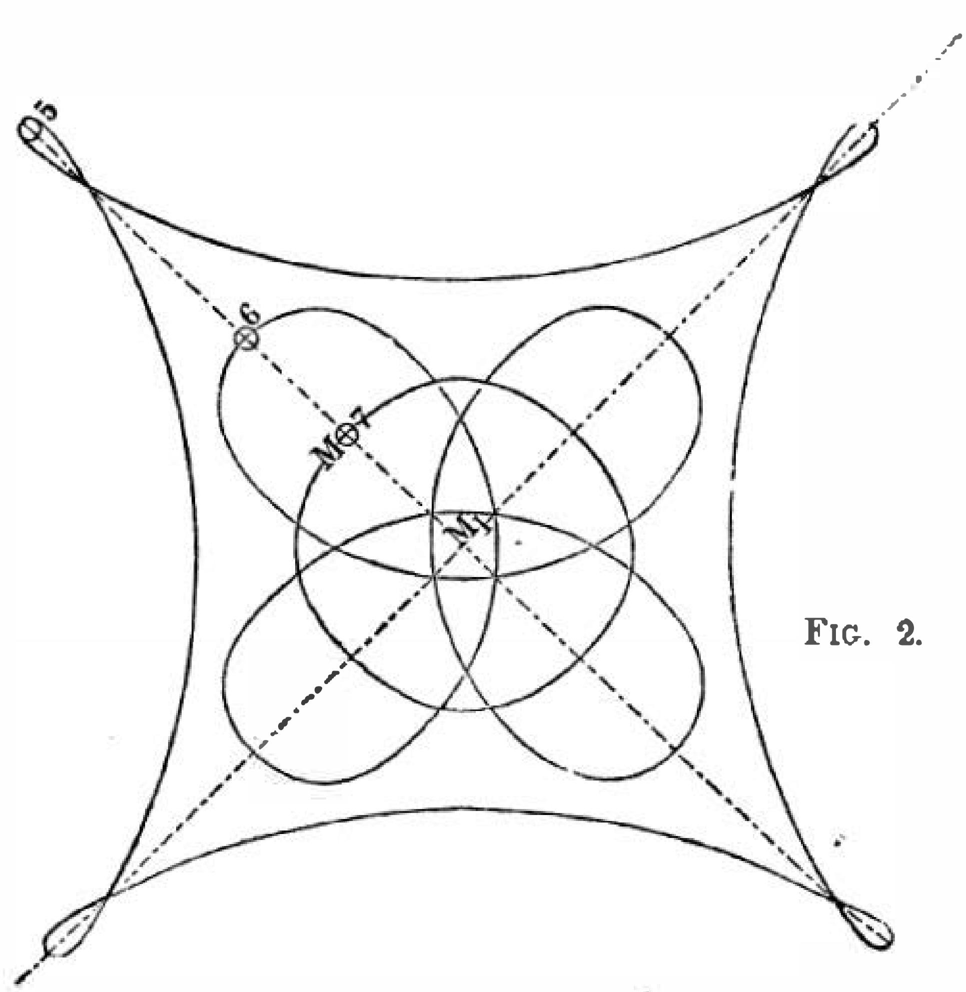


FIG. 2.

PLATE VI.

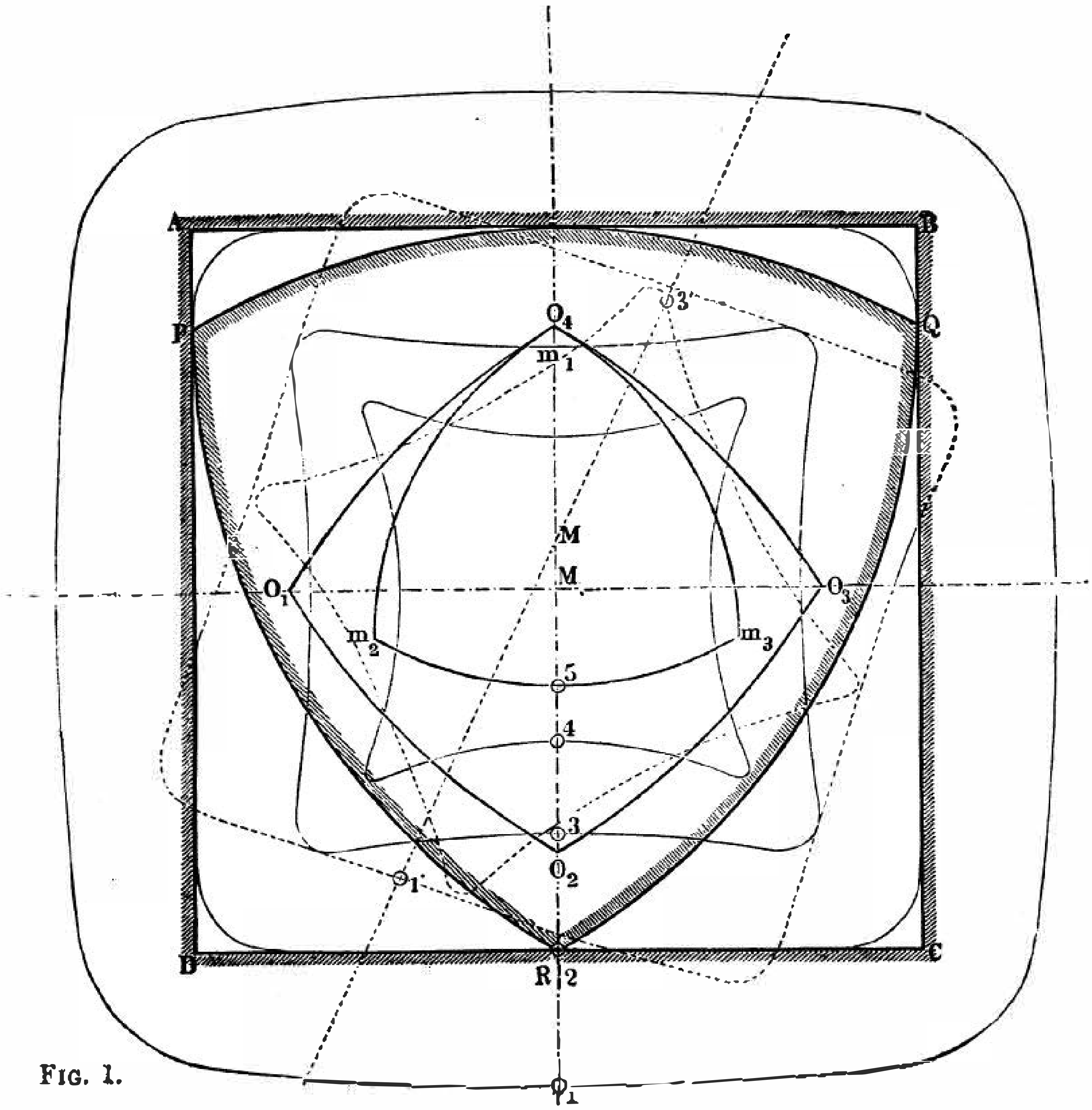


FIG. 1.

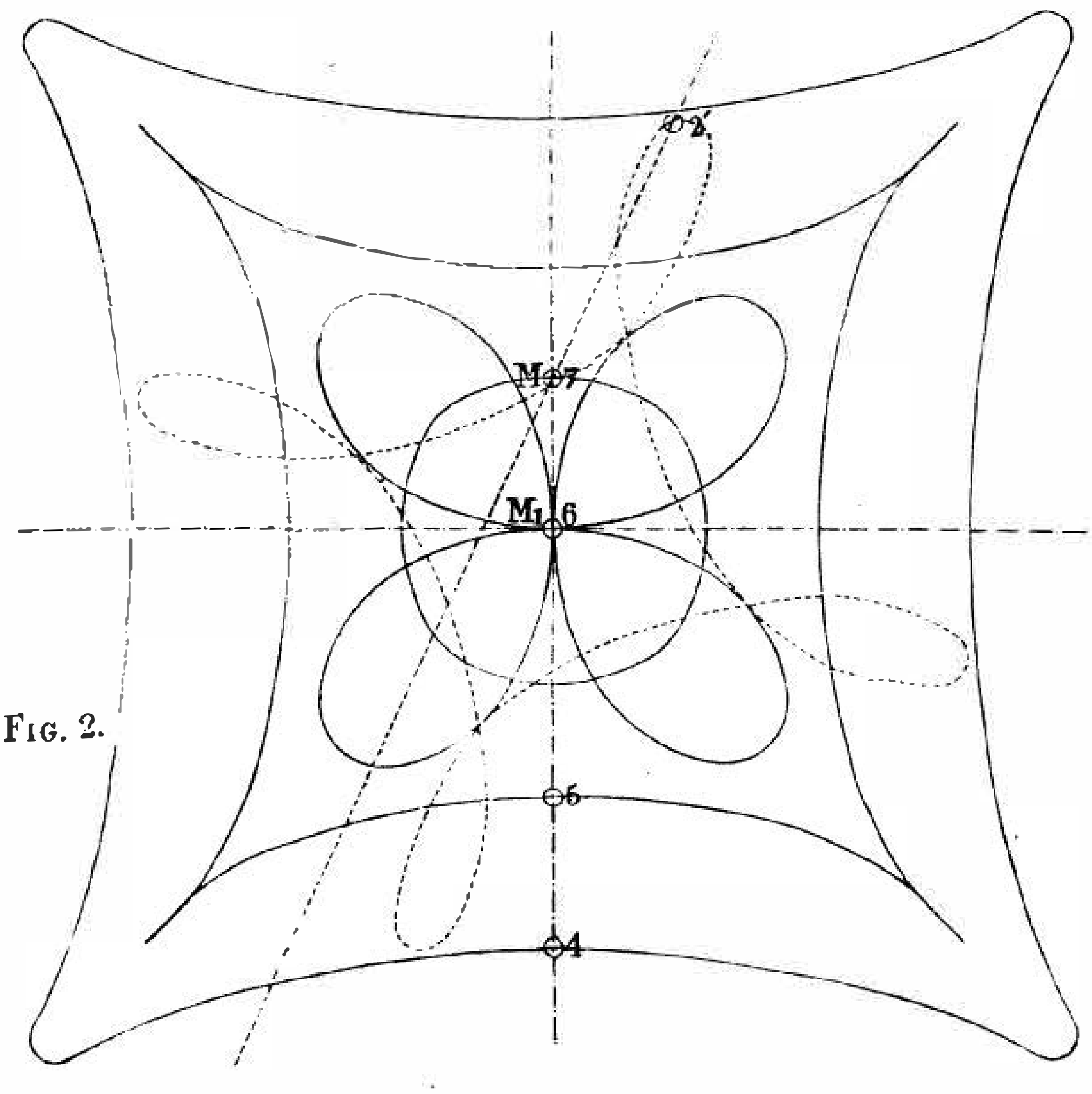
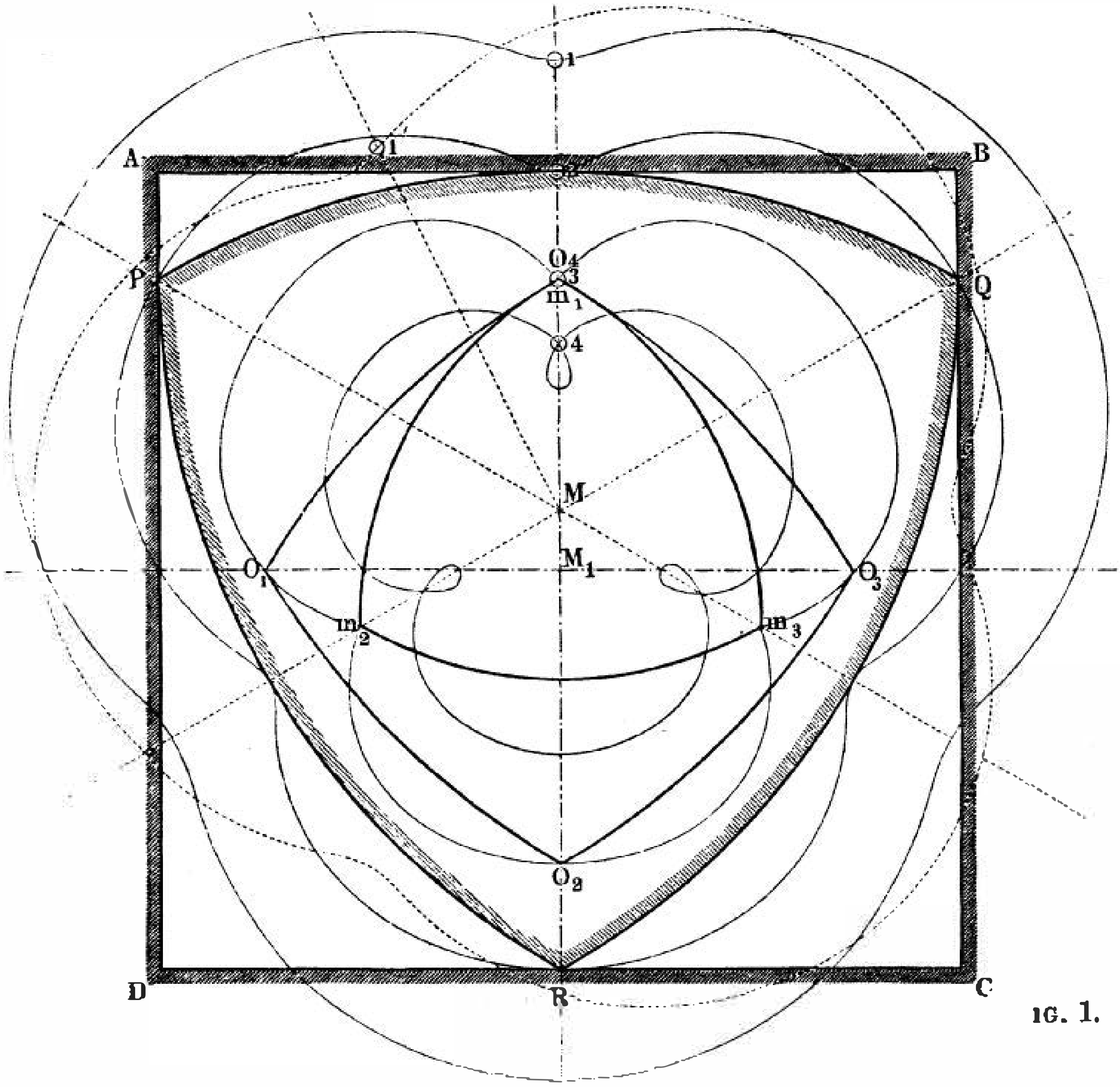


FIG. 2.



IG. 1.

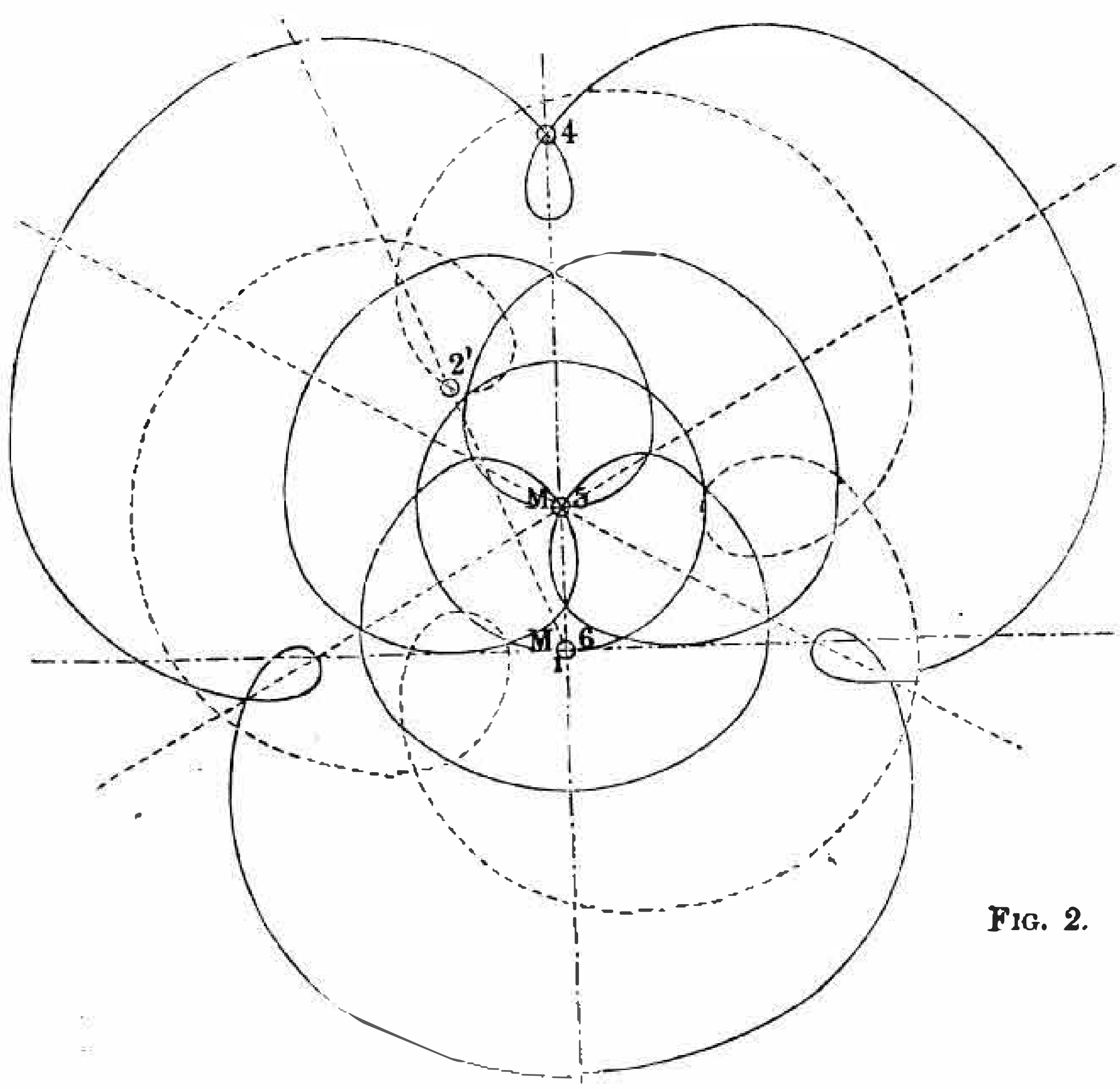


FIG. 2.

Plate VIII.

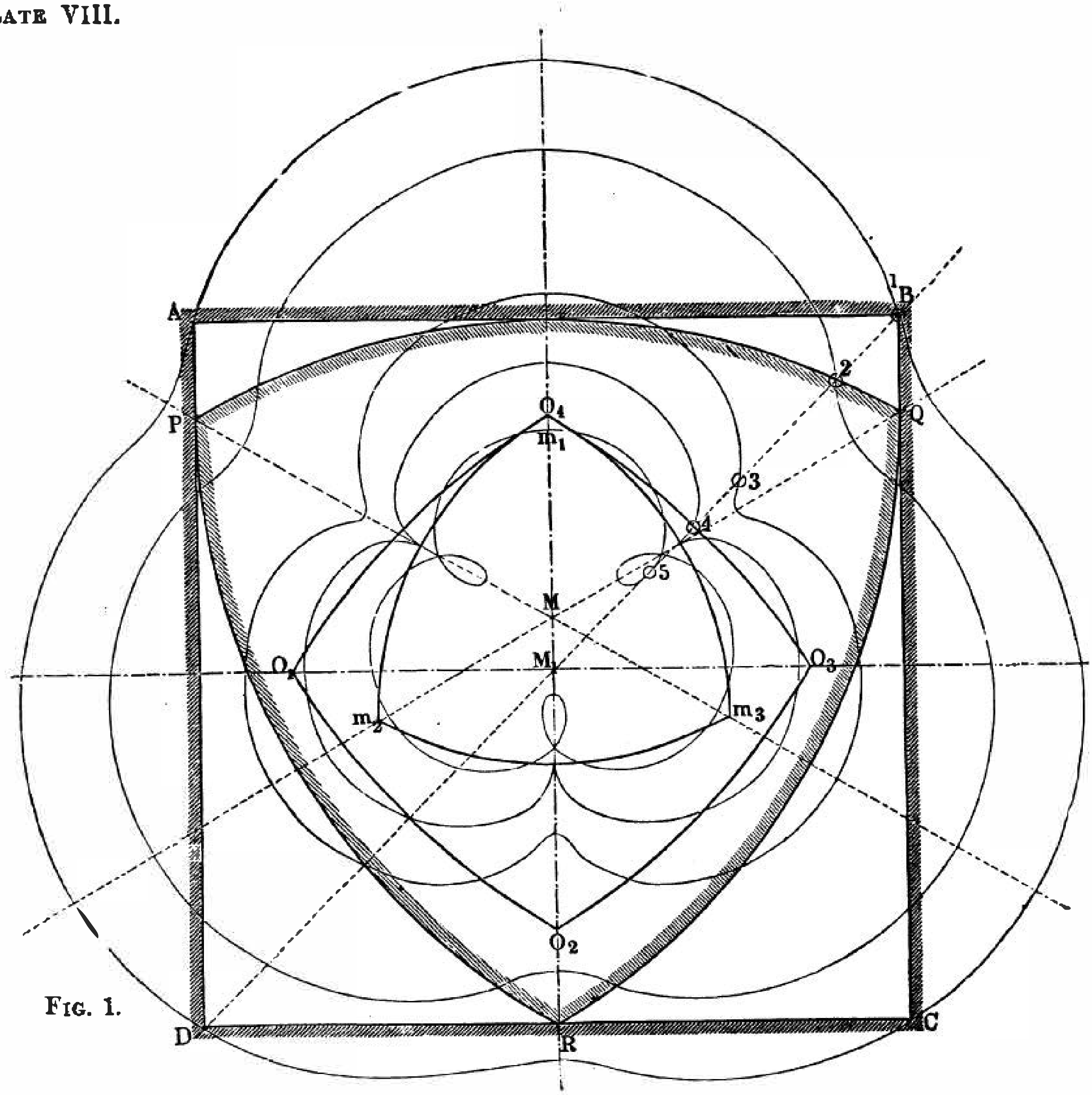


Fig. 1.

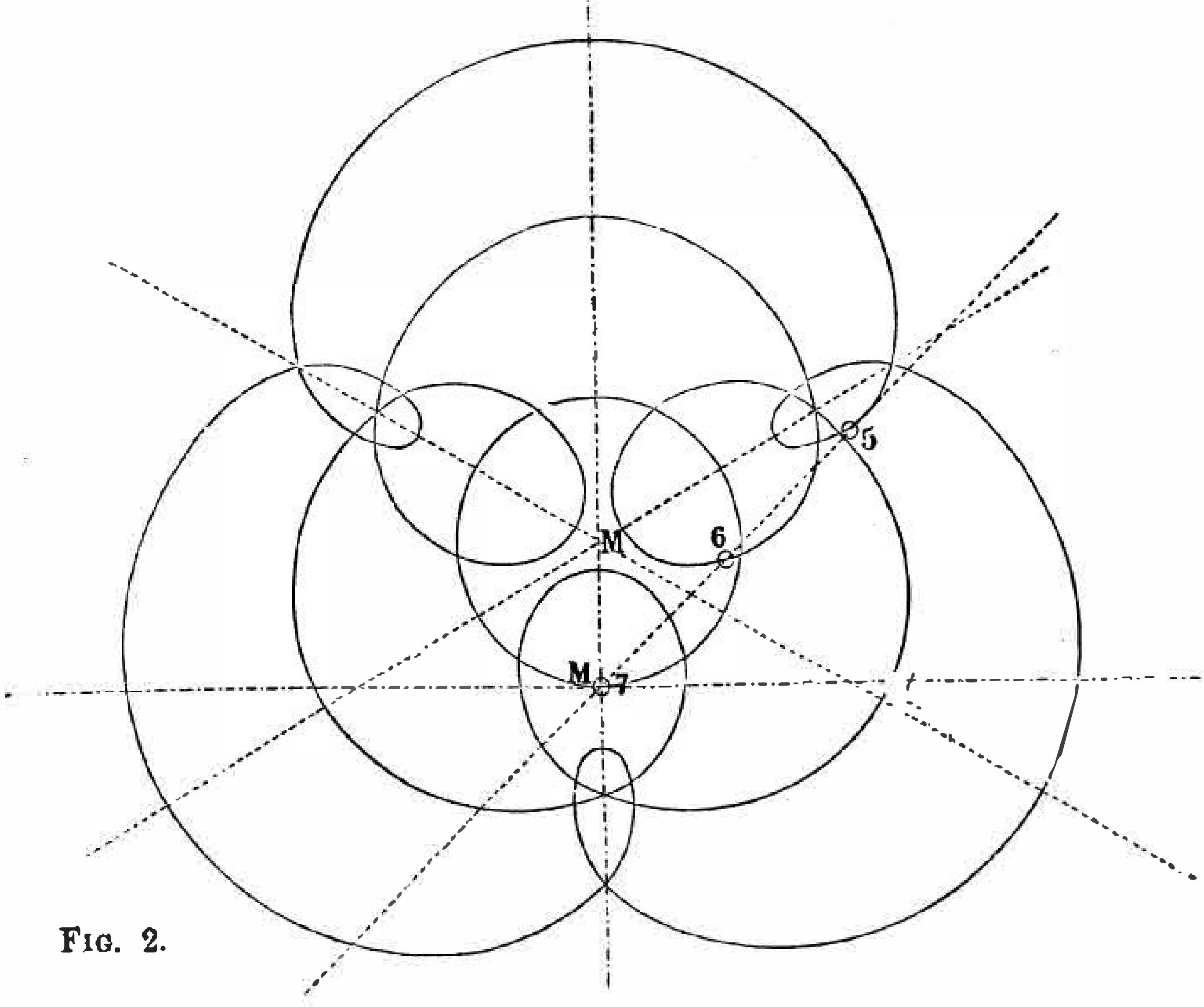


Fig. 2.

§ 28.

Paths described by Points of the Square relatively to the Curve-triangle.

(Plates VII. and VIII.)

Let now the triangle $P\ Q\ R$ be fixed, and the square $A\ B\ C\ D$ moved upon it. In Plate VII., six paths described by points upon the line $M\ O_4$ are shown. Nos. 1 and 2 give curtate roulettes built up of peri-trochoidal arcs,—No. 3 is the common form of the curve, No. 4 a prolate curve. This is repeated on a double scale in Fig. 2. No. 5 is the homocentral, No. 6 the concentral curve. The last resembles a circle very closely, but consists of peri-trochoidal arcs which so cover one another that the curve is fourfold,—being traversed by its describing point four times in every period.

The curves 1′ and 2′ belong to points in a line lying between two principal axes, the first is a curtate, the second a prolate roulette.

Plate VIII. shows seven roulettes corresponding to points on the line $M_1 B$. Nos. 1, 2, and 3 are curtate roulettes, No. 4 the common form of the series, Nos. 5 and 6 prolate roulettes, No. 7 the concentral form, the same as No. 6 in Plate VII., 2.

§ 29.

Higher Pairs of Elements:—other Curved Figures of Constant Breadth.

(Plates IX. to XIII.)

We found in § 25 that every figure of constant breadth can be constrained in a circumscribed rhombus, so that a pair of elements may be made from it and the rhombus. Eight examples of such pairs are given in Plates IX. to XIII.,—they are chosen as specially suited for showing the extraordinary variety of coustrained motions to which this proposition leads us. On account, however, of the completeness with which we have examined the pairs already considered we shall be able to dismiss these more shortly.

In Plate IX. we have the already known equilateral curve-triangle enclosed in a rhombus with angles of 60° and 120°. The form of the centroids here differs very greatly from that of the centroids in Plates V. to VIII. The centroid for the curve-triangle is a three-rayed figure, built up of three circular arcs of a radius $C\,Q = C\,R =$ half the length of the side of the rhombus;—the centroid of the rhombus is an equilateral duangle, its sides arcs described with the radius $B\,A = B\,C =$ the side of the rhombus;—or twice the magnitude of the radius with which the sides of the first figure were described. The centroids are therefore again arcs of Cardanic circles, and the point-paths built up of trochoidal arcs.

Some of the paths described by points of the triangle relatively to the rhombus are given. Point I, on the centre of the perpendicular $A\,Q$ to one of the sides of the rhombus, gives a figure symmetrical about two axes, and resembling in profile a double-headed rail;—the centre, II, of the triangle describes its concentral point-path, which is nothing else than the diameter $E\,F$, of the duangle—(in the direction $D\,B$)—this line being traversed three times in each whole period. This concentral point-path coincides with the homocentral,—and is at the same time (as the path of a point on the centroid) a common form of the curve. The point-path I′ is a curtate roulette of the rhombus; the point-path II″ is a prolate roulette for the same figure. Every point in the diameter $E\,F$ describes a homocentral curve in the curve-triangle;—one of these—that corresponding to the points E and F—is given. The variety of the forms here taken by the roulettes shows that it is impossible to draw conclusions from analogy alone as to the general character of the forms of any series of point-paths.

Plate X. 1. Equilateral curve-pentagon in Square.—The curve pentagon is constructed by describing arcs of circles having a radius equal to the diagonal about each of the corners of a regular pentagon, and a figure of constant breadth is thus obtained. The centroids are:—for the square $A\,B\,C\,D$ the four-cornered figure 1′ 2′ 3′ 4′, consisting of arcs of circles having the corners of the square for their centres and the side length $P\,Q$ of the pentagon for radius;—for the pentagon another equilateral curve-pentagon, described with radii equal to the half side-length of the pentagon from the centres m_1, m_2, m_3, m_4, m_5 of its sides. The centroid of the square rolls within that of the pentagon, and in every period each side of the one centroid must roll upon every side of the other, so that the instantaneous centre traverses the one five times

PLATE IX.

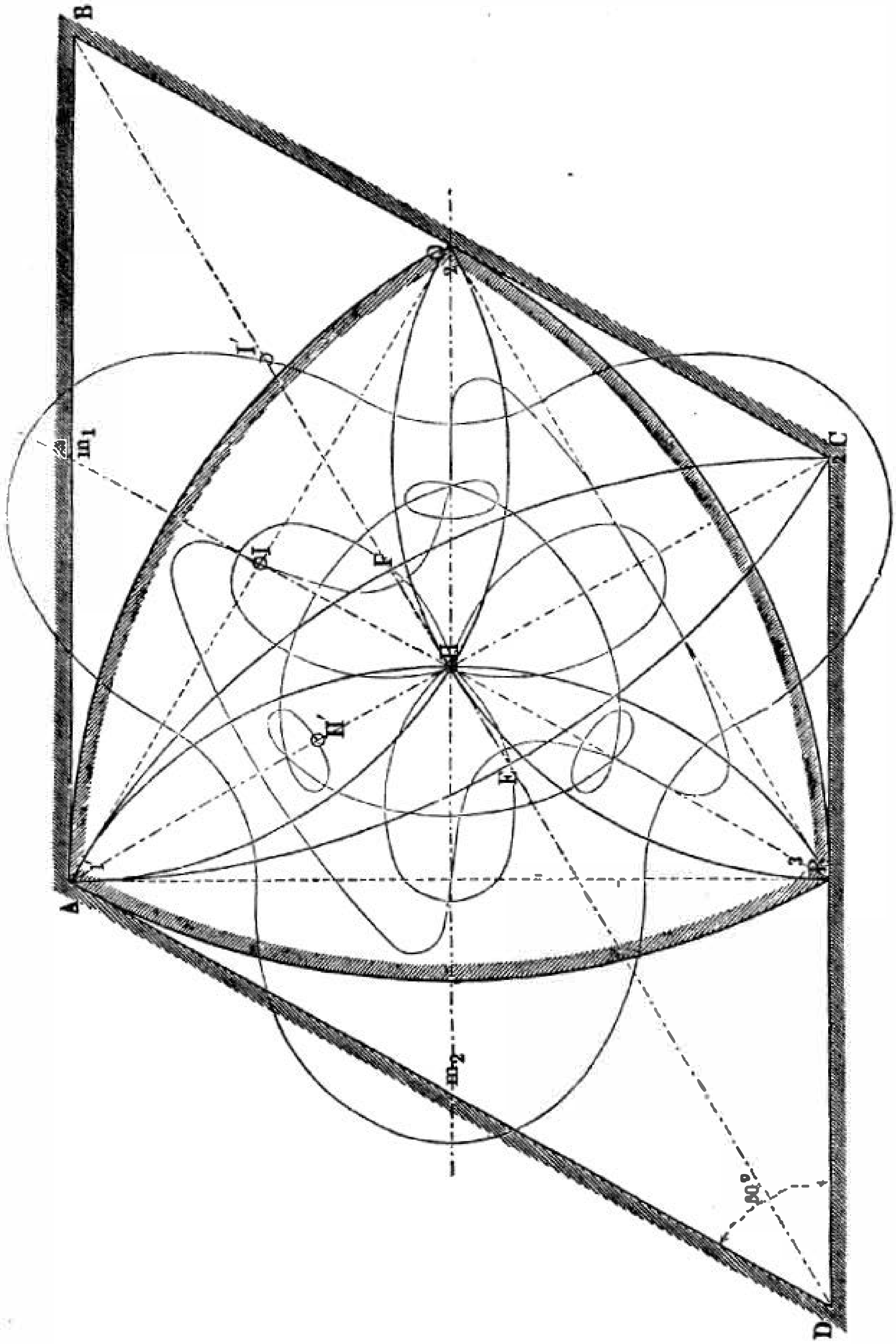


PLATE X.

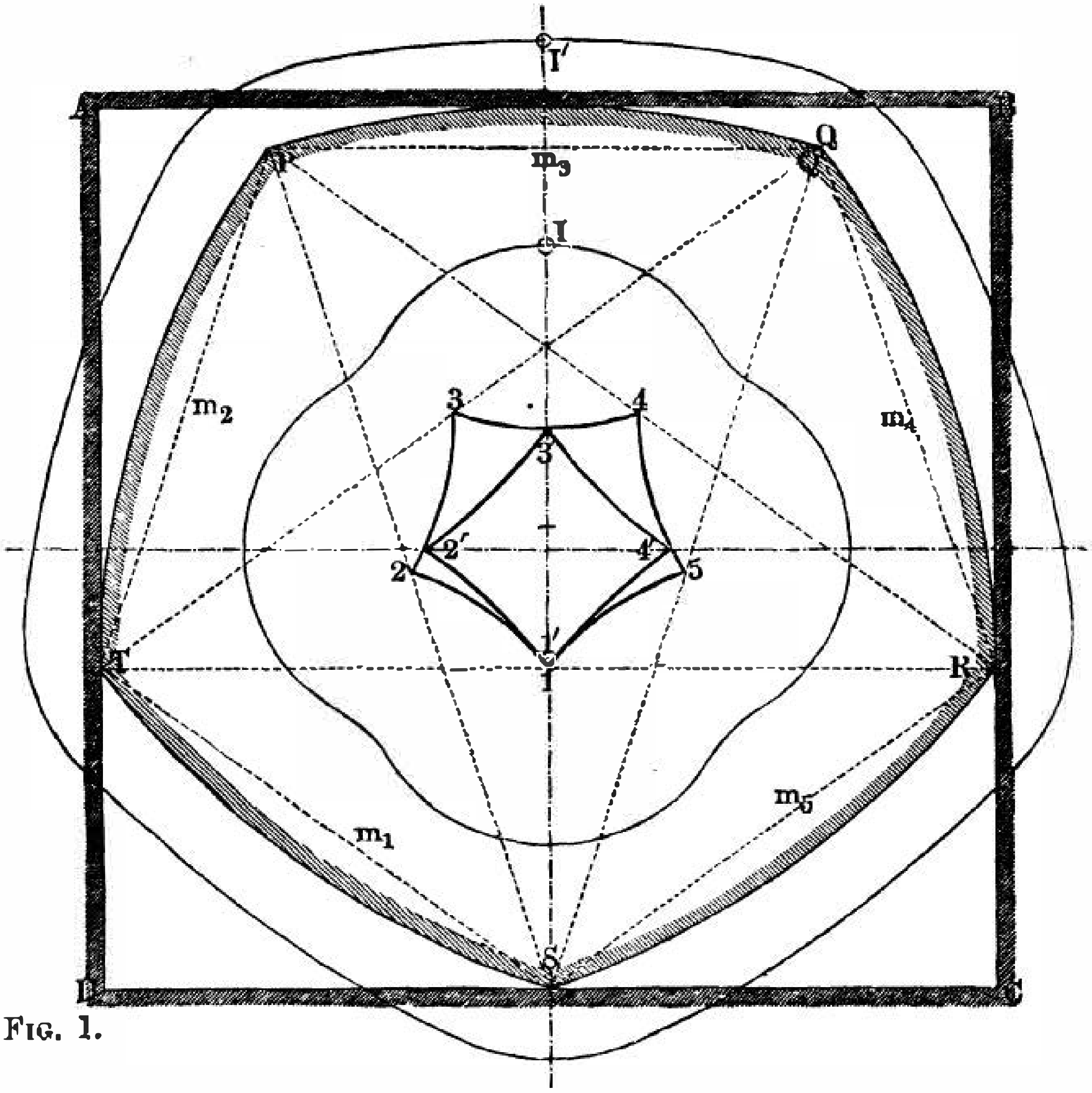


FIG. 1.

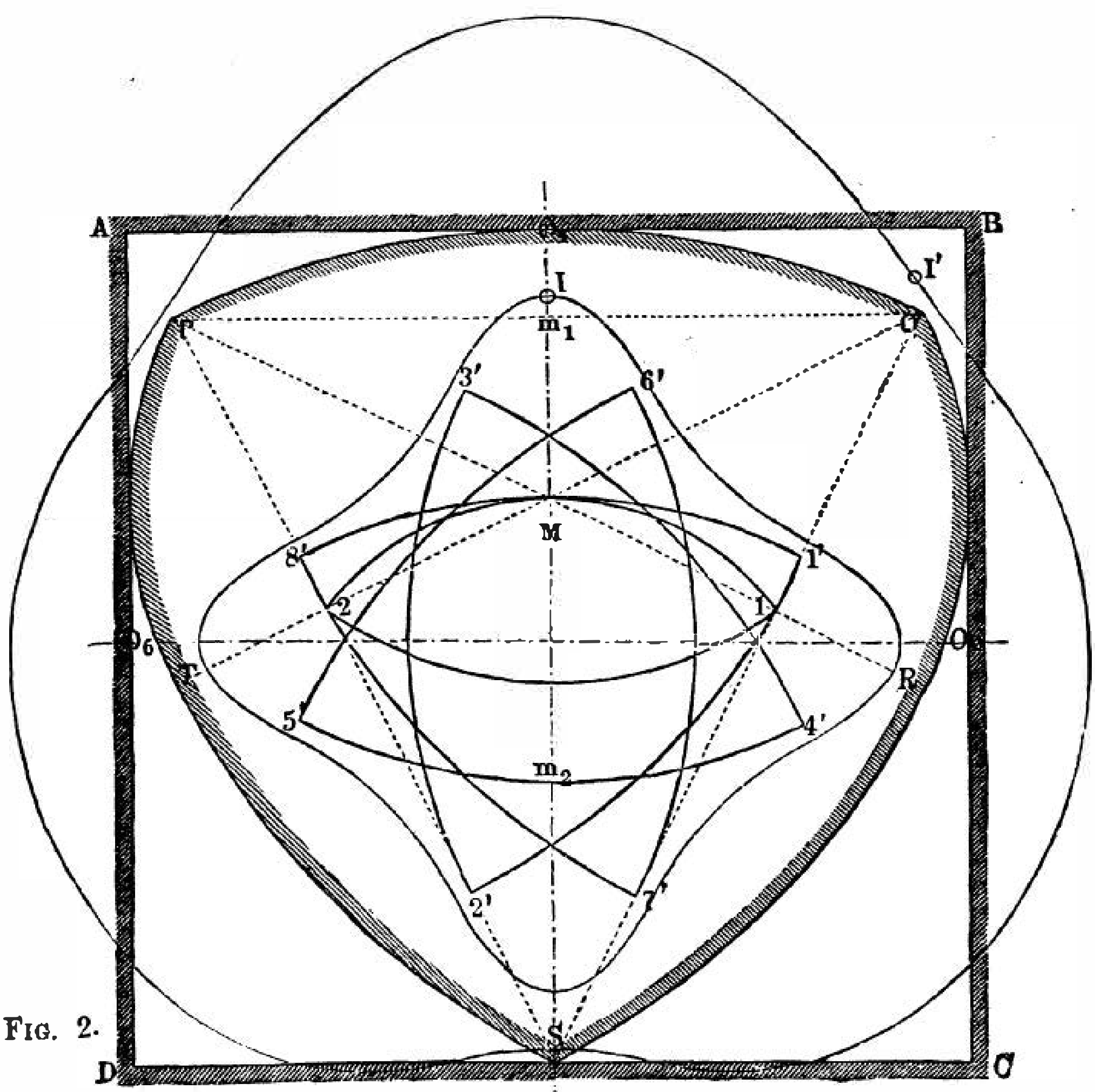


FIG. 2.

PLATE XI.

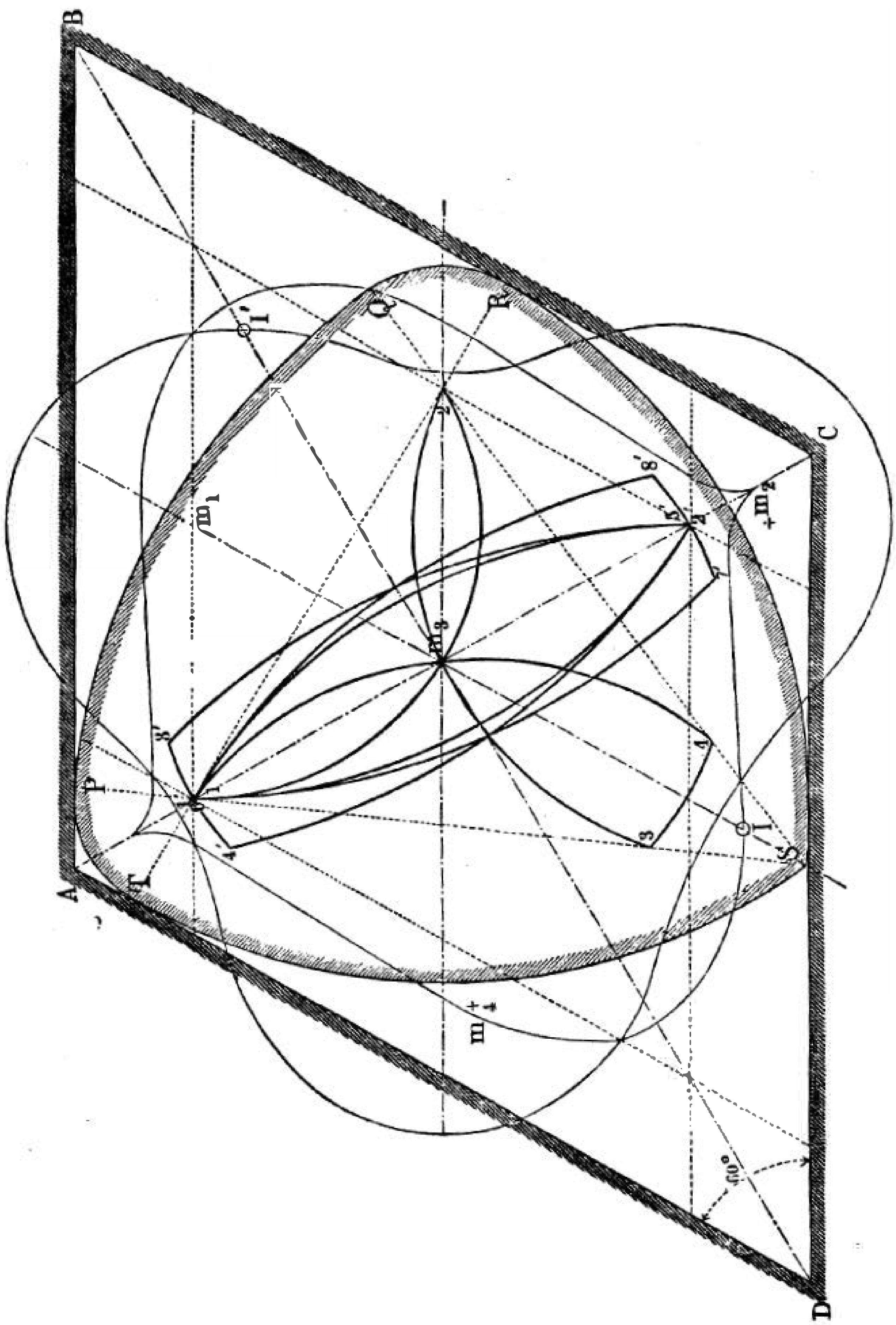


Plate XII.

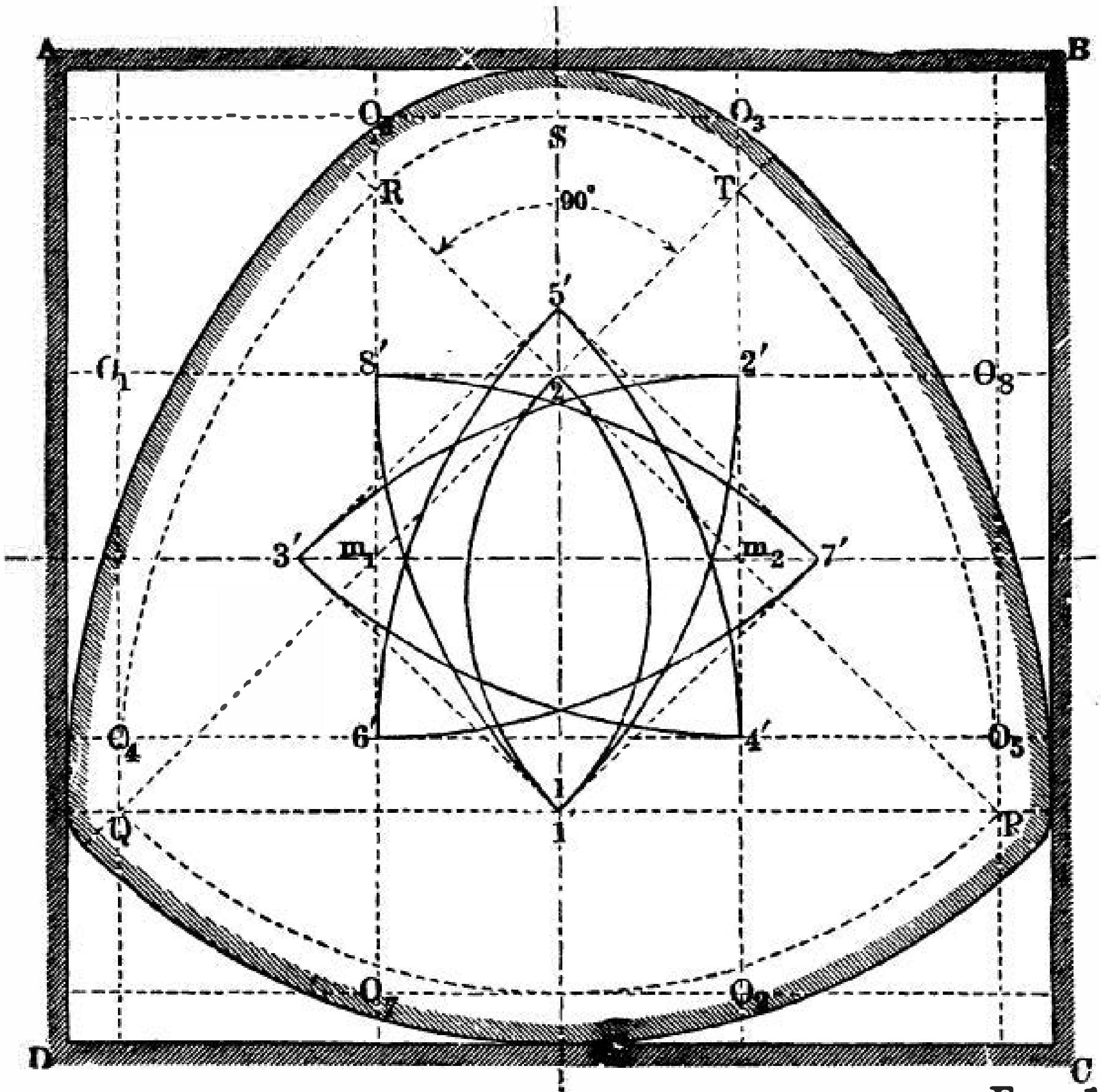


Fig. 1.

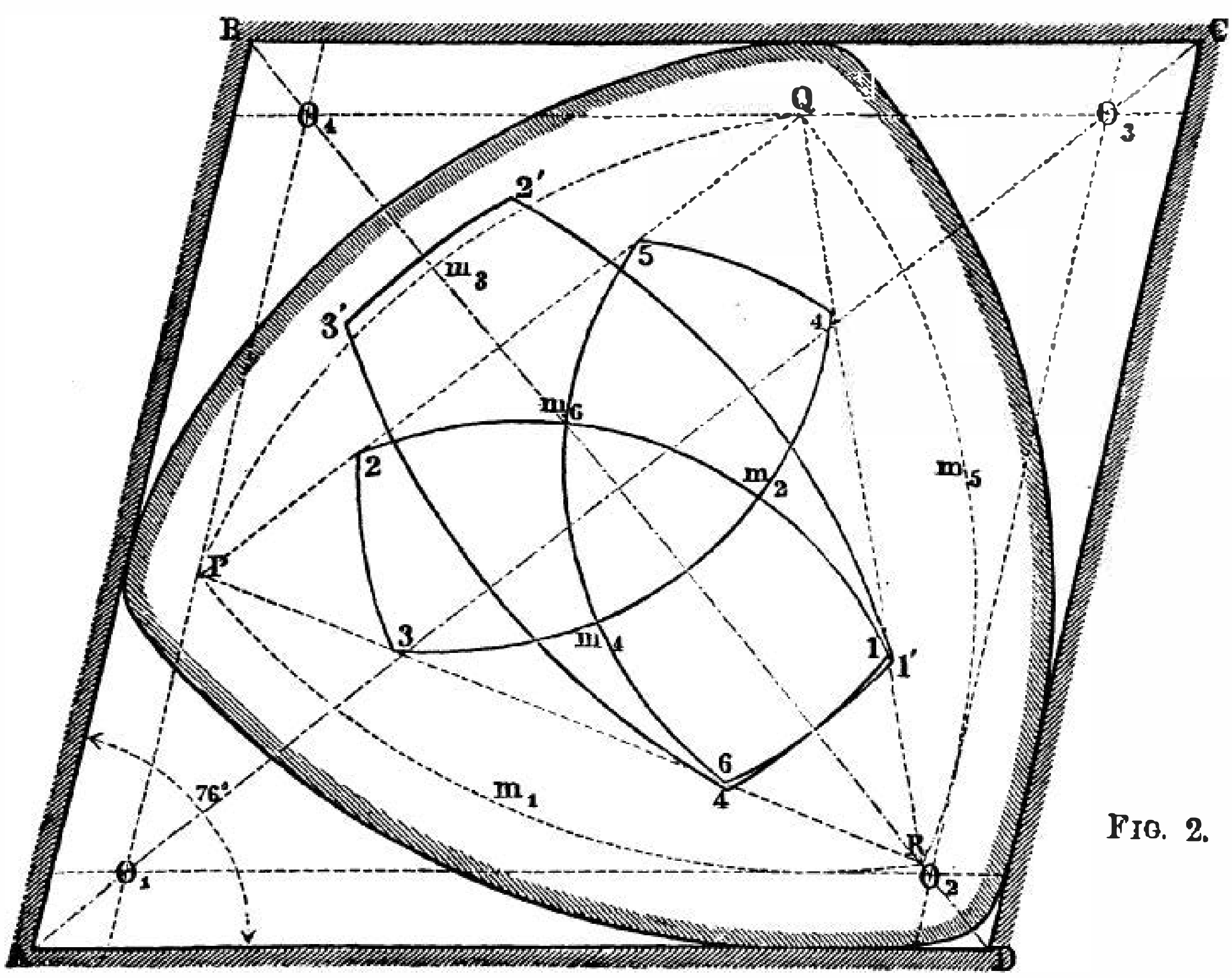


Fig. 2.

and the other four times in each period. The point-path marked I is described by the pentagon moving relatively to the square and is (the describing point lying beyond the centroid) a curtate curve; the I′ is also a curtate roulette, it is however described by a point of the square moving relatively to the pentagon.

Plate X.,2. Heart-shaped figure, made up of five circular arcs, in square.—PSQ is an isosceles triangle with vertex angle $PSQ = 53°$. The arcs PQ, ST, and SR are described from S, Q, and P with a radius equal to the side AB of the square,—the arcs TP and QR from the intersection M of the lines PR and QT, with radii equal to half the side of the square. The figure has therefore the constant breadth AB. The centroids for figures of these proportions are: for the curved figure a duangle 1.2 not equilateral; for the square an eight-rayed star of curves of different radii. The arcs of the centroids which roll upon each other belong always to Cardanic circles. Two point-paths I and I' are shown. The roulettes in their common form,—described by the corners 1 and 2 of the duangle (centroid) are specially characteristic. They are squares having for their corners the points $1'\ 3'\ 5'\ 7'$ and $2'\ 4'\ 6'\ 8'$.[19]

Plate XI. Isosceles curve-triangle in rhombus.—Upon an isosceles triangle $1\,S\,2$, having a vertex angle $< 60°$, the circular arcs ST and SR, having radii $S2$ and $S1$, are drawn to their intersections T and R with 1.2 produced;—from the same centres, and with radii $1\,T$ and $2\,R$, the arcs TP and RQ are drawn until they intersect in P and Q the sides $S1$ and $S2$ of the triangle produced; lastly, P and Q are united by a circular arc drawn from the centre S. The figure thus inclosed has the constant breadth QS. It is here paired with a rhombus having angles of 60° and 120°. The centroids are somewhat complex, but consist as before of arcs of Cardanic circles, the centroid of the triangle having four such arcs, that of the rhombus eight. Two point-paths I and I', belonging respectively to the triangle and the rhombus, are shown.

Plate XII., 1, shows another curve-triangle in a rhombus. The former is equilateral as in Fig. 1, but the radii of its sides (which are as before arcs described from the three corners PQR) are somewhat longer than the sides of the triangle,—and the corners are rounded off with radii equal to this excess of length. The motion which occurs is exactly the same as would be given by a pair consisting of the dotted curve-triangle PQR of the normal form and the enveloping rhombus $O_1O_2O_3O_4$. The rhombus is drawn with angles of 76° and 104° instead of 60° and 120° as before. This difference makes a notable alteration in the centroids, which resemble those of Plate IX. with the corners removed. The nature of the changes of form in the point-paths from those obtained before can thus be readily traced.

Plates XII., 2, and XIII. show three more pairs constructed analogously to those we have already examined. The first of these is remarkable, both on account of the regularity of its centroids and because the end points 1 and 2 of the smaller centroid again describe squares. The curved element in Plate XIII., 1, is formed like that in Plate XI., but with a smaller vertex angle at $\mathfrak{S}$; that in Plate XIII., 2, is similar to the one in Plate X., 2. The difference between the centroids in Figs. 3 and 8 is very remarkable. The manner in which the one centroid rolls upon the other is indicated as distinctly as has been possible by the numerals. These examples show clearly the multitude of motions which can be obtained by means of the higher pairs, and show at the same time how wonderfully the use of centroids simplifies the comprehension of these complicated motions.

§ 30.

General Determination of Profiles of Elements for a given Motion.

The forms of the pairs of elements considered in the foregoing investigation were found synthetically;—starting from the general solution of the problem of restraint we built up constrained pairs in accordance with its conditions, and afterwards ascertained their relative motions by the construction of their centroids. This latter part of our investigation was again analytical. It furnishes us however with the means of solving a further synthetical problem,—the construction, namely, of pairs of elements for a given motion,—*i.e.* for given centroids. For the forms which we found necessary for the continuous reciprocal restraint of the elements are reciprocally envelopes for one relative motion, and for that one only which is determined by the centroids. We have chosen hitherto forms conditioned only by the property of reciprocal restraint, and have from them determined their centroids. We may now reverse the problem, by assuming the centroids as given

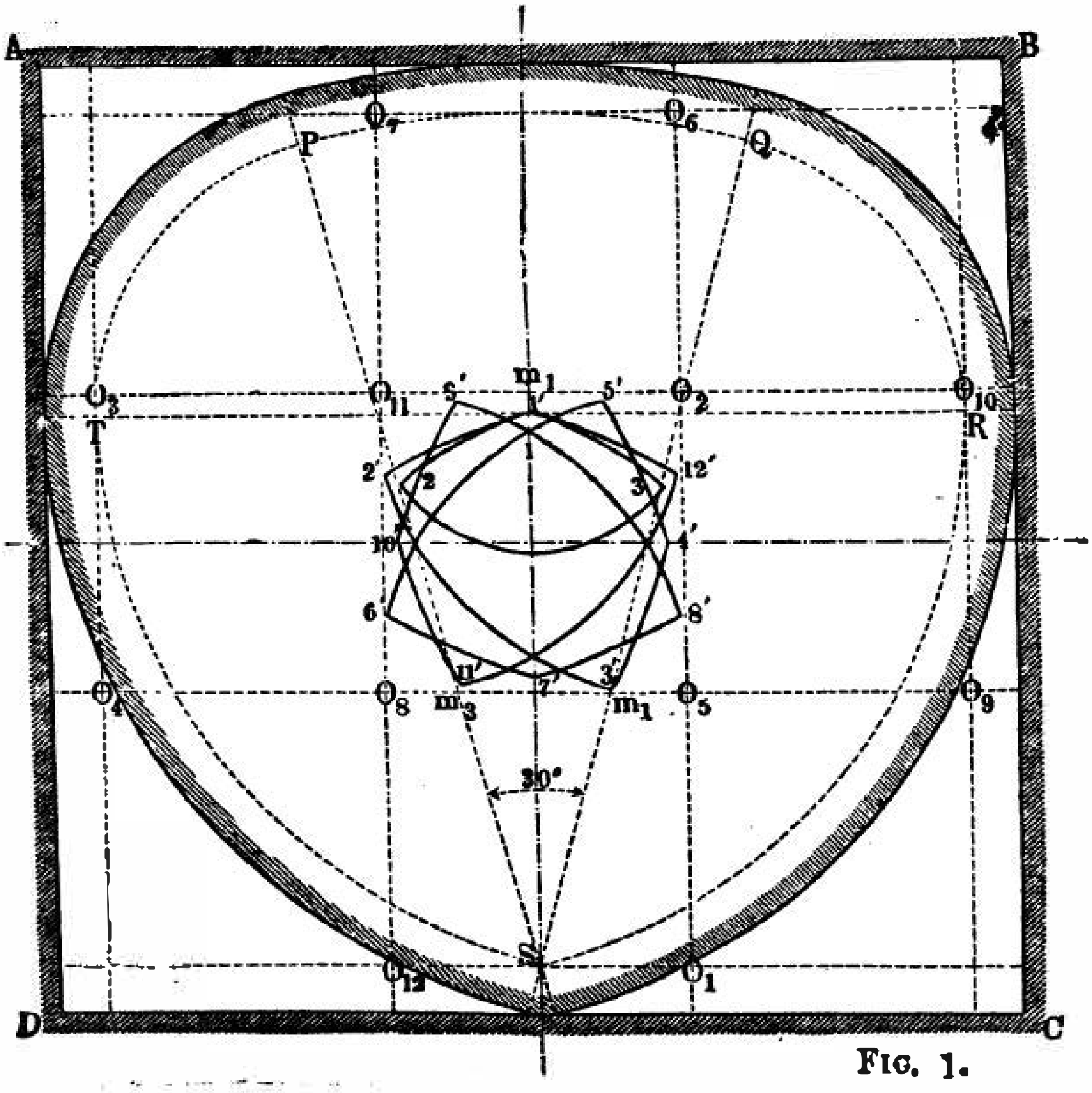

A
B
C
D
P
Q
R
T
S
m_1
m_3
m_1
O_1
O_2
O_4
O_5
O_6
O_7
O_8
O_9
O_{10}
O_{11}
O_{12}
30°

FIG. 1.

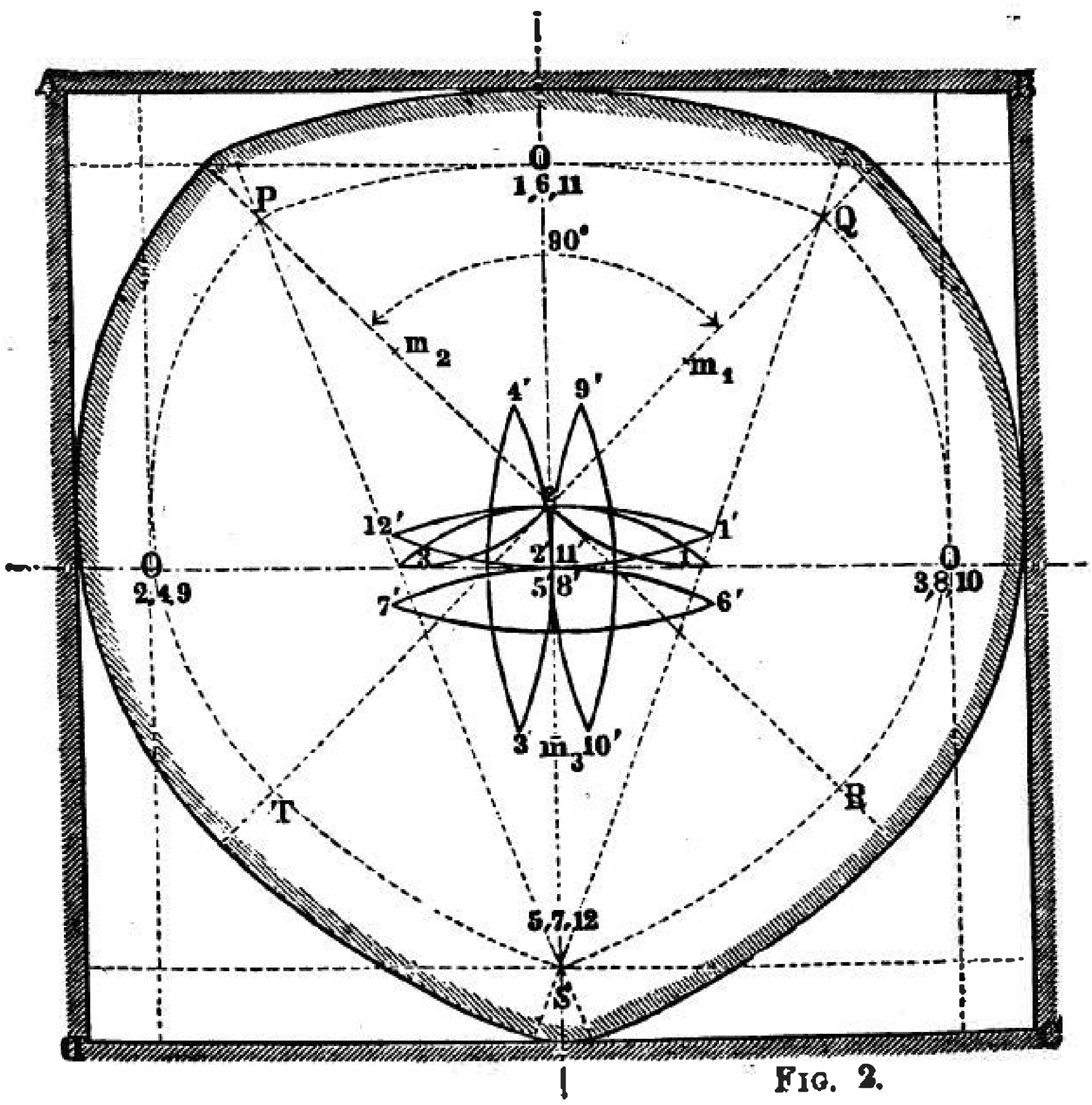

A
B
P
Q
R
T
S
O
1,6,11
90°
m_2
m_1
m_3
O
2,4,9
O
3,8,10
5,7,12

FIG. 2.

and determining by their means the reciprocally restraining figures. This problem is the one which occurs by far the most often in connection with machine design, and has frequently to be solved both for simple and for complex motions.

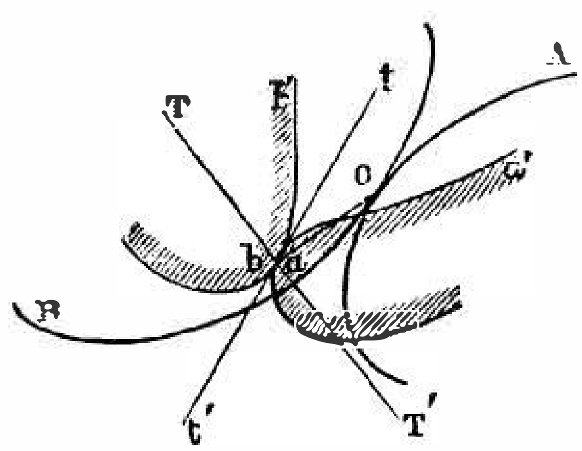
We shall here simply indicate the general methods of procedure in this case. These are very numerous, but admit of being classed under the seven headings examined in the following paragraphs, in which we shall consider cylindric rolling only in the first place.*

§ 31.

First Method.—Determination of the Profile of one Element, that of the other being arbitrarily assumed.

If the profile of one element of a pair of which the centroids are known be arbitrarily assumed, the centroid of the unknown element may be brought to rest, and that of the assumed one rolled

* The following note may make clearer to some readers the nature of the problems treated by Prof. Reuleaux in §§ 31–37. Let A and B be any two centroids, and $a\,a'$ and $b\,b'$ the profiles of bodies whose relative motions the centroids represent. It is required so to form these profiles that during the rolling of the centroids they shall remain continuously in contact. The necessary condition for this may be thus shown. Let O be the point of contact of the centroids, and let the two profiles be touching at a; draw their common tangent tt', join $O\,a$ and draw $T\,T'$ perpendicular to it. Then suppose B fixed, and A free to roll upon it. The instantaneous motion of the point a can only take place about the instantaneous centre O, that is in the direction $T\,T'$ perpendicular to $O\,a$. But if a move towards T' it leaves the profile b, while it cannot move towards T because the point b restrains motion in that direction. Hence the assumed profiles $a\,a'$ and $b\,b'$, in no way fulfil the requirements of the problem. They show very clearly, however, the condition necessary for this fulfilment, for it is obvious that the point a, moving about O, can remain in contact with b only if the tangent $t\,t'$ to that profile coincide with the line $T\,T'$. $O\,a$ is normal to $T\,T'$ by construction,—we may therefore express the condition generally by saying :— in order that two elements may remain in contact during the rolling of their centroids the normal to the common tangent of their profiles must always pass through the instantaneous centre, or point of contact of the centroids. It must be remembered that (except in one special case), the profiles themselves do not roll upon one another, but slip or grind to a greater or less extent.



upon it,—any number of consecutive positions of the latter may be drawn, and a curve enveloping these, if rigidly connected to the stationary centroid, gives a figure with which the known element remains in continuous contact during its motion. Such a figure will serve as the profile of the stationary element, if it can be provided with a sufficient number of points of restraint. This may readily happen, indeed the figure may have even more such points than are necessary. In these cases only such portions of the figure need be constructed as suffice to make the restraint continuous. We have seen this in the case of the curve-triangle and square, and also with the duangle and triangle, where the enveloping curves were not drawn in the corners of the square and triangle, their omission not affecting the restraint. The method is therefore available;—we must examine the manner in which it can be practically carried out.

If A and B (Fig. 103) be the two given centroids, and $a\,b$ the assumed profile of the element corresponding to A, then if any point, as b, of the assumed profile be also a point in the centroid, the corresponding point of contact O of the centroid B gives at once one point in the profile to be found. In order to determine a second point in it,—that for instance which shall correspond to a,—let a normal $a\,c$ be drawn to the given profile at a. It cuts the corresponding centroid A in c. If we roll the centroid B upon A until c becomes the point of contact,—*i.e.* the instantaneous centre,—then a must be the point of contact of the profiles, and $a\ c$, which we may call the central distance of a, must be the common distance of the two touching profile points from the contact points of their respective centroids. The distance at which the point of the centroid B originally at O must then be from a can at once be seen when B is in the position $b_1\,c$ (in which contact occurs at c); it is simply $a\ b_1$, the centroidal arc $b_1\,c$ being $= b\,c$. If now we have $b\,c_1 = b_1\,c = b\,c$, and describe from c_1 with

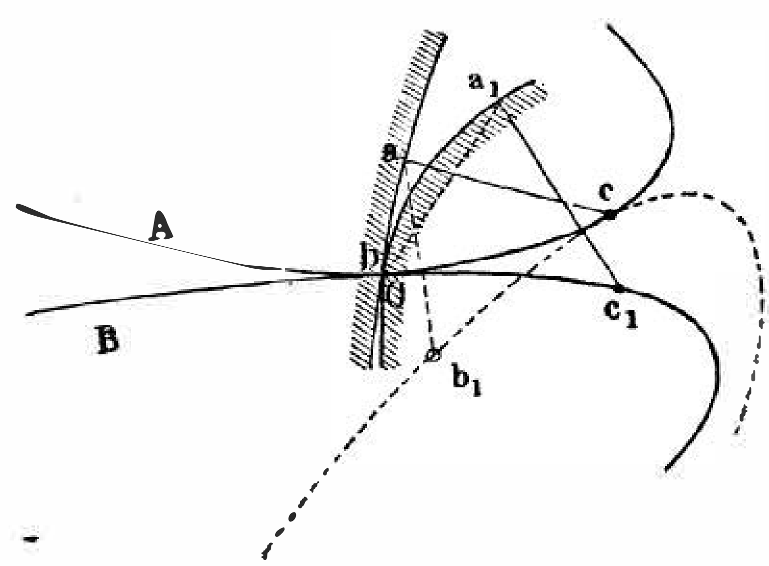


FIG. 103.

radius = the central distance $c\,a$, and from b with radius = $b_1\,a$ arcs of circles, the intersection of these gives us the required profile-point, a_1. In this way the profile $a_1\,b$ . . . can be exactly determined point by point. If great accuracy be not required the following approximate method (by Poncelet), can be used. Erect normals in a sufficient number of points a, c, e . . . of the assumed profile, and continue these to their intersections b, d, f . . . with the corresponding centroid. Find the corresponding points b_1, d_1, f_1 . . on the centroid B, and describe from them circular arcs with radii equal to the central distances $a\,b$, $c\,d$,

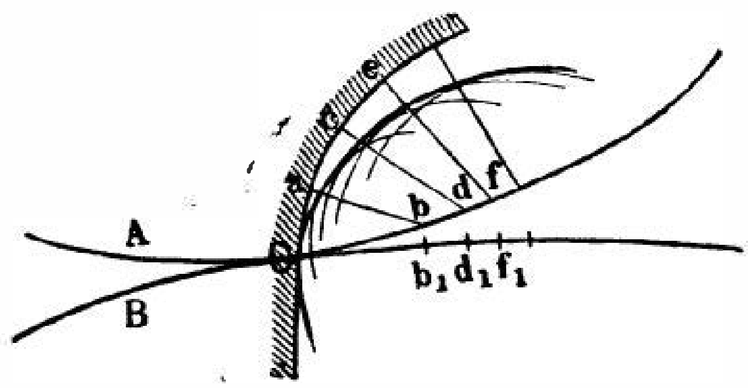


FIG. 104.

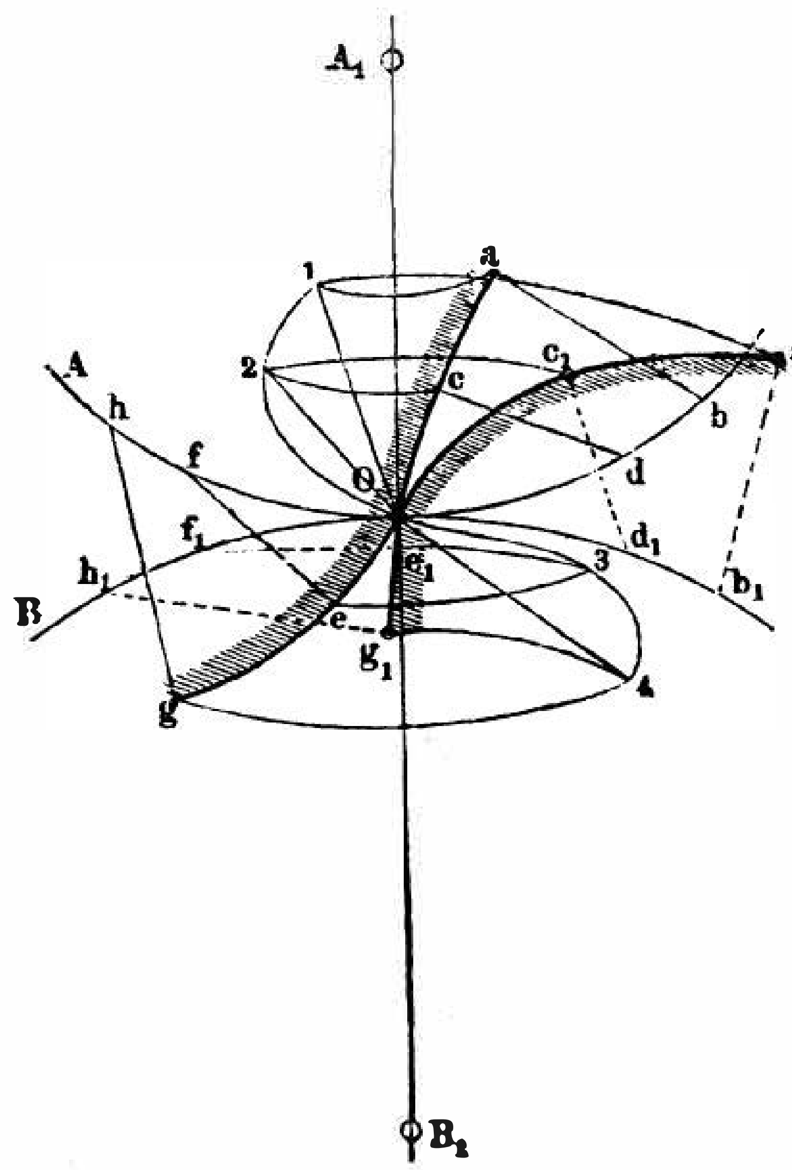


FIG. 105.

$e\,f$, etc.; a curve enveloping these arcs gives a near approximation to the required profile.

The foregoing method in its application to spur-wheels is known in Germany as the "general method" of drawing teeth, (*allgemeine Verzahnung*). In this case the centroids are commonly circles, which renders some simplification possible. The following is the way in which I have used* this method for the construction of wheel-teeth upon circular centroids.

In Fig. 105 $a c e g$ is the given profile, A the corresponding centroid drawn about the centre A_1; B is the centroid, drawn about B_1, of the element whose profile is to be found. After drawing the normal $a b$ to any point a in the profile we must next find the position in which a will be when it itself becomes the point of contact with the as yet unknown profile, for which purpose we may suppose both centroids to be turning about their centres A_1 and B_1, assumed to be fixed points in the plane of the paper. The point of contact 1 lies necessarily at the intersection of a circle described about A_1 with a radius $A_1 a$, with a second circle described about the instantaneous centre O with the central distance $b a$, for at the moment of contact the normal $a b$ must pass through O. B has meanwhile turned through an arc $O b_1 = O b$. The new profile point a_1 which corresponds to a must therefore be at the intersection of a circle drawn from B_1 with radius $B_1 1$ with another circle drawn from b_1 with radius $O 1$. In the same way the points of contact 2, 3, 4 . . and the corresponding points c_1, e_1, g_1 . . . in the profile, can be found. The series of points 1, 2, 3, 4 . . . give us the line of contact, or locus of all the successive points of contact of the two profiles. The line joining the point of contact with the instantaneous centre O is for each instant both the direction of restraint, and the direction of the pressure between the two profiles.

The method of which we have here given three applications furnishes an immense variety of profile forms, among them many which are of little or no practical use. Those curves especially which contain cusps or loops, or form contracted spirals etc.,—(see Fig. 106)—are commonly unsuitable; they are not useful although they are geometrically correct,—fulfilling the required conditions as to continuous restraint in motions determined by the given pair of centroids. If the application of this method furnish us in any case with such impracticable profile-curves it becomes necessary to choose some more suitable

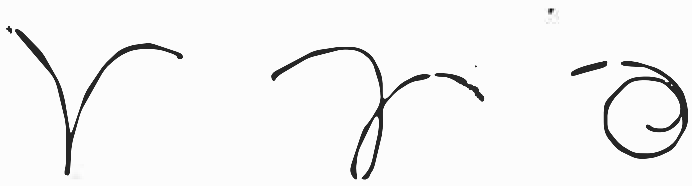


FIG. 106.

* Published first in *Der Constructeur*, 2nd Ed. 1865.

form for the assumed profile,—proceeding thus by trial. Absolute freedom in the choice of the form of the first profile is to this extent limited, and a further limitation arises from the fact that such curves as have normals which cut the centroid at the point of contact at too great an angle (as *e.g.* the normal 01 in Fig. 105) are not suitable for the profile, for with these most injurious friction will occur, if not complete "jamming." Those parts of profiles, lastly, whose normals do not pass through the centroid, and therefore cannot be normals of restraint, are entirely unusable. Thus in the employment of this method in Applied Kinematics a number of unsuitable and unusable forms must be withdrawn from those which can be used for the (otherwise) arbitrarily chosen form of the assumed profile.

§ 32.

Second Method.—Auxiliary Centroids.

The method just described gives the profile of a single element only; by that which we have now to examine the two profile forms possessing the required property of continuousn restraint are determined at the same time.

Let A and B—Fig. 107—be again the pair of centroids in contact at O. If any third curve C touch A and B at the same point, and roll with them as they roll, the three curves will always have a common point of contact, and that point will always be the instantaneous centre. Then any point D fixed to C describes a roulette relatively to each of the centroids A and B. The two curves thus obtained,—$a\ D$ . . . and $b\ D$. . . . have in any position a common point, as D, and also a common normal, as $D\,O$. They may therefore serve as profiles for elements,—for their common normal always passes through the point of contact of the centroids. Their practical usefulness depends on the same conditions as those mentioned in the last section.

The centroid C, by the help of which we have obtained the profiles, we may call an a u x i l i a r y c e n t r o i d. If the describing point D be taken on this curve or within it, the roulettes obtained remain always upon one side of the primary centroid to which they belong; there is then always space upon the opposite side of the same curve to construct similarly another pair of roulette profiles. The whole procedure may therefore be there repeated,—a second auxiliary centroid, similar or dissimilar to the first, being employed to give the two new profiles *c* and *d*—Fig. 108.

If the describing points are points upon the auxiliary centroids, all the roulettes must extend to the primary centroids,—we can therefore join the profiles *a* and *c*, and also the profiles *b* and *d*, into one piece. Then by repeating the process for a number of positions on each of the centroids we can obtain profiles which may serve as those of tooth-formed projections upon the element to which they belong. A regular series of such projections with

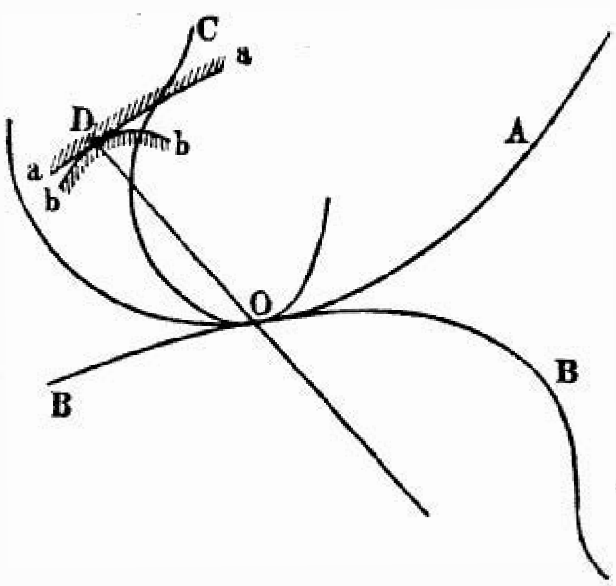


FIG. 107.

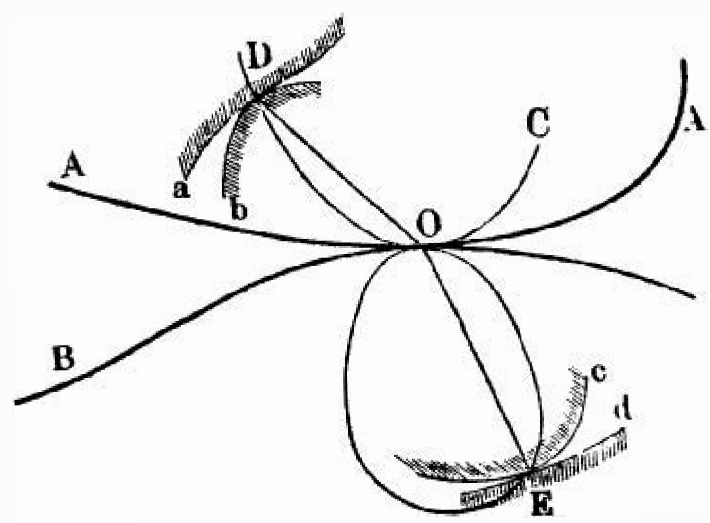


FIG. 108.

corresponding hollows between them gives us the familiar spur-wheel. The portion of the centroid lying between homologous points of two consecutive teeth is there called the pitch, and the centroid itself is the pitch-line, or if circular the pitch-circle. Circles ("describing" circles) are used as the auxiliary centroids. The teeth must carry such portions of the roulette profiles that the restraint is never interrupted; such portions must be at least so large that the restraint by each tooth lasts while the centroids roll through a distance equal to the pitch, the contact between each pair of teeth, in other words, must continue for at least this period.

With spur-wheels or toothed-wheels having cylindric axoids it may be further required that all wheels of the same pitch should gear with each other, that is that their tooth-profiles should communicate a motion to which the corresponding centroids (pitch lines) are circular.* Wheels so arranged we may call Set-

* If wrongly formed profiles be set to work with each other the motion of the

wheels ;* any pair of them, equally pitched, will gear truly together. Willis† appears to have been the first to point out both this problemh and its solution. Looking at it from the general point of view which we have here reached, it is evident that the solution of the problem is the use of similar auxiliary centroids for those shown on opposite sides of the primary centroids in Fig. 108.

The delineation of wheel-teeth very early led geometricians to the use of roulettes as profiles of elements. Camus laid down its fundamental principles very clearly in 1733 in a little-known treatise, as Willis has shown.[20]

Camus' predecessor, De la Hire, had apparently used this method before him, and he himself refers to the still earlier Desargues ‡ (1593—1662) as having used epicycloidal teeth, and thus preceded by many years Römer (1664—1710), so often mentioned as their inventor.

In the distinctness and completeness of its results, this method of forming element profiles by roulettes greatly excels the method first described, which indeed, in a certain sense, it includes. For we may conceive of the assumed profile of the first method as having been itself found by means of an auxiliary centroid. The second profile then becomes a roulette drawn by the same curve, which thus may be considered as actually the describing curve of the profiles, although it has not itself been drawn.

De la Hire also enunciated the general proposition as to the describing of roulettes which we are here applying, and which is of so great importance in Kinematics; and it is to the same geometrician that we owe their name.[21] The methods and propositions relating to them have hitherto hardly received their due development. The delineation of the auxiliary centroid in the method of § 31 is interesting—but not necessary,—the method itself is practically useful chiefly where a single result is all that is required.

bodies to which they belong will have different centroids from those originally assumed. Thus in the special case mentioned above, if the wheel teeth be not of the right shape, the wheels will not have a constant, but a varying angular velocity ratio. Their centroids will therefore no longer be circles, but irregular figures more or less nearly resembling them.

* I cannot find that any name has hitherto been used for them in this country. In Germany they are called Satz-räder.

† *Trans. of Inst of Civil Engineers*, 1837, vol. ii. p. 91.

‡ Chasles, *Geschichte der Geometrie*, p. 83. (Sohncke.)

§ 33.

Third Method.—Profiles described by Secondary Centroids.

We have already mentioned (§ 9) the employment of secondary centroids instead of the original primaries, and have found that by their means problems which otherwise contained some difficulties could be easily solved,—that in certain cases we could use the secondary centroids interchangeably with the principal ones. Among these secondary centroids one class especially is useful in the delineation of element profiles. This class consists of those in which two curves and their tangent are used to represent the motion. Such centroids are obtained if through a sufficient number of points in the two primary centroids we draw a series of secants making a constant angle to the tangents at the points through which they are drawn;—these secants envelope a pair of curves which, touched by a line rolling upon them, form together with it secondary centroids.

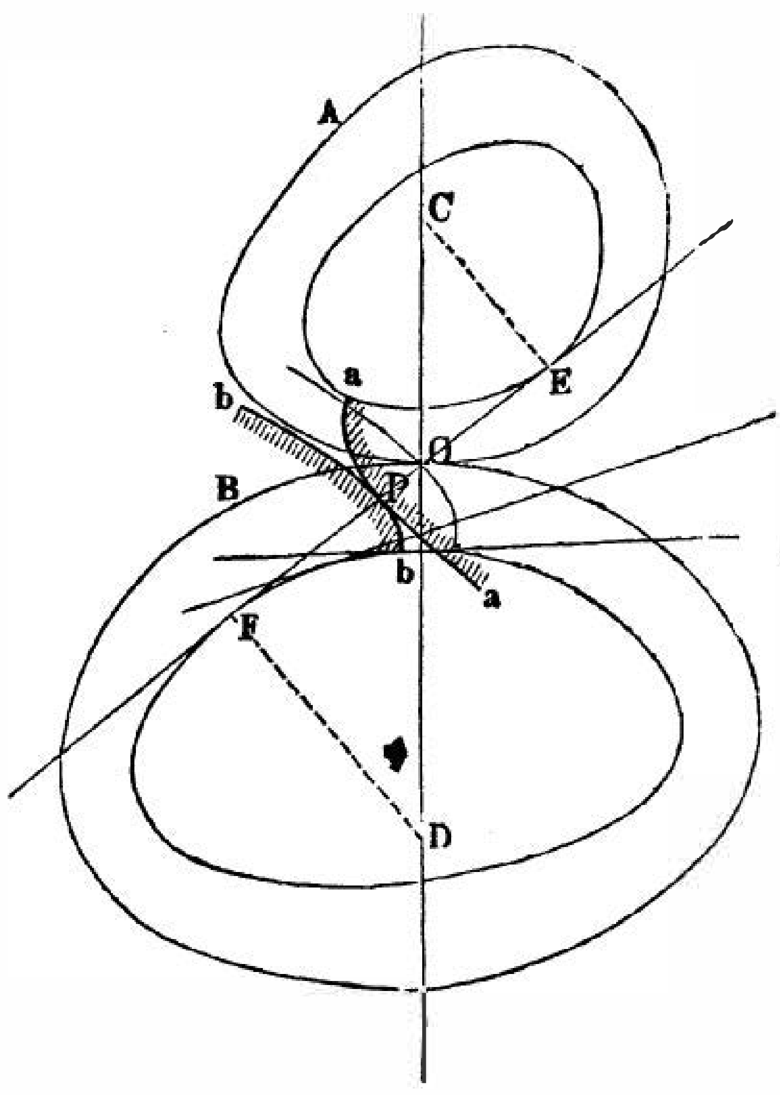


FIG. 109.

If for example C and D be the centres of curvature for the portions of the centroids touching at O, the ratio $\frac{CE}{DF}$ of the perpendiculars is equal to that $\frac{CO}{OD}$ into which the point O divides the line of centres; for any very small motion of the line EF, therefore, the same small angular motion occurs as if the primary centroids were rolling on each other. If now any point P in the straight line describe a curve relatively to A and another upon

B,—$a P$ and $b P$ in Fig. 109 —these must by construction have a common normal passing through the instantaneous centre—the describing line $E F$ itself. Thus the two curves may serve as portions of profiles for the pair of elements which it is desired to construct. In the special case in which the primary centroids A and B are circular, the secondaries E and F are circles also, and the profile-curves $a P$ and $b P$ are involutes of those circles. This gives in spur-wheels the involute teeth which have been sometimes employed. Set-wheels can be made by making the angle $F O D$ constant for the whole series of wheels.

The profile-curves $a P$ and $b P$ are in this case roulettes obtained by rolling a straight line upon the two curves E and F. It must, however, be possible to describe them, as in the former case, as roulettes upon the primary centroids A and B. For circular centroids the auxiliary curve by rolling which upon A and B the involutes $a P$ and $b P$ can be respectively obtained is a logarithmic spiral.* If the middle curve of the three secondary centroids be not a straight line, the roulettes described by its points have not a common normal passing through the point of contact, and therefore are unsuitable for the profiles of elements.

§ 34.

Fourth Method.—Point-paths of Elements used as Profiles.

The auxiliary centroids employed in the second of the methods which we have discussed may take the most various forms. A special case occurs when the auxiliary centroid coincides with one of the primaries. Here it no longer describes a curve in the latter, but each point in it describes there one other point only;—relatively to the other centroid, however, it describes some point-path. If the latter be taken as the profile of an element, the profile of the element with which it works must be a point. This method of constructing profiles has also been used for wheel-teeth. Fig. 110 is an illustration of the contact between teeth profiled in this

* This can be seen without difficulty. A demonstration is given (*e.g.*) by Willis p. 92, another by Haton (*Mécanismes*), p. 101.

The two auxiliary centroids coincide with the two circular primary centroids A and B; $a\,b$ and $b\,c$ are point-paths (here epicycloids) described by the points a and c of the circle B; $d\,e$ and $e\,f$ are epicycloids described by the points d and g of the circle A; $g\,h\,d$ is a portion of a curtate epitrochoid described by the point e of the wheel B; $a\,i\,d$ a similar curve described by the point b of A. Simultaneous contact takes place in a, b and c, the normals to these points of restraint all passing through the point O; after a very small rotation in the one or the other direction the point e comes into contact with $g\,h$, or k with $a\,l$.

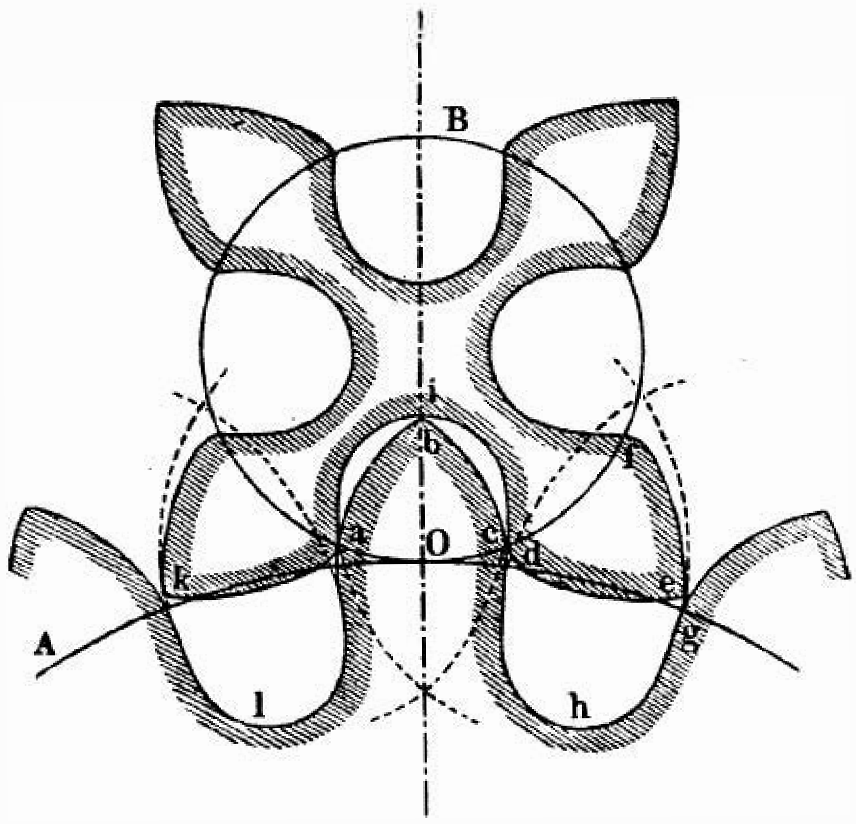


FIG. 110.

It may sometimes be required to combine this method of constructing profiles with one of the others,—such mixed methods are occasionally used in drawing the teeth of wheels.

§ 35.

Fifth Method.—Parallels or Equidistants to the Roulettes as Profiles.

If we have obtained by any of the methods now described the profiles $a\,P$ and $b\,P$ corresponding to the centroids A and B, and from the centre of curvature of the element P of $a\,P$ describe a circle with a radius larger by any amount, $P\,P_1$, than the radius of curvature of that element, and from the centre of curvature of the corresponding element of $b\,P$ a circle with a radius smaller by the same amount, we obtain two circular arcs touching each other on the normal at P_1, and having their normal passing through the instantaneous centre O, in common with the elements at P of the original profiles. This procedure, carried out for every point in the two profiles, furnishes two new profiles, $a_1\,P_1$ and $b_1\,P_1$, which are equidistants or parallels to the first curves, and may equally

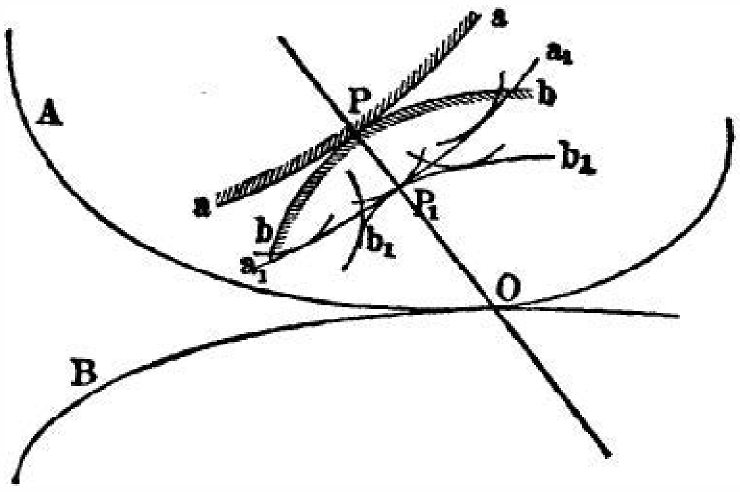


FIG. 111.

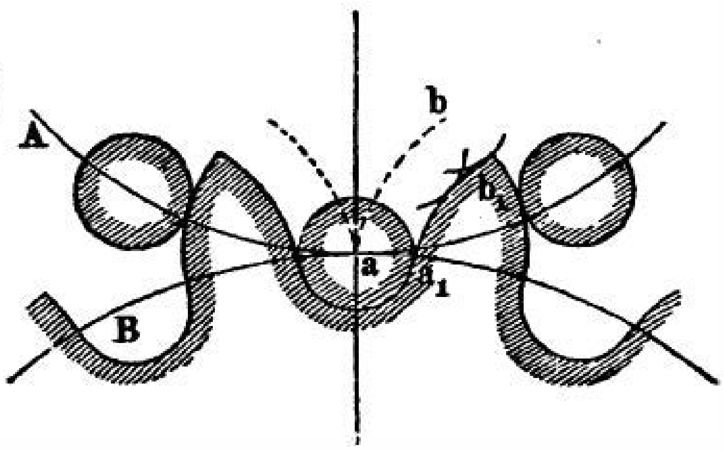


FIG. 112.

serve as profiles for elements.* This gives us a further immense variety of profile forms, which are only limited by the conditions as to usefulness mentioned in section 31. These equidistant curves often possess advantages,—as in the case when they are employed to represent the point-profiles already described. They give us then a circle or circular arc for a profile instead of a point. We have this applied practically in the "pine" teeth of "danterne" pinions, which once were frequently used, and even now are occasionally seen. Fig. 112 is an example. Instead of the point a and the epicycloid $a\,b$, the circle of radius $a\,a_1$ and the line $a_1\,b_1$ equidistant from the epicycloid are employed.

As a further illustration we may employ the forms already treated in another way, the curve-triangle and square. In Fig. 113 $O\,2'\,3'$ is the centroid of the one and $O\,2\,3\,4$ that of the other element of a higher pair, whose profiles we wish to determine. Using the method of § 34 we place in $O\,2'\,3'$ an auxiliary centroid of the same figure as itself. We choose a describing point at R, a point

* In the figure the equidistants are found by drawing arcs about points in the profile with radii equal to the difference between the original and the intended radii, and not by drawing arcs from the centres of curvature of each element with the actual increased or decreased radii in the way described. The method of the figure gives the same result as the latter, and is obviously much more simple.

upon the normal bisector O 3 of the arc $2'\,3'$, at the intersection of the curves $O\,2'$ and $O\,3'$. Relatively to the three-cornered centroid R describes a point only; relatively to the other centroid however it describes a straight line right and left of R, being a point in the circumference of a smaller Cardanic circle rolling within a larger. $R\,C$ and $R\,D$ are these lines and their prolongations. If we complete the rolling of the inner centroid upon the other we obtain the four sides of the square as the profile for the outer element. Only the necessary restraint for the inner element is now required. For this purpose we must find a point homologous to R for each of the two remaining sides of the inner centroid; such points we obviously have in P and Q (as shown

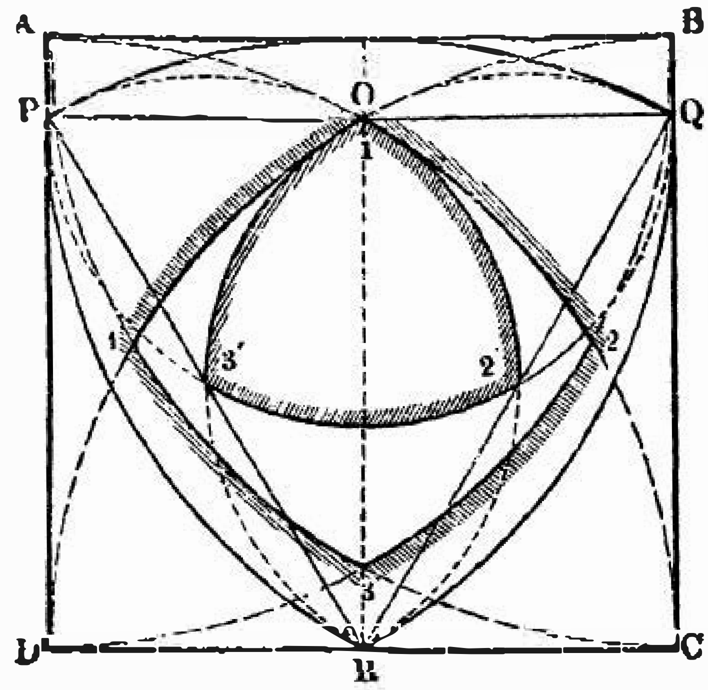


FIG. 113.

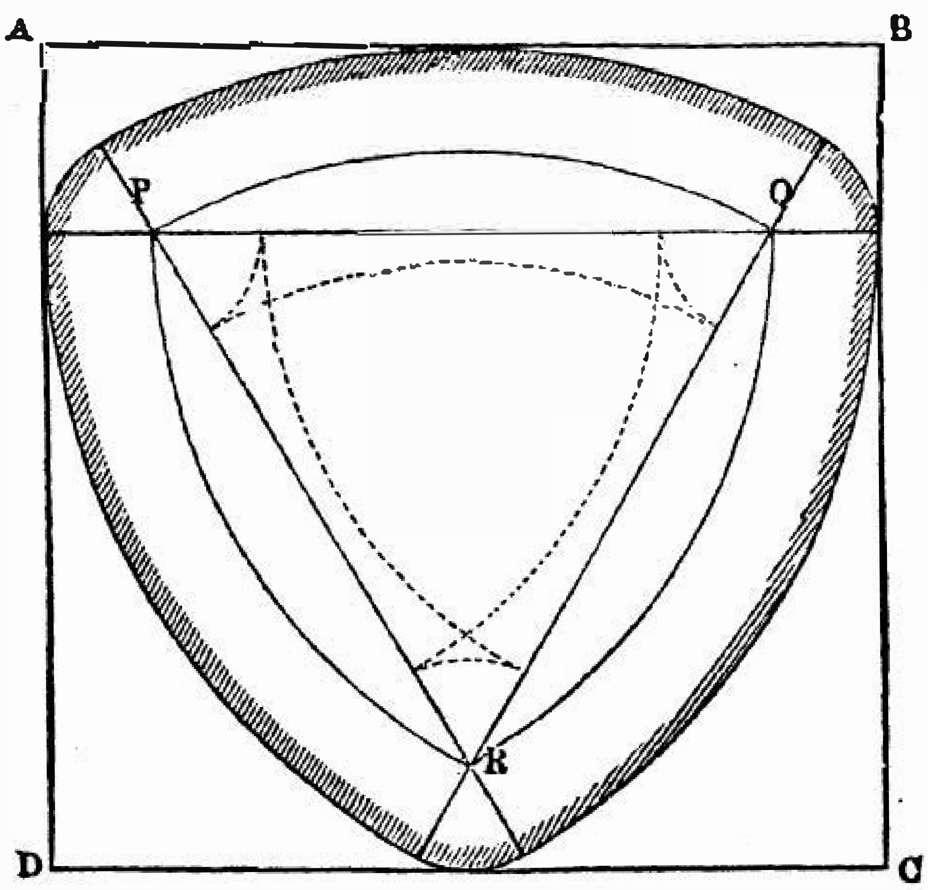


FIG. 114.

by the dotted arcs), and these like R have their paths along the central portions of the sides of the square. To obtain the restraint we have then only to draw from R, P and Q the equi-distant curves $P\,Q$, $Q\,R$, and $R\,P$, and we obtain the curve triangle as a profile.

There is nothing to prevent our choosing larger radii for these equidistants,—in Fig. 114, for instance, they are $1\frac{1}{2}$ times as long as the chord PQ. The profile form obtained only differs from the

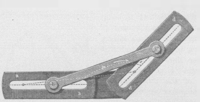


Fig. 113.

former one in having the vertex angles at P, Q and R rounded by arcs having those points as centres. From a practical point of view the resulting profile is a great improvement on the former one, on account of the removal of its sharp edges at P, Q and R. We have before used a similar construction to this in Plates XII. and XIII. If the radii of the equidistants be chosen less than PQ we get unusable forms, such as the one shown in dotted lines.

A third illustration of the use of equidistant profiles is furnished to us by the higher pair of elements shown in Fig. 115, which has already been described. Here the nature of the motion is known from two given point-paths,—the straight lines in which the points b and c move. Equidistants to these lines give us the profiles of the prismatic slots in the piece $a\ a\ d\ d$, while the equidistants to the two points are the circular profiles of the pins b and c. Here we do not even require to know the centroids in order to construct the pair of profiles. They have, however, as follows from § 22, the form of Cardanic circles, or their arcs.

§ 36.

Sixth Method. Approximations to Curved Profiles by Circular Arcs. Willis' Method.

If the profiles of elements be curves of varying radius their construction is somewhat troublesome, and it may be very convenient to represent them by circular arcs. This can always be done where an approximation to the true curve will suffice, and when only a small portion of each curve is used, as is in general the case with wheel-teeth. As substitutes for such portions of the curve suitably chosen arcs of circles having the same curvature are employed: for finding these there are several methods in use. For set-wheels with cycloidal teeth I have recommended* the following method:—*A* in Fig. 116 is a circular centroide—the pitch circle of the wheel for which teeth are to be constructed,—*B* its centre,—*C* and *D* the centres of two equal describing circles (auxiliary centroids) by which the cycloidal arcs *a* and *b*, which it is desired to represent by circles, are drawn; their radius is equal to 0·875 of the pitch. *O* is the point of contact of the circles *A*, *C* and *D*. Make the angle $O\ C\ a$ and $O\ D\ b = 30°$, find the peripheral points a' and b' on the auxiliary centroids opposite *a* and *b*; draw through *a* and *b* a straight line, which it is evident must pass through the point of contact and must therefore be a normal to the elements of the curves at *a* and *b*,—and join a' and b' with the centre *B*,—then the lines $B\,a'$ and $B\,b'$ (produced if necessary) cut the normal in the required centres of curvature *P* and *Q*. The circular arcs for the faces and flanks of the teeth are then drawn beyond and within the pitch circle respectively, joined together on *A* to make a fair profile (as in the figure), and repeated symmetrically round the pitch circle.

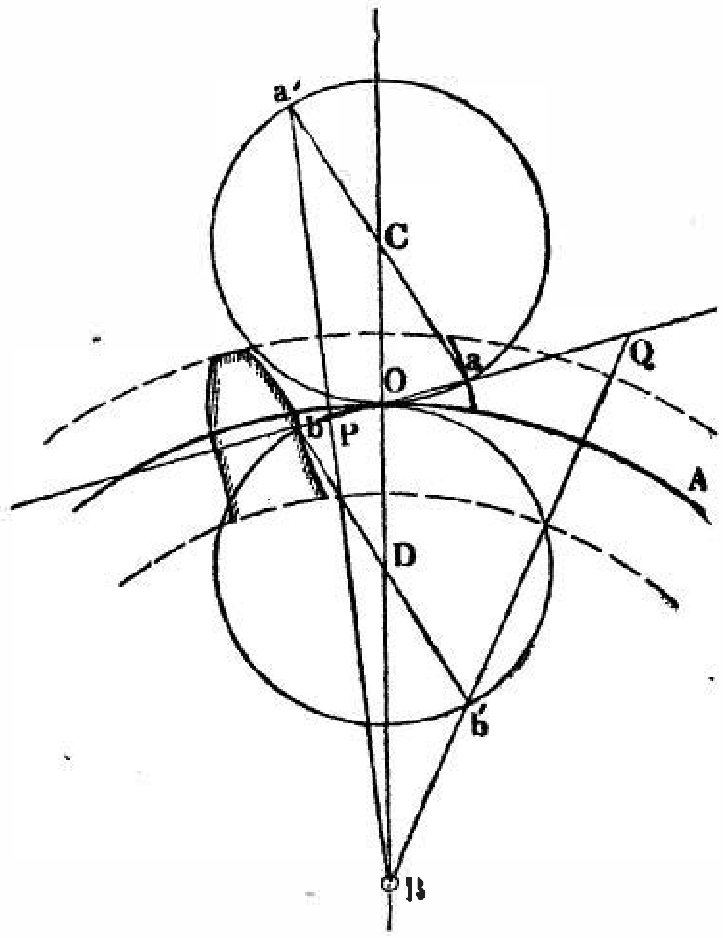


FIG. 116.

In the well-known method given by Willis, he attempts to determine the circles best suited for the teeth profiles directly, that is without the use of auxiliary centroids or roulettes. The nature of his approximation, in which he follows out some suggestions

* *Der Constructeur*, 3rd Ed. pp. 419.

of Euler, is shortly as follows. Let A and B (Fig. 117) be the centres of rotation of two bodies which can drive each other by means of the circular profiles touching in R, and drawn from the centres P and Q; then the intersection O of the two lines of centres $P\,Q$ and $A\,B$ is the point of contact of the centroids corresponding to the relative motion of the figures A and B (see § 8), which therefore have for their angular velocity ratio $O\,B : O\,A$. In order that this may remain nearly constant for a short interval of time, $P\,Q$ must in its motion continue to pass as nearly as possible through O. The instantaneous centre, however, of $P\,Q$ relatively to $A\,B$ is the point C at the intersection of $P\,A$ and $B\,Q$ produced, and if C be a point upon a perpendicular to $P.Q$ at O, as C' in our figure, then the instantaneous motion of $P\,Q$ will be in fact through the point O. If therefore one of the centres, as P, be chosen, the position of the other must be the intersection Q' of $P\,Q$ and the line $B\,C'$. We thus obtain in the distance $P\,Q$, or rather $P\,Q'$, the sum of the required radii of curvature, but may take the point of contact R in any position, as R' for example, as follows from what we have said in § 35.

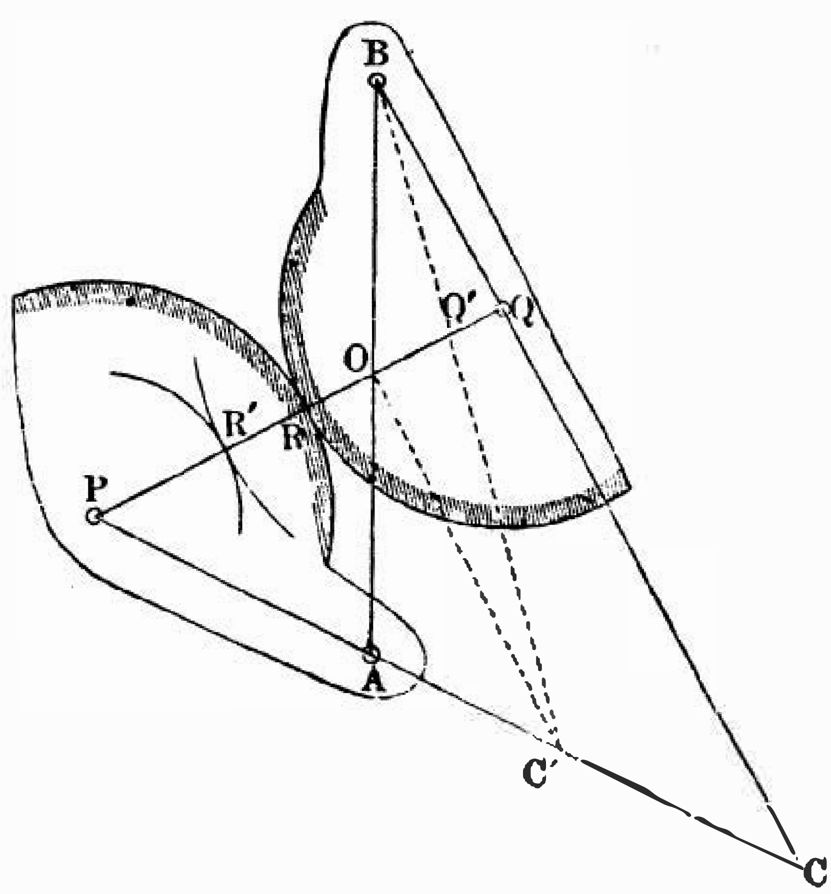


FIG. 117.

In order to adapt this elegant method to set-wheels, Willis chose three constant magnitudes, the distances $O\,C'$ and $O\,R$ and the angle $P\,O\,A$; the latter he made 75°. If the teeth were to be profiled by one arc only he took $O\,C' = \infty$, $O\,R = 0$, the circular arcs becoming approximations to involutes (see § 33). If two circular arcs were to be used, joining at the pitch line into an S-shaped figure,—as is usual,—the method was applied twice over, once for the portion of the tooth on each side of the pitch circle. Fig. 118 shows this;—$O\,R'$ and $O\,R''$ are each equal to half the pitch, and $O\,C'$, $O\,C''$ are so taken that for a pinion of twelve teeth the flanks become radii. This occurs if $O\ C' =$ the pitch $\times \frac{6}{\pi}$ sin 75°. A closer examination shows this construction to be identical with that given in Fig. 116, when equal constants are used. $R'\ C'$ and $R''\ C''$ are the diameters of our auxiliary centroids, which are here dotted so that the constructions may

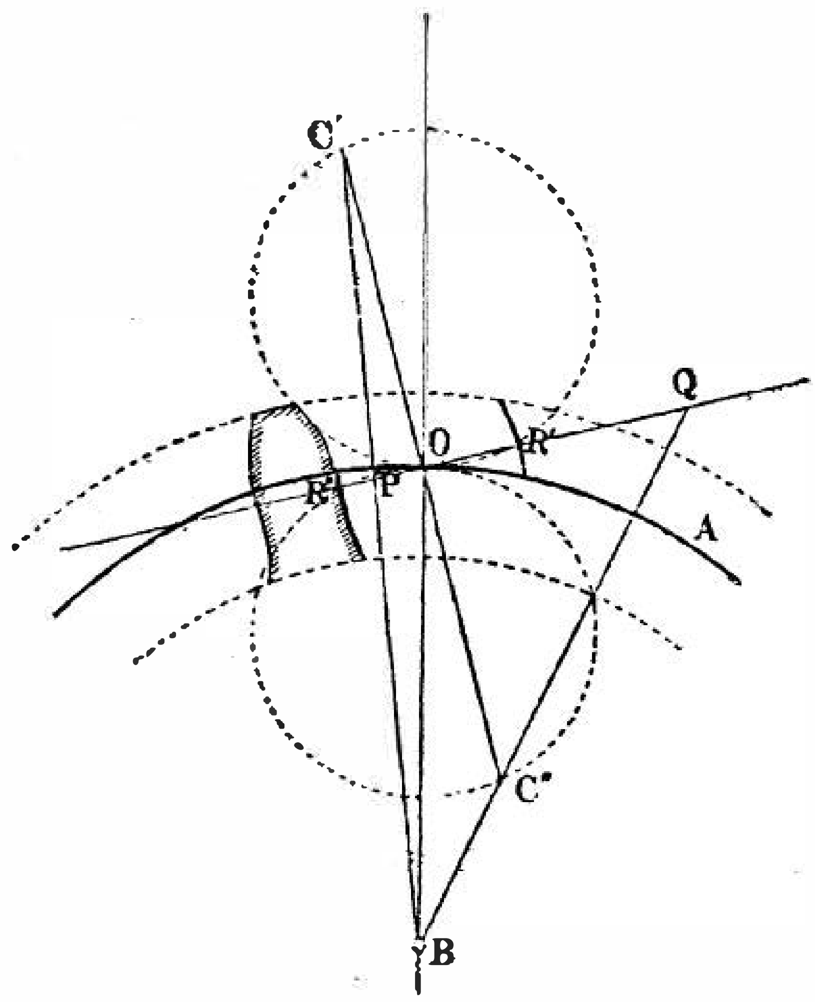


FIG. 118.

be compared; the points C' and C'' correspond to a' and b', P and Q are found as before. Willis * himself recognised and mentioned this coincidence.†

§ 37.

Seventh Method.—The Centroids themselves as Profiles of Elements.

If we suppose the auxiliary centroids of the method of § 32 to be closed figures, and to be made smaller and smaller until they

* *Principles of Mechanism*, 1st Ed. p. 107, 2nd Ed. p. 142.

† The approximation proposed by Professor Unwin, and described by him in *Engineering*, May 29, 1874, is perhaps more exact than either of these. He finds two points in each roulette (epi- or hypo-cycloid), one upon the pitch circle, and one

become points only, the paths they describe in rolling become the centroids themselves. These may be used as profiles if the cylindric axoids bearing them be so pressed together that sliding at the point of contact is prevented by friction, so that they are compelled to roll on one another. This is the only case where the profiles of elements have a pure rolling motion. Circular centroids give us cylindric wheels, like those known as friction wheels (Fig. 119). The applications of this method are not few, the most important and most familiar being perhaps the wheels of railway carriages. We shall in the next chapter consider in detail the subject of axoids pressed together, or restrained, by a force.

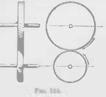


Fig. 119.

§ 38

Generalisation of the foregoing Methods.

In the foregoing paragraphs we have throughout limited ourselves to axoids for general cylindric rolling, but the methods employed are equally applicable to the case of non-cylindric axoids. For conic axoids this is easily seen, but not so easily with the higher rolling and twisting axoids (see § 13). Considerable difficulties appear here in the theoretic examination even of motions occurring according to simple laws, and still greater difficulties in their practical presentation. It is a part of Applied Kinematics to consider so far as may be necessary the more important cases. It must be admitted that in general the actual forming of profiles for the higher axoids, even those for instance for the teeth of hyperboloidal wheels, presents as yet considerable

upon a circle larger or smaller than that by a distance equal to two-thirds of the intended depth of the face or the flank of the tooth. The first of these points is assumed and the second found from it by a very accurate approximation. A normal to the roulette at the second point is drawn, and the intersection of this line with the normal bisector of a line joining the two points gives the centre for the circular arc to be substituted for the roulette. This arc passes through both the points, and has a common tangent with the roulette at one of them difficulties, a circumstance which helps to explain the existing imperfect comprehension of those even which have been hitherto made. The modern sewing-machine manufacture, and in part also that of agricultural machinery, have empirically and unknown to themselves made very satisfactory progress in the employment of the higher axoids;—the former with special skill, for it has already brought to considerable perfection the formation of complex enveloping surfaces.

The illustrations which we have used in the foregoing paragraphs have been in great part, although not entirely, drawn from the methods used for constructing wheel-teeth, and must be therefore more or less known, if not entirely familiar to those readers who have made machine-construction a subject of scientific study. The methods of procedure, however, deserve renewed attention, for they have now been developed in the special light of the general fundamental principles upon which they rest. The question has here been treated as one not of rules for constructing wheel-teeth, but of their general correspondence to a great principle. We found that by a quite small extension of the ideas contained in them, methods are available generally which are commonly stated and understood as very limited rules. I trust therefore that such previous familiarity with particular instances will only have made it more easy to understand the general case and their relation to it.

I hope now to have made completely intelligible the fact that the construction of pairs of elements is possible for any motion, however complex,—that is, that in all cases suitable profiles can be determined for those elements. We have seen also that this problem may be solved in an immense variety of different ways, even in the case of the simpler motions and those more often occurring. While in former centuries the most distinguished geometrician occupied himself with separate solutions of detached problems, and necessarily regarded them as important propositions, he has to-day presented to him a limitless perspective, which appears almost more simple in its universality than the single case appeared before, and which affords rich opportunities to the practical mind in the determination of the best solution among the immense number of possible ones.

Perhaps I must fear that I have wearied my readers by these investigations, in which we have apparently progressed more point by point than step by step, and have rather courted than shunned the difficulties of the problem. By degrees, however, we have completely proved the special laws upon which constrained pairs of elements can be constructed, and this was the end we had in view. These laws are not simple,—not lying on the surface,—but they are fixed—and within the assumed limits universally applicable—general laws. It may be well on this account to pause for a moment and to refer the reader once more to the nature of the ideas on this subject which have hitherto been held and which were sketched in the introduction,—and especially to the somewhat extended system of Laboulaye of which I there promised further mention.

Let us ask ourselves what it is that Laboulaye's three systems levier, tour, and plan, which bear so great an apparent impress of geometric generalization, really represent. We must look at this question in the light of the special acquaintance with the subject which it has been our object in the foregoing chapters to obtain.

In the first system the moving body has one fixed point,—(*le corps a un point fixe*), in the second two fixed points (*le corps a deux points ou une droite fixe*), in the third three fixed, or rather restrained points (*l'obstacle consiste en trois points fixes ou en un plan passant par ces trois points*). We remark in the first place that Laboulaye bases his classification not upon kinematic chains, into which, as we have seen, the machine separates itself, but upon the pairs of elements themselves. For he speaks always of the restraint of a single body, not of one forming a part of a whole sytem of bodies. Let us then limit ourselves to this, although Laboulaye himself considers his system to represent machines generally. Which pair, however, has only one fixed point? We have seen (§ 5. IV.) that the fixing of one point only leaves the motion of the body to which it belongs quite indeterminate; that neither a constrained chain nor a constrained pair of elements can be so formed. Laboulaye, however, chose as his illustration a swinging lever, one, that is, which turns about an axis. His system "levier" therefore coincides with the second of our lower pairs of elements, the turning-pair (§ 15). But such a pair requires not one, but at least six points of restraint. It may of course be said that the fixed point in the système levier represents a geometrical axis, and that Laboulaye's meaning, strictly rendered, is that two points in this axis,—which might be considered as a kind of idealisation of the body,—should be prevented from altering their position; the two points coinciding of course in their normal projection. This cannot, however, be the meaning, for it is the description Laboulaye himself gives of his second system, the système tour,—he has therefore not fallen into this error; he always speaks, too, of bodies, and not of their ideal representation by axes. But if any body is to be restrained we know that it must have a definite form, and that, being suitably formed, it must be restrained at at least six points. If we have one fixed point only the body must be spherical,—it will require at least four points of restraint, and the motion which occurs is only so far constrained that the centre of the sphere cannot alter its position, and that the other points must move on spheric surfaces;—with this limitation however they may have any possible motion.

To proceed:—Laboulaye includes under système levier those pairs of elements which move by conic rolling, § 11. That also is said distinctly. He does not however leave his chosen illustration the lever. "Le mouvement d'un point quelconque, appartenant au levier, sera de nature circulaire, en chaque instant et de plus en général alternatif dans une machine, se produisant le plus souvent dans un plan." We see that this definition fails altogether in clearness and certainty. Apparently it shadows forth in dim outline a pair of elements with a swinging motion,—its appearance of deep and categoric generalisation has sometimes brought it into favour with mathematicians; it falls altogether to pieces, however, on a closer examination.

Nor can the two other systems, tour and plan, fare better. In no one of the three systems is it made completely clear on the one hand what in strictness is meant by point fixé or plan inébranlable, or on the other hand what it is that distinguishes absolutely one system from another. Let us reverse the question, and attempt to find in which of Laboulaye's classes one or other of our higher pairs of elements must be placed. We may choose the curve-triangle and square for instance. About this pair we know that if the square be fixed, then on motion taking place no point of the triangle remains in its original position. Every point in it moves. According to Laboulaye at least one point must remain stationary. The pair might perhaps be best placed under the système plan, the surfaces of the triangle being supposed to be caused to move always in the same planes,—which are therefore such plans inébranlable as are considered to be peculiar to this third system. We might look in this direction for the real essence of the system, but then the turning pair would also on these grounds have to be put in the same category, while it must be claimed on the other hand as distinctly belonging to système tour. So we lose our clue, and fail in discovering what the real difference between things varying so essentially is,—the very thing which it was and must be the object of the "system" to point out.

But I will stop: it is evident that Laboulaye's position is untenable. The question here, too, is not one of criticism of Laboulaye himself; many other writers have followed him without putting his ideas sufficiently to the proof, and they would therefore be open to the same criticism. That I do not undervalue Laboulaye will be seen from the Introduction, where I have paid my tribute of recognition to him as an investigator. I do this the more freely, that he has not drawn any special deductions in the applied part of his work from his first propositions,—deductions which would necessarily have led to error. I have been desirous only to show upon what insecure and feeble supports it has been attempted to build up a science of Kinematics, a science having ostensibly a logical basis of its own. I wished to place again before the reader, in a form that could be readily grasped, a proof that if anything whatever is to be accomplished by means of axiomatic propositions, they must be subjected to, and be able to bear, the most inexorably strict, exact and penetrating examination.