CHAPTER II.

PHORONOMIC PROPOSITIONS.

§ 4.

Preliminary Remarks.

THE science of Mechanism is a derived science, and has—as we have already said — its foundations in Applied Mechanics and Mathematics. It chooses from the first a special part, in considering chiefly those motions which occur through latent forces, and are therefore conditioned by the geometric properties of the pieces transmitting these forces. Its problems therefore lie, for the most part, in a well defined portion of Mechanics—the geometrical. This department does not always receive the same name in scientific books; it is best called Phoronomy. Frequently it is simply called Kinematics, but this seems to be a misunderstanding of the word. Ampère at least, who invented the name, did not intend it to be used in this sense. It is at the same time unnecessary so to employ it, for Phoronomy is quite sufficient, and is besides more distinctive than Kinematics. Phoronomy is, therefore, the study of the measurement of the motions of bodies of every kind, and has become specially the study of the geometric representation of motions. We therefore retain the designation Phoronomy, and as we shall frequently have to employ its propositions, especially so far as they concern rigid bodies, it will be well here to review a portion of them.

The purely mathematical part of the phoronomic propositions coming thus into consideration forms what Professor Aronhold has called "Kinematic Geometry." It is, however, really a part of Phoronomy, and there does not appear any reason why it should not receive that name. If along with this the masses of the moving point-systems be also taken into consideration, we have "Phoronomic Mechanics." This is unquestionably what the later French writers (see p. 16) mean by "pure Kinematics." I shall be glad if my view of the matter be generally accepted, so that some sort of common understanding may be reached as to the general directions of these studies. It is greatly to be wished that some end could be brought to this multitude of new names.

Until recently only the lower and simpler part of Phoronomy, that namely which relates to the motion of a point, has been systematically taught in our German Polytechnic Schools; and this has been so frequently treated in text-books as to be familiar to all practical mechanicians who take any interest in the theory of their subject. Problems connected with point-systems have been only occasionally treated—in the text-books familiar to practical men they appear but seldom, and then rather as interesting corollaries than as important problems. These problems, however, are of the highest importance in Kinematics, and it is only to them that we need turn our attention here. I must therefore suppose that the following propositions, here taking their place in our investigations, will be in great part new to my Engineer readers. One of the most important characteristics of the method of treatment we shall employ is that it enables us to make the progressive changes of position visible in form to the imagination; I have tried, wherever it has appeared possible, to develope this conception still more fully than has hitherto been done.

§ 5.

Relative Motion in a Plane.

We are unable to grasp with our senses the absolute motion of a point; we observe only the relation of its successive positions to other points or bodies in our neighbourhood. This relation, when known for every instant, is called the relative motion of the point—or if instead of a point we have a body, the relative motion of the body,—to the portion of space surrounding us. If this were itself motionless, the absolute and relative motions of the point would be identical, but if not, they must differ. Absolute motion in the universe being of no importance in our investigations, we may limit the meaning of the term absolute motion, and understand by it only motion relative to the portion of space in which our observations are made,—to the earth, for instance, or to a ship or a train.

We shall examine first the case in which this portion of space is a plane merely, the motions to be considered being thus motions in a plane.

Prop. I.—The motion of a point P relatively to another point Q in the plane $P\,Q$ takes place along the line $P\,Q$ which joins those

FIG. 13.

points, no matter what motion the point may have relatively to the plane itself. The motion of P relatively to Q and of Q to P is known when the distance $P\,Q$ is known for every instant. This first proposition is not limited to motion in a plane, but is entirely applicable to the general motion of two points in space.

Example.—The motion of the centre (P) of a planet relatively to that (Q) of a body around which it revolves in any orbit, is an oscillation along the line $P\,Q$ joining their centres.

Prop. II.—The motion of a point P relatively to a plane in which it moves* is known if its motion relatively to two fixed points A and B in the plane of motion be given.

The path of motion is then the locus of the vertex P of the triangle $A\,P\,B$, which takes, for instance, the position $A\,P'\,B$ (Fig. 14).

Example.—The motion of any point P in an ordinary connecting-rod relatively to the plane in which it swings, is a curve which can be

* In Fig. 14 the plane of the paper.

determined by distances measured from any two points in the section of the frame traversed by that plane.

Prop. III.—The motion of a plane figure relatively to a plane in which it moves is known if the motions of any two of its points P and Q (Fig. 15) relatively to two fixed points A and B in the plane of motion be given. For when the positions of P and Q are known, the positions of every other point in the moving figure

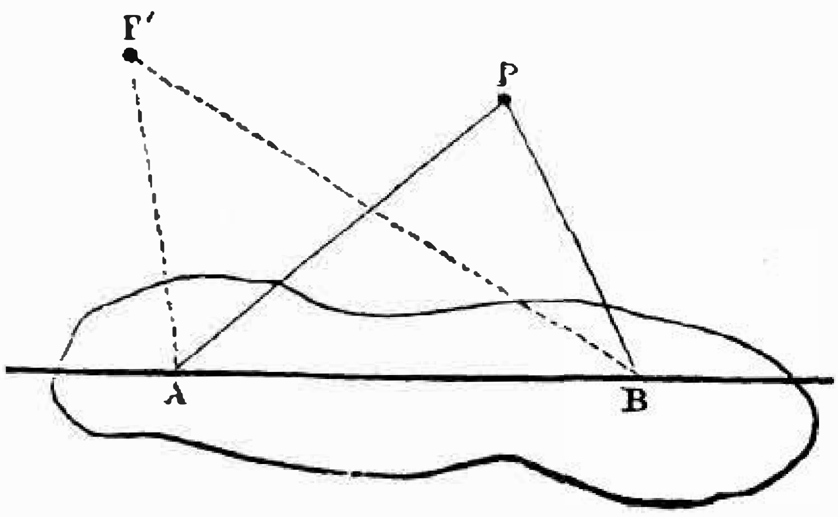

FIG. 14.

can be found by considering it as the vertex of a triangle of which the position of the base and the lengths of all three sides are known. The motion of any plane figure may therefore be expressed by that of any line in it. The motion of the line PQ relatively to the line AB is therefore the same as that of the figure PQ to the figure AB.

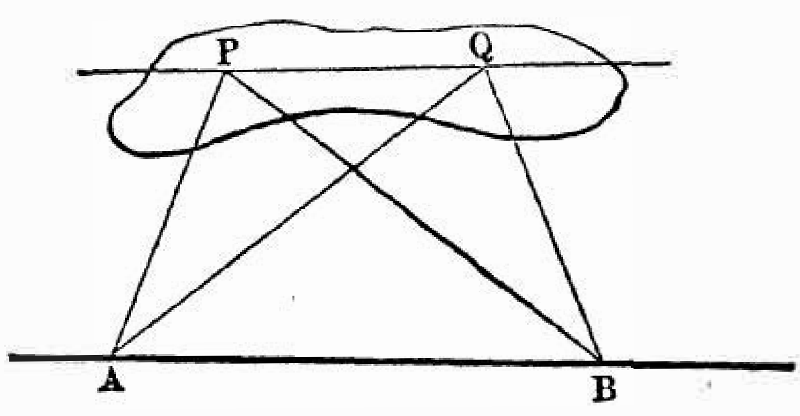

FIG. 15.

Example.—In order to determine the motion of a plane equatorial section of any planet relatively to the plane of the solar system (assuming the section to move in that plane), the motions of at least two points in that section relatively to the plane of motion must be known. Visible inequalities on the surface of a star, or a spot on the sun, may be supposed to furnish points relatively to which the motion of the section can be observed.

Prop. IV.—The motion of a plane figure PQ relatively to one point A in its plane of motion is expressed by that of the points P and Q relatively to A, and this, as we have seen (Prop. I), takes place along the lines joining P and Q with A. But the position of these lines in the plane remains indeterminate, so that complex motions in the plane may occur without any alteration in the lines defining the motion relatively to the given point. [Thus in Fig. 16 the triangle PQA is similar and equal to $P'Q'A$, so that the position of PQ relatively to A is the same in both cases; its position in the plane is, however, very different. The same is true of P^2Q^2 and P^3Q^3.]

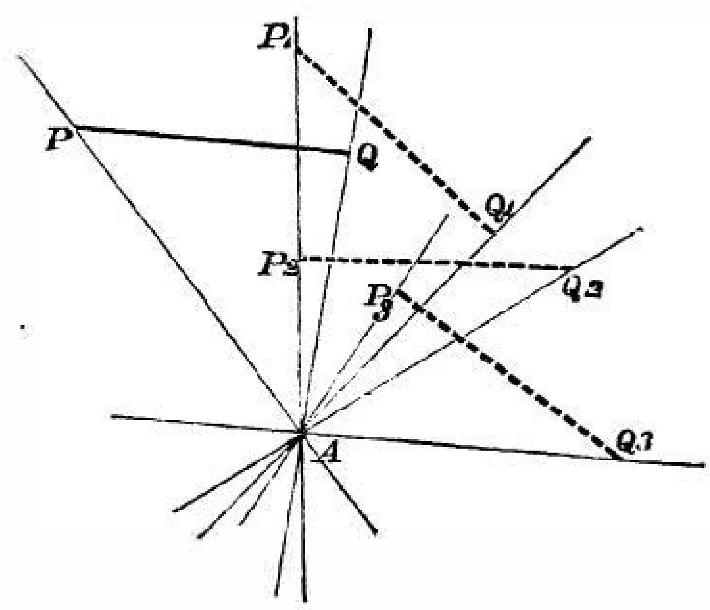

FIG 16.

Example.—A kinematic chain adapted for motion in a plane, but having one point only fixed in that plane, gives no determinate absolute motion, although the motion of every point in it relatively to the chain may be absolutely fixed, the chain being closed. The apparent contradiction that certain recent parallel motions require only one fixed point arises from a misapprehension of their nature. If only one point, or more strictly one axis, of such mechanisms be fixed, no "constrained" motion of the other parts occurs.

§ 6.

Temporary Centre; the Central Polygon.

The foregoing four propositions constitute the groundwork of the Phoronomics of point-systems, and are in certain respects exhaustive. They give, however, no distinct idea of the way in which the relative positions assumed by the moving point or body follow one another. We must now examine this more closely.

If for any plane figure two positions PQ and $P_1 Q_1$ in the same plane be given, the figure can in every case be moved from the one position to the other by turning about some point O in the plane, which can be determined by joining PP_1 and QQ_1 and finding the intersection of perpendiculars drawn from the middle points of these two lines. This intersection, O, is the required point, because the two triangles $O P Q$ and $O P_1 Q_1$ are similar and equal, $O P$ being equal to $O P_1$ and $O Q$ to $O Q_1$. The point O is called the temporary centre for the given change of position.

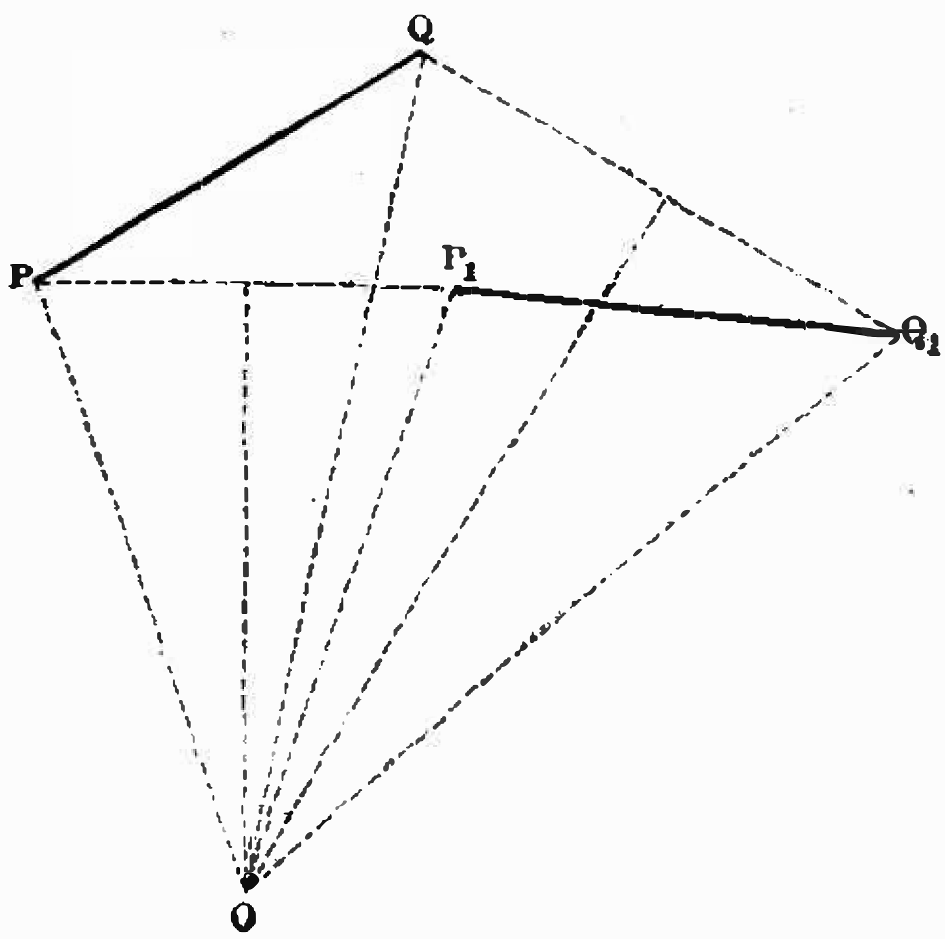

Fig. 17.

If the temporary centres for further changes of position,—from $P_1 Q_1$ to $P_2 Q_2$, $P_3 Q_3$ and so on, be determined in the same way, we obtain a series of points O O_1 O_2 O_3 etc., which may be joined together by straight lines. We thus obtain a polygon which has the centres for its corners, and which we may therefore call a central polygon. If the figure PQ return into its original place after a series of changes of position, the polygon is closed, otherwise it is open. The figure itself in every case makes a series of turnings about the temporary centres,—its points, that is, move always in arcs of circles; these are completely determined if the angle through which each single turning takes place be given. This angle we must therefore look at more closely.

Rotation about the centre O extends through the angle $P\,O\,P_1 = \phi_1$. It will help us in our examination of the matter if we suppose the line $M\,M_1$,—which is equal to $O_1\,O$, and which is so placed that $\angle\,O_1\,O\,M_1 = \phi_1$, the point M coinciding with O,—to be rigidly connected with $P\,Q$. Then in the first turning the line

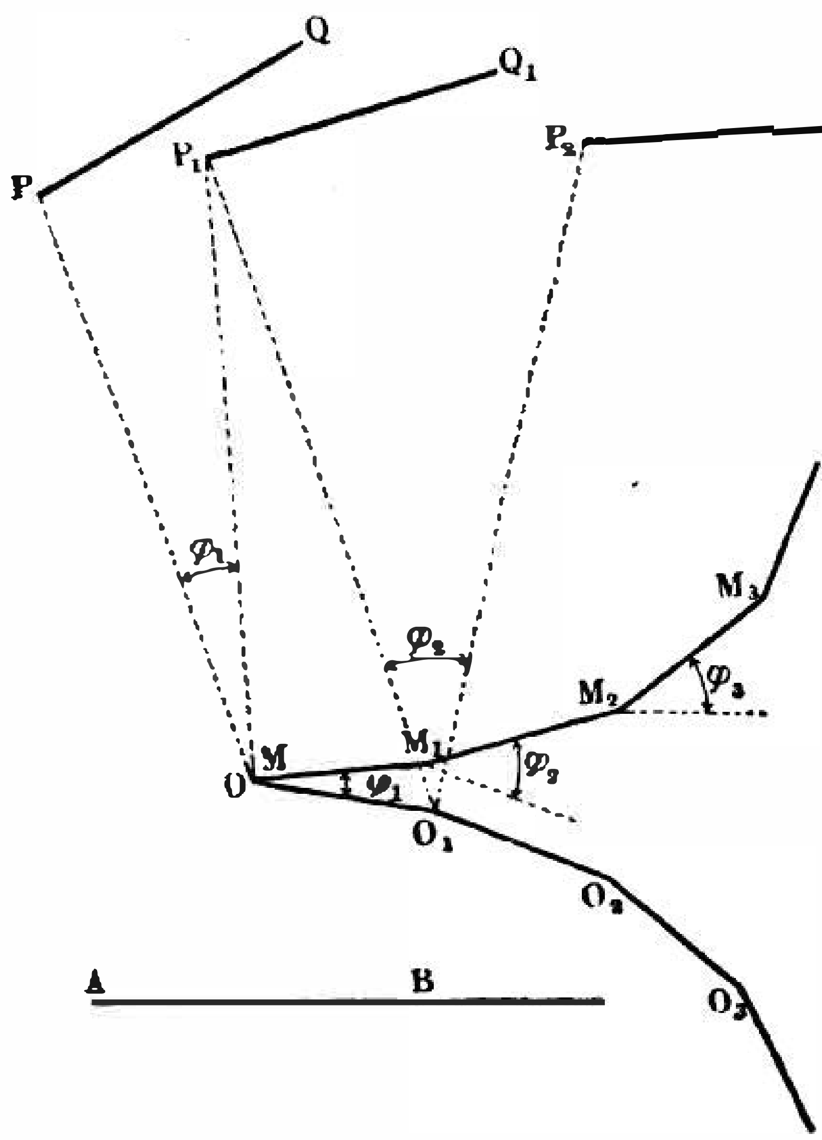

FIG. 18.

$M\,M_1$, turning about O, will take the position $O\,O_1$, and at the same time $P\,Q$ (with which it is rigidly connected) will be moved as before into the position $P_1\,Q_1$, so that so far as the determination of the motion is concerned, $M\,M_1$ may replace $P\,Q$. If we repeat the whole process for the rotation about O_1, by joining the line $M_1\,M_2$ $(=O_1\,O_2)$ to $M\,M_1$ in such a way that when M_1 coincides with O_1, the angle $O_2\,O_1\,M_2$ is equal to ϕ_2, we can again replace the figure $P\,Q$ by $M_1\,M_2$, or rather by the polygon $M\,M_1\,M_2$. In this way a second polygon, $M\,M_1\,M_2\,M_3$, can be found, which by the consecutive turnings of its corners about the corresponding corners of the first polygon will give to $P\,Q$ the required changes of position relative to the fixed plane, or to any stationary figure, as $A\,B$, lying in it.

If we examine the relation of the two polygons to each other we notice the special and important peculiarity, that each has the same properties in reference to the other, that is, that they are reciprocal. Thus in any one of the positions in which two corresponding sides coincide, the polygons show not only the position of the figure supposed to be movable relatively to the fixed figure, but also conversely the position of the fixed relatively to the movable figure. (Prop. III. of § 5.) We can thus determine as many relative positions of the two figures by means of the central polygons as the latter have pairs of corresponding sides.

§ 7.

Centroids; Cylindric Rolling.

The method above described affords us the means of representing a succession of given separate positions of two figures. It leaves, however, the actual changes of position undetermined, or substitutes for them a series of rotations about isolated points. But if we suppose the assumed positions $P\,Q$, $P_1\,Q_1$, $P_2\,Q_2$, etc., taken nearer and nearer together until at last the intervals between them disappear, we shall have a complete representation of the whole motion. The corners of both the central polygons will at the same time have approached each other until each is removed from its neighbour by only an indefinitely small distance, and thus the two polygons become curves, of which infinitely small parts of equal length continually fall together after infinitely small rotations about their end points,—which, that is to say, turn or roll upon each other during the continuous alteration in the relative positions of the two figures. The turning which takes place about each point in the curves is not now, as before, temporary, but in general for an instant only, and each point is therefore called an instantaneous centre. The curves into which the polygons are transformed both pass through the whole series of instantaneous centres point by point, and on this account we may call them the centroids of the moving figures. If these be known for any given pair of figures, their relative motions for a series of positions infinitely near together are also known,—their changes of position are completely determined, and can be found by rolling one of the centroids upon the other.

It will be evident from the foregoing that in general the relative motions of two plane figures to each other are not alike, for none of the conditions of the problem necessitate the similarity of the centroids;—whenever the centroids are similar, however, the relative motions become the same.

Example 1.—The construction of trochoids illustrates the relative motion of plane figures of which the centroids are known. If a circular cylinder roll upon a plane, the normal sections of both figures move in a common plane, and therefore come within the conditions of our problem. The circle PQ and the straight line AB (the forms of these sections) are the centroids both of the two figures and of all figures or points connected with them. All points of PQ describe linear trochoids[13] relatively to AB; these being common, curtate, or prolate, according to whether the point lies upon, without, or within the circle. All points connected with AB describe involute srelatively to PQ; these again being common, curtate, or prolate, according as the describing point lies upon, beyond, or within the straight line.

Example 2.—Two equal circles rolling upon one another have the same relative motions; points in both at equal distances from their centres describe equal epicycloids.

Our examination applies generally to the relative motions of plane figures in a common plane, or, as we shall in future call them shortly, con-plane figures, and the result of it may be summed up as follows:—

All relative motions of con-plane figures may be considered to be rolling motions, and the motion of any points in them can be determined so soon as the centroids of the figures are known.

If solid bodies be laid through the supposed figures PQ and AB, and rigidly connected with them, then every pair of sections of such bodies which (like the pair of figures) lie parallel to the plane of motion have a pair of centroids identical with theirs. The series of centroids, which we may suppose in this way to be lying closely one behind the other, form together two cylinders (in general non-circular), which always touch along one line, and turn or roll upon one another. Each line in which the cylinders come in contact for an instant is for that instant the axis of rotation, and is called therefore the instantaneous axis of the motion. The motion itself in such a case is called a cylindric rolling. We may extend the law just enunciated for plane figures equally to this relative motion of solids. The characteristic of a series of reciprocal positions of a body undergoing cylindric rolling is that its sections normal to the instantaneous axis are figures which remain always con-plane during the motion. We can therefore say:—Every relative motion of two con-plane bodies may be considered to be a cylindric rolling, and the motions of any points in them may be determined so soon as their cylinders of instantaneous axes are known.

§ 8.

The Determination of Centroids.

With the transition from irregular to continuous motion the perpendiculars (see Fig. 17) upon the lines joining pairs of consecutive positions of the points P and P_1, Q and Q_1, become normals to the curve-elements in which at the given instant the points P and Q are moving. In order therefore that the centroid for the motion of a figure $P\,Q$ relatively to another $A\,B$ may be known, there must be known for every position of $P\,Q$ the directions in which at least two of its points are moving, that is the position of the tangents to their paths. The normals to such tangents for any number of points in the moving figure all intersect in the same point, from which it follows that only one pair of centroids is possible for any relative motion of con-plane figures.

Centroids can always be found by determining separately a sufficient number of points in them, and often by a general investigation into the nature of the curves to which they correspond. We shall examine briefly both methods of determination.

Let the relative motion of any two con-plane figures $P Q$ and $A B$ be known. In order to find the corresponding centroids we must first convert this given relative motion into an absolute one (in the limited sense in which we use the word), by supposing the system as a whole to receive such a motion that one of the figures, *e.g.* $A B$. (Fig. 19), comes to rest in reference to ourselves. We can then find the paths or curves in which

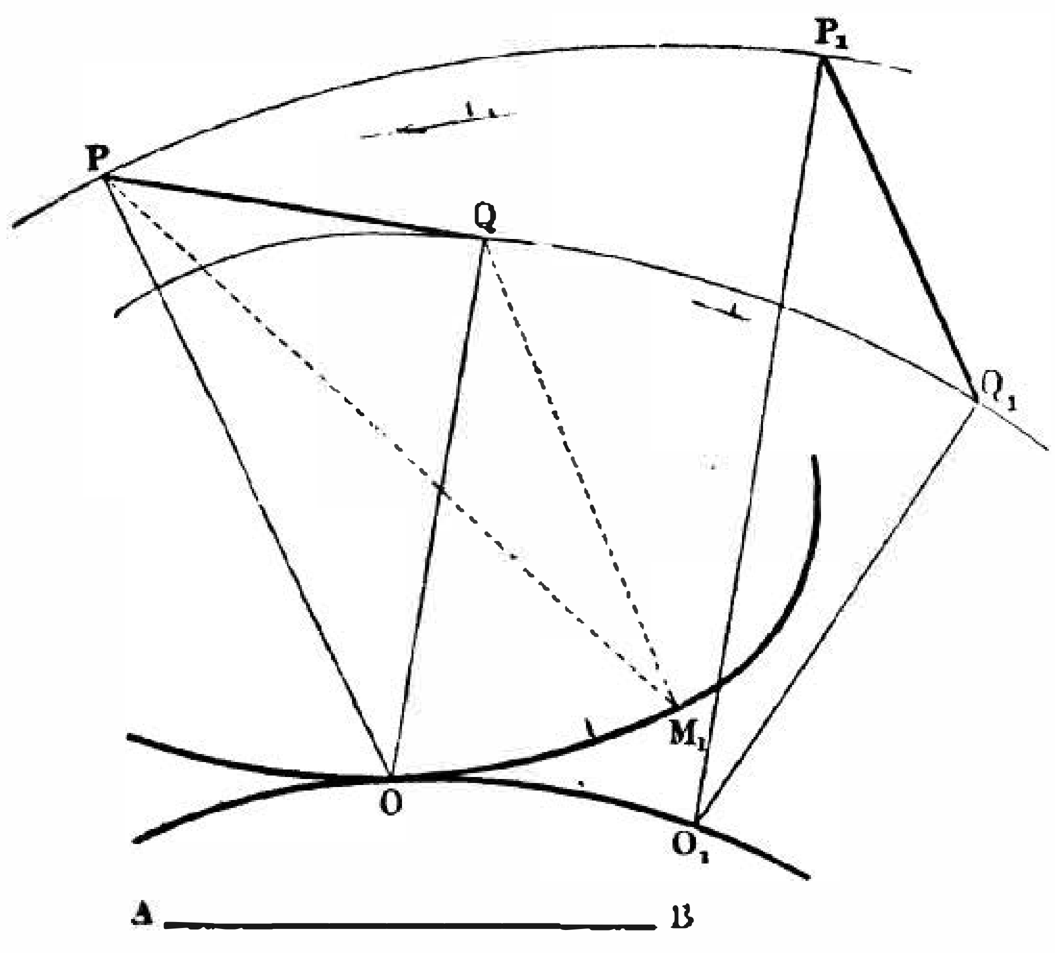

FIG. 19.

two points P and Q move, and draw normals to those paths at P and at Q; their intersection gives a point O in the centroid belonging to $A B$. Another pair of normals, drawn from P_1 and Q_1 give another point O_1 of the same centroid, and so on. The second centroid, $O M_1$ . . . . . can be found in a similar way by bringing $P Q$ to rest and making $A B$ the moving figure,—but it may be determined more easily as follows. The pole O_1 is a point in both curves when the moving figure is in the position $P_1 Q_1$, its distance from the point P in that figure is $O_1 P_1$ and from Q, $O_1 Q_1$;—it therefore is necessary only to describe arcs of circles about P and Q with the radii $= O_1 P_1$ and $O_1 Q_1$ in order to find the point M_1 in the second centroid which corresponds to O_1, and which must be at their intersection.

Example 1.—The nature of the kinematic chain (Fig 20) discussed in § 3 allows none but con-plane motions; it may therefore serve us here for an illustration. Let it be required to find its centroids. As each of its links can have a motion relatively to all the three others, there are in all six pairs of centroids belonging to the chain, four for the motions of adjacent and two for those of opposite links. The four first are very simple, each curve being a point only; the two others are not so readily found. We will here examine the pair of centroids belonging to the links a —— h and d —— e. For this purpose we first bring the link a —— h to rest (we may suppose it connected with a fixed pedestal, as shown in Fig. 21); then a —— d rotates about a, while e —— h swings to and fro in circular arcs about h. The centres of the elements c and f

FIG. 21.

describe therefore paths to which the normals are always radii passing through the centres of a and h. By producing these radii until they intersect we can consequently obtain any number of points in the centroid of the fixed link a —— h. The curve found in this way is shown in Fig. 22. O or M is the pole for the original position a —— d —— e —— h obtained by producing a —— d and h —— e until they cut each other. The whole figure $O\,O_1$ . . . . O_4 is not simple in form. It contains four infinitely distant points, corresponding to the two parallel positions of a —— d and h —— e. The second centroid, that of the link d —— e, drawn in the way above described, is shown in $M M_1$ . . . . M_4; it also contains, necessarily, four infinitely distant points. The two centroids which in the engraving touch at $O\,M$, roll upon one another as the mechanism moves ($O\,O_1$ . . . . O_4 remaining stationary), and supply completely the means of examining the whole complicated motion of the link d —— e. As regards easy comprehension this geometric representation still leaves something to be wished,—the infinitely distant points impair its clearness not a little. But the principal question here is not whether the solution be easy or difficult to comprehend, but whether it be a real and complete solution of the problem. We shall have occasion to point out, further on, the way in which very intricate cases may often be made very simple and easy to realise.

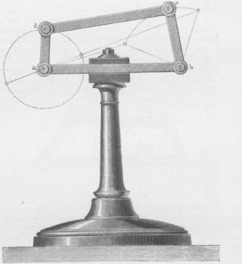

Fig. 23.

Example 2.—A very simple illustration of finding centroids by general investigation occurs in common spur-gearing, or generally in any two bodies which turn about parallel axes at a fixed distance apart with a uniform velocity-ratio. If a and b (Fig. 23) be two con-plane sections of such bodies, and c and d the two centres about which they revolve, then if we consider a as fixed, the point d must move in a circle round c, the distance $d\,c$ being unalterable, and at the same time b must be turning about the as yet unknown pole. But the normal to the path of d must always coincide with the line of centres $d\,c$; the instantaneous centre must therefore lie in that line or its prolongation. Assume, provisionally, that O is the instantaneous centre, and suppose the line $c\,d$ to be fixed, so that both a and b may revolve as they did at first, then the two centroids, both moving, roll upon one another in O, and have therefore at that point the

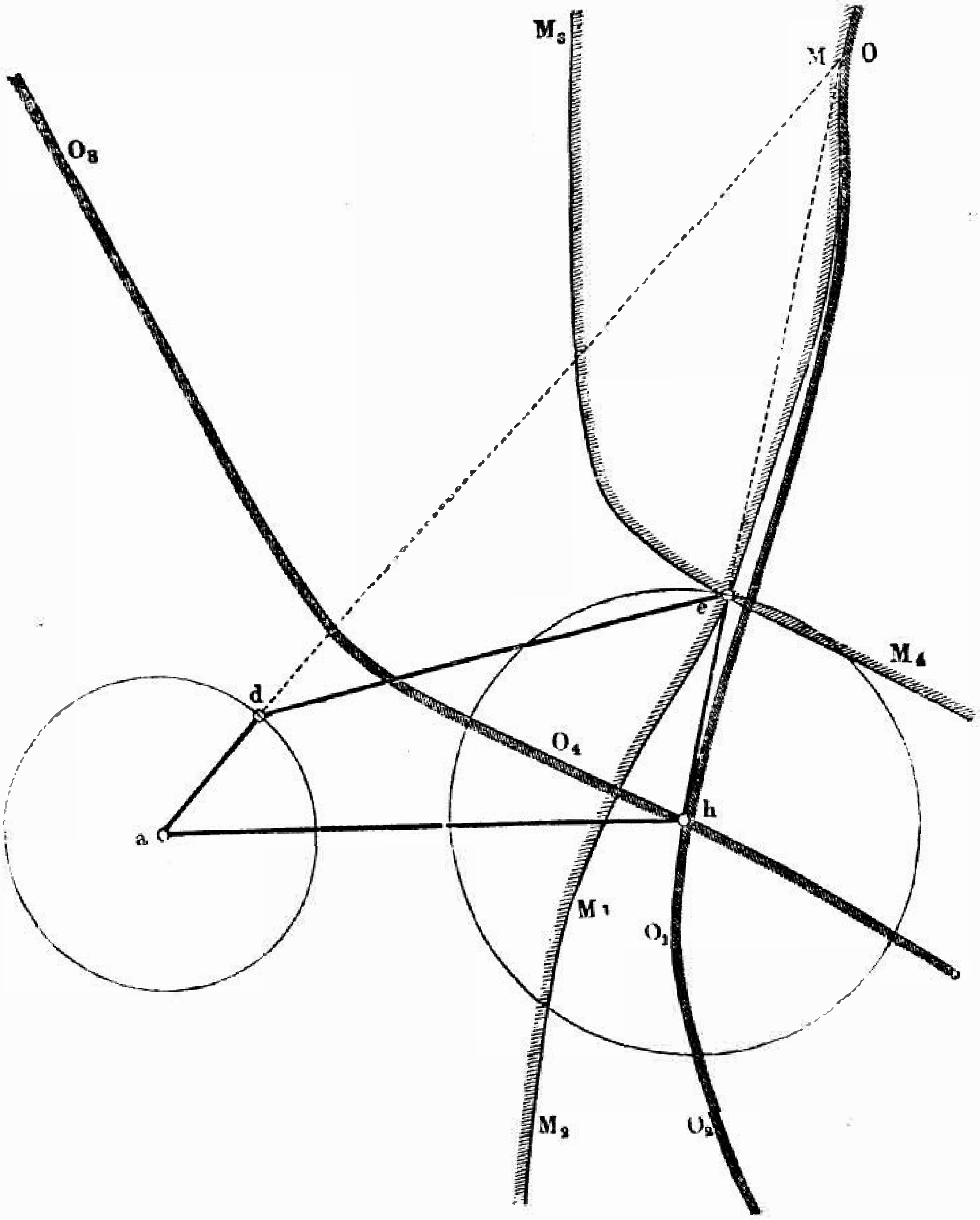

Fig. 22.

same peripheral velocities. Their angular velocities will have to each other the ratio $d\,O : c\,O$. But by hypothesis this ratio is constant, therefore also $d\,O$ and $c\,O$ themselves must be constant; and the centroids become circles described about c and d with radii which are in inverse proportion to the angular velocities of the bodies. The pitch circles of spur-wheels are thus simply the centroids of their normal sections. We

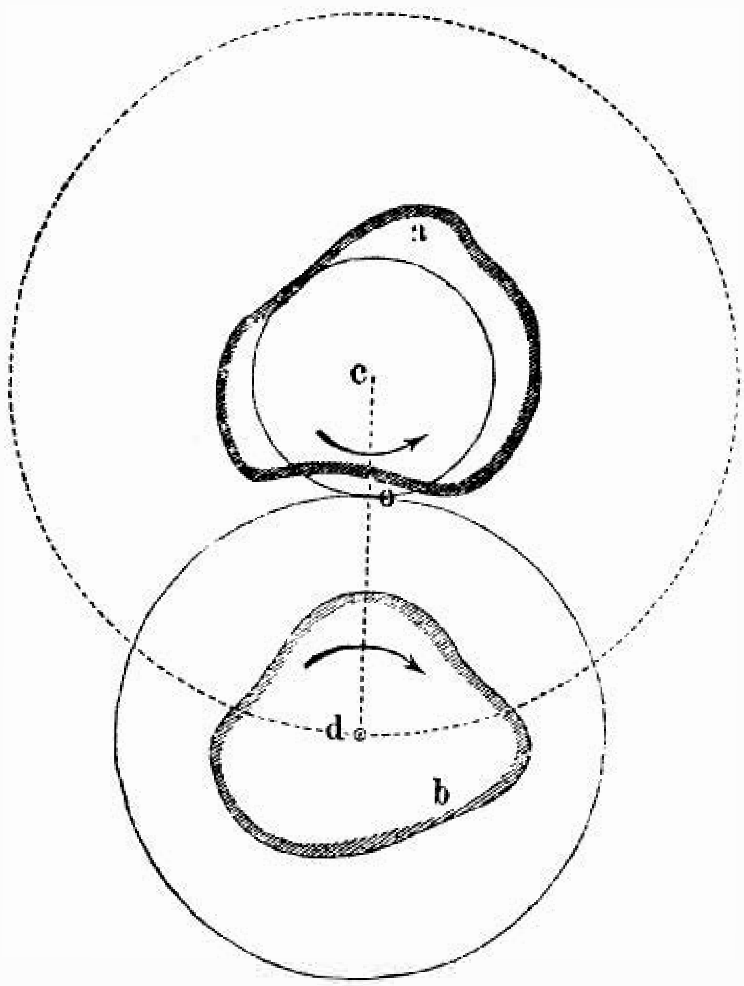

FIG. 23.

touch here, consequently, upon a case in which the centroids are of peculiar simplicity, and which moreover has the most extended practical application in the construction of machines.

§ 9.

Reduction of Centroids.

The pair of centroids which we found in the first example of § 8 completely determines the relative motion between the links *d* —— *e* and *a* —— *h* of the given kinematic chain, and must therefore equally express that motion, whichever link of the chain be fixed or in whatever way it be set in motion. Let us suppose the chain to be arranged in the first of the four methods given on page 41, that is by making *b* —— *c* the fixed link; we can obtain in this way another important mechanism, one in which the links, *a* —— *h* and *d* —— *e* both describe circles, connected with each other always by the link f —— g. The mechanism is that known as a drag-link coupling. The arms or cranks d —— e and a —— h revolve with a varying velocity ratio which can be

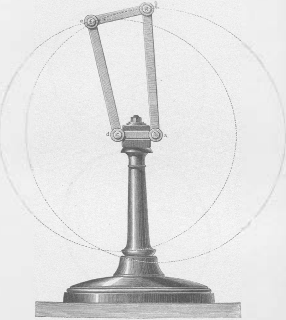

FIG. [illegible]

ascertained for each particular position from the corresponding radii of the centroids. If we imagine, for instance, the centroids shown in Fig. 22 to be fixed to the arms to which they belong, weecanesee thatetheywill both turneabout theefixed pointsea andeď,eandesoeroll upon each otherelikeetheepitch linesseofenon-circular spur-wheels.[9] It is very difficult, however, to realize

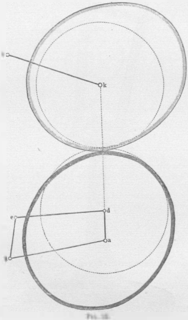

Fig. 33.

the matter from this point of view on account of the peculiar formóf the centroids, and some method which can be more easily comprehended,eforeshowingeateleastetheevelocity-ratioeofetheearms, iserequired. Forethis purpose leteusesuppose aecylindricespurwheel fixed to the axis of one of the arms, as d —— e, and gearing with another equal and similar wheel upon a third axis k. This axis then turns with precisely the same velocity as d —— e but in the opposite direction. If now the centroids be found for the motion of an arm $i\,k$ upon the axis k relatively to the first arm a —— h, these can evidently,—so far as the velocity-ratio between a —— h and d —— e goes,—take the place of the less easily comprehended centroids of Fig. 22. We transform, as it were, the first two centroids into two new ones. Fig. 25 shows these transformed or reduced centroids for this special case. If these be considered as the pitch lines of non-circular spur-

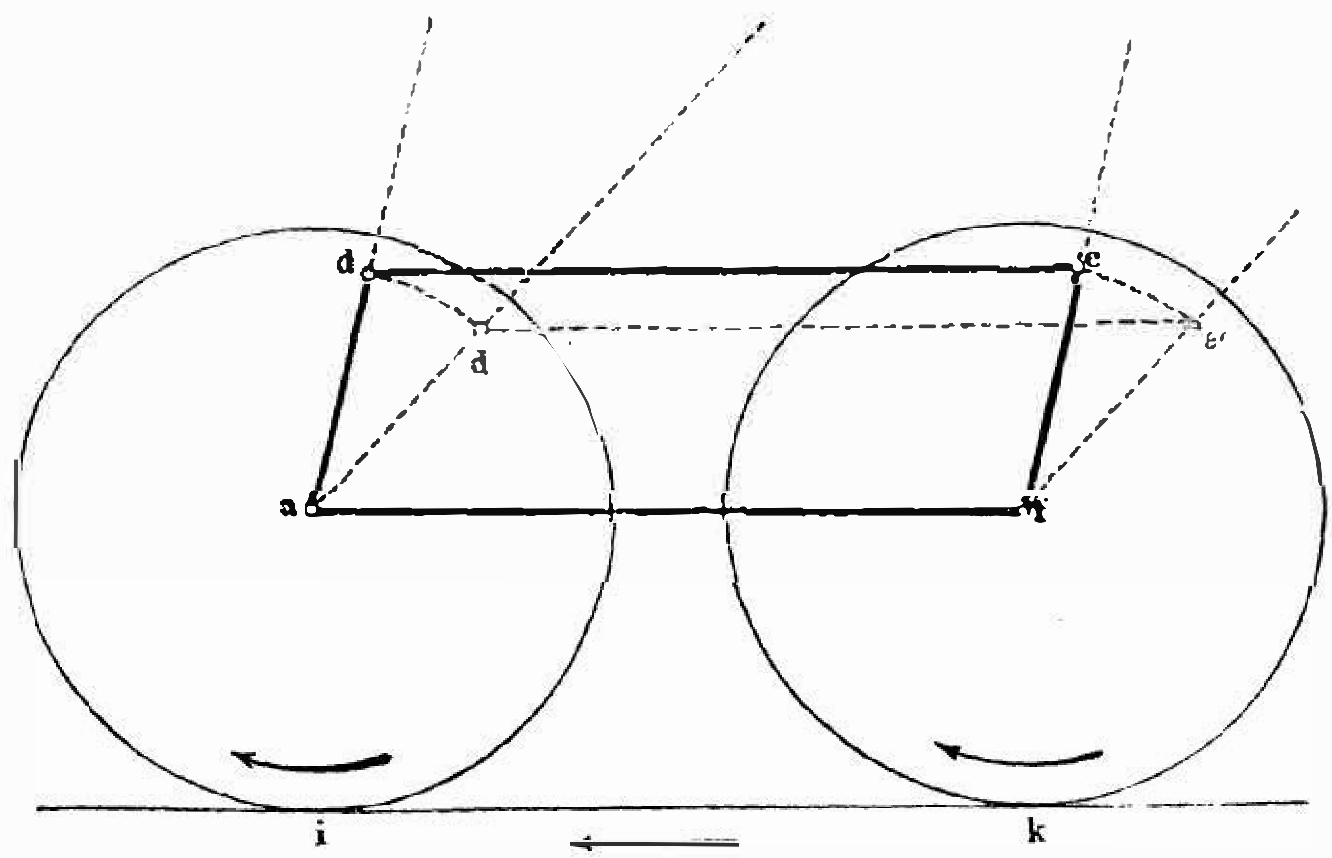

FIG. 26.

wheels, we have at once an easily understood representation of the communication of rotation between a —— h and d —— e. We shall return later on to the methods of drawing such reduced curves. It is sufficient just now to point out that here the sum of the instantaneous radii is constant (being $= a\,k$), while with the original centroids—Fig. 22—their difference was constant (being $= Oa\text{-}Od = a\,d$). The infinitely distant points have disappeared, as will be seen, and the whole representation is very simple and can very frequently be employed.

The infinitely distant parts of centroids may under some circumstances be even more troublesome than in the case we have supposed, where they can to some extent be used through their asymptotes. If for example the opposite links of the mechanism were made of equal length (Fig. 26), the centre lines of the four arms would form a parallelogram, and their intersections would always be at an infinite distance; so that both centroids become infinite and therefore cannot be drawn. The method of reduction which we have just used, if applied to this case, gives us two rolling circles of equal diameter. We can here, however, obtain the same result in a still clearer manner. For this purpose let equal circles *i* and *k*, Fig. 26, be described about *a* and *f*, with radii less than half as long as *a* ——*f*, and let a straight line *i k* touch them both externally. If now this line move without sliding upon the circles, they will turn about their centres *a* and *f* in the same way as they would if they were connected with the arms of the

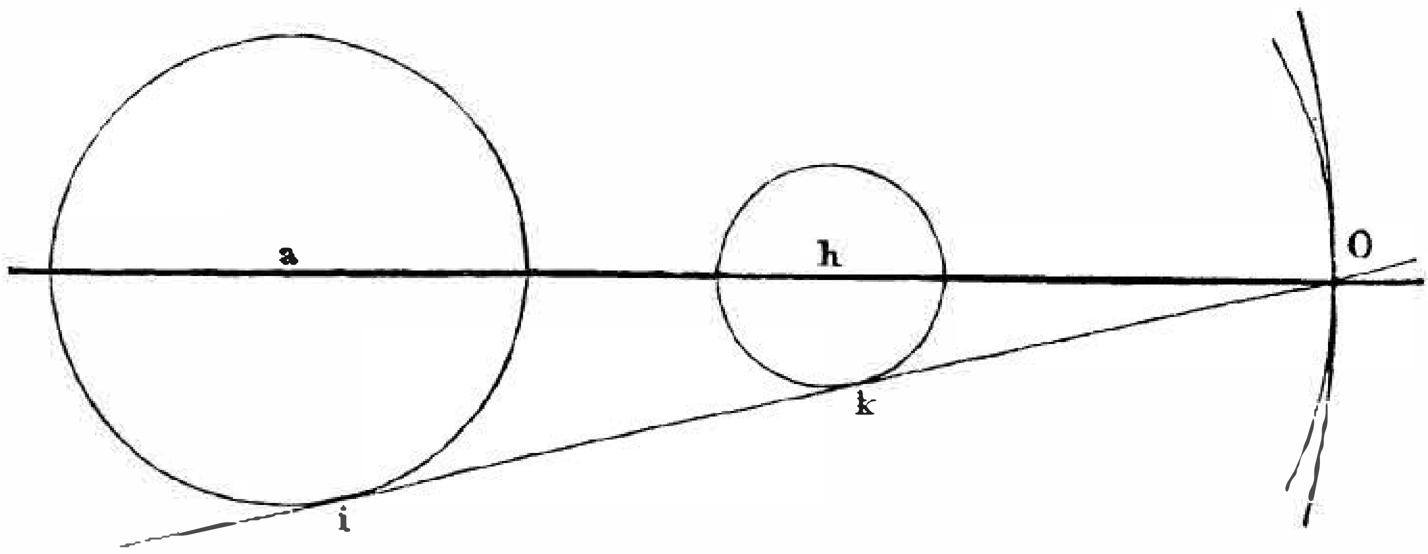

Fig. 27.

parallelogram,—that is, with a uniform angular velocity-ratio equal, in this case, to unity. Thus the three centroids, to which we have here reduced the original infinitely large curves, completely express the relative motion which it was desired to represent. Further, we are able to imagine them existing and moving simultaneously with the original curves. They second, as it were, the movement which has been set up, for which reason we may call them secondary centroids. It will be noticed that in this case we have not two but three connected figures, which is noteworthy, for we already know (§ 8 above) that only a pair of (primary) centroids accompanies any relative motion of con-plane figures. That more than a pair of mutually rolling figures should result from the secondary representation of such a motion is not peculiar to this particular case, but is general.

Secondary centroids are of service to us also in many cases where it is possible, and even easy, to draw the primary curves, but where they would be inconveniently large; as for instance in the case of a pair of bodies which revolve in the same direction with a uniform angular velocity-ratio not equal to unity. Such bodies would have as centroids a pair of circles of which one would touch the other internally. Their secondary centroids would be circles whose diameters bear the same ratio to each other as those of the original curves,* by which means the position of the tangent

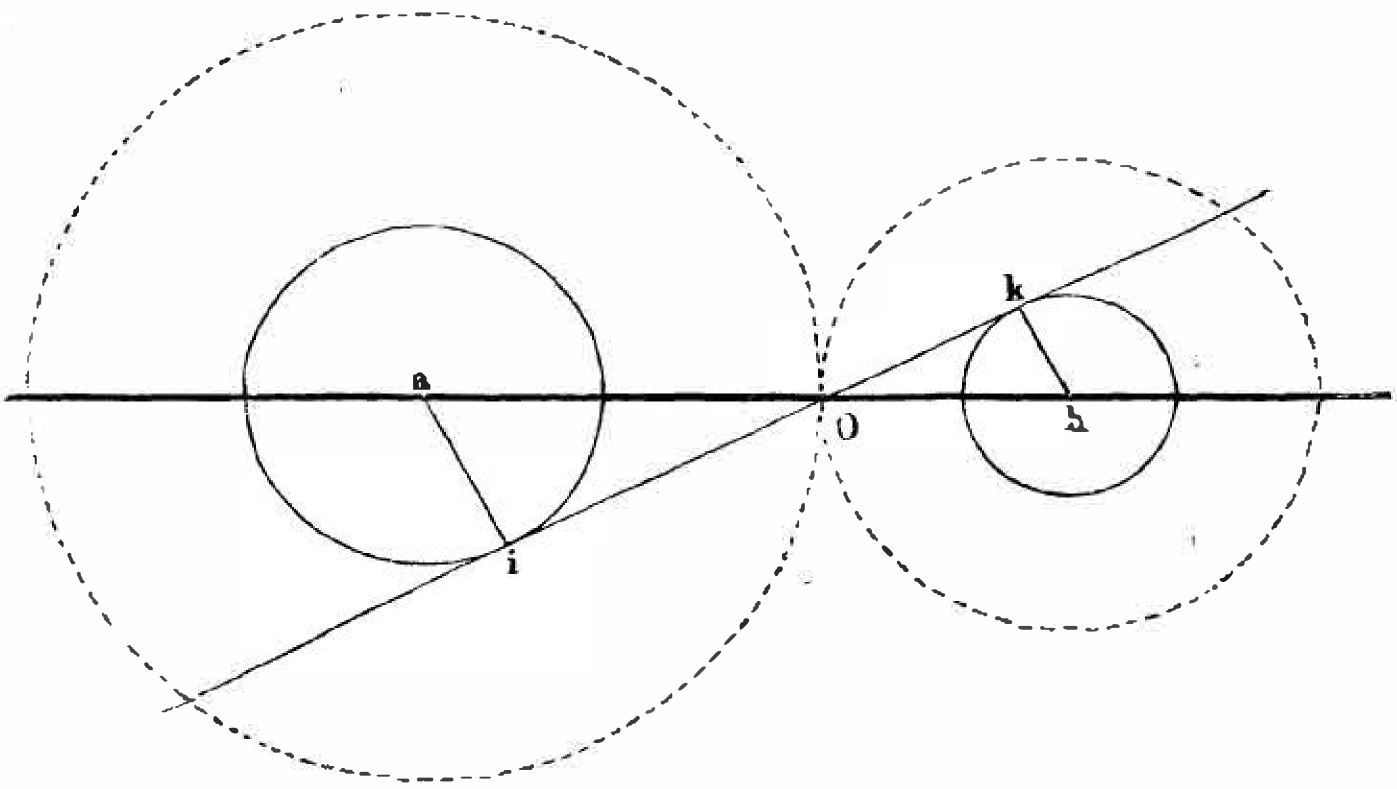

Fig. 28.

i k can be determined, as in Fig. 27. These centroids are also suitable for cases in which rotation with constant-velocity-ratio takes place in opposite directions, as in Fig. 28. In the various methods for drawing the forms of the teeth of spur-wheels,[10] these secondary centroids often play an important and acknowledged part. They have thus already made their way into practice, and enable us to reduce a much used practical method to general phoronomic principles.

* The secondary circles being drawn, the point O in which their tangent cuts the line of centres is always the point of contact of the primary centroids; and from this, conversely, the secondary circles can be drawn as circles touching any line *i k* passing through it. It must be remembered that the point of contact of the centroids is always the instantaneous centre for the motion actually occurring.

§ 10.

Rotation about a Point.

Having now investigated the general methods of representing relative motion in a plane, we must proceed to consider the more difficult problem of relative motion in space, in the first place, however, with the limitation that one point of the body which we are considering be supposed not to alter its position in space in reference to us. When a body moves in such a way that each of its points remains always at a constant distance from some one fixed point, it is said to turn about that point. In order to find the motion which thus takes place relatively to a stationary body rigidly connected with a fixed point, let us describe about that point A, Fig. 29, a sphere of such a size that it shall pass through

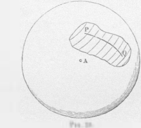

Fig. 29.

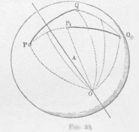

Fig. 30.

the moving body, giving us a spheric section of it, PQ. If we then know the motion of such a sectional figure upon the sphere, it is evident that we shall know the motion of the body itself. But the motion of the figure $R\,Q$ is known if all the positions of two of its points, as P and Q, or of the great circles connected with them, be known. For from the position of this curved line the positions of all other points in the figure may be found by considering them as vertices of spherical triangles, of which the position of the base (PQ) and the magnitudes of all three sides are known. Thus on investigation we see that the motion of the figure about a fixed point reduces itself simply to that of a circular arc, PQ, upon a sphere, and, generally, that we may determine the motion of any spheric figure by that of any arc lying within it, just as we have already found that in con-plane motions we could replace a figure by a line.

Every spheric figure, as PQ in Fig. 30, which moves upon the surface of a sphere, can be moved from one position PQ into another $P_1 Q_1$ through spheric turning about some point O on the surface of the sphere; and this point can be determined by finding the intersection of two great circles passing through the centres of the lines $P P_1$ and $Q Q_1$, and perpendicular to those lines. The point of intersection is then the required pole O, because the spherical triangles $O P Q$ and $O P_1 Q_1$ are similar and equal-sided. The point O is the temporary centre for the supposed spherical turning. The two great circles cut each other twice, once at O, and once at the other end of the diameter passing through O.

But by hypothesis the distance of the figure $P Q$ from the fixed point A is constant, and therefore the diameter passing through O and A is stationary during the turning relatively to the figure, and so becomes the temporary axis of the assumed motion.

A new turning supplies a second pole O_1, another, a third pole O_2, and so on, and by joining these with arcs of great circles we have a spheric polar polygon. A second spheric polar polygon, rigidly connected with the moving figure, corresponds to the first. If a series of straight lines be drawn passing through the corners of these polygons and the fixed point A,—that is, a series of diameters of the sphere passing through these corners,—we obtain two pyramids, about the angles of which the separate turnings take place.

§ 11.

Conic Rolling.

It will be seen that the method we have here used bears the most complete analogy to that employed in the consideration of motions in a plane. If we continue the process further by supposing the consecutive positions of PQ to be infinitely near together, the spheric polar polygons become spheric centroids, the temporary axes become instantaneous, and the pyramids become cones (in general non-circular), which have a common vertex at A, and which roll or turn upon one another. The cones are cones of instantaneous axes, and the whole motion is called conic rolling. We arrive therefore at the following law, connecting together the phenomena we have been considering: All relative motions of two bodies which have during their motion a common point, may be considered as conic rolling, and the motions of any points in the bodies are known so soon as the corresponding cones of instantaneous axes are determined.

It is evident that our former examination of the methods of determining centroids and reducing them applies equally to conic and to cylindric rolling, so that it is not necessary to reconsider these matters here.

§ 12.

Most general Form of the Relative Motion of Rigid Bodies.

If the positions of three points in a body be known, the positions of all other points may be found by making them the vertices of triangular pyramids of which the magnitude and position of the base are fixed and the length of the edges known. We can therefore determine the relative motion of any two rigid bodies by means of two fixed triangles, PQR and ABC, in them. Let the body ABC be brought to rest, so that only PQR moves relatively to us, and let the latter move into any position, as $P_1\,Q_1\,R_1$, Fig. 31. This change of position may take place in many ways. If, for example, we join P and P_1 by a straight line, and cause PQR to undergo translation parallel to it until P falls upon P_1 and the figure takes the position $P_1\,Q'\,R'$,—we have only further to turn PQR about an axis SS (which may be found in every case), passing through P_1, in order that it may assume the required position $P_1\,Q_1\,R_1$. Thus in the most general case any motion of

$P\,Q\,R$ relatively to $A\,B\,C$ may be obtained by combining a simple translation and a simple rotation about an axis; and this in an endless number of ways. It is in no way essential that the translation should be parallel to the line joining the two positions of one of the points (as $P\,P_1$ above). Among the infinite number of possible methods just mentioned one is of special simplicity, that, namely, in which the translation takes place

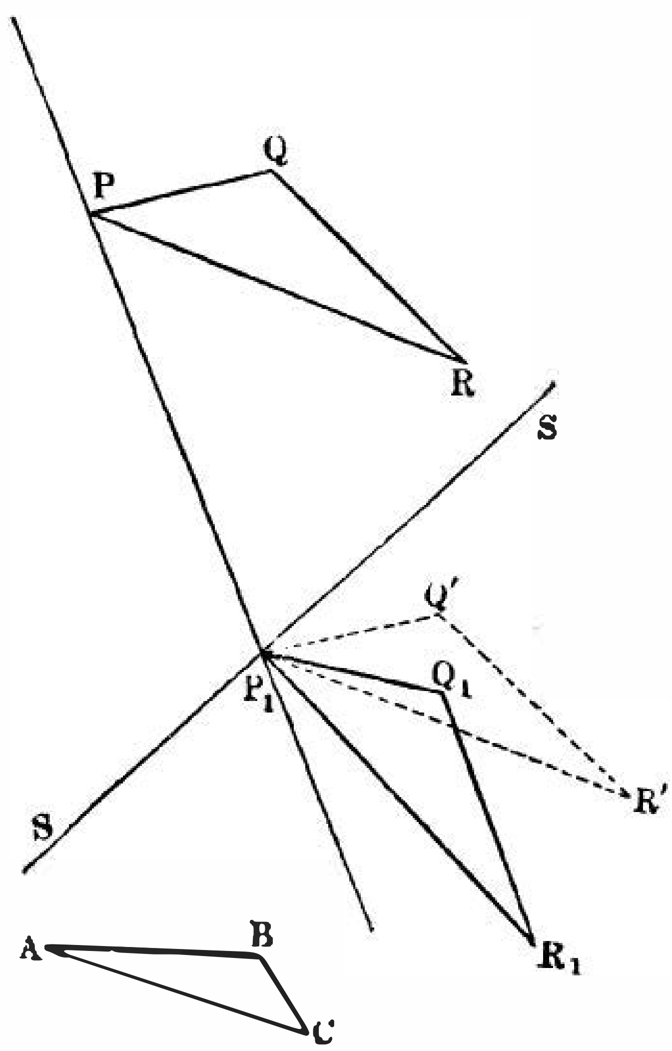

FIG. 31.

parallel to the axis of rotation. Here the motion resolves itself into an endlong sliding upon and rotation about one and the same axis; and if in such a case the changes of position of $P\,Q\,R$ in reference to $A\,B\,C$ be taken indefinitely small, the instantaneous axes of rotation, along which also simultaneous sliding takes place, become infinitely near together.

§ 13.

Twisting and Rolling of Ruled Surfaces.

Many attempts have been made to render easy the comprehension of the kind of motion just described, but the task is far from being a simple one. Poinsot proposed that the fixed body (or body assumed to be stationary relatively to the observer) should be imagined to be a screw, and the moving body a nut, in which case the sliding would take place along, and the rotation about, the axis of the screw, as above supposed. But as the motion takes place with variable velocity, both as regards the sliding and the rotation, the angle of the thread in the screw and the nut must be imagined to be continually changing. It is difficult to realize this distinctly: a form so irregular is no longer a body, it cannot be realized even with the most determined effort,—indeed, things of so varying a nature as are these screws and nuts can scarcely be grasped better than the abstract idea of rotation and translation in space.[11]

Belanger makes two proposals. The first is to imagine a pair of bodies having conic rolling (as above, § 11), in which both cones have a motion of translation in space. The a rotation then takes place through the conic rolling, and the sliding through the translation of the pair of bodies. By this means the motion may certainly be realized, but only by using three bodies to find the relative motions of two, a thing which is in certain cases advisable, and even necessary (§ 9), but which can only be justified when no simpler method is equally satisfactory.

Belanger's second proposition is to consider the consecutive positions of the axis as forming a pair of ruled surfaces, one for each body, so that the motion is reduced to a rolling of the two ruled surfaces upon each other, with a simultaneous endlong sliding upon each other of the generators which are in contact. Other later investigations have attached themselves to this method of representation. Indeed it follows, as a direct conclusion from what we found above, that the consecutive positions of the instantaneous axes of turning and sliding in each of the two bodies enclose such ruled-surface forms as solids of instantaneous axes.[12]

The special motion in which translation or sliding along a straight line and turning about it take place simultaneously is called twisting. We may now also, as we have arrived at the most general standpoint, indicate with a common name the bodies we have found, which, by their motions upon each other, determine the relative motions of the bodies with which they are connected. As the surfaces of these solids are always the loci of a series of axes, they may be called axoids. What we have

found may then be collectively expressed in the following law:—
All relative motions of two bodies may be considered as the twisting or rolling of ruled surfaces or axoids.

The laws already found separately must follow from this general proposition by a diminution and simplification of its conditions. There is indeed no difficulty in realizing the transformation of a ruled surface into a cone or a cylinder (of any form) where instead of the twisting motion only rolling occurs. Nevertheless the conclusion must not be drawn from this that a pure rolling motion

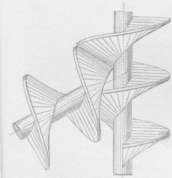

Fig. 35.

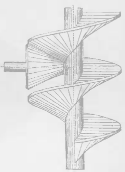

Fig. 36.

can be ascribed only to these two cases (as has been generally done so far as I know). The condition determining the absence of the sliding of the edges or generators is neither that all the axes should intersect in one point, as in the cone, nor that they should be parallel, as in the cylinder, but the higher condition that the two ruled surfaces should be so formed that their infinitely near generators in homologous positions include surfaces of the same figure, or, in geometrical language, that they can be developed upon one another.

The surface of any cone or cylinder can be developed upon that of any other cone or cylinder (respectively), because the portions of surface included between infinitely near generators in corresponding or homologous positions vary similarly. For the same reason many other ruled surfaces can roll upon one another, their forms being such as can be developed upon each other. Thus, for example, two screw surfaces may be mutually developable, and if placed suitably, as in Fig. 32, will roll upon each other; and similarly a screw surface and a hyperboloid, Fig. 33. Forms very closely related to these are in actual use in machines;*—the consideration of them is therefore of practical as well as theoretical interest.

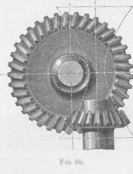

Fig. 34.

Ruled surfaces which twist upon one another are also employed in machine construction; the axoids of hyperboloidal spur wheels, as in Fig 34, belong to this class. These axoids are not themselves constructed, although in the case before us their nature is indicated by the form and position of the teeth upon the wheel. This however is not the case in the pair of elements shown in Fig. 35, the worm wheel and endless screw. The axoids representing the motions of these elements are again hyperboloids,— but this is not

* See for example Johnson's *Imperial Cyclopædia*, "Steam Engine," Moison's regulator, page 49.

recognisable in anyway from their constructive form. Even the ratio between the pitch radii R and R_1 does not give us the relative diameters of the throats of the axoids. The wheels which I have elsewhere described* as hyperboloidal face gear, form a case in which the twisting motion becomes very visible. Here the axoid is a circular cone, the other a plane hyperboloid, or one in which the generator moves always in a plane normal to the axis of rotation (Fig. 36). The sliding of the instantaneous axes as they pass the line of contact is here very distinct, especially near the vertex of the cone.

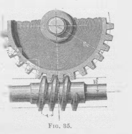

FIG. 35.

We have seen that all axoids belong necessarily to one particular class of geometric forms,— viz. ruled surfaces,— that pairs of such surfaces can, therefore, express all possible motions. We are therefore justified in considering these, in preference to any of the other geometric forms above mentioned (p. 80), as the general representatives of the motions occurring in machines. The most general characteristic of the relative motions of the axoids is their rolling. This exists even in the special case where of the two motions constituting the twist the turning becomes infinitely small and the sliding only remains, for the latter may itself be considered as a turning about an infinitely distant axis,† and is therefore only a particular case of cylindric rolling.

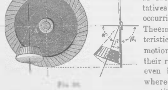

FIG. 36.

* *Der Constructeur*; 3rd Edition, p. 451.

† This is the special case of the proposition given on p. 61, where the two positions PQ and $P_1 Q_1$ are parallel, and where therefore the normals to $P P_1$ and $Q Q_1$ intersect only at infinity.

Every motion which occurs in the machine thus connects itself with one leading idea, of which the single propositions considered contain special applications. Just as the old philosopher compared the constant gradual alteration of things to a flowing, and condensed it into the sentence: "Everything flows;" so we may express the numberless motions in that wonderful production of the human brain which we call a machine in one word, "Everything rolls." Through the whole machine, hidden or apparent, the same fundamental law of rolling applies to the mutual motions of the parts. The same idea, as we have already seen occasionally, can be extended to all the phenomena of kosmic motion, for our investigations do not merely cover the movements of the different parts of a machine, but are applicable generally to all moving bodies.

But the rolling geometrical figures which we can imagine to connect themselves with kosmic bodies are not in themselves constant. They are parts of the universal "flowing;" they alter themselves incessantly as the motions vary, now disappearing into nothing, now remodelling themselves into other continually changing forms, exactly determinable at every instant only in the rolling point itself. In the planetary motions also, that regularity which is capable of exact representation by axoids exists only approximately. In the machine, on the other hand, the rolling figures are rendered constant by artificial limitations of the motions,—this end at least is sought by all possible means, and practically attained,—so that, considered in the abstract, we are entitled to say that here this constancy exists.

Here these figures pass periodically through their mutual changes of position unnumbered times; they rest when the machine stops, and commence their play anew exactly as before so soon as the driving force again infuses life into the whole; one part only remains stationary, that which serves as a connecting piece between the machine itself and the unmoving space surrounding it.

For the practical mechanician, who has made himself familiar with modern Phoronomy, and still more for the theorist, the machine becomes instinct with a life of its own through the rolling geometrical forms everywhere connected with it. Some of these stand out in bodily form, as in belt-pullies or friction-wheels (as for example those of a railway carriage); others, such as spur-wheels, are slightly disguised in a very transparent veil; others still are closely drawn together in the interior of solid bodies which in their exterior forms scarcely give any indication of them, as in the case of the curved discs which we shall shortly have to examine more closely; and others, lastly, such as those of the link-work mechanisms, are widely extended, encircling the bodies to which they belong at a great distance, their branches indeed stretching to infinity, their outward forms quite undiscernible. These all carry on, partly before the bodily eye of the student and partly before the eye of his imagination, the same never tiring play. In the midst of the distracting noise of their material representatives they carry on their noiseless life-work of rolling. They are as it were the soul of the machine, ruling its utterances—the bodily motions themselves—and giving them intelligible expression. They form the geometrical abstraction of the machine, and confer upon it, besides its outer meaning, an inner one, which gives it an intellectual interest to us far greater than any it could otherwise possess.