CHAPTER II.

OF THE PERFORMANCE OF WORK BY MACHINES.

SECTION I.—*Of Resistance and Work.*

297. The **Action of a Machine** is to produce Motion against Resistance. For example, if the machine is one for lifting solid bodies, such as a crane, or fluid bodies, such as a pump, its action is to produce upward motion of the lifted body against the resistance arising from gravity; that is, against its own weight: if the machine is one for propulsion, such as a locomotive engine, its action is to produce horizontal or inclined motion of a load against the resistance arising from friction, or from friction and gravity combined: if it is one for shaping materials, such as a planing machine, its action is to produce relative motion of the tool and of the piece of material shaped by it, against the resistance which that material offers to having part of its surface removed; and so of other machines.

298. **Work.** (*A. M.*, 513.)—The action of a machine is measured, or expressed as a definite quantity, by multiplying the motion which it produces into the resistance, or force directly opposed to that motion, which it overcomes; the product resulting from that multiplication being called WORK.

In Britain, the distances moved through by pieces of mechanism are usually expressed in feet; the resistances overcome, in pounds avoirdupois; and quantities of work, found by multiplying distances in feet by resistances in pounds, are said to consist of so many *foot-pounds*. Thus the work done in lifting a weight of one pound, through a height of one foot, is *one foot-pound;* the work done in lifting a weight of twenty pounds, through a height of one hundred feet, is $20 \times 100 = 2{,}000$ foot-pounds.

In France, distances are expressed in mètres, resistances overcome in kilogrammes, and quantities of work in what are called *kilogrammètres*, one kilogrammètre being the work performed in lifting a weight of one kilogramme through a height of one mètre.

The following are the proportions amongst those units of distance, resistance, and work, with their logarithms:—

		Logarithms.
One mètre	= 3·2808693 feet,..............	0·515989
One foot	= 0·30479721 mètres,............	$\bar{1}$·484011
One kilogramme	= 2·20462 lbs. avoirdupois,......	0·343334
One lb. avoirdupois	= 0·453593 kilogramme,.........	$\bar{1}$·656666
One kilogrammètre	= 7·23308 foot-pounds, ...	0·859323
One foot-pound	= 0·138254 kilogrammètres,.....	$\bar{1}$·140677 1

299. The **Rate of Work** of a machine means, the quantity of work which it performs in some given interval of time, such as a second, a minute, or an hour (*A. M.*, 661). It may be expressed in units of work (such as foot-pounds) per second, per minute, or per hour, as the case may be; but there is a peculiar unit of power appropriated to its expression, called a HORSE-POWER, which is, in Britain, 2.

550 foot-pounds per second,
or 33,000 foot-pounds per minute,
or 1,980,000 foot-pounds per hour.

This is also called an *actual* or *real* horse-power, to distinguish it from a *nominal* horse-power, the meaning of which will afterwards be explained. It is greater than the performance of any ordinary horse, its name having a conventional value attached to it.

In France, the term FORCE DE CHEVAL, or CHEVAL-VAPEUR, is applied to the following rate of work:—

		Foot-lbs.
75 kilogrammètres per second	=	542½
or 4,500 kilogrammètres per minute	=	32,549
or 270,000 kilogrammètres per hour	=	1,952,932

being about one-seventieth part less than the British horse-power.

300. **Velocity.**—If the *velocity of the motion* which a machine causes to be performed against a given resistance be given, then the product of that velocity into the resistance obviously gives the rate of work, or effective power. If the velocity is given in feet per second, and the resistance in pounds, then their product is the rate of work in foot-pounds per second, and so of minutes, or hours, or other units of time.

It is usually most convenient, for purposes of calculation, to express the velocities of the parts of machines either in feet per second or in feet per minute. For certain dynamical calculations to be afterwards referred to, the second is the more convenient unit of time: in stating the performance of machines for practical purposes, the minute is the unit most commonly employed.

Comparison of Different Measures of Velocity.

	Miles per hour.	Feet per second.	Feet per minute.	Feet per hour.
	1	= $1{\cdot}4\dot{6}$	= 88	= 5280·
	$0{\cdot}68\dot{1}\dot{8}$	= 1	= 60	= 3600
	$0{\cdot}011\dot{3}\dot{6}$	= $0{\cdot}01\dot{6}$	= 1	= 60
	$0{\cdot}00018\dot{9}\dot{3}$	= $0{\cdot}0002\dot{7}$	= $0{\cdot}01\dot{6}$	= 1
1 nautical mile per hour. or "knot,"......	= 1·1508	= 1·688	= 101·27	= 6076

The units of time being the same in all civilized countries, the proportions amongst their units of velocity are the same with those amongst their linear measures.

301. **Work in Terms of Angular Motion.** (*A. M.*, 593.)—When a resisting force opposes the motion of a part of a machine which moves round a fixed axis, such as a wheel, an axle, or a crank, the product of the amount of that resistance into its *leverage* (that is, the perpendicular distance of the line along which it acts from the fixed axis) is called the *moment*, or *statical moment*, of the resistance. If the resistance is expressed in pounds, and its leverage in feet, then its moment is expressed in terms of a measure which may be called a *foot-pound*, but which, nevertheless, is a quantity of an entirely different kind from a foot-pound of work. (See p. 321.)

Suppose now that the body to whose motion the resistance is opposed turns through any number of revolutions, or parts of a revolution; and let T denote the angle through which it turns, expressed in revolutions, and parts of a revolution; also, let

$$2\pi = 6{\cdot}2832$$

denote, as is customary, the ratio of the circumference of a circle to its radius. Then the distance through which the given resistance is overcome is expressed by

$$\text{the leverage} \times 2\pi \times \mathrm{T};$$

that is, by the product of the circumference of a circle whose radius is the leverage, into the number of turns and fractions of a turn made by the rotating body.

The distance thus found being multiplied by the resistance overcome, gives the work performed; that is to say,

$$\textit{The work performed}$$
$$= \textit{the resistance} \times \textit{the leverage} \times 2\pi \times \mathrm{T}.$$

But the product of the resistance into the leverage is what is called the *moment* of the resistance, and the product $2\pi T$ is called the *angular motion* of the rotating body; consequently,

The work performed
= the moment of the resistance × the angular motion.

The mode of computing the work indicated by this last equation is often more convenient than the direct mode already explained in Article 298.

The angular motion $2\pi T$ of a body during some definite unit of time, as a second or a minute, is called its *angular velocity;* that is to say, *angular velocity is the product of the turns and fractions of a turn made in an unit of time into the ratio* ($2\pi = 6{\cdot}2832$) *of the circumference of a circle to its radius.* Hence it appears that

The rate of work
= the moment of the resistance × the angular velocity.

302. **Work in Terms of Pressure and Volume.** (*A. M.*, 517.)—If the resistance overcome be a pressure uniformly distributed over an area, as when a piston drives a fluid before it, then the amount of that resistance is equal to the intensity of the pressure, expressed in units of force on each unit of area (for example, in pounds on the square inch, or pounds on the square foot) multiplied by the area of the surface at which the pressure acts, if that area is perpendicular to the direction of the motion; or, if not, then by the projection of that area on a plane perpendicular to the direction of motion. In practice, when the *area of a piston* is spoken of, it is always understood to mean the projection above mentioned.

Now, when a plane area is multiplied into the distance through which it moves in a direction perpendicular to itself, if its motion is straight, or into the distance through which its centre of gravity moves, if its motion is curved, the product is the *volume of the space traversed* by the piston.

Hence the work performed by a piston in driving a fluid before it, or by a fluid in driving a piston before it, may be expressed in either of the following ways:—

Resistance × distance traversed
= intensity of pressure × area × distance traversed;
= intensity of pressure × volume traversed.

In order to compute the work in foot-pounds, if the pressure is stated in pounds on the square foot, the area should be stated in square feet, and the volume in cubic feet; if the pressure is stated in pounds on the square inch, the area should be stated in square inches, and the volume in units, each of which is a prism of one foot in length and one square inch in area; that is, of $\frac{1}{144}$ of a cubic foot in volume.

The following table gives a comparison of various units in which the intensities of pressures are commonly expressed. (*A. M.*, 86.)

	Pounds on the square foot.	Pounds on the square inch.
One pound on the square inch,.....	144	1
One pound on the square foot,.....	1	$\frac{1}{144}$
One inch of mercury (that is, weight of a column of mercury at 32° Fahr., one inch high),............	70·73	0·4912
One foot of water (at 39°·1 Fahr.),	62·425	0·4335
One inch of water,....................	5·2021	0·036125
One atmosphere, of 29·922 inches of mercury, or 760 millimètres,	2116·3	14·7
One foot of air, at 32° Fahr., and under the pressure of one atmosphere,............a................	0·080728	0·0005606
One kilogramme on the square mètre, ...	0·204813	0·00142231
One kilogramme on the square millimètre, ...	204813	1422·31
One millimètre of mercury,.........	2·7847	0·01934

303. **Algebraical Expressions for Work.** (*A. M.*, 515, 517, 593.)—To express the results of the preceding articles in algebraical symbols, let

s denote the distance in feet through which a resistance is overcome in a given time;

R, the amount of the resistance overcome in pounds.

Also, supposing the resistance to be overcome by a piece which turns about an axis, let

T be the number of turns and fractions of a turn made in the given time, and $i = 2\pi T = 6·2832\ T$ the angular motion in the given time; and let

l be the leverage of the resistance; that is, the perpendicular distance of the line along which it acts from the axis of motion; so that $s = i\,l$, and R l is the statical moment of the resistance. Supposing the resistance to be a pressure, exerted between a piston and a fluid, let A be the area or projected area of the piston, and p the intensity of the pressure in pounds per unit of area.

Then the following expressions all give quantities of work in the given time in foot-pounds:—

$$\mathrm{R}\,s;\ i\,\mathrm{R}\,l;\ p\,\mathrm{A}\,s;\ i\,p\,\mathrm{A}\,l.$$

The last of these expressions is applicable to a piston turning on an axis, for which l denotes the distance from the axis to the centre of gravity of the area A.

304. **Work against an Oblique Force.** (*A. M.*, 511.)—The resistance directly due to a force which acts against a moving body in a direction oblique to that in which the body moves, is found by resolving that force into two components, one at right angles to the direction of motion, which may be called a *lateral force*, and which must be balanced by an equal and opposite lateral force, unless it takes effect by altering the direction of the body's motion, and the other component directly opposed to the body's motion, which is the *resistance* required. That resolution is effected by means of the well known principle of the parallelogram of forces as follows:—

In fig. 239, let A represent the point at which a resistance is overcome, A B the direction in which that point is moving, and let $\overline{\mathrm{A\,F}}$ be a line whose direction and length represent the direction and magnitude of a force obliquely opposed to the motion of A.

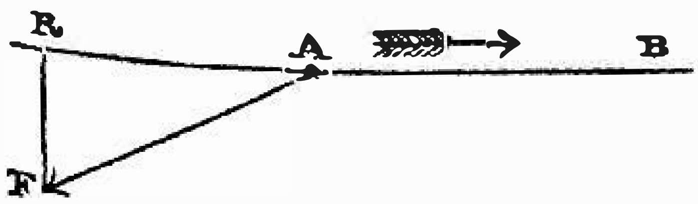

Fig. 239.

From F upon B A produced, let fall the perpendicular $\overline{\mathrm{F\,R}}$; the length of that perpendicular will represent the magnitude of the lateral component of the oblique force, and the length $\overline{\mathrm{A\,R}}$ will represent the direct component or resistance.

The work done against an oblique resisting force may also be calculated by resolving the motion into a direct component in the line of action of the force, and a transverse component, and multiplying the whole force by the direct component of the motion.

305. **Summation of Quantities of Work.**—In every machine, resistances are overcome during the same interval of time, by different moving pieces, and at different points in the same moving piece; and the whole work performed during the given interval is found by adding together the several products of the resistances into the respective distances through which they are simultaneously overcome. It is convenient, in algebraical symbols, to denote the result of that summation by the symbol—

$$\Sigma \cdot \mathrm{R}\,s; \qquad \text{(1.)}$$

in which Σ denotes the operation of taking the sum of a set of quantities of the kind denoted by the symbols to which it is prefixed.

When the resistances are overcome by pieces turning upon axes, the above sum may be expressed in the form—

$$\Sigma \cdot i \, R \, l; \text{..........................} (2.)$$

and so of other modes of expressing quantities of work.

The following are particular cases of the summation of quantities of work performed at different points:—

I. In a *shifting piece*, or one which has the kind of movement called *translation* only, the velocities of every point at a given instant are equal and parallel; hence, in a given interval of time, the motions of all the points are equal; and the work performed is to be found by multiplying the *sum of the resistances* into the motion as a common factor; an operation expressed algebraically thus—

$$s \, \Sigma \, R; \text{..........................} (3.)$$

II. For a *turning piece*, the angular motions of all the points during a given interval of time are equal; and the work performed is to be found by multiplying the *sum of the moments of the resistances relatively to the axis* into the angular motion as a common factor—an operation expressed algebraically thus—

$$i \, \Sigma \cdot R \, l; \text{..........................} (4.)$$

The sum denoted by $\Sigma \cdot R \, l$ is the *total moment of resistance* of the piece in question.

III. In every *train of mechanism*, the *proportions* amongst the motions performed during a given interval of time by the several moving pieces, can be determined from the mode of connection of those pieces, independently of the absolute magnitudes of those motions, by the aid of the science called by Mr. Willis, *Pure Mechanism*. This enables a calculation to be performed which is called *reducing the resistances to the driving point;* that is to say, determining the resistances, which, if they acted directly at the point where the motive power is applied to the machine, would require the same quantity of work to overcome them with the actual resistances.

Suppose, for example, that by the principles of pure mechanism it is found, that a certain point in a machine, where a resistance R is to be overcome, moves with a velocity bearing the ratio $n : 1$ to the velocity of the driving point. Then the work performed in overcoming that resistance will be the same as if a resistance $n \, R$ were overcome directly at the driving point. If a similar calculation be made for each point in the machine where resistance is overcome, and the results added together, as the following symbol denotes:—

$$\Sigma \cdot n \mathrm{R}, \ldots\ldots\ldots\ldots (5.)$$

that sum is the *equivalent resistance at the driving point;* and if in a given interval of time the driving point moves through the distance s, then the work performed in that time is—

$$s \, \Sigma \cdot n \mathrm{R}. \ldots\ldots\ldots\ldots (6.)$$

The process above described is often applied to the steam engine, by reducing all the resistances overcome to equivalent resistances acting directly against the motion of the piston.

A similar method may be applied to the moments of resistances overcome by rotating pieces, so as to reduce them to *equivalent moments at the driving axle.* Thus, let a resistance R, with the leverage l, be overcome by a piece whose angular velocity of rotation bears the ratio $n : 1$ to that of the driving axle. Then the equivalent moment of resistance at the driving axle is $n \mathrm{R} l$; and if a similar calculation be made for each rotating piece in the machine which overcomes resistance, and the results added together, the sum—

$$\Sigma \cdot n \mathrm{R} l \ldots\ldots\ldots\ldots (7.)$$

is the total *equivalent moment of resistance* at the driving axle; and if in a given interval of time the driving axle turns through the arc i to radius unity, the work performed in that time is—

$$i \, \Sigma \cdot n \mathrm{R} l. \ldots\ldots\ldots\ldots (8.)$$

IV. *Centre of Gravity.*—The work performed in lifting a body is *the product of the weight of the body into the height through which its centre of gravity is lifted.* (See Article 282, page 328.)

If a machine lifts the centres of gravity of several bodies at once to heights either the same or different, the whole quantity of work performed in so doing is the sum of the several products of the weights and heights; but that quantity can also be computed by *multiplying the sum of all the weights into the height through which their common centre of gravity is lifted.*

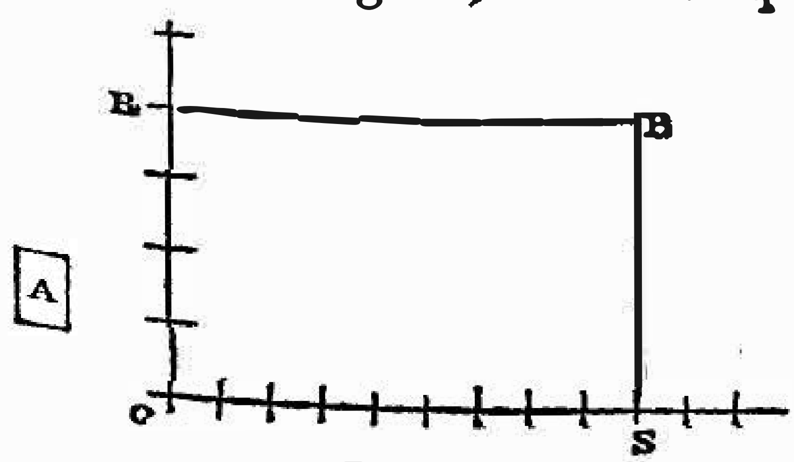

Fig. 240.

306. **Representation of Work by an Area.**—As a quantity of work is the product of two quantities, a force and a motion, it may be represented by the area of a plane figure, which is the product of two dimensions.

Let the base of the rectangle A, fig. 240, represent *one foot* of motion, and its height *one pound* of resistance; then will its area represent one foot-pound of work.

In the larger rectangle, let the base $\overline{\mathrm{OS}}$ represent a certain motion s, on the same scale with the base of the unit-area A; and let the height $\overline{\mathrm{OR}}$ represent a certain resistance R, on the same scale with the height of the unit-area A; then will the number of times that the rectangle $\overline{\mathrm{OS}} \cdot \overline{\mathrm{OR}}$ contains the unit-rectangle A, express the number of foot-pounds in the quantity of work Rs, which is performed in overcoming the resistance R through the distance s.

307. **Work against Varying Resistance.** (*A. M.*, 515.)—In fig. 241, let distances as before, be represented by lengths measured along the base line O X of the figure; and let the magnitudes of the resistance overcome at each instant be represented by the lengths of ordinates drawn perpendicular to O X, and parallel to O Y:—For example, when the working body has moved through the distance represented by $\overline{\mathrm{OS}}$, let the resistance be represented by the ordinate $\overline{\mathrm{SR}}$.

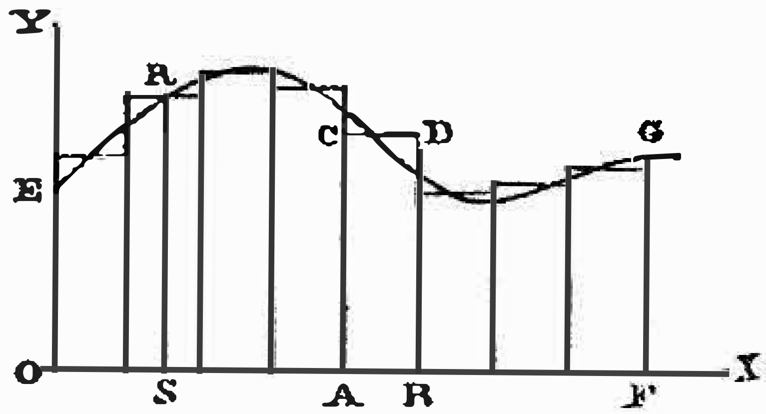

Fig. 241.

If the resistance were constant, the summits of those ordinates would lie in a straight line parallel to O X, like R B in fig. 240; but if the resistance varies continuously as the motion goes on, the summits of the ordinates will lie in a line, straight or curved, such as that marked E R G, fig. 241, which is not parallel to O X.

The values of the resistance at each instant being represented by the ordinates of a given line E R G, let it now be required to determine the work performed against that resistance during a motion represented by $\overline{\mathrm{OF}} = s$.

Suppose the area O E G F to be divided into bands by a series of parallel ordinates, such as A C and B D, and between the upper ends of those ordinates let a series of short lines, such as C D, be drawn parallel to O X, so as to form a stepped or serrated outline, consisting of lines parallel to O X and O Y alternately, and *approximating* to the given continuous line E G.

Now conceive the resistance, instead of varying continuously, to remain constant during each of the series of divisions into which the motion is divided by the parallel ordinates, and to change abruptly at the instants between those divisions, being represented for each division by the height of the rectangle which stands on that division: for example, during the division of the motion represented by A B, let the resistance be represented by A C, and so for other divisions.

Then the work performed during the division of the motion represented by A B, on the supposition of alternate constancy and abrupt variation of the resistance, is represented by the rectangle AB·AC; and the whole work performed, on the same supposition, during the whole motion O F, is represented by the sum of all the rectangles lying between the parallel ordinates; and inasmuch as the supposed mode of variation of the resistance represented by the stepped outline of those rectangles is an approximation to the real mode of variation represented by the continuous line E G, and is a closer approximation the closer and the more numerous the parallel ordinates are, so the sum of the rectangles is an approximation to the exact representation of the work performed against the continuously varying resistance, and is a closer approximation the closer and more numerous the ordinates are, and by making the ordinates numerous and close enough, can be made to differ from the exact representation by an amount less than any given difference.

But the sum of those rectangles is also an approximation to the area O E G F, bounded above by the continuous line E G, and is a closer approximation the closer and the more numerous the ordinates are, and by making the ordinates numerous and close enough, can be made to differ from the area O E G F by an amount less than any given difference.

Therefore the area O E G F, *bounded by the straight line* O F, *which represents the motion, by the line* E G, *whose ordinates represent the values of the resistance, and by the two ordinates* O E *and* F G, *represents exactly the work performed.* (See Article 289, page 331.)

The MEAN RESISTANCE during the motion is found by dividing the area O E G F by the motion O F.

308. **Useful Work and Lost Work.**—The useful work of a machine is that which is performed in effecting the purpose for which the machine is designed. The lost work is that which is performed in producing effects foreign to that purpose. The resistances overcome in performing those two kinds of work are called respectively *useful resistance* and *prejudicial resistance.*

The useful work and the lost work of a machine together make up its *total* or *gross work.*

In a pumping engine, for example, the useful work in a given time is the product of the weight of water lifted in that time into the height to which it is lifted; the lost work is that performed in overcoming the friction of the water in the pumps and pipes, the friction of the plungers, pistons, valves, and mechanism, and the resistance of the air pump and other parts of the engine.

For example, the useful work of a marine steam engine in a given time is the product of the resistance opposed by the water to the motion of the ship, into the distance through which she moves: the lost work is that performed in overcoming the resistance of the water to the motion of the propeller through it, the friction of the mechanism, and the other resistances of the engine, and in raising the temperature of the condensation water, of the gases which escape by the chimney, and of adjoining bodies.

There are some cases, such as those of muscular power and of windmills, in which the useful work of a prime mover can be determined, but not the lost work.

309. **Friction.** (Partly extracted and abridged from *A. M.*, 189, 190, 191, 204, and 669 to 685).—The most frequent cause of loss of work in machines is friction—being that force which acts between two bodies at their surface of contact so as to resist their sliding on each other, and which depends on the force with which the bodies are pressed together. The following law respecting the friction of solid bodies has been ascertained by experiment:—

The friction which a given pair of solid bodies, with their surfaces in a given condition, are capable of exerting, is simply proportional to the force with which they are pressed together.

There is a limit to the exactness of the above law, when the pressure becomes so intense as to crush or grind the parts of the bodies at and near their surface of contact. At and beyond that limit the friction increases more rapidly than the pressure; but that limit ought never to be attained at the bearings of any machine.[1] For some substances, especially those whose surfaces are sensibly indented by a moderate pressure, such as timber, the friction between a pair of surfaces which have remained for some time at rest relatively to each other, is somewhat greater than that between the same pair of surfaces when sliding on each other. That excess, however, of the *friction of rest* over the *friction of motion*, is instantly destroyed by a slight vibration; so that the *friction of motion* is alone to be taken into account as causing continuous loss of work.

As to *materials for bearings*, see pages 462, 463, 464.

The friction between a pair of bearing surfaces is calculated by multiplying the force with which they are directly pressed together, by a factor called the *co-efficient of friction*, which has a special value depending on the nature of the materials and the state of the surfaces as to smoothness and lubrication. Thus, let R denote the friction between a pair of surfaces; Q, the force, in a direction perpendicular to the surfaces, with which they are pressed together; and f the co-efficient of friction; then

$$R = fQ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

The co-efficient of friction of a given pair of surfaces is the tangent of an angle called the *angle of repose*, being the greatest angle which an oblique pressure between the surfaces can make with a perpendicular to them, without making them slide on each other.

The following is a table of the angle of repose φ, the co-efficient of friction $f = \tan \varphi$, and its reciprocal $1 : f$, for the materials of mechanism—condensed from the tables of General Morin, and other sources, and arranged in a few comprehensive classes. The values of those constants which are given in the table have reference to the *friction of motion.** (See page 399.)

No.	SURFACES.	φ	f	$1 : f$
1	Wood on wood, dry,	14° to 26½°	·25 to ·5	4 to 2
2	" " soaped,	11½° to 2°	·2 to ·04	5 to 25
3	Metals on oak, dry,	26½° to 31°	·5 to ·6	2 to 1·67
4	" " wet,	13½° to 14½°	·24 to ·26	4·17 to 3·85
5	" " soapy,	11½°	·2	5
6	Metals on elm, dry,	11½° to 14°	·2 to ·25	5 to 4
7	Hemp on oak, dry,	28°	·53	1·89
8	" " wet,	18½°	·33	3
9	Leather on oak,	15° to 19½°	·27 to ·38	3·7 to 2·86
10	Leather on metals, dry,	29½°	·56	1·79
11	" " wet,	20°	·36	2·78
12	" " greasy,	13°	·23	4·35
13	" " oily,	8½°	·15	6·67
14	Metals on metals, dry,	8½° to 11½°	·15 to ·2	6·67 to 5
15	" " wet,	16½°	·3	3·33
16	Smooth surfaces, occasionally greased,	4° to 4½°	·07 to ·08	14·3 to 12·5
17	" " continually greased,	3°	·05	20
18	" " best results,	1¾° to 2°	·03 to ·036	33·3 to 27·6
19	Bronze on lignum vitæ, constantly wet,	3°.?	·05 ?	20 ?

* In a paper, of which an abstract appeared in the *Comptes Rendus* of the French Academy of Sciences for the 26th of April, 1858, M. H. Bochet describes a series of experiments which have led him to the conclusion, that the friction between a pair of surfaces of iron is not, as had formerly been believed, absolutely independent of the velocity of sliding, but that it diminishes slowly as that velocity increases, according to a law expressed by the following formula. Let

R denote the friction;

Q, the pressure;

v, the velocity of sliding, in mètres per second = velocity in feet per second $\times$ 0·3048;

f, a, γ, constant co-efficients; then

$$\frac{R}{Q} = \frac{f + \gamma a v}{1 + a v}.$$

The following are the values of the co-efficients deduced by M. Bochet from

310. **Unguents.**—Three results in the preceding table, Nos. 16, 17, and 18, have reference to smooth firm surfaces of any kind, greased or lubricated to such an extent that the friction depends chiefly on the continual supply of unguent, and not sensibly on the nature of the solid surfaces; and this ought almost always to be the case in machinery. Unguents should be thick for heavy pressures, that they may resist being forced out, and thin for light pressures, that their viscidity may not add to the resistance.

Unguents may be divided into four classes, as follows :—

I. *Water*, which acts as an unguent on surfaces of wood and leather. It is not, however, an unguent for a *pair* of metallic surfaces; for when applied to them, it increases their friction.

II. *Oily unguents*, consisting of animal and vegetable fixed oils, as tallow, lard, lard oil, seal oil, whale oil, olive oil. The vegetable drying oils, such as linseed oil, are unfit for unguents, as they absorb oxygen, and become hard. The animal oils are on the whole better than the vegetable oils.

III. *Soapy unguents*, composed of oil, alkali, and water. For a temporary purpose, such as lubricating the ways for the launch of a ship, one of the best unguents of this class is soft soap, made from whale oil and potash, and used either alone or mixed with tallow. But for a permanent purpose, such as lubricating railway carriage axles, it is necessary that the unguent should contain less water and more oil or fatty matter than soft soap does, otherwise it would dry and become stiff by the evaporation of the water. The best grease for such purposes does not contain more than from 25 to 30 per cent. of water; that which contains 40 or 50 per cent. is bad.

IV. *Bituminous unguents*, in which liquid mineral hydrocarbons are used to dissolve and dilute oily and fatty substances.

The *intensity of the pressure* between a pair of greased surfaces ought not to be so great as to force out the unguent. The following formula agrees very fairly with the results of practice:—

Let v be the velocity of sliding, in feet per second; p, the greatest proper intensity of pressure, in lbs. on the square inch; then 1

$$p = \frac{44800}{60v + 20} = \frac{2240}{3v+1}$$

p ought not in any case to exceed 1200. 2

his experiments, for iron surfaces of wheels and skids rubbing longitudinally on iron rails:—

f, for dry surfaces, 0·3, 0·25, 0·2; for damp surfaces, 0·14.

a, for wheels sliding on rails, 0·03; for skids sliding on rails, 0·07.

γ, not yet determined, but treated meanwhile as inappreciably small.

As to the measurement of friction, see Article 348, page 395.

310 A. **Friction of a Band.**—A flexible band may be used either to exert an effort or a resistance upon a drum or pulley. In either case, the tangential force, whether effort or resistance, exerted between the band and the pulley, is their mutual friction, caused by and proportional to the normal pressure between them.

In fig. 242, let C be the axis of a pulley A B, round an arc of which there is wrapped a band, T_1 A B T_2; let the outer arrow represent the direction in which the band slides, or tends to slide, relatively to the pulley, and the inner arrow the direction in which the pulley slides, or tends to slide, relatively to the band.

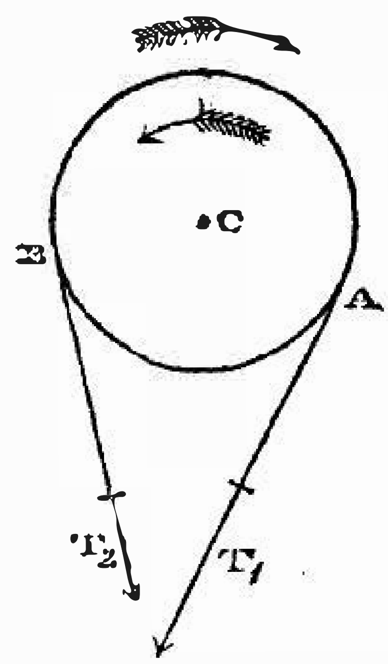

Fig. 242.

Let T_1 be the tension of the free part of the band at that side *towards* which it tends to draw the pulley, or *from* which the pulley tends to draw it; T_2, the tension of the free part at the other side; T, the tension of the band at any intermediate point of its arc of contact with the pulley; θ, the ratio of the length of that arc to the radius of the pulley; $d\,\theta$, the ratio of an indefinitely small element of that arc to the radius; $R = T_1 - T_2$, the total friction between the band and the pulley; $d\,R$, the elementary portion of that friction due to the elementary arc $d\,\theta$; f, the co-efficient of friction between the materials of the band and pulley.

Then it is known that the normal pressure at the elementary arc $d\,\theta$ is $T\,d\,\theta$; T being the mean tension of the band at that elementary arc; consequently, the friction on that arc is

$$d\,R = f\,T\,d\,\theta.$$

Now, that friction is also the difference between the tensions of the band at the two ends of the elementary arc; or

$$d\,T = d\,R = f\,T\,d\,\theta;$$

which equation being integrated throughout the entire arc of contact, gives the following formulæ:—

$$\left.\begin{array}{c} \text{hyp log}\,\dfrac{T_1}{T_2} = f\,\theta;\ T_1 \div T_2 = e^{f\theta}; \\ R = T_1 - T_2 = T_1\,(1 - e^{-f\theta}) = T_2\,(e^{f\theta} - 1). \end{array}\right\} \ldots\ldots(1.)$$

When a belt connecting a pair of pulleys has the tensions of its two sides originally equal, the pulleys being at rest; and when the pulleys are set in motion, so that one of them drives the other by means of the belt; it is found that the advancing side of the belt is exactly as much tightened as the returning side is slackened, so that the *mean* tension remains unchanged. The ratio which it bears to the force, R, to be transmitted, is given by this formula:—

$$\frac{T_1 + T_2}{2 R} = \frac{e^{f\theta} + 1}{2(e^{f\theta} - 1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

If the arc of contact between the band and pulley, expressed in turns and fractions of a turn, be denoted by n,

$$\theta = 2 \pi n;\ e^{f\theta} = 10^{2 \cdot 7288 f n}; \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

that is to say, $e^{f\theta}$ is the *antilogarithm*, or natural number, corresponding to the common logarithm $2 \cdot 7288 f n$.

The value of the co-efficient of friction, f, depends on the state and material of the rubbing surfaces. For leather belts on iron pulleys, the table of Article 309, page 349, shows that it ranges from ·56 to ·15. In calculating, by equation 2 of this Article, the proper mean tension for a belt, the smallest value $f = \cdot 15$, is to be taken, if there is a probability of the belt becoming wet with oil. The experiments of Messrs. Henry R. Towne and Robert Briggs, however (published in the Journal of the Franklin Institute for 1868), show that such a state of lubrication is not of ordinary occurrence; and that in designing machinery, we may in most cases safely take $f = 0 \cdot 42$. Professor Reuleaux (*Constructionslehre für Maschinenbau*) takes $f = 0 \cdot 25$. The following table shows the values of the co-efficient $2 \cdot 7288 f$, by which n is multiplied in equation 3, corresponding to different values of f; also the corresponding values of various ratios amongst the forces, when the arc of contact is half a circumference:—

$f =$	0·15	0·25	0·42	0·56
$2 \cdot 7288 f =$	0·41	0·68	1·15	1·53

Let $\theta = \pi$, and $n = \frac{1}{2}$; then

$T_1 \div T_2 =$	1·603	2·188	3·758	5·821
$T_1 \div R =$	2·66	1·84	1·36	1·21
$T_1 + T_2 \div 2 R =$	2·16	1·34	0·86	0·71

In ordinary practice, it is usual to assume $T_2 = R$; $T_1 = 2 R$; $T_1 + T_2 \div 2 R = 1 \cdot 5$. This corresponds to $f = 0 \cdot 22$ nearly.

For a wire rope on cast-iron, f may be taken as $= 0 \cdot 15$ nearly; and if the groove of the pulley is bottomed with gutta-percha, 0·25.

When an endless band runs at a very high velocity, its centrifugal force has an indirect effect on the friction, which will be considered further on. (See page 441.)

311. **The Work Performed against Friction** in a given time, between a pair of rubbing surfaces, is the product of that friction into the distance through which one surface slides over the other.

When the motion of one surface relatively to the other consists in rotation about an axis, the work performed may also be calculated by multiplying the relative *angular motion* of the surfaces to radius unity into the *moment of friction;* that is, the product of the friction into its leverage, which is the mean distance of the rubbing surfaces from the axis.

For a cylindrical journal, the leverage of the friction is simply the radius of the journal.

For a *flat pivot*, the leverage is two-thirds of the radius of the pivot.

For a *collar*, let r and r' be the inner and outer radii; then the leverage of the friction is

$$\frac{2}{3} \cdot \frac{r^3 - r'^3}{r^2 - r'^2} \quad \text{(1.)}$$

For "*Schiele's anti-friction pivot*," whose longitudinal section is the curve called the "tractrix," the moment of friction is $f \times$ the load $\times$ the external radius. This is greater than the moment for an equally smooth flat pivot of the same radius; but the anti-friction pivot has the advantage, inasmuch as the wear of the surfaces is uniform at every point, so that they always fit each other accurately, and the pressure is always uniformly distributed, and never becomes, as is the case in other pivots, so intense at certain points as to force out the unguent and grind the surfaces.

In the *cup and ball* pivot, the end of the shaft, and the step on which it presses, present two recesses facing each other, into which are fitted two shallow cups of steel or hard bronze. Between the concave spherical surfaces of those cups is placed a steel ball, being either a complete sphere, or a lens having convex surfaces of a somewhat less radius than the concave surfaces of the cups. The moment of friction of this pivot is at first almost inappreciable, from the extreme smallness of the radius of the circles of contact of the ball and cups; but as they wear, that radius and the moment of friction increase.

By the rolling of two surfaces over each other without sliding, a resistance is caused, which is called sometimes "rolling friction," but more correctly *rolling resistance*. It is of the nature of a *couple* resisting rotation; its *moment* is found by multiplying the normal pressure between the rolling surfaces by an *arm* whose length depends on the nature of the rolling surfaces; and the work lost in an unit of time in overcoming it is the product of its moment by the *angular velocity* of the rolling surfaces relatively to each other. The following are approximate values of the arm in *decimals of a foot*—

Oak upon oak, ... h........0·006 (Coulomb).
Lignum-vitæ on oak, ... 0·004 —
Cast-iron on cast-iron, ... 0·002 (Tredgold).

The work lost in friction produces HEAT in the proportion of one British thermal unit, being so much heat as raises the temperature of a pound of water one degree of Fahrenheit, for every 772 foot-pounds of lost work.

The heat produced by friction, when moderate in amount, is useful in softening and liquefying unguents; but when excessive, it is prejudicial by decomposing the unguents, and sometimes even by softening the metal of the bearings, and raising their temperature so high, as to set fire to neighbouring combustible matters.

Excessive heating is prevented by a constant and copious supply of a good unguent. When the velocity of rubbing is about four or five feet per second, the elevation of temperature is found to be, with good fatty and soapy unguents, 40° to 50° Fahrenheit; with good mineral unguents, about 30°. The effect of friction upon the efficiency of machines will be considered further on, in Section IV.

312. **Work of Acceleration.** (*A. M.*, 12, 521-33, 536, 547, 549, 554, 589, 591, 593, 595-7.)—In order that the velocity of a body's motion may be changed, it must be acted upon by some other body with a force in the direction of the change of velocity, which force is proportional directly to the change of velocity, and to the mass of the body acted upon, and inversely to the time occupied in producing the change. If the change is an acceleration or increase of velocity, let the first body be called the *driven body*, and the second the *driving body*. Then the force must act upon the driven body in the direction of its motion. Every force being a pair of equal and opposite actions between a pair of bodies, the same force which accelerates the driven body is a *resistance* as respects the driving body. (See Article 287, page 329.)

For example, during the commencement of the stroke of the piston of a steam engine, the velocity of the piston and of its rod is accelerated; and that acceleration is produced by a certain part of the pressure between the steam and the piston, being the excess of that pressure above the whole resistance which the piston has to overcome. The piston and its rod constitute the driven body; the steam is the driving body; and the same part of the pressure which accelerates the piston, acts as a *resistance* to the motion of the steam, in addition to the resistance which would have to be overcome if the velocity of the piston were uniform.

The resistance due to acceleration is computed in the following manner:—It is known by experiment, that if a body near the earth's surface is accelerated by the attraction of the earth,—that is, by its own weight, or by a force equal to its own weight, its velocity goes on continually increasing very nearly at the rate of 32·2 *feet per second of additional velocity, for each second during which the force acts.* This quantity varies in different latitudes, and at different elevations, but the value just given is near enough to the truth for purposes of mechanical engineering. For brevity's sake, it is usually denoted by the symbol g; so that if at a given instant the velocity of a body is v_1 feet per second, and if its own weight, or an equal force, acts freely on it in the direction of its motion for t seconds, its velocity at the end of that time will have increased to

$$v_2 = v_1 + g\,t \text{ .... (1.)}$$

If the acceleration be at any different rate per second, *the force necessary to produce that acceleration, being the resistance on the driving body due to the acceleration of the driven body, bears the same proportion to the driven body's weight which the actual rate of acceleration bears to the rate of acceleration produced by gravity acting freely.* (In mètres per second, $g = 9·81$ nearly.)

To express this by symbols, let the weight of the driven body be denoted by W. Let its velocity at a given instant be v_1 feet per second; and let that velocity increase at an uniform rate, so that at an instant t seconds later, it is v_2 feet per second.

Let f denote the rate of acceleration; then

$$f = \frac{v_2 - v_1}{t}; \text{ .... (2.)}$$

and the force R necessary to produce it will be given by the proportion,

$$g : f :: \mathrm{W} : \mathrm{R};$$

that is to say,

$$\mathrm{R} = \frac{f\,\mathrm{W}}{g} = \frac{\mathrm{W}\,(v_2 - v_1)}{g\,t} \text{ .... (3.)}$$

The factor $\frac{\mathrm{W}}{g}$, in the above expression, is called the MASS of the driven body; and being the same for the same body, in what place soever it may be, is held to represent the *quantity of matter* in the body. (See Article 277A, page 318.)

The product $\frac{\mathrm{W}\,v}{g}$ of the mass of a body into its velocity at any instant, is called its MOMENTUM; so that the resistance due to a given acceleration is equal to *the increase of momentum divided by the time which that increase occupies.*

If the product of the force by which a body is accelerated, equal and opposite to the resistance due to acceleration, into the time during which it acts, be called IMPULSE, the same principle may be otherwise stated by saying, that the *increase of momentum is equal to the impulse by which it is caused.*

If the rate of acceleration is not constant, but variable, the force R varies along with it. In this case, the value, at a given instant of the rate of acceleration, is represented by $f = \frac{dv}{dt}$, and the corresponding value of the force is

$$R = \frac{fW}{g} = \frac{W}{g} \cdot \frac{dv}{dt} \text{...t..............(4.)}$$

The WORK PERFORMED in accelerating a body is the product of the resistance due to the rate of acceleration into the distance moved through by the driven body while the acceleration is going on. The resistance is equal to the mass of the body, multiplied by the increase of velocity, and divided by the time which that increase occupies. The distance moved through is the product of the mean velocity into the same time. Therefore, the work performed is equal to the mass of the body multiplied by the increase of the velocity, and by the mean velocity; that is, *to the mass of the body, multiplied by the increase of the half-square of its velocity.*

To express this by symbols, in the case of an uniform rate of acceleration, let s denote the distance moved through by the driven body during the acceleration; then

$$s = \frac{v_2 + v_1}{2} t; \text{........................(5.)}$$

which being multiplied by equation 3, gives for the work of acceleration,

$$R s = \frac{W}{g} \cdot \frac{v_2 - v_1}{t} \cdot \frac{v_2 + v_1}{2} \cdot t = \frac{W}{g} \cdot \frac{v_2^2 - v_1^2}{2} \text{........(6.)}$$

In the case of a variable rate of acceleration, let v denote the mean velocity, and ds the distance moved through, in an interval of time dt so short that the increase of velocity dv is indefinitely small compared with the mean velocity. Then

$$ds = v\, dt; \text{..........................(7.)}$$

which being multiplied by equation 4, gives for the work of acceleration during the interval $d\,t$,

$$R\,d\,s = \frac{W}{g}\,s\cdot\frac{d\,v}{d\,t}\cdot v\,d\,t$$

$$= \frac{W}{g}\cdot v\,d\,v; \qquad (8.)$$

and the *integration* of this expression (see Article 11 A) gives for the work of acceleration during a finite interval,

$$\int R\,d\,s = \frac{W}{g}\int v\,d\,v s = \frac{W}{g}\cdot\frac{v_2^2 - v_1^2}{2} \qquad (9.)$$

being the same with the result already arrived at in equation 6.

From equation 9 it appears that *the work performed in producing a given acceleration depends on the initial and final velocities,* v_1 *and* v_2, *and not on the intermediate changes of velocity.*

If a body falls freely under the action of gravity from a state of rest through a height h, so that its initial velocity is 0, and its final velocity v, the work of acceleration performed by the earth on the body is simply the product W h of the weight of the body into the height of fall. Comparing this with equation 6, we find—

$$h = \frac{v^2}{2g} \qquad (10.)$$

This quantity is called the *height*, or *fall, due to the velocity* v; and from equations 6 and 9 it appears that *the work performed in producing a given acceleration is the same with that performed in lifting the driven body through the difference of the heights due to its initial and final velocities.*

If work of acceleration is performed by a prime mover upon bodies which neither form part of the prime mover itself, nor of the machines which it is intended to drive, that work is lost; as when a marine engine performs work of acceleration on the water that is struck by the propeller.

Work of acceleration performed on the moving pieces of the prime mover itself, or of the machinery driven by it, is not necessarily lost, as will afterwards appear.

313. **Summation of Work of Acceleration—Moment of Inertia.**— If several pieces of a machine have their velocities increased at the same time, the work performed in accelerating them is the sum of the several quantities of work due to the acceleration of the respective pieces; a result expressed in symbols by

$$\Sigma\left\{\frac{W}{g}\cdot\frac{v_2^2 - v_1^2}{2}\right\} \qquad (1.)$$

The process of finding that sum is facilitated and abridged in certain cases by special methods.

I. *Accelerated Rotation—Moment of Inertia.*—Let a denote the angular velocity of a solid body rotating about a fixed axis;—that is, as explained in Article 46, the velocity of a point in the body whose radius-vector, or distance from the axis, is unity.

Then the velocity of a particle whose distance from the axis is r, is

$$v = a\,r; \text{..............................(2.)}$$

and if in a given interval of time the angular velocity is accelerated from the value a_1 to the value a_2, the increase of the velocity of the particle in question is

$$v_2 - v_1 = r\,(a_2 - a_1) \text{..............................(3.)}$$

Let w denote the weight, and $\frac{w}{g}$ the mass of the particle in question. Then the work performed in accelerating it, being equal to the product of its mass into the increase of the half-square of its velocity, is also equal to *the product of its mass into the square of its radius-vector, and into the increase of the half-square of the angular velocity;* that is to say, in symbols,

$$\frac{w}{g} \cdot \frac{v_2^2 - v_1^2}{2} = \frac{w\,r^2}{g} \cdot \frac{a_2^2 - a_1^2}{2} \text{.................(4.)}$$

To find the work of acceleration for the whole body, it is to be conceived to be divided into small particles, whose velocities at any given instant, and also their accelerations, are proportional to their distances from the axis; then the work of acceleration is to be found for each particle, and the results added together. But in the sum so obtained, the increase of the half-square of the angular velocity is a common factor, having the same value for each particle of the body; and the rate of acceleration produced by gravity, $g = 32{\cdot}2$, is a common divisor. It is therefore sufficient *to add together the products of the weight of each particle* (w) *into the square of its radius-vector* (r^2), *and to multiply the sum so obtained* ($\Sigma \cdot w\,r^2$) *by the increase of the half-square of the angular velocity* $\left(\frac{1}{2}\,(a_2^2 - a_1^2)\right)$, *and divide by the rate of acceleration due to gravity* (g). The result, viz. :—

$$\Sigma\left\{\frac{w}{g} \cdot \frac{v_2^2 - v_1^2}{2}\right\} = \frac{a_2^2 - a_1^2}{2\,g} \cdot \Sigma\,w\,r^2 \text{.........(5.)}$$

is the work of acceleration sought. In fact, the sum $\Sigma\,w\,r^2$ is *the weight of a body, which, if concentrated at the distance unity from* *the axis of rotation, would require the same work to produce a given increase of angular velocity which the actual body requires.*

The term MOMENT OF INERTIA is applied in some writings to the sum $\Sigma w r^2$, and in others to the corresponding *mass* $\Sigma w r^2 \div g$. For purposes of mechanical engineering, the sum $\Sigma w r^2$ is, on the whole, the most convenient, bearing as it does the same relation to angular acceleration which *weight* does to acceleration of linear velocity.

The *Radius of Gyration*, or *Mean Radius* of a rotating body, is a line whose square is the mean of the squares of the distances of its particles from the axis; and its value is given by the following equation:—

$$\varrho^2 = \frac{\Sigma w r^2}{\Sigma w} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.)$$

so that if we put $W = \Sigma w$ for the weight of the whole body, the moment of inertia may be represented by

$$I = W \varrho^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

The following are additional rules relating to moments of inertia and radii of gyration :—

RULE II.—Given, the moment of inertia of a body about an axis traversing its centre of gravity in a given direction; to find its moment of inertia about another axis parallel to the first; multiply the mass (or weight) of the body by the square of the perpendicular distance between the two axes, and to the product add the given moment of inertia.

RULE III.—Given, the separate moments of inertia of a set of bodies about parallel axes traversing their several centres of gravity; required, the combined moment of inertia of those bodies about a common axis parallel to their separate axes; multiply the mass (or weight) of each body by the square of the perpendicular distance of its centre of gravity from the common axis; add together all the products, and all the separate moments of inertia; the sum will be the combined moment of inertia.

RULE IV.—To find the square of the radius of gyration of a body about a given axis; divide the moment of inertia of the body about the given axis by the mass (or weight) of the body.

RULE V.—Given, the square of the radius of gyration of a body about an axis traversing its centre of gravity in a given direction; to find the square of the radius of gyration of the same body about another axis parallel to the first; to the given square add the square of the perpendicular distance between the two axes.

Table of Squares of Radii of Gyration.

Body.	Axis.	Radius.[2]
I. Sphere of radius r,	Diameter	$\frac{2r^2}{5}$
II. Spheroid of revolution—polar semi-axis a, equatorial radius r,	Polar axis	$\frac{2r^2}{5}$
III. Ellipsoid—semi-axes a, b, c,	Axis, $2a$	$\frac{b^2+c^2}{5}$
IV. Spherical shell—external radius r, internal r',	Diameter	$\frac{2(r^5-r'^5)}{5(r^3-r'^3)}$
V. Spherical shell, insensibly thin—radius r, thickness dr,	Diameter	$\frac{2r^2}{3}$
VI. Circular cylinder—length $2a$, radius r,	Longitudinal axis, $2a$	$\frac{r^2}{2}$
VII. Elliptic cylinder—length $2a$, transverse semi-axes b, c,	Longitudinal axis, $2a$	$\frac{b^2+c^2}{4}$
VIII. Hollow circular cylinder—length $2a$, external radius r, internal r',	Longitudinal axis, $2a$	$\frac{r^2+r'^2}{2}$
IX. Hollow circular cylinder, insensibly thin—length $2a$, radius r, thickness dr,	Longitudinal axis, $2a$	r^2
X. Circular cylinder—length $2a$, radius r,	Transverse diameter	$\frac{r^2}{4}+\frac{a^2}{3}$
XI. Elliptic cylinder—length $2a$, transverse semi-axes b, c,	Transverse axis, $2b$	$\frac{c^2}{4}+\frac{a^2}{3}$
XII. Hollow circular cylinder—length $2a$, external radius r, internal r',	Transverse diameter	$\frac{r^2+r'^2}{4}+\frac{a^2}{3}$
XIII. Hollow circular cylinder, insensibly thin—radius r, thickness dr,	Transverse diameter	$\frac{r^2}{2}+\frac{a^2}{3}$
XIV. Rectangular prism—dimensions $2a$, $2b$, $2c$,	Axis, $2a$	$\frac{b^2+c^2}{3}$
XV. Rhombic prism—length $2a$, diagonals $2b$, $2c$,	Axis, $2a$	$\frac{b^2+c^2}{6}$
XVI. Rhombic prism, as above,	Diagonal, $2b$	$\frac{c^2}{6}+\frac{a^2}{3}$

314. **Centre of Percussion—Equivalent Simple Pendulum—Equivalent Fly-wheel.**—In calculations respecting the rotation of a rigid body about a given axis, it is often convenient to conceive that for the actual body there is substituted its *equivalent simple pendulum;* that is, a body having the same total mass, but concentrated at two points, of which one is in the axis: also the same statical moment, and the same moment of inertia.

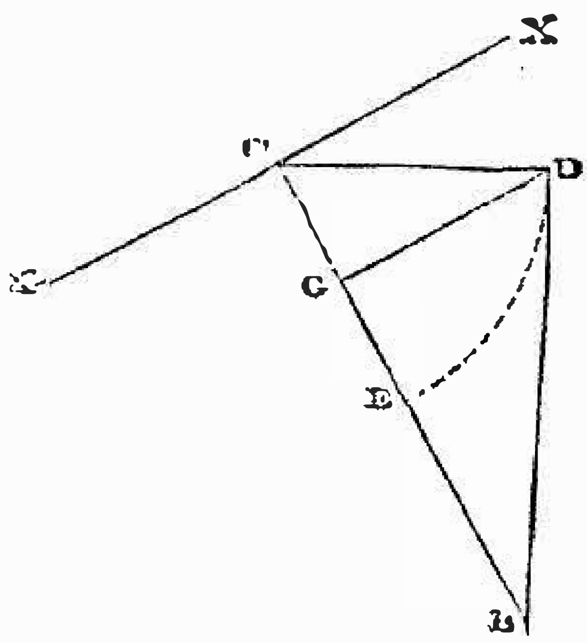

Fig. 243.

The following are the rules for doing this:—

I. To find the centre of percussion of a given body turning about a given axis.

In fig. 243, let X X be the given axis, and G the centre of gravity of the body. From G let fall G C perpendicular to X X. Through G draw G D parallel to X X, and equal to the radius of gyration of the body about the axis G D. Join C D. Then will $C E = C D = \sqrt{G D^2 + C G^2}$ = the radius of gyration of the body about X X. From D draw D B perpendicular to C D, cutting C G produced in B. Then will B be the centre of percussion of the body for the axis X X.

To find B by calculation; make $G B = \frac{G D^2}{G C}$.

C is the centre of percussion for an axis traversing B parallel to X X.

II. To convert the body into an "equivalent simple pendulum" for the axis X X, or for an axis through B parallel to X X; divide the mass of the body into two parts inversely proportional to G C and G B, and conceive those parts to be concentrated at C and B respectively, and rigidly connected together.

(Let W be the whole mass, and C and B the two parts; then $C = \frac{W \cdot G B}{C B}$; $B = \frac{W \cdot G C}{C B}$.)

(The "equivalent simple pendulum" has the same weight with the given body, and also the same moment of weight, and the same moment of inertia, with the given body, relatively to an axis in the given direction X X, traversing either C or B.)

III. *Equivalent Ring, or Equivalent Fly-wheel.*—When the given axis traverses the centre of gravity, G, there is no centre of percussion. The moment of the body's weight is nothing, and its moment of inertia is the same as if its whole mass were concentrated in a ring of a radius equal to the radius of gyration of the body. That ring may be called the "equivalent ring," or "equivalent fly-wheel."

315. **Reduced Inertia.**—If in a certain machine, a moving piece whose weight is W has a velocity always bearing the ratio $n : 1$ to the velocity of the driving point, it is evident that when the driving point undergoes a given acceleration, the work performed in producing the corresponding acceleration in the piece in question is the same with that which would have been ~~required~~ if a weight n^2 W had been concentrated at the driving point.

If a similar calculation be performed for each moving piece in the machine, and the results added together, the sum

$$\Sigma \cdot n^2 \mathrm{W} \ldots\ldots\ldots\ldots \mathrm{h} \ldots\ldots\ldots \mathrm{h} \ldots\ldots (1.)$$

gives the weight which, being concentrated at the driving point, would require the same work for a given acceleration of the driving point that the actual machine requires; so that if v_1 is the initial, and v_2 the final velocity of the driving point, the work of acceleration of the whole machine is

$$\frac{v_2^2 - v_1^2}{2g} \cdot \Sigma \cdot n^2 \mathrm{W} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

This operation may be called *the reduction of the inertia to the driving point.* Mr. Moseley, by whom it was first introduced into the theory of machines, calls the expression (1.) the "*co-efficient of steadiness,*" for reasons which will afterwards appear.

In finding the reduced inertia of a machine, the mass of each rotating piece is to be treated as if concentrated at a distance from its axis equal to its radius of gyration ρ; so that if v represents the velocity of the driving point at any instant, and a the corresponding angular velocity of the rotating piece in question, we are to make

$$n^2 = \frac{a^2 \rho^2}{v^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

in performing the calculation expressed by the formula (1.)

316. **Summary of Various Kinds of Work.**—In order to present at one view the symbolical expression of the various modes of performing work described in the preceding articles, let it be supposed that in a certain interval of time dt the driving point of a machine moves through the distance ds; that during the same time its centre of gravity is elevated through the height dh; that resistances, any one of which is represented by R, are overcome at points, the respective ratios of whose velocities to that of the driving point are denoted by n; that the weight of any piece of the mechanism is W, and that n' denotes the ratio of its velocity (or if it rotates, the ratio of the velocity of the end of its radius of gyration) to the velocity of the driving point; and that the driving point, whose mean velocity

is $v = \frac{d s}{d t}$, undergoes the acceleration $d v$. Then the *whole work performed* during the interval in question is

$$d h \cdot \Sigma \mathrm{W} + d s \cdot \Sigma n \mathrm{R} + \frac{v d v}{g} \cdot \Sigma n'^2 \mathrm{W} \ldots (1.)$$

The *mean total resistance, reduced to the driving point,* may be computed by dividing the above expression by the motion of the driving point $d s = v d t$, giving the following result:—

$$\frac{d h}{d s} \cdot \Sigma \mathrm{W} + \Sigma n \mathrm{R} + \frac{d v}{g d t} \cdot \Sigma n'^2 \mathrm{W} \ldots\ldots (2.)$$

SECTION II.—*Of Deviating and Centrifugal Force.*

317. **Deviating Force of a Single Body.** (*A. M.*, 537.)—It is part of the first law of motion, that if a body moves under no force, or balanced forces, it moves in a straight line. (*A. M.*, 510, 512.)

It is one consequence of the second law of motion, that in order that a body may move in a curved path, it must be continually acted upon by an unbalanced force at right angles to the direction of its motion, the direction of the force being that towards which the path of the body is curved, and its magnitude bearing the same ratio to the weight of the body that the height due to the body's velocity bears to half the radius of curvature of its path.

This principle is expressed symbolically as follows:—

Half radius of curvature.		Height due to velocity.		Body's weight.		Deviating force.
$\frac{r}{2}$	:	$\frac{v^2}{2g}$	::	W	:	$\mathrm{Q} = \frac{\mathrm{W} v^2}{g r} \ldots\ldots (1.)$

In the case of projectiles and of the heavenly bodies, deviating force is supplied by that component of the mutual attraction of two masses which acts perpendicular to the direction of their relative motion. In machines, deviating force is supplied by the strength or rigidity of some body, which *guides* the revolving mass, making it move in a curve.

A pair of free bodies attracting each other have both deviated motions, the attraction of each guiding the other; and their deviations of motion relatively to their common centre of gravity are inversely as their masses.

In a machine, each revolving body tends to press or draw the body which guides it away from its position, in a direction from the centre of curvature of the path of the revolving body; and that tendency is resisted by the strength and stiffness of the guiding body, and of the frame with which it is connected.

318. **Centrifugal Force** (*A. M.*, 538) is the force with which a revolving body reacts on the body that guides it, and is equal and opposite to the deviating force with which the guiding body acts on the revolving body.

In fact, as has been already stated, every force is an action between two bodies; and *deviating force* and *centrifugal force* are but two different names for the same force, applied to it according as the condition of the revolving body or that of the guiding body is under consideration at the time.

319. **A Revolving Pendulum** is one of the simplest practical applications of the principles of deviating force, and is described here because its use in regulating the speed of prime movers will afterwards have to be referred to. It consists of a ball A, suspended from a point C by a rod C A of small weight as compared with the ball, and revolving in a circle about a vertical axis C B. The tension of the rod is the resultant of the weight of the ball A, acting vertically, and of its centrifugal force, acting horizontally; and therefore the rod will assume such an inclination that

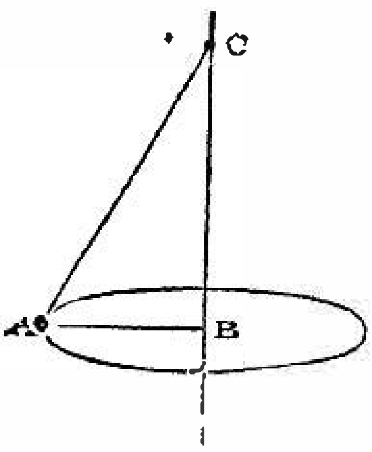

Fig. 244.

$$\frac{\text{height } \overline{\text{B C}}}{\text{radius } \overline{\text{A B}}} = \frac{\text{weight}}{\text{centrifugal force}} = \frac{g\,r}{v^2} \ldots (1.)$$

where r = A B. Let T be the *number of turns per second* of the pendulum, and a its angular velocity; then

$$v = a\,r = 2\,\pi\,\text{T}\,r;$$

and therefore, making $\overline{\text{B C}} = h$,

$$h = \frac{g\,r^2}{v^2} = \frac{g}{a^2} = \frac{g}{4\,\pi^2\,\text{T}^2}$$

$$= \text{(in the latitude of London)}\ \frac{0{\cdot}8154 \text{ foot}}{\text{T}^2} = \frac{9{\cdot}7848 \text{ inches}}{\text{T}^2} \ldots (2.)$$

320. **Deviating Force in Terms of Angular Velocity.** (*A. M.*, 540.) —When a body revolves in a circular path round a fixed axis, as is almost always the case with the revolving parts of machines, the radius of curvature of its path, being the perpendicular distance of the body from the axis, is constant; and the velocity v of the body is the product of that radius into the angular velocity; or symbolically,

$$v = a\,r = 2\,\pi\,\text{T}\,r.$$

If these values of the velocity be substituted for v in equation 1 of Article 317, it becomes—

$$Q = \frac{W a^2 r}{g} = \frac{W \cdot 4 \pi^2 T^2 r}{g} \ldots\ldots\ldots\ldots\ldots (1.)$$

321. **Resultant Centrifugal Force.** (*A. M.*, 603.)—The whole centrifugal force of a body of any figure, or of a system of connected bodies, rotating about an axis, is the same in *amount* and *direction* as if the whole mass were concentrated at the centre of gravity of the system. That is to say, in the formula of Article 320, W is to be held to represent the weight of the whole body or system, and r the perpendicular distance of its centre of gravity from the axis; and the line of action of the resultant centrifugal force Q is always *parallel* to r, although it does not in every case *coincide* with r.

When the axis of rotation *traverses* the centre of gravity of the body or system, the amount of the centrifugal force is *nothing;* that is to say, the rotating body does not tend to pull its axis as a whole out of its place.

The centrifugal forces exerted by the various rotating pieces of a machine against the bearings of their axles are to be taken into account in determining the lateral pressures which cause friction, and the strength of the axles and framework.

As those centrifugal forces cause increased friction and stress, and sometimes, also, by reason of their continual change of direction, produce detrimental or dangerous vibration, it is desirable to reduce them to the smallest possible amount; and for that purpose, unless there is some special reason to the contrary, the axis of rotation of every piece which rotates rapidly ought to traverse its centre of gravity, that the resultant centrifugal force may be nothing.

322. **Centrifugal Couple—Permanent Axis.**—It is not, however, sufficient to annul the effect of centrifugal force, that there should be no tendency to *shift* the axis as a whole; there should also be no tendency to *turn* it into a new angular position.

To show, by the simplest possible example, that the latter tendency may exist without the former, let the axis of rotation of the system shown in fig. 245 be the centre line of an axle resting in bearings at E and F. At B and D let two arms project perpendicularly to that axle, in opposite directions in the same plane, carrying at their extremities two heavy bodies A and C. Let the weights of the arms be

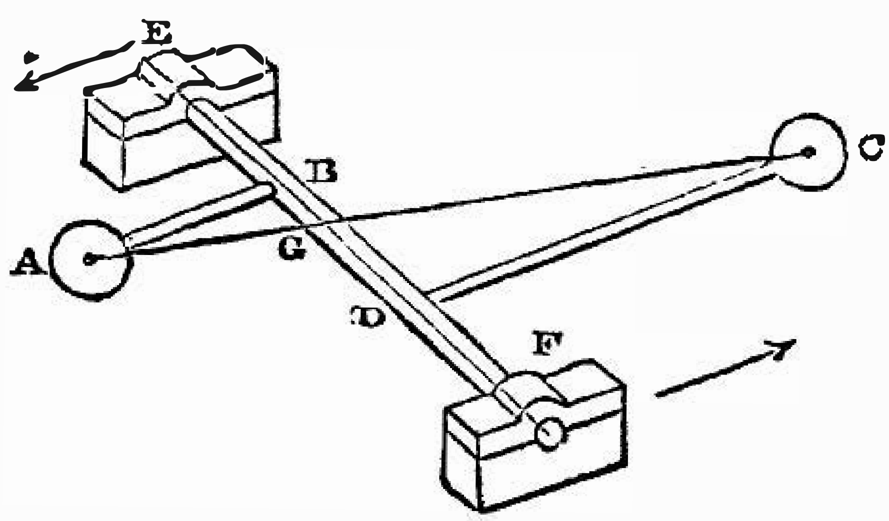

Fig. 245.

insensible as compared with the weights of those bodies; and let the weights of the bodies be inversely as their distances from the axis; that is, let

$$A \cdot \overline{AB} = C \cdot \overline{CD}.$$

Let A C be a straight line joining the centres of gravity of A and C, and cutting the axis in G; then G is the common centre of gravity of A and C, and being in the axis, the resultant centrifugal force is nothing.

In other words, let a be the angular velocity of the rotation; then

The centrifugal force exerted on the axis by A

$$= \frac{a^2 A \cdot \overline{AB}}{g};$$

The centrifugal force exerted on the axis by C

$$= \frac{a^2 C \cdot \overline{CD}}{g};$$

and those forces are equal in magnitude and opposite in direction; so that there is no tendency to remove the point G in any direction.

There is, however, a tendency to *turn the axis about* the point G, being the product of the common magnitude of the *couple* of centrifugal forces above stated, into their leverage; that is, the perpendicular distance $\overline{BD}$ between their lines of action. That product is called *the moment of the centrifugal couple;* and is represented by

$$Q \cdot \overline{BD}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots h \ldots (1.)$$

Q being the common magnitude of the equal and opposite centrifugal forces.

That couple causes a couple of equal and opposite pressures of the journals of the axle against their bearings at E and F, in the directions represented by the arrows, and of the magnitude given by the formula—

$$Q \cdot \frac{\overline{BD}}{\overline{EF}}; \ldots h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

these pressures continually change their directions as the bodies A and C revolve; and they are resisted by the strength and rigidity of the bearings and frame. It is desirable, when practicable, to reduce them to nothing; and for that purpose, the points B, G, and D should coincide; in which case the centre line of the axle E F is said to be *a permanent axis.*

When there are more than two bodies in the rotating system, the centrifugal couple is found as follows:—

LetsX X′, fig. 246, represent the axis of rotation; G, the centre of gravity of the rotating body or system, situated in that axis; so that the resultant centrifugal force is nothing.

Let W be any one of the parts of which the body or system is composed, so that, the weight of that part being denoted by W, the weight of the whole body or system may be denoted by $\Sigma \cdot W$.

Let r denote the perpendicular distance of the centre of W from the axis; then

$$\frac{W a^2 r o}{g},$$

is the centrifugal *force* of W, pulling the axis in the direction x W.

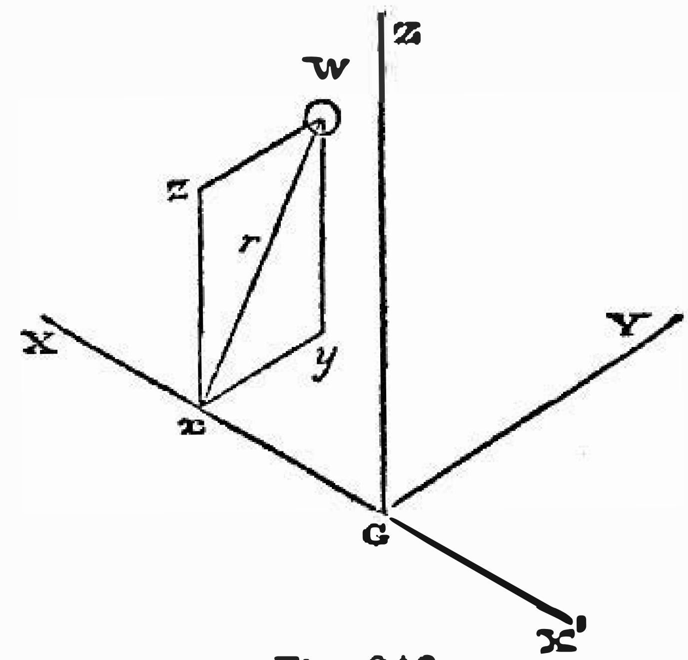

Fig. 246.

Assume a pair of axes of co-ordinates, G Z, G Y, perpendicular to X X′ and to each other, and fixed relatively to the rotating body or system—that is, rotating along with it.

From W let fall $\overline{Wy}$ perpendicular to the plane of G X and G Y, and parallel to G Z; also $\overline{Wz}$, perpendicular to the plane of G X and G Z, and parallel to G Y; and make

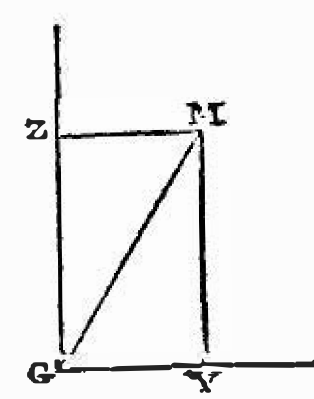

Fig. 247.

$$\overline{xy} = \overline{Wz} = y;\quad \overline{xz} = \overline{Wy} = z;\quad \overline{Gx} = x.$$

Then the centrifugal force which W exerts on the axis, and which is proportional tosr, may be resolved into two components, in the direction of, and proportional to, y and z respectively, viz.:—

$$\frac{W a^2 y}{g} \text{ parallel to G Y, and}$$

$$\frac{W a^2 z}{g} \text{ parallel to G Z;}$$

and those two component forces, being both applied at the end of the lever $\overline{Gx} = x$, exert *moments*, or tendencies to turn the axis X X′ about the point Z, expressed as follows:—

$$\frac{W a^2 y x}{g}, \text{ tending to turn G X about G Z towards G Y;}$$

$$\frac{W a^2 z x}{g}, \text{ tending to turn G X about G Y towards G Z.}$$

In the same manner are to be found the several moments of the centrifugal forces of all the other parts of which the body or system consists; and care is to be taken to distinguish moments which tend to turn the axis *towards* G Y or G Z from those which tend to turn it *from* those positions, by treating one of these classes of quantities as positive, and the other as negative.

Then by adding together the positive moments and subtracting the negative moments for all the parts of the body or system, are to be found the two sums,

$$\frac{a^2}{g} \cdot \Sigma \cdot \mathrm{W}\, y\, x\,;\ \frac{a^2}{g} \cdot \Sigma \cdot \mathrm{W}\, z\, x\,;\ldots\ldots\ldots\ldots\ldots (3.)$$

which represent the total tendencies of all the centrifugal forces to turn the axis in the planes of G Y and G Z respectively.

In fig. 247 lay down $\overline{\mathrm{G\,Y}}$ to represent the former moment, and $\overline{\mathrm{G\,Z}}$, perpendicular to G Y, to represent the latter. Then the diagonal $\overline{\mathrm{G\,M}}$ of the rectangle G Z M Y will represent the resultant moment of what is called the CENTRIFUGAL COUPLE, and the direction of that line will indicate the direction in which that couple tends to turn the axis G X about the point G. Its value, and its angular position, are given by the equations,

$$\left.\begin{array}{l} \overline{\mathrm{G\,M}} = \sqrt{(\overline{\mathrm{G\,Y}}^2 + \overline{\mathrm{G\,Z}}^2)}\,; \\ \tan \angle\, \mathrm{Y\,G\,M} = \overline{\mathrm{G\,Z}} \div \overline{\mathrm{G\,Y}} \end{array}\right\} \ldots\ldots\ldots\ldots (4.)$$

The condition which it is desirable to fulfil in all rapidly rotating pieces of machines, that the axis of rotation shall be a *permanent axis*, is fulfilled when each of the sums in the formula 3 is nothing; that is, when

$$\Sigma \cdot \mathrm{W}\, y\, x = 0\,;\ \Sigma \cdot \mathrm{W}\, z\, x = 0,\ldots\ldots\ldots\ldots (5.)$$

The question, whether the axis of a rotating piece is a permanent axis or not, is tested experimentally by making the piece spin round rapidly with its shaft resting in bearings which are suspended by chains or cords, so as to be at liberty to swing. If the axis is not a permanent axis, it oscillates; if it is, it remains steady.

When the axis of rotation traverses the centre of gravity of the piece, there is said to be a STANDING BALANCE; when it is also a permanent axis, there is said to be a RUNNING BALANCE.

SECTION III.—*Of Effort, Energy, Power, and Efficiency.*

323. **Effort** is a name applied to a force which acts on a body in the direction of its motion (*A. M.*, 511).

If a force is applied to a body in a direction making an acute angle with the direction of the body's motion, the component of that oblique force along the direction of the body's motion is an effort. That is to say, in fig. 248, let A B represent the direction in which A is moving; let $\overline{AF}$ represent a force applied to A, obliquely to that direction; from F draw $\overline{FP}$ perpendicular to A B; then $\overline{AP}$ is the *effort* due to the force $\overline{AF}$. The transverse component $\overline{PF}$ is a *lateral force*, like the transverse component of the oblique resisting force in Article 304.

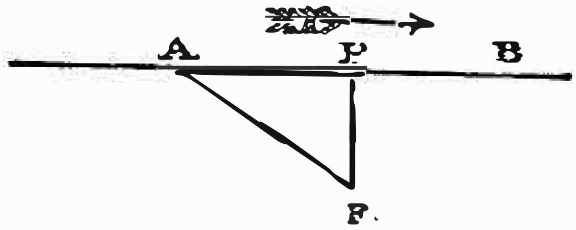

Fig. 248.

To express this algebraically, let the entire force $\overline{AF} = F$, the effort $\overline{AP} = P$, the lateral force $\overline{PF} = Q$, and the angle of obliquity $PAF = \theta$. Then

$$\left.\begin{array}{l} P = F \cdot \cos\theta; \\ Q = F \cdot \sin\theta \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

324. **Condition of Uniform Speed.** (*A. M.*, 510, 512, 537.)—According to the first law of motion, in order that a body may move uniformly, the forces applied to it, if any, must balance each other; and the same principle holds for a machine consisting of any number of bodies.

When the *direction* of a body's motion varies, but not the *velocity*, the lateral force required to produce the change of direction depends on the principles set forth in Section 2; but the condition of balance still holds for the forces which act *along* the direction of the body's motion, that is, for the *efforts* and *resistances*; so that, whether for a single body or for a machine, the condition of *uniform velocity* is, that the *efforts shall balance the resistances.*

In a machine, this condition must be fulfilled for each of the single moving pieces of which it consists.

It can be shown from the principles of statics (that is, the science of balanced forces), that in any body, system, or machine, that condition is fulfilled when *the sum of the products of the efforts into the velocities of their respective points of action is equal to the sum of the products of the resistances into the velocities of the points where they are overcome.*

Thus, let v be the velocity of a *driving point*, or point where an effort P is applied; v' the velocity of a *working point*, or point where a resistance R is overcome; the condition of uniform velocity for any body, system, or machine is

$$\Sigma \cdot P v = \Sigma \cdot R v' \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

If there be only one driving point, or if the velocities of all the driving points be alike, then P being the total effort, the single product $P v$ may be put in in place of the sum $\Sigma \cdot P v$; reducing the above equation to

$$P v = \Sigma \cdot R v' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Referring now to Article 305, let the machine be one in which the *comparative* or *proportionate* velocities of all the points at a given instant are known independently of their absolute velocities, from the construction of the machine; so that, for example, the velocity of the point where the resistance R is overcome bears to that of the driving point the ratio

$$\frac{v'}{v} = n;$$

then the condition of uniform speed may be thus expressed :—

$$P = \Sigma \cdot n R; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

that is, *the total effort is equal to the sum of the resistances reduced to the driving point.*

325. **Energy—Potential Energy.** (*A. M.*, 514, 517, 593, 660.)—*Energy* means *capacity for performing work,* and is expressed, like work, by the product of a force into a space.

The energy of an effort, sometimes called "*potential energy*" (to distinguish it from another form of energy to be afterwards referred to), is the *product of the effort into the distance through which it is capable of acting.* Thus, if a weight of 100 pounds be placed at an elevation of 20 feet above the ground, or above the lowest plane to which the circumstances of the case admit of its descending, that weight is said to possess potential energy to the amount of $100 \times 20 = 2{,}000$ *foot-pounds;* which means, that in descending from its actual elevation to the lowest point of its course, the weight is *capable of performing work* to that amount.

To take another example, let there be a reservoir containing 10,000,000 gallons of water, in such a position that the centre of gravity of the mass of water in the reservoir is 100 feet above the lowest point to which it can be made to descend while overcoming resistance. Then as a gallon of water weighs 10 lbs., the weight of the store of water is 100,000,000 lbs., which being multiplied by the height through which that weight is capable of acting, 100 feet, gives 10,000,000,000 foot-pounds for the potential energy of the weight of the store of water.

326. **Equality of Energy Exerted and Work Performed.**—When an effort actually does drive its point of application through a certain distance, energy to the amount of the product of the effort into that distance is said to be *exerted;* and the potential energy,

or energy which remains *capable of being exerted*, is to that amount diminished.

When the energy is exerted in driving a machine at an uniform speed, it is *equal to the work performed.*

To express this algebraically, let t denote the time during which the energy is exerted, v the velocity of a driving point at which an effort P is applied, s the distance through which it is driven, v' the velocity of any working point at which a resistance R is overcome, s' the distance through which it is driven; then

$$s = v\,t;\ s' = v'\,t;$$

and multiplying equation 1 of Article 324 by the time t, we obtain the following equation:—

$$\Sigma \cdot \mathrm{P}\,v\,t = \Sigma \cdot \mathrm{R}\,v'\,t = \Sigma \cdot \mathrm{P}\,s = \Sigma \cdot \mathrm{R}\,s';\ldots\ldots\ldots(1.)$$

which expresses the equality of energy exerted, and work performed, for constant efforts and resistances.

When the efforts and resistances vary, it is sufficient to refer to Article 307, to show that the same principle is expressed as follows:—

$$\Sigma \int \mathrm{P}\,d\,s = \Sigma \int \mathrm{R}\,d\,s';\ldots\ldots\ldots\ldots\ldots(2.)$$

where the symbol $\int$ expresses the operation of finding the work performed against a varying resistance, or the energy exerted by a varying effort, as the case may be; and the symbol Σ expresses the operation of adding together the quantities of energy exerted, or work performed, as the case may be, at different points of the machine.

327. **Various Factors of Energy.**—A quantity of energy, like a quantity of work, may be computed by multiplying either a force into a distance, or a statical moment into an angular motion, or the intensity of a pressure into a volume. These processes have already been explained in detail in Articles 301 and 302, pages 340 to 341.

328. The **Energy Exerted in Producing Acceleration** (*A. M.*, 549) is equal to the work of acceleration, whose amount has been investigated in Articles 312 and 313, pages 354 to 357.

329. The **Accelerating Effort** (*A. M.*, 554) by which a given increase of velocity in a given mass is produced, and which is exerted by the *driving body* against the *driven body*, is equal and opposite to the resistance due to acceleration which the driven body exerts against the driving body, and whose amount has been given in Articles 312 and 313. Referring, therefore, to equations 4 and 8 of Article 312, we find the two following expressions, the first of which gives the accelerating effort required to produce a given

acceleration $d\,v$ in a body whose weight is W, when the *time* $d\,t$ in which that acceleration is to be produced is given, and the second, the same accelerating effort, when the *distance* $d\,s = v\,d\,t$ in which the acceleration is to be produced is given:—

$$\mathrm{P} = \frac{\mathrm{W}}{g} \cdot \frac{d\,v}{d\,t} \quad \text{......................(1.)}$$

$$= \frac{\mathrm{W}}{g} \cdot \frac{v\,d\,v}{d\,s} = \frac{\mathrm{W}}{g} \cdot \frac{d\,(v^2)}{2\,d\,s} \quad \text{..................(2.)}$$

Referring next to Article 313, page 35[illegible], we find from equations 5, 6, and 7, that the work of acceleration corresponding to an increase $d\,a$ in the angular velocity of a rotating body whose moment of inertia is I, is

$$\frac{\mathrm{I} \cdot d\,(a^2)}{2\,g} = \frac{\mathrm{I}\,a\,d\,a}{g}.$$

Let $d\,t$ be the *time*, and $d\,i = a\,d\,t$ the *angular motion* in which that acceleration is to be produced; let P be the accelerating effort, and l its *leverage*, or the perpendicular distance of its line of action from the axis; then, according as the time $d\,t$, or the angle $d\,i$, is given, we have the two following expressions for the *accelerating couple*:—

$$\mathrm{P}\,l = \frac{\mathrm{I}}{g} \cdot \frac{d\,a}{d\,t} \quad \text{......................(3.)}$$

$$= \frac{\mathrm{I}}{g} \cdot \frac{a\,d\,a}{d\,i} = \frac{\mathrm{I}}{g} \cdot \frac{d\,(a^2)}{2\,d\,i} \quad \text{..................(4.)}$$

Lastly, referring to Article 315, page 362, equation 2, we find, that if a train of mechanism consists of various parts, and if W be the weight of any one of those parts, whose velocity v' bears to that of the driving point v the ratio $\frac{v'}{v} = n$, then the accelerating effort which must be applied to the driving point, in order that, during the interval $d\,t$, in which the driving point moves through the distance $d\,s = v\,d\,t$, that point may undergo the acceleration $d\,v$, and each weight W the corresponding acceleration $n\,d\,v$, is given by one or other of the two formulæ—

$$\mathrm{P} = \frac{\Sigma\,n^2\,\mathrm{W}}{g} \cdot \frac{d\,v}{d\,t} \quad \text{......................(5.)}$$

$$= \frac{\Sigma\,n^2\,\mathrm{W}}{g} \cdot \frac{v\,d\,v}{d\,s} = \frac{\Sigma\,n^2\,\mathrm{W}}{g} \cdot \frac{d\,(v^2)}{2\,d\,s} \quad \text{............(6.)}$$

330. **Work During Retardation—Energy Stored and Restored.** (*A. M.*, 528, 549, 550.)—In order to cause a given retardation, or diminution of the velocity of a given body, in a given time, or while it traverses a given distance, resistance must be opposed to its motion equal to the effort which would be required to produce in the same time, or in the same distance, an acceleration equal to the retardation.

A moving body, therefore, while being retarded, *overcomes resistance* and *performs work;* and that work is equal to the energy exerted in producing an acceleration of the same body equal to the retardation.

It is for this reason that it has been stated, in Article 312, that the work performed in accelerating the speed of the moving pieces of a machine is not necessarily lost; for those moving pieces, by returning to their original speed, are capable of performing an equal amount of work in overcoming resistance; so that the performance of such work is not prevented, but only deferred. Hence energy exerted in acceleration is said to be *stored;* and when by a subsequent and equal retardation an equal amount of work is performed, that energy is said to be *restored.*

The algebraical expressions for the relations between a retarding resistance, and the retardation which it produces in a given body by acting during a given time or through a given space, are obtained from the equations of Article 329 simply by putting R, the symbol for a resistance, instead of P, the symbol for an effort, and $-\,d\,v$, the symbol for a retardation, instead of $d\,v$, the symbol fo an acceleration.

331. The **Actual Energy** (*A. M.*, 547, 589) of a moving body is the work which it is capable of performing against a retarding resistance before being brought to rest, and is equal to the energy which must be exerted on the body to bring it from a state of rest to its actual velocity. The value of that quantity is the *product of the weight of the body into the height from which it must fall to acquire its actual velocity;* that is to say,

$$\frac{\mathrm{W}\,v^2}{2\,g} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1.)}$$

The total actual energy of a system of bodies, each moving with its own velocity, is denoted by

$$\frac{\Sigma \cdot \mathrm{W}\,v^2}{2\,g}\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.)}$$

and when those bodies are the pieces of a machine, whose velocities bear definite ratios (any one of which is denoted by n) to the velocity of the driving point v, their total actual energy is

$$\frac{v^2}{2g} \cdot \Sigma n^2 \mathrm{W}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

being *the product of the reduced inertia* (or co-efficient of steadiness, as Mr. Moseley calls it) *into the height due to the velocity of the driving point.*

The actual energy of a rotating body whose angular velocity is a, and moment of inertia $\Sigma \mathrm{W} r^2 = \mathrm{I}$, is

$$\frac{a^2 \mathrm{I}}{2g}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.)$$

that is, *the product of the moment of inertia into the height due to the velocity, a, of a point, whose distance from the axis of rotation is unity.*

When a given amount of energy is alternately stored and restored by alternate increase and diminution in the speed of a machine, the actual energy of the machine is alternately increased and diminished by that amount.

Actual energy, like motion, is *relative* only. That is to say, in computing the actual energy of a body, which is the capacity it possesses of performing work upon certain other bodies by *reason of its motion*, it is the motion *relatively to those other bodies* that is to be taken into account.

For example, if it be wished to determine how many turns a wheel of a locomotive engine, rotating with a given velocity, would make, before being stopped *by the friction of its bearings only*, supposing it lifted out of contact with the rails,—the actual energy of that wheel is to be taken *relatively to the frame of the engine* to which those bearings are fixed, and is simply the actual energy due to the rotation. But if the wheel be supposed to be detached from the engine, and it is inquired *how high it will ascend up a perfectly smooth inclined plane before being stopped by the attraction of the earth*, then its actual energy is to be taken *relatively to the earth*; that is to say, to the energy of rotation already mentioned, is to be added the energy due to the *translation* or forward motion of the wheel along with its axis.

332. **A Reciprocating Force** (*A. M.*, 556) is a force which acts alternately as an effort and as an equal and opposite resistance, according to the direction of motion of the body. Such a force is the weight of a moving piece whose centre of gravity alternately rises and falls; and such is the elasticity of a perfectly elastic body.

The work which a body performs in moving against a reciprocating force is employed in increasing its own potential energy, and is not lost by the body; so that by the motion of a body alternately against and with a reciprocating force, energy is *stored* and *restored*, as well as by alternate acceleration and retardation.

Let ΣW denote the weight of the whole of the moving pieces of any machine, and h a height through which the common centre of gravity of them all is alternately raised and lowered. Then the quantity of energy—

$$h \Sigma W,$$

is stored while the centre of gravity is rising, and restored while it is falling.

These principles are illustrated by the action of the plungers of a single acting pumping steam engine. The weight of those plungers acts as a resistance while they are being lifted by the pressure of the steam on the piston; and the same weight acts as effort when the plungers descend and drive before them the water with which the pump barrels have been filled. Thus, the energy exerted by the steam on the piston is stored during the up-stroke of the plungers; and during their down-stroke the same amount of energy is restored, and employed in performing the work of raising water and overcoming its friction.

333. **Periodical Motion.** (*A. M.*, 553.)—If a body moves in such a manner that it periodically returns to its original velocity, then at the end of each period, the entire variation of its actual energy is nothing; and if, during any part of the period of motion, energy has been stored by acceleration of the body, the same quantity of energy exactly must have been during another part of the period restored by retardation of the body.

If the body also returns in the course of the same period to the same position relatively to all bodies which exert reciprocating forces on it—for example, if it returns periodically to the same elevation relatively to the earth's surface—any quantity of energy which has been stored during one part of the period by moving against reciprocating forces must have been exactly restored during another part of the period.

Hence *at the end of each period, the equality of energy and work, and the balance of* mean effort *and* mean resistance, *holds with respect to the driving effort and the resistances, exactly as if the speed were uniform and the reciprocating forces null;* and all the equations of Articles 324 and 326 are applicable to periodic motion, provided that in the equations of Article 324, and equation 1 of Article 326, P, R, and v are held to denote the *mean values* of the efforts, resistances, and velocities,—that s and s' are held to denote spaces moved through in one or more *entire periods*,—and that in equation 2 of Article 326, the integrations denoted by $\int$ be held to extend to one or more *entire periods.*

These principles are illustrated by the steam engine. The velocities of its moving parts are continually varying, and those of some of them, such as the piston, are periodically reversed in direction. But at the end of each period, called a *revolution*, or *double-stroke*, every part returns to its original position and velocity; so that the *equality of energy and work*, and the *equality of the mean effort to the mean resistance reduced to the driving point*,—that is, the equality of the mean effective pressure of the steam on the piston to the mean total resistance reduced to the piston—hold for one or any whole number of *complete revolutions*, exactly as for uniform speed.

It thus appears that (as stated at the commencement of this Part) there are two fundamentally different ways of considering a periodically moving machine, each of which must be employed in succession, in order to obtain a complete knowledge of its working.

"I. In the first place is considered the action of the machine during one or more whole periods, with a view to the determination of the relation between the mean resistances and mean efforts, and of the EFFICIENCY; that is, the ratio which the *useful* part of its work bears to the whole expenditure of energy. The motion of every ordinary machine is either uniform or periodical.

"II. In the second place is to be considered the action of the machine during intervals of time less than its period, in order to determine the law of the periodic changes in the motions of the pieces of which the machine consists, and of the periodic or reciprocating forces by which such changes are produced."

334. **Starting and Stopping.** (*A. M.*, 691.)—The *starting* of a machine consists in setting it in motion from a state of rest, and bringing it up to its proper mean velocity. This operation requires the exertion, besides the energy required to overcome the mean resistance, of an additional quantity of energy equal to the actual energy of the machine when moving with its mean velocity, as found according to the principles of Article 331, page 373.

If, in order to *stop* a machine, the effort of the prime mover is simply suspended, the machine will continue to go until work has been performed in overcoming resistances equal to the actual energy due to the speed of the machine at the time of suspending the effort of the prime mover.

In order to diminish the time required by this operation, the resistance may be increased by means of the friction of a *brake*. Brakes will be further described in the sequel.

335. The **Efficiency** of a machine is a fraction expressing the ratio

of the useful work to the whole work, which is equal to the energy expended. The COUNTER-EFFICIENCY is the reciprocal of the efficiency, and is the ratio in which the energy expended is greater than the useful work. The object of improvements in machines is to bring their efficiency and counter-efficiency as near to unity as possible.

As to useful and lost work, see Article 308. The algebraical expression of the efficiency of a machine having uniform or periodical motion, is obtained by introducing the distinction between useful and lost work into the equations of the conservation of energy. Thus, let P denote the mean effort at the driving point; s, the space described by it in a given interval of time, being a whole number of periods of revolutions; R_1, the mean useful resistance; s_1, the space through which it is overcome in the same interval; R_2, any one of the wasteful resistances; s_2, the space through which it is overcome; then

$$P s = R_1 s_1 + \Sigma \cdot R_2 s_2; \text{.................}(1.)$$

and the efficiency of the machine is expressed by

$$\frac{R_1 s_1}{P s} = \frac{R_1 s_1}{R_1 s_1 + \Sigma \cdot R_2 s_2} \text{.................}(2.)$$

In many cases the lost work of a machine, $R_2 s_2$, consists of a constant part, and of a part bearing to the useful work a proportion depending in some definite manner on the sizes, figures, arrangement, and connection of the pieces of the train, on which also depends the constant part of the lost work. In such cases the whole energy expended and the efficiency of the machine are expressed by the equations

$$\left.\begin{array}{l} P s = (1 + A) R_1 s_1 + B; \\ \dfrac{R_1 s_1}{P s} = \dfrac{1}{1 + A + \dfrac{B}{R_1 s_1}} \end{array}\right\} \text{................}(3.)$$

and the first of these is the mathematical expression of what Mr. Moseley calls the "modulus" of a machine.

The useful work of an intermediate piece in a train of mechanism consists in driving the piece which follows it, and is less than the energy exerted upon it by the amount of the work lost in overcoming its own friction. Hence the efficiency of such an intermediate piece is the ratio of the work performed by it in driving the following piece, to the energy exerted on it by the preceding piece; and it is evident that *the efficiency of a machine is the product of the efficiencies of the series of moving pieces which transmit energy from the driving point to the working point.* The same principle applies to a train of *successive machines*, each driving that which follows it; and to counter-efficiency as well as to efficiency.

336. **Power and Effect—Horse-Power.**—The *power* of a machine is the energy exerted, and the *effect*, the useful work performed, in some interval of time of definite length, such as a second, a minute, an hour, or a day.

The unit of power called conventionally a *horse-power*, is 550 foot-pounds per second, or 33,000 foot-pounds per minute, or 1,980,000 foot-pounds per hour. The effect is equal to the power multiplied by the efficiency; and the power is equal to the effect multiplied by the counter-efficiency. The loss of power is the difference between the effect and the power. As to the French "Force de Cheval," see Article 299, page 339. It is equal to 0·9863 of a British horse-power; and a British horse-power is 1·0139 force de cheval.

337. **General Equation.**—The following general equation presents at one view the principles of the action of machines, whether moving uniformly, periodically, or otherwise:—

$$\int P\, d\, s = \Sigma \int R\, d\, s' \pm h\, \Sigma\, W + \Sigma \cdot \frac{W\,(v_2^2 - v_1^2)}{2\, g};$$

where W is the weight of any moving piece of the machine;

h, when positive, the elevation, and when negative, the depression, which the common centre of gravity of all the moving pieces undergoes in the interval of time under consideration; v_1 the velocity at the beginning, and v_2 the velocity at the end, of the interval in question, with which a given particle of the machine of the weight W is moving;

g, the acceleration which gravity causes in a second, or 32·2 feet per second, or 9·81 mètres per second.

$\int R\, d\, s'$, the work performed in overcoming any resistance during the interval in question;

$\int P\, d\, s$, the energy exerted during the interval in question.

The second and third terms of the right-hand side, when positive, are *energy stored*; when negative, *energy restored*.

The principle represented by the equation is expressed in words as follows:—

The energy exerted, added to the energy restored, is equal to the energy stored added to the work performed.

338. **The Principle of Virtual Velocities**, when applied to the uniform motion of a machine, is expressed by equation 3 of Article 324, already given in page 369; or in words as follows:—*The effort is equal to the sum of the resistances reduced to the driving point;*

that is, each multiplied by the ratio of the velocity of its working point to the velocity of the driving point. The same principle, when applied to reciprocating forces and to re-actions due to varying speed, as well as to passive resistances, is expressed by means of a modified form of the general equation of Article 337, obtained in the following manner:—Let n denote either the ratio borne at a given instant by the velocity of a given working point, where the resistance R is overcome, to the velocity of the driving point, or the mean value of that ratio during a given interval of time; let n'' denote the corresponding ratio for the vertical ascent or descent (according as it is positive or negative) of a moving pieces whose weight is W; let n' denote the corresponding ratio for the mean velocity of a mass whose weight is W, undergoing acceleration or retardation, and $\frac{d v'}{d t}$ either the rate of acceleration of that mass, if the calculation relates to an instant, or the mean value of that rate, if to a finite interval of time. Then the effort at the instant, or the mean effort during the given interval, as the case may be, is given by the following equation:—

$$P = \Sigma \cdot n R + \Sigma \cdot n'' W + \Sigma \cdot \frac{n' W d v'}{g d t}.$$

If the ratio n', which the velocity of the mass W bears to that of the driving point, is constant, we may put $\frac{d v'}{d t} = \frac{n' d v}{d t}$, where $\frac{d v}{d t}$ denotes the rate of acceleration of the driving point; and then the third term of the foregoing expression becomes $\frac{d v}{g d t} \Sigma \cdot n'^2 W$, as in formula 2 of Article 216, page 229.

339. **Forces in the Mechanical Powers, neglecting Friction—Purchase.**—The mechanical powers, considered as means of modifying motion only, have been considered in Articles 221 to 224, pages 231 to 234. When friction is neglected, any one of the mechanical powers may be regarded as *an uniformly-moving simple machine, in which one effort balances one resistance;* and in which, consequently, according to the principle of virtual velocities, or of the equality of energy exerted and work done, *the effort and resistance are to each other inversely as the velocities along their lines of action of the points where they are applied.*

In the older writings on mechanics, the effort is called the *power*, and the resistance the *weight;* but it is desirable to avoid the use of the word "powers" in this sense, because of its being very commonly used in a different sense—viz., the rate at which energy is exerted by a prime mover; and the substitution of "resistance" for "weight" is made in order to express the fact, that the principle just stated applies to the overcoming of all sorts of resistance, and not to the lifting of weights only.

The weight of the moving piece itself in a mechanical power may either be wholly supported at the bearing, if the piece is balanced; or if not, it is to be regarded as divided into two parallel components, one supported directly at the bearing, and the other being included in the effort or in the resistance, as the case may be.

The relation between the effort and the resistance in any mechanical power may be deduced from the principles of statics; viz.:—In the case of the LEVER (including the *wheel and axle*), from the balance of couples of equal and opposite moments; in the case of the INCLINED PLANE (including the *wedge* and the *screw*), from the parallelogram of forces; and in the case of the pulley, from the composition of parallel forces. The principle of virtual velocities, however, is more convenient in calculation.

The *total load* in a mechanical power is the resultant of the effort, the resistance, the lateral components of the forces acting at the driving and working points, and the weight directly carrieds at the bearings; and it is equal and directly opposed to the re-action of the bearings or supports of the machine.

By the *purchase* of a mechanical power is to be understood the ratio borne by the resistance to the effort, which is equal to the ratio borne by the velocity of the driving point to that of the working point. This term has already been explained in connection with the pulley, in Article 201, pages 215, 216.

The following are the results of the principle of virtual velocities, as applied to determine the purchase in the several mechanical powers:—

I. LEVER.—The effort and resistance are to each other in the inverse ratio of the perpendicular distances of their lines of action from the axis of rotation or fulcrum; so that the *purchase* is the ratio which the perpendicular distance of the effort from the axis bears to the perpendicular distance of the resistance from the axis.

Under the head of the lever may be comprehended all turning or rocking primary pieces in mechanism which are connected with their drivers and followers by linkwork.

II. WHEEL AND AXLE.—The purchase is the same as in the case of the lever; and the perpendicular distances of the lines of action of the effort and of the resistance from the axis are the radii of the pitch-circles of the wheel and of the axle respectively.

Under the head of the wheel and axle may be comprehended all turning or rocking primary pieces in mechanism which are connected with their drivers and followers by means of rolling contact, of teeth, or of bands. By thes" wheel" is to be understood the pitch-cylinder of that part of the piece which is driven; and by the "axle," the pitch-cylinder of that part of the piece which drives.

III. INCLINED PLANE, and IV. WEDGE.—Here the purchase, or ratio of the resistance to the effort, is the ratio borne by the whole velocity of the sliding body (represented by B C in fig. 165, page 233, and C c in fig. 166, page 234) to that component of the velocity (represented by B D in fig. 165, page 233, and C e in fig. 166, page 234) which is directly opposed to the resistance: it being understood that the effort is exerted in the direction of motion of the sliding body.

The term *inclined plane* may be used when the resistance to the motion of a body that slides along a guiding surface consists of its own weight, or of a force applied to a point in it by means of a link; and the term *wedge*, when that resistance consists of a pressure applied to a plane surface of the moving body, oblique to its direction of motion.

V. SCREW.—Let the resistance (R) to the motion of a screw be a force acting along its axis, and directly opposed to its advance; and let the effort (P) which drives the screw be applied to a point rigidly attached to the screw, and at the distance r from the axis, and be exerted in the direction of motion of that point. Then, while the screw makes one revolution, the working point advances against the resistance through a distance equal to the pitch (p); and at the same time the driving point moves in its helical path through the distance $\sqrt{(4\pi^2 r^2 + p^2)}$; therefore the purchase of the screw, neglecting friction, is expressed as follows:—

$$\frac{\mathrm{R}}{\mathrm{P}} = \frac{\sqrt{4\pi^2 r^2 + p^2}}{p}$$

$$= \frac{\text{length of one coil of path of driving point}}{\text{pitch}}.$$

VI. PULLEY. (See Articles 200 and 201, pages 214 to 216.)—In the pulley without friction, the purchase is the ratio borne by the resistance which opposes the advance of the running block to the effort exerted on the hauling part of the rope; and it is expressed by the number of plies of rope by which the running block is connected with the fixed block.

VII. The HYDRAULIC PRESS, when friction is neglected, may be included amongst the mechanical powers, agreeably to the definition of them given at the beginning of this Article. By the resistance is to be understood the force which opposes the outward motion of the press-plunger, A, fig. 159, page 224; and by the effort, the force which drives inward the pump-plunger, A′. The intensity of the pressure exerted between each of the two plungers and the fluid is the same; therefore the amount of the pressure exerted between each plunger and the fluid is proportional to the area of that plunger; so that the purchase of the hydraulic press is expressed as follows:—

$$\frac{R}{P} = \frac{A}{A'} = \frac{\text{transverse area of press-plunger}}{\text{transverse area of pump-plunger}};$$

and this is the reciprocal of the ratio of the velocities of those plungers, as already shown in Article 209, page 223.

The purchase of a train of mechanical powers is the product of the purchases of the several elementary parts of that train.

The object of producing a purchase expressed by a number greater than unity is, to enable a resistance to be overcome by means of an effort smaller than itself, but acting through a greater distance; and the use of such a purchase is found chiefly in machines driven by muscular power, because of the effort being limited in amount.

SECTION IV.—*Of Dynamometers.*

340. **Dynamometers** are instruments for measuring and recording the energy exerted and work performed by machines. They may be classed as follows:—

I. Instruments which merely *indicate the force* exerted between a driving body and a driven body, leaving the *distance* through which that force is exerted to be observed independently. The following are examples of this class:—

a. The weight of a solid body may be so suspended as to balance the resistance, as in Scott Russell's experiments on the resistance of boats. (*Edin. Trans.*, xiv.)

b. The weight of a column of liquid may be employed to balance and measure the effort required to drag a carriage or other body, as in Milne's mercurial dynamometer.

c. The available energy of a prime mover may be wholly expended in overcoming friction, which is measured by a weight, as in Prony's dynamometer (described further on).

d. A spring balance may be interposed between a prime mover and a body whose resistance it overcomes.

II. Instruments which *record* at once the *force, motion,* and *work* of a machine, by drawing a line, straight or curved, as the case may be, whose abscissæ represent the distances moved through, its ordinates the resistances overcome, and its area the work performed (as in fig. 241, page 346).

A dynamometer of this class consists essentially of two principal parts: a spring whose deflection indicates the force exerted between a driving body and a driven body; and a band of paper, or a card, moving at right angles to the direction of deflection of the spring with a velocity bearing a known constant proportion to the velocity with which the resistance is overcome. The spring carries a pen or pencil, which marks on the paper or card the required line. The following are examples of this class of instruments:—

a. Morin's Traction Dynamometer.

b. Morin's and Hirn's Rotatory Dynamometers.

c. The Steam Engine Indicator.

III. Instruments called *Integrating Dynamometers*, which record the work performed, but not the resistance and motion separately.

341. **Prony's Friction Dynamometer** measures the useful work performed by a prime mover, by causing the whole of that work to be expended in overcoming the friction of a brake. In fig. 249, A represents a cylindrical drum, driven by the prime mover. The block D, attached to the lever B C, and the smaller blocks with which the chain E is shod, form a brake which embraces the drum, and which is tightened by means of the screws F, F, until its friction is sufficient to cause the drum to rotate at an uniform speed. The end C of the lever carries a scale G, in which weights are placed to an amount just sufficient to balance the friction, and keep the lever horizontal. The lever ought to be so loaded at B that when there are no weights in the scale, it shall be balanced upon the axis. The lever is prevented from deviating to any inconvenient extent from a horizontal position by means of safety-stops or guards, H, K.

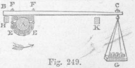

Fig. 249.

The weight of the load in the scale which balances the friction being multiplied into the horizontal distance of the point of suspension C from the axis, gives the *moment of friction*, which being multiplied into the angular velocity of the drum, gives the *rate of useful work* or *effective power* of the prime mover.

As the whole of that power is expended in overcoming the friction between the drum and the brake, the heat produced is in general considerable; and a stream of water must be directed on the rubbing surfaces to abstract that heat.

The friction dynamometer is simple and easily made; but it is ill adapted to measure a variable effort; and it requires that when the power of a prime mover is measured, its ordinary work should be interrupted, which is inconvenient and sometimes impracticable.

342. **Morin's Traction Dynamometer.**—The descriptions of this and some other dynamometers invented by General Morin are abridged from his works, entitled *Sur quelques Appareils dynamométriques* and *Notions fondamentales de Mécanique*.

Fig. 250 is a plan and fig. 250 *a* an elevation of a dynamometer for recording by a diagram the work of dragging a load horizontally. a a, b b are a pair of steel springs, through which the tractive force is transmitted, and which serve by their deflection to measure

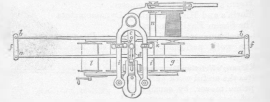

Fig. 250.

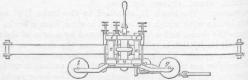

Fig. 250 A.

that force. They are connected together at the ends by the steel links f, f. The effort of the prime mover is applied, through the link r, to the gland d, which is fixed on the middle of the foremost spring; the equal and opposite resistance of the vehicle is applied to the gland c, which is fixed on the middle of the aftermost spring. When no tractive force is exerted, the inward faces of the springs are straight and parallel; when a force is exerted, the springs are bent, and are drawn apart, through a distance proportional to the force. The springs are protected against being bent so far as to injure them by means of the safety bridles i, i, with their bolts e, e. Those bridles are carried by the after-gland, and their bolts serve to stop the foremost spring when it is drawn forward as far as is consistent with the preservation of elasticity and strength.

The frame of the apparatus for giving motion to the paper band is carried by the after-gland. The principal parts of that apparatus are the following:—

l, store drum on which the paper band is rolled, before the commencement of the experiment, and off which it is drawn as the experiment proceeds;

g, taking-up drum, to which one end of the paper band is glued, and which draws along and rolls up the paper band with a velocity proportional to that of the vehicle. Fixed on the axis of this drum is a fusee having a spiral groove round it, whose radius gradually increases at the same rate as that at which the effective radius of the drum g is increased during its motion by the rolling of successive coils of paper upon it. The object of this is to prevent that increase of the effective radius of the drum from accelerating the speed of the paper band;

n is a drum which receives through a train of wheelwork and endless screws a velocity proportional to that of the wheels of the vehicle, and which, by means of a cord, drives the fusee. The mechanism is usually so designed that the paper moves at one-fiftieth of the speed of the vehicle.

Between the drums l and g there are three small rollers to support the paper band and keep it steady.

One of the safety bridles carries a pencil, k, which, being at rest relatively to the frame of the recording apparatus, traces a straight line on the band of paper as the latter travels below the pencil. That line is called the *zero line*, and corresponds to O X in fig. 241, page 346.

An arm fixed to the forward gland carries another pencil, whose position is adjusted before the experiment, so that when there is no tractive force its point rests on the zero line. During the experiment, this pencil traces on the paper band a line such as E R G, fig. 241, whose ordinate or distance from any given point in the zero line is the deflection of the pair of springs, and proportional to the tractive force, at the corresponding point in the journey of the vehicle.

The areas of the diagrams drawn by this apparatus, representing quantities of work, may be found either by the method described in Article 289, page 331, or by an instrument for measuring the areas of plane figures, called the *Planimeter*, or *Platometer*, of which various forms have been invented by Ernst, Sang, Clerk Maxwell, Amstler, and others.

A third pencil, actuated by a clock, is sometimes caused to mark a series of dots on the paper band at equal intervals of time, and so to record the changes of velocity.

When one vehicle (such as a locomotive engine) drags one or more others, the apparatus may, if convenient, be turned hind side before, and carried by the foremost vehicle. In such a case the motion of the band of paper ought to be derived, not from a driving-wheel, which is liable to slip, but from a bearing-wheel.

When the apparatus is used to record the tractive force and work performed in towing a vessel, the apparatus for moving the paper band may be driven by means of a wheel or fan, acted upon by the water; in which case the ratio of the velocity of the band to that of the vessel should be determined by experiment.

Owing to the varieties which exist in the elasticity of steel, the relation between the deflections of the springs and the tractive forces can only be roughly calculated beforehand, and should be determined exactly by direct experiment—that is, by hanging known weights to the springs, and noting the deflections.

The best form of longitudinal section for each spring is that which gives the greatest flexibility for a given strength, and consists of two parabolas, having their vertices at the two ends of the spring, and meeting base to base in the middle; that is to say, the thickness of the spring at any given point of its length should be proportional to the square root of the distance of that point from the nearest end of the spring. To express this by a formula, let

c be the half-length of the spring; 1.

h, the thickness in the middle;

x, the distance of any point in the spring from the end nearest to it;

h', the thickness at that point; then

$$h' = h \cdot \sqrt{\frac{x}{c}} \quad \text{.........................(1.)}$$

The breadth of each spring should be uniform, and, according to General Morin, should not exceed from $1\frac{1}{2}$ to 2 inches. Let it be denoted by b.

The following is the formula for calculating beforehand the *probable* joint deflection of a given pair of springs under a given tractive force:—

Let the dimensions c, h, b be stated in inches, and the force P in pounds.

Let y denote the deflection in inches.

Let E denote the *modulus of elasticity* of steel, in pounds on the square inch. Its value, for different specimens of steel, varies from 29,000,000 to 42,000,000, the smaller values being the most common. Then

$$y = \frac{8 \text{ P } c^3}{\text{E } b \, h^3} \quad \text{..........................(2.)}$$

The deflection should not be permitted to exceed about one-tenth part of the length of the springs.

343. **Morin's Rotatory Dynamometer** is represented in figs. 251, 251 *a*, and is designed to record the work performed by a prime mover in transmitting rotatory motion to any machine. A is a fast pulley, and C a loose pulley, on the same shaft. A belt transmits motion from the prime mover to one or other of those pulleys according as it is desired to transmit motion to the shaft or not.

A third pulley, B, on the same shaft, carries the belt which transmits motion to the machine to be driven. This pulley is also loose on the shaft to a certain extent, so that it is capable of mov-

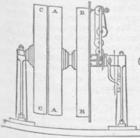
Fig. 251.

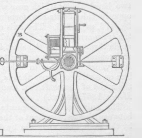
Fig. 251 A.

ing, relatively to the shaft, backwards and forwards through a small arc, sufficient to admit of the deflection of a steel spring by which motion is transmitted from the shaft to the pulley.

One end of that spring is fixed to the shaft, so that the spring projects from the shaft like an arm, and rotates along with it. The other end of the spring is connected with the pulley B near its circumference, and is the means of driving that pulley; so that the spring undergoes deflection proportional to the effort exerted by the shaft on the pulley.

A frame projecting radially like an arm from the shaft, and rotating along with it, carries an apparatus, similar to that used in the traction dynamometer, for making a band of paper move radially with respect to the shaft with a velocity proportional to the speed with which the shaft rotates. A pencil carried by this frame traces a zero line on the paper band; and another pencil carried by the end of the spring, traces a line whose ordinates represent the forces exerted, just as in the traction dynamometer.

The mechanism for moving the paper band is driven by a toothed ring surrounding the shaft, and kept at rest while the shaft rotates by means of a catch. When that catch is drawn back, the toothed ring is set free, rotates along with the shaft, and ceases to drive the mechanism; and thus the motion of the paper band can be stopped if necessary. (See page 446.)

344. In the **Torsion Dynamometer** (otherwise called "Pandynamometer") of M. G. A. Hirn, the torsion of the rotating shaft which transmits power is made the means of measuring and record-

ing, by a self-acting apparatus, the moment of the couple by which the shaft is driven, and the work done by that couple. Two trains of wheels, driven from the shaft at two different points, communicate rotations of equal speed in opposite directions about one axis to two bevel-wheels which gear with an intermediate bevel-wheel at opposite sides of its rim, forming a combination like that shown in fig. 176, page 245. The axis of the third wheel, corresponding to the train-arm A in fig. 176, indicates by its position one-half of the angle through which the shaft is twisted between the spur-wheels, and communicates its motion to the pencil of the recording apparatus; which pencil, as in other recording dynamometers, draws a line on a strip of paper that is moved at a speed proportional to the speed of the shaft that transmits the power. (See *Annales des Mines*, 1867, vol. xi.)

The only perfectly accurate way of determining the relation between the displacement of the pencil and the moment transmitted by the shaft, is to ascertain by direct experiment the twisting effect of a known couple when applied to the shaft. But a probable approximate value of that relation may be calculated as follows:— Let M be the twisting moment; x, the length of that part of the shaft whose angular torsion is to be determined; h, its diameter; C, the co-efficient of transverse elasticity of the material; θ, the angle of torsion, in circular measure; then,*

$$\frac{\mathrm{M}}{\theta} = \frac{\pi}{32} \cdot \frac{\mathrm{C}\,h^4}{x} = 0{\cdot}098\,\frac{\mathrm{C}\,h^4}{x} \qquad \text{(1.)}$$

Let n be the ratio which each of the contrary angular velocities of the bevel-wheels corresponding to B and C in fig. 176 bears to the angular velocity of the shaft, and y the length of an index corresponding to the train-arm, A, in that figure; then the angular displacement of that index is $\frac{n\,\theta}{2}$; and the linear displacement of its end (which may be denoted by z) is

$$z = \frac{n\,\theta\,y}{2};$$

therefore the following formula expresses the relation between the moment M, and the displacement z;

$$\frac{\mathrm{M}}{z} = \frac{2\,\mathrm{M}}{n\,y\,\theta} = \frac{\pi}{16} \cdot \frac{\mathrm{C}\,h^4}{n\,x\,y} = 0{\cdot}196\,\frac{\mathrm{C}\,h^4}{n\,x\,y} \qquad \text{(2.)}$$

Should the shaft be hollow, let h' be its internal diameter; then in each of the preceding formulæ for h^4 substitute $h^4 - h'^4$.

The following are values of the co-efficient C:—

* *Manual of Applied Mechanics*, Article 322, page 357.

Dimensions in... / Forces in...	Inches... / Lbs...	Millimètres. / Kilogrammes.
Cast Iron, about	2,850,000	2,000
Wrought Iron, from	8,500,000	6,000
" to	10,000,000	7,000
Steel, from	10,000,000	7,000
" to	12,000,000	8,400

Calculation may be used preliminary to the designing of the apparatus, in order to find approximately the extent of the displacement of the recording pencil; but the exact relation of that displacement to the twisting moment exerted through the shaft ought always to be determined by experiment.

345. **Elasticity of Spiral Springs.**—As spiral or helical springs are much used in dynamometric apparatus, it is convenient here to state the laws of their resistance to extension and compression.

In order that such a spring may be an accurate instrument for measuring forces—that is, in order that the proportion borne by the load acting on the spring to the extension or compression which it produces may be constant—the figure of the spring should be a true helix, as described in Article 58, page 36.

It is more favourable to accuracy to measure a force by the extension than by the compression of a spiral spring; because during extension it preserves almost exactly a truly helical form, and the coils remain in a cylindrical surface; whereas during compression the middle coils are apt to swerve sideways, so as to make the spring lose the proper figure. There are cases, however, in which the use of the compression of the spring is unavoidable; and then it is kept approximately in its proper figure by being enclosed in a cylindrical casing, which ought to be so large as not to impede the longitudinal motion of the spring.

The pair of equal and opposite forces by which a spiral spring is stretched should act exactly along the axis of the helix; for which purpose the ends of the spring should be made fast to a pair of strong and stiff arms, each of which should be perpendicular to the helix, and should lie along a radius of the cylinder on which the helix is described, so that the inner ends of the arms may be in the axis of the helix; and at those inner ends the forces to be measured should be applied. The best form of section for the wire of which the spring is made is circular; because the extension of the spring depends on the torsion of the wire; and the laws of torsion are known with greater precision for a circular form of section than for any other.

The following formulæ show the relations between the load and the extension or compression of the spring:—

Let r be the radius of the cylinder containing the helical centre line of the spiral spring, as measured from the axis to the centre of the wire; n, the number of coils of which the spring consists; d, the diameter of the wire; C, the co-efficient of rigidity or transverse elasticity of the material; f, the greatest safe shearing stress upon it; W, any load not exceeding the greatest safe load; v, the corresponding extension or compression; W_1, the greatest safe *steady* load; v_1, the greatest safe extension or compression; then

$$\text{(1)}\quad \frac{W}{v}=\frac{C\,d^4}{64\,n\,r^3};\quad \text{(2)}\quad W_1=\frac{0{\cdot}196\,f\,d^3}{r};\quad v_1=\frac{12{\cdot}566\,n\,f\,r^2}{C\,d}.\quad \text{(3.)}$$

The greatest safe *sudden* load is $\frac{W_1}{2}$.

If the wire of which the spring is made is square, and of the dimensions $d \times d$, the load for a given deflection is greater than for a round wire of the diameter d, in the ratio of 281 to 196, or of 1·43 to 1, or of 10 to 7, nearly.

The values of the co-efficient, C, of transverse elasticity of steel and charcoal iron wire, in lbs. on the square inch, range between 10,500,000 and 12,000,000; and in kilogrammes on the square millimètre, from, 7,400 to 8,400, nearly.

By the greatest safe stress is to be understood the greatest stress which is certain not to impair the elasticity of the spring by frequent repetition; say 30,000 lbs. on the square inch.

The value of the ratio $\frac{W}{v}$ borne by the load to the extension ought to be ascertained by direct experiment for every spring that is used in dynamometers or indicators.

346. **Steam Engine Indicator.**—This instrument was invented by Watt, and has been improved by other inventors, especially M'Naught and Richards. Its object is to record, by means of a diagram, the intensity of the pressure exerted by steam against one of the faces of a piston at each point of the piston's motion, and so to afford the means of computing, according to the principles of Articles 302 and 307, first, the energy exerted by the steam in driving the piston during the forward stroke; secondly, the work lost by the piston in expelling the steam from the cylinder during the return stroke; and thirdly, the difference of those quantities, which is the *available* or *effective* energy exerted by the steam on the piston, and which, being multiplied by the number of strokes per minute and divided by 33,000 foot-pounds, gives the INDICATED HORSE-POWER.

The indicator in a common form is represented by fig. 252. A B is a cylindrical case. Its lower end, A, contains a small cylinder, fitted with a piston, which cylinder, by means of the screwed nozzle at its lower end, can be fixed in any convenient position on a tube communicating with that end of the engine-cylinder where the work of the steam is determined. The communication between the engine-cylinder and the indicator-cylinder can be opened and shut at will by means of the cock K. When it is open, the intensity of the pressure of the steam on the engine-piston and on the indicator-piston is the same, or nearly the same.

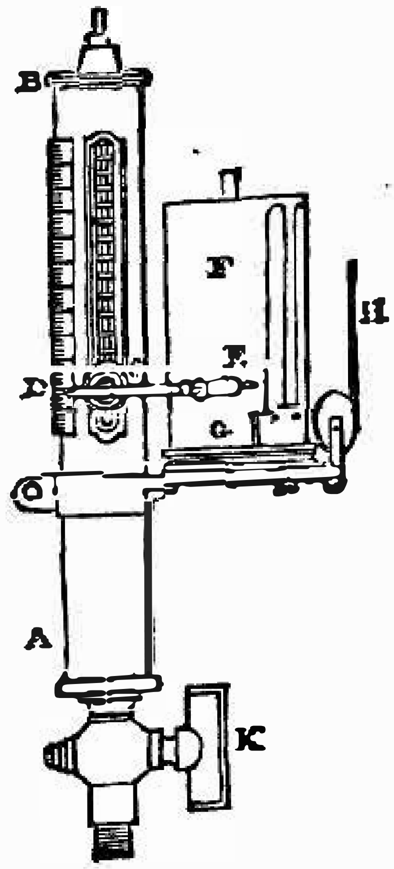

Fig. 252.

The upper end, B, of the cylindrical case contains a spiral spring, one end of which is attached to the piston, or to its rod, and the other to the top of the casing. The indicator-piston is pressed from below by the steam, and from above by the atmosphere. When the pressure of the steam is equal to that of the atmosphere, the spring retains its unstrained length, and the piston its original position. When the pressure of the steam exceeds that of the atmosphere, the piston is driven outwards, and the spring compressed; when the pressure of the steam is less than that of the atmosphere, the piston is driven inwards, and the spring extended. The compression or extension of the spring indicates the difference, upward or downward, between the pressure of the steam and that of the atmosphere.

A short arm, C, projecting from the indicator piston-rod carries at one side a pointer, D, which shows the pressure on a scale whose zero denotes the *pressure of the atmosphere,* and which is graduated into pounds on the square inch both upwards and downwards from that zero. At the other side the short arm has a longer arm jointed to it, carrying a pencil, E.

F is a brass drum, which rotates backward and forward about a vertical axis, and which, when about to be used, is covered witht a piece of paper called a "card." It is alternately pulled round in one direction by the cord H, which wraps on the pulley G, and pulled back to its original position by a spring contained within itself. The cord H is to be connected with the mechanism of the steam engine in any convenient manner which shall ensure that the velocity of rotation of the drum shall at every instant bear a constant ratio to that of the steam engine piston: the back and forward motion of the surface of the drum representing that of the steam engine piston on a reduced scale. This having been done, and before opening the cock K, the pencil is to be placed in contact with the drum during a few strokes, when it will mark on the card a line which, when the card is afterwards spread out flat, becomes a straight line. This line, whose position indicates the pressure of the atmosphere, is called the *atmospheric line.* In fig. 253 it is represented by A A.

Then the cock K is opened, and the pencil, moving up and down with the variations of the pressure of the steam, traces on the card during each complete or double stroke, a curve such as B C D E B. The ordinates drawn to that curve from any point in the atmospheric line, such as $\overline{\text{H K}}$ and $\overline{\text{H G}}$, indicate the differences between the pressure of the steam and the atmospheric pressure at the corresponding point of the motion of the piston. The ordinates of the part B C D E represent the pressures of the steam during the forward stroke, when it is driving the piston; those of the part E B represent the pressures of the steam when the piston is expelling it from the cylinder.

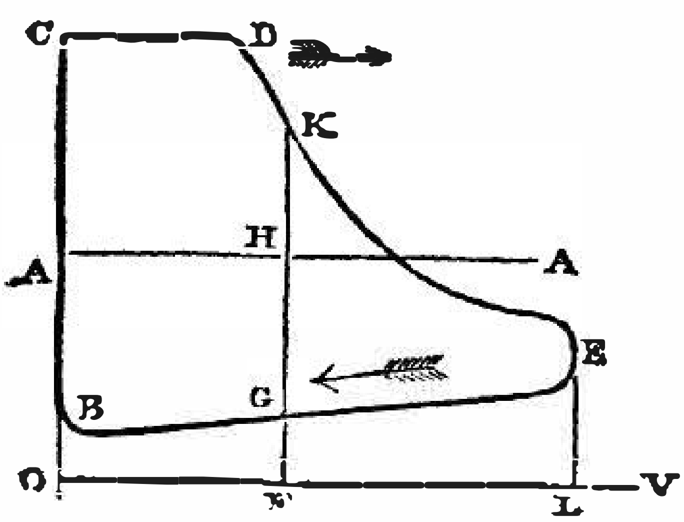

Fig. 253.

To found exact investigations on the indicator-diagrams of steam engines, the atmospheric pressure at the time of the experiment ought to be ascertained by means of a barometer; but this is generally omitted; in which case the atmospheric pressure may be assumed at its mean value, being 14·7 lbs. on the square inch, or 2116·3 lbs. on the square foot, at and near the level of the sea.

Let $\overline{\text{A O}} = \overline{\text{H F}}$ be ordinates representing the pressure of the atmosphere. Then O F V parallel to A A is the *absolute* or *true* zero line of the diagram, corresponding to *no pressure;* and ordinates drawn to the curve from that line represent the absolute intensities of the pressure of the steam. Let O B and L E be ordinates touching the ends of the diagram; then

$\overline{\text{O L}}$ represents the *volume* traversed by the piston at each single stroke ($= s$ A, where s is the length of the stroke and A the area of the piston);

The area O B C D E L O represents the energy exerted by the steam on the piston during the forward stroke;

The area O B E L O represents the work lost in expelling the steam during the return stroke;

The area B C D E B, being the difference of the above areas, represents the *effective work* of the steam on the piston, during the complete stroke.

Those areas can be found by the Rules of Article 289, page 331; and the common trapezoidal rule, D, page 333, is in general sufficiently accurate. The number of intervals is usually ten, and of ordinates eleven.

The *mean forward pressure*, the *mean back pressure*, and the *mean effective pressure*, are found by dividing those three areas respectively by the volume s A, which is represented by $\overline{\text{O L}}$.

Those mean pressures, however, can be found by a direct process, without first measuring the areas, viz.:—having multiplied each ordinate, or breadth, of the area under consideration by the proper multiplier, divide the sum of the products by the sum of the multipliers, which process, when the common trapezoidal rule is used, takes the following formh add together the halves of the endmost ordinates, and the whole of the other ordinates, and divide by the number of intervals. That is, let b_0 be the first, b_n the last, and b_1, b_2, &c., the intermediate breadths; then let n be the number of intervals, and b_m the mean breadth; then

$$b_m = \frac{1}{n}\left(\frac{b_0 + b_n}{2} + b_1 + b_2 + \&c.\right); \text{............} (1.)$$

and this represents the mean forward pressure, mean back pressure, or mean effective pressure, as the case may be. Let p_e be the mean effective pressure; then the effective energy exerted by the steam on the piston during each double stroke is the product of the mean effective pressure, the area of the piston, and the length of stroke, or

$$p_e \text{ A } s; \text{............................} (2.)$$

and if N be the number of double strokes in a minute, the *indicated power in foot-pounds per minute*, in a single-acting engine, is

$$p_e \text{ A N } s; \text{..........................} (3.)$$

from which the *indicated horse-power* is found by dividing by 33,000.

In a *double-acting engine* the steam acts alternately on either side of the piston; and to measure the power accurately, two indicators should be used at the same time, communicating respectively with the two ends of the cylinder. Thus a pair of diagrams will be obtained, one representing the action of the steam on each face of the piston. The mean effective pressure is to be found as above for each diagram separately, and then, if the areas of the two faces of the piston are sensibly equal, *the mean of those two results* is to be taken as the *general mean effective pressure;* which being multiplied by the area of the piston, the length of stroke, and *twice* the number of double strokes or revolutions in a minute, gives the indicated power per minute; that is to say, if p'' denotes the general mean effective pressure, the indicated power per minute is

$$p'' \text{ A} \cdot 2 \text{ N } s; \text{........................} (4.)$$

If the two faces of the piston are sensibly of unequal areas (as in "trunk engines"), the indicated power is to be computed separately for each face, and the results added together.

If hthere are two or more cylinders, the quantities of power indicated by their respective diagrams are to be added together.

The re-actions of the moving parts of the indicator, combined with the elasticity of the spring, cause oscillations of its piston. In order that the errors thus produced in the indicated pressures at particular instants may be as small as possible, and may neutralise each other's effects on the whole indicated power, the moving masses ought to be as small as practicable, and the spring as stiff as is consistent with showing the pressures on a visible scale. In Richards's indicator this is effected by the help of a train of very light linkwork, which causes the pencil to show the movements of the spring on a magnified scale.

The *friction* of the moving parts of the indicator tends on the whole to make the indicated power and indicated mean effective pressure less than the truth, but to what extent is uncertain.

Every indicator should have the accuracy of the graduation of its scale of pressures frequently tested by comparison with a standard pressure gauge.

The indicator may obviously be used for measuring the energy exerted by any fluid, whether liquid or gaseous, in driving a piston; or the work performed by a pump, in lifting, propelling, or compressing any fluid.

347. **Integrating Dynamometers** record simply the work performed in dragging a vehicle or driving a machine, without recording separately the force and the motion. In that of Morin this is effected by means of a combination which has already been described in Article 270, page 311, and illustrated in fig. 221. In that figure (which see) A represents a plane circular disc, made to rotate with an angular velocity proportional to the speed of the motion of the vehicle or machine, and B a small wheel driven by the friction of the disc against its edge, and having its axis parallel to a radius of the disc. The wheel B, and some mechanism which it drives, are carried by a frame which is carried by a dynamometer spring, and so adjusted that the distance of the edge of B from the centre of A is equal to the deflection of the spring, and proportional to the effort.

The velocity of the edge of B at any instant being the product of its distance from the centre of A into the angular velocity of A, is proportional to the product of the effort into the velocity of the vehicle or machine—that is, to the *rate at which work is performed*; therefore the motion of the wheel B, in any interval of time, is *proportional to the work performed in that time;* and that work can be recorded by means of dial-plates, with indexes moved by a train of wheelwork driven by the wheel B.

In Moison's integrating dynamometer a ratchet-wheel is driven by the strokes of a click. (See Articles 194 to 196, pages 206 to 211.) The number of these strokes in a given time is proportional to the speed of the machine whose work is to be measured; and by means of a dynamometer-spring the length of each stroke of the click is adjusted so as to be proportional to the effort exerted at the time. The result is that the total extent of motion of the ratchet-wheel in a given time is proportional to the work performed. It is obvious that the frictional catch might be applied to this apparatus (Article 197, page 211).

348. **Measurement of Friction.**—Under the head of Dynamometers may be classed apparatus for the experimental measurement of friction.

If by means of any kind of dynamometer whose use does not involve the interruption of the performance of the ordinary work of a train of mechanism, we measure the power transmitted at two parts of that train, the difference will be the power expended in overcoming the friction of the intermediate parts. Hirn's Pandynamometer (Article 344, page 387) seems well adapted for experiments of this class. The power of a steam engine, as exerted in the cylinder, may be measured by means of the indicator, and the power transmitted to machinery which that engine drives, by a suitable dynamometer; and the difference will be power expended chiefly in overcoming the friction of the intermediate mechanism.

Special apparatus for measuring the friction of axles is used, not only for purposes of scientific investigation as to the co-efficients of friction of different pairs of surfaces in different states, but for practically testing the lubricating properties of oil and grease. Two forms of apparatus may be described.

I. *Statical Apparatus.*—A short cylindrical axle, of a convenient diameter (say 2, 3, or 4 inches), is supported at its ends by bearings on the top of a pair of strong fixed standards. The ends of the axle overhang their bearings, and carry a pair of equal and similar pulleys, by means of which it is driven at a speed equal, or nearly equal, to the greatest intended working speed of the axles with which the unguents to be tested are to be used in practice. The object of driving the axle at both ends is to ensure great steadiness of motion. The driving-gear ought to be capable of reversing the direction of rotation. At the middle of its length the axle is turned so as to form a very accurate and smooth journal, of a length equal to from $1\frac{1}{2}$ to $2\frac{1}{2}$ times its diameter. Upon that journal there hangs a plumber-block or axle-box, fitted with a suitable bush or bearing. That plumber-block is rigidly connected with a heavy mass of suitable material, such as cast iron, so as to form as it were a *pendulum* hanging from the journal in the middle of the axle, and of a weight suited to produce a pressure on the journal equal to the greatest pressure to which the unguent is to be exposed in practice (see Article 310, page 353). The pendulum is furnished with an index and graduated arc, to show its deviation from a vertical position.

The hanging plumber-block having been supplied with the unguent to be tested, the axle is to be driven at full speed, first in one direction, and then in the contrary direction, and the two contrary deviations of the pendulum observed. Let θ denote the *half-sum* of those deviations, expressed in circular measure to radius unity; c, the distance from the axis of rotation to the centre of gravity of the pendulum; r, the radius of the journal; let W be the weight of the pendulum; then the mean statical moment of the pendulum is

$$\mathrm{W}\, c \sin \theta = \mathrm{W}\, c\, \theta \text{ nearly};$$

and that moment balances the moment of friction (Article 311, page 35[illegible]), whose value is $f\,\mathrm{W}\,r$ nearly, and will be afterwards shown to be exactly

$$\mathrm{W}\, r \sin \varphi,$$

φ being the angle of repose. Equating, therefore, those two equal moments, we find

$$r \sin \varphi = c \sin \theta; \text{ and}$$

$$\sin \varphi = f \text{ nearly} = \frac{c}{r} \sin \theta = \frac{c\,\theta}{r} \text{ nearly} \ldots\ldots\ldots\ldots (1.)$$

The distance, c, of the centre of gravity of the pendulum from the axis may be found experimentally, by applying a known weight at a known horizontal distance from the axis, so as to make the pendulum deviate, and observing the deviation. Let P be the weight so applied, x its leverage, Θ the deviation which it produces; then, if there were no friction, we should have

$$c = \frac{\mathrm{P}\,x}{\mathrm{W} \sin \Theta}.$$

In order to eliminate the effects of friction from the determination of c, the load P with the leverage x should be applied at contrary sides, so as to increase the deviation of the pendulum, while the axle is rotating in the two contrary directions.

Let $\sin \theta$ be the mean of the sines of the deviations produced by friction alone, and $\sin \Theta$ the mean of the sines of the deviations produced by the friction and the load P together; then we shall have

$$c = \frac{\mathrm{P}\,x}{\mathrm{W}\,(\sin \Theta - \sin \theta)} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

II. *Dynamic* or *Kinetic Apparatus.*—To measure the friction of an axle by means of its retarding effect upon a rotating mass, the axle is supported on suitable bearings at its ends, as in the Statical Apparatus just described; and at the middle of its length it has fitted on it, and accurately balanced, a round disc acting as a fly-wheel, of weight sufficient to produce the required pressure on the bearings. (See Article 310, page 353.) The numbers of turns made by the axle are counted and indicated by means of a light and easily-driven train of small wheels, with dial-plates and indexes.

The axle is provided with driving-gear of a kind which can be instantly disengaged when required; for example, a fast pulley on one overhanging end, with a loose pulley alongside of it, the loose pulley being carried, not by the fly-axle itself, but by a separate axle in the same straight line with the fly-axle.

After the axle with its fly-disc has been set in motion at a speed greater than the working speed of the axles to which the unguent to be tested is to be applied in practice, the driving-gear is to be disengaged; when the speed of rotation will undergo a gradual retardation through the friction of the journals. The numbers of turns made in a series of equal intervals of time (for example, intervals of thirty seconds, or of sixty seconds, or of a hundred seconds) are to be observed on the counting dials, and noted down.

Let W denote the weight of the whole rotating mass, consisting of the axle with its fly-disc; ϱ, the radius of gyration of that mass. (See Article 313, page 357). Let t be the uniform length in seconds of the intervals of time during which the numbers of revolutions are recorded; and in one of those intervals let the disc make n revolutions, and in the next interval n' revolutions. Then the mean angular velocity is, during the first interval, $\frac{2\pi n}{t}$, and during the second interval, $\frac{2\pi n'}{t}$; and treating the rate of retardation as sensibly uniform, the retardation which takes place during the t seconds which elapse from the middle of the first interval to the middle of the second interval is

$$\frac{2\pi(n-n')}{t},$$

and to produce that retardation in the course of t seconds in a body whose moment of inertia is $W\varrho^2$, there is required a retarding moment of the following value:—

$$M = \frac{2\pi(n-n')W\varrho^2}{g\,t^2} \dots\dots\dots\dots\dots\dots(1.)$$

Part of the retarding moment is due to the resistance of the air; but if the fly is a smooth round disc without arms, this may be neglected for the purpose of the experiments, and the whole moment treated as due to axle-friction. Let r be the radius of the journals, and f the co-efficient of friction: then, as before, the moment of friction is very nearly $f \mathrm{W} r$; and by equating this to the retarding moment, and dividing both sides of the equation by $\mathrm{W} r$, we obtain the following formula for the co-efficient of friction:—

$$f = \frac{2 \pi (n - n') \rho^2}{g t^2 r}. \quad \text{(2.)}$$

When the numbers of revolutions have been observed during a series of more than two equal intervals of time, the formula 2 for the co-efficient of friction is to be applied to each consecutive pair of intervals, and a mean of the results taken.

The radius of gyration ρ and the radius of the journals r should of course be expressed in the same units of measure. In British measures, feet are the most convenient for the present purpose. The constant factor has the following values:—

$$\frac{2 \pi}{g} = \frac{1}{5{\cdot}125 \text{ feet}} = \frac{1}{1{\cdot}56 \text{ mètre}} \quad \text{(3.)}$$

Similar experiments may be made with a disc rotating about a vertical axis, and supported by a pivot; regard being had to the value of the moment of friction of a pivot, as stated in Article 311, page 356.

To find the square ρ^2 of the radius of gyration by experiment, fix a pair of slender pins in the two faces of the disc at two points opposite each other, and near its circumference; hang up the disc with its axle by these pins, and make it swing like a pendulum in a plane perpendicular to its axis; count the number of single swings in some convenient interval of time; calculate their number per second, and let N denote that number. Then calculate the length L of the equivalent simple pendulum, by the following formula:—

$$\mathrm{L} = \frac{g}{\pi^2 \mathrm{N}^2} \quad \text{(4.)}$$

The constant factor of this expression, being the length of the seconds pendulum, has approximately the following values:—

$$\frac{g}{\pi^2} = 3{\cdot}26 \text{ feet} = 0{\cdot}992 \text{ mètre} \quad \text{(5.)}$$

Let C be the distance from the point of suspension to the axis of figure of the disc and axle; then the square of the radius of gyration is calculated as follows:—

$$\rho^2 = \mathrm{C}(\mathrm{L} - \mathrm{C}). \quad \text{(6.)}$$

When the object of the experiments is not to obtain absolute values of the co-efficient of friction, but merely to compare one specimen of unguent with another, it is sufficient to compare together the rates of retardation with the two unguents in equal intervals of time.

III. *Comparison of Heating Effects.*—For the same purpose of comparing unguents with each other, without measuring the friction absolutely, the heating effects of the friction with different unguents are sometimes compared together. The apparatus used is similar to that described under the head of (I.) Static Apparatus; except that there is no reversing-gear, and that the pendulum, or loaded plumber-block, has no index nor graduated arc, and is provided with a thermometer, having its bulb immersed in the passage through which the unguent flows from the grease-box to the journal. Another thermometer, hung on the wall of the room, shows the temperature of the air. The axle is driven at its proper speed, until the temperature shown by the first-mentioned thermometer ceases to rise; and then the elevation of that temperature above the temperature of the air is noted. (See Article 310, page 353.) 2

In all experiments for the purpose of comparing unguents with each other, care should be taken to remove one sort of unguent completely from the rubbing surfaces, grease-box, and passages, before beginning to test the effect of another sort, lest the mixture of different sorts of unguents should make the experiments inconclusive.

Addendum to Article 309, Page 348.

Friction of Pistons and Plungers.—From experiments made by Mr. William More and others, it appears that the friction of ordinary pistons and plungers may be estimated at about one-tenth of the amount of the effective pressure exerted by the fluid on the piston. 1

The friction of a plunger working through a cupped leather collar is equal to the pressure of the fluid upon a ring equal in circumference to the collar, and of a breadth which, according to Mr. More's experiments, is about 0·4 of the depth of bearing-surface of the collar; and according to the experiments of Messrs. Hick and Luthy, from ·01 to ·015 inch (= from ·25 to ·375 millimètres), according to the state of lubrication of the collar.