CHAPTER IV.

OF ELEMENTARY COMBINATIONS IN MECHANISM.

SECTION I.—*Definitions, General Principles, and Classification.*

89. **Elementary Combinations Defined.** (*A. M.*, 431.)—An "Elementary Combination" in Mechanism (a term introduced by Mr. Willis) consists of a pair of primary moving pieces, so connected that one transmits motion to the other. In other words, (to quote the Article *Mechanics* (*Applied*), in the eighth edition of the *Encyc. Brit.*)—

"An *elementary combination* in mechanism consists of two pieces whose kinds of motion are determined by their connection with the frame, and their comparative motion by their connection with each other; that connection being effected either by direct contact of the pieces, or by a connecting" (secondary) "piece" (such as a band, or a link, or a mass of fluid), "which is not connected with the frame, and whose motion depends entirely on the motions of the pieces which it connects."

"The piece whose motion is the cause is called the *driver;* the piece whose motion is the effect, the *follower.*"

"The connection of each of those two pieces with the frame is in general such as to determine the path of every moving point. In the investigation, therefore, of the comparative motion of the driver and follower, in an elementary combination, it is unnecessary to consider relations of angular direction, which are already fixed by the connection of each piece with the frame; so that the inquiry is confined to the determination of the velocity-ratio, and of the directional-relation so far only as it expresses the connection between *forward* and *backward* movements of the driver and follower. When a continuous motion of the driver produces a continuous motion of the follower, forward or backward, and a reciprocating motion a motion reciprocating at the same instant, the directional-relation is said to be *constant.* When a continuous motion produces a reciprocating motion, or *vice versâ;* or when a reciprocating motion produces a motion not reciprocating at the same instant, the directional-relation is said to be *variable.*"

90. **Line of Connection.**—In every class of elementary combinations, except those in which the connection is made by reduplication of cords, or by an intervening fluid, there is at least one straight

line called the *line of connection* of the driver and follower; being a line traversing a pair of points in the driver and follower respectively, which points are so connected that the component of their velocity relatively to each other, resolved along the line of connection, is null.

91. **Comparative Motions of Connected Points and Pieces.**—From the definition of a line of connection it follows, that *the components of the velocities of a pair of connected points along their line of connection are equal.* And from this, and from the property of a rigid body already stated in Article 54, page 32, it follows, that *the components, along a line of connection, of all the points traversed by that line, whether in the driver or in the follower, are equal.*

The general principle which has just been stated serves to solve every problem in which—the mode of connection of a pair of pieces being given—it is required to find their comparative motion at a given instant, or *vice versâ.*

The following are the rules for applying that principle to the three classes of problems which most frequently occur with reference to elementary combinations:—

I. *Pair of Points; or Pair of Sliding Pieces.*—In fig. 57, let A B be a line of connection; and let it be taken as the axis of projection. Let A be a point in the driver, and B a point in the follower, both in the line of connection. Let A a', A a'' be the two projections of the direction of motion of A at a given instant; and let B b', B b'' be the two projections of the direction of motion of B at the same instant. Lay off, along the line of connection and in the same direction, the equal distances A a = B b; draw a'' a a', b'' b b' perpendicular to the line of connection; then A a' and A a'', B b' and B b'' will be the projections of a pair of lines proportional respectively to the velocities of A and B at that instant. The lengths of those lines may be found by the Rule of Article 19, page 7.

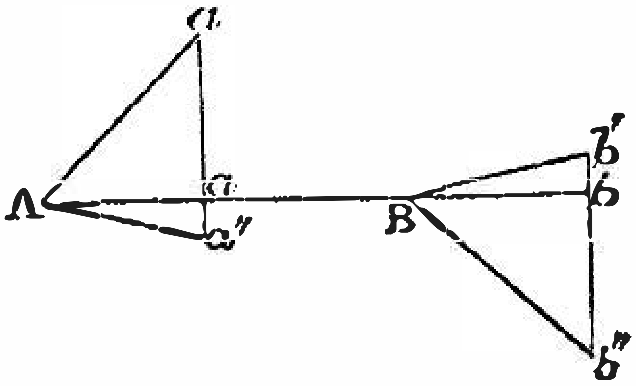

Fig. 57.

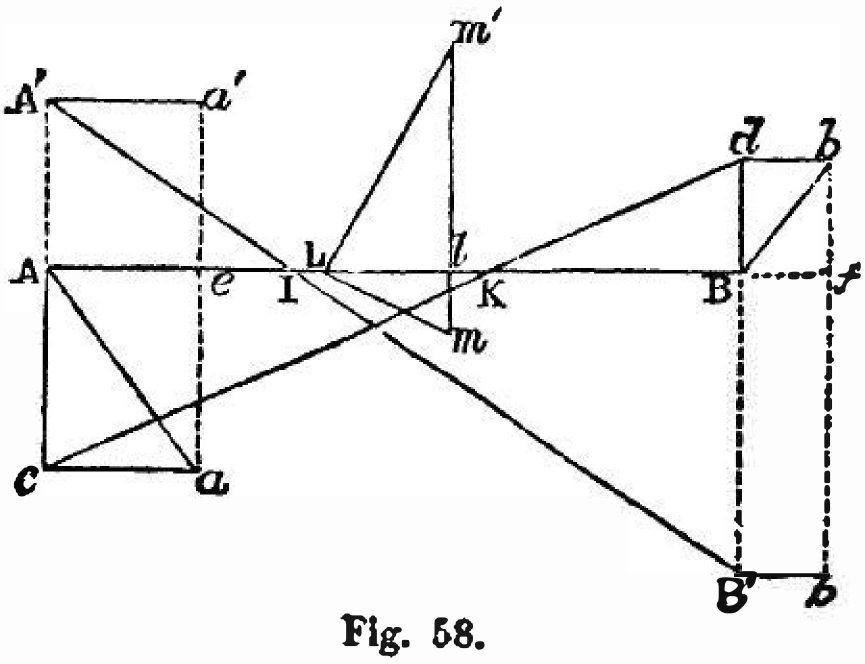

Fig. 58.

II. *Pair of Turning Pieces.*—In fig. 58, let A B be the line of connection of a pair of turning primary pieces. Let A and B be the points where that line is met by the common perpendiculars from the axes of rotation of the two pieces. (As to finding such common perpendiculars, see Article 36, page 15.) Let A A′ and B B′ be the *rabatments* of those two perpendiculars, drawn in opposite directions. Draw the straight line A′ B′ (called the *line of centres*), cutting the line of connection in I.

Then, because the component velocities of A and B along A B are equal, the angular velocities (or the component angular velocities) of the driver and follower about axes perpendicular to A B must be to each other *in the inverse ratio of the perpendiculars* A A′ *and* B B′; or, what is the same thing, *in the inverse ratio of the segments* I A′ *and* I B′ *into which the line of centres is cut by the line of connection.*

Hence the following construction:—In A B take A K = B I (or B K = A I); and through K draw an oblique straight line in any convenient direction, so as to cut A′ A produced in c and B′ B produced in d; then the component angular velocities of the pieces about two axes, A c and B d, perpendicular to the line of connection, will be to each other in the direct ratio of A c to B d. Also lay off, in opposite directions, the angles B A a and f B b, equal to the angles which the two axes of rotation respectively make with the line of connection, and draw c a and d b parallel to A B, cutting A a and B b in a and b respectively. Then *the ratio of* A a to B b *will be that of the resultant angular velocities of the two pieces.*

Through A′ and B′ draw A′ a' and B′ b' parallel to A B; and through a and b draw a e a' and b f b' perpendicular to A B. Then the proportion borne by $c\,a$ = A e = A′ a' to $d\,b$ = B f = B′ b' is that of the component angular velocities of the two pieces about axes parallel to the line of connection A B. Also A a and A′ a' represent the projections of the axis of rotation of the first piece upon a pair of planes which cut each other in A e, one perpendicular and the other parallel to the common perpendicular whose rabatment is A A′; and B b and B′ b' represent the projections of the axis of rotation of the second piece upon a pair of planes which cut each other in B f, one perpendicular and the other parallel to the common perpendicular whose rabatment is B B′

III. *Turning Piece and Sliding Piece.*—In fig. 58, let A L be the line of connection of a turning piece and a sliding piece, and let it be taken for the axis of projection; and let one of the planes of projection be parallel to the axis of the turning piece. Let A a and A a' be the projections of that axis; so that A A′ perpendicular to A L is the common perpendicular of the axis and the line of connection. Take A a to represent the angular velocity of the turning piece, and from a draw a c parallel to L A, cutting A′ A (produced if necessary) in c. Then A c will represent the component angular velocity of the turning piece about an axis, A c, perpendicular to A L; and the product A A′ · A c will represent *the component velocity of any point in* A L *along that line.*

Let L be a point in the line of connection and in the sliding piece; and let L m and L m' be the projections of the direction of motion of that piece. Lay off any convenient length, L l, to represent the component velocity of the sliding piece along the line of connection, and draw $m\ l\ m'$ perpendicular to that line; then L m and L m' will represent *the two projections of the velocity of the sliding piece.*

Another construction is as follows:—Having determined the angle which the direction of motion of the sliding piece makes with the line of connection A L, draw A′ I, making the angle A A′ I equal to that angle; then the velocity of the sliding piece will be equal to that of *a point revolving at the end of the arm* A′ I, *with the angular velocity represented by* A c.

92. **Adjustments of Speed.**—The velocity-ratio of a driver and its follower is sometimes made capable of being changed at will, by means of apparatus for varying the position of their line of connection: as when a pair of rotating cones are embraced by a belt which can be shifted so as to connect portions of their surfaces of different diameters. Various such contrivances will be described in a later chapter.

93. A **Train of Mechanism** consists of a series of moving pieces, each of which is follower to that which drives it, and driver to that which follows it. In the case of a train of elementary combinations the comparative motion of the last follower and first driver is found by multiplying together all the velocity-ratios of the several elementary combinations of which the train consists, each ratio having the directional-relation with which it is connected denoted by means of the positive or negative algebraical sign, as the case may be. The product is the velocity-ratio of the last follower and first driver; and their directional-relation is indicated by the algebraical sign of that product, found by the rules, that any number of positive factors, and any even number of negative factors, give a positive product; and that any odd number of negative factors give a negative product.

94. **Elementary Combinations Classed.**—The only classification of elementary combinations that is founded, as it ought to be, on comparative motion, as expressed by velocity-ratio and directional-relation, is that first given by Mr. Willis in his Treatise on *Pure Mechanism.* Its general plan is as follows:—

Class A: Directional-relation constant; velocity-ratio constant.

Class B: Directional-relation constant; velocity-ratio varying.

Class C: Directional-relation changing periodically; velocity-ratio constant or varying.

Each of those classes is subdivided by Mr. Willis into five divisions, of which the characters are as follows:—

Division I.—Connection by *rolling contact* of surfaces, as in toothless wheels.
— II.—Connection by *sliding contact* of surfaces, as in toothed wheels, cams, &c.
— III.—Connection by *wrapping connectors* or *bands*, as in pulleys connected by belts, cords, or chains.
— IV.—Connection by *link-work*, as in levers and cranks connected by means of rods, &c.
— V.—Connection by *reduplication* of cords, as in blocks and tackle used on board ship;

and to those five divisions may be added—

Division VI.—Connection by an *intervening fluid*, as in the hydraulic press.

In the present treatise the principle of the classification of Mr. Willis is followed; but the arrangement (as in a *Manual of Applied Mechanics*, already referred to) is modified by taking the *mode of connection* as the basis of the primary classification.

With reference to classes B and C, in which the velocity-ratio is or may be varying, it is to be observed that two kinds of problems arise respecting velocity-ratioe the determination of the *instantaneous velocity-ratio* at the instant when the pieces are in one given position; and the determination of the *mean velocity-ratio* during the interval between two such instants: the latter quantity is the ratio of the entire motions of the pieces during the interval.

Section II.—*Of Rolling Contact and Pitch Surfaces.*

95. **Pitch Surfaces** are those surfaces of a pair of moving pieces which touch each other when motion is communicated by rolling contact. The LINE OF CONTACT is that line which at each instant traverses all the pairs of points of the pair of pitch surfaces which are in contact.

The motion, relatively to the line of contact of their surfaces, of a pair of primary pieces which move in rolling contact, is the same with that of a secondary piece and a fixed piece, of which the former rolls upon the latter, as already described in Article 72, page 51; Articles 74 and 75, pages 53, 54; Article 77, page 56; Article 82, page 68, and Articles 84, 85, 86, pages 70 to 74; and therefore the proper figures for the pitch surfaces of such primary pieces are the same; that is to say, cylinders, cones, and hyperboloïds.

96. **Toothless Wheels, Rollers, Toothless Racks.**—Of a pair of

primary moving pieces in rolling contact, both may rotate, or one may rotate and the other have a motion of straight sliding. A rotating piece, in rolling contact, is called a *toothless wheel*, and sometimes a *roller;* a sliding piece may be called a *toothless rack*.

97. **Ideal Pitch Surfaces.**—The designing of pitch surfaces is used not only with a view to the making of toothless wheels and toothless racks (which are seldom employed), but much oftener as a step towards determining the proper figures for wheels and racks provided with teeth.

The pitch surface of a toothed wheel or of a toothed rack is an ideal smooth surface, intermediate between the crests of the teeth and the bottoms of the spaces between them, which, by rolling contact with the pitch surface of another wheel, would communicate the same velocity-ratio that the teeth communicate by their sliding contact. In designing toothed wheels and racks the forms of the ideal pitch surfaces are first determined, and from them are deduced the forms of the teeth.

Wheels with cylindrical pitch surfaces are called *spur wheels;* those with conical pitch surfaces, *bevel wheels;* and those with hyperboloidal pitch surfaces, *skew-bevel wheels*.

98. The **Pitch Line** of a wheel, or, in circular wheels, the PITCH CIRCLE, is the trace of the pitch surface upon a surface perpendicular to it and to the axis; that is, in spur wheels, upon a plane perpendicular to the axis; in bevel wheels, upon a sphere described about the apex of the conical pitch surface; and in skew-bevel wheels, upon an oblate spheroid generated by the rotation of an ellipse whose foci are the same with those of the hyperbola that generates the pitch surface. The pitch line might be otherwise defined, in most cases which occur in practice, simply as the trace of the pitch surface upon a plane perpendicular to the axis of rotation.

The PITCH POINT of a pair of wheels is the point of contact of their pitch lines; that is, the trace of the line of contact upon the surface or surfaces on which the pitch lines are traced.

The *pitch line of a rack* is the trace of its pitch surface on a plane parallel to its direction of motion, and containing its line of connection with the wheel with which it works.

A SECTOR is a name given to a wheel whose pitch-line forms only part of a circumference; sectors are used where the motion required is reciprocating or "rocking," and does not extend to a complete revolution. Everything stated respecting the figures of complete wheels applies also to the figures of sectors.

99. **General Conditions of Perfect Rolling Contact.** (*A. M.*, 439.)—The whole of the principles which regulate the motions of a pair of primary pieces in perfect rolling contact follow from the single principle, *that each pair of points in the pitch surfaces which are in*

contact at a given instant must at that instant be moving in the same direction with the same velocity.

The direction of motion of a point in a rotating body being perpendicular to a plane passing through its axis, the condition, that each pair of points in contact with each other must move in the same direction, leads to the following consequences:—

I. That when both pieces rotate, their axes, and all their points of contact, lie in the same plane.

II. That when one piece rotates and the other slides, the axis of the rotating piece, and all the points of contact, lie in a plane perpendicular to the direction of motion of the sliding piece.

The condition, that the velocities of each pair of points of contact must be equal, leads to the following consequences:—

III. That the angular velocities of a pair of wheels, in rolling contact, must be inversely as the perpendicular distances of any pair of points of contact from the respective axes.

IV. That the linear velocity of a rack in rolling contact with a wheel is equal to the product of the angular velocity of the wheel by the perpendicular distance from its axis to a pair of points of contact.

Respecting the line of contact, the above principles III. and IV. lead to the following conclusions:—

V. That for a pair of wheels with parallel axes, and for a wheel and rack, the line of contact is straight, and parallel to the axes or axis; and hence that the pitch surfaces are either cylindrical or plane (the term "cylindrical" including all surfaces generated by the motion of a straight line parallel to itself).

VI. That for a pair of wheels with intersecting axes the line of contact is also straight, and traverses the point of intersection of the axes; and hence that the rolling surfaces are conical, with a common apex (the term "conical" including all surfaces generated by the motion of a straight line which traverses a fixed point).

There is a sort of *imperfect rolling contact* which takes place between hyperboloidal pitch surfaces; that is to say, there is a sliding motion, but along the line of contact of the surfaces only; so that the component motions of points in directions perpendicular to the line of contact are the same as in perfect rolling contact. This kind of motion will be considered in treating specially of skew-bevel wheels.

100. **Wheels with Parallel Axes.**—Given, the positions of the parallel axes of a pair of wheels, and their velocity-ratio at a given instant, to find the pitch-point. Fig. 59 represents the case in which the directions of the rotations are contrary; fig. 60 that in which they are the same. Let the plane of projection be perpendicular to the two axes, and let A and B be their traces; so that A B is the line of centres. Perpendicular to A B draw A a and B b

proportional to the intended angular velocities. Draw the straight line *a b*, cutting A B (produced if necessary) in K. Lay off B I =

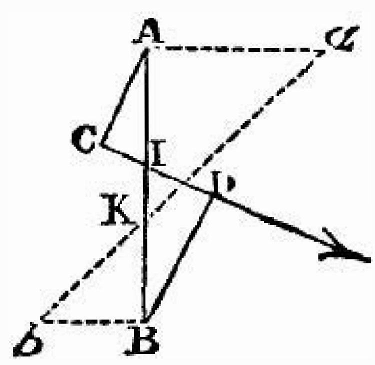

Fig. 59.

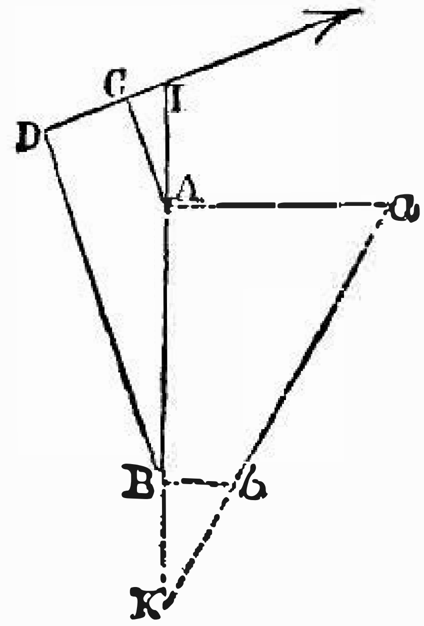

Fig. 60.

A K (or A I = B K); I will be the *required pitch-point* or trace of the line of contact.

The line of connection may be any straight line which traverses I, or whose projection traverses I; as C D. Let A C and B D be perpendicular to the line of connection; then the velocities of the points C and D are identical; and the perpendiculars A C and B D are inversely as the angular velocities of the pieces.

101. **Wheel and Rack.**—Given, at a given instant, the angular velocity of a wheel and the linear velocity of a rack, to find their pitch-point. In fig. 61, let the plane of projection be perpendicular to the axis of the wheel, and let A be the trace of that axis. Draw A I perpendicular to the direction of motion of the rack, and make its length such that a point in the wheel at I shall revolve with a velocity equal to that of the rack; that is to say, make $\mathrm{A\,I} = \dfrac{\text{linear velocity of rack}}{\text{angular velocity of wheel}}$; then I will be *the required pitch-point.*

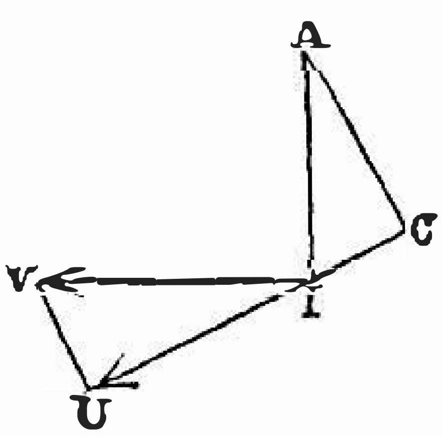

Fig. 61.

The line of connection may be any straight line which traverses I, or whose projection traverses I; as C U. Let fall the perpendicular A C; then the velocity of the point C in the wheel is equal to the component velocity of the rack along C U. Draw I V perpendicular to A I, to represent the whole velocity of the rack, and from V draw V U perpendicular to C U; it is evident that I U is the component velocity along the line of connection; and that A I : A C : : I V : I U.

102. **Circular Wheels in General.**—In order that, in an elementary combination of wheels, or of a wheel and rack, the velocity-

ratio may be constant (so that the combination shall belong to Mr. Willis's class A), it is obviously necessary that the pitch-point during the entire revolution of each wheel should occupy an invariable position in the line of centres; in other words, the pitch-line of each wheel must be a circle, and that of a rack a straight line. The corresponding forms of pitch-surface are:—for a spur-wheel, a circular cylinder; for a bevel-wheel, a cone with a circular base, and sometimes a plane circular disc; for a rack, a plane; for a skew-bevel wheel, a hyperboloïd of revolution. Circular wheels are by far the most common, the cases in which non-circular wheels are used being comparatively rare.

103. **Circular Spur-Wheels.**—Given, a pair of parallel axes and the constant velocity-ratio of a pair of wheels which are to turn about them, to draw the pitch-circles of those wheels. Fig. 62 represents the case in which the directions of rotation are contrary; fig. 63 that in which they are the same. Let A and B, as before, be the traces of the axes on a plane perpendicular to them. Find

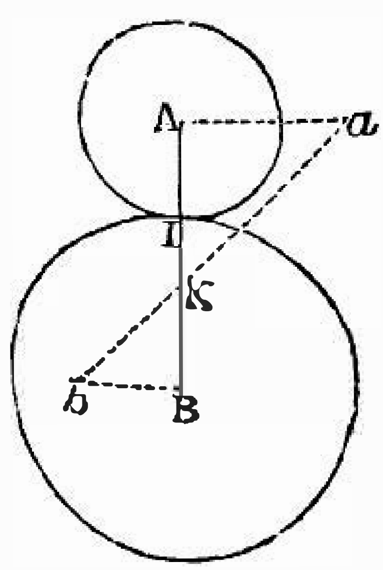

Fig. 62.

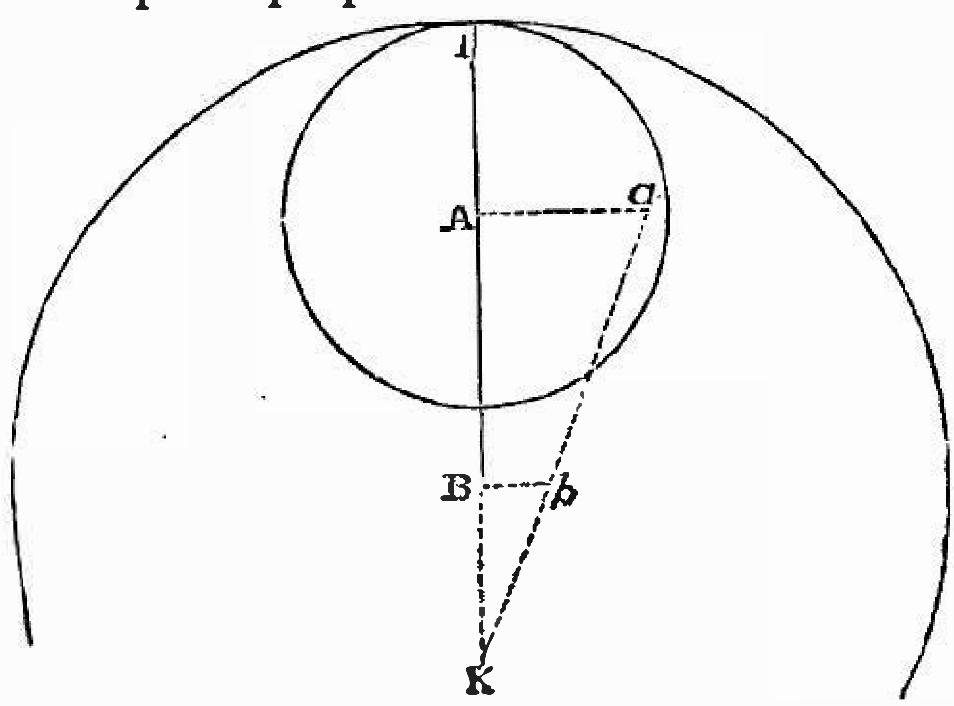

Fig. 63.

the pitch-point, I, as in Article 100, page 83. Then, about A and B, with the radii A I and B I respectively, draw two circles; these will be the pitch-circles required.

In fig. 62, where the rotations are contrary, and the pitch-point between the axes, the pitch-surfaces are both convex, and are said to be in "*outside gearing.*" In fig. 63, where the rotations are in the same direction, and the pitch-point beyond the axis of most rapid rotation, the smaller pitch-surface is convex and the larger concave; and these are said to be in "*inside gearing.*"

104. **Circular Wheel and Straight Rack.**—Given, the axis of a wheel, the direction of motion of a rack perpendicular to that axis, and the distance from the axis of a point in the wheel whose velocity is to be equal to that of the rack, to draw the pitch-lines of the wheel and rack. In fig. 64, let A be the trace

of the axis on a plane perpendicular to it. Draw A I perpendicular to the direction in which the rack is to move, and of a length equal to the given distance; then, about A, with the radius A I, draw a circle, and through I draw a straight line, M N, touching that circle; these will be the required pitch-lines.

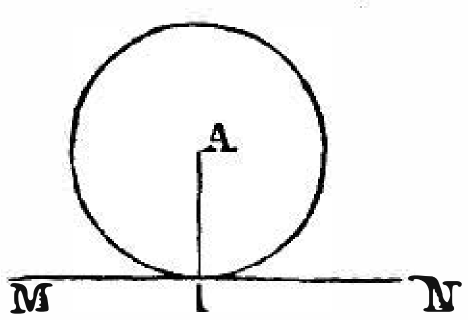

Fig. 64.

105. **Circular Bevel Wheels.**—Given, a pair of axes which intersect each other in a point, and the constant velocity-ratio of two wheels which are to turn about those axes, to draw projections of the pitch-surfaces of those wheels. Let the plane of fig. 65 represent the plane of the two axes; let O A and O B be their positions, and O their point of intersection. Lay off, on any convenient scale, along those axes, the distances O a and O b respectively proportional to the intended angular velocities (which, in the example shown, are contrary).

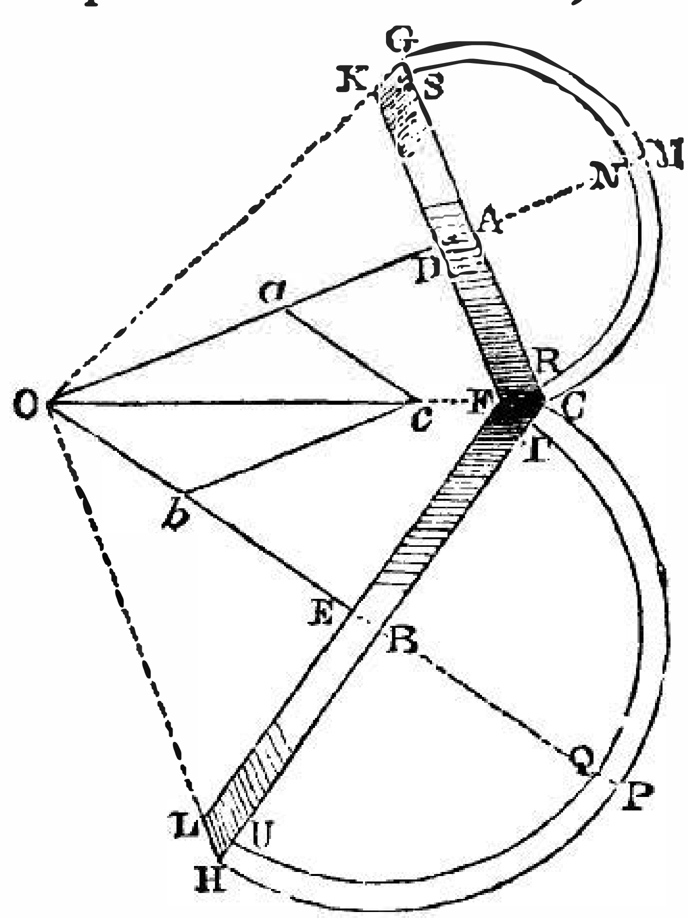

Fig. 65.

Draw a c parallel to O b, and b c parallel to O a, cutting each other in c; draw the diagonal O c C; this will be the line of contact; and the required pitch-surfaces will be parts of two cones described by making O C sweep round O A and O B respectively, and having their common summit at O. O C will be one of the traces of both these cones; and their other traces will be O G, making the angle A O G = A O C; and O H, making the angle B O H = B O C.

In any convenient position on the line of contact, mark a convenient breadth, C F, for the rims of both wheels, so that C F shall be their actual line of contact. Draw C A G and F D K perpendicular to O A, and C B H and F E L perpendicular to O B; then C G K F and C H L F will be the projections of the two wheels on the plane of their axes.

To draw the projection of one-half of each of those wheels on a plane perpendicular to its axis, about A, with the radius A C, draw the semicircle C M G, and with the radius A R = D F draw the semicircle R N S; these will be parts of the pitch-circles of which C A G and F D K are projections, and will form the required projection of one-half of the rim of the wheel whose axis is O A; then, about B, with the radius B C, draw the semicircle

C P H, and with the radius B T = E F draw the semicircle T Q U; these will be parts of the pitch-circles of which C B H and F E L are projections, and will form the required projection of one-half of the rim of the wheel whose axis is O B.

The proper widths for the rims of wheels will be considered farther on.

The perspective sketch, fig. 66, illustrates the case in which one of the pitch-surfaces becomes a flat disc or ring.

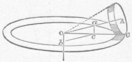

Fig. 66.

106. **Circular Skew-Bevel Wheels** are used when a constant velocity-ratio is to be communicated between two pieces which turn about axes that are neither parallel nor intersecting. Their pitch-surfaces are *rolling hyperboloïds;* and the figures and principal dimensions of such hyperboloïds are determined by the method already described in Article 84, page 70, and shown in fig. 54, page 72; it being understood that, in that figure, O a and O b represent the intended angular velocities *in contrary directions* of the two wheels.

For the actual wheels, narrow zones or frusta only of the hyperboloïds are used, as shown in fig. 67. Where approximate accuracy of form is sufficient, frusta of a pair of tangent cones (or of tangent cylinders, if the pitch-circles are the throats of the hyperboloïds) may be used; the figures of such cones and cylinders being determined as described in Articles 85 and 86, and shown in figs. 55 and 56, pages 73, 74.

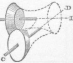

Fig. 67.

In all skew-bevel wheels the rolling motion is combined with *a relative sliding motion along the line of contact*, at a rate equal to the *rate of advance* described in Article 83, page 70.

The present Article contains some additional rules, which may have to be used in the designing and execution of skew-bevel wheels.

In fig. 68, let the vertical line through O represent the axis of a skew-bevel wheel, O A = O a the radius of its throat, and O C' a generating line, or line of contact, in that position in which it is parallel to the plane of projection, which plane is supposed to pass through the axis.

Draw the semicircle A B a; this will be the projection of half the throat of the hyperboloïd on a second plane of projection, perpendicular to the axis of the wheel.

Let C' G' be the projection and trace of a plane perpendicular to the axis, and chosen as a convenient plane for the pitch-circle in the middle of the breadth of the rim of the intended wheel, and let that projection cut O C′ in C′.

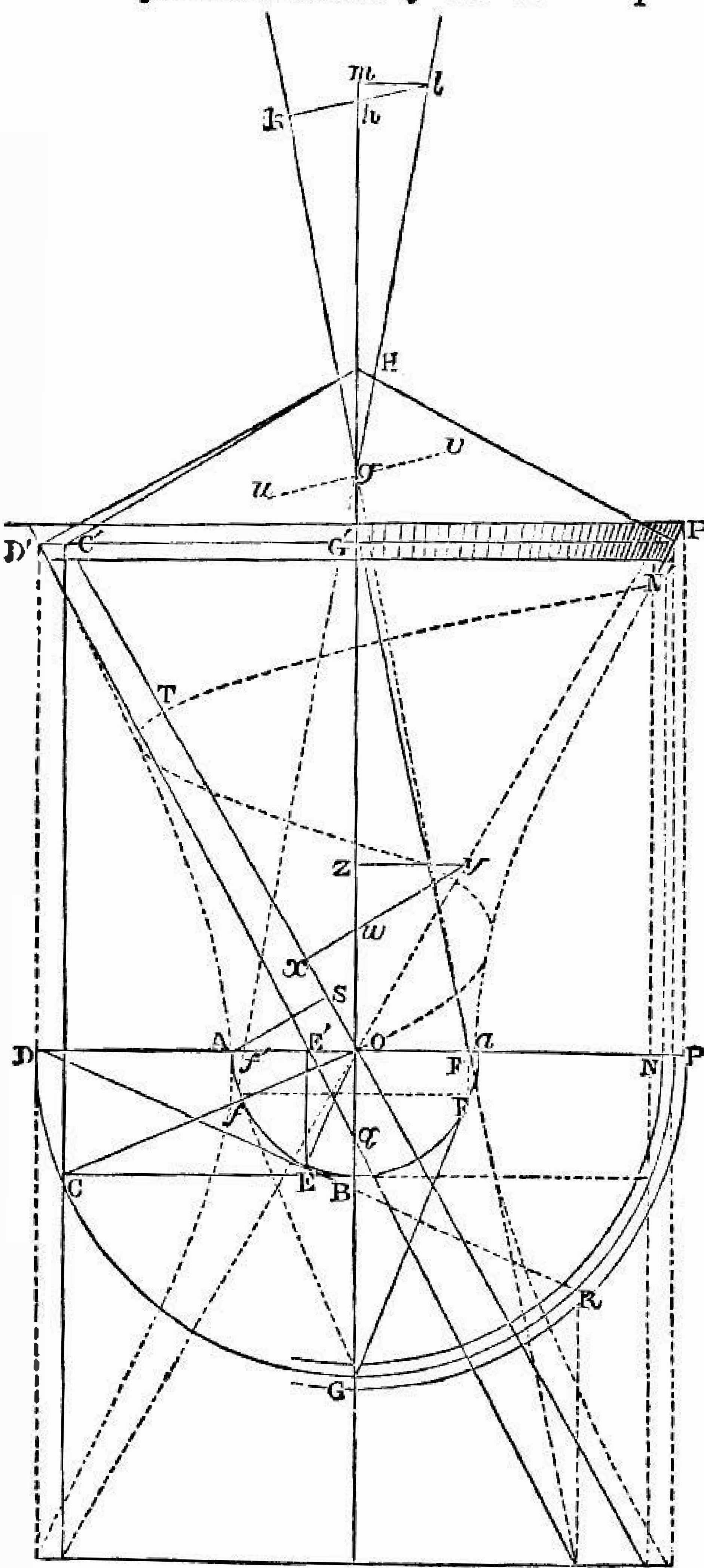

Fig. 68.

I. *To find the radius of the middle pitch-circle, and to draw its projections.* Through B draw B C parallel to O A; through C′ draw C′ C parallel to the axis, cutting B C in C. Join O C; this will be the required radius, and the circle D C G will be the projection of the pitch-circle on the second plane; in G′ C′ produced take G′ D′ = O D = O G = O C; G′ D′ will be the projection of the pitch-circle on the first plane.

D′ is a point in the hyperbolic trace of the hyperboloïd on the first plane; and by the same process any number of points in that trace may be found.

II. *To draw a normal to the pitch-surface in the first plane of projection.* Perpendicular to O C′ draw C′ H, cutting the axis of the wheel in H. This line and O C will be the projections of a normal to the pitch-surface at the point whose projections are C′ and C. Join H D′; this line and O D will be the projections of a normal to the pitch-surface at the point whose projections are D′ and D.

III. *To draw the traces of a tangent plane to the pitch surface at the point* D′, D. The line D′ D is the trace on the second plane of projection; and the trace on the first plane is D′ q perpendicular to D′ H.

Another process for finding the trace D′ q is as follows:—About D, with a radius, D E, equal to C B, draw a circular arc, cutting the circle A B a in E. Through E draw E E′ parallel to B O, cutting O A in E′. The straight line D E′ q will be the trace required.

D E and D′ E′ are also the projections of a generating line of the hyperboloïd.

IV. *Tangent cone.*—The summit of the tangent cone at the pitch-circle D′ G′ is at the point q, and D′ q is the trace of that cone on the first plane of projection. When extreme accuracy of form is not required, a portion of that cone, having the pitch-circle D′ G′ at the middle of its breadth, may be used instead of the exact hyperboloidal surface (Article 86, p. 73).

V. *Normal spiral.*—The normal spiral is a curve on the hyperboloidal surface which cuts all the generating straight lines, such as those whose projections are E′ D′, O C′, &c., at right angles. Its general form is indicated by the winding dotted curve which traverses O and T in fig. 68. It has an *uniform normal pitch*, O T, which is found as follows:—From A let fall A S perpendicular to O C′; then the normal pitch of the normal spiral is equal to the circumference of a circle whose radius is O S; that is to say,

$$\mathrm{O\,T} = \frac{710}{113}\,\mathrm{O\,S}.$$

It is not necessary to draw precisely the projections of the normal spiral; but for purposes connected with the designing of teeth for skew-bevel wheels it is useful to know its radius of curvature at the pitch-circle chosen for the wheel. That is done as follows:—

About G, with the radius G F = G f = B C, describe a circle, cutting the circle A B a in F and f; from which two points draw F F′ and $f f'$ parallel to B O. (Or otherwise, lay off O F′ = $o f'$ = E E′. F′ G′ and F G will be the two projections of a generating line.) In O H take O g = E′ D′; join F′ g, f' g, and produce both these lines as far as may be necessary. F′ g will be the rabatment of the triangle whose projection is O F′ G′. In O H produced, take g h = H D′; through h draw k h l perpendicular to F′ G′ k, and cutting f' g l in l: through l draw l m parallel to O A, cutting O H produced in m; then g m will be *the radius of curvature of every normal spiral at the point where it crosses the pitch-circle* G′ D′.

(The object of making this construction above instead of below the point g is merely to avoid confusion in the figure.)

Through g draw $u\,g\,v$ parallel to $k\,h\,l$; this will be the rabatment of a tangent to the normal spiral at the point G′.

To find the radius of curvature of a normal spiral at the throat of the hyperboloïd, in O H take O w = O A; draw $x\,w\,y$ perpendicular to O C′, and $y\,z$ parallel to O A; O z will be the required radius of curvature.

The lower part of the figure shows the projection on a plane through the axis, of a pitch-circle equal to the pitch-circle G′ D′, and at the same distance from the throat along the axis in the opposite direction. D E R and D′ E′ R′ are the two projections of one generating line extending from one of those pitch-circles to the other. G′ F′ R′ is the projection of another such generating line. The drawing of a pair of equal pitch-circles may sometimes be useful in the making of patterns for the wheel and for its teeth.

P, P′ and N, N′ are the projections of points in the two edges of the rim of the wheel. When the exact hyperboloidal pitch-surface is to be used, and not merely a tangent cone, those points are to be found by a process similar to that by which the projections D, D′ are found. When a tangent cone is used as an approximation, they are simply the intersections of two planes perpendicular to the axis, with a tangent in the plane of the axis.

VI. *Radius of curvature of hyperbolic trace.*—In constructing the pitch-surface of a skew-bevel wheel, it is sometimes useful to determine the radius of curvature of the hyperbolic trace of that surface on a plane traversing the axis, at the point where that trace cuts the pitch-circle, in order that a circular arc of that radius may, if required, be used as an approximation to a small arc of the hyperbolic curve.

In fig. 68 A, let O X be the axis of the hyperboloïd, O A the radius of its throat, O D an asymptote (being, as before, the projection of a line of contact that is parallel to the plane of projection), and X Y the trace of the plane of the intended pitch-circle. Part of the following process has already been described, but it will be described again here, to make the explanation complete:—Let D be the point where X Y cuts the asymptote. Lay off X E = O A; join D E; and make X Y = D E; then X Y will be the radius of the pitch-circle, and Y a point in the hyperbola. Perpendicular to O D draw D F, cutting the axis in F; join F Y; this will be a normal to the hyperbola at the point Y. Thus far the process has already been described.

Through A draw A B parallel to the axis, cutting the asymptote in B. From B, perpendicular to O B, draw B C, cutting O A produced in C. Then C will be *the centre of curvature*, and A C *the radius of curvature* of the hyperbola at A; that is, at the throat of the hyperboloïd.

In X Y, produced both ways as far as may be required, take

Y H = A O, Y L = A C, and Y G = Y F. In Y F take Y K = A O: join H F and K G. Through L, parallel to F H,

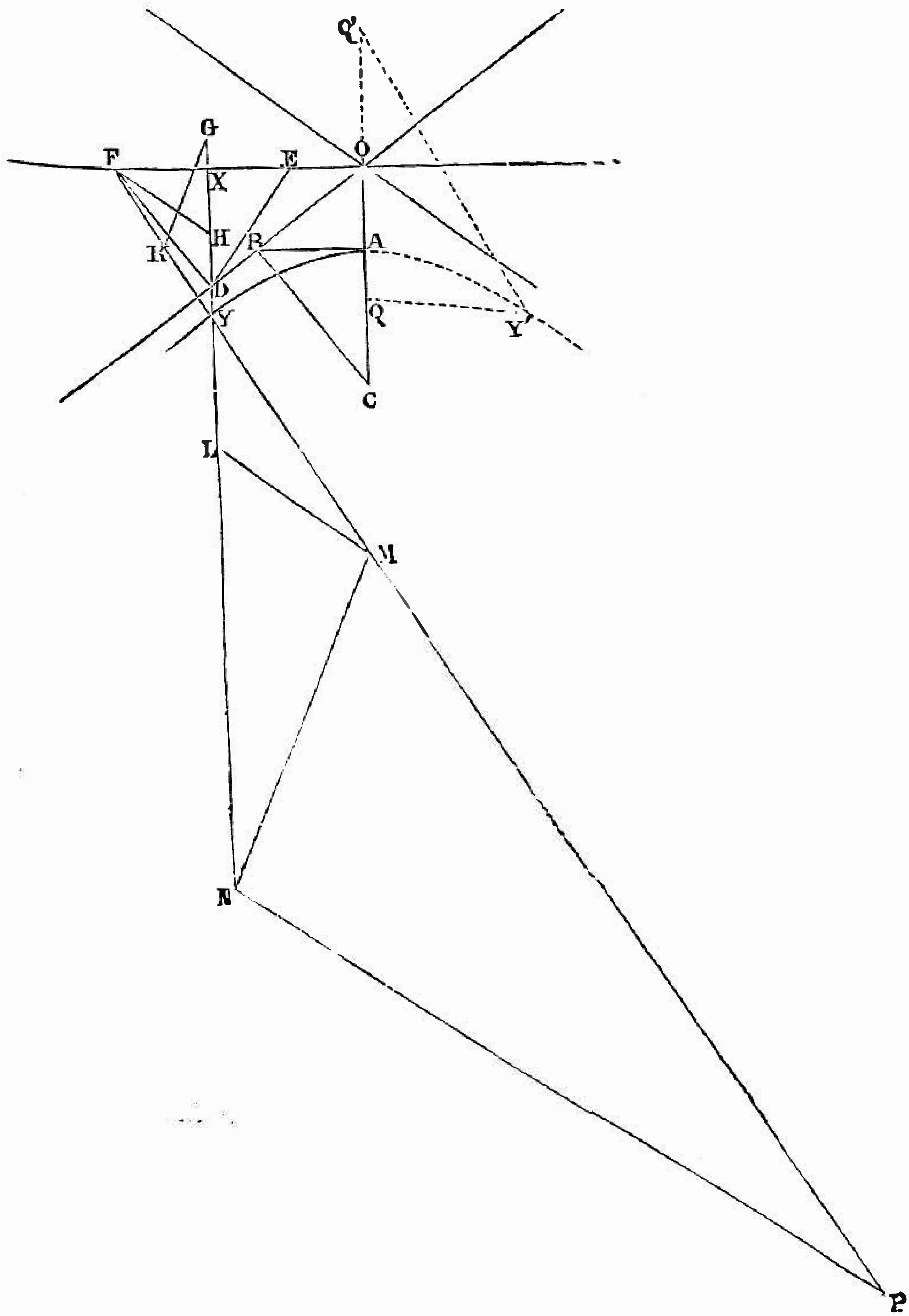

Fig. 68 A.

draw L M, cutting F Y produced in M; through M, parallel to G K, draw M N, cutting X Y L produced in N; and through N parallel to F H, draw N P, cutting F Y M produced in P; then P will be *the centre of curvature*, and Y P *the radius of curvature* of the hyperbola at Y.*

VII. *Foci of hyperbolic trace.*—To find, if required, the foci of the hyperbolic trace of the pitch-surface; produce, in fig. 68 A, the straight line O A, in both directions, as far as may be required, and lay off in it O Q = O Q′ = O B. Then Q and Q′ will be the two foci. The well-known property of a hyperbola, by means of which it can be drawn when one point in it and the two foci are given, is, that the difference of the distances from any point in the curve to the foci is a constant quantity; for example, Y′ Q′ — Y′ Q = A Q′ — A Q = 2 A O. Instruments founded on this principle are used for drawing hyperbolas.

107. **Non-Circular Wheels in General.** (*A. M.*, 443.)—Non-circular wheels are used to transmit a variable velocity-ratio between a pair of parallel axes. In fig. 69, let C_1, C_2 represent the axes of such a pair of wheels; T_1, T_2, a pair of points which at a given instant touch each other in the line of contact (which line is parallel to the axes and in the same plane with them); and U_1, U_2, another pair of points which touch each other at another instant of the motion; and let the four points, T_1, T_2, U_1, U_2, be in one plane perpendicular to the two axes and to the line of contact. Then, for every such set of four points, the two following equations must be fulfilled:—

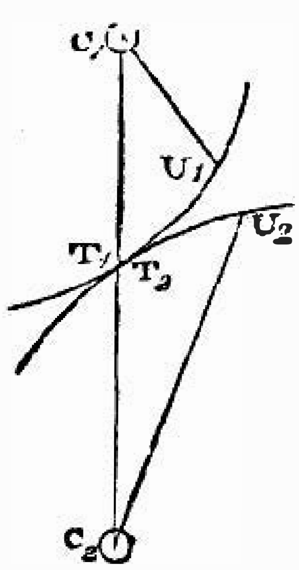

Fig. 69.

$$\left.\begin{array}{c} C_1 U_1 + C_2 U_2 = C_1 T_1 + C_2 T_2 = C_1 C_2; \\ \text{arc } T_1 U_1 = \text{arc } T_2 U_2; \end{array}\right\} (1.)$$

and those equations show the geometrical relations which must exist between a pair of rotating surfaces in order that they may move in rolling contact round fixed axes.

If one of the wheels be fixed and the other be rolled upon it, a point in the axis of the rolling wheel describes a circle of the radius $C_1 C_2$ round the axis of the fixed wheel.

The equations are made applicable to *inside gearing*, by putting − instead of + and + instead of −.

* The algebraical expressions of these operations are as follows:—

Let O A = b; A B = a; O X = x; X Y = y;

X F = m; Y F = n; Y P = ρ; A C = ρ_0; then

$$\rho_0 = \frac{a^2}{b}; \; y = \frac{b}{a}\sqrt{\left(a^2 + x^2\right)};$$

$$m = \frac{b^2 x}{a^2}; \; n = \sqrt{\left(y^2 + m^2\right)}; \; \rho = \rho_0 \frac{n^3}{b^3}.$$

The angular velocity-ratio at a given instant has the value

$$C_1 T_1 : C_2 T_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Non-circular wheels, when without teeth, may be called **Rolling Cams**; and in order that motion may be communicated by means of a pair of rolling cams, and of a suitably shaped smooth rack or sliding bar, it is necessary that the forces exerted by the two pieces on each other should be such as to press their pitch-surfaces together.

The following are the general problems to be solved in designing non-circular wheels:—

I. *Given, the axis and pitch-line of a non-circular wheel; to find approximately the axis of another non-circular wheel which shall turn in rolling contact with the first wheel, and of which an arc of a given length on the pitch-line shall subtend a given angle.*

In fig. 70, let the plane of projection be a plane perpendicular to the axes of the wheels. Let A be the axis of the given wheel, B C its pitch-line, and B its pitch-point at a given instant; and let A B be part of the line of centres. Also, let B D be a straight tangent to B C at B; and let the length of B D be the length of the arc on the pitch-line of the second wheel which is to subtend a given angle.

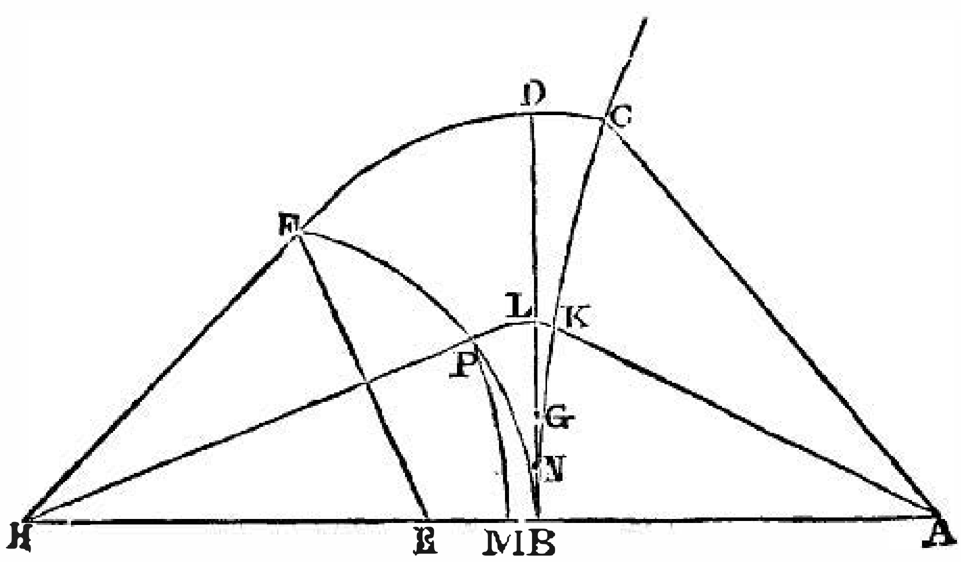

Fig. 70.

In B D take B G = ¼ B D, and about G, with the radius G D = ¾ B D, draw a circle, C D F, cutting the first pitch-line in C. Then, according to Rule IV., Article 51, page 29, the arc B C will be approximately equal in length to B D. Draw and measure the straight line A C; and in the line of centres take A E = A C. Then draw the straight line E F, making, with the line of centres, the angle H E F = the complement of half the angle that the arc of a length equal to B D is to subtend, and cutting the circle C D F in F. F will be approximately a point in the required pitch-line of the second wheel; and B and F will be the two ends of an arc approximately equal in length to B D and B C. To find the axis of that wheel, find, by plane geometry, in the line of centres, H B E, the centre, H, of a circle which shall traverse F and E; H will be approximately the trace of the required axis.

II. *To find a point in the second pitch-line whose distance from B, as measured on that pitch-line, shall be approximately equal to any given straight tangent,* B L. Take B N = ¼ B L; and about N, with a radius N L $= \frac{3}{4}$ B L, draw a circular arc, cutting the first pitch-line in K. Then B K will be approximately equal in length to B L. Join and measure A K, and in the line of centres take A M = A K. About H, with the radius H M, draw a circular arc, M P, cutting the arc K L P in P; P will be approximately the required point in the second pitch-line.

By repeating the same process, any number of points in the required pitch-line of the second wheel may be found approximately. The error of the two preceding rules, in what may be considered an extreme case—viz., where the pitch-line of the first wheel coincides with the straight tangent B D, and the angles B H F and B A C are each half a right angle (as in designing a roller to roll with a square roller)—is about 0·003 of the length B D to be laid off, and is in excess; the arc B F being too long by that fraction of its length; and the error, in fractions of the arc, varies nearly as the fourth power of the angle subtended by the arc. To ascertain whether the error is sensible, and to correct it by a second approximation, proceed as follows:—

III. *To obtain a closer approximation to the required axis and pitch-line.* Having drawn the pitch-line B F by Rule II., measure its length in subdivisions by Rule I. of Article 51, page 28, and compare that length with B D, so as to ascertain the error. Divide that error by B D, so as to express it in fractions of the required length. Multiply the half-sum of the greatest and least radii by the fraction expressing the ratio of the error to the required length; the product will be a *correction*, which is to be applied to the lengths of the line of centres, A H, and of each of the radii H B, H F, H P, &c., of the second pitch-line; that is to say, if the pitch-line, as at first drawn, is too long, each of those straight lines is to be shortened by having the correction subtracted from it.

For example, in the extreme case already cited, where the first pitch-line is a straight line, B D, perpendicular to A B, and subtending half a right angle at A, and the second pitch-line is to subtend also half a right angle at its axis H, let A B be taken as unity; then we have (to three places of decimals)

$$\text{B C} = \text{B D} = \text{A B} = 1{\cdot}000;$$

A C (coinciding with a straight line from A to D) = 1·414; and the application of Rule I. of this Article gives the following results as first approximations:—

$$\text{A H} = 2{\cdot}267;\ \text{H B} = 1{\cdot}267;\ \text{H F} = 0{\cdot}853.$$

Upon drawing the second pitch-line, B F, by Rule II. of this Article, and measuring it in subdivisions, it is found to be too long by 0·003 of its own length; which being multiplied by $\frac{HB + HF}{2} = \frac{2·120}{2} = 1·060$, gives 0·003 for the correction to be subtracted from the line of centres and from each of the radii of the second pitch-line. Thus are obtained the second approximations,

$$AH = 2·264; \; HB = 1·264; \; HF = 0·850.$$

As examples of non-circular wheels, the following may be mentioned:—

I. An ellipse rotating about one focus rolls completely round in outside gearing with an equal and similar ellipse also rotating about one focus, the distance between the axes of rotation being equal to the major axis of the ellipses, and the velocity-ratio varying from

$$\frac{1 - \text{eccentricity}}{1 + \text{eccentricity}} \text{ to } \frac{1 + \text{eccentricity}}{1 - \text{eccentricity}} \text{ (see Article 108).}$$

II. Lobed wheels, of forms derived from the ellipse, roll completely round in outside gearing with each other (see Article 109).

III. A hyperbola rotating about its farther focus rolls in inside gearing, through a limited arc, with an equal and similar hyperbola rotating about its nearer focus, the distance between the axes of rotation being equal to the axis of the hyperbolas, and the velocity-ratio varying between

$$\frac{\text{eccentricity} + 1}{\text{eccentricity} - 1} \text{ and unity.}$$

IV. Two logarithmic spiral sectors of equal obliquity rotate in rolling contact with each other; or one logarithmic spiral sector rotates in rolling contact with the oblique plane surface of a sliding piece (see Article 110).

108. **Elliptic Wheels.**—The following rules are applicable to the drawing of the pitch-lines of elliptic wheels, and the determination of their comparative motions:—

I. *Given, the angle by which each wheel is alternately to overtake and to fall behind the other, and the length of the line of centres, to draw the ellipse which is the figure of both pitch-lines.*

From a point, B, draw two straight lines, B F = B F′ = half the line of centres, making with each other the given angle F B F′. Join F F′, bisect it in O, produce it both ways, and make O A = O A′ = half the line of centres. Draw B B′ perpendicular to A A″, making O B′ = O B. Then A A′ is the major axis, B B′ the minor axis, O the centre, and F, F′, the two foci of the required ellipse, which may be drawn by means of a suitable instrument or machine, or by the well-known process of putting an endless thread, of a length = 2 A F′ = 2 F A′, round two pins at the foci, and a pencil equal in diameter to those pins, and moving the pencil round so as to keep the thread tight. In the workshop ellipses of given dimensions can be drawn with great precision by means of the turning lathe, fitted with apparatus to be afterwards referred to.

The wheels are to be centred, as shown in fig. 72, each upon one of its foci. The *fixed foci*, which are thus placed in the axes of the wheels, are marked A, B, in this figure, and the *revolving foci*, C, D. The ellipses in fig. 72 are similar to that in fig. 71, but drawn on one-half of the scale.

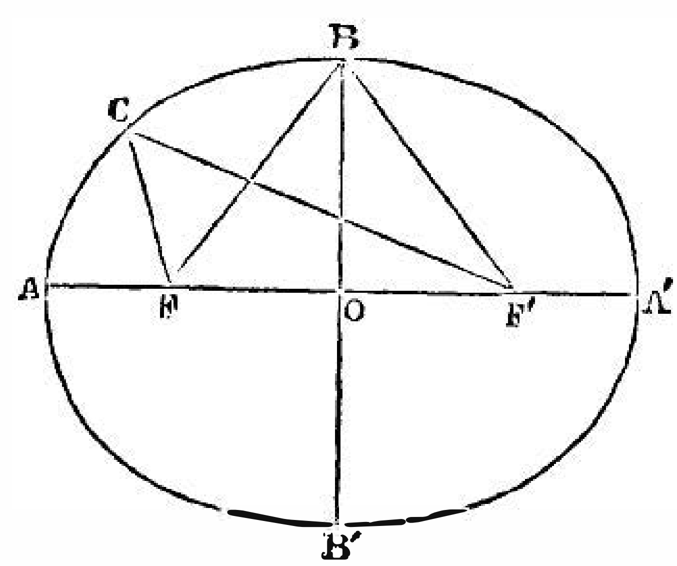

Fig. 71.

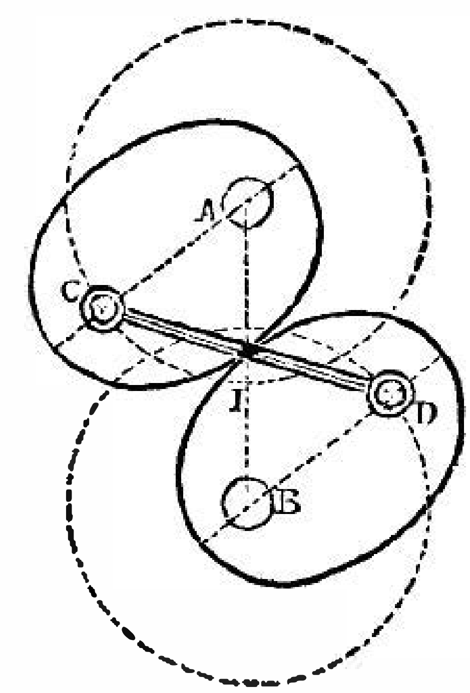

Fig. 72.

II. *To find the angular motions and the angular velocity-ratio corresponding to a given position of the pitch-point.* Suppose both wheels to have started from a position in which A, fig. 71, is the pitch-point, being at the distance A F from the axis of one wheel, and A F′ from that of the other, so that the angular velocity-ratio of the second wheel to the first is A F ÷ A F′. Let C be a new position of the pitch-point. Draw C F, C F′. Then the *angular motion* of the first wheel is A F C; that of the second wheel A F′ C; the first wheel has *overtaken* the second wheel to the extent represented by the angle F C F′ = A F C − A F′ C; and the *velocity-ratio* of the second wheel to the first is C F ÷ C F′.

The angular velocity-ratio ranges between the limits $\frac{A\,F}{A\,F'}$ and $\frac{A\,F'}{A\,F}$; and its mean value in each half-revolution is *unity;* because each half-revolution is made in the same time by both wheels. The instantaneous velocity-ratio is unity when the pitch-point is at B or B′; because B F = B F′.

III. *Given, at any instant, the position of one of the revolving foci, to find the position of the other revolving focus, and of the pitch-point.* In fig. 72, let A and B be the fixed foci. With a radius equal to the distance between the foci, or double eccentricity (F F′ in fig.

71), draw circles about A and B. Let C be the given position of one of the revolving foci. Then, with a radius C D = A B (the line of centres), draw a circular arc about C, cutting the circle round B in D; this will be the other revolving focus. Join C D, cutting A B in I; this will be the pitch-point.

If the wheels and their axles overhang the bearings, the revolving foci, being at a constant distance apart, may be connected by means of a link, C D, as shown in fig. 72. This is useful in elliptic toothed wheels of great eccentricity, because of the teeth in certain positions of the wheel being apt to lose hold of each other.

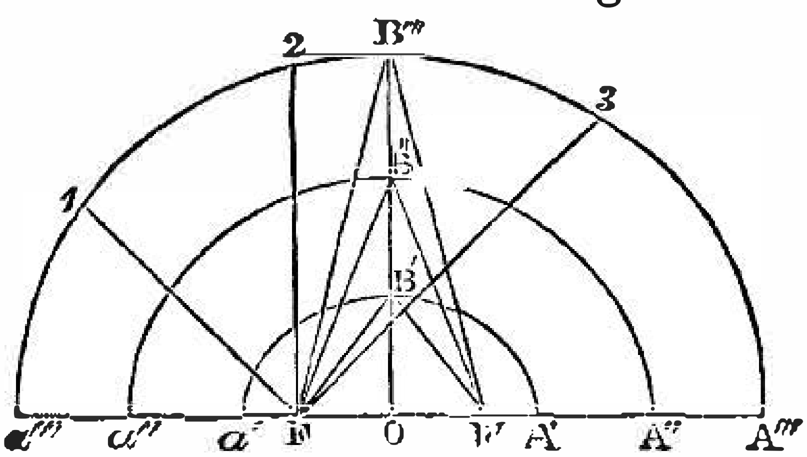

Fig. 73.

109. **Lobed Wheels** * are wheels such as those shown in figs. 74 and 75, having two, three, or any greater number of equal greatest radii (such as those marked F A'' in fig. 74, and F A''' in fig. 75), and also of least radii (such as those marked F a'' in fig. 74, and F a''' in fig. 75). Fig. 74 represents a two-lobed wheel, and fig. 75 a three-lobed wheel. An elliptic wheel may be regarded as a *one-lobed wheel*. Let the difference between the

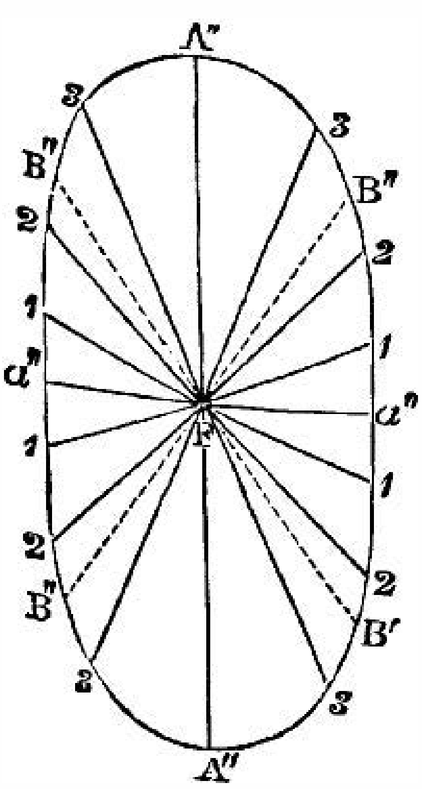

Fig. 74.

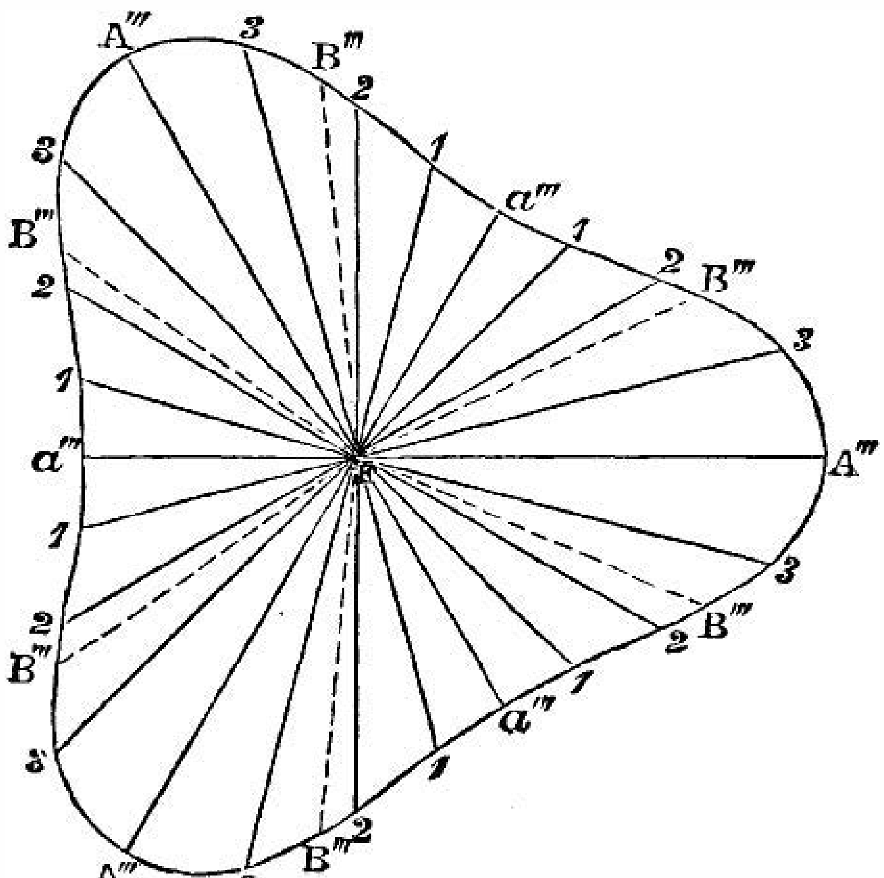

Fig. 75.

greatest and least radii of a lobed wheel be called the *inequality*; so that in an elliptic wheel (fig. 71) the inequality is the distance

* The properties of these wheels were discovered by the Reverend W. Holditch.

between the foci, F F′. Then any pair of lobed wheels in which the inequality is the same will, if properly shaped, work together in rolling contact, and that whether their numbers of lobes are many or few, the same or different; and this statement includes one-lobed or elliptic wheels.

The advantage of wheels with two or more lobes is their being self-balanced, which elliptic wheels are not.

The following are the rules for designing lobed wheels:—

I. *Given, in a pair of equal and similar lobed wheels, the angle by which each wheel is alternately to overtake and to fall behind the other wheel, the number of lobes, and the mean radius, to find the inequality, and thence the greatest and least radii.* Multiply the given angle by the number of lobes; then, from a point B‴, fig. 73, draw two lines, B‴ F, B‴ F′, making with each other an angle equal to the product, and make the length of each of them equal to the given mean radius. Draw the straight line F F′; this will be the required *inequality.* Bisect F F′ in O, and produce it both ways; then lay off O A‴ = O a‴ = B‴ F = B‴ F′, the mean radius; then F A‴ = F′ a‴ will be the greatest radius, and F′ A‴ = F a‴ the least radius.

II. *To find any number of points in the pitch-line.* In fig. 73, with the major axis A‴ a‴, and the foci F and F′, draw a semi-ellipse A‴ B‴ a‴. Then, in fig. 75, draw from the centre, F, the straight lines marked F A‴, dividing a complete revolution into as many equal parts as there are to be lobes (in the present case, three). Make each of these lines equal to the greatest radius (F A‴, fig. 73). Bisect the angles between them with the straight lines marked F a‴, fig. 75, and make each of the latter set of lines equal to the least radius (F a‴, fig. 73). Divide the half-revolution in fig. 73 into any convenient number of equal angles by the radiating lines F 1, F 2, &c., meeting the ellipse at 1, 2, &c. Divide each of the angles marked A‴ F a‴ in fig. 75 into the same number of equal parts by radiating lines, and lay off upon them lengths, F 1, F 2, &c., equal to those of the corresponding lines in fig. 73; the points 1, 2, &c., in fig. 75, thus found, will be *points in the required pitch-line.*

III. *To find the positions of the mean radii of the required pitch-line.* Divide the angle A‴ F B‴, in fig. 73 by the number of lobes, and lay off the quotient for each of the angles marked A‴ F B‴ in fig. 75; then make each of the radii F B‴ in fig. 75 equal to F B‴, in fig. 73; these will be the required mean radii.

Remark.—The example in fig. 75 is a three-lobed wheel. The two-lobed wheel of fig. 74 is drawn by a similar process; the ellipse used for finding the radii being A″ B″ a″ in fig. 73; the inequality F F′; and the angle by which each wheel alternately overtakes and falls behind another equal and similar wheel being one-half of F B″ F′, fig. 73.

IV. *To draw the pitch-lines of a set of wheels of different numbers of lobes, all of which shall work with each other in rolling contact.* The inequality must be the same in each wheel. Let F F′, fig. 73, be that inequality; and let O be the centre, A‴ a‴ the major axis, and O B‴ the semi-minor axis of the ellipse which serves for finding the radii of one of the set of wheels, which one wheel is given. Divide O B‴ into as many equal parts as there are lobes in the given wheel; say, for example, three. To find the figure of a wheel having any other number of lobes (say two), take the point B″ at that number of divisions from O; join F B″, F′ B″; lay off O A″ = O a″ = F B″ = F′ B″; draw the ellipse A″ B″ a″ with A″ a″ for its major axis, and F and F′ for its foci; this will be the ellipse for determining the lengths of the radii of the new (two-lobed) wheel.

The ellipse A′ B′ a′ with the same foci, F F′, whose minor semi-axis, O B′, is *one* division of O B‴, is itself the pitch-line of the one-lobed wheel, which will work in rolling contact with any wheel of the set.

110. **Logarithmic Spiral Sectors or Rolling Cams.**—A pair of logarithmic spiral sectors may be used as rolling cams, to communicate by rolling contact an angular motion of limited extent, in the course of which it is desired that the velocity-ratio shall range between certain limits. The general nature of the figure and position of such a pair of sectors may be represented by fig. 69, page 90.

The only cases in which the dimensions and figures of such sectors can be determined by plane geometry alone, without the aid of calculation, are two, viz. when the two sectors are equal and similar, so that the sum of the greatest and least radii of each of the two sectors is equal to the line of centres; and when the combination consists of one sector, working with a sliding bar or smooth rack. The following are the rules applicable to such cases:—

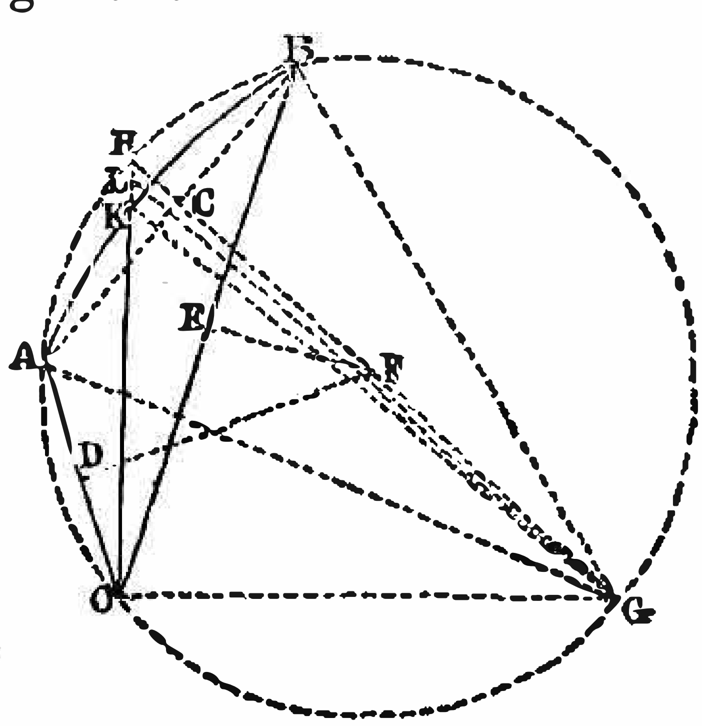

Fig. 76.

I. *Given, in fig. 76, the least and greatest radii,* O A *and* O B, *of a logarithmic spiral sector, and the angle* O A B *between them, to find intermediate points in the pitch-line of such a sector, and to draw that pitch-line approximately by means of one or more circular arcs.*

Describe a circle about the triangle O A B; that is to say, bisect any two of the sides of that triangle (C, E, and D being the three points of bisection), and from the points of bisection draw perpendiculars to the sides, meeting in F, which will be the centre of the circle through O, A, and B. Draw the diameter G F C H, bisecting the arc A H B in H and the arc A O G B in G. Join O H (which will be perpendicular to O G, and will bisect the angle A O B); and about G, with the radius G A = G B, draw the circular arc A K B, cutting O H in K. Then K will be a point in the required spiral; and A K B will be the nearest approximation to the spiral arc traversing the three points, A, K and B, that it is possible to make by means of one circular arc only.

To find two additional points, and a closer approximation to the curve, treat each of the triangles O A K and O K B as the triangle O A B was treated; the result will be the finding of two more points in the spiral, situated respectively in the radii which bisect the angles A O K and K O B; and the drawing of two circular arcs, one extending from A to K, and the other from K to B, which will make a closer approximation to the spiral arc than a single circular arc does.

The next repetition of the process will give four additional points and four circular arcs; the next, eight additional points and eight circular arcs; and so on to any degree of precision that may be required.

The radius O K is *a mean proportional* between O A and O B; and this property enables its length to be found by calculation, if required.

The *obliquity* of a logarithmic spiral, being the angle which a tangent to the spiral at a given point makes with a tangent to a circle described about the axis through that point, or the equal angle which a normal to the spiral at the same point makes with a radius drawn from that point to the axis, is uniform in a given spiral. In fig. 76 the equal angles, O A G, O H G, and O B G, are each of them less than the true obliquity of the spiral, and the angle O K G is greater than the true obliquity. To obtain the closest approximation to the true obliquity possible without further subdividing the angle A O B, proceed as follows:—

II. *To find the approximate obliquity.* In H K take H L = $\frac{1}{3}$ H K; join L G; then O L G will be the obliquity, very nearly. In other words, L G will be very nearly parallel to a normal, and perpendicular to a tangent, to the true spiral at the point K.

II A. *To find the approximate obliquity* (Another method). By Rule I. or Rule II. of Article 51, page 28, measure the approximate length of the arc A B in fig. 76. Then, in fig. 77, draw the straight line M N = O B − O A; draw M P perpendicular to M N; and about N, with a radius equal to the approximate length of the arc A B, draw a circular arc, cutting M P in P; join N P; then the angle M P N will be approximately the required obliquity.

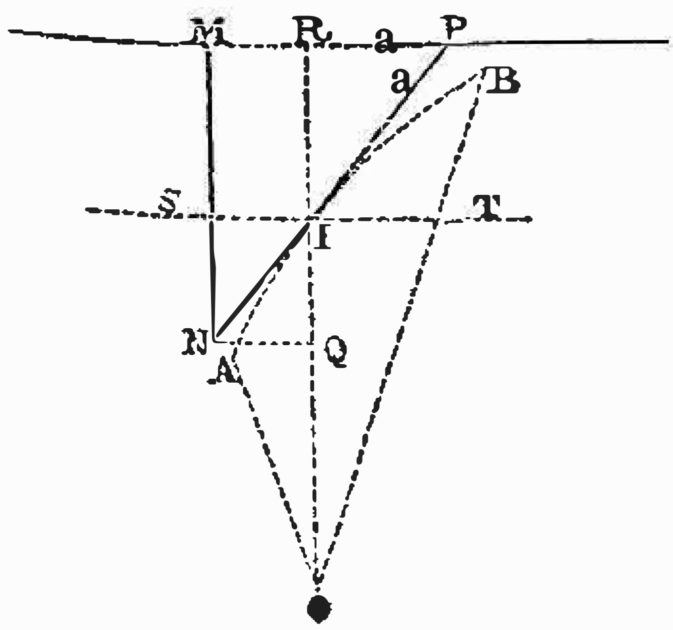

Fig. 77.

III. *Given* (in fig. 76), *one radius,* O K, *in a logarithmic spiral of a given obliquity, to draw approximately a short arc of that spiral through* K. Draw O G perpendicular to O K; draw K G, making O K G = the angle of obliquity, and cutting O K G in G; then, with the radius G K, draw a short circular arc through K.

IV. *To draw the pitch-line of a sliding bar which shall work in rolling contact with a given logarithmic spiral sector,* A O B (fig. 77). From the trace of the axis O draw O Q R perpendicular to the direction in which the bar is to slide, making O Q = O A, and O R = O B. Find the obliquity of the sector by means of one or other of the preceding rules. Let I be any given position of the pitch-point, and let T I S, traversing I perpendicularly to O Q R, be parallel to the direction in which the bar is to slide. Draw the straight line N I P, making the angle S I N = T I P = the obliquity; then draw Q N and R P parallel to T I S, and cutting N I P in N and P respectively. The straight line N I P will be the required pitch-line; and N and P will be the points in it corresponding to A and B respectively in the pitch-line of the sector.

At the instant when the pitch-point is at I, the velocity of the sliding bar is equal to that of the point I in the sector; that is to say, angular velocity × O I; agreeably to the general principle of Article 101, page 84.

The following rules relate to the solution of questions respecting logarithmic spiral sectors by calculation.

V. *Given, two radii of a logarithmic spiral sector* (as O A and O B, fig. 76), *and the angle between them* (A O B), *to find the obliquity of the spiral.* Take the hyperbolic logarithm * of the ratio $\frac{\text{O B}}{\text{O A}}$; divide it by the angle A O B in

* Hyperbolic logarithm of a ratio = common logarithm × 2·3026 nearly.

circular measure;* the quotient will be the tangent of the obliquity.

VI. *Given, the least and greatest radii of a logarithmic spiral sector, and the angle between them, to find the lengths of a series of intermediate radii, which shall divide that angle into a given number of equal smaller angles.* Take the difference between the logarithms of the greatest and least radii; divide it by the given number; then, commencing with the logarithm of the least radius, compute by successive additions of the quotient a series of logarithms, increasing by uniform differences up to the logarithm of the greatest radius; these will be the logarithms of the required intermediate radii.

VII. *Given, one radius and the obliquity of a logarithmic spiral, to find the length of a radius making a given angle with the given radius.* Multiply the given angle in circular measure (see first footnote below) by the tangent of the obliquity; to the product add the hyperbolic logarithm of the given radius; the sum will be the hyperbolic logarithm of the required radius;—or otherwise, multiply the product by 0·4343, and to the new product add the common logarithm of the given radius; the sum will be the common logarithm of the required radius.

VIII. *Given, the difference between the greatest and least radii of a logarithmic spiral sector, and the obliquity of its pitch-line, to find the length of its pitch-line.* Multiply the difference of the radii by the cosecant of the obliquity. †

111. **Frictional Gearing.**—To increase that friction or adhesion between a pair of wheels which is the means of transmitting force and motion from one to the other, their surfaces of contact are sometimes formed into alternate ridges and grooves parallel to the

* Reduction of angles to circular measure—

1 degree	= 0·0174533	radius length, nearly.
30 degrees	= 0·5236	,, ,, ,,
60 degrees	= 1·0472	,, ,, ,,
90 degrees	= 1·5708	,, ,, ,,

† In symbols, the equations of a logarithmic spiral are as follows:—Let a be the radius from whose directions angles are reckoned; r, any other radius; θ, the angle which r makes with a, in circular measure; ϕ, the obliquity of the spiral; s, the length of the arc from a to r; ρ, the radius of curvature at the end of the radius r. Then

$$r = a\,e^{\theta \tan \phi}; \quad \tan \phi = \frac{1}{\theta}\,\text{hyp. log.}\,\frac{r}{a}$$

$$\theta = \text{cotan}\,\phi \cdot \text{hyp. log.}\,\frac{r}{a};$$

$$s = (r - a)\,\text{cosec}\,\phi = a\,\text{cosec}\,\phi\left(e^{\theta \tan \phi} - 1\right);$$

$$\rho = r \tan \phi.$$

plane of rotation, constituting what is called *frictional gearing.* Fig. 78 is a cross-section of the rim of a wheel, illustrating the kind of frictional gearing invented by Mr. Robertson. The comparative motion of a pair of wheels thus ridged and grooved is nearly the same with that of a pair of smooth wheels in rolling contact, having cylindrical or conical pitch-surfaces lying midway between the tops of the ridges and bottoms of the grooves.

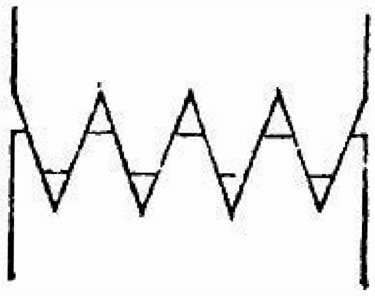

Fig. 78.

The relative motion of the surfaces of contact of the ridges and grooves is a rotatory sliding or grinding motion about the line of contact of the ideal pitch-surfaces as an instantaneous axis; and the angular velocity of that relative grinding motion is equal to the angular velocity of one wheel considered as rolling upon the other as a fixed wheel; which may be found by the principles of Article 77, page 56, and Article 82, page 68.

The angle between the sides of each groove is about 40°; and it is stated that the mutual friction of the wheels is about once and a-half the force with which their axes are pressed towards each other.

SECTION III.—*Of the Pitch and Number of the Teeth of Wheels.*

112. **Relation between Teeth and Pitch-Surfaces—Nature of the Subject.** (*A. M.*, 446.)—The most usual method of communicating motion between a pair of wheels, or a wheel and a rack, and the only method which, by preventing the possibility of the rotation of one wheel unless accompanied by the other, insures the preservation of a given velocity-ratio exactly, is by means of a series of alternate ridges and hollows parallel, or nearly parallel, to the successive lines of contact of the ideal toothless wheels or *pitch-surfaces,* whose velocity-ratio would be the same with that of the toothed wheels. The ridges are called *teeth;* the hollows, *spaces.* The teeth of the driver push those of the follower before them, and in so doing sliding takes place between them in a direction across their lines of contact.

The properties of pitch-surfaces and pitch-lines, and the art of designing them, have been explained in the preceding section. The figures of teeth depend on the principles of sliding contact, which belong to the ensuing section. The present section relates to questions connected with the manner in which the pitch-line of a wheel is divided by the acting surfaces of its teeth, without reference to the figures of those surfaces; for such questions do not require the principles of sliding contact for their solution.

113. **Pitch Defined.** (*A. M.*, 447.)—The distance, measured

along the pitch-line, from the front of one tooth to the front of the next, is called the PITCH.

114. **General Principles.** (*A. M.*, 447.)—The pitch, and the number of teeth in wheels, are regulated by the following principles:—

I. In wheels which rotate continuously for one revolution or more, it is obviously necessary *that the pitch should be an aliquot part of the circumference of the pitch-line.*

In racks, and in wheels which reciprocate without performing a complete revolution, this condition is not necessary. Such wheels are called *sectors*, as has been already stated.

II. In order that a pair of wheels, or a wheel and a rack, may work correctly together, it is in all cases essential *that the pitch should be the same in each.*

III. Hence, in any pair of wheels which work together, *the numbers of teeth in a complete circumference are inversely as the numbers of whole revolutions in a given time;* or, in other words, *directly as the times of revolution.*

IV. Hence, also, in any pair of wheels which rotate continuously for one revolution or more, the ratio of the times of revolution (being equal to that of the numbers of teeth), and its reciprocal, the ratio of the numbers of revolutions in a given time, *must both be expressible in whole numbers.*

V. In circular wheels, everything stated in the preceding principles regarding the ratio of the numbers of revolutions in a given time (in other words, of the *mean angular velocity-ratio*) is true also of the angular velocity-ratio at every instant.

115. **Frequency of Contact—Hunting-Cog.**—Let n, N, be the respective numbers of teeth in a pair of wheels, N being the greater. Let t, T, be a pair of teeth in the smaller and larger wheel respectively, which at a particular instant work together. It is required to find, first, how many pairs of teeth must pass the pitch-point before t and T work together again (let this number be called a); secondly, with how many different teeth of the larger wheel the tooth t will work at different times (let this number be called b); and thirdly, with how many different teeth of the smaller wheel the tooth T will work at different times (let this be called c).

CASE 1. If n is a divisor of N,

$$a = \mathrm{N}; \; b = \frac{\mathrm{N}}{n}; \; c = 1 \ldots\ldots\ldots\ldots (1.)$$

CASE 2. If the greatest common divisor of N and n be d, a number less than n, so that $n = m\,d$, $\mathrm{N} = \mathrm{M}\,d$, then

$$a = m\,\mathrm{N} = \mathrm{M}\,n = \mathrm{M}\,m\,d; \; b = \mathrm{M}; \; c = m \ldots\ldots (2.)$$

CASE 3. If N and n be prime to each other,

$$a = \mathrm{N}\,n; \; b = \mathrm{N}; \; c = n \ldots\ldots\ldots\ldots (3.)$$

It is considered desirable by millwrights, with a view to the preservation of the uniformity of shape of the teeth of a pair of wheels, that each given tooth in one wheel should work with as many different teeth in the other wheel as possible. They therefore study to make the numbers of teeth in each pair of wheels which work together such as to be either prime to each other, or to have their greatest common divisor as small as is possible consistently with the purposes of the machine.

When the ratio of the angular velocities of two wheels, being reduced to its least terms, is expressed by numbers less than those which can be given to wheels in practice, and it becomes necessary to employ multiples of those numbers by a common multiplier, which becomes a common divisor of the numbers of teeth in the wheels, millwrights and engine-makers avoid the evil of frequent contact between the same pairs of teeth, by giving one additional tooth, called a *hunting-cog*, to the larger of the two wheels. This expedient causes the velocity-ratio to be not exactly but only approximately equal to that which was at first contemplated; and therefore it cannot be used where the exactness of certain velocity-ratios amongst the wheels is of importance, as in clockwork.

116. **Smallest Pinion.**—The *smallest* number of teeth which it is practicable to give to a pinion (that is, a small wheel), is regulated by principles which will appear when the forms of teeth are considered. The following are the least numbers of teeth which can be usually employed in pinions having teeth of the three classes of figures named below, whose properties will be explained in the sequel:—

I.	Involute teeth,.............	h...........	25
II.	Epicycloidal teeth,...	h..................	12
III.	Round teeth, or *staves*,.........	t........	6

117. **Arithmetical Rules.**—For convenience sake the following arithmetical rules are here given, as being useful in the designing of toothed gearing.

I. *To find the prime factors of a given number.* Try the prime numbers, 2, 3, 5, 7, 11, &c., as divisors in succession, until a prime number has been found to divide the given number without a remainder; then try whether and how many times over the quotient is again divisible by the same prime number, so as to obtain a quotient not divisible again by the same prime number; then try the division of that quotient by the next greater prime number; and so on until a quotient is obtained which is itself a prime number; that is, a number not divisible by any other number except 1. This final quotient and the series of divisors will be the prime factors of the given number. To test the accuracy of the process, multiply all the prime factors together; the product should be the given number.

II. *To find the greatest common measure* (otherwise called the *greatest common divisor*) *of two numbers.* Divide the greater number by the less, so as to obtain a quotient, and a remainder less than the divisor; divide the divisor by the remainder as a new divisor; that new divisor by the new remainder; and so on, until a remainder is obtained which divides the previous divisor without a remainder. That last remainder will be the required greatest common measure.

If the last remainder is 1, the two numbers are said to be "prime to each other."

Example.—Required, the greatest common measure of 1420 and 1808.

```
Divisor, 1420) 1808 (1, Quotient.
               1420
               ----
Remainder,      388) 1420 (3, Quotient.
                     1164
                     ----
     Remainder,       256) 388 (1, Quotient.
                           256
                           ---
          Remainder,       132) 256 (1, Quotient.
                                132
                                ---
               Remainder,       124) 132 (1, Quotient.
                                     124
                                     ---
                    Remainder,         8) 124 (15, Quotient.
                                          120
                                          ---
                         Remainder,         4) 8 (2, Quotient.
                                               ==
```

The last remainder, 4, is the required greatest common measure.

III. To reduce the ratio of two numbers to its least terms, divide both numbers by their greatest common measure.

For example, $\dfrac{1808 \div 4}{1420 \div 4} = \dfrac{452}{355}$.

IV. *To express the ratio of two numbers in the form of a continued fraction.* Let A be the lesser of the two numbers, and B the greater; and let a, b, c, d, &c., be the quotients obtained during the process of finding the greatest common measure of A and B. Then, in the equation

$$\frac{B}{A} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \&c.,}}}$$

the right-hand side is the continued fraction required.

To save space in printing, a continued fraction is often arranged as follows:—

$$a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+}\,\&c.$$

The ratio of two incommensurable quantities is expressed by an endless continued fraction. For example, the ratio of the diagonal to the side of a square is expressed by $1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}$ &c., without end.

V. *To form a series of approximations to a given ratio.* Express the ratio in the form of a continued fraction. Then write the quotients in their order; and in a line below them write $\frac{0}{1}$ to the left of the first quotient, and $\frac{1}{0}$ directly under the first quotient. Then calculate a series of fractions by the following rule:—Multiply the first quotient by the numerator of the fraction that is below it, and add the numerator of the fraction next to the left; the sum will be the numerator of a new fraction: multiply the first quotient by the denominator of the fraction that is below it, and add the denominator of the fraction that is next to the left; the sum will be the denominator of the new fraction; then write that new fraction under the second quotient, and treat the second quotient, the fraction below it, and the fraction next to the left, as before, to find a fraction which is to be written under the third quotient, and so on. For example:

$$\text{Quotients,} \ldots\ldots a,\ \ b,\ \ c,\ \ d,\ \&c.$$

$$\text{Fractions,}\ \frac{0}{1},\ \frac{1}{0},\ \frac{n}{m},\ \frac{n'}{m'},\ \frac{n''}{m''};$$

$$\frac{n}{m} = \frac{0 + a}{1 + 0} = \frac{a}{1};\ \frac{n'}{m'} = \frac{1 + b\,n}{0 + b\,m};\ \frac{n''}{m''} = \frac{n + c\,n'}{m + c\,m'};\ \&c.$$

To take a particular case; let the given ratio be as before, $\frac{452}{355}$. then we have the following series:—

Quotients,........e...........		1	3	1	1	1	15	2	
Fractions,................	$\frac{0}{1}$	$\frac{1}{0}$	$\frac{1}{1}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{9}{7}$	$\frac{14}{11}$	$\frac{219}{172}$	$\frac{452}{355}$
Less or greater than given ratio,.........	L	G	L	G	L	G	L	G	

The fractions in a series formed in the manner just described are called *converging fractions*, and they have the following properties:— *First*, each of them is in its least terms; *secondly*, the difference between any pair of consecutive converging fractions is equal to unity divided by the product of their denominators; for example, $\frac{9}{7} - \frac{5}{4} = \frac{36 - 35}{7 \times 4} = \frac{1}{28}$; $\frac{9}{7} - \frac{14}{11} = \frac{99 - 98}{7 \times 11} = \frac{1}{77}$; *thirdly*, they are alternately less and greater than the given ratio towards which they approximate, as indicated by the letters L and G in the example; and, *fourthly*, the difference between any one of them and the given ratio is less than the difference between that one and the next fraction of the series.

Fractions intermediate between the converging fractions may be found by means of the formula $\frac{h n + k n'}{h m + k m'}$; where $\frac{n}{m}$ and $\frac{n'}{m'}$ are any two of the converging fractions, and h and k are any two whole numbers, positive or negative, that are prime to each other.

118. A **Train of Wheelwork** (*A. M.*, 449,) consists of a series of axes, each having upon it two wheels, one of which is driven by a wheel on the preceding axis, while the other drives a wheel on the following axis. If the wheels are all in outside gearing, the direction of rotation of each axis is contrary to that of the adjoining axes. In some cases a single wheel upon one axis answers the purpose both of receiving motion from a wheel on the preceding axis and giving motion to a wheel on the following axis. Such a wheel is called an *idle wheel*: it affects the direction of rotation only, and not the velocity-ratio.

Let the series of axes be distinguished by numbers 1, 2, 3, &c. .... m; let the numbers of teeth in the *driving wheels* be denoted by N's, each with the number of its axis affixed; thus, N_1, N_2, &c. .... N_{m-1}; and let the numbers of teeth in the *driven* or *following* wheels be denoted by n's, each with the number of its axis affixed; thus, n_2, n_3, &c. .... n_m. Then the ratio of the angular velocity a_m of the m^{th} axis to the angular velocity a_1 of the first axis is the product of the $m - 1$ velocity-ratios of the successive elementary combinations, viz.:—

$$\frac{a_m}{a_1} = \frac{N_1 \cdot N_2 \cdot \&c. \ldots . N_{m-1}}{n_2 \cdot n_3 \cdot \&c. \ldots . n_m};$$

that is to say, the velocity-ratio of the last and first axes is the ratio of the product of the numbers of teeth in the drivers to the product of the numbers of teeth in the followers; and it is obvious, that so long as the same drivers and followers constitute the train, the *order* in which they succeed each other does not affect the resultant velocity-ratio.

Supposing all the wheels to be in outside gearing, then, as each elementary combination reverses the direction of rotation, and as the number of elementary combinations, $m - 1$, is one less than the number of axes, m, it is evident that if m is odd, the direction of rotation is preserved, and if even, reversed.

It is often a question of importance to determine the numbers of teeth in a train of wheels best suited for giving a determinate velocity-ratio to two axes. It was shown by Young, that to do this with the *least total number of teeth*, the velocity-ratio of each elementary combination should approximate as nearly as possible to 3·59. This would in some cases give too many axes; and as a convenient practical rule it may be laid down, that from 3 to 6 ought to be the range of the velocity-ratio of an elementary combination in wheelwork. *

Let $\frac{B}{C}$ be the velocity-ratio required, reduced to its least terms, and let B be greater than C.

If $\frac{B}{C}$ is not greater than 6, and C lies between the prescribed minimum number of teeth (which may be called t), and its double $2t$, then one pair of wheels will answer the purpose, and B and C will themselves be the numbers required. Should B and C be inconveniently large, they are if possible to be resolved into factors, and those factors, or, if they are too small, multiples of them, used for the numbers of teeth. Should B or C, or both, be at once inconveniently large, and prime, or should they contain inconveniently large prime factors, then, instead of the exact ratio $\frac{B}{C}$,

* The following are some examples of the results of Young's rule, the first line containing velocity-ratios, and the second, the numbers of elementary combinations of wheels suited to give velocity-ratios intermediate between the numbers in the first line:—

1		7		24		88		315		1132		4064		14596
	1		2		3		4		5		6		7	

The following are examples of the results of the modified rule, that the lowest of the velocity-ratios for each elementary combination should range from 3 to 6:—

1		6		36		216		1296		7776
	1		2		3		4		5	

some ratio approximating to that ratio, and capable of resolution into convenient factors, is to be found by the method of continued fractions (see Article 117, page 106); also Willis *On Mechanism*, pages 223 to 238).

Should $\frac{B}{C}$ be greater than 6, the best number of elementary combinations is found by dividing by 6 again and again till a quotient is obtained less than unity, when the number of divisions will be the required number of combinations, $m - 1$.

Then, if possible, B and C themselves are to be resolved each into $m - 1$ factors, which factors, or multiples of them, shall be not less than t, nor greater than $6t$; or if B and C contain inconveniently large prime factors, an approximate velocity-ratio, found by the method of continued fractions, is to be substituted for $\frac{B}{C}$, as before. When the prime factors of either B or C are fewer in number than $m - 1$, the required number of factors is to be made up by inserting 1 as often as may be necessary. In multiplying factors that are too small to serve for numbers of teeth, prime numbers differing from those already amongst the factors are to be preferred as multipliers; and in general, where two or more factors require to be multiplied, different prime numbers should be used for the different factors.

So far as the resultant velocity-ratio is concerned, the *order* of the drivers N, and of the followers n, is immaterial; but to secure equable wear of the teeth, as explained in Article 115, page 104, the wheels ought to be so arranged that for each elementary combination the greatest common divisor of N and n shall be either 1, or as small as possible; and if the preceding rules have been observed in the choice of multipliers, this will be ensured by so placing each driving wheel that it shall work with a following wheel whose number of teeth does not contain any of the same multipliers; for the original numbers B and C contain no common factor except 1.

The following is an example of a case requiring the use of additional multipliers:—Let the required velocity-ratio, in its least terms, be

$$\frac{B}{C} = \frac{360}{7}.$$

To get a quotient less than 1, this ratio must be divided by 6 three times, therefore $m - 1 = 3$. The prime factors of 360 are $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$; these may be combined so as to make three factors in various different ways; and the preference is to be given to that which makes these factors least unequal, viz., $5 \cdot 8 \cdot 9$. Hence, resolving numerator and denominator into three factors each, we have

$$\frac{B}{C} = \frac{5 \cdot 8 \cdot 9}{1 \cdot 1 \cdot 7}.$$

It is next necessary to multiply the factors of the numerator and denominator by a set of three multipliers. Suppose that the wheels to be used are of such a class that the smallest pinion has 12 teeth, then those multipliers must be such that none of their products by the existing factors shall be less than 12; and for reasons already given, it is advisable that they should be different prime numbers. Take the prime numbers, 2, 13, 17 (2 being taken to multiply 7); then the numbers of teeth in the followers will be

$$13 \times 1 = 13;\ 17 \times 1 = 17;\ 2 \times 7 = 14.$$

In distributing the multipliers amongst the factors of the numerator, let the smallest multiplier be combined with the largest factor, and so on; then we have

$$17 \times 5 = 85;\ 13 \times 8 = 104;\ 2 \times 9 = 18.$$

Finally, in combining the drivers with the followers, those numbers are to be combined which have no common factor; the result being the following train of wheels:—

$$\frac{85}{14} \cdot \frac{18}{13} \cdot \frac{104}{17} = \frac{360}{7}.$$

119. **Diametral and Radial Pitch.**—The *diametral pitch* of a circular wheel is a length bearing the same proportion to the pitch proper, or *circular pitch*, that the diameter of a circle bears to its circumference; and the *radial pitch* is half the diametral pitch. In other words, the diametral pitch is to be found by dividing the diameter of the pitch-circle by the number of teeth in the whole circumference, and the radial pitch by dividing the radius by the same number. In symbols, let p be the pitch, properly so called, or circular pitch, as measured on the pitch circle, r the radius of the pitch circle, or *geometrical radius*, and n the number of teeth; q the diametral pitch, and $\frac{q}{2}$ the radial pitch; then

$$q = \frac{113}{355} p = \frac{2r}{n};\ 2r = nq;\ p = \frac{355}{113} q;$$

$$\frac{q}{2} = \frac{113}{710} p = \frac{r}{n};\ r = \frac{nq}{2};\ p = \frac{710}{113} \cdot \frac{q}{2}.$$

Wheels are sometimes described by stating how many teeth they have for each inch of diameter; that is to say, by stating *the reciprocal of the diametral pitch in inches* $\left(\frac{1}{q} = \frac{n}{2\,r}\right)$; and the phrases used in so describing them are such as the following:—*A three-pitch wheel* is a wheel having three teeth for each inch of diameter; so that $q = \frac{1}{3}$ inch, and $p = \frac{355}{113 \times 3}$ inch $= 1{\cdot}0472$ inch; a *ten-pitch wheel* is a wheel having ten teeth for each inch of diameter; so that $q = 0{\cdot}1$ inch, and $p = \frac{35{\cdot}5}{113}$ inch $= 0{\cdot}31416$ inch; and so on.

The following are rules for solving questions regarding radial and circular pitch by graphic construction.

I. *Given, the circular pitch of a wheel, to find the radial pitch.* Draw a straight line equal to one-sixth part of the given circular pitch, and then, by Rule IV. of Article 51, page 29, find the two ends of a circular arc approximately equal in length to that straight line, and subtending 60°. The chord of that arc will be the required radial pitch very nearly, being too long by about one-900th part only.

This may be expressed in other words, as follows (see fig. 79):—Let A B be a straight line equal to one-sixth of the given circular pitch. Draw the equilateral triangle A B C, bisect B C in D, and join A D; in A B, take A E $= \frac{1}{4}$ A B, and about E, with the radius E B $= \frac{3}{4}$ A B, draw the circular arc B F, cutting A D produced in F; A F will be the required approximate radial pitch.

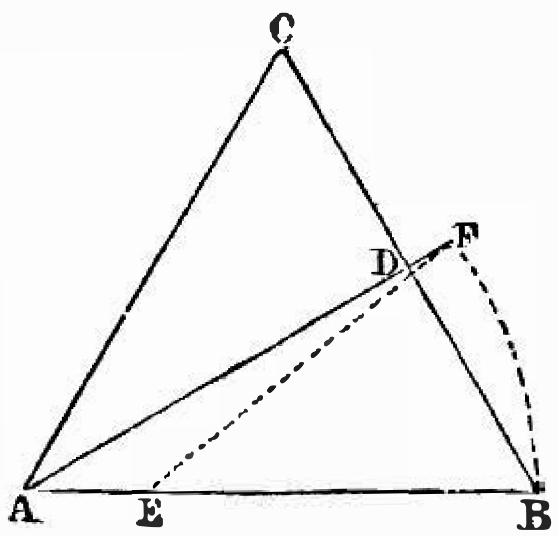

Fig. 79.

If greater accuracy is required, make the straight line equal to one-twelfth of the circular pitch, and let the angle subtended by the arc be 30°; the *radius* of that arc will be the required radial pitch, correct to one-14,400th part.

II. *Given, the radial pitch of a wheel, to find the circular pitch.* With a radius equal to six times the given radial pitch describe a circle: mark upon the circumference of that circle a chord equal to the radius, so as to lay off an arc equal to one-sixth part of the circumference; then, by Rule I. or II. of Article 51, page 28, draw a straight line approximately equal in length to that arc; the length of that straight line will be the required circular pitch, very nearly.

If Rule I. is used, the straight line will be too short by about one-900th part; if Rule II. is used, it will be too long by about one-3,600th part. If a closer approximation is required, measure the circular pitch by both rules; then to the length, as measured by Rule I., add *four times* the length as measured by Rule II., and divide the sum by *five;* the quotient will be the required circular pitch, correct to about one-40,000th part.

120. **Relative Positions of Parallel Axes in Wheelwork.**—I. Given, the radial pitch and the numbers of teeth of a pair of wheels with parallel axes, to find the length of the line of centres, or distance between the axes. Multiply the radial pitch by the sum or by the difference of the numbers of teeth, according as the wheels are in outside or inside gearing.

II. Given, the length of the line of centres, and the numbers of teeth, to find the radial pitch. Divide the given length by the sum or the difference of the numbers, according as the wheels are in outside or inside gearing.

III. Given, in fig 80, the perpendicular distance A A‴ between the first and last axes of a train of wheels, which are to turn about parallel axes all in one plane, and the numbers of teeth of the wheels; required, the positions of the several pitch-points and intermediate axes. From one end, A, of the straight line A A‴, draw, in any convenient different direction, another straight line A a''', on which lay off, on any convenient scale, a series of lengths proportional to the numbers of teeth, viz.: A i for the first driver, $i\ a'$ for the first follower; $a'\ i'$ for the second driver, $i'\ a''$ for the second follower; and so on. Let a''' be the end of that series of lengths. Draw the straight line a''' A‴, and parallel to that line draw a series of straight lines, i I, a' A′, &c., through the points of division of A a''', cutting A A‴ in a corresponding series of points of division. Then A′, A″, &c., will represent the intermediate axes, and I, I′, &c., the pitch-points.

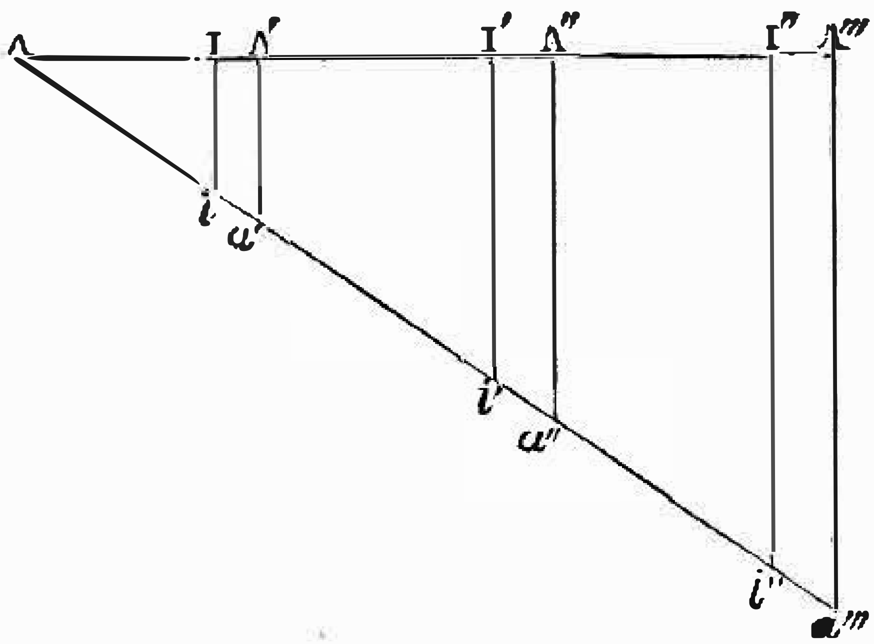

Fig. 80.

121. **Laying-off Pitch, and Subdivision of Pitch-Lines.**—The laying-off of the pitch, or of any multiple of the pitch, on the pitch-line of a wheel, is to be performed by means of Rule III. of Article 51, page 29. The laying-off of the same length upon several different pitch-lines, so as to find *corresponding pitch-points* upon them, may be performed at one operation, as follows:—Let the straight line A G represent the given length. In A G take A C = $\frac{1}{4}$ A G; and about C, with the radius C G = $\frac{3}{4}$ A G, draw a circular arc,

D G D.''''' Let A D, A D′, &c., be arcs of different pitch-lines, touching A G in A, and cut off by the dotted circular arc; each of these arcs will be approximately equal in length to A G.

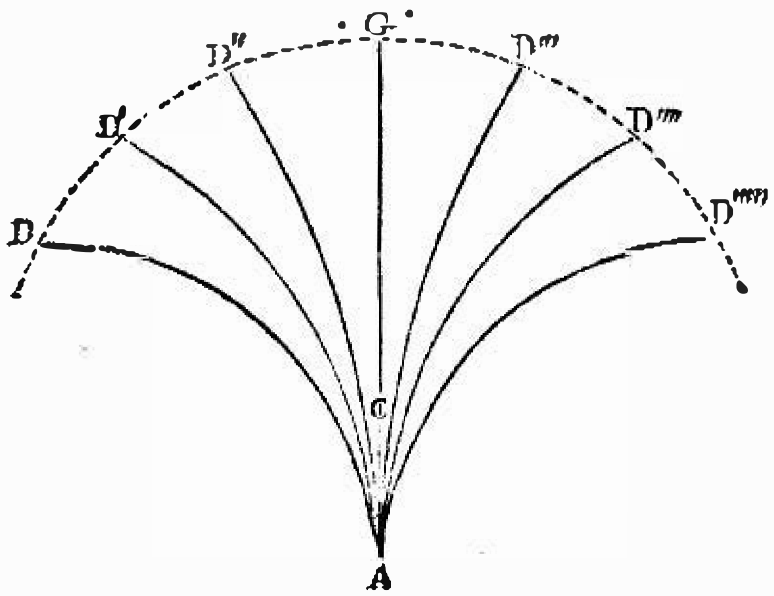

Fig. 81.

As to the operation of "*pitching*"—that is, division of a pitch-circle or other pitch-line, or of any part of a pitch line, into any required number of parts, each equal to the pitch—see Article 51, Rules V. and VI., pages 29 and 30.

Circular and straight pitch-lines may be subdivided by means of "Dividing Engines." In a dividing engine the piece upon which divisions are to be marked is fixed upon a suitable support, capable of turning about an axis or of sliding in a straight line, as the case may be, and moved by means of a screw. By turning the screw a motion of any required extent can be given to the piece, and repeated as often as may be necessary; and after each such movement, a mark is made on the surface to be divided by means of a sharp point or edge, having a movement transverse to that of the piece to be divided.*

Machines are used by mechanical engineers, with movements on the principle of dividing engines, which serve both to *pitch* wheels or divide their pitch-circles, and to cut their teeth to the proper shape. Such machines will be again mentioned further on.

SECTION IV.—*Sliding Contact—Teeth, Screw-Gearing, and Cams.*

122. **General Principle of Sliding Contact.**—The *line of connection*, in the case of sliding contact of two moving pieces, is the common normal to their surfaces at the point where they touch; and the principle of their comparative motion is, that *the components, along that normal, of the velocities of any two points traversed by it, are equal.* This being borne in mind, all questions of the comparative motion of a pair of primary pieces in sliding contact may be solved by means of the Rules of Article 91, pages 78 to 80.

The acting surfaces of a pair of pieces in sliding contact may be both plane or both convex, or one convex and one plane; but one

* For descriptions of dividing engines for purposes of great precision, see Ramsden's *Description of an Engine for Dividing Mathematical Instruments*, 1777; Ramsden's *Description of an Engine for Dividing Straight Lines*, 1779; Holtzapffel *On Turning and Mechanical Manipulation*, vol. ii., pages 639 to 654.

of them only can be concave; and in that case the other must be convex, and of a curvature not flatter than that of the concave surface.

123. **Teeth of Wheels and Racks. General Principle.** (*A. M.*, 451.)—The figures of the teeth of wheels and racks are regulated by the principle, *that the teeth shall give the same velocity-ratio by their sliding contact which the ideal toothless pitch-surfaces would give by their rolling contact.*

Let B_1, B_2, in fig. 82, be parts of the pitch-lines of a pair of wheels, I the pitch-point, and C_1, C_2 the traces of the axes. According to Article 91, pages 78 to 80, the comparative velocity of two connected pieces depends on the position of the point where the line of connection cuts the line of centres. For a pair of smooth pitch-surfaces, that point is the pitch-point I; and for a pair of surfaces in sliding contact, it is the point where the line of connection of these surfaces (being, as stated in the preceding Article, their common normal at the point where they touch) cuts the plane of the axes. Hence the condition of the correct working of the teeth of wheels and racks is the following:—

The line of connection of the teeth should always traverse the pitch-point.

For example, in fig. 82, A_1 T_1 and A_2 T_2 may represent the traces of parts of the acting surfaces of a pair of teeth belonging to the driver and follower respectively, T_1 and T_2 a pair of particles in these surfaces, which at a given instant touch each other in one point, and P_1 T_1 T_2^e P_2 the common normal at that point; then that normal ought always to traverse the pitch-point I.

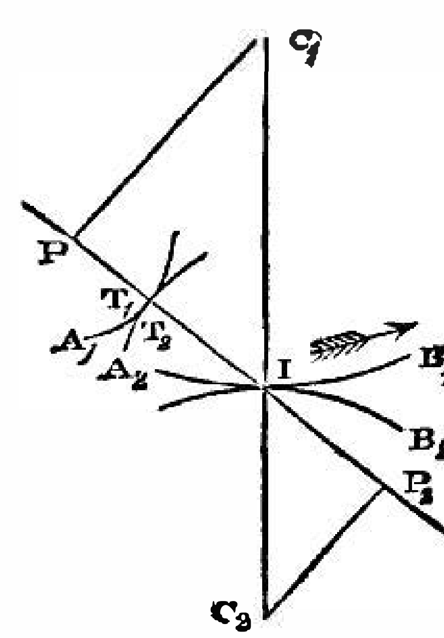

Fig. 82.

At the instant of passing the line of centres the point of contact of a pair of teeth coincides with the pitch-point.

124. **Teeth—Definitions of their Parts.**—That part of the FRONT or acting surface of a tooth which projects beyond the pitch-surface is called the FACE; that part which lies within the pitch-surface, the FLANK. The flanks of the teeth of the driver drive the faces of the teeth of the follower, and the faces of the teeth of the driver drive the flanks of the teeth of the follower. The corresponding divisions of the BACK of a tooth may be called the BACK-FACE and BACK-FLANK. The face of a tooth in outside gearing is always convex; the flank may be convex, plane, or concave.

When the motion of a pair of wheels is reversed, the backs of the teeth become the acting surfaces.

By the PITCH-POINT OF A TOOTH is meant the point where the pitch-line of the wheel cuts the front of the tooth. At the instant of passing the line of centres, the pitch-points of a pair of teeth coincide with each other, with the point of contact, and with the pitch-point of the pitch-lines.

The DEPTH of a tooth is the distance in the direction of a radius from root to crest; the extent to which the crest of a tooth projects beyond the pitch-surface is called the ADDENDUM; and a line parallel to the pitch-line, and touching the crests of all the teeth of a wheel or rack, is called the ADDENDUM-LINE, or, in a circular wheel, the ADDENDUM-CIRCLE. The radius of the addendum-circle of a circular wheel is called the REAL RADIUS, to distinguish it from the radius of the pitch-circle, which is called the GEOMETRICAL RADIUS.

CLEARANCE OR FREEDOM is the excess of the total depth above the working depth, or, in other words, the least distance between the crest of a tooth of one wheel and the bottom of the hollow between two teeth of another wheel, with which the first wheel gears.

The pitch of a pitch-line is divided by the fronts and backs of the teeth into THICKNESS and SPACE. The excess of the space between the teeth of one wheel above the thickness of the teeth of another wheel with which the first wheel gears is called PLAY or BACK-LASH; because it is the distance through which the pitch-line of the driver moves after having its motion reversed before the backs of the teeth begin to act.

125. **Customary Dimensions of Teeth.**—The following are *customary dimensions* for teeth, taken from a table which Mr. Fairbairn gives in his treatise *On Millwork* (see fig. 83).

It is to be understood that these customary dimensions may be departed from when there is any sufficient reason for doing so. Examples of this will appear in the sequel.

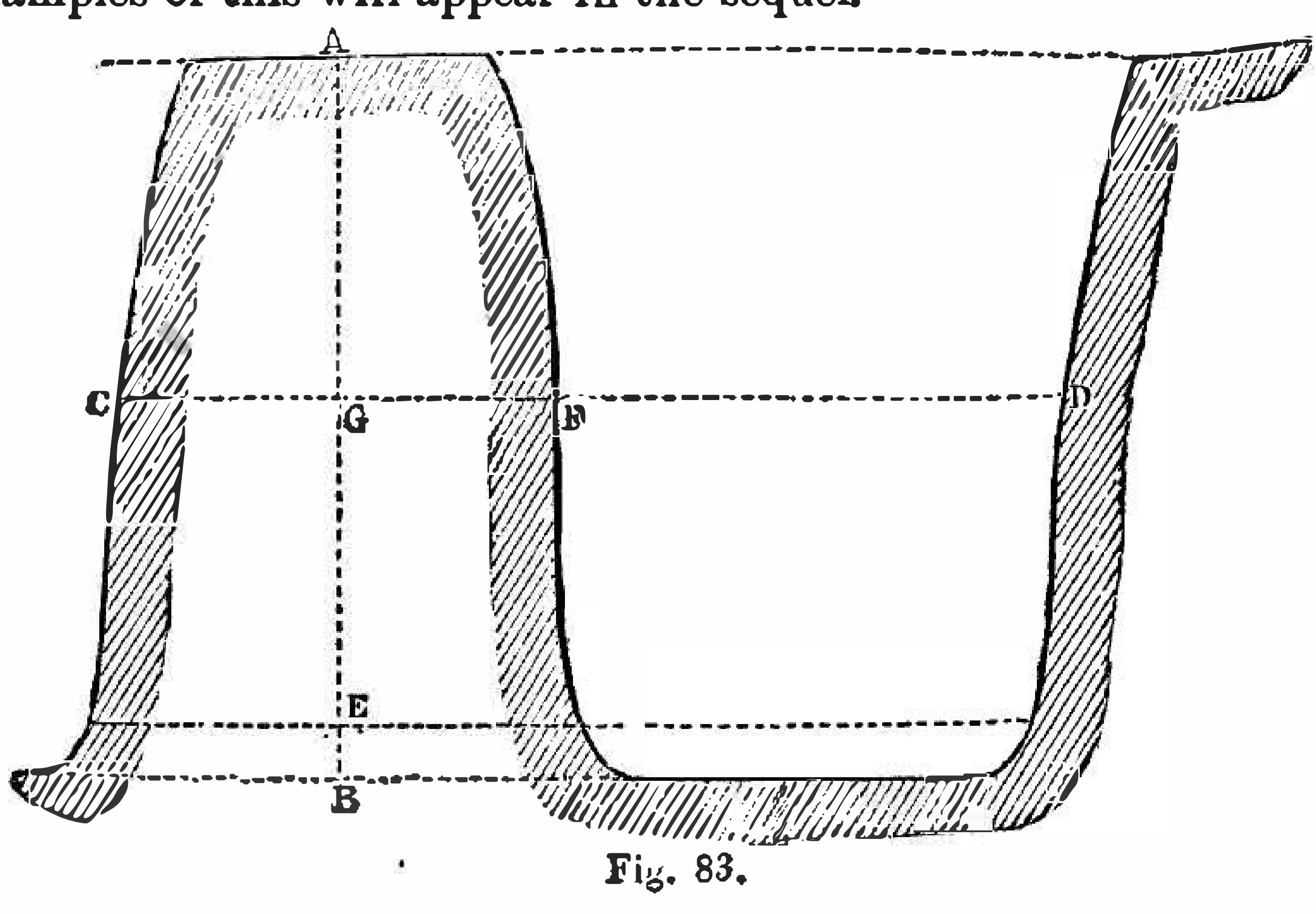

Fig. 83.

Let the pitch C D $= p$; then

Depth, total, A B $= 0{\cdot}75\,p$;

$$\left.\begin{array}{l}\text{Clearance or freedom, E B,} \ldots\ldots\ldots\ldots \\ \text{Also, play or back-lash, F D} - \text{C F}\end{array}\right\} = f = 0{\cdot}06\,p + 0{\cdot}04 \text{ inch;*}$$

Depth, working, A E $= 0{\cdot}75\,p - f$;

Addendum, A G $= \frac{1}{2}$ A E;

Thickness, C F $= \frac{p - f}{2}$;

Space, F D $= \frac{p + f}{2}$.

Thickness of *ring* which carries the teeth (in a cast-iron wheel) = thickness of tooth at root.

The *least thickness sufficient* for the teeth of a given pair of wheels is a question of strength, depending on the force to be exerted; and although such questions properly belong to a later division of this treatise, it may be convenient to state here the rule generally relied on:—*Divide the greatest pressure to be exerted between a pair of teeth in pounds by* 1,500; *the square root of the quotient will be the least proper thickness in inches.*

For pressures expressed in kilogrammes, and thicknesses in millimètres, the divisor becomes 1·055; the rule being in other respects the same.

The *least breadth sufficient* for the fronts of teeth is a quantity depending on dynamical principles, and belonging properly to the next division; but for convenience it may here be stated that an ordinary rule is as follows:—*Divide the greatest pressure to be exerted in pounds by the pitch in inches, and by* 160; *the quotient will be the breadth in inches.*

For pressures in kilogrammes and dimensions in millimètres, instead of dividing by 160, multiply by 9.

126. **Teeth for Inside Gearing.**—The figures of the acting surfaces of teeth for a pitch-circle in inside gearing are exactly the same with those suited for the same pitch-circle in outside gearing; but the relative positions of teeth and spaces, and those of faces and flanks, are reversed; and the addendum-circle is of less radius than the pitch-circle. All the rules in the ensuing Articles, with these modifications, may be applied to inside gearing.

127. **Common Velocity and Relative Velocity of Teeth—Approach and Recess—Path of Contact.**—The *common velocity* of a pair of

* 0·04 inch = 1 millimètre, nearly.

teeth is that component velocity along the line of connection which is common to the pair of particles that touch each other at a given instant. In fig. 82, page 115, let $C_1 P_1$ and $C_2 P_2$ be the two common perpendiculars of the line of connection and the two axes respectively, and let a_1 and a_2 denote the angular velocities about those axes; then the common component in question has the value

$$a_1 a \cdot C_1 P_1 = a_2 \cdot C_2 P_2 \dots\dots\dots\dots\dots\dots\dots (1.)$$

The *relative velocity* of a pair of teeth is the velocity with which their acting surfaces slide over each other; and it is found as follows:—Conceive one of the pitch-surfaces to be fixed, and the other to roll upon it, so that the line of contact (I, fig. 82, page 115) becomes an instantaneous axis; find the resultant angular velocity (see Articles 73 to 77, pages 52 to 56, and Articles 81 and 82, pages 66 to 68), and multiply it by the perpendicular distance of the point of contact of the teeth (T, fig. 82) from the instantaneous axis; the product will be the relative velocity required. That is to say, let c denote the resultant angular velocity about the instantaneous axis of the pitch-surface which is supposed to roll; and in fig. 82 let I T be the perpendicular distance of the point of contact from the instantaneous axis; then the relative velocity of sliding is

$$c \cdot \mathrm{I\,T} \dots \mathrm{a} \dots\dots\dots\dots\dots\dots\dots\dots (2.)$$

The values of the resultant angular velocity c (as has been shown in the previous Articles, already referred to) are, for parallel axes in outside gearing, $c = a_1 + a_2$; for parallel axes in inside gearing, $c = a_1 - a_2$; and for intersecting axes, the diagonal of a parallelogram, of which a_1 and a_2 are the sides.

While the point of contact, T, is advancing towards the pitch-point I, the roots of the teeth are sliding towards each other; and this relative motion is called the APPROACH.

The relative velocity gradually diminishes as the approach goes on, and vanishes at the instant when I T = 0; that is, when the point of contact coincides with the pitch-point; so that at that precise instant the pair of teeth are in rolling contact.

After the point of contact has passed the pitch-point, the roots of the teeth are sliding away from each other with a gradually increasing relative velocity, and this relative motion is called the RECESS.

During the approach the flank of the driver drives the face of the follower; during the recess the face of the driver drives the flank of the follower.

The *extent of the sliding motion* of a pair of teeth is equal, during the approach, to the excess of the length of the face of the driven tooth above the length of the flank of the driving tooth; and during the recess, to the excess of the length of the face of the driving tooth above the length of the flank of the driven tooth.

The PATH OF CONTACT is the line traversing the various positions of the point of contact, T (fig. 82, page 115). If the line of connection preserves always the same position, the path of contact coincides with it, and is straight; in other cases the path of contact is curved.

It is divided by the pitch-point I into two parts: the *path of approach*, described by T in approaching the pitch-point; and the *path of recess*, described by T after having passed the pitch-point.

The path of contact is bounded where the approach commences by the addendum-line of the follower; and where the recess terminates, by the addendum-line of the driver. The length of the path of contact must be such that there shall always be at least one pair of teeth in contact; and it is better still, when practicable, to make it so long that there shall always be at least two pairs of teeth in contact; but this is not always possible.

128. **Arc of Contact.** (*A. M.*, 454.)—The arc of contact on a pitch-line is that part of the pitch-line which passes the pitch-point during the action of one given tooth with the corresponding tooth of the other wheel.

In order that one pair of teeth at least may be in action at each instant, the length of the arc of contact must be *greater than the pitch*; and when practicable, it should be *double the pitch*, in order that two pairs of teeth, at least, may be in action at each instant; but this is not always practicable; and the most common values are from 1·4 to 1·8 times the pitch. It is divided by the front of the tooth to which it belongs into two parts: the *arc of approach*, lying in advance of the front of the tooth; and the *arc of recess*, lying behind the front of the tooth. It is usual to make the arcs of approach and of recess of equal length; and in that case each of them must be *greater than half the pitch*, and should, if practicable, be made *equal to the pitch*. For a given pitch-line, and a given pitch and figure of tooth, the length of those arcs depends on the addendum, in a manner to be afterwards described.

129. **Obliquity of Action.**—The obliquity of action of a pair of teeth is the angle which the line of connection makes at any instant with a tangent plane to the two pitch-surfaces; for example, in fig. 82, page 115, the *complement* of the angle at I. When the path of contact is a straight line, coinciding at every instant with the line of connection, the obliquity is constant; in other cases it is variable; and its mode of variation is usually such that it diminishes during the approach, and increases again during recess. In a dynamical point of view, it is advantageous to make

the obliquity as small as possible; and, on the other hand, there is a connection between the obliquity of action and the number of teeth which makes it impracticable to use pinions of fewer than a certain number of teeth with less than a certain maximum obliquity of action. Mr. Willis, from an examination of the results of ordinary practice, concludes that the best value on the whole for the *mean obliquity* of action in toothed gearing is between 14° and 15°. Such an angle may be easily constructed by drawing a right-angled triangle whose three sides bear to each other the proportion of the numbers

$$65 : 63 : 16;$$

when the required angle will lie opposite to the shortest side of the triangle. The values of its chief trigonometrical functions are—

sine,................16 ÷ 65 = 0·2461538, nearly.
cosine,..............63 ÷ 65 = 0·9692308, nearly.
tangent,......h....16 ÷ 63 = 0·2539683, nearly.
cosecant,...........65 ÷ 16 = 4·0625.
cotangent,.........63 ÷ 16 = 3·9375.

The corresponding angle is 14° 15'; being a little less than one-25th part of a revolution.

130. The **Teeth of Spur-Wheels and Racks** have acting surfaces of the class called *cylindrical surfaces,* in the comprehensive sense of that term; and their figures are designed by drawing the traces of their surfaces on a plane perpendicular to the axes of the wheels (or, in the case of a rack, to the axis of the wheel that is to gear with the rack); which plane contains the pitch-lines and the line of connection, and may be represented by the plane of the paper in fig. 82, page 115. The path of contact, also, is situated in the same plane; and the angle of obliquity of action is at each instant equal to the angle I C P, which the common perpendicular, C P, of the line of connection and one of the axes makes with the line of centres, C_1 I C_2. Because of the comparative simplicity of the rules for drawing the figures of the teeth of spur-wheels, those rules are used, with the aid of certain devices to be afterwards described, for drawing the figures of the teeth of bevel wheels and skew-bevel wheels also.

131. **Involute Teeth for Circular Wheels.** (*A. M.*, 457.)—The simplest of all forms for the teeth of circular wheels is that in which the path of contact is a straight line always coinciding with the line of connection, which makes a constant angle with the line of centres, and is inclined at a constant angle of obliquity to the common tangent of the pitch-lines.

In fig. 84, let C_1, C_2, be the centres of two circular wheels, whose pitch-circles are marked B_1, B_2. Through the pitch-point I draw the intended *line of connection*, P_1 P_2, making, with the line of centres, the angle C I P = the complement of the intended obliquity.

From C_1 and C_2 draw C_1 P_1 and C_2 P_2 perpendicular to P_1 P_2, with which two perpendiculars as radii describe circles (called *base-circles*) marked D_1, D_2.

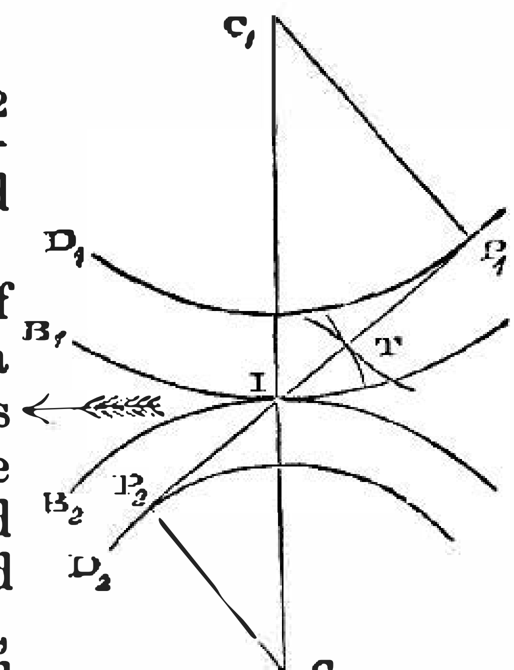

Fig. 84.

Suppose the base-circles to be a pair of circular pulleys, connected by means of a cord whose course from pulley to pulley is P_1 I P_2. As the line of connection of those pulleys is the same with that of the proposed teeth, they will rotate with the required velocity-ratio. Now, suppose a tracing point, T, to be fixed to the cord, so as to be carried along the path of contact, P_1 I P_2. That point will trace, on a plane rotating along with the wheel 1, part of the involute of the base-circle D_1, and on a plane rotating along with the wheel 2, part of the involute of the base-circle D_2, and the two curves so traced will always cut the line of connection at right angles, and touch each other in the required point of contact T, and will therefore fulfil the condition required by Article 122, page 114. The teeth thus traced are called *Involute Teeth.*

All involute teeth of the same pitch work smoothly together.

The following is the process by which the figures of involute teeth are to be drawn in practice:—

In fig. 85, let C represent the centre of the wheel, I the pitch-point, C I the geometrical radius, B I B the pitch-circle, and let the intended angle of obliquity of action be given, and also the pitch. (In the example represented by the figure, the obliquity is supposed to be $14\frac{1}{4}°$, as stated in Article 129, page 120; and the wheel has 30 teeth.) Then proceed by the following rules:—

I. *To draw the base-circle and the line of connection.* About C, with the radius C P = C I × cosine of obliquity (that is to say, in the present example, $\frac{63}{65}$ C I), draw a circle, D P D; this is the *base-circle*. Then about I, with a radius I P = C I × sine of obliquity (that is to say, in the present example, $\frac{16}{65}$ C I), draw a short circular arc, cutting the base-circle in P. Draw the straight line P F I E; this will be the line of connection; and it will touch the base-circle at P.

II. *To find the normal pitch, the addendum, and the real* *radius, and to draw the addendum-circle and the flank-circle.* At the pitch-point, I, draw the straight line I A, touching the pitch-

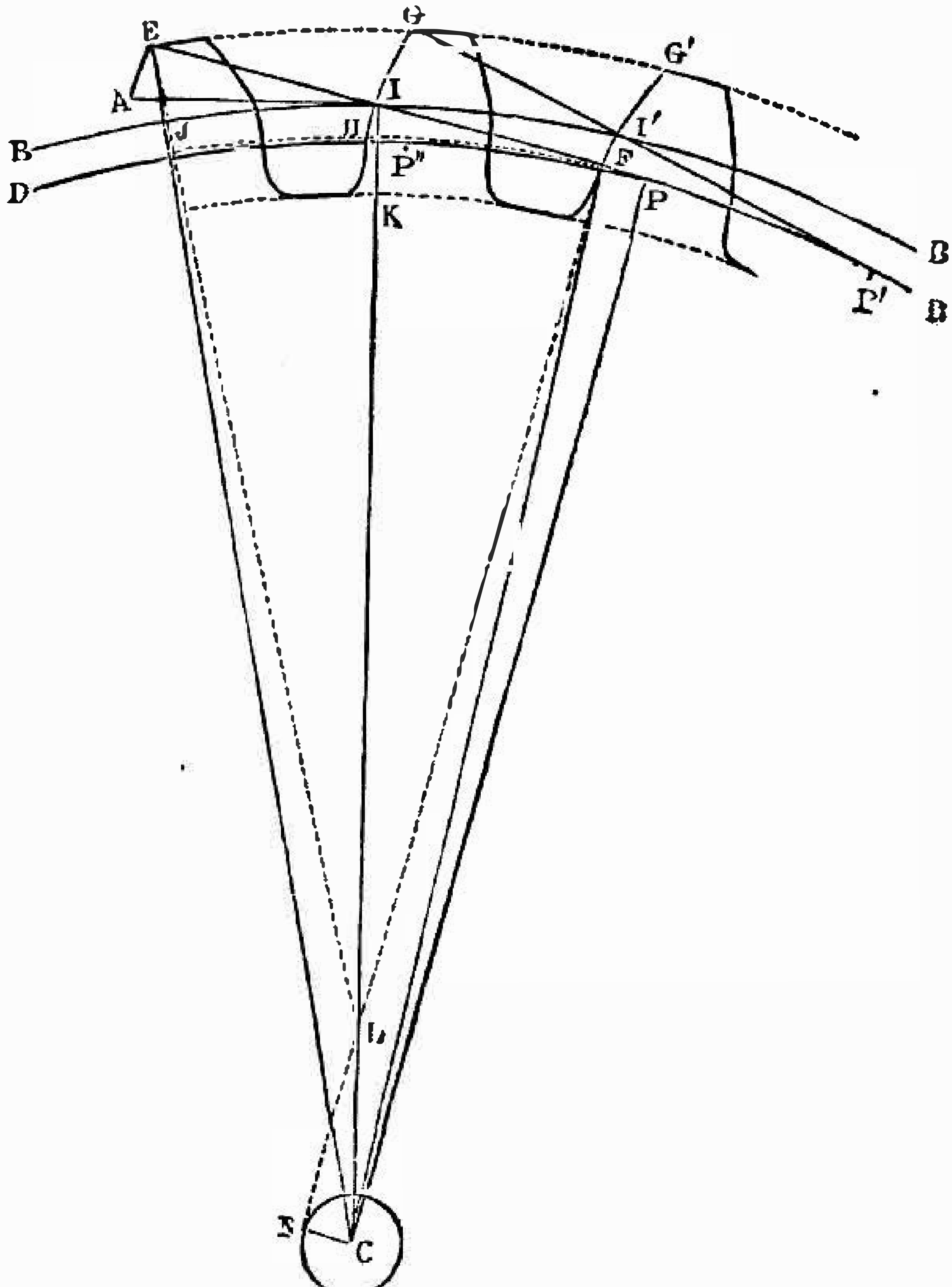

Fig. 85.

circle, and lay off upon it the length I A equal to the pitch. From A let fall A E perpendicular to I E. Then I E will be what may be called the NORMAL PITCH, being the distance, as measured along the line of connection, from the front of one tooth to the front of the next tooth.

The normal pitch is also the *pitch on the base-circle;* that is, the distance, as measured on the base-circle, between the front of one tooth and the front of the next.

The ratio of the normal pitch of involute teeth to the circular pitch is equal to the ratio of the radius of the base-circle to that of the pitch-circle; that is to say,

$$\frac{\text{I E}}{\text{I A}} = \frac{\text{C P}}{\text{C I}} = \text{cosine of obliquity} \left(= \frac{63}{65} \right.$$

in the present example).

In order that two pairs of teeth at least may always be in action, the *arc of contact* is to consist of two halves, each equal to the pitch (see Article 128, page 119). Lay off on the line of connection, E P, the distance I F = I E. Then E F will be the *path of contact* (Article 127, page 119), consisting of two halves, each equal to the normal pitch.

Draw the straight line C E; this will be the *real radius*, and the circle E G G′, drawn with that radius, will be the *addendum-circle*, which all the crests of the teeth are to touch. Then, with the radius C F, draw the circle F H (marked with dots in the figure); this may be called the FLANK-CIRCLE, for it marks the inner ends of the flanks of all the teeth.

The *addendum* is C E − C I.

III. *To draw the* ROOT CIRCLE; that is, the circle which the bottoms of all the hollows between the teeth (or CLEARING CURVES, as they are called) are to touch. First find, by drawing or by calculation, the *greatest addendum* of any wheel with which the given wheel may have to gear; that is, the addendum of the smallest practicable pinion of the same pitch and obliquity; that is, the addendum of a pinion in which the pitch subtends at the centre an angle approximately equal to the obliquity. With the obliquity already stated, such a pinion has 25 teeth. To find the addendum of such a pinion by drawing:—Through F, parallel to P C, draw F L, as cutting I C in L. Join L E; then L E − L I will be the required *greatest addendum*. To find the greatest addendum by calculation, let ϕ denote the obliquity, and p the pitch; then

$$\mathrm{L\,E} - \mathrm{L\,I} = p \operatorname{cotan} \phi \left\{ \sqrt{(3 \sin^2 \phi + 1)} - 1 \right\}.$$

With the angle of obliquity already stated, this gives

$$\mathrm{L\,E} - \mathrm{L\,I} = 0{\cdot}343\, p, \text{ very nearly;}$$

and this is the origin of the value $0{\cdot}35\, p$, which is very commonly used for the addendum of teeth.

To the greatest addendum, thus found, add a suitable allowance for clearance (Article 125, page 116), and lay off the sum I K inwards from the pitch-circle along the radius. Then C K will be the radius of the required root-circle.

IV. *To draw the traces of the teeth.* Mark the pitch-points of the fronts of the teeth (I, I′, &c.), according to the principles of Article 121, page 113, and those of their backs, by laying off a suitable thickness on the pitch-circle (see Article 125, page 116). Obtain a "*templet,*" or thin flat disc of wood or metal, having its edge accurately shaped to the figure of the base-circle. Such a templet is represented in plan by C D D, fig. 86, and in elevation by D′ D′. A piece of watch-spring, marked P M in plan, and P′ M′ in elevation, is to have its edges filed so as to leave a pair of sharp

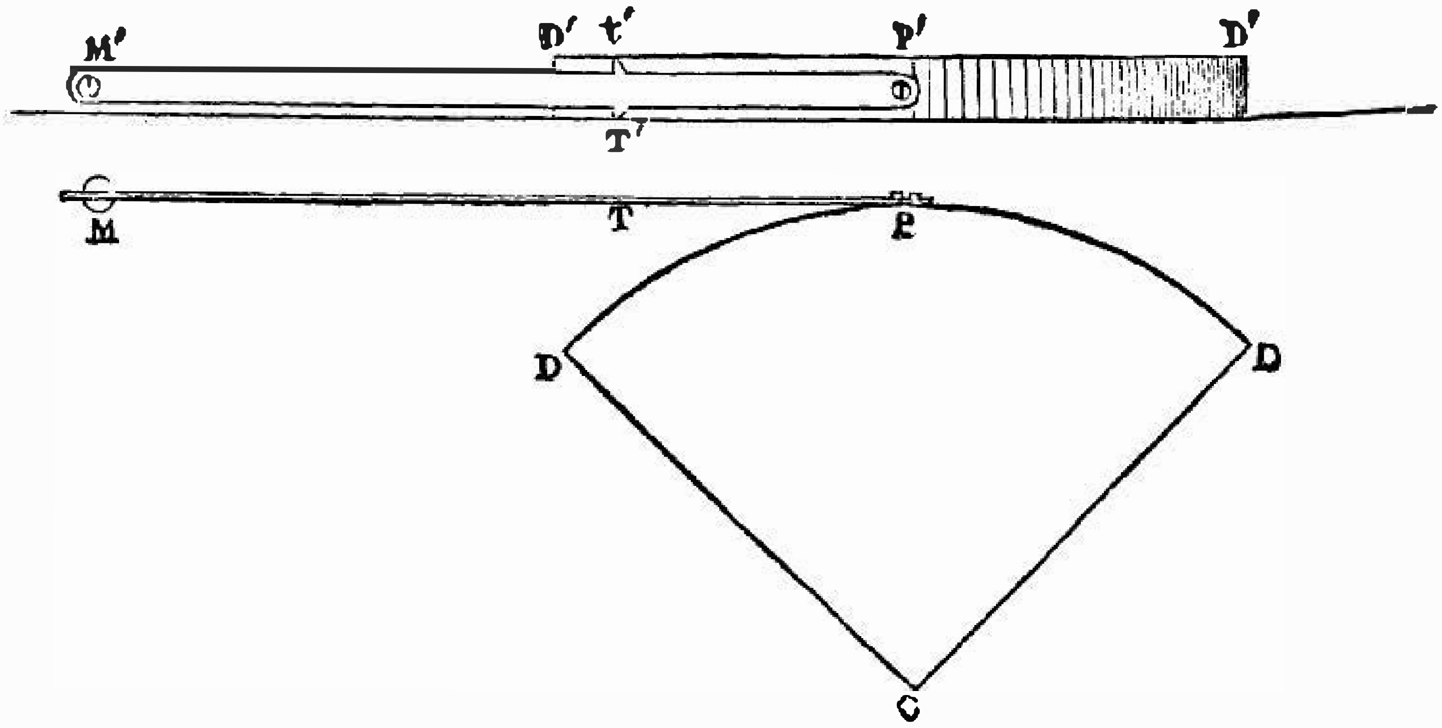

Fig. 86.

projecting tracing-points, marked T′, t′, in elevation, and T in plan. One end of that spring, P, P′, is to have a round hole drilled in it, and to be fixed to the middle of the edge of the templet by means of a screw, about which the spring is to be free to turn; and the other end, M, M′, is to be fitted with a knob to hold it by. Place the templet on the drawing (or pattern, as the case may be), so that C shall coincide with the centre of the wheel, and D D with the base-circle; and also so that the lower of the two tracing-points, when the spring is moved to and fro, shall pass through the pitch-point of a tooth; then that tracing-point will draw the trace of the front of the tooth; and by turning the templet about C, and repeating the process, the traces of the fronts of any required number of teeth may be drawn.

To draw the traces of the backs of the teeth, the position of the spring relatively to the templet is to be reversed, by turning it about the screw at P, so as to use the tracing-point that was previously uppermost.

The distance, P T, from the screw to the tracing-points should not be less than *twice the normal pitch.*

V. The *Clearing Curves* are the traces of the hollows which lie inside the flank-circle, F H, fig. 85. Their side parts ought to be tangents to the inner ends of the flanks of the teeth (at F and H, for example), and their bottom parts ought to coincide with the root-circle through K. Those different parts may be joined to each other by means of small circular arcs. In connection with the figures of the side parts of those clearing curves, it may be observed, that F L is a tangent to the inner end of the flank I′ F, and therefore to the clearing curve at that point; and that tangents to the inner ends of other flanks may be drawn by repeating the process by which F L is drawn, or by the following process:—About C, with the radius C N = P F, draw a circle; F L N will be a straight tangent to that circle; and so also will all the tangents to the flanks at their inner ends. Therefore, from the inner ends of all the flanks, both front and back, draw straight lines touching the circle C N, and so placed that the straight lines from the front and back flanks of the same tooth shall not cross each other; these lines will show the proper positions for the side parts of the clearing curves. When the flank-circle coincides with the base-circle (as in the smallest pinion of a given pitch), the side parts of the clearing curves coincide with the radii drawn from the centre C to the inner ends of the flanks.

132. **Involute Teeth for Racks.**—The following is the process of designing the teeth of a straight rack which is to gear with an involute-toothed wheel of a given pitch and a given obliquity:—In fig. 87, let A B be the pitch-line of the rack, and let A I = I I′ be the pitch. Lay off A I E = the given angle of obliquity, and from A let fall A E perpendicular to I E; then I E will be the *normal pitch*; further, if the path of contact is to consist of two halves, each equal to half the normal pitch, I E will be one of those halves; then in E I produced make I F = I E, and I F will be the other half of the path of contact. Through E, parallel to A B, draw E G G′; this will be the *addendum line;* through F, parallel to B A, draw F H; this will be the *flank-line,* marking the inner ends of the acting surfaces of the teeth. Perpendicular to A B draw I K, equal to the greatest addendum in the set of wheels of the given pitch and obliquity with an allowance for clearance added, as in Rule III. of Article 131, page 123; through K, parallel to A B, draw a straight line; this will be the *root-line,* with which the bottoms of all the hollows between the teeth are to coincide.

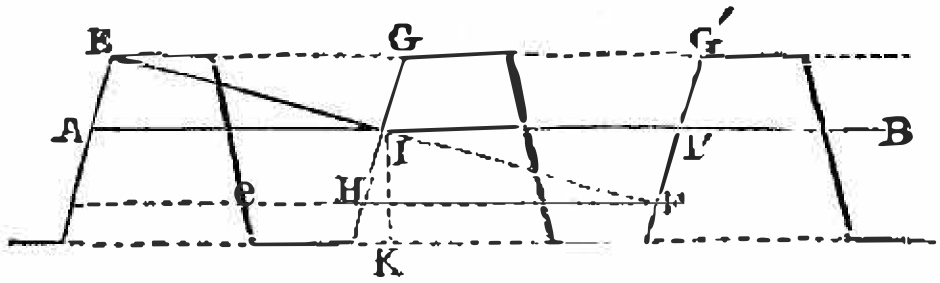

Fig. 87.

The traces of the fronts of the teeth are straight lines perpendicular to E F, and the fronts themselves are planes perpendicular to E F. The backs of the teeth are planes inclined at the same angle to A B in the contrary direction.

133. **Peculiar Properties of Involute Teeth.**—Involute teeth have some peculiar properties not possessed by teeth of other figures.

I. Sets of involute teeth have *a definite and constant normal pitch*, being, as already explained, the distance between the fronts of successive teeth, measured on the path of contact, or on the circumference of the base-circle; and *all wheels and racks with involute teeth of the same normal pitch gear correctly with each other.*

II. The length of the line of centres, or perpendicular distance between the axes, of a pair of wheels with involute teeth of the same normal pitch, or the perpendicular distance from the axis of a wheel with involute teeth to the addendum-line of a rack with which it gears, *may be altered;* and so long as the wheels, or wheel and rack, are sufficiently near together to make the path of contact longer than the normal pitch, and sufficiently far asunder for the crests of each set of teeth to clear the hollows between the teeth of the other set, the wheels, or the wheel and rack, will continue to work correctly together, and to preserve their velocity-ratio; although, in the case of a pair of wheels, the pitch-lines, the pitch as measured on the pitch-lines, and the obliquity, will all be altered when the length of the line of centres is altered. In other words, the velocity-ratio of a pair of wheels with involute teeth of the same normal pitch is the reciprocal of the ratio of the radii of their

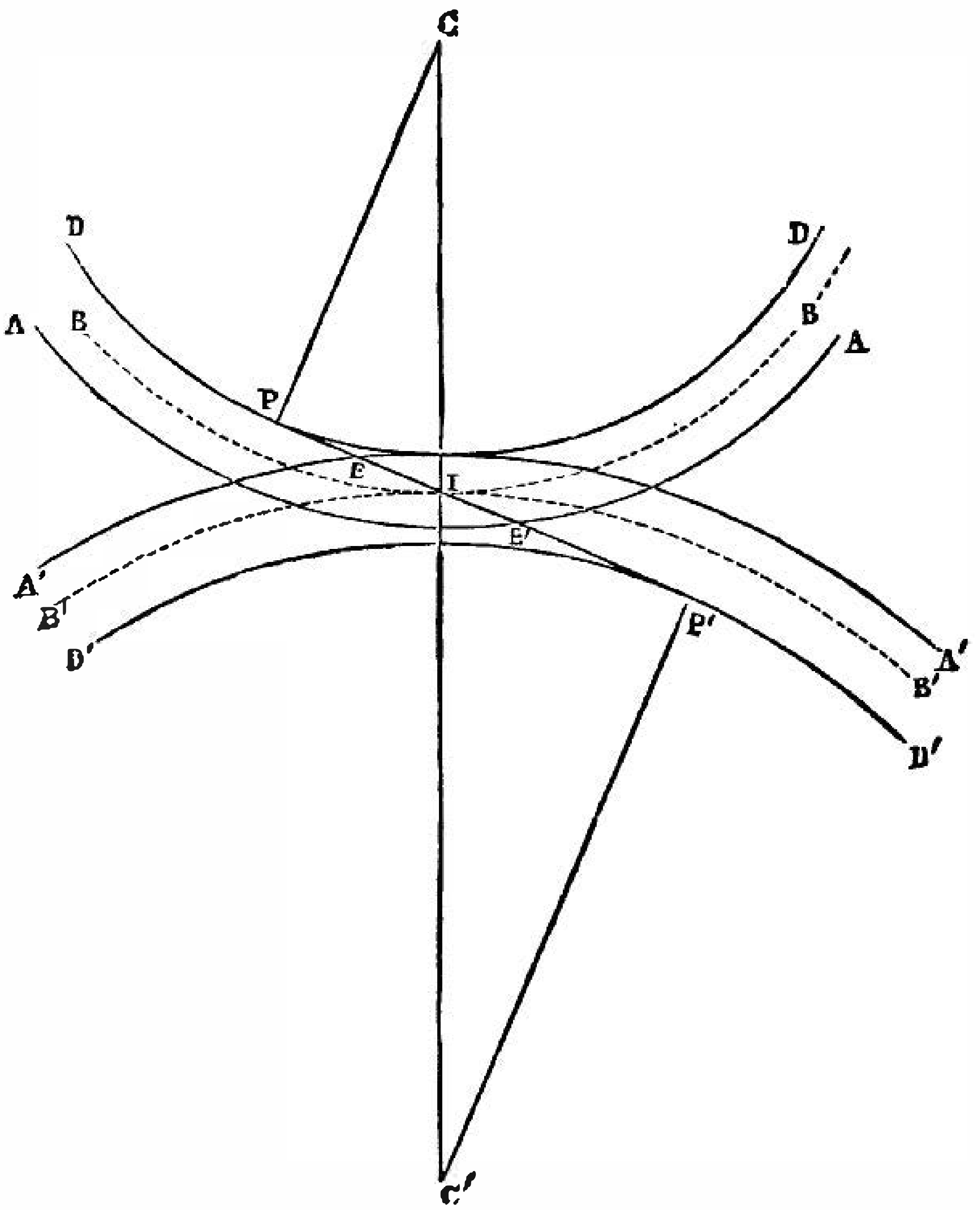

Fig. 88.

base-circles, and depends on this ratio alone; and the velocity-ratio of a wheel and rack with involute teeth of the same normal pitch depends solely on the radius of the base-circle of the wheel and on the angle of obliquity of the line of connection.

Another way of stating this property of involute teeth is, that the pitch-lines of wheels and racks with such teeth are arbitrary to an extent limited only by the necessity of having a path of contact of a certain length.

One practical result of this is (as Mr. Willis first pointed out), that the *back-lash* of involute teeth is variable at will, being capable of being increased or diminished by moving the wheels, or the wheel and rack, farther from or nearer to each other, and may thus be adjusted so as to be no greater than is absolutely necessary in order to prevent jamming of the teeth—a property not possessed by teeth of any other figure.

III. *Given* (in fig. 88), *the centres*, C, C′, *the base-circles*, D D, D′ D′, *and the addendum-circles*, A A, A′ A′, *of a pair of spur-wheels with involute teeth of a given normal pitch, to find the line of connection, the pitch-point, the pitch-circles, the pitch on the pitch-circles, and the path of contact.*

Draw a common tangent, P P′, to the two base-circles in such a position as to run from the driver to the follower in the direction of motion. That common tangent will be the line of connection: the point I, where it cuts the line of centres, will be the pitch-point: two circles, B B and B′ B′, described about C and C′ respectively, and touching each other in I, will be the pitch-circlese the pitch on the pitch-circles will be greater than the normal pitch in the ratio $\frac{\mathrm{C\,I}}{\mathrm{C\,P}} = \frac{\mathrm{C'\,I}}{\mathrm{C'\,P'}}$; and the part E E′ of the line of connection which lies between the two addendum-circles will be the path of contact.

IV. *Given* (in fig. 89), *the centre*, C, *the base-circle*, D D, *and the addendum-circle*, A A, *of a spur-wheel with involute teeth of a given*

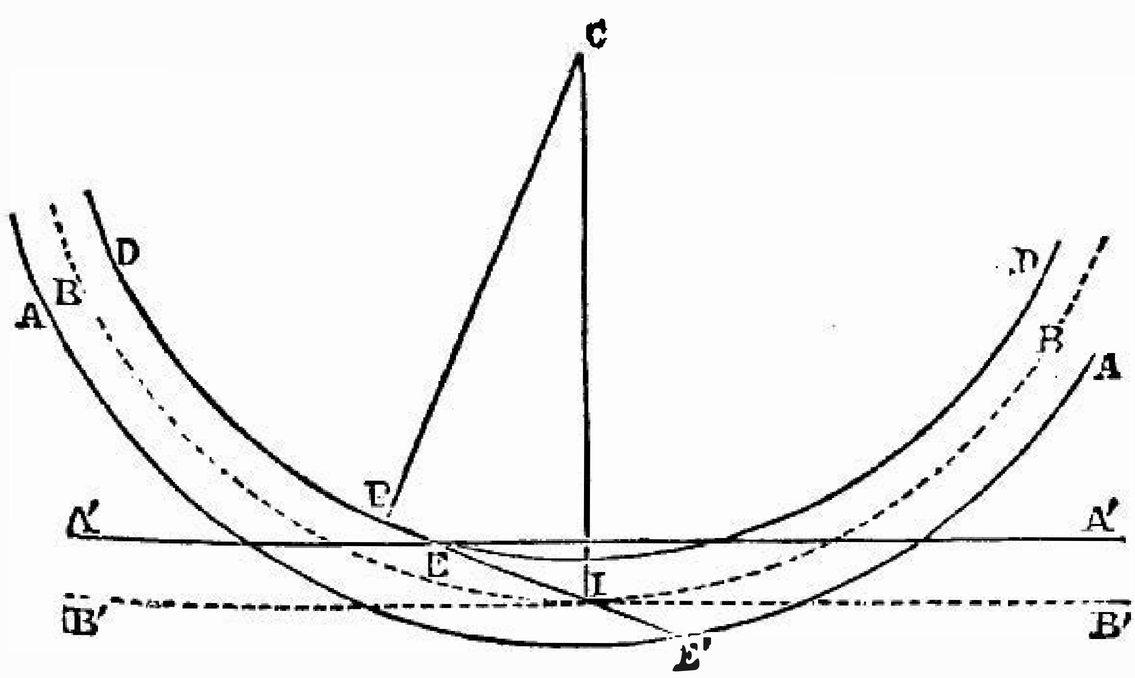

Fig. 89.

normal pitch; also the pitch-line, B′ B′, *and the addendum-line*, A′ A′, *of a rack which is to have involute teeth of the same normal pitch; to* *find the pitch-point, the pitch-circle of the wheel, the line of connection, the pitch, as measured on the pitch-lines, the path of contact, and the position of the fronts of the teeth of the rack.*

From C let fall C I perpendicular to B′ B′; then I will be the pitch-point; and a circle, B B, of the radius C I, will be the pitch-circle of the wheel. From I draw I P, touching the base-circle D D; I P will be the line of connection. The pitch, as measured on the pitch-lines, will be greater than the normal pitch, in the ratio $\frac{\mathrm{C\ I}}{\mathrm{C\ P}}$ of the radius of the pitch-circle to that of the base-circle. The path of contact will be the part E E′ of the line of connection, which is contained between the addendum-line of the rack, A′ A′, and the addendum-circle of the wheel, A A. The fronts of the teeth of the rack are to be planes perpendicular to I P, or, in other words, parallel to P C.

V. By the application of the preceding principles, two or more wheels of different numbers of teeth, turning about one axis, can be made to gear correctly with one wheel or with one rack; or two or more parallel racks, with different obliquities of action, may be made to gear correctly with one wheel, the normal pitches in each case being the same; and thus *differential movements* of various sorts may be obtained. This is not possible with teeth of any other form.

The obliquity of the action of involute teeth is by many considered an objection to their use; and that is the reason why, notwithstanding their simplicity and their other advantages, they are not so often used as other forms. In anticipation of the subject of the dynamics of machinery it may be stated, that the principal effect of the obliquity of the action of involute teeth is to increase the pressure exerted between the acting surfaces of the teeth, and also the pressure exerted between the axles of the wheels and their bearings, nearly in the ratio in which the radius of the pitch-circle of each wheel is greater than the radius of the base-circle, and that a corresponding increase of friction is produced by that increase of pressure. In the example of Article 131, that ratio is 65 : 63.

134. **Teeth for a Given Path of Contact.**—In the three preceding Articles the forms of the teeth are found by assuming a figure for the path of contact—viz., the straight line. Any other convenient figure may be assumed for the path of contact, and the corresponding forms of the teeth found, by determining what curves a point moving along the assumed path of contact will trace on two discs, rotating round the centres of the wheels with angular velocities, which bear that relation to the component velocity of the tracing-point along the line of connection which is given by the principles of Article 127, page 118. This method of finding the forms of the teeth of wheels is the subject of an interesting treatise by Mr. Edward Sang.

All wheels having teeth of the same pitch, traced from the same path of contact, work correctly together, and are said to belong to the *same set.*

135. **Teeth Traced by Rolling Curves.** (*A. M.*, 452.)—From the Principles of Articles 122 and 123, pages 114, 115, it appears that at every instant the position of the point of contact, T, of the acting surfaces of a pair of teeth (fig. 82, page 115), and the corresponding position of the pitch-point I in the pitch-lines of the wheels to which those teeth belong, are so related, that the line, I T, which joins them, is normal to the surface of each of the teeth at the point T. Now this is the relation which exists between the *tracing-point* T, and the *instantaneous axis or line of contact* I, in a rolling curve of such a figure, that, being rolled upon the pitch-line, its tracing-point T traces the outline of a tooth. (As to rolling curves and rolled curves, see Articles 72, 74, 75, 77, 78, 79, pages 51 to 62.)

In order that a pair of teeth may work correctly together, it is necessary and sufficient that the *instantaneous normals* from the pitch-point to the acting surfaces of the two teeth should coincide at each instant; and this condition is fulfilled *if the outlines of the two teeth be traced by the motion of the same tracing-point, in rolling the same rolling curve on the same side of the pitch-lines of the respective wheels.*

The *flank* of a tooth is traced while the rolling curve rolls *inside* of the pitch-line; the *face*, while it rolls *outside.*

To illustrate this more fully, the following explanation is quoted from the Article "*Mechanics (Applied),*" in the *Encyclopædia Britannica* (see fig. 90):—"If any curve, R, be rolled on the inside of the pitch-line, B B, of a wheel, the instantaneous axis of the rolling curve at any instant will be at the point I, where it touches the pitch-line for the moment; and consequently the line A T, traced by a tracing-point T, fixed to the rolling curve, will be everywhere perpendicular to the straight line T I; so that the traced curve A T will be suitable for the flank of a tooth, in which T is the point of contact corresponding to the position I of the pitch-point. If the same rolling curve R, with the same tracing-point T, be rolled on the *outside* of any other pitch-line, it will trace the *face* of a tooth suitable to work with the *flank* A T.

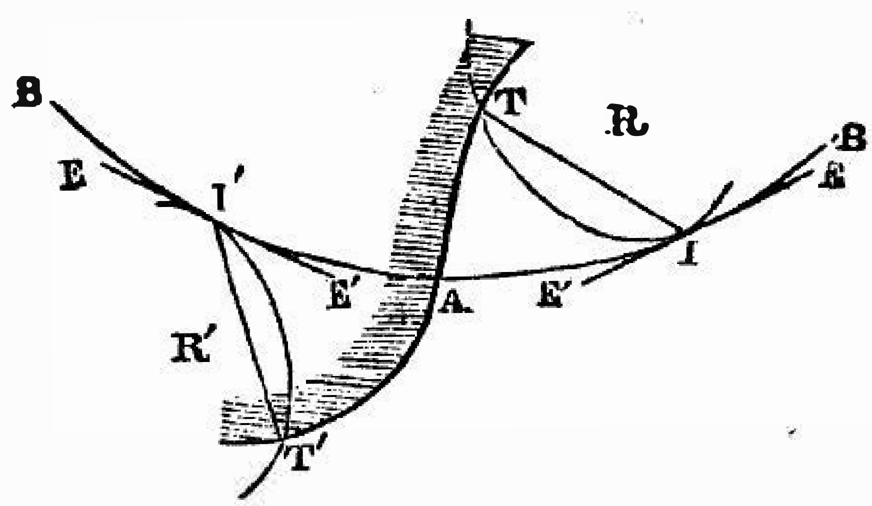

Fig. 90.

"In like manner, if either the same or any other rolling curve

R′ be rolled the opposite way, on the *outside* of the pitch-line B B, so that the tracing-point T′ shall start from A, it will trace the *face* A T′ of a tooth suitable to work with a *flank* traced by rolling the same curve R′ with the same tracing-point T′ *inside* any other pitch-line.

"The figure of the *path of contact* is that traced on a fixed plane by the tracing-point, when the rolling curve is rotated in such a manner as always to touch a fixed straight line E I E (or E′ I′ E′, as the case may be) at a fixed point I (or I′).

"If the same rolling curve and tracing-point be used to trace both the faces and the flanks of the teeth of a number of wheels of different sizes, but of the same pitch, all those wheels will work correctly together, and will form a *set*. The teeth of a *rack* of the same set are traced by rolling the rolling curve on both sides of a straight line.

"The teeth of wheels of any figure, as well as of circular wheels, may be traced by rolling curves on their pitch-lines; and all teeth of the same pitch, traced by the same rolling curve with the same tracing-point, will work together correctly if the pitch-surfaces are in rolling contact."

Involute teeth themselves might be traced by rolling a logarithmic spiral on the pitch-circle; but it is unnecessary to explain this in detail, as the ordinary method of tracing them is much more simple.

136. **Epicycloidal Teeth in General.**—For tracing the figures of teeth, the most convenient rolling curve is the circle. The path of contact which a point in its circumference traces is identical with the circle itself; the flanks of the teeth for circular wheels are internal epicycloids, and their faces external epicycloids, and both flanks and faces are cycloids for a straight rack. (See Article 74, page 53, and Article 77, page 56.)

Wheels of the same pitch, with epicycloidal teeth traced by the same rolling circle, all work correctly with each other, whatsoever may be the numbers of their teeth; and they are said to belong to *the same set*.

For a pitch-circle of twice the radius of the rolling or *describing* circle (as it is called), the internal epicycloid is a straight line, being a diameter of the pitch-circle; so that the flanks of the teeth for such a pitch-circle are planes radiating from the axis. For a smaller pitch-circle, the flanks would be convex, and *incurved* or *under-cut*, which would be inconvenient; therefore the smallest wheel of a set should have its pitch-circle of twice the radius of the describing circle, so that the flanks may be either straight or concave.

In fig. 91, let B B be the pitch-circle of a wheel, C C the line of centres, I the pitch-point, R the internal describing circle, and R′

the external describing circle, so placed as to touch the pitch-circle and each other at I; let E E be a straight tangent to the pitch-circle at the pitch-point; and let T I T′ be the path of contact, consisting of the path of approach, T I, and the path of recess, I T′. Each of those arcs should be equal to the pitch when practicable, in order that there may be always at least two pairs of teeth in action; but this is not always possible; and the length of each of them in many cases is only from 0·7 to 0·9 of the pitch, being regulated by the customary practice of making the addendum from 0·3 to 0·35 of the pitch.

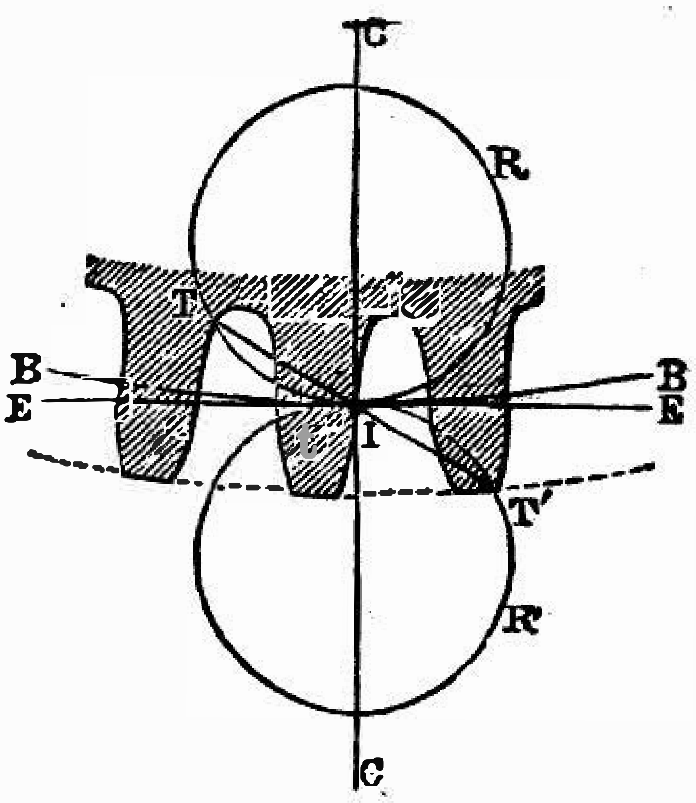

Fig. 91.

The *real radius* of the wheel is the distance from its centre to the point T′; at the outer end of the face of a tooth; the dotted circle traversing T′ is the *addendum-circle*, and the perpendicular distance from T′ to the pitch-circle B B is the *addendum*.

The *flank-circle* is a circle described about the centre of the wheel, and traversing the point T; and the *clearing curves* (as in the case of involute teeth, Article 131, Rule V., page 124) must have a depth sufficient to clear the greatest addendum given to the teeth of any one of the set of wheels that are capable of gearing with the wheel under consideration.

In passing the line of centres, the line of connection coincides with the tangent E E, and the *obliquity* is nothing. The greatest angle of obliquity of action is, during the approach, E I T, and during the recess E I T′; and the mean angles of obliquity during the approach and recess are the halves of those greatest angles respectively. From the results of practical experience, Mr. Willis deduces the rule that the mean obliquity should not exceed 15°, or one-twenty-fourth of a revolution; therefore the maximum obliquity should not exceed 30°, or one-twelfth of a revolution; therefore the arcs I T and I T′ should neither of them in any case exceed one-sixth of the circumference of the describing circles to which they respectively belong; from which it follows, that if either of those arcs is to be equal to the pitch, the circumference of the describing circle ought *not to be less than six times the pitch;* therefore the smallest pinion of a set should have *twelve teeth.*

137. **Tracing Epicycloidal Teeth by Templets.**—The face of an epicycloidal tooth may be traced by rolling a templet of the form of the describing circle upon a convex templet of the form of the pitch-circle; and the flank, by rolling a templet of the form of the

describing circle upon a concave templet of the form of the pitch-circle.

When the fixed templet is either convex (as when the face of the tooth of a wheel is to be traced) or straight (as when either the face or the flank of the tooth of a rack is to be traced), the rolling templet may be prevented from slipping on the fixed templet by connecting them together by means of a slender piece of watch-spring, as follows:—In fig. 92, C B B represents the fixed templet

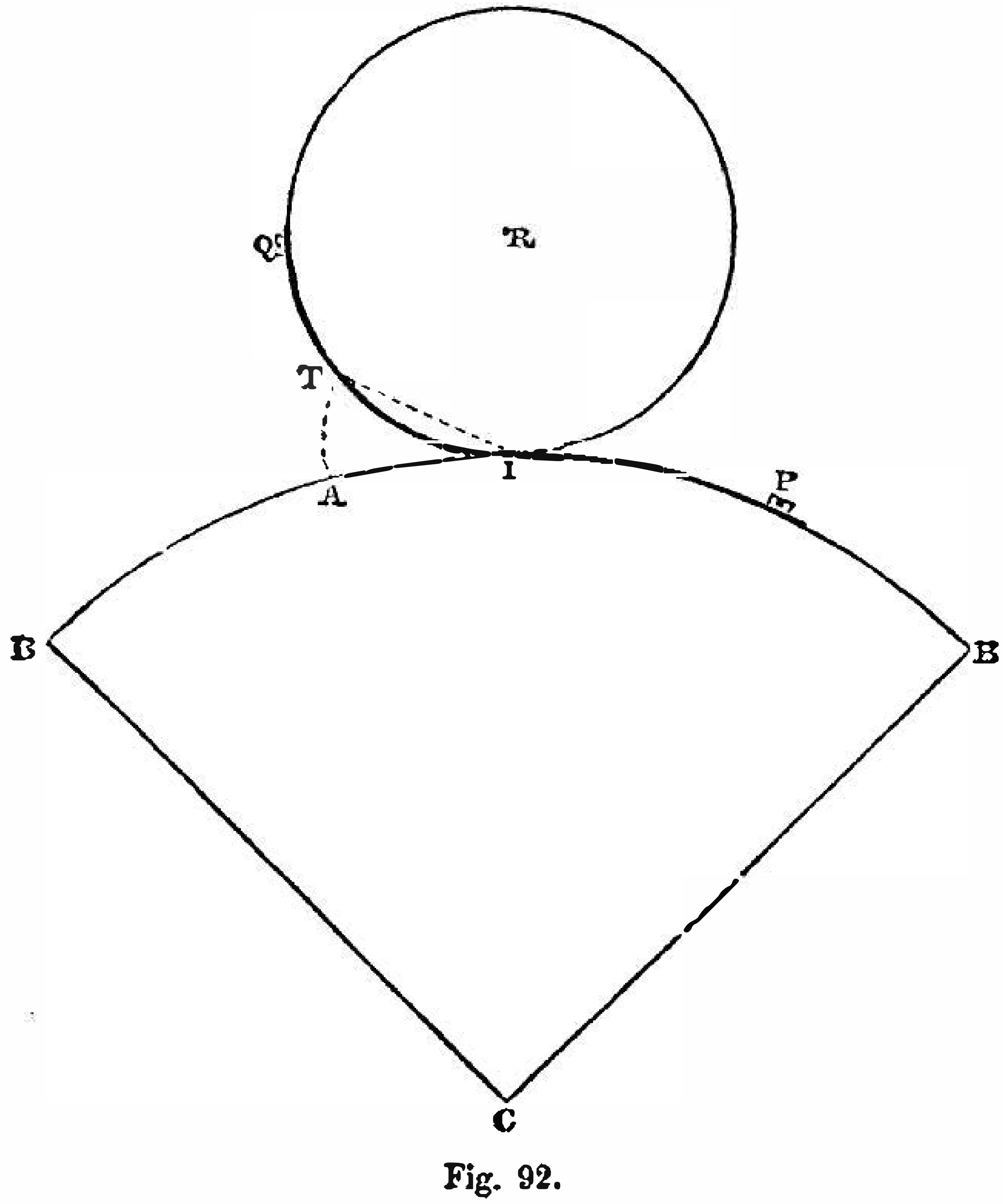

Fig. 92.

of the form of the pitch-circle, and R the rolling templet of the form of the describing circle. Q I P is a slender piece of watch-spring, fastened by a screw at P to the edge of the fixed templet, and by a screw at Q to the edge of the rolling templet. The spring may have a sharp tracing-point formed at T on one of its edges, as already described in Article 131, Rule IV., and shown in fig. 86, page 124. A T, in fig. 92, represents part of the epicycloid traced by the point T, and I the point of contact of the pitch-circle and describing circle. The radius of each of the templets ought to be made less than the radius of the circle which it represents, by *half the thickness* of the spring P Q.

When the fixed templet is concave (for tracing the flanks of teeth) this method of preventing the rolling templet from slipping is not available.

138. **Straight-Flanked Epicycloidal Teeth.**—In the oldest form of epicycloidal teeth, the traces of the flanks are straight lines radiating from the centre of the wheel, being the lines which would be traced by a describing circle, of half the radius of the pitch-circle, rolling inside the pitch-circle. Hence, in order that a pair of wheels with teeth described according to this principle may gear correctly together, the faces of the teeth of each wheel must be traced by rolling upon the outside of its pitch-circle a describing circle of *half the radius of the other pitch-circle.*

For example, in fig. 93, let C and C′ be the centres of a pair of

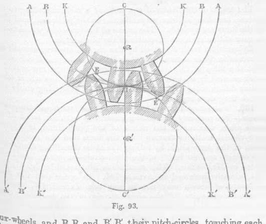

Fig. 93.

spur-wheels, and B B and B′ B′ their pitch-circles, touching each other in the pitch-point I. Lay off the pitch-points of the fronts and backs of the teeth on each of the pitch-circles, and draw straight lines from the centres of the wheels to the points of division of their respective pitch-circles; these lines will be the traces of the flanks of the teeth. Bisect C I in R, and C′ I in R′, and about R and R′ respectively describe circles traversing I; these will be the two describing circles for the faces of the teeth. Lay off, on those two circles, sufficient lengths, I E and I E′, for the two divisions of the path of contact; that is to say, each of these lengths must be greater than half the pitch, and should be made as nearly equal to the pitch as practicable. Then a circle, A A, described about C through E′, will be the addendum-circle of the first wheel, and a circle, A′ A′, described about C′, through E, will be the addendum-circle of the second wheel. The two root-circles, K K and K′ K′, are to be drawn so as to leave a sufficient clearance between each of them and the opposite addendum-circle.

To trace the front and back faces of the teeth of the first wheel, roll the describing circle R′ on the pitch-circle B B; to trace the front and back faces of the teeth of the second wheel, roll the describing circle R on the pitch-circle B′ B′. This may be done with the aid of templets connected together by means of a spring, as described in Article 137, page 131.

The traces of the flanks of the teeth of a rack, according to this system, are straight lines perpendicular to the pitch-line, and those of the faces are cycloids. The traces of the faces of a wheel that is to gear with a rack are involutes of the pitch-circle.

Epicycloidal teeth described by this method are very smooth and accurate in their action; but they labour under the disadvantage that the faces of the teeth of any given wheel are not suited to work accurately with the flanks of the teeth of any wheel whose radius differs from double the radius of the describing circle with which they were traced.

139. **Epicycloidal Teeth Traced by an Uniform Describing Circle.**—The property of working accurately with all teeth of the same pitch, whatsoever the radius of the pitch-circle, is given to epicycloidal teeth by tracing both the faces and the flanks of all teeth of the same pitch, by rolling the same describing circle upon the outside and the inside of the pitch-circle—a system first introduced by Mr. Willis. This method is illustrated by fig. 91, page 131, already described in Article 136. In order that the mean obliquity of action may not in any case exceed 15°, nore the maximum obliquity 30°, the circumference of the describing circle employed is *six times the pitch;* so that its radius is *six times the radial pitch* (see Article 119, page 111). According to this system, the traces of both the flanks and the faces of the teeth of a rack are cycloids.

140. **Approximate Drawing of Epicycloidal Teeth.**—Various approximate methods of drawing epicycloids have already been described in Article 79, pages 59 to 62. The following are the additional explanations required in order to show the application of those methods to epicycloidal teeth:—

I. *By two pairs of Circular Arcs.* In fig. 94, let I A be part of the pitch-circle of a wheel. Draw the describing circle touching the pitch-circle at any convenient point, I, and outside or inside,

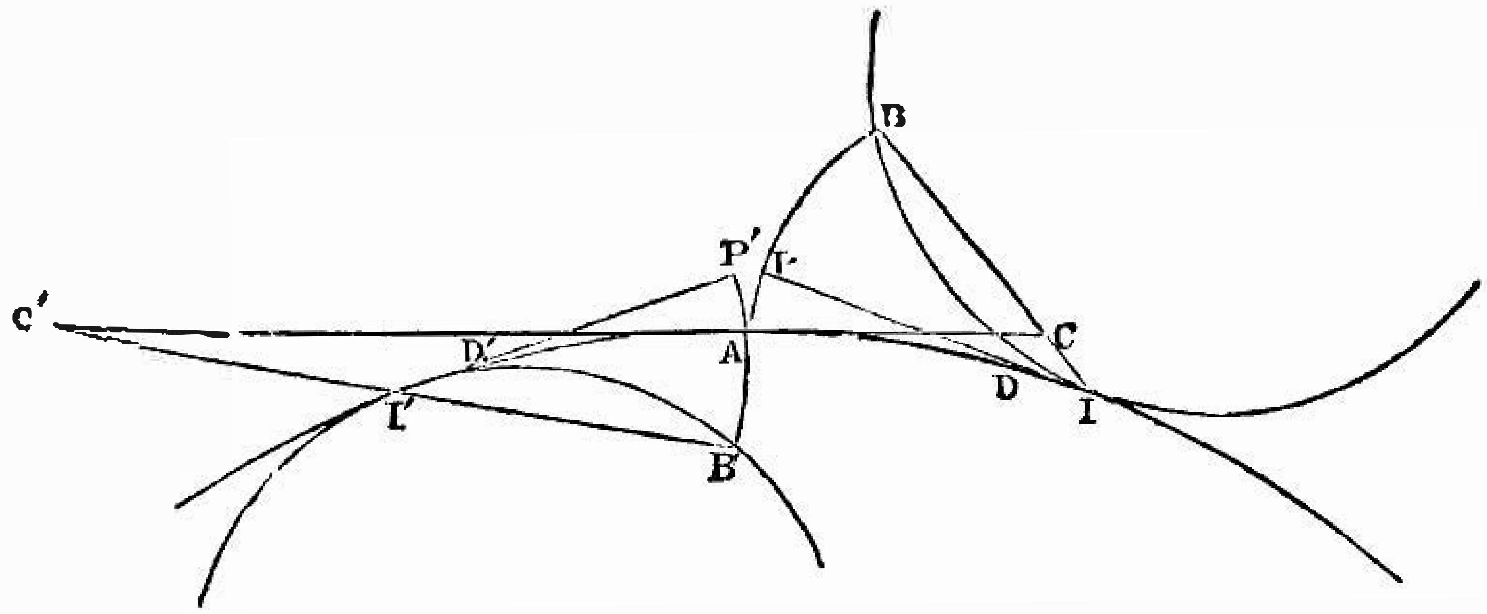

Fig. 94.

according as the face or the flank of a tooth is to be traced. (Letters without an accent refer to the face; letters with an accent, to the flank.)

Draw the straight tangent I P, equal in length to one of the two divisions of the arc of contact, and in it take I D = $\frac{1}{4}$ I P. Then, with the radius D P = $\frac{3}{4}$ I P, draw the circular arc A B; A and B will be the two ends of the required epicycloidal arc. Join B I; and from A draw the straight tangent A C, cutting B I in C. Then A C and B C will be the normals at the two ends of the epicycloidal arc. Then proceed, according to Rule IV. of Article 79, pages 61 and 62, fig. 48, to draw two circular arcs approximating to the required curve; and perform the same operation both for the face and for the flank of the tooth.

According to this method, the traces of the face and flank of a tooth consist each of a pair of circular arcs, and the two arcs which join each other at the pitch-point, A, of a tooth have a common tangent there; because their centres are in the straight line C′ A C.

II. *By one pair of Circular Arcs—Mr. Willis's Method.* Mr. Willis first showed how to approximate to the figures of epicycloidal teeth by means of two circular arcs—one concave, for the flank, the other convex, for the face; and each having for its radius the mean radius of curvature of the epicycloidal arc. Mr. Willis's rules may be deduced from the formula for finding the centre of curvature of an epicycloid, which is given in Article 78, equation 2, page 59; that formula being applied to the point in the epicycloid whose normal meets the pitch-circle at a distance from the pitch-point of the tooth to be traced equal to one-half of the pitch, and the obliquity of that normal to the pitch-circle being 15°.

In fig. 95, let B C be part of the pitch-circle, and A the pitch-point of a tooth whose front is to be traced. Lay off, in opposite directions from the point A, the arcs A B and A C, each equal to one-half of the pitch. Draw the radii of the pitch-circle, B D and C E, and through the points B and C draw the straight lines B F and C G, making angles of 75° with the radii respectively; these lines are normals to the face and to the flank of the tooth respectively. Let n denote the number of teeth in the wheel. Lay off along the two normals the distances B F and C G, as calculated by the following formulæ:—

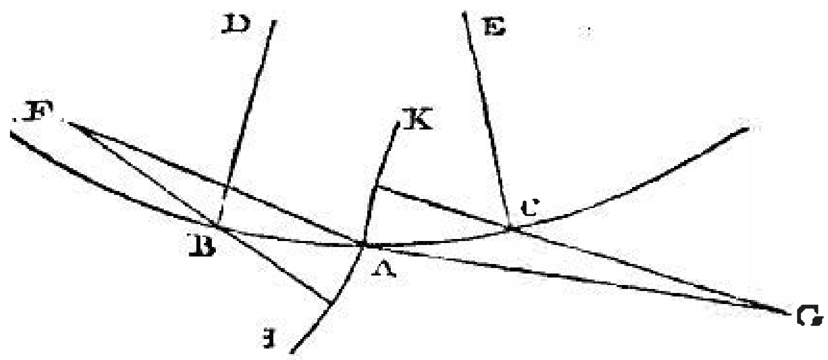

Fig. 95.

$$\text{B F} = \frac{\text{pitch}}{2} \cdot \frac{n}{n+12}; \ \text{C G} = \frac{\text{pitch}}{2} \cdot \frac{n}{n-12};$$

then F will be the centre of curvature for the face, and G the centre of curvature for the flank.

About F, with the radius F A, draw the circular arc A H; this will be the trace of the face of the tooth. About G, with the radius G A, draw the circular arc A K; this will be the trace of the flank of the tooth.

To facilitate the application of this rule, Mr. Willis has published tables of the values of B F and C G, and invented an instrument called the "*Odontograph.*" That instrument is an oblong piece of card-board, F G K H, fig. 96, measuring about 13 inches by $7\frac{1}{2}$ inches. The oblique edge, L H, makes an angle of 75° with the edge G F; so that when the edge L H is laid along a radius, O I, of a pitch-circle, B B, the edge G I F shows the positions of normals to acting surfaces of teeth whose pitch-points are at a distance from I equal to half the pitch. Along the edge G I F two scales of equal parts are laid off in opposite directions from the point I, where the straight line coinciding with H L meets G F; the scale I F serving to mark the centres for faces, and the scale I G the centres for flanks, at distances

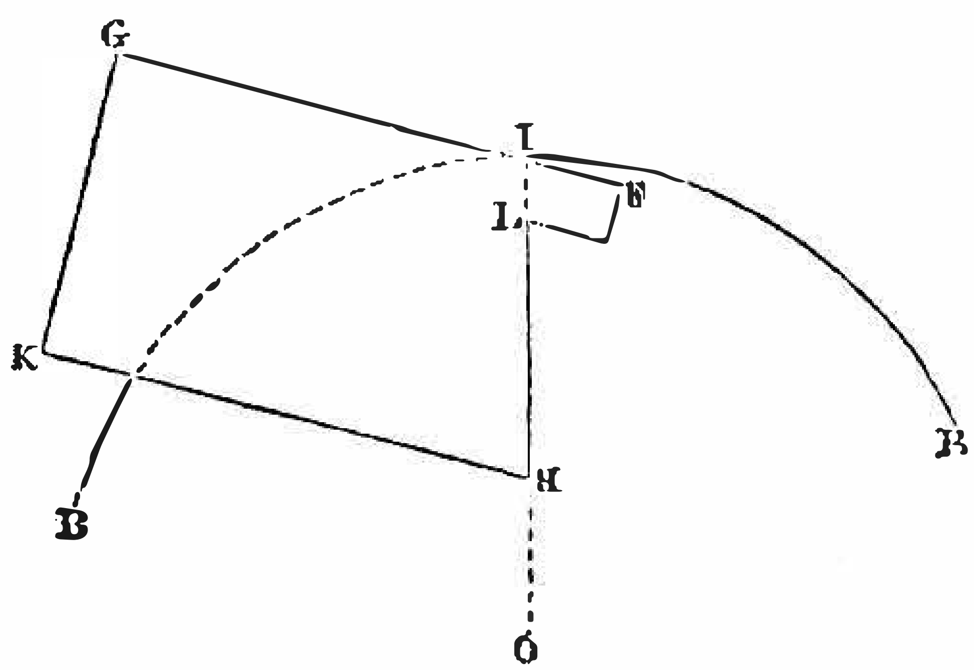

Fig. 96.

from I computed by the formulæ. Values of those distances for different pitches and numbers of teeth, and other useful dimensions, are given in tables which are printed on the sides of the card-board.

141. **Teeth Gearing with Round Staves—Trundles and Pin-wheels.** —When two wheels gear together, and one of them has cylindrical pins (called *staves*) for teeth, that one is called, if it is the larger of the two, a *pin-wheel*, and if the smaller, a *trundle*. The traces of the teeth of the other wheel are drawn in the following manner:—In fig. 97, let B_2 be the pitch-circle and C_2 the centre of the trundle or pin-wheel, and let B_1 B_1 be the pitch-circle of the other wheel. Divide the pitch-circle, B_1 B_1, into arcs equal to the pitch, and through the points of division trace a set of external epicycloids by rolling the pitch-circle B_2 on the pitch-circle B_1, with the centre of a stave for a tracing-point, as shown by the dotted lines; then draw curves parallel to and within the epicycloids, at a distance from them equal to the radius of a stave. These will be the fronts and backs of the required teeth. The clearing curves are circular arcs of a radius equal to that of the staves.

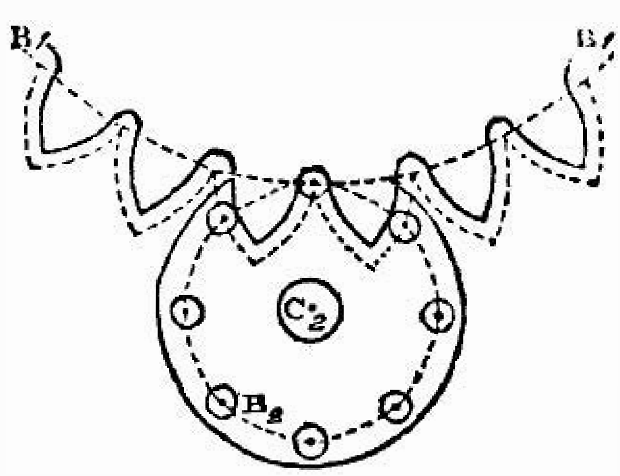

Fig. 97.

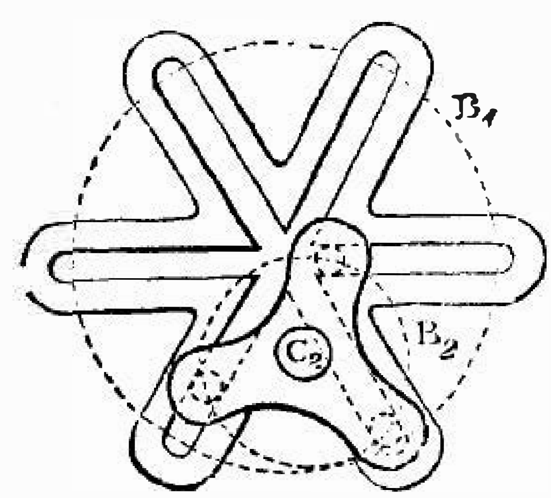

Fig. 98.

When the teeth drive the staves, the whole path of contact consists of *recess*, and there is no approach; for the teeth begin to act on the staves at the instant of passing the line of centres. When the staves drive the teeth, the whole path of contact consists of *approach*, and there is no recess; for the staves cease to act on the teeth at the instant of passing the line of centres. The latter mode of action is avoided where economy of power is studied, because it tends to produce increased friction, for reasons to be stated under the head of the Dynamics of Machines.

To drive a trundle in *inside gearing*, the outlines of the teeth of the wheel should be curves parallel to internal epicycloids. A peculiar case of this is represented in fig. 98, where the radius of the pitch-circle of the trundle is exactly one-half of that of the pitch-circle of the wheel; the trundle has three equidistant staves; and the internal epicycloids described by their centres, while the pitch-circle of the trundle is rolling within that of the wheel, are three straight lines, diameters of the wheel, making angles of 60° with each other. Hence the surfaces of the teeth of the wheel form three straight grooves intersecting each other at the centre, each being of a width equal to the diameter of a stave of the trundle, with a sufficient addition for back-lash.

The following is the construction given by Mr. Willis for finding in pin-wheels and trundles *what is the greatest radius of stave consistent with having an arc of contact not less than the pitch* (see fig. 99):—

Let C be the centre of the wheel with teeth, and C′ that of the wheel with staves. On their two respective pitch-circles lay off the arcs I D and I B, each equal to the pitch. Draw the straight line I B; draw also the straight line C E, bisecting the angle I C D, and cutting I B in E; then B E will be the greatest radius that can be given to the staves consistently with having an arc of contact not less than the pitch.

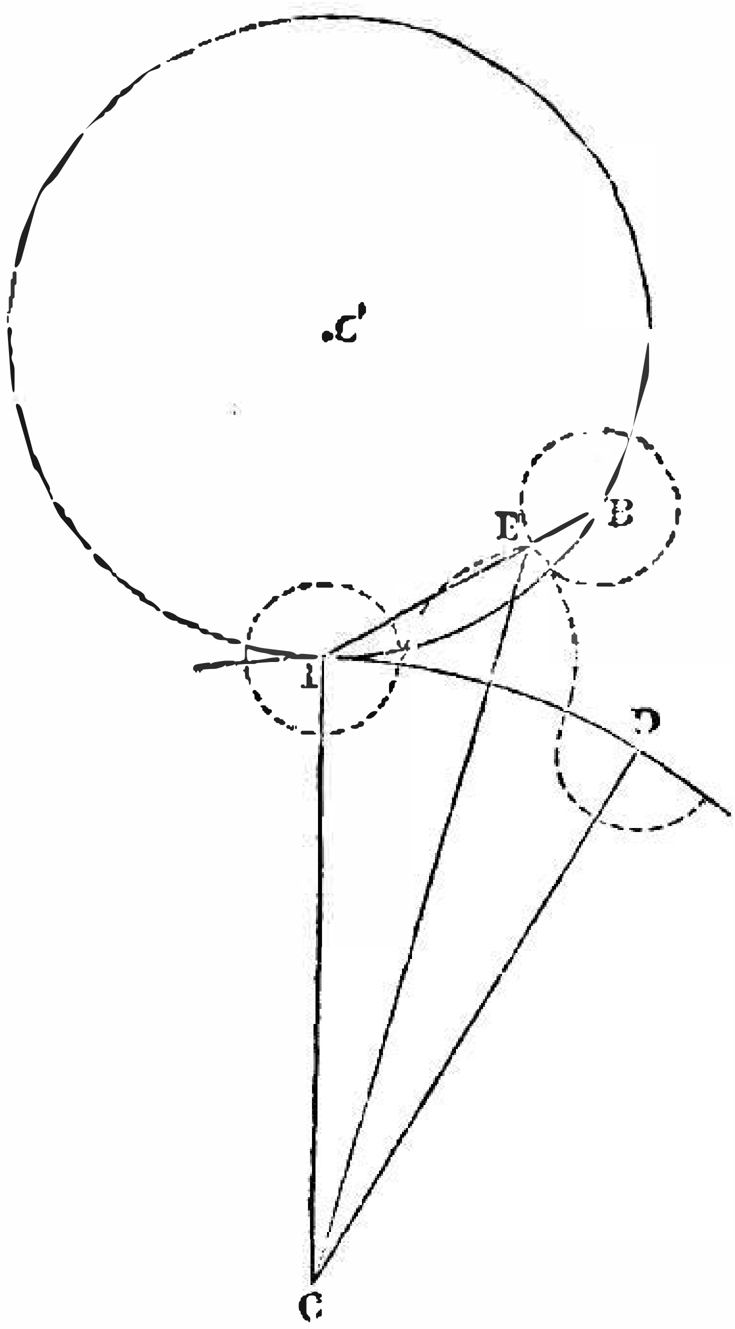

Fig. 99.

The proof is as follows:—Because the fronts and backs of the teeth are similar, the crest of the tooth that acts on a stave at B must be in the straight line C E, that bisects the angle I C D. When the centre of a stave is at B, the point of contact of the stave and tooth must be in the line of connection I B. When the staves have the greatest radius consistent with the continuance of action, while the centre of a stave moves from I to B, the point of contact and the crest of the tooth coincide, and are therefore at the point E, where I B and C E intersect.

Should C E pass beyond B, the proposed pair of wheels will not work, and the design must be altered; and such is also the case when C E either traverses the point B or cuts I B so near to B as to give a radius too small for strength.

In practice, the radius B E ought to be made a little less than that given by the Rule, in order that there may be no risk of imperfect working through the effects of tear and wear.

The smallest number of staves commonly met with in a trundle is five.

A straight rack may have staves instead of teeth; it is then called a *pin-rack;* and it is evident that the fronts and backs of the teeth of a wheel to gear with it should be parallel to involutes of the pitch-circle of that wheel. On the other hand, a toothed straight rack may gear with a trundle, and then the teeth of the rack are to be traced by first rolling the pitch-circle of the trundle on the pitch-line of the rack, so as to draw cycloids, and then drawing curves parallel to and inside those cycloids, at a distance equal to the radius of the staves.

142. **Intermittent Gearing.**—The action of a pair of wheels is said to be *intermittent* when there are certain parts of the revolution of the driver during which the follower stands still. This is effected by having a *dead arc*, or portion without teeth, such as A E, fig. 100, in the circumference of the driver, to which there corresponds a suitable *gap* in the series of teeth of the follower, as between C and D; and in most cases there are also required a *guide-plate*, G H, fixed to one side of the follower, which, when the connection of the wheels is renewed, is acted upon by a *pin*, F, in the driver.

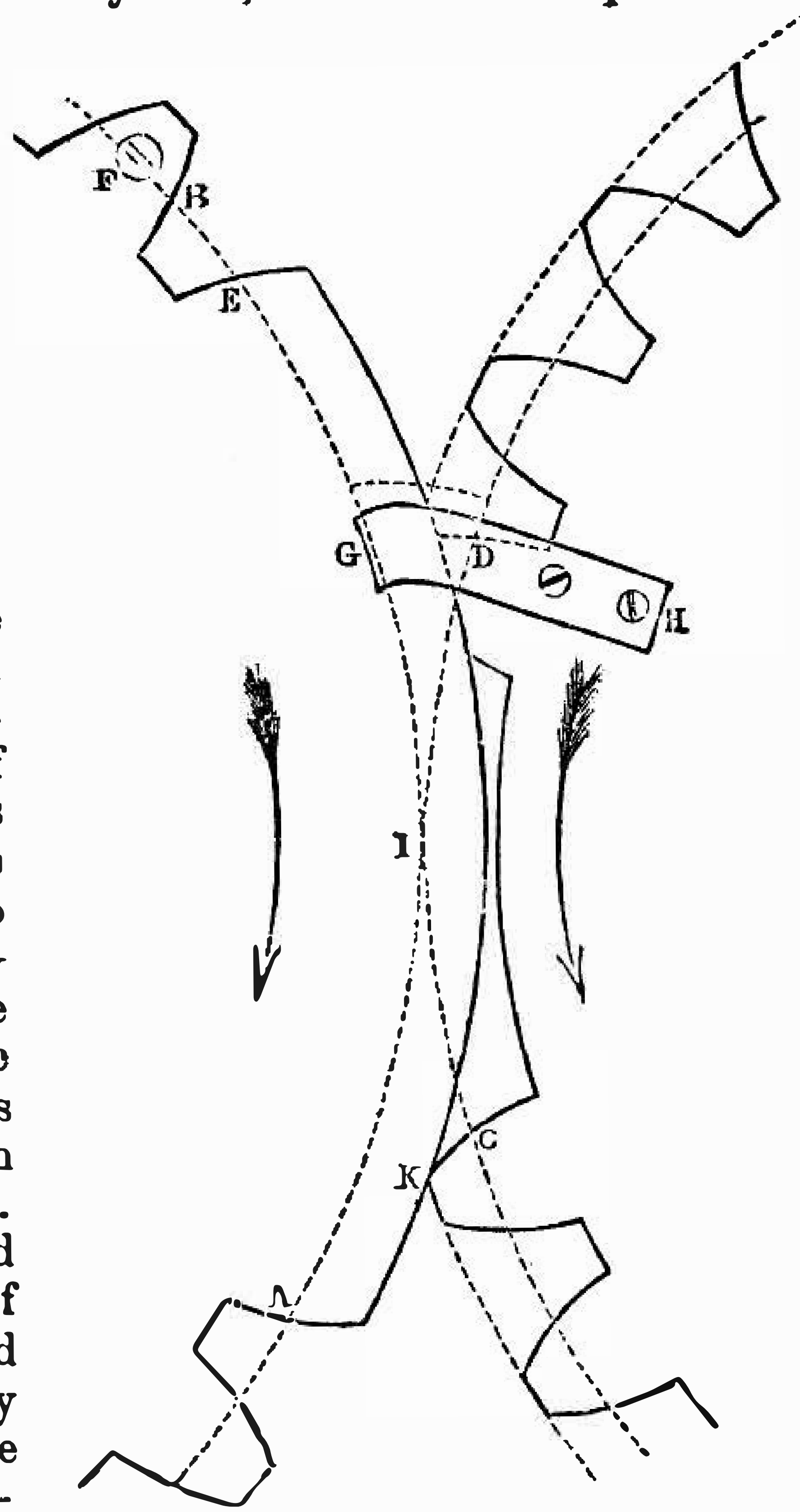

Fig. 100.

Supposing the radii and the pitch of a pair of wheels to be given, and also the *arc of repose*—by which term is meant the length upon the pitch-circle of the driver of that part which is to pass during the pause in the movement of the follower—the method of designing those wheels so as to work with smoothness and precision is as follows:—

Draw the pitch-circles, and divide them as usual; draw also the addendum-circles and root-circles according to the ordinary rules. Mark the point, K, where the addendum-circles cut each other at the *receding* side; this will be the point at which the action of the teeth will terminate, at the instant when the pause begins. Through K draw a curve suited for the front of a tooth of the follower, and let C be the pitch-point of that tooth. Then, starting from C, lay off the pitch-points of the fronts and backs of the teeth of the follower, and draw those fronts and backs. Then, looking towards the *approaching* side, mark the furthest tooth from the line of centres, which is cut by the addendum-circle of the driver; let D be the pitch-point of the face of that tooth. The crest of the tooth D is to be cut away so as exactly to fit the addendum-circle of the driver, and the teeth between it and the tooth C are to be omitted, leaving a smooth part of the root-circle between the front of C and the back of D; this is the required *gap*.

Measure the arc C D on the pitch-circle of the follower between the fronts of the teeth C and D, and to its length add the length of the intended *arc of repose;* from the sum subtract the *space*, B E, that is to be left between each pair of teeth on the pitch-circle of the driver; the remainder will be the *dead arc*, A E, which is to be laid off on the pitch-circle of the driver. The two ends of that arc are to be bounded by curves like the front and back of a tooth of the driver respectively: a front at A, a back at E; and the intervening part of the rim of the driver is to have a smooth edge coinciding with its addendum-circle.

For the purpose of renewing the connection between the driver and follower, the cylindrical pin F is to be fixed with its centre in the pitch-circle of the driver, and the guide-plate G H is to be fixed to the corresponding side of the follower. The acting edge of the guide-plate is to be shaped like the front of a tooth for working with the pin F (as in Article 141, page 137); and the distance on the pitch-circle of the follower from the front of that edge to the front of the tooth D is to be equal to the distance on the pitch-circle of the driver from the front of the pin F to the front of the tooth B; so that when B is driving D, F shall at the same time be driving G H. The end G of the guide-plate in the position of repose should project just far enough inside the pitch-circle of the driver to insure that the pin F shall meet it.

The action in working is as follows:—Just before the pause, the front, A, of the dead-arc drives the front of the tooth, C, in the usual way throughout the ordinary path of contact; and then, as there is a gap following C, the crest of the front A continues to drive C until the crest of C reaches the position K, and clears the addendum-circle of the driver. At that instant the driver loses hold of the follower, and at the same instant the top of the tooth D comes in contact with the rim of the dead arc, which it is shaped to fit; and this prevents the follower from moving until the dead arc has passed clear of the tooth D. At this instant the pin F begins to drive the guide-plate G H, and continues to do so until the tooth B has begun to drive the tooth D, and the connection is renewed.

If the pressure to be exerted is considerable, there may be a pair of pins at F, one at each side of the driver, and a pair of guide-plates at G H, one at each side of the follower.

The shortest arc through which the follower can be driven in the interval between two pauses is C D; and such is the case when B is the front of a second dead arc, and the tooth D is immediately followed by a second gap. In this case it may be necessary to cut away part of the outer side of the pin F, in order to insure its clearing the tip, G, of the guide-plate when the next pause begins.

It is easy to see how the same principles may be applied to the designing of a *wheel and rack* with intermittent action. When the rack is the follower, a pair of similar and parallel racks, rigidly framed together, may be made to gear with opposite edges of a spur-wheel, having a toothed arc and a dead arc so arranged as to drive the two racks alternately in opposite directions, and thus produce a reciprocating motion of the piece of which they are parts. This combination belongs to Class C of Mr. Willis's arrangement. As to the form which it takes when one tooth only acts at a time, see Article 164, further on. (See also Addendum, page 286.)

143. **The Teeth of Non-Circular Wheels** may be traced by rolling circles or other curves on the pitch-lines; and when those teeth are small, compared with the wheels to which they belong, each tooth is nearly similar to the tooth of a circular wheel whose pitch-circle has a radius equal to the radius of curvature of the pitch-line of the actual wheel at the point where the tooth is situated; the tooth being traced by means of the same describing circle which is used for the circular wheel.

It is obvious that the use of an uniform describing circle for teeth of a given pitch (as explained in Article 139, page 134) is the most easily practicable method of tracing teeth for a non-circular wheel. It may be carried out by means of templets, as in Article 137, page 131.

The operation is necessarily much more laborious than the corresponding operation for a circular wheel; because in a non-circular wheel the teeth have figures varying with the curvature of the pitch-line.

If the pitch-line of a non-circular wheel is one whose radii of curvature at a series of points can be easily found, a series of figures may be used for the teeth similar to the figures suited for circular wheels of those radii; and in drawing those figures the approximate methods of Article 140, pages 134 to 136, may be employed.*

* The following relation between the radii of curvature at a pair of corresponding points of a pair of pitch-lines that roll together, may be useful to determine one of those radii of curvature when the other is known. Let r and r' be the two segments into which the pitch-point divides the line of centres at the instant when the pair of corresponding points in question are in contact; let ρ and ρ' be the two radii of curvature at these points, and let θ be the angle which those radii make with the line of centres at the instant before mentioned; then

$$\frac{1}{\rho} + \frac{1}{\rho'} = \left(\frac{1}{r} + \frac{1}{r'}\right) \cos \theta \ldots\ldots (1.)$$

When the pitch-lines are in inside gearing, the greater of the two segments, r, r', is to be made negative, and each radius of curvature is to be considered as positive for a convex and negative for a concave pitch-line.

For *a pair of equal elliptic pitch-lines*, as in Article 108, page 93, the radii of curvature at a pair of corresponding points are equal, and are therefore both given by the following formulæ:—

$$\frac{1}{\rho} = \frac{1}{\rho'} = \frac{\cos \theta}{2}\left(\frac{1}{r} + \frac{1}{r'}\right); \ldots\ldots (2.)$$

or,

$$\rho = \rho' = \frac{2\, r\, r'}{(r + r') \cos \theta}; \ldots\ldots (2\text{ A.})$$

and the same formulæ apply to any pair of *equal and similar lobed pitch-lines* of the class described in Article 109, page 97.

For a *logarithmic spiral pitch-line* (Article 110, page 99) the radius of curvature at any point is given by the formula

$$\rho = \frac{r}{\cos \theta}; \ldots\ldots (3.)$$

and may be found approximately by construction, as already described in the article referred to.

If *one of the pitch-lines is straight* (a case already used as an example in Article 107, page 92), the reciprocal of the radius of curvature of that line is at every point equal to nothing; so that equation 1 of this note becomes (for the other pitch-line)

$$\frac{1}{\rho} = \left(\frac{1}{r} + \frac{1}{r'}\right) \cos \theta \ldots\ldots (4.)$$

Let c denote the length of the line of centres, and a the shortest distance of the straight pitch-line from its own axis of motion; then $r' = \frac{a}{\cos \theta}$; and $r = c - r' = c - \frac{a}{\cos \theta}$; consequently equation 4 becomes

$$\frac{1}{\rho} = \frac{c \cos^3 \theta}{a\, c \cos \theta - a^2}; \ldots\ldots (4\text{ A.})$$

or

$$\rho = \frac{a}{\cos^2 \theta} - \frac{a^2}{c \cos^3 \theta}. \ldots\ldots (4\text{ B.})$$

When a pair of non-circular wheels are connected by means of teeth alone, care must be taken that the obliquity of the action of the teeth does not become too great in certain positions of the wheels. That obliquity is greatest at the instant when the obliquity of the common tangent of the two pitch-lines at their pitch-point to a perpendicular to the line of centres at that point is greatest, such obliquity being in the direction of rotation of the follower; for, as that is also the direction of the obliquity of the line of connection of the teeth to the pitch-lines, those two obliquities are added together at the instant in question; their sum being the total obliquity of the line of connection to a perpendicular to the line of centres. Excessive obliquity of action tends to produce great friction, and involves also the risk of the teeth either getting jammed or losing hold of each other. In practice, the total obliquity of action of the teeth of non-circular wheels is seldom allowed to exceed about 50°; or say, about 15° for the obliquity of the line of connection of the teeth to the pitch-lines, and 35° for the greatest obliquity of the pitch-lines to a line perpendicular to the line of centres.

There is one case, however, in which it is not necessary to confine the obliquity within such narrow limits; and that is when the wheels have a pair of equal and similar elliptic pitch-lines centred on two of their foci, and it is practicable to link the revolving foci together, as shown in Article 108, fig. 72, page 96; for the link preserves the connection accurately at the time when the obliquity of the pitch-lines is greatest. In this case, indeed, the teeth may be omitted throughout a pair of arcs at the two sides of each elliptic pitch-line, each such toothless arc having a smooth rim of the form of the pitch-line, and extending both ways from the end of the minor axis to a pair of points perpendicularly opposite the foci, or nearly so. (See page 292.)

144. **Teeth of Bevel-Wheels.** (*A. M.*, 467.)—The teeth of a bevel-wheel have acting surfaces of the conical kind, generated by the motion of a line traversing the apex of the conical pitch-surface, while a point in it is carried round the traces of the teeth upon a spherical surface described about that apex.

The operations of drawing the traces of the teeth of bevel-wheels exactly, whether by involutes or by rolling curves, are in every respect analogous to those for drawing the traces of the teeth of spur-wheels; except that in the case of bevel-wheels all those operations are to be performed on the surface of a sphere described about the apex, instead of on a plane, substituting *poles* for *centres*, and *great circles* for *straight lines*.

In consideration of the practical difficulty, especially in the case of large wheels, of obtaining an accurate spherical surface, and of drawing upon it when obtained, the following *approximate* method, proposed originally by Tredgold, is generally used:—

I. *Development of Teeth.*—Let O, fig. 101, be the common apex of the pitch-cones, O B I, O B′ I, of a pair of bevel-wheels; O C, O C′, the axes of those cones; O I their line of contact. Perpendicular to O I draw A I A′, cutting the axes in A, A′; make the outer rims of the patterns and of the wheels portions of the cones A B I, A′ B′ I, of which the narrow zones occupied by the teeth will be sufficiently near for practical purposes to a spherical surface described about O. As the cones, A B I, A′ B′ I, cut the pitch-cones at right angles in the outer pitch-circles, I B, I B′, they may be called the *normal cones.* To find the traces of the teeth upon the normal cones, draw on a flat surface circular arcs, I D, I D′, with the radii A I, A′ I; those arcs will be the *developments* of arcs of the pitch-circles, I B, I B′, when the conical surfaces, A B I, A′ B′ I, are spread out flat. Describe the traces of teeth for the developed arcs as for a pair of spur-wheels, then wrap the developed arcs on the normal cones, so as to make them coincide with the pitch-circles, and trace the teeth on the conical surfaces.

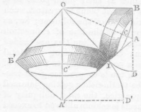

Fig. 101.

II. *Traces and Projections of Teeth.*—Fig. 102 illustrates the process of drawing the *projection of a tooth of a bevel-wheel on a plane perpendicular to the axis.* In the first place, let A C represent the common axis of the pitch-cone and normal cone; A being the apex of the normal cone. Let A I be the trace of the normal cone on a plane traversing the axis; and let I I′, perpendicular to I A, be part of the trace of the pitch-cone on the same plane, of a length equal to the intended breadth of the toothed rim of the wheel. C I perpendicular to A C is the radius of the pitch-circle in which the pitch-cone and normal cone intersect each other. About A, with the radius A I, draw the circular arc D I D, making D I = I D = half the pitch; D I D will be the development of an arc of the pitch-circle of a length equal to the pitch. On the arc D I D lay off I G = I G = half the thickness of a tooth on the outer pitch-circle. Then, by the rules for spur-wheels, draw the trace, H G E G H, of one tooth and a pair of half-spaces, with a suitable addendum-circle through E, and a suitable root-circle, H F H.

The straight line F I E will be the trace, upon a plane traversing the axis, of the outer side of a tooth; and E and F will be the traces, on that plane, of the outer addendum-circle and root-circle respectively. From E and F draw straight lines, E E′ and

F F′, converging towards the apex of the pitch-cone; these will be the traces of the *addendum-cone* and *root-cone* respectively. (For want of space, the apex of the pitch-cone is not shown in fig. 102.) Through I′, parallel to F I E, draw F′ I′ E′; this will be the trace,

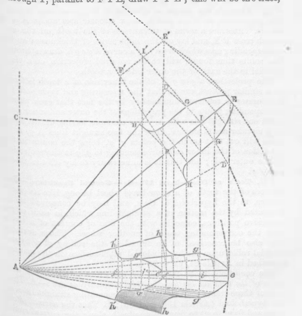

Fig. 102.

on a plane traversing the axis, of the inner side of a tooth; and the points E′, I′, and F′ will be respectively the traces of the inner addendum-circle, inner pitch-circle, and inner root-circle.

Through A, parallel to C I, draw the straight line A *i e*, and conceive this line to be traversed by a plane perpendicular to the axis, as a new plane of projection. Through the points F, I, E, F′, I′, E′, draw straight lines parallel to C A, cutting A *e* in *f*, *i*, *e*, *f′*, *i′*, *e′*; these points, marked with small letters, will be the projections, on the new plane, of the points marked with the capital letters.

Divide the depth, F E, of the tooth at its outer side into any convenient number of intervals. Through the points of division draw straight lines parallel to C A; these will cut $f\,e$ in a series of points, which will be the projections of the points of division of F E. Through the points of division of F E, and also through the projections of those points, draw circular arcs about A as a centre. Measure a series of thicknesses of the tooth on the arcs which cross F E, and lay off the same series of thicknesses on the corresponding arcs which cross $f\,e$; a curve, $h\,g\,e\,g\,h$, drawn through the points thus found, will be the required projection, on a plane parallel to the axis, of the outer side of a tooth.

The projection, $h'\,g'\,e'\,g'\,h'$, of the inner side of a tooth is found by a similar process, except that the measuring and laying-off the thicknesses is rendered unnecessary by the fact that each pair of corresponding points in the projections of the outer and inner sides lie in one straight line with A. For example, having drawn about A a circular arc through i', draw the two straight lines A g, A g; these will cut that arc in the points g', g', being the points in the projection of the inner side corresponding to g, g in the projection of the outer side; and thus it is unnecessary to lay off the thickness $g'\,g'$.

145. **Teeth of Skew-bevel Wheels — General Conditions.**—The surfaces of the teeth of a skew-bevel wheel belong, like its pitch-surface, to the hyperboloïdal class, and may be conceived to be generated by the motion of a straight line which, in each of its successive positions, coincides with the line of contact of a tooth with the corresponding tooth of another wheel. Those surfaces may also be conceived to be traced by the rolling of a hyperboloïdal roller upon the hyperboloïdal pitch-surface, in the manner described in Article 84, pages 70 to 73.

The conditions to be fulfilled by the *traces of the fronts and backs of the teeth on the hyperboloïdal pitch-surface* are :—A. That each of those traces shall be one of the generating straight lines of the hyperboloïd (Article 106, page 89); B. That the *normal pitch*, measured from front to front of the teeth along the *normal spiral* (Article 106, page 89), shall be the same in two wheels that gear together—(this second condition is always fulfilled if the two pitch-surfaces are correctly designed, and the numbers of teeth made inversely proportional to the angular velocities); and C. That the teeth, if in outside gearing, shall be *right-handed* on both wheels, or *left-handed* on both wheels; and if in inside gearing, contrary-handed on the two wheels.

Skew-bevel teeth may be said to be RIGHT-HANDED or LEFT-HANDED, according to the direction in which the generating lines of the teeth appear to deviate from the axis when looked at with the axis upright, as in fig. 103, page 147. For example, the wheel in that figure has left-handed teeth; for the generating line I′ I deviates to the left of the axis A′ A. The same rule applies to the direction in which the crests of the teeth appear to deviate from the radii of the wheel, when looked at as in the upper part of fig. 105, page 150.

Right-handed teeth have left-handed normal spirals, and left-handed teeth right-handed normal spirals.

146. **Skew-bevel Teeth—Rules.**—I. *Normal Section of a Tooth.*—In fig. 103, let A *a* A′ be the axis of a skew-bevel wheel: let *a* be the centre of the *throat* of its hyperboloïdal pitch-surface; let the dotted curve through I be the trace of that surface on a plane traversing the axis; and let C I = *ai* be the radius of the pitch-circle at the middle of the breadth of the rim of the intended

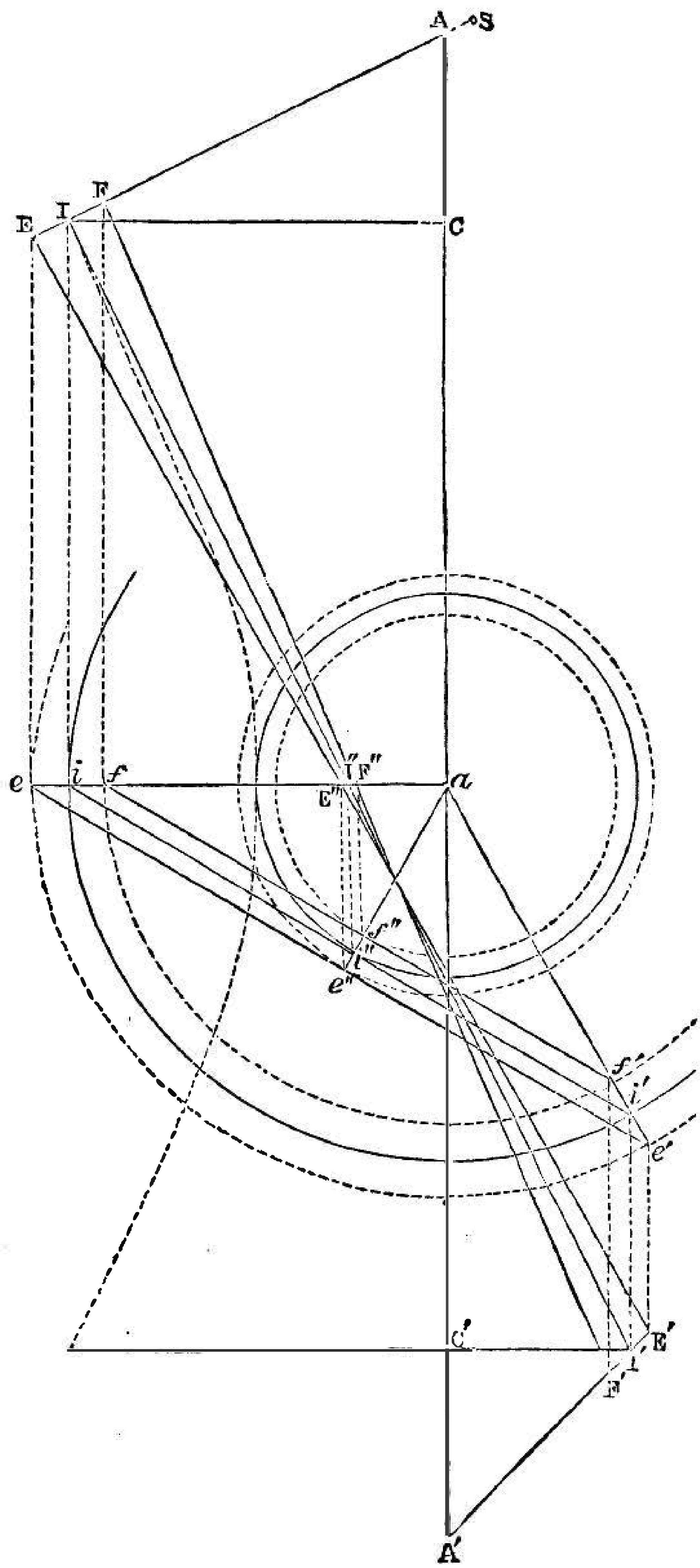

Fig. 103.

wheel, as found by Rule I. of Article 106, page 88. Draw by Rules II. and III. of that Article, pages 88, 89, the normal I A

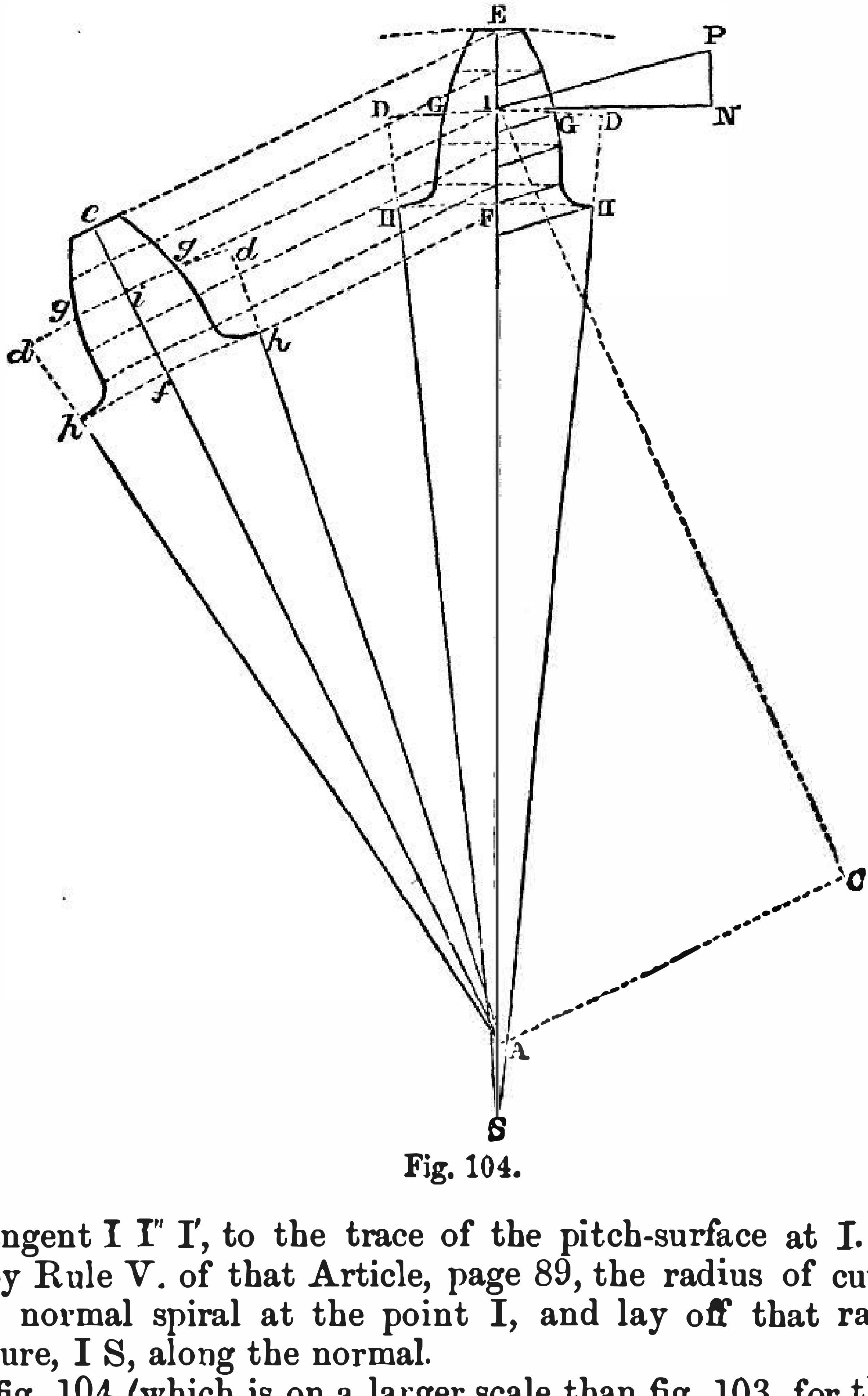

Fig. 104.

and tangent I I'' I', to the trace of the pitch-surface at I. Then find, by Rule V. of that Article, page 89, the radius of curvature of the normal spiral at the point I, and lay off that radius of curvature, I S, along the normal.

In fig. 104 (which is on a larger scale than fig. 103, for the sake of distinctness), let A C, as before, be the axis of the wheel, C I the radius of the middle pitch-circle, I A the normal, and I S the radius of curvature of the normal spiral; draw I N perpendicular to I S. Then, by Rule V. of Article 106, page 89, find the angle (= O *g* F' in fig. 68, page 88) which a tangent to the normal spiral makes with a tangent to the pitch-circle, and draw I P, making that angle with I N. Lay off I P equal to *the pitch as measured on the middle pitch-circle;* let fall P N perpendicular to I N; then

I N will be the *normal pitch at the middle pitch-circle.* About S, with the radius S I, draw a circular arc, and lay off on that arc the distance, D D, equal to the normal pitch, one-half to each side of I. Lay off the intended middle thickness, G G, of a tooth, one-half to each side of I. Then draw, by the rules for spur-wheels, the *normal section,* H G E G H, of a tooth, being its trace upon a surface which cuts it normally at the middle of the breadth of the rim of the wheel.

II. *Trace of a Tooth on the Normal Cone.*—Through A in fig. 104 draw A *i* parallel and equal to C I; and through I draw I *i* parallel and equal to C A. About A, with the radius A *i*, draw the circular arc *d d*, equal in length to I P, the pitch on the pitch-circle, and having the middle of its length at the point *i*. This will be the arc on the pitch-circle corresponding to the arc D D on the normal spiral.

Divide E F, the middle depth of the tooth, into any convenient number of intervals; and through E and F and the points of division draw straight lines parallel to I *i*, cutting A *i e* in a series of corresponding points. Through the points in E F draw circular arcs about S. Through the corresponding points of *e f* draw circular arcs about A. From the points where the arcs cut the trace E G H measure *oblique half-thicknesses* to the centre line, E F, of the tooth, *along oblique lines drawn parallel to* P I; and lay off those half-thicknesses at both sides of *e f*, along the arcs which cross it. Through the points thus found draw the curve *h g e g h;* this will be *the projection, on a plane perpendicular to the axis, of the trace of a tooth upon the normal cone of the pitch-surface at the middle of its breadth;* that is, upon the cone whose trace is A I in fig. 103. (If it be desired to draw the *development* of that trace, lay off the *oblique half-thicknesses* along arcs drawn about A, through the points of division of the radius A F I E. The result is the drawing of an outline outside of, and nearly parallel to, H G E G H. To prevent confusion, it is not shown in the figure.)

If the pitch-circle chosen is at the *throat* of the hyperboloïd, the normal cone becomes simply the plane of that circle; and in fig. 104, A *f i e* coincides with A F I E.

III. *Projections of the Middle Lines of a Tooth.*—In fig. 103, let F I E and *f i e*, as before, represent the projections of the central depth of a tooth, being part of a normal (E I F A, *e i f a*) to the pitch-surface at a point, I *i*, in the middle pitch-circle, whose radius is C I = *a i;* so that F, *f*, I, *i*, and E, *e* are the projections of the *middle points* of the tooth at the root, at the pitch-surface, and at the crest respectively; and let it be required to find the projections of the *middle lines* of that tooth at the root, pitch-surface, and crest respectively.

About *a* draw the circles *f f'*, *i i'*, and *e e'*; being the projections, on a plane perpendicular to the axis, of the root-circle through F, the pitch-circle through I, and the addendum-circle through E. Draw also about a the pitch-circle at the throat of the hyperboloïd, and let $a\ i''$ be its radius. Through i draw a straight line, $i\ i''\ i'$, so as to touch this *throat pitch-circle*, and let that straight line cut the circle $i\ i'$ in i and i'. Draw the straight lines $f\ f''\ f'$ and $e\ e''\ e'$ parallel to $i\ i''\ i'$. Then these three parallel lines will be the *projections of the three middle lines* before mentioned, *on a plane perpendicular to the axis.*

Describe about a two circles touching $f\ f''\ f'$ and $e\ e''\ e'$ respectively. These will be respectively *the root-circle and the addendum-circle at the throat of the hyperboloïd.* The roots and crests of all the teeth lie in a pair of hyperboloïdal surfaces traversing this pair of circles, and traversing also the pair of circles through F and E.

The projection, on a plane traversing the axis, of the middle line of the tooth on the pitch-surface is the tangent I I$''$ already found, the points I$''$ and i'' being in one straight line parallel to a A.e To find the corresponding projections of the other two middle lines, there are two methods.

First Method.—From the points of contact f'' and e'', parallel to a A, draw f'' F$''$ and e'' E$''$, cutting $a\ i$ in F$''$ and E$''$ respectively. Join F F$''$ and E E$''$. These will be the required projections.

Second Method.—Lay off on the axis, a C$' = a$ C, and a A$' = a$ A, and draw C$'$ I$'$ parallel to C I: then C$'$ I$'$ will be part of the projection of a pitch-circle equal to C I. From i', parallel to A a A$'$, draw i' I$'$, cutting C$'$ I$'$ in I$'$. Then i' and I$'$ will be the two projections of one pitch-point, and II$''$ I$'$ will be one straight line. Join A$'$ I$'$. This will be the projection of a normal to the pitch-surface at I$'$. Through f' and e' (which lie in one radius, $a\ f'\ i'\ e'$) draw f' F$'$ and e' E$'$ parallel to A a A$'$, cutting A$'$ I$'$ in F$'$ and E$'$ respectively. Join F F$'$ and E E$'$. These will be the required projections of the middle lines at the root and crest of the tooth respectively.

IV. *Complete Projection of a Tooth on a Plane Normal to the Axis.*—Let the plane of projection in fig. 105 be normal to the axis of the wheel, and (as in fig. 103) let a be the axis; let the circles $e\ e'$, $i\ i'$, and $f\ f'$, be the projections of the middle addendum-circle, middle pitch-circle, and middle root-circle of the intended wheel; let the circles through e'', i'', and f'' be the corresponding circles at the throat of the pitch-surface; and let the parallel straight lines $e\ e''\ e'$, $i\ i'\ i''$, $f\ f''\ f'$, be the projections of the middle lines of a tooth at the crest, pitch-surface, and root, drawn according to the preceding rules.

At the end of the radius $a\ f\ i\ e$ construct, by the rules already given, the projection of the trace of the tooth upon the middle normal cone, being the curve marked $h\,g\,e\,g\,h$ in fig. 104; and at the end of the radius $a\,f'\,i'\,e'$ construct a similar and equal figure. From a series of points in the figure at $f\,i\,e$ draw straight lines to the corresponding points in the figure at $f'\,i'\,e'$; each of those straight lines will be the projection of a *generating line* of the surface of the tooth. For example, the straight lines from the corners of the crest at e to the corresponding corners of the crest at e' (both of which lines touch the circle through e'') will be the projections of the two edges of the crest; the straight lines from the pair of points where the curve at $f\,i\,e$ cuts the pitch-circle to the corresponding pair of points near $f'\,i'\,e'$ (both of which lines touch the circle through i'') will be the projections of the lines in which the front and back of the tooth respectively cut the pitch-surface; and the straight lines from the bottoms of the clearing curves near f to the corresponding points near f' (both of which lines touch the circle through f') will be the projections of the lines marking the bottoms of the hollows of which these curves are the traces.

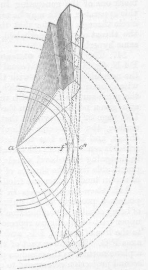

Fig. 105.

The projections of the outer and inner sides of the tooth (being portions of the outer and inner sides of the rim of the wheel) are figures similar to the curve at $f\,i\,e$, and constructed by the same method; the dimensions of the former being larger and those of the latter smaller than the dimensions of that middle figure, in the proportion in which the radii of the outer and inner pitch-circles are respectively greater and smaller than those of the middle pitch-circle. (As to those two circles, see Article 106, page 90.) The tooth in fig. 105 is drawn of an exaggerated breadth, in order to show more clearly the construction of the figure.

V. *Modelling Skew-bevel Teeth.*—Construct a frame of rods to represent the axis A A′ in fig. 103, the equal radii C I and C′ I′, the equal normals A F I E and A′ F′ I′ E′, and the generating line I I′. Make a pair of equal and similar templets, each of the shape and dimensions of the normal section of a tooth, H G E G H, fig. 104, Fix those two templets to the frame at F I E and F′ I′ E′, fig. 103, with their flat surfaces parallel to each other and normal to the rod II′. Then a straight edge or a stretched wire, made to touch the edges of the templets at a pair of corresponding points, will mark one of the generating lines of a tooth; and by the help of this apparatus, teeth may be modelled suitable either for the pitch-circle through C, or for that through C′, or for the pitch-circle at the throat of the hyperboloïd, or for any other pitch-circle on the same hyperboloïd.

147. The **Transverse Obliquity of Skew-bevel Teeth** is the angle, P I N, fig. 104, page 148 (equal to O g F′ in fig. 68, page 88), which the normal spiral makes with the pitch-circle; or it may be otherwise defined as the angle which the generating line of a tooth on the pitch-surface makes with the generating line of a tangent cone at a given point, I. From the rule for finding that angle (Article 106, page 89), it is evident that, with a given hyperboloïdal pitch-surface, the transverse obliquity of the teeth is greatest at the throat, and is the less the farther the middle pitch-circle of the wheel is removed from the throat. Hence, it is generally advisable, in designing skew-bevel wheels, to place the pitch-circles as far as practicable from the throats of the hyperboloïds, because obliquity of action tends to increase friction.

148. **Skew-bevel Wheels in Double Pairs.**—Skew-bevel wheels possess a property which ordinary bevel wheels do not—viz., that of being capable of combination by *double pairs*, as in fig. 106. The upper part of the figure represents a projection on a plane parallel to the line of contact, I I′, and to the common perpendicular of the axes F G. The lower part of the figure represents a projection on a plane normal to the common perpendicular. Small letters in the second projection correspond to capital letters in the first projection.

B and B′ are two equal and similar wheels fixed on the shaft A A′, with pitch-surfaces forming parts of the same hyperboloïd, and at equal distances from its throat. They have equal and similar teeth, with equal obliquities in the same direction; and, in short, both wheels may be cast from the same pattern. In the example given, the teeth of both wheels are right-handed. In like manner, D and D′ are two equal and similar wheels fixed on the shaft C C′; B gears with D, and B′ with D′.

This arrangement may be useful where it is desired, for the sake of strength or of steadiness of motion, to divide the force exerted in transmitting the motion between two pairs of wheels.

149. **Teeth with Sloping Backs.**—The teeth described in the preceding Articles of this Section have their backs similar to their fronts, so that the motion of the wheels may be reversed, the backs then acting as the fronts did during the forward motion. There are many cases in mechanism in which it is not necessary that the

motion of the wheels should ever be reversed; and in such cases the backs of the teeth of a pair of wheels are required simply to be

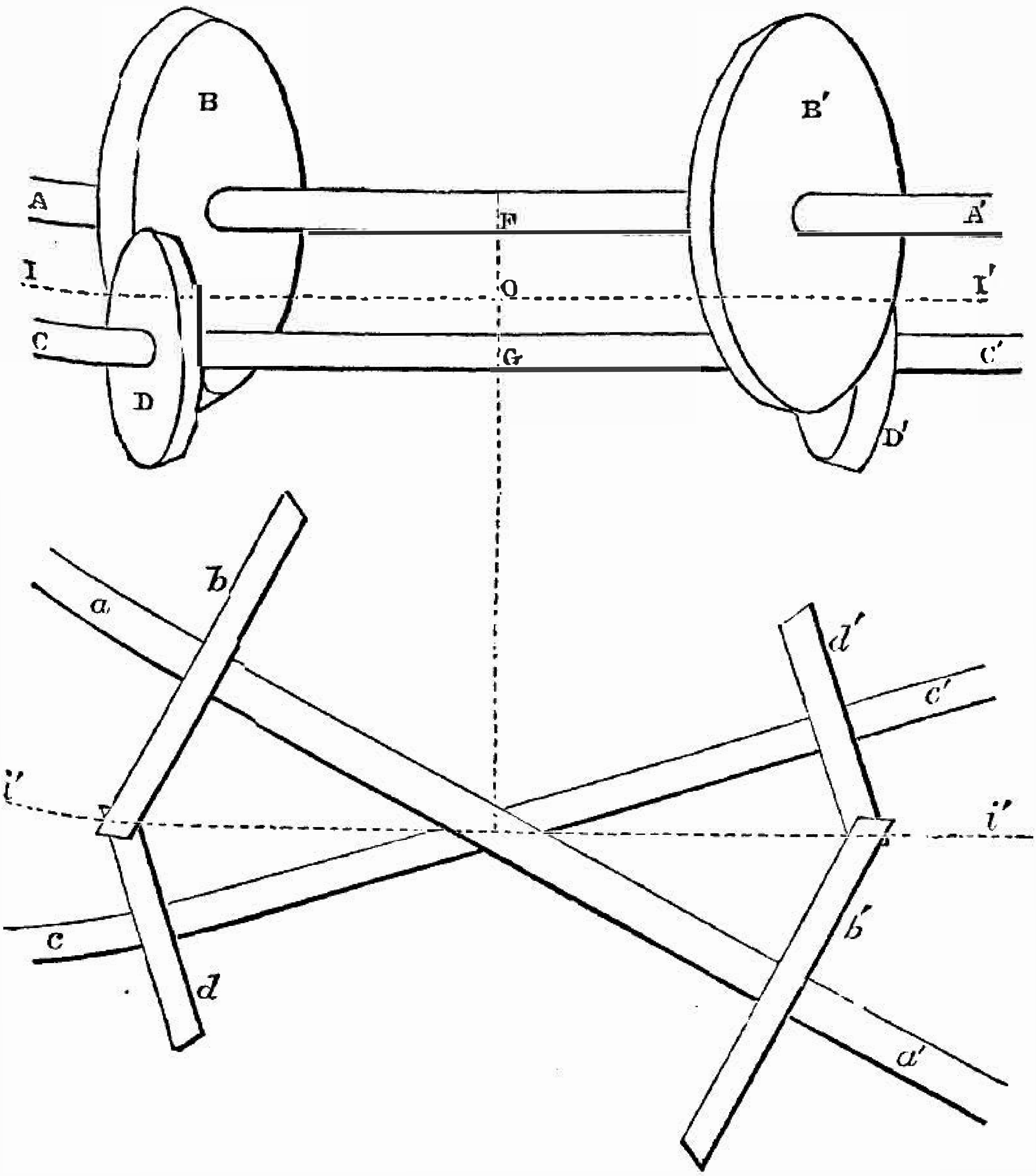

Fig. 106.

of such shapes as to clear each other, without reference to the transmission of motion. The consequence of this is, that although the traces of the backs must still belong to the same class of curves with the traces of the fronts, their *obliquity* may be considerably increased, the effect being to strengthen the teeth at their roots.*

The most convenient curves for the traces of the backs of teeth under those circumstances are involutes of a circle, for which there may be substituted in practice circular arcs approximating to them;

* This was first pointed out by Professor Willis.

and the method of drawing those arcs is as follows:—Let fig. 107 represent the trace of part of a wheel with its teeth, that wheel being the smallest wheel of a set that are to be capable of gearing together; because the smallest wheel of such a set requires the greatest addendum: let C be the centre, A A the addendum-circle,

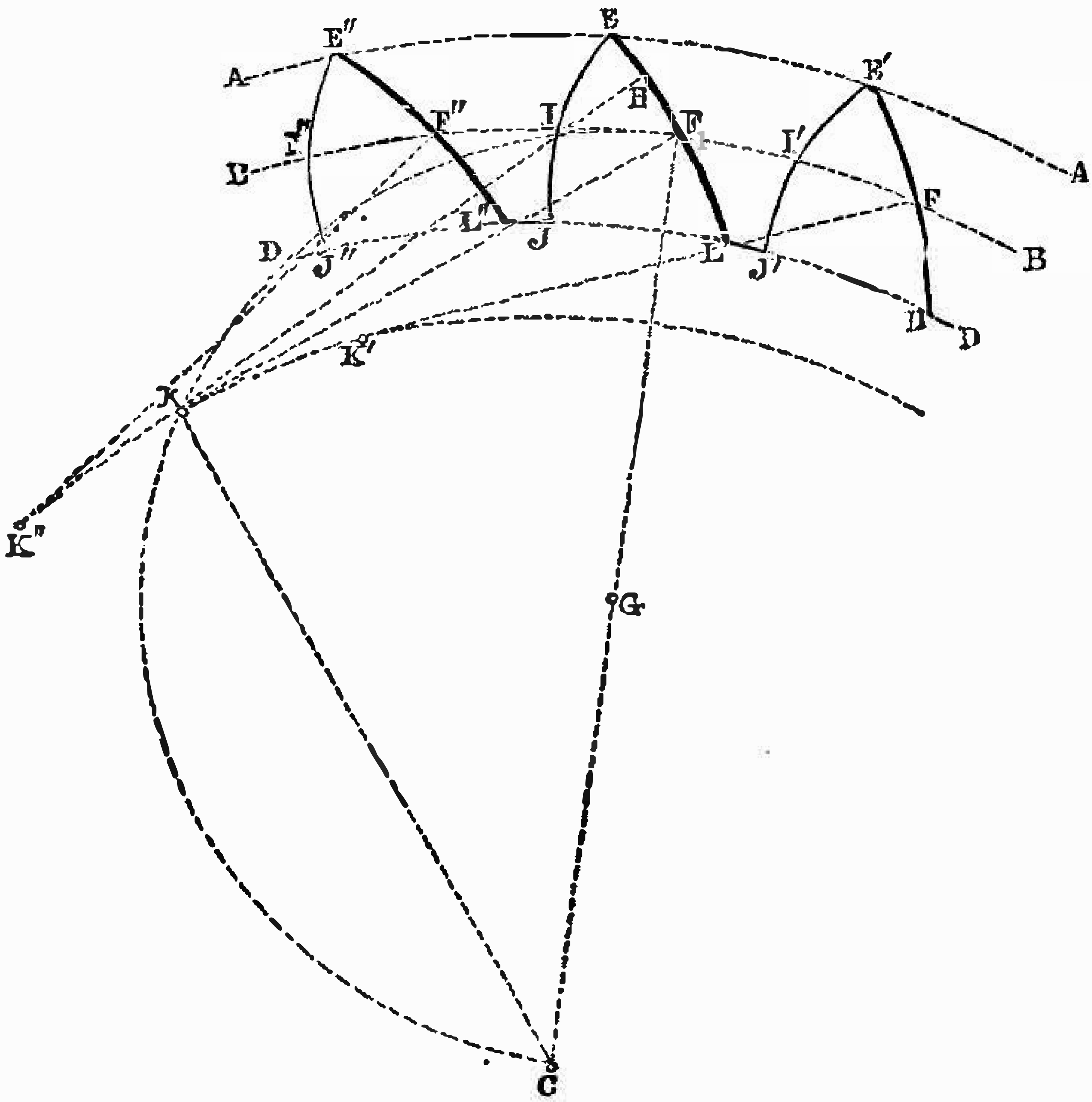

Fig. 107.

B B the pitch-circle, D D the root-circle, and let E'' I'' J'', E I J, E' I' J', be the fronts of teeth designed according to the proper rules, and F'', F, F', the pitch-points of the backs of those teeth. To any one of those *back pitch-points*, as F, draw the radius C F; bisect C F in G, and about G draw the semicircle F K C. Draw a straight line, H K, perpendicular to and bisecting the distance, E F, between the crest E and back pitch-point F; and let that straight line cut the semicircle in K. About the centre C, with the radius C K, draw the circle K'' K K'; this will be the *base-circle* of the required involutes (see Article 131, page 121).

To draw the circular arcs approximating to those involutes, lay off, from the back pitch-points to the base-circle, the equal distances F'' K'' = F' K' = F K, &c; and about the respective centres, K, K′, K″, &c., draw the circular arcs E F L, E′ F′ L′, E″ F″ L″, &c.

In each of the larger wheels of the set, the radius of the base-circle for the backs is to bear to the radius of the pitch-circle the constant proportion $\frac{CK}{CF}$, in order that the backs of the teeth of all the wheels of the set may have the same obliquity—viz., the angle K C F.

In a straight rack capable of gearing with any wheel of the set, the traces of the backs of the teeth are to be straight lines, making with the pitch-line an angle equal to C F K.

150. **Stepped Teeth.**—In order to increase the smoothness of the action of toothed wheels, Dr. Hooke invented the making of the fronts of teeth in a series of steps, as shown in fig. 108, where the

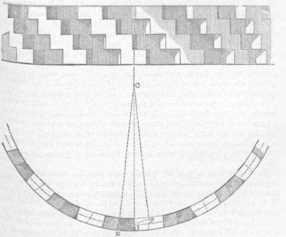

Fig. 108.

Fig. 108 A.

upper part of the figure is a projection of the rim of a wheel with stepped teeth on a plane parallelhto the axis, and the lower part is a projection on a plane perpendicular to the axis. A wheel thus formed resembles in shape a series of equal and similar toothed discs placed side by side, with the teeth of each a little behind those of the preceding disc. In such a wheel, let p be the circular pitch, and n the number of steps. Then the path of contact, the addendum, and the extent of sliding, are those due to the *divided* pitch $\frac{p}{n}$, while the strength of the teeth is that due to the thickness corresponding to the *total* pitch p; so that the smooth action of small teeth and the strength of large teeth are combined. The action of small teeth is smoother and steadier than that of large teeth, because they can be made to approximate more closely to the exact theoretical figure; and also because the sliding motion of one tooth upon another is of less extent. In the example shown in fig. 108 there are four steps, so that the divided pitch is one-fourth of the total pitch; and the path of contact (E I F, in the lower part of the figure) is of the length suited to the divided pitch, being only one-fourth of the length which would have been required had the fronts of the teeth not been stepped.

151. **Helical Teeth**, also invented by Dr. Hooke with the same object, are teeth whose fronts, instead of being parallel to the line of contact of the pitch-cylinders of a pair of spur-wheels, cross that line obliquely, so as to be of a screw-like or helical form: in other words, they are teeth of the figure of short portions of *screw-threads* (Article 58, page 36); the trace of each thread on a plane perpendicular to the axis being similar to that of a stepped tooth, as shown in the lower part of fig. 108. Fig. 108 A shows a projection of the rim of a wheel with helical teeth on a plane parallel to the axis.

In order that a pair of wheels with parallel axes and helical teeth may gear correctly together, the teeth, besides being of the same circular pitch, must have the same transverse obliquityh and if in outside gearing, they must be right-handed on one wheel and left-handed on the other. If in inside gearing, they must be either right-handed or left-handed on both wheels. In fig. 108 A the teeth are left-handed. In wheel-work of this kind the contact of each pair of teeth commences at the foremost end of the helical fronts, and terminates at the aftermost end; and the rims of the wheels are to be made of such a breadth that the contact of one pair of teeth shall not terminate until that of the next pair has commenced.

Helical teeth are open to the objection that they exert a laterally oblique pressure, which tends to increase friction.

When, in designing a skew-bevel wheel, a portion of the tangent cylinder at the throat of the hyperboloïd (Article 106, page 87;

and Article 85, page 73) is used as an approximation to the true pitch-surface, the teeth of that wheel become screw-threads, having a transverse obliquity determined by the principles of Article 147, page 152; and, as has been already stated in the article referred to, they are either right-handed or left-handed in both wheels.

152. **Screw and Nut.**—The figure of a true screw, external or internal, and the motion of a screw working in a corresponding screw-shaped bearing, have been described in Articles 57 to 66, pages 36 to 42. In the elementary combination of an *external and internal screw,* more commonly called a *screw and nut,* the two pieces have threads, one external and the other internal, of similar figures and equal dimensions, so as to fit each other truly; and one of them turns about their common axis without translation, while the other slides parallel to that axis without rotation. The best form of section for the threads is rectangular. The comparative motion is, that the sliding piece advances through a distance equal to the pitch (viz., the "*total axial pitch*") during each revolution of the turning piece. If the threads are $\left\{\begin{matrix}\text{right-handed,}\\ \text{left-handed,}\end{matrix}\right\}$ the sliding piece is made to move towards an observer at one end of the axis by $\left\{\begin{matrix}\text{right-handed}\\ \text{left-handed}\end{matrix}\right\}$ rotation, and to move from him by $\left\{\begin{matrix}\text{left-handed}\\ \text{right-handed}\end{matrix}\right\}$ rotation, of the turning piece. The combination belongs to Mr. Willis's Class A, because the velocity-ratio is constant; and the extent of the motion is limited by the length of the screw.

153. **Screw Wheel-Work in General.**—Screw wheel-work consists of wheels with cylindrical pitch-surfaces, having screw-threads or helical teeth instead of ordinary teeth. One case of screw-gearing has been described in Article 151, page 156—viz., that in which the axes are parallel. The cases to which this and the following articles relate are those in which the axes are not parallel; so that the pitch-surfaces in an elementary combination are a pair cylinders touching each other in *one pitch-point,* like those represented in Article 85, fig. 55, page 73. The pitch-point (O′, fig. 55) is obviously in the common perpendicular of the two axes (F′ G′, fig. 55); and there is one straight line traversing the pitch-point (O′ C′, fig. 55), which is a tangent at once to the two pitch-cylinders and to the acting surfaces or fronts of each pair of threads at the instant when those surfaces touch each other at the pitch-point: that straight line may be called the LINE OF CONTACT. The *angles of inclination* of the screw-threads to the two axes (see Article 63, page 40) are equal respectively to the angles made by the line of contact with those axes. The PITCH-CIRCLES of the two screws are the two circular sections of the pitch-cylinders which traverse the pitch-point. The PLANE OF CONNECTION, OR PLANE OF

ACTION, is a plane traversing the pitch-point normal to the line of contact: that plane, of course, traverses the common perpendicular of the axes.

When the line of contact is found by the rule given in Article 84, page 71, the cylindrical pitch-surfaces represent the tangent-cylinders at the throats of a pair of hyperboloïds; and the screw-threads are approximations to the skew-bevel teeth suited for that combination, as already stated in Article 151, page 156. But in many cases the line of contact has positions greatly differing from this; and then the comparative motion becomes different from that of a pair of skew-bevel wheels; the object of screw-gearing in such cases being to obtain, with a given pair of cylindrical pitch-surfaces, a velocity-ratio of rotation independent of the radii of those surfaces; and such is the difference between approximate skew-bevel gearing and screw gearing in general.

In every elementary combination in screw wheel-work, each of the two pieces is at once a screw and a wheel; but it is customary, when their diameters are very different, to call that which has the smaller diameter the ENDLESS SCREW, or WORM, and that which has the greater diameter the WORM-WHEEL. For example, in fig. 111 (farther on) a' is the worm, or endless screw, and A' the worm-wheel. The word "endless" is used because of the extent of the motion being unlimited.

Screw wheel-work belongs to Mr. Willis's Class A, the velocity-ratio being constant.

The following are the general principles of elementary combinations in screw wheel-work:—

I. The angular velocities of the two screws are inversely, and their times of revolution directly, as the numbers of threads; whence it follows that the angular velocity-ratio must be expressible in whole numbers, as in the case of ordinary toothed wheels.

II. The *divided normal pitch* (see Article 66, page 42), as measured on the pitch-cylinders, must be the same in two screws that gear together.

III. The *common component* of the velocities of a pair of points in the two screws at the instant when those two points touch each other and pass the pitch-point, is perpendicular to the line of contact and to the common perpendicular of the axes; in other words, it coincides with the intersection of the plane of connection and the common tangent-plane of the two pitch-cylinders.

IV. The *circular* or *circumferential pitches* of the two screws (Article 42, page 66), as measured on their pitch-cylinders, are proportional to the total velocities of points (called the *surface velocities*) in those cylinders; and they bear the same proportion to the divided normal pitch that those total velocities bear to their common component.

V. The *relative transverse sliding* of a pair of threads that are in action takes place along the line of contact.

It will be shown in the next article that for a given pair of axes and a given angular velocity-ratio the relative transverse sliding is least when the pitch-cylinders are the tangent-cylinders at the throats of a pair of skew-bevel hyperboloïds.

154. **Screw Wheel-Work—Rules for Drawing.**—In figs. 109 and 110 the plane of projection is supposed to be the common tangent-plane of the two pitch-cylinders; and I represents the pitch-point; which is also the trace and projection of the common perpendicular of the two axes.

I. *Given, the projections of the two axes, the angular velocity-ratio, and the radii of the two pitch-cylinders, to find the proportionate values of their surface-velocities, and the proportionate value and direction of the velocity of transverse sliding.* The two cylinders may be called respectively A and a.

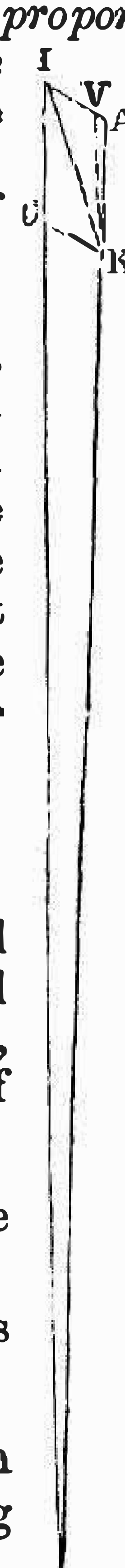

Fig. 109.

In fig. 109, let I A and I a represent the projections of the two axes. Along those projections lay off lengths I A, I a, proportional to the two angular velocities of rotation, and pointing in the direction in which an observer must look from I in order to make both rotations seem right-handed. Draw the straight line A a, and divide it at K into two parts inversely proportional to the radii of the two pitch-cylinders; in other words, let B and b denote the radii of the cylinders A and a respectively, so that B + b is the length of the common perpendicular, or line of centres; and let K divide Aa in the following proportion:—

$$B + b : B : b$$
$$:: Aa : Ka : KA$$

Complete the parallelogram I V K v; then I V, I v, and the diagonal I K, will be respectively proportional and perpendicular to the surface velocity of the cylinder A, the surface velocity of the cylinder a, and the velocity of relative transverse sliding at the pitch-point I.

Or otherwise, by calculation; let $\frac{a}{A}$ be the ratio of the angular velocities, and $\frac{b}{B}$ that of the radii; then $\frac{a\,b}{A\,B}$ is obviously the ratio of the surface velocities.

It is obvious that for a given pair of axes and a given pair of angular velocities the velocity of transverse sliding is least when I K is perpendicular to A a. But A a is parallel to the line of contact of a pair of hyperboloïdal pitch-surfaces for skew-bevel wheels having the given velocity-ratio; and this is the demonstration of the statement in the preceding article, that screws which coincide approximately with skew-bevel wheels give the least possible transverse sliding of the threads for a given pair of axes and a given velocity-ratio (see page 159).

The proportionate value of the *common component of the surface velocities* may be represented by the length of a perpendicular let fall from either V or v upon I K; but the next rule gives a more convenient way of representing both it and the transverse sliding.

II. *To draw the line of contact, and to find the proportions borne to the surface velocities by their common component, and by the transverse sliding; also the proportions borne to each other by the circular pitches, the divided axial pitches, and the divided normal pitch.*

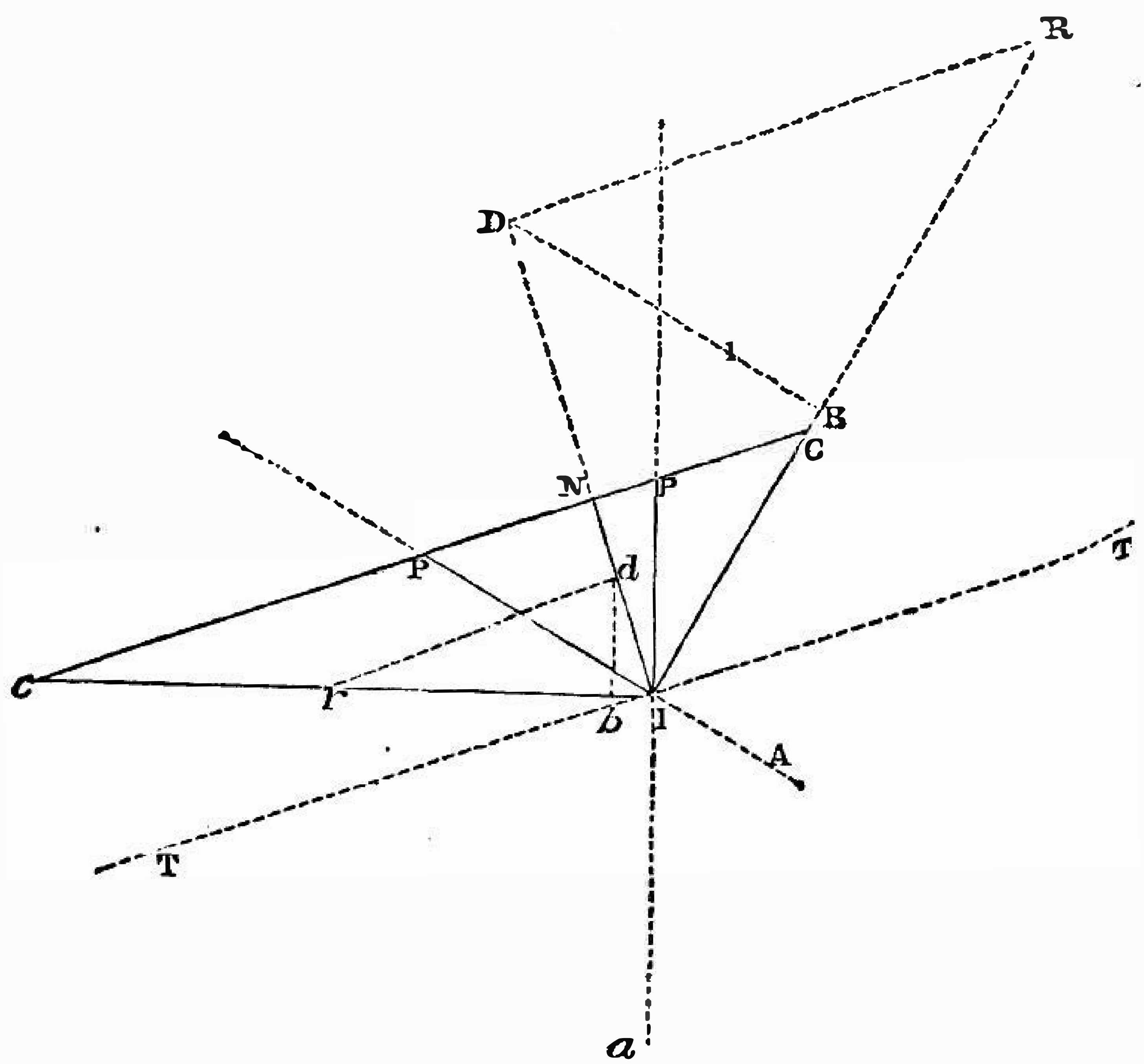

Fig. 110.

In fig. 110 (as in fig. 109), let I represent the pitch-point, and I A and I a the projections of the two axes. Perpendicular to I A and I a respectively, draw I C and I c, of the proper lengths, and in the proper directions, to represent the surface velocities of the two pitch-cylinders at the point I; draw the straight line C c, cutting the projections of the two axes in P and p respectively;

and upon C c let fall the perpendicular I N (which will obviously be parallel to I K in fig. 109). Through I draw T I T parallel to C c.

Then T I T will be *the line of contact;* I N will represent *the common component of the surface velocities* (and will also be the trace of the plane of connection); C c will represent *the velocity of transverse sliding;* and the proportions of the several divided pitches will be as follows:—

I C : circular pitch of A.
: : I c : circular pitch of a.
: : I P : divided axial pitch of A.
: : I p : divided axial pitch of a.
: : I N : divided normal pitch of both screws.

The figure may be regarded as part of the *development* of both screws upon the common tangent plane of their pitch-cylinders. (See Article 63, page 40. As to RACKS, see Addendum, page 289.)

The *absolute lengths* of the circular pitches are found by dividing the pitch-circles into suitable numbers of equal parts, precisely as in the case of spur-wheels (see Articles 112 to 121, pages 103 to 114); and from them, by the aid of the proportions given by fig. 110, the absolute lengths of the divided axial pitches and of the divided normal pitch are easily found. For the total axial pitch of either screw, multiply the divided axial pitch by the number of threads.

III. *To find the radii of curvature of the normal screw-lines.* The normal helix, or normal screw-line (see Article 65, page 41), of each of the two screws touches I N at the pitch-point I; and the plane of connection of which I N is the trace is the common osculating plane of the two normal screw-lines at I. Their radii of curvature at that point both coincide with the common perpendicular of the axes. The rule for finding such radii (Articles 64 and 65, page 41), when applied to this case, takes the following form:—On I C lay off I B to represent the radius of the pitch-cylinder A; then perpendicular to I C draw B D parallel to I A, cutting I N in D; then perpendicular to I N draw D R, cutting I C in R; I R will be the radius of curvature of the normal helix of the screw A. A similar construction, substituting small for capital letters, serves to find I r, the radius of curvature of the normal helix of the screw a.

Fig. 111 represents two projections of the pitch-cylinders of a pair of screws designed by the rules which have just been given, and shows also the helical lines in which the fronts of the threads cut those pitch-cylinders. The upper part of the figure is a projection on the plane of action, whose trace, in fig. 110, is I N. A′ a' is the common perpendicular of the two axes, and I′ the pitch-point; N′N′ is the trace of the common tangent plane of the two pitch-cylinders; and the arrow shows the direction of the common component of their surface velocities at the point I′. R and r are the centres of curvature of the two normal screw-lines at the point I′; and S S and $s\ s$, described about R and r respectively, are their two osculating circles, whose radii, I′ R and I′ r, are found by Rule III.

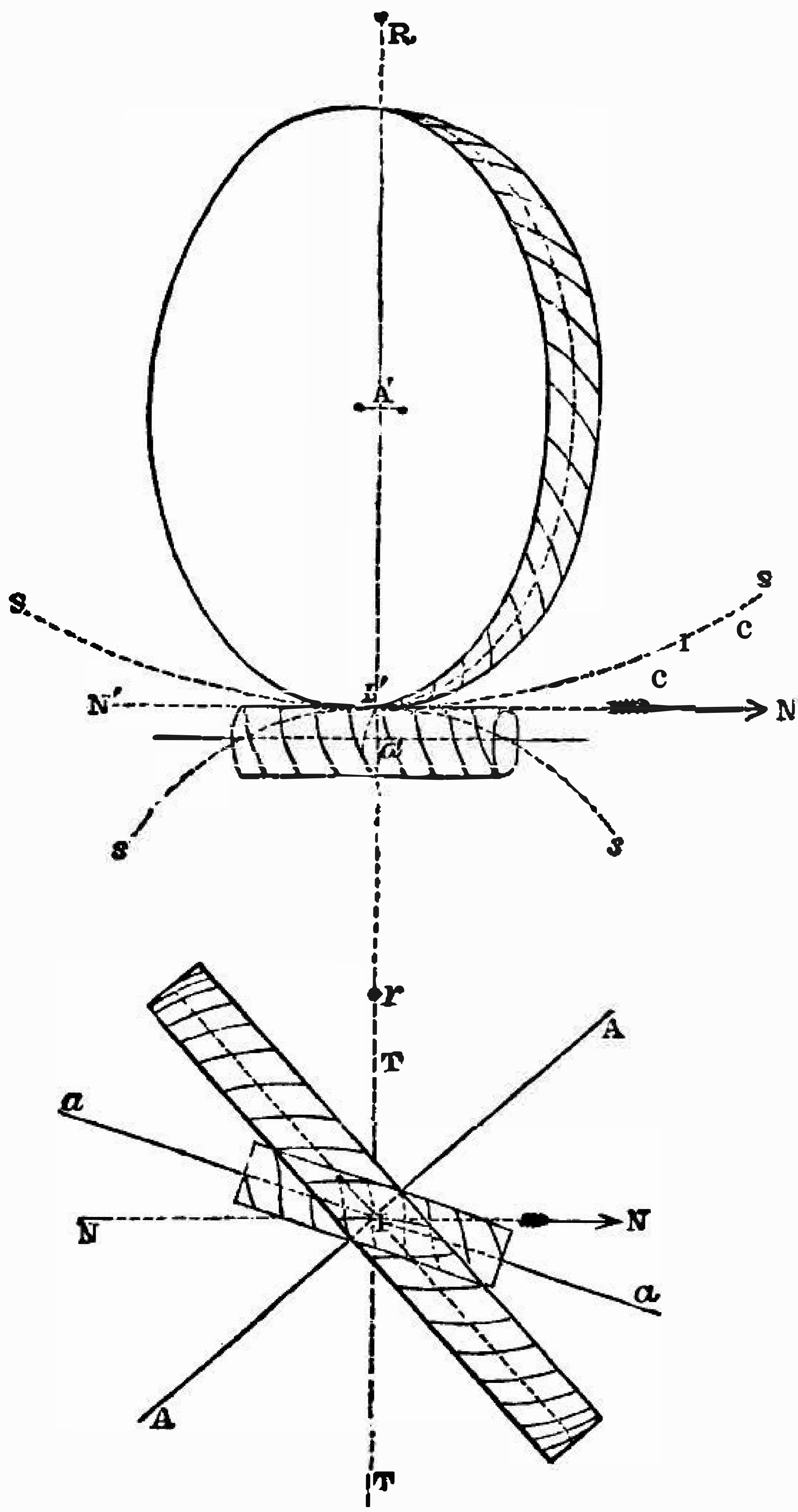

Fig. 111.

The lower part of the figure is a projection on the common tangent plane of the pitch-cylinders. A A and $a\ a$ are the projections of their two axes; T I T is the line of contact; N I N is the trace of the plane of action; and the arrow marks the direction of the common component of the surface velocities at the pitch-point I.

In the particular example represented by figs. 109, 110, and 111, the following are the principal data and proportions:—

$$\text{Velocity-ratio } \frac{a}{A} = 20;$$

Number of threads of A, 40; of a, 2;

Ratios of radii and line of centres,

$$B + b : B : b$$
$$:: \ 11 \ : 10 \ : 1$$

Both screws right-handed.

155. **Figures of Threads found by Means of Normal Screw-Lines.**—By the following process threads may be designed for any gearing screw, so that they shall gear correctly with threads designed on the same principle for any other screw of the same normal pitch.

Let the screw to be provided with threads be, for example, the screw A of fig. 111. Draw, by Rule III. of Article 154, page 161, the osculating circle, S I′ S, of its normal screw-line. Lay off the normal pitch upon that osculating circle, and design the figure of a tooth and two half-spaces of that pitch, with the proper addendum and depth, as if the osculating circle were the pitch-circle of a spur-wheel; the figure so drawn will be the *normal section* of a thread, being the trace of the thread upon a surface which cuts it at right angles; and by the help of that section the threads may be made of the correct figure.

The normal sections of the acting surfaces of a thread may be either involutes of circles (Articles 131, 133, pages 120 to 128), or epicycloïds (Articles 136 to 140, pages 130 to 137). All screws with *involute threads* of the same divided normal pitch gear correctly together, and may be said to belong to *one set;* and they have the same property with involute toothed wheels, of admitting of some alteration of the distance between the axes. All screws of the same divided normal pitch having epicycloïdal teeth described by the same rolling circle gear correctly together, and may be said to belong to *one set*.

This method of designing the threads of gearing screws is believed to be now published for the first time.

156. **Figures of Threads designed on a Plane Normal to one Axis.**—In many cases which occur in practice the axes of the two screws are perpendicular to each other; so that, in fig. 110, page 160, A I P and *a* I *p* are at right angles, I C coincides with I*p*, and I *c* coincides with I P; and therefore the *divided axial pitch of either screw is equal to the circular pitch of the other*. In such cases, and especially where the diameters of the pitch-cylinders are very unequal, so that the larger screw is called a *worm-wheel*, and the smaller an *endless screw,* it is often convenient to design the traces of the threads on a plane normal to the axis of the worm-wheel, and traversing the axis of the endless screw; and then it is evident (as Mr. Willis appears to have been the first to show) that if the traces of the threads of the worm-wheel be made like those of a spur-wheel of the same radius and pitch, and those of the threads of the screw like the traces of the teeth of a rack suited to gear

with that spur-wheel, the worm-wheel and screw will gear correctly together.

Fig. 112 represents a worm-wheel and endless screw.

The lower part of the figure is a diagram drawn on the common tangent plane of the pitch-cylinders. I is the pitch-point; I C is

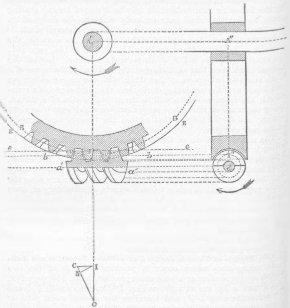

Fig. 112.

the divided axial pitch of the endless screw, being also the development of the circular pitch of the worm-wheel; I c is the divided axial pitch of the worm-wheel, being also the development of the circular pitch of the endless screw. I N, perpendicular to C c, is the development of the divided normal pitch of both screws; and C c is the extent of transverse sliding which takes place while an arc equal to the pitch passes the pitch-point.

In the left-hand division of the upper part of the figure the plane of projection is normal to the axis, A′, of the worm-wheel, and traverses the axis, a' a', of the endless screw. The circle B B is the trace of the pitch-cylinder of the wheel; the straight line b b is the trace of the upper side of the pitch-cylinder of the screw; and those traces touch each other in the pitch-point I′. The threads of the wheel, and those at the upper side of the screw, are shown in section; the traces of the threads of the wheel are like those of the teeth of a spur-wheel having the same circular pitch, and B B for a pitch-circle; the traces of the threads of the screw are like those of the teeth of a rack suited to gear with that spur-wheel, and having b b for its pitch-line. The addendum-circle, E E, of the worm-wheel, and the addendum-line, e e, of the endless screw, are drawn as for a spur-wheel and rack. The lower parts of the threads of the endless screw are shown in projection. In the example given, both wheel and screw have right-handed threads; the number of threads of the screw is two; of the wheel, 40; and the screw is represented as driving the wheel. The right-hand division of the upper part of the figure shows the wheel in section and the screw in projection; and the plane of projection traverses the axis, A″, of the wheel, and is normal to the axis, a'', of the screw; I″ is the pitch-point.

The traces of the threads of the wheel in the left-hand division of the upper part of the figure are involutes of a circle, and those of the threads of the screw are straight lines. That shape, as in the case of spur-wheels, enables the distance between the axes to be varied to a certain extent without affecting the accuracy of the action. But any shapes suited for the teeth of wheels and racks may be employed.

If a set of worm-wheels be made of the same circular pitch and obliquity of thread, and having the traces of the threads all involutes or all epicycloïds, traced by the same rolling circle; and if a set of endless screws be made, all of the same divided axial pitch, equal to the circular pitch of the wheels, and of an obliquity of thread equal to the complement of the obliquity of the threads of the wheels, and having the traces of the teeth, as the case may be, all straight lines of the proper obliquity, or all epicycloïds traced by the same rolling circle that is used to trace the threads of the wheels, then any one of the wheels will gear correctly with any one of the screws.

157. **Close-Fitting Tangent Screws.**—In many cases the object of screw-gearing is not the economical transmission of motive power, but the production of small angular motions with great accuracy: as, for example, when the principal wheel of a dividing engine, or that of a machine for pitching and cutting the teeth of wheels, or the wheel or sector which adjusts the direction of stroke of a

cutting tool in a shaping machine, is driven by a "tangent-screw" situated relatively to the wheel in the manner already shown in fig. 112. In such cases the screw has not only to move the wheel into any required position, but to hold it there; and therefore it is essential that there should be no back-lash. In order to ensure this, together with the requisite precision of action, an exact copy of the tangent-screw is made of steel, the edges of its threads are notched, and it is hardened, so that it becomes a cutting tool: it is then mounted in a suitable frame, so as to gear with the roughly formed teeth or threads of the wheel, and turned so as to drive them; in the course of which operation it cuts them to the proper figure. The axis of the cutting screw is placed at first at a distance from the axis of the wheel somewhat greater than the intended permanent distance; and after each complete revolution of the wheel the axes are brought a little nearer together, until the permanent distance is attained; and by turning the screw in this last position the shaping of the teeth or wheel-threads is finished. From the property of threads with traces similar to those of involute teeth, which has already been mentioned in Article 156, page 165, it is evident that this class of figures is peculiarly well suited to cases in which the tangent-screw is made to cut the wheel, because of the gradual diminution of the distance between the axes which takes place during the process of cutting.

158. **Oldham's Coupling.**—A *coupling* is a mode of connecting a pair of shafts so that they shall rotate in the same direction, with the same mean angular velocity. If the axes of the shafts are in the same straight line, the coupling consists in so connecting their contiguous ends that they shall rotate as one piece; but if the axes are not in the same straight line, combinations of mechanism are required. Various sorts of couplings will be described and compared together in a later division of this treatise. The present Article relates to a coupling for parallel shafts, invented by Oldham, which acts by *sliding contact*. It is represented in fig. 113. C_1, C_2 are the axes of the two parallel shafts; D_1, D_2, two discs facing each other, fixed on the ends of the two shafts respectively; E_1 E_1, a bar sliding in a diametral groove in the face of D_1; E_2 E_2, a bar sliding in a diametral groove in the face of D_2: those bars are fixed together at A, at right angles to each other, so as to form a rigid cross. The angular velocities of the two discs and of the cross are all equal at every instant; the middle point of the cross, at A, revolves in the dotted

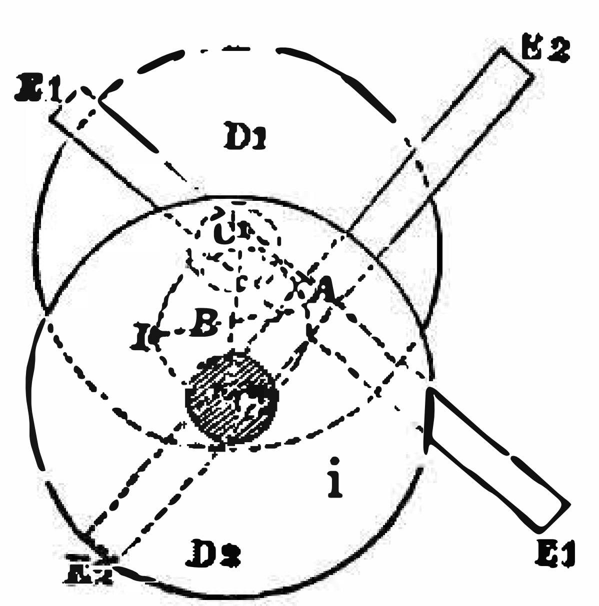

Fig. 113.

circle described upon the line of centres, C_1 C_2, as a diameter, twice for each turn of the discs and cross; the instantaneous axis of rotation of the cross at any instant is at I, the point in the circle C_1 C_2 diametrically opposite to A; and each arm of the cross slides in its groove through a distance equal to *twice the line of centres* during each half revolution, or twice the line of centres and back again—that is, four times the line of centres—during each revolution.

Oldham's coupling belongs to Mr. Willis's Class A. The cross may be strengthened by making its two bars take the form of projecting diametral ridges on opposite sides of a third circular disc. Or the cross may consist of two grooves in the opposite sides of such a disc, and instead of grooved discs, the two shafts may carry cross bars fitting the grooves of the cross.

159. **Pin and Straight Slot.**—The communication of a uniform velocity-ratio by the sliding contact of a round pin with the sides of a slot or groove has already been described in Article 141, page 137. A velocity-ratio varying in any manner may be communicated by making the slot of a suitable figure, the principle of the combination being, that the line of connection is a normal to the centre line of the slot, traversing the centre line of the pin. The present Article relates to cases in which the slot is straight and the velocity-ratio variable. Three such cases are illustrated by figs. 114, 115, and 116, further on. Fig. 114 represents a *coupling*, belonging to Mr. Willis's Class B, where two shafts turn about the parallel axes A and B with equal mean angular velocities, though the angular velocity-ratio at each instant is variable. Fig. 115 shows a crank turning continuously about the axis A, and carrying a pin, C, which, by means of the slot F G, drives a lever which rocks or oscillates about the axis B. Fig. 116 shows a crank turning continuously about the axis A, and carrying a pin, C, which, by means of the slot F G in the cross-head of the rod B, gives a reciprocating sliding motion to that rod. The last two combinations belong to Mr. Willis's Class C.

In practice, for the purpose of diminishing friction and preventing back-lash, it is usual to make the pin turn in a bush which slides in the slot; but that bush is not shown in the figures.

The following are the principles of the action of those three combinations:—

I. *Coupling* (fig. 114).—In order that the directional relation of the rotations may be constant, the *crank-arm*, A C, must be greater than the line of centres, A B.

With a given crank-arm, A C, to find *the position of the axis* B *of the slot-lever*, so that the crank and slot-lever shall alternately overtake and fall behind each other by a given angle:—With the radius A C describe the circle D C E, and draw the diameter D A E, with which the line of centres is to coincide. Lay off

E A H = E A *h* = the complement of the given angle, and draw H B *h* perpendicular to D A E. B will be the trace of the required axis.

At the instant when the centre of the pin is at H or *h*, the angular velocities are equal; and A H B = A *h* B is the given angle beforementioned.

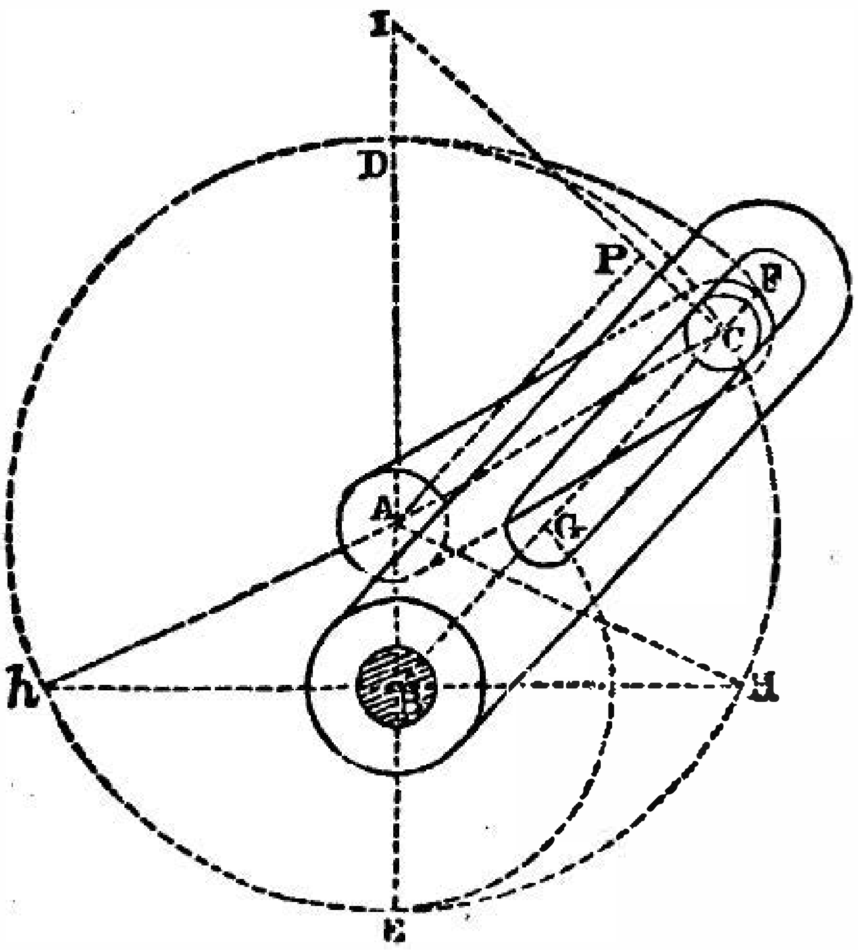

Fig. 114.

With a given position, C, of the centre of the pin, to find the *angular velocity - ratio*:—From C, perpendicular to the centre line, B C, of the slot, draw the *line* of connection, C I, cutting the line of centres in I; then

$$\frac{\text{Angular velocity of B}}{\text{Angular velocity of A}} = \frac{\text{A I}}{\text{B I}};$$

or otherwise: draw A P parallel to B C and perpendicular to C I; then

$$\frac{\text{Angular velocity of B}}{\text{Angular velocity of A}} = \frac{\text{A P}}{\text{B C}}.$$

Thee $\left\{\begin{matrix}\text{greatest}\\\text{least}\end{matrix}\right\}$ values of this ratio occur when the pin is at $\left\{\begin{matrix}\text{E}\\\text{D}\end{matrix}\right\}$ respectively; and they are as followse—

$$\text{Greatest, } \frac{\text{A E}}{\text{B E}} = \frac{\text{A C}}{\text{A C} - \text{A B}};$$

$$\text{Least, } \frac{\text{A D}}{\text{B D}} = \frac{\text{A C}}{\text{A C} + \text{A B}}.$$

The *travel* or *length of sliding of the pin in the slot* is

$$\text{F G} = \text{B F} - \text{B G} = \text{B D} - \text{B E};$$

and this takes place twice in each revolution.

II. *Crank and Slotted Lever* (fig. 115).—As the crank-arm, A C, in fig. 115, is shorter than the line of centres, A B, the slotted lever, B G F, has a reciprocating or rocking motion.

With a given line of centres, A B, and a given *semi-amplitude* or angular half-stroke of the rocking motion of the lever, A B K = A B *k*, to find *the length of crank-arm*:—From A let fall A K perpendicular to B K, or A *k* perpendicular to B *k*; A K = A *k* will be the required crank-arm.

K and *k* will be the two *dead points;* that is to say, the positions of the centre of the pin at the two instants when the lever has no velocity, having just ceased to move in one direction, and being just about to begin to move in the opposite direction.

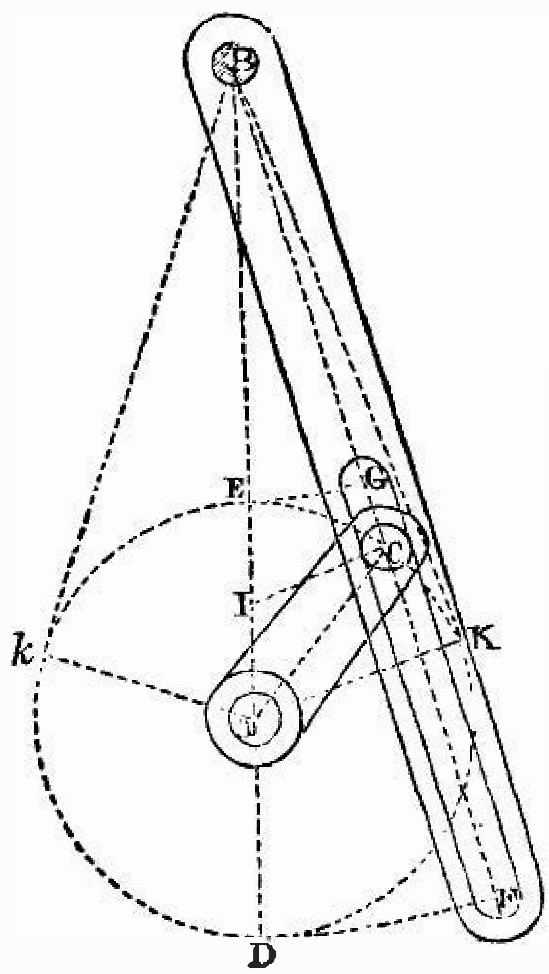

Fig. 115.

To find the *angular velocity-ratio* at the instant when the centre of the pin is in a given position, C:—Draw the corresponding position, B C F, of the centre line of the slot, and perpendicular to it draw C I, cutting the line of centres in I; then

$$\frac{\text{Angular velocity of lever}}{\text{Angular velocity of crank}} = \frac{\text{A I}}{\text{B I}}.$$

To find the travel of the pin in the slot, lay off B G = B E, and B F = B D; G and F will be the two ends of the travel of the centre of the pin; and F G = D E = 2 A C will be the length of travel.

III. *Crank and Slot-headed Sliding Rod* (fig. 116).—The crank-arm, A C, in this case is to be made equal to one-half of the intended *length of stroke* of the sliding rod, B. Draw the circle described by C, the centre of the pin, and let *k* A K be the diameter of that circle which is parallel to the direction of motion of the rod; then K and *k* will be the *dead points*, or positions of the centre of the pin at the two instants when the rod has no velocity. To find the *velocity-ratio* of the rod and crank-pin when the centre of the crank-pin is in a given position, C: perpendicular to the direction of motion of the rod draw the diameter D A E; this line will correspond to the line of centres in the preceding problems; then through C, and perpendicular to the centre-line, F G, of the slot, draw the line of connection, C I, cutting D A E in I; the following will be the required velocity-ratio:—

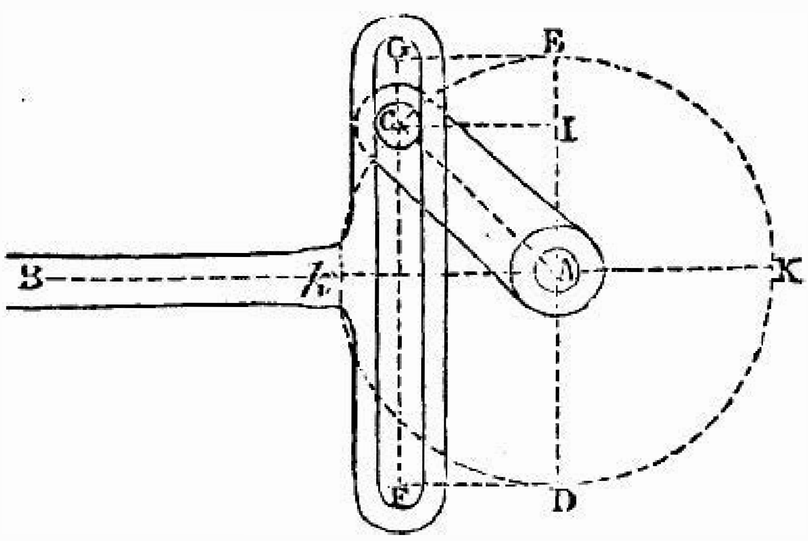

Fig. 116.

$$\frac{\text{Velocity of rod, B}}{\text{Velocity of centre of pin, C}} = \frac{\text{A I}}{\text{A C}}.$$

The *extent of travel of the pin in the slot* is F G = D E = 2 A C.

160. **Cams and Wipers in General.**—Cams and wipers are those primary pieces, with curved acting surfaces, which work in sliding contact without being related to imaginary pitch-surfaces, as the teeth of wheels and threads of screws are. The distinction between a cam and a wiper is, that a cam in most cases is continuous in its action, and a wiper is always intermittent; but a wiper is sometimes called a cam notwithstanding. A cam is often like a non-circular sector or wheel in appearance; a wiper is often like a solitary tooth. (As to "rolling cams," see Article 110, page 99.)

The solutions of all problems respecting the velocity-ratio and directional relation in the action of cams and wipers are obtained by properly applying the general principle of Article 122, page 114.

In most cases which occur in practice, the condition to be fulfilled in designing a cam or a wiper does not directly involve the velocity-ratio, but assigns a certain series of definite positions which the follower is to assume when the driver is in a corresponding series of definite positions. Examples of such problems will be given in the following Articles.

161. **Cam with Groove and Pin.**—Throughout the present Article it will be supposed that the acting surface of the follower, which is to be driven by the cam, is the cylindrical surface of a pin. It is easy to see that without in any respect altering the action, a cylindrical roller turning about a smaller pin may be substituted for a pin in order to diminish friction. If the pin is to be driven by the cam in one direction only, being made to return at the proper time by the force of gravity or by the elasticity of a spring, the cam may have only one acting edge; but if the pin is to be driven back as well as forward by the cam, the cam must have two acting edges, with the pin between them, so as to form a groove or a slot of a uniform width equal to the diameter of the pin, with clearance just sufficient to prevent jamming or undue friction. The centre of the pin may be treated as practically coinciding at all times with the centre-line of such a groove, which centre-line may be called the *pitch-line* of the cam. The most convenient way to design a cam is usually to draw, in the first place, its pitch-line, and then to lay off the half-breadth of the groove on both sides of the pitch-line. When one acting edge only is required, it is to be laid off on one side of a groove, the other side being omitted.

The *line of connection* at any instant is a straight line normal to the pitch-line at the centre of the pin.

The surface in which the groove is made may be either a plane or a surface of revolution; a plane for a *cam-plate* which either turns about an axis normal to its own plane or slides in a straight line, and acts upon a pin whose centre moves in a plane parallel to that of the cam-plate; a solid of revolution, being either a cylinder, a cone, or a hyperboloïd, for a cam which turns about an axis, and acts on a pin whose centre has a reciprocating motion in a straight line coinciding with a generating line of the surface of revolution.

The following example is a case of a rotating plane cam, giving motion through a pin and lever to a rocking shaft whose axis is parallel to the axis of rotation of the cam.

In fig. 117 the plane of projection is that of the cam-plate, and is normal to the axes of the cam and of the lever. In the lower part of the figure, A′ represents the trace of the axis of the rocking

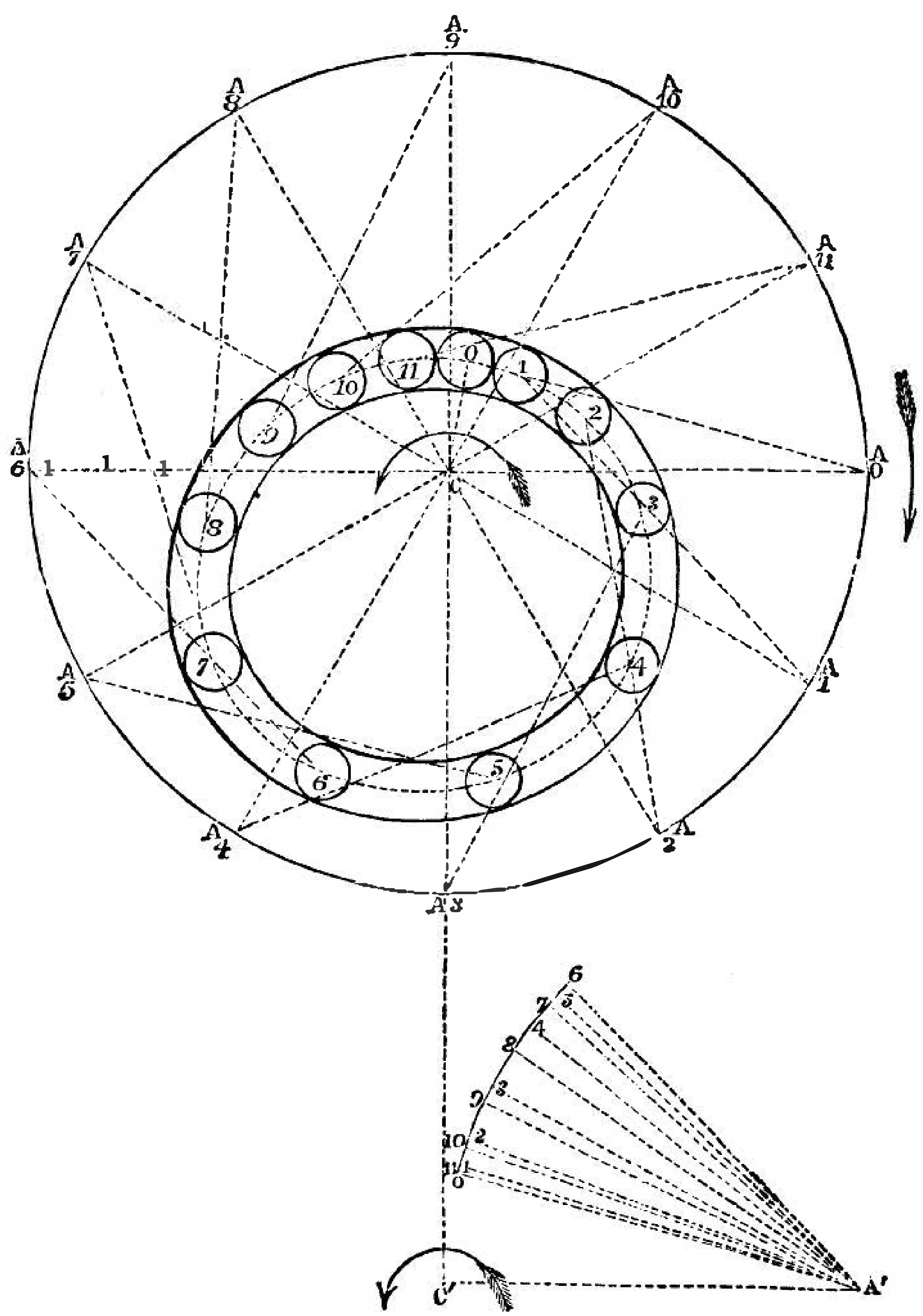

Fig. 117.

shaft, and C' the trace of the axis of the cam, so that $A' C'$ is the line of centres. The direction of rotation of the cam is shown by an arrow. In the example, the direction is left-handed. The circular arc, 0 6, described about A' with the radius $A' 0$, is the path to be described by the centre of the pin; and the twelve points in that arc, marked with numbers from 0 to 11, are twelve positions which the centre of the pin is to occupy at the end of twelve equal divisions of a revolution of the cam. It is required to find the form of the cam which will produce that motion in the pin.

In the upper part of the figure, let C represent the axis of the cam; suppose that the cam is fixed, and that the line of centres, C A, rotates about C, carrying the axis, A, of the rocking shaft along with it, with an angular velocity equal and contrary to the actual angular velocity of the cam. That supposition will not alter the relative motions of the working pieces. With the radius C A describe a circle to represent the supposed path of A relatively to C; divide its circumference into twelve equal parts, and to the points of division draw radii, $C A_0$, $C A_1$, $C A_2$, &c., to represent twelve successive positions of the line of centres relatively to the cam, as supposed to be fixed. Lay off the angles $C A_0$ 0, $C A_1$ 1, $C A_2$ 2, &c., in the upper part of the figure respectively, equal to the angles $C' A'$ 0, $C' A'$ 1, $C' A'$ 2, &c., in the lower part of the figure; and make each of the straight lines A_0 0, A_1 1, A_2 2, &c., equal to the lever arm A' 0. The points thus found, 0, 1, 2, &c., will be points in the pitch-line of the cam, and a curve drawn through them will be the required pitch-line.

About each of the points 0, 1, 2, &c., draw a circle of a radius equal to that of the pin: a pair of curves touching those circles so as to be parallel to the pitch-line will mark the two sides of the groove, without allowance for clearance. Clearance may be provided either by slightly diminishing the diameter of the pin or by slightly increasing the width of the groove. If the lever is to be raised by the cam, but brought down again by gravity, the outer side of the groove may be omitted, and the cam will become a disc bounded by the innermost of the three parallel curves shown in the figure.

The number of parts into which the revolution of the cam is divided may be made more or less numerous according to the degree of precision required.

It is easy to see how a similar method may be applied to the designing of a cam-disc which shall produce a given motion in a follower whose acting surface is of any given form. A figure is to be constructed like the upper part of fig. 117, on the supposition that the cam is fixed, and that the frame of the machine rotates about the axis of the cam with an angular velocity equal and contrary to the actual angular velocity of the cam. Then, just as the pin in the upper part of fig. 117 is drawn in its several positions, 0, 1, 2, &c., the trace of the acting surface of the follower is to be drawn in its several successive positions; and a line touching that trace in all its positions will be the trace of the required cam-disc.

The *dead points* of a cam are the points in its pitch-line which are at the greatest and least distances from its axis. In the example shown in fig. 117 the dead points are 0 and 6. When the centre of the pin is at those points it has no velocity. Any part of the pitch-line which is an arc of a circle about C corresponds to a *pause* in the motion of the pin.

162. **Drawing a Cam by Circular Arcs.**—In many cases in which cams have to be designed, the dead points alone are given by the conditions of the problem, leaving the parts of the pitch-line between those points to be drawn according to convenience. For example, in fig. 118,

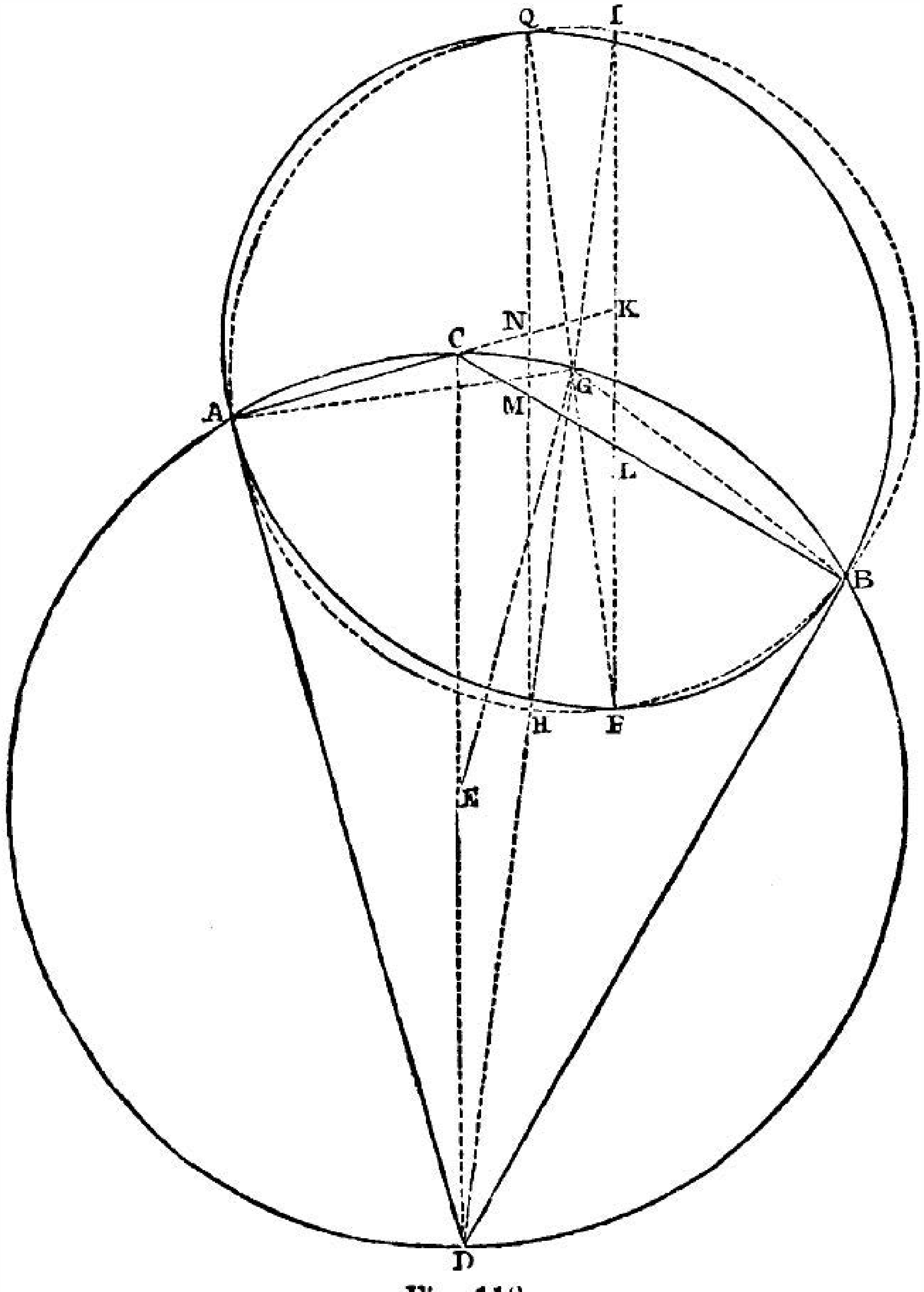

Fig. 118.

C is the axis of the cam, and A and B are dead points; so that C B and C A are respectively the least and greatest radii drawn from the axis to the pitch-line; and the pitch-line at A and B is normal to those radii respectively. The intermediate arcs of the pitch-line are to be drawn of any convenient form, so as to traverse A and B, and be normal to C A and C B.

The easiest way to draw such curves is by means of arcs of circles.

The simplest case is when C A and C B are parts of one straight line. The required pitch-line is then an *eccentric circle*, described upon the straight line A C B as a diameter.

When C A and C B, as in the figure, are not parts of one straight line, the following method may be used, being an extension of Rule IV. of Article 79, page 61, and having the effect of giving a pitch-line made up of four circular arcs, whose radii deviate less from equality than those of any other combination of four circular arcs which would answer the same purpose.

From A and B, perpendicular to A C and B C respectively, draw A D and B D, cutting each other in D. These will be tangents to the required pitch-line. Join C D; bisect it in E; and about E, with the radius E C = E D, describe a circle which will traverse the four points A, C, B, D. Bisect the arc A C B in G. About G, with the radius G A = G B, describe a circle; and draw the straight line D H G I, cutting that circle in H and I. Through the points H and I, and parallel to D C, draw the straight lines H Q and I P, cutting the circle A I B H in P and Q (the ends of one diameter), and cutting also the straight line C B in M and L, and the straight line A C produced in N and K. Then draw four circular arcs, as follows:—

The arc	A P,	described about	the point	K,
„	P B,	„	„	L,
„	B Q,	„	„	M,
„	Q A,	„	„	N;

and those arcs will make up a pitch-line having C B and C A for its greatest and least distances from the axis C, as required; and also having its radii of curvature less unequal than is possible with any other combination of four circular arcs, and no more, fulfilling the required conditions.

When a cam is to have more than two dead points, each pair of adjacent dead points are to be connected with each other by means of two circular arcs, drawn according to Rule IV. of Article 79, pages 61 and 62, fig. 48.

163. **Many-coiled Cams; Spiral and Conoidal Cams.**—When the complete series of movements of a piece that is to be driven by a cam extends over more than one revolution of the cam, there are

cases in which the required result may be effected by means of a groove in a cam-plate having a pitch-line of more than one coil; but difficulties in working may arise from the fact that the coils of the groove must intersect each other. There are other cases in which the motion required in the follower is of a kind that may be produced by means of a spiral cam, such as that shown in fig. 119. The upper part of the figure is a projection on a plane normal to the axis; the lower part, a projection on a plane parallel to the axis. A A′ is the spiral cam; B, a screw of an axial pitch exactly equal to the axial pitch of the cam. This screw, resting in a fixed nut, forms one of the bearings of the cam-shaft, and causes that shaft and cam together to advance along the axis at each revolution through a distance equal to the pitch, thus bringing a new coil of the cam into action. The cam, A, may also be made with a continuous conoidal surface, of which different parts are brought into action at each revolution by the advance caused by the screw B. It is evident that in spiral and conoidal cams the extent of the motion is limited.

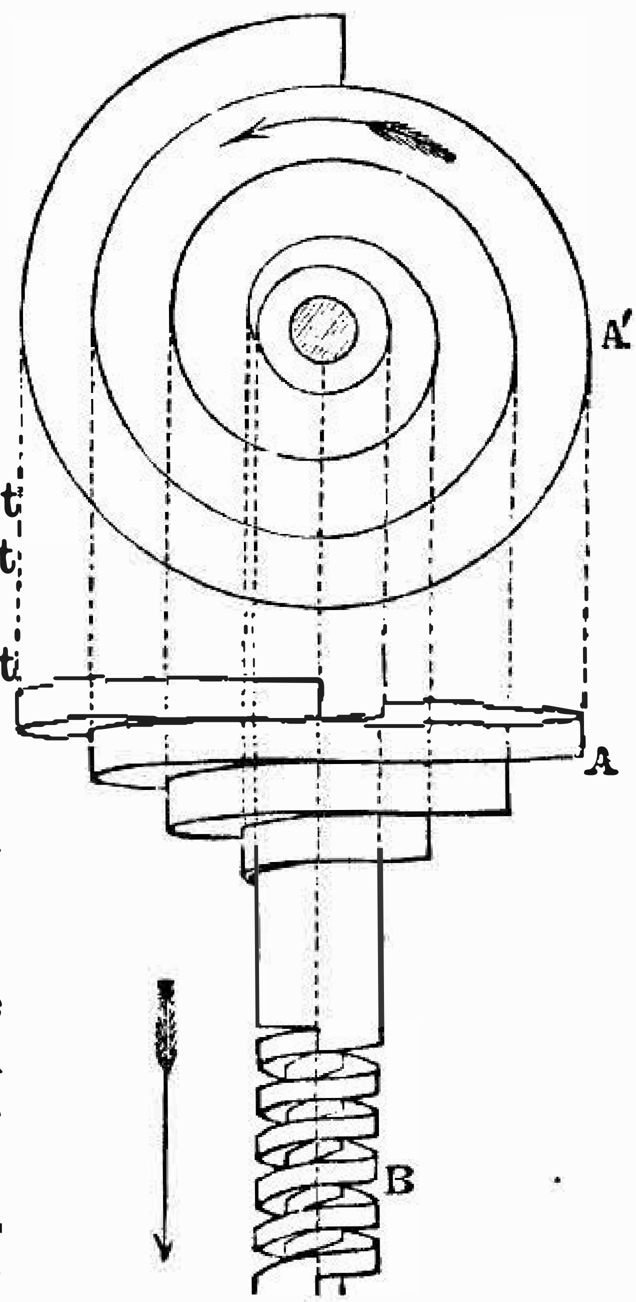

Fig. 119.

164. **Wipers and Pallets — Escapements.**—In fig. 120 a shaft rotating about the axis A is provided with one or more solitary teeth called *wipers*, such as E. The action of the wipers upon the projecting parts of the piece that they drive (which, for the sake of a general term, may be called *pallets*) may be either *intermittent* or *reciprocating*.

I. As an example of *intermittent* action, one of the wipers represented in fig. 120, in moving from the position H to the position E, is supposed to have driven before it a pallet from the position G to the position F. The pallet projects from a vertical sliding bar, or *stamper*, C.

B B is the addendum-circle of the wipers, and D D the addendum-line of the pallets. Those lines cut each other at the *point of escape*, E; and just at that point the pallet *escapes* from the wiper, and the stamper, with its pallet, falls back to its original position, and is ready to be lifted again by the next wiper.

The stamper and pallet referred to in this case are shaded.

II. As an example of *reciprocating* action, let the sliding bar, C, of the preceding example is supposed to have attached to it a frame,

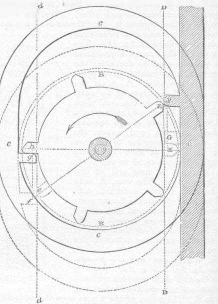

Fig. 120.

c c c, at the opposite side of which is another pallet, g; and this pallet is so placed that immediately after the escape of the former pallet, F, from the wiper at E, another wiper at h begins to act upon the pallet g, and so to produce the *return stroke* of the frame, C c c c. The point, e, where the addendum-line, d d, of the pallet g cuts the addendum-circle, B B, of the wipers, is the *point of escape* of the second pallet (whose position at the instant of escape is marked f); and immediately afterwards a third wiper, arriving at the position H, begins to produce a new forward stroke.

The *length of stroke* is represented in the figure by F G $= f\,g$. It is evident that the number of wipers must be odd.

This is the combination already referred to in Article 142, page 141. It belongs to a class of contrivances called *escapements*, because of the *escape* of the follower from the action of the driver at certain instants. There are many escapements which do not belong to the subject of pure mechanism; and amongst these are found most of the escapements that are used in clocks and watches, as being well suited to the regulation of those machines; for in such escapements the driver and follower are disconnected from each other during the greater part of the movement. Only two more escapements, therefore, will be described here.

III. *Anchor Recoil Escapement.*—This escapement, though not well suited to the exact keeping of time, is used in old clockwork. It is also used in vertical roasting jacks. The driver is a wheel called the *scape wheel*, and the trace of its axis is represented by the point A, fig. 121. E I F is its pitch-circle, cutting the line

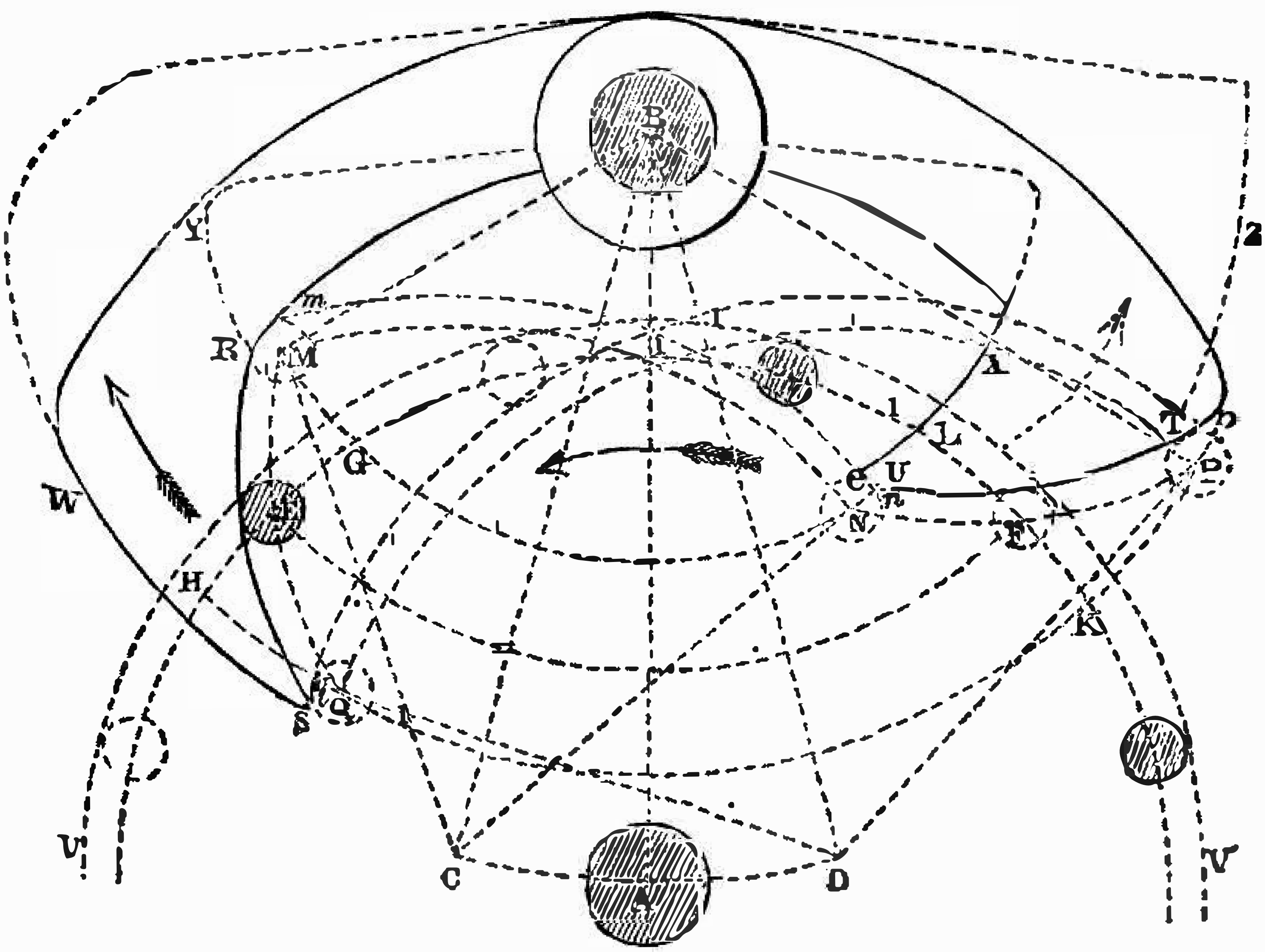

Fig. 121.

of centres, A B, in I. V V is its addendum-circle. In the figure the teeth are represented as cylindrical pins; in any case their acting surfaces may be regarded as parts of cylinders, which, if the teeth are sharp-pointed, are of insensible diameter. The arrow near I shows the direction of rotation of the wheel. The point B is the trace of the axis of the *verge*, or rocking shaft, to which a reciprocating movement is to be given through the alternate action of the teeth on the *pallets*, R S and T U, which are the acting surfaces of the *crutch*, S R T U. At the instant represented in the figure, the crutch is at the middle of its swing, and in the act of moving towards the left, through the action of the tooth E on the pallet R S. The swing of the crutch takes place while the wheel moves through half the pitch, at the end of which interval the tooth E and pallet R S *escape* from each other, and another tooth begins to act on the pallet T U, so as to make the crutch swing towards the right, and so on alternately. The dotted circle at F represents a tooth in the act of driving the pallet T U, at the middle of the swing, towards the right.

To design the figures of the pallets, a method is to be employed analogous to that described in Article 161, page 172; that is to say, the crutch is to be supposed fixed, and the line of centres, B A, is to be supposed to swing to and fro about the axis B, carrying with it the axis A, through an angle equal to the angle through which the crutch is actually to swing.

Lay off the angles A B C = A B D = the *semi-amplitude*, or half angle of swing; and make B C = B D = B A. Then C and D are the two extreme positions of the axis A in its supposed swinging motion. With a radius equal to that of the pitch-circle, draw the arcs M N about C, and P Q about D; and with a radius equal to that of the addendum-circle, draw the arcs *m n* about C, and *p q* about D. From the point I lay off upon the pitch-circle the arcs I E = I F = *an odd number of times the quarter-pitch;* so that E I F shall be an odd number of half pitches. The points E and F should be as near as practicable to the points where two straight lines from B touch the pitch-circle. About E and F draw circles to represent the traces of the acting surfaces of the pins or teeth. Lay off, on the pitch-circle, the arcs E G = F K = the quarter-pitch, with the radius of the acting surface of a tooth deducted: this deduction is to ensure that between the escape of a tooth from one pallet and the commencement of the action of another tooth on the opposite pallet there shall be an interval sufficient to enable the tooth that has just escaped to move clear of the pallet which it has quitted.

About the centre B, through the point G, draw the circular arc M G N, cutting the arc M N, already described about C, in the points M and N. About the centre B, through the point K, draw the circular arc P K Q, cutting the arc P Q, already described about D, in the points P and Q. Through M, E, and Q draw a continuous curve; this will be the pitch-line of the pallet R S. Through N, F and P draw a continuous curve: this will be the pitch-line of the pallet T U. Then, parallel to those pitch-lines respectively, and at a distance from them equal to the radius of the acting surface of a tooth, draw the traces, R S and T U, of the acting surfaces of the pallets.

The points of the pallets, at S and U, are to be cut off, so as not to project within the circles $q\ p$ and $n\ m$ respectively. The traces of the backs of the pallets, S W and U X, are to be circular arcs described about B.

IV. *Dead-beat Escapement.*—In the dead-beat escapement the crutch swings each way through an arc of indefinite extent, in addition to that through which it is driven by the action of the teeth of the scape wheel; and the scape wheel is made to pause in its motion during each such additional swing, by its teeth bearing against parts of the pallets whose surfaces are cylinders described about the axis of the verge. The traces of these may be called the *dead arcs* of the pallets. The recoil escapement shown in fig. 121, may be converted into a dead-beat escapement, as follows:—About B, with a radius equal to B M added to the radius of the acting surface of a tooth, draw the circular arc R Y; and also about B, with a radius equal to B P, deducting the radius of the acting surface of a tooth, draw the circular arc T Z: those two arcs will be the required dead arcs of the pallets.

In order that a dead-beat escapement may go on working, there must be a force, such as gravity or the elasticity of a spring, continually tending to bring the crutch to its middle position, at and near which the pallets are driven by the teeth; hence its principles are to a certain extent beyond the province of pure mechanism.

In the dead-beat escapements of accurate clocks, the angle through which the crutch swings is very small, and the angle through which the teeth act on the pallets is still smaller; so that in fig. 121 those angles may be looked upon as greatly exaggerated, for the sake of distinctly showing the geometrical principles of the combination.

SECTION V. — *Connection by Bands.*

165. **Bands and Pulleys Classed.** (*A. M.*, 478.)—The word *bands* may be used as a general term to denote all kinds of flexible connecting pieces; and the word *pulleys*, when not otherwise qualified, to denote all kinds of rotating pieces which are connected with each other by means of bands. Bands maybe classed in the following manner; which also involves a classification of the pulleys to which the bands are suited:—

I. *Belts*, which are made of leather, gutta percha, woven fabrics, &c., are flat and thin, and require nearly cylindrical pulleys with smooth surfaces. A belt tends to move towards that part of a pulley whose radius is greatest. Pulleys for belts, therefore, are slightly swelled in the middle, in order that the belt may remain on the pulley unless forcibly shifted, and are in general without ledges. A belt when in motion is shifted off a pulley, or from one pulley on to another of equal size alongside of it, by pressing against the "advancing side" of the belt; that is, that part of the belt which is moving *towards* the pulley. Amongst belts may be classed *flat ropes.*

II. *Cords*, made of catgut, leather, hempen or other fibres, or wire, are nearly cylindrical in section, and require either drums with ledges, or grooved pulleys.

III. *Chains*, which are composed of links or bars jointed together, require wheels or drums, grooved, notched, and toothed, so as to fit the links of the chains. Chains suited for this purpose are called *gearing chains.*

Bands for communicating motion of indefinite extent are *endless.*

Bands for communicating reciprocating motion have usually their ends made fast to the pulleys or drums which they connect, and which, when the extent of motion is less than a revolution, may be sectors.

166. **Principles of Connection by Bands.**—The *line of connection* of a pair of pulleys connected by means of a band is the central line or axis of that part of the band whose tension transmits the motion.

The *pitch-surface* of a pulley over which a band passes is the surface to which the line of connection is always a tangent; that is to say, an imaginary surface whose distance from all parts of the acting surface of the pulley that the band touches is equal to the distance from the acting surface of the band to its centre line. The pitch-surface of a pulley cannot be anywhere concave; for where the acting surface is concave, the band stretches in a straight line across the hollow, and the pitch-surface is plane. In ordinary pulleys for communicating a constant velocity-ratio the pitch-surface is a circular cylinder; and its radius (called the *effective radius*) is equal to the real radius of the pulley added to half the thickness of the band.

The *pitch-line* of a pulley is the line on its pitch-surface in which the centre-line lies of that part of the band which touches the pulley. The line of connection is a tangent to the pitch-line. When the line of connection is in a plane perpendicular to the axis of the pulley, the pitch-line is the trace of the pitch-surface on that plane: for example, the circular section of a cylindrical pulley. When the line of connection is oblique to the axis, the pitch-line is *helical*, or screw-like.

Problems respecting the comparative motion of pieces connected by bands are solved by applying the principles of Article 91, page 78, taking A B in fig. 58 of that Article to represent the centre line of that part of the band whose tension transmits the motion, and A A′ and B B′ to represent the common perpendiculars from that line to the axes of the pulleys. When the pitch-surfaces of the pulleys are circular cylinders, A A′ and B B′ represent their effective radii. Rule II. of Article 91 shows how to find the angular velocity-ratio of two pulleys whose proportionate dimensions are given. The following is the converse rule for finding the proportionate radii of two pulleys which are to transmit a given angular velocity-ratio. In fig. 58, page 78, draw A *a* to represent the projection of the axis of one pulley upon a plane parallel to that axis traversing the line of connection, A B; and draw B *b* to represent a similar projection of the axis of the other pulley. Lay off the distances A *a* and B *b* to opposite sides of A B, to represent the intended angular velocities of the two pulleys. Draw A *c* and B *d* perpendicular to A B; and draw *a c* and *b d* parallel to A B, cutting A *c* and B *d* in *c* and *d* respectively. Then the lengths A *c* and B *d* will represent the component angular velocities of the pulleys about axes perpendicular to the line of connection, A B. (In most cases which occur in practice, both the axes lie in planes perpendicular to the line of connection; and then A *a* and B *b* coincide with A *c* and B *d* respectively.)

Draw the straight line *c d*, cutting the line of connection, A B, in K. Then we have the proportion

$$\text{B K} : \text{A K}$$

$$: : \text{effective radius of A} : \text{effective radius of B};$$

and if one of those radii—for example, that of A—is given, the other is found as follows:—From A lay off A I = B K (or otherwise, from B lay off B I = A K). Perpendicular to A B draw A A′ and B B′; lay off A A′ = the given radius of the pulley A, and draw the straight line A′ I B′, cutting B B′ in B′; B B′ will be the required radius of B.

In the ordinary case, in which both axes lie in planes perpendicular to the line of connection, it is evident that the velocities of a pair of circular pulleys are *inversely as their effective radii.*

It is to be borne in mind that, especially as regards cases in which the axes do not both lie in planes perpendicular to the line of connection, everything stated in the present Article is based on the supposition that *the band is perfectly flexible in all directions.* In the case of flat belts connecting pulleys whose axes are not both in planes perpendicular to the line of connection, there are certain effects of the lateral stiffness of the belt which will be considered farther on.

The *velocity of the band* is equal to that of a point revolving at the end of the radius A A′, fig. 58, page 78, with the angular velocity represented by A *c*, and also to that of a point revolving at the end of the radius B B′, with the angular velocity represented by B *d*. When a band connects a pulley with a sliding piece, the comparative motion is given by Rule III. of Article 91, page 79.

Smooth bands, such as belts and cords, are not suited to communicate a velocity-ratio *with precision*, as teeth are, because of their being free to slip on the pulleys; but the freedom to slip is advantageous in swift and powerful machinery, because of its preventing the shocks which take place when mechanism which is at rest is suddenly *thrown into gear*, or put in connection with the prime mover. A band at a certain tension is not capable of exerting more than a certain definite force upon a pulley over which it passes; and therefore occupies, in communicating its own speed to the rim of that pulley, a certain definite time, depending on the masses that are set in motion along with the pulley and the speed to be impressed upon them; and until that time has elapsed the band has a slipping motion on the pulley; thus avoiding shocks, which consist in the too rapid communication of changes of speed. This will be further considered under the head of the Dynamics of Machines.

167. **Pulleys with Equal Angular Velocities.**—When a pair of pulleys turn about parallel axes in the same direction, with equal angular velocities, their pitch-lines may be of any figure whatsoever, curved or polygonal, provided they are equal and similar, and not concave. Each of the two straight parts of the band is equal and parallel to the line of centres; and those parts, if the pulleys are circular and not eccentric, remain at a constant distance from the line of centres; but have a reciprocating motion towards and from that line if the pulleys are either eccentric or non-circular. A *reel* is virtually a pulley whose pitch-line is a polygon with rounded angles; and such is also the case with the *expanding pulley*, consisting of four quadrants of a circle, which can be separated to a greater or less distance from each other by means of screws.

168. **Bands and Pulleys for a Constant Velocity-Ratio.**—In order that the velocity-ratio of a pair of pulleys may be constant, their pitch-lines must be circular (except in the particular case specified in the preceding Article, when the figure is not restricted to the circle alone).

The band may be *open* or *uncrossed*, as in fig. 122; or it may be *crossed*, as in fig. 123. With an open band the directions of rotation

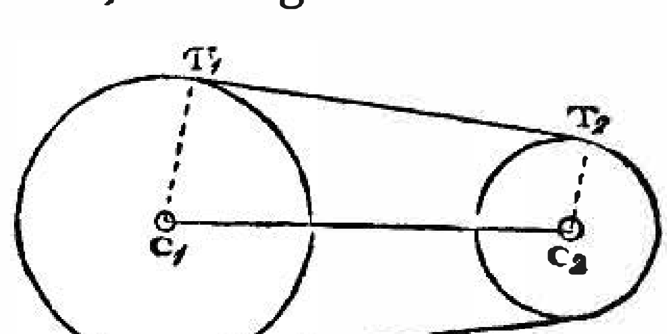

Fig. 122.

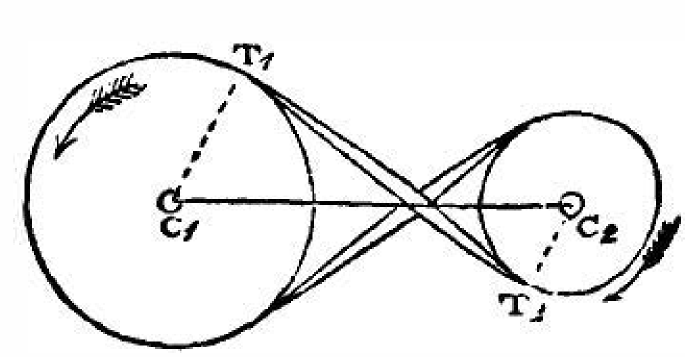

Fig. 123.

are the same; with a crossed band, contrary. In each of these figures, 1 denotes the driving pulley, and 2 the following pulley; $C_1 C_2$ is the line of centres, and $T_1 T_2$ the line of connection; and the angular velocity-ratio is expressed by

$$\frac{a_2}{a_1} = \frac{C_1 T_1}{C_2 T_2}.$$

169. The **Length of an Endless Band**, such as those shown in figs. 122 and 123, consists of two straight parts, each equal to the line of connection, and two circular arcs. When the band is crossed, as in fig. 123, the circular arcs are of equal angular extent; when the band is open, as in fig. 122, the angles subtended by the two arcs make up one revolution. When the length of a band is to be measured on a drawing, the circular parts may be rectified graphically by Rule I. or Rule II. of Article 51, page 28.

To find the length of an endless band by calculation, let the line of centres, $\overline{C_1 C_2} = c$, and the *effective radii* of the pulleys, $\overline{C_1 T_1} = r_1$; $\overline{C_2 T_2} = r_2$; r_1 being the greater. Then each of the two equal straight parts of the band is evidently of the length

$$\left.\begin{array}{l} T_1 T_2 = \sqrt{c^2 - (r_1 + r_2)^2} \text{ for a crossed band;} \\ T_1 T_2 = \sqrt{c^2 - (r_1 - r_2)^2} \text{ for an open band.} \end{array}\right\} \ldots\ldots(1.)$$

Let i_1 be the arc to radius unity of the greater pulley, and i_2 that of the less pulley, with which the band is in contact; then for a crossed band

$$i_1 = i_2 = \pi + 2 \operatorname{arc} \cdot \sin \frac{r_1 + r_2}{c};$$

and for an open band

$$i_1 = \pi + 2 \operatorname{arc} \cdot \sin \frac{r_1 - r_2}{c}; \; i_2 = \pi - 2 \operatorname{arc} \cdot \sin \frac{r_1 - r_2}{c}; \quad (2.)$$

and the addition of the lengths of the straight and curved parts gives the following total length:—

For a crossed band,

$$L = 2\sqrt{c^2 - (r_1 + r_2)^2} + (r_1 + r_2) \cdot \left(\pi + 2 \operatorname{arc} \cdot \sin \cdot \frac{r_1 + r_2}{c}\right);$$

and for an open band,

$$L = 2\sqrt{c^2 - (r_1 - r_2)^2} + \pi(r_1 + r_2) + 2(r_1 - r_2) \operatorname{arc} \cdot \sin \cdot \frac{r_1 - r_2}{c}. \quad (3.)$$

As the last of these equations would be troublesome to use in a practical application to be mentioned in Article 171, an approximation to it, sufficiently close for practical purposes, is obtained by considering, that if $r_1 - r_2$ ist smalltcompared with c,

$$\sqrt{c^2 - (r_1 - r_2)^2} = c - \frac{(r_1 - r_2)^2}{2c} \text{ nearly, and arc} \cdot \sin \cdot \frac{r_1 - r_2}{c} = \frac{r_1 - r_2}{c}$$

nearly; whence,tforantopentband,

$$\text{L nearly} = 2c + \pi (r_1 + r_2) + \frac{(r_1 - r_2)^2}{c}, \ldots\ldots (3 \text{ A.})$$

intwhich it istsufficientlyaccuratetfort practicalt purposesttotmake $\pi = 3\frac{1}{7}$.

170. **Pulleys withtFlat Belts.**—It hasalready beenstatedin Article 165, page 179, that a flattbelttends to movetowards that part of the pulley whose radius is greatest, or to "climb," as the phrase is; and that pulleys for such belts are therefore made without ledges, and with a slight swell or convexity at the middle of the rim, in orderthatthetbelt maytend totremainthere. Thetswell usually allowedtintthe rimstoftpulleystist *onettwenty-fourtht part oftthe breadth.*

Thettendency totclimb istproducedtbytthetlateraltstiffnesstof the belt, in the following manner:—When the part of the belt which touches the pulley deviates towards one side, as in fig. 124, the part which is approaching the pulley istmadettotdeviatettowardstthetopposite side; and thus, after the pulley has turned through a small angle, the deviation of the belt is corrected.

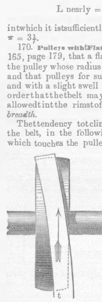

Fig. 124.

A crossed belt is twisted half round in passing from one pulley to another, as shown in fig. 123, so as to bring the same side oftthetbelt into contact with both pulleys. The principal object of this is,thatthettwotstraightpartstofthe belttmay pass eachtotherflatwisewhere they cross, so as not to resistteachtother's motion. Another object, in the case of leather belts, is to bring the rougher side of the leather into contact with both pulleys.

It has already been stated that the positiontwhich a beltassumestupon a pulley istdetermined by the position of its *advancing side;* that is, of the part of the belt which is approaching the pulley. In the contrivance called the "*fast and loose pulley,*" for engaging and disengaging machinery, a belt driven by a suitable driving pulley is provided with two similar and equal following pulleys, mounted side by side upon one axis; one of these pulleys is made fast to the shaft; the other turns loosely upon it. The belt, when in motion, can be shifted by means of a fork, that guides its advancing side to the fast pulley or to the loose pulley at will, so as to engage or disengage the shaft on which those pulleys are fitted. The driving pulley is made of a breadth equal to the breadths of the fast and loose pulleys together.

The lateral stiffness of a belt is also made available for the purpose of keeping it in its place on the pulleys when their axes are not parallel, as in fig. 125, which is sketched in isometrical perspective. C_1 C_1 and C_2 C_2 are the axes; E_1 E_2, their common

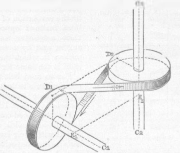

Fig. 125.

perpendicular. In order that the belt may remain on the pulleys, *the central plane of each pulley must pass through the* point of delivery *of the other* pulley—that is, the point where the belt leaves the other pulley; or, in other words, *the central planes of the two pulleys should intersect in the straight line which connects the two points of delivery*. In fig. 125, D_1 and D_2 are the two points of delivery; and the pulleys are so placed that D_1 D_2 is the line of intersection of their central planes. It is easy to see that this arrangement does not admit of the motion being reversed; for when that takes place, D_1 and D_2 cease to be the points of delivery, and become the points where the belt is received; and it is at once thrown off the pulleys.

171. **Speed Cones** (*A. M.*, 483) are a contrivance for varying and adjusting the velocity-ratio communicated between a pair of parallel shafts by means of a belt, and may be either continuous cones or conoids, as in fig. 126, A, B, whose velocity-ratio can be

varied gradually while they are in motion by shifting the belt; or sets of pulleys whose radii vary by steps, as in fig. 126, C, D—in which case the velocity-ratio can be changed by shifting the belt from one pair of pulleys to another while the machine is at rest.

In order that the belt may be equally tight in every possible position on a pair of speed-cones, the quantity L in the equations of Article 169, pages 183, 184, must be constant.

For a *crossed* belt, as at A and C, L depends solely on the line of centres, c, and on the sum of the radii, $r_1 + r_2$. Now c is constant because the axes are parallel; therefore the *sum of the radii* of the pitch-circles connected in every position of the belt is to be constant.

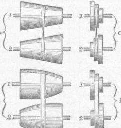

Fig. 126.

That condition is fulfilled by a pair of continuous cones, generated by the revolution of two straight lines inclined opposite ways to their respective axes at equal angles, and by a set of pairs of pulleys in which the sum of the radii is the same for each pair.

For an *open* belt the following practical rule is deduced from the approximate equation (3 A.) of Article 169, page 184 :—

Let the speed-cones be equal and similar conoids, as in fig. 126, B, but with their large and small ends turned opposite ways. Let r_1 be the radius of the large end of each, r_2 that of the small end, r_0 that of the middle; and let y be the *swell* or *convexity*, measured perpendicular to the axis, of the arc by whose revolution each of the conoids is generated; then

$$y = \frac{(r_1 - r_2)^2}{2 \pi c};\text{.........................}(1.)$$

and

$$r_0 = \frac{r_1 + r_2}{2} + y;\text{.........................}(2.)$$

$\pi = 3\frac{1}{7}$ nearly enough for the present purpose.

To find the swell, y, by graphic construction: in fig. 126 E, draw A B = $3\frac{1}{7}$ times the line of centres; from B, perpendicular to A B, draw B C = the difference between the greatest and least radii; join A C, and cut off from it A D = A B; D C will be the required swell.

Fig. 126 E.

The radii at the middle and ends being thus determined, make the generating curve an arc either of a circle or of a parabola.

For a pair of stepped cones, as in fig. 126 D, let a series of *differences* of the radii, or values of $r_1 - r_2$, be assumed; then, for each pair of pulleys, the half-sum of the radii is to be computed from the difference by the formula—

$$\frac{r_1 + r_2}{2} = r_0 - y; \qquad (3.)$$

r_0 being the value of that half-sum when the radii are equal; and finally, the radii are to be computed from their half-sum and half-difference, as follows:—

$$\left.\begin{aligned} r_1 &= \frac{r_1 + r_2}{2} + \frac{r_1 - r_2}{2}; \\ r_2 &= \frac{r_1 + r_2}{2} - \frac{r_1 - r_2}{2} \end{aligned}\right\} \qquad (4.)$$

172. **Pulleys for Ropes and Cords** require ledges to prevent the band from slipping off; for even flat ropes have not sufficient lateral stiffness to make them remain, of themselves, on the convexity of a pulley. A cord, in passing round a pulley, lies in a groove, sometimes called the *gorge* of the pulley; if the object of the pulley is merely to support, guide, or strain the cord, the gorge may be considerably wider than the cord; if the pulley is to drive or to be driven by the cord, so as to transmit motive power, the gorge must in general fit the cord closely, or even be of a triangular shape, so as to hold it tight. Sometimes the gorge of a pulley which is to be driven by a cord at a low speed has radial ribs on its sides, in order to give it a firmer hold of the cord.

The groove of a pulley for a wire rope should not grasp it tightly, lest the rope be injured; and the motion must be communicated by means of the ordinary friction alone. M. C. F. Hirn has introduced, with good success, the practice of filling the bottoms of the grooves of iron pulleys for wire ropes moving at a high speed with gutta percha, jammed in tight. This will be again referred to in treating of the dynamics of machinery, and of its construction.

When a cord does not merely pass over a pulley, but is made fast to it at one end, and wound upon it, the pulley usually becomes what is called a *drum* or a *barrel*. A drum for a round rope is cylindrical, and the rope is wound upon it in helical coils. Each layer of coils increases the effective radius of the drum by an amount equal to the diameter of the rope. A drum for a flat rope is of a breadth simply equal to the breadth of the rope, which is wound upon it in single coils, each of which increases the effective radius by an amount equal to the thickness of the rope; and instead of ledges it often has pairs of arms, forming as it were skeleton ledges.

173. **Guide Pulleys.**—A guide pulley merely changes the direction of a band on the way from the pulley which drives the band to the pulley which is driven by it. Guide pulleys are useful chiefly to change the direction of a round cord which communicates motion between two other pulleys whose pitch-circles are not in the same plane. In a case of that kind the following is the rule for finding a proper position for a guide pulley:—By the Rule of Article 27, page 10, find the line of intersection of the planes of the pitch-circles of the driving and following pulley respectively. From any convenient point in that line draw tangents to the proper sides of the two pitch-circles, to represent the centre-lines of two straight parts of the band; then, by the rule of Article 22, page 8, draw the rabatment of the angle which these straight lines make with each other. Let A C B in fig. 127 represent that rabatted angle; draw a straight line, C D, bisecting it; and about any convenient point, D, in that straight line describe a circle touching the two straight lines, C A, C B: this will be the pitch-circle of a suitable guide pulley.

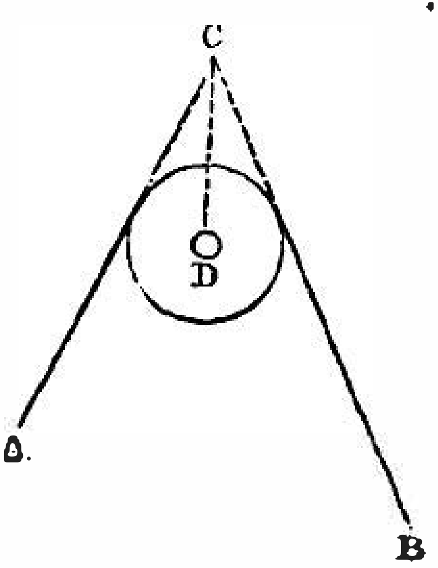

Fig. 127.

174. **Straining Pulleys.**—A straining pulley is used to bring a band to the degree of tension which is necessary in order to enable it to transmit motion from a driving pulley to a following pulley. A straining pulley, as applied to a flat belt, is usually pressed, by means of a lever, against one of the parts of the belt which extends between the driving and following pulleys, so as to push that part of the belt towards the line of centres. The effect of this is to tighten the belt and increase the friction exerted between it and the pulleys which it connects. This is one of the contrivances used for engaging and disengaging machinery. The straining or tightening pulley is usually applied to the *returning* part of the belt; that is, the part which moves from the driving pulley towards the following pulley.

Sometimes a straining pulley hangs in a loop or bight of a cord, and is loaded with a weight, as in fig. 128, farther on.

175. **Eccentric and Non-Circular Pulleys** are used for transmitting a varying velocity-ratio. For example, in fig. 128 the pitch-line of the pulley A is an eccentric circle, and might be a curve of any figure presenting no concavity; the pitch-line of B is circular and centred on its axis in the figure; but it, too, might be eccentric or non-circular. D E is the line of connection, being the centre-line

of the driving part of the cord, and a tangent to both pitch-lines; and the cord is kept tight by a loaded straining pulley at C. The angular velocities of the pulleys A and B at any given instant are inversely as the perpendicular distances A D and B E of their axes from the line of connection; or in symbols, let a and b be those angular velocities; then

$$\frac{b}{a} = \frac{\text{A D}}{\text{B E}}.$$

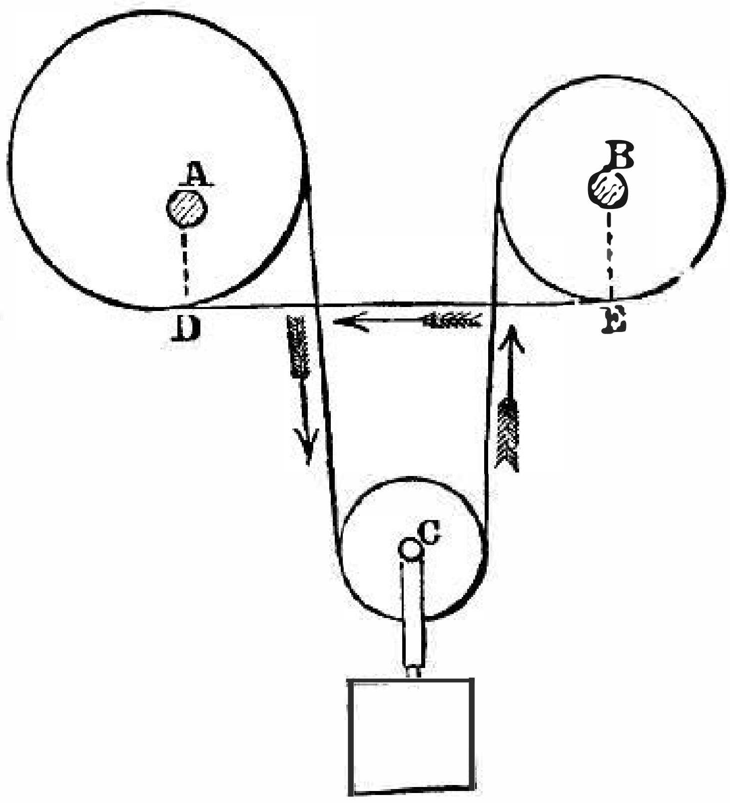

Fig. 128.

There is one instance in which no straining pulley is required; and that is when the pitch-lines of the driving pulley and of the following pulley are a pair of equal and similar ellipses, centred on two of their foci, A, A′, as shown in fig. 129, and connected by means of a crossed cord. The *mean angular velocities* are equal and opposite, each entire revolution being performed in the same

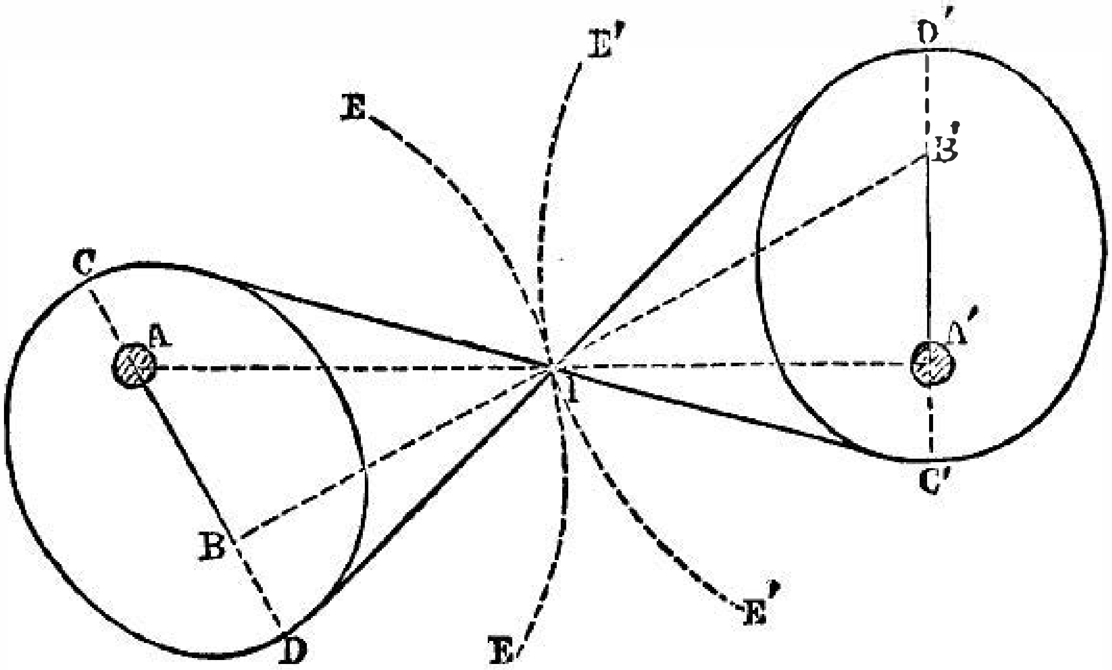

Fig. 129.

time by both pulleys; and the velocity-ratios at different instants are the same as in a combination of a pair of elliptic wheels having the same foci and the same line of centres. In the figure, E I E and E′ I E′ represent the pitch-lines of such a pair of elliptic wheels: the pitch point being always at the intersection, I, of the two straight parts of the cord.

To design such a pair of elliptic pulleys, the data required are the line of centres, A A′, and the angle by which each pulley is alternately to overtake and to fall behind the other pulley. Then, by Rule I. of Article 108, page 95, find the foci; and about those foci draw any ellipse that is not larger than the ellipse suited, according to the same rule, for the pitch-line of a wheel to work in rolling contact; the ellipse so drawn will be suitable for the pitch-lines of both pulleys, C D and C′ D′. The pulleys, like the wheels described in Article 108, will rotate in the same manner as if the revolving foci were connected with each other by a straight link, B B′, equal to the line of centres, A A′; and their corresponding positions and velocity-ratio at any given instant may be found by Rules II. and III. of Article 108, pages 96, 97.

Amongst non-circular pulleys are *fusees*, used in watch-work; in which the pitch-line is a spiral described on a conoidal surface.

Non-circular pulleys may be indefinitely varied in figure without difficulty; for the possibility of keeping the band tight by means of a straining pulley removes the necessity of preserving certain relations between the pitch-lines, as in rolling contact.

176. **Chain Pulleys and Gearing Chains.**—A chain pulley in some cases is merely a circular grooved pulley for guiding a chain: or a cylindrical barrel on which a chain is wound, being made fast at one end to the barrel, as in cranes; and those need no special description. But when a chain is to drive or to be driven by a pulley to which it is not made fast, the acting surface of the pulley must be adapted to the figure of the chain, so as to insure a sufficient hold between them. Amongst chain pulleys of this kind are included *capstans and windlasses.*

The pitch-line of a true chain pulley is a polygon, as exemplified in figs. 130 and 131, in each of which figures the angles of the pitch polygon are marked by black spots, and its sides by dotted lines. Each side of the pitch polygon is equal to what may be called the *pitch*, or *effective length*, of a link of the chain. When the links consist of flat bars of equal length, connected by means of cylindrical pins, as in fig. 130, the pitch of each link is the same, being the distance between the centres of two pins; and the pitch-line accordingly is an equilateral polygon (in the figure a regular hexagon). When the chain consists of oval links, like those of a chain-cable, as in fig. 131, the pitch of a link which lies flatwise on the rim of the pulley is equal to its longer internal diameter *plus* the diameter of the iron, and the pitch of a link which stands edgewise on the rim of the pulley is equal to its longer interval diameter *minus* the diameter of the iron; so that the pitch polygon has long and short sides alternately (in the figure there are twelve sides—six long and six short; and the length of a long side is to that of a short side as 5 to 3). In fig. 130 the pulley is simply a polygonal prism; in fig. 131 it has hollows to fit those links which stand edgewise.

Each of the pulleys shown in these figures has teeth; and the traces of the acting surfaces of the teeth are circular arcs, described about the adjacent angles of the pitch polygons. In fig. 130 the

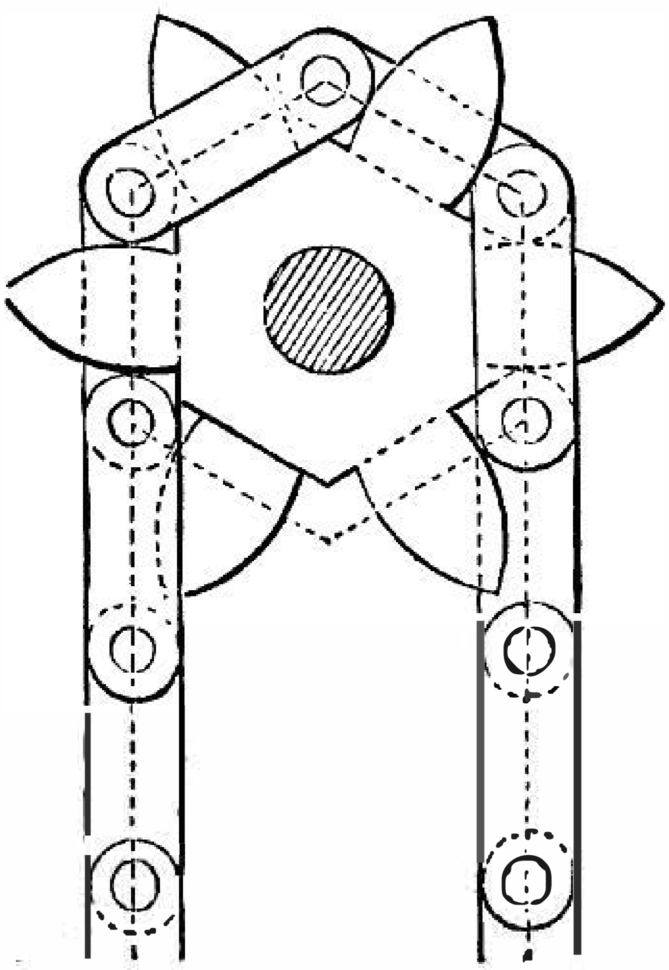

Fig. 130.

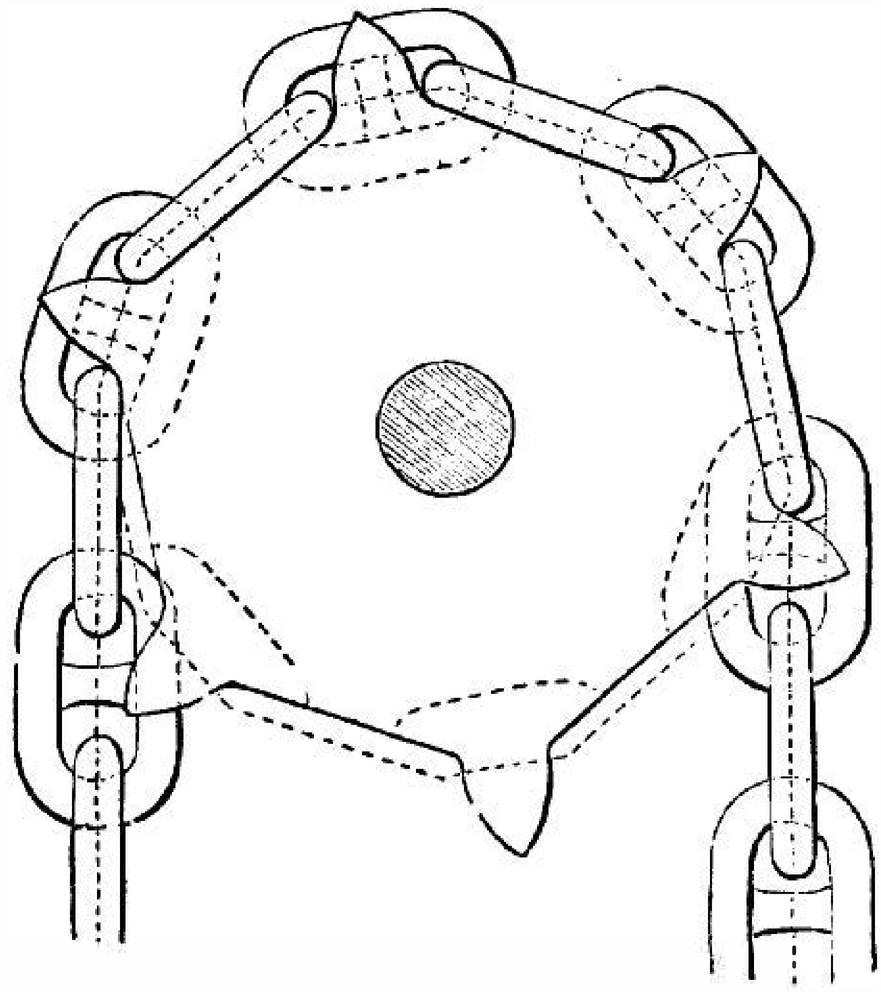

Fig. 131.

chain consists of double and single links alternately; and the sides of the pulley are provided alternately with single teeth and with pairs—a single tooth where each double link lies, and a pair of teeth for each single link to lie between. Sometimes the pulley is provided with single teeth only—one in the middle of each side on which a double link lies. Chains of the shape shown in fig. 130 are made with various numbers of parallel and similar bars in each link, according to the strength required. Of course, the number of bars in a link is even and odd alternately. Such chains are also sometimes made with links of leather, connected together by brass pins, and are used to communicate motion between cylindrical drums. The object of this is to have greater flexibility than is possessed by a flat leather belt. In fig. 131 each short side only of the polygon is provided with a pair of teeth, which receive a link standing edgewise between them, and press against the end of a link that lies flatwise.

Sometimes a chain pulley consists of a number of radiating forks, forming as it were a reel; this is called a *sprocket-wheel.* Sometimes it has a triangular gorge, with radiating ribs on the inner surface of each of the ledges.

177. **Suspended Pulleys.**—When rotation is transmitted, by means of two pairs of pulleys connected by cords, from one shaft through

an intermediate shaft to a third shaft, having its axis in one straight line with the first shaft, the waste of work in overcoming friction may be diminished by supporting the intermediate shaft without bearingsa its weight being simply hung by means of the cords from the pulleys on the other two shafts; and care being taken to load the intermediate shaft so as to produce the tension on the cords which is required in order to transmit the motion. What that tension ought to be is a question belonging to the dynamics of machinery. This contrivance appears to have been first introduced by Sir William Thomson. In fig. 132 A is the first and C the third shaft, and B is the intermediate shaft, suspended by means of the cords that pass round its pulleys; D, D are heavy round discs, of the weights required in order to give sufficient tension to the cords. The shaft B, and all the pieces which it carries, should be very accurately balanced.

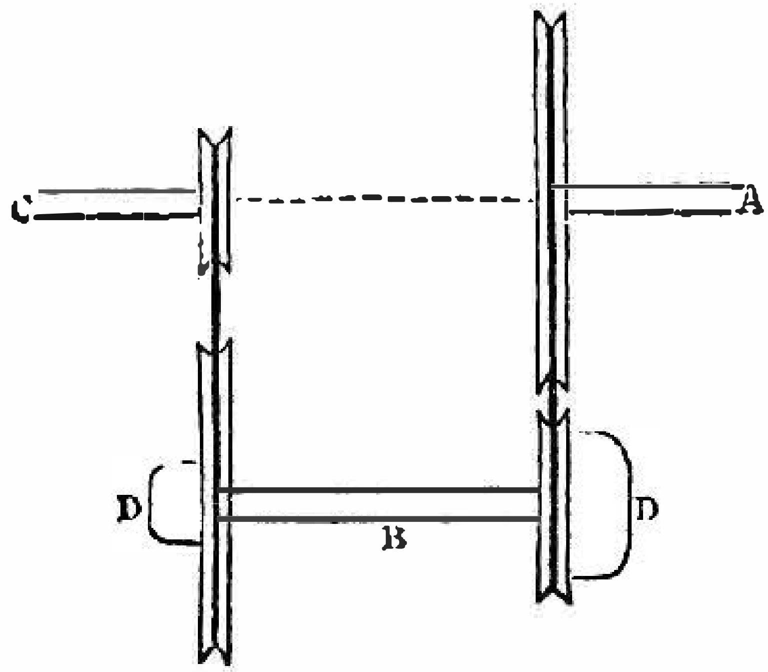

Fig. 132.

SECTION VI.—*Connection by Linkwork.*

178. **Definitions.** (*A. M.*, 484.)—The pieces which are connected by linkwork, if they rotate or oscillate, are usually called *cranks*, *beams*, and *levers*. The *link* by which they are connected is a rigid bar, which may be straight or of any other figure: the straight figure, being the most favourable to strength, is used when there is no special reason to the contrary. The link is known by various names under various circumstances, such as *coupling-rod*, *connecting-rod*, *crank-rod*, *eccentric-rod*, &c. It is attached to the pieces which it connects by two pins, about which it is free to turn. The effect of the link is to maintain the distance between the centres of those pins invariable; hence the line joining the centres of the pins is *the line of connection;* and those centres may be called the *connected points*. In a turning piece the perpendicular let fall from its connected point upon its axis of rotation is the *arm* or *crank-arm*. If the motions of the pieces are performed parallel to one plane, or about one central point, the pins are almost always cylindrical, with their axes perpendicular to the plane, or traversing the point, as the case may be. In all other cases the acting surfaces of the pins must be portions of spheres described about the connected points—making what are called *ball-and-*

socket joints; unless *universal joints* are used, which will be described further on.

179. **Principles of Connection.** (*A. M.* 485.)—All questions as to the comparative motions of a pair of pieces connected by a link may be solved by means of the general principles and rules given in Article 91, pages 78 to 80, and illustrated by figs. 57 and 58. The axes of rotation of a pair of turning pieces connected by a link are almost always parallel to each other, and perpendicular to the line of connection; in which case the angular velocity-ratio at any instant is the reciprocal of the ratio of the common perpendiculars let fall from the line of connection upon the axes of rotation.

Another method of treating questions of linkwork is to find, by the principles of Article 69, pages 46 to 50, the instantaneous axis of the link; for the two connected points move in the same manner with two points in the link, considered as a rigid body.

If a connected point belongs to a turning piece, the direction of its motion at a given instant is perpendicular to the plane containing the axis and crank-arm of the piece. If a connected point belongs to a shifting piece, the direction of its motion at any instant is given, and a plane can be drawn perpendicular to that direction.

The line of intersection of the planes perpendicular to the paths of the two connected points at a given instant is the *instantaneous axis of the link* at that instant; and the *velocities of the connected points are directly as their distances from that axis.*

In drawing on a plane surface, the two planes perpendicular to the paths of the connected points are represented by two lines (being their traces on a plane normal to them), and the instantaneous axis by a point; and should the length of the two lines render it impracticable to produce them until they actually intersect, the velocity-ratio of the connected points may be found by the principle, that it is equal to the ratio of the segments which a line parallel to the line of connection cuts off from any two lines drawn from a given point perpendicular respectively to the paths of the connected points. Examples will be given further on.

180. **Dead Points.** (*A. M.*, 486.)—If at any instant the plane traversing one of the crank-arms and its axis of rotation coincides with the line of connection, the common perpendicular of the line of connection and the axis of that crank-arm vanishes, and the directional relation of the motions becomes indeterminate. The position of the connected point of the crank arm in question at such an instant is called a *dead point.* The velocity of the other connected point at that instant is null; unless it

also reaches a dead point at the same instant, so that the line of connection is in the plane of the two axes of rotation; in which case the velocity-ratio is indeterminate.

181. **Coupled Parallel Shafts.** (*A. M.*, 487.)—There are only two cases in which an uniform angular velocity-ratio (being that of equality) is communicated by linkwork. One of them is that in which two or more parallel shafts (such as those of the driving wheels of a locomotive engine) are made to rotate with constantly equal angular velocities, by having equal cranks, which are maintained parallel by a coupling rod of such a length that the line of connection is equal to the distance between the axes. The cranks pass their dead points simultaneously. To obviate the unsteadiness of motion which this tends to cause, the shafts are provided with a second set of cranks at right angles to the first, connected by means of a similar coupling rod, so that one set of cranks pass their dead points at the instant when the other set are farthest from theirs. (See fig. 32, page 44.)

This elementary combination belongs to Willis's Class A.

182. **Drag-Link.**—The term *drag-link* is applied to a link, as C D, fig. 133, which connects together two cranks, A C and B D, so as to make them perform a complete revolution in the same time and in the same direction. The cranks may be equal or unequal. If the two axes (whose traces in the figure are A and B) are parts of one straight line (that is, if the points A and B coincide), the angular velocities of the cranks are equal at every instant, and the combination belongs to Willis's Class A; and such is the action of the drag-link when used as a coupling. If the axes are not parts of one straight line (so that A and B are different points), the velocity-ratio varies, and the combination belongs to Class B.

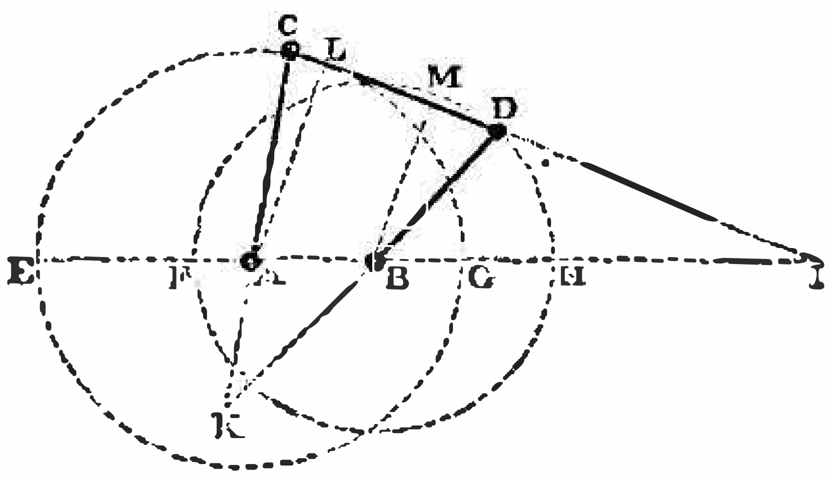

Fig. 133.

In most cases the crank which is the driver goes foremost, and pulls the follower after it; and hence the name of "drag-link."

The following are rules to be observed in determining the dimensions of the parts.

I. In order that the directional relation may be constant, *each of the crank-arms*, A C, B D, *should be longer than the line of centres*, A B.

II. For the same reason, and also in order that there may be no dead-points, *the length of the line of connection*, C D, *should be greater than the lesser segment*, E F, *and less than the greater* *segment*, F G, *into which the diameter*, E G, *of the greater of the two circles described by the connected points is divided by the other circle.* This principle holds also when those circles are equal, and is then applicable to the diameter of either of them. In other words, C D is to be made

$$\text{Greater than } A\,B + A\,C - B\,D,$$
$$\text{and less than } A\,C + B\,D - A\,B.$$

The comparative motions may be found by either of the following rules :—

III. *To find the angular velocity-ratio* in a given position of the cranks: on the line of connection, C D, let fall from the axes the perpendiculars, A L, B M ; then

$$\frac{\text{Angular velocity of } B\,D}{\text{Angular velocity of } A\,C} = \frac{A\,L}{B\,M};$$

Or otherwise: produce the line of connection, C D, till it cuts the line of centres in I ; then.

$$\frac{\text{Angular velocity of } B\,D}{\text{Angular velocity of } A\,C} = \frac{I\,A}{I\,B}.$$

When C D is parallel to A B the angular velocities are equal.

IV. *To find the linear velocity-ratio of the connected points:* in a given position of the cranks produce the crank-arms until they intersect; their point of intersection, K, will be the trace of the instantaneous axis of the link ; then

$$\frac{\text{Velocity of } D}{\text{Velocity of } C} = \frac{K\,D}{K\,C}.$$

The limits between which that velocity-ratio fluctuates are $\frac{B\,D - A\,B}{A\,C}$, when B D traverses A, and $\frac{B\,D}{A\,C - A\,B}$, when A C traverses B.

The two shafts, in their rotation, may be regarded as alternately overtaking and falling behind each other by an angle which we may call the *angular displacement.* The complete angular displacement is attained in two opposite directions alternately, at the two instants when the angular velocities of the shafts are equal: that is, when the line of connection is parallel to the line of centres. The following is a rule for designing a *drag-link motion with equal cranks, which shall produce a given angular displacement;* and although not the only rule by which that problem might be solved, it appears to be the simplest in its application.

V. In fig. 134 draw two straight lines, C O *c*, D O *d*, cutting each other at right angles in the point O; lay off along those lines the equal lengths O C = O D. From C and D draw the straight lines C A, D B, making the angles O C A = O D B = *half the given angular displacement*, and cutting O *d* and O *c* respectively in A and B. Join A B and C D. Then A B will represent the line of centres; A C and B D the two crank-arms; and C D the line of connection.

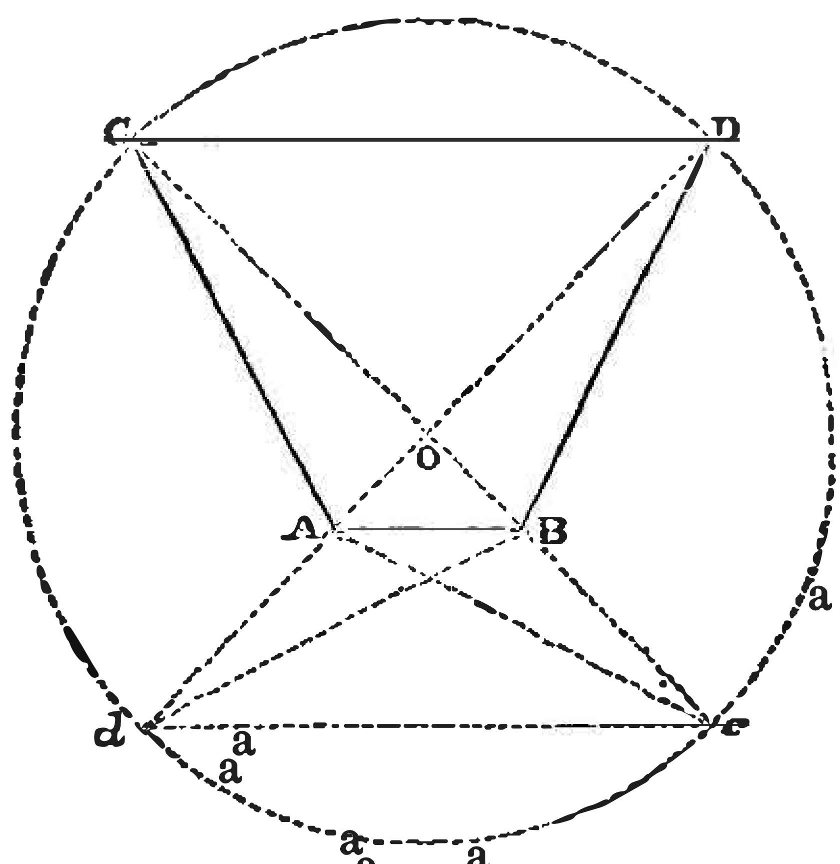

Fig. 134.

The position of the parts represented will be that in which the angle between the crank-arms is least. To show, if required, the position of the parts when that angle is greatest, lay off O *c* and O *d* equal to O C and O D, and join A *c*, B *d*, and *c d*.

183. **Link for Contrary Rotations.**—The only other elementary combination by linkwork which belongs to Willis's Class B is that in which two equal cranks, rotating about parallel axes in contrary directions, are connected by means of a link equal in length to the line of centres. This has been already described in Article 108, page 97, and represented in fig. 72, page 96, as a contrivance to aid the action of elliptic wheels. There are two dead points in each revolution which the pins pass at the instant when the line of connection coincides with the line of centres; consequently the link is not well adapted to act alone, and requires a pair of elliptic wheels, or of elliptic pulleys (Article 175, page 189), to ensure the accurate transmission of the motion.

184. **Linkwork with Reciprocating Motion—Crank and Beam—Crank and Piston-Rod.** (*A. M.*, 488.)—The following are examples of the most frequent cases in practice of linkwork belonging to Willis's Class C, in which the directional relation is reciprocating; and in determining the comparative motion, they are treated by the method of instantaneous axes, already referred to in Article 179, page 193:—

Example I. *Two Turning Pieces with Parallel Axes*, such as a beam and crank (fig. 135).—Let C_1, C_2, be the parallel axes of the pieces; T_1, T_2, their connected points; C_1 T_1, C_2 T_2, their crank arms; T_1 T_2, the link. At a given instant let v_1 be the velocity of T_1; v_2 that of T_2.

To find the ratio of those velocities, produce C_1 T_1, C_2 T_2, till they intersect in K; K is the instantaneous axis of the link or connecting-rod, and the velocity-ratio is

$$v_1 : v_2 :: \text{K T}_1 : \text{K T}_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

Should K be inconveniently far off, draw any triangle with its sides respectively parallel to C_1 T_1, C_2 T_2, and T_1 T_2; the ratio of the two sides first mentioned will be the velocity-ratio required. For example, draw C_2 A parallel to C_1 T_1, cutting T_1 T_2 in A; then

$$v_1 : v_2 :: \text{C}_2 \text{A} : \text{C}_2 \text{T}_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

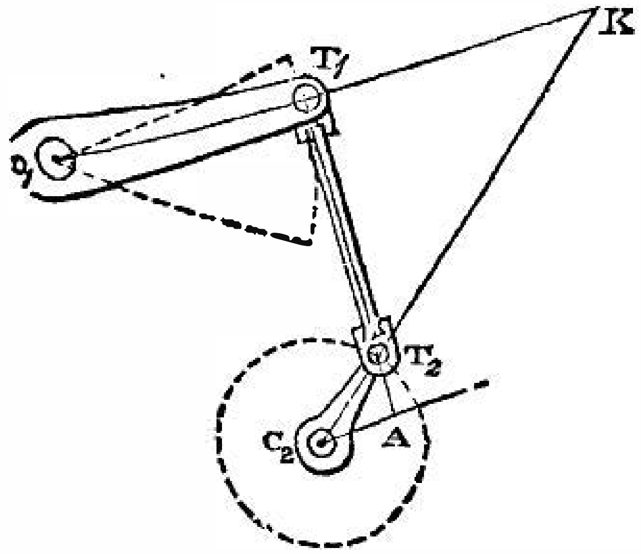

Fig. 135.

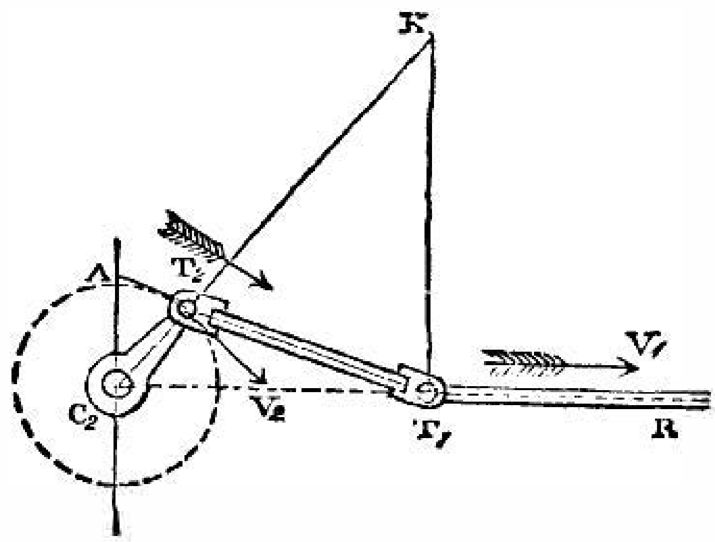

Fig. 136.

Example II. *Rotating Piece and Sliding Piece*, such as a piston-rod and crank (fig. 136).—Let C_2 be the axis of a rotating piece, and T_1 R the straight line along which a sliding piece moves. Let T_1, T_2, be the connected points; C_2 T_2, the crank arm of the rotating piece; and T_1 T_2, the link or connecting rod. The points T_1, T_2, and the line T_1 R, are supposed to be in one plane, perpendicular to theeaxis C_2. Draw T_1 K perpendicular to T_1 R, intersecting C_2 T_2 in K; K is the instantaneous axis of the link; and the rest of the solution is the same as in Example I.

185. (*A. M.*, 489.) An **Eccentric** (fig. 137) being a circular disc keyed on a shaft, with whose axis its centre does not coincide, and used to give a reciprocatinge motion to a rod, is equivalent to a crank whose connected point is T, the centre of the eccentric disc, and whose crank arm is C T, the distance of that point from the axis of the shaft, called the *eccentricity*.

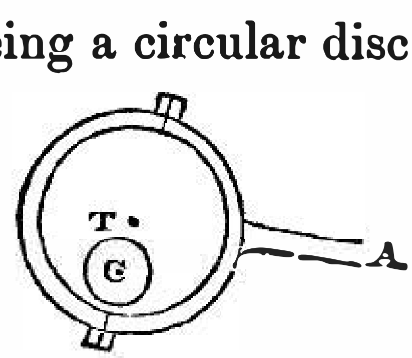

Fig. 137.

An eccentric may be made capable of having its eccentricity altered by means of an adjusting screw, so as to vary the extent of the reciprocating motion which it communicates, and which is called the *throw*, or *travel*, or *length of stroke.*

186. (*A. M.*, 490.) **The Length of Stroke** of a point in a reciprocating piece is the distance between the two ends of the path in

which that point moves. When it is connected by a link with a point in a continuously rotating piece, the ends of the stroke of the reciprocating point correspond with the *dead points* of the continuously rotating piece (Article 180, page 193).

I. *When the crank-arm and the path of the connected point in the reciprocating piece are given, to find the stroke and the dead points.* If the connected point in the reciprocating piece moves in a straight line traversing and perpendicular to the axis of the turning piece, the length of stroke is obviously twice the crank-arm. If that connected point moves in any other path, let F F, in fig. 138, represent that path, A the trace of the crank-axis, and A D = A E the crank-arm. From the point A to the path F F lay off the distances A B = the line of connection − the crank-arm, and A C = the line of connection + the crank-arm; then B C will be the stroke of the connected point in the reciprocating piece. Draw the straight lines C E A and B A D, cutting the circular path of the crank-pin in the points E and D: these will be the dead points.

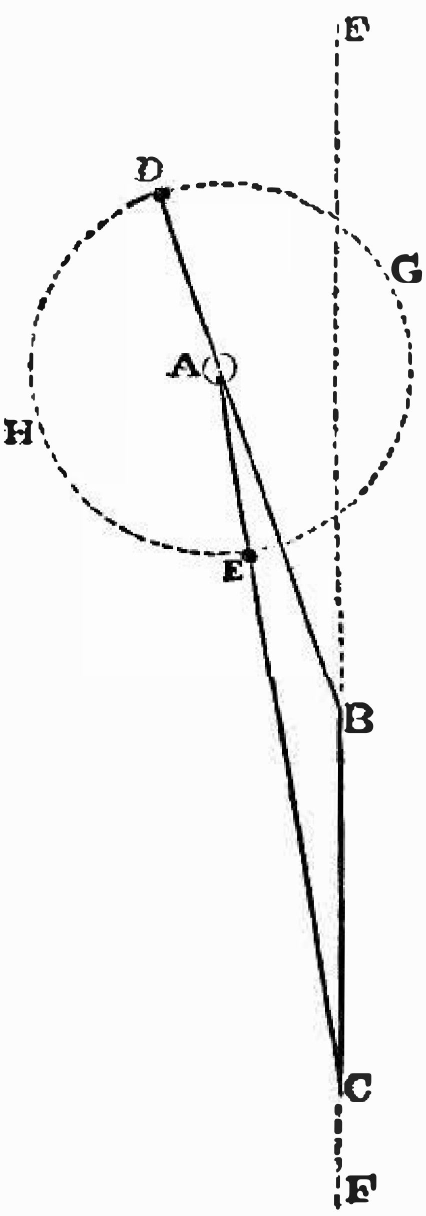

Fig. 138.

II. *When the crank-arm,* A D = A E, *the length of the line of connection, and the dead points,* D *and* E, *are given, to find the two ends of the stroke of the connected point in the reciprocating piece.* In D A and A E produced, make D B and E C each equal to the length of the line of connection; B and C will be the required ends of the stroke.

When the path of the connected point in the reciprocating piece is a straight line, the preceding principles may be thus expressed in algebraical symbols:—

Let S be the length of stroke, L the length of the line of connection, and R the crank-arm. Then, if the two ends of the stroke are in one straight line with the axis of the crank,

$$S = 2\,R; \qquad \text{(1.)}$$

and if their ends are not in one straight line with that axis, then S, L − R, and L + R, are the three sides of a triangle, having the angle opposite S at that axis; so that if θ be the supplement of the arc between the dead points,

$$\left.\begin{aligned} S^2 &= 2\,(L^2 + R^2) - 2\,(L^2 - R^2)\cos\theta; \\ \cos\theta &= \frac{2\,L^2 + 2\,R^2 - S^2}{2\,(L^2 - R^2)}. \end{aligned}\right\} \qquad \text{(2.)}$$

187. **Mean Velocity Ratio.**—In dynamical questions respecting machines, especially when the mode of connection is by linkwork, it is often requisite to determine the *mean ratio* of the linear velocities of a pair of connected points during some definite period; which mean ratio is simply the ratio of the distances moved through by those points in that period. Three cases may be distinguished, according as the combination of linkwork belongs to Willis's Class A, Class B, or Class C.

In Class A the mean velocity-ratio is identical with the velocity-ratio at each instant. For examples, see Article 181, page 194, and Article 182, page 194.

In Class B the mean velocity-ratio of the connected points during each complete revolution is that of the circumferences of the circles in which they move. For examples, see Article 182, page 194, and Article 183, page 196.

In Class C the mean velocity-ratio of the connected points may be taken either for a whole revolution of the revolving point and double stroke of the reciprocating point, or it may be taken separately for the forward stroke and return stroke of the reciprocating point, where it has different values for these two parts of the motion. In the former case it is expressed by the ratio of twice the length of stroke of the reciprocating point to the circumference of the circle described by the revolving point; that is to say, for example, in fig. 138, page 198, by the ratio

$$\frac{2\,\mathrm{B\,C}}{\text{Circumference }\mathrm{D\,G\,E\,H}}.$$

In the latter case, the two mean velocity-ratios are expressed by the proportions borne by the length of stroke of the reciprocating point, to the two arcs into which the dead points divide the path of the revolving point. For example, in fig. 138, those two ratios are respectively—

$$\frac{\mathrm{B\,C}}{\text{Arc }\mathrm{D\,G\,E}}, \text{ and } \frac{\mathrm{B\,C}}{\text{Arc }\mathrm{E\,H\,D}}.$$

The most frequent case in practice is that represented in fig. 136, page 197, where the reciprocating point moves in a straight line traversing the axis about which the revolving point moves; and in that case the mean velocity-ratio for each single stroke and for a whole revolution is

$$\frac{2}{\pi} = 0{\cdot}63662 \text{ nearly.}$$

188. **Extreme Velocity-Ratios.**—In those cases in which one of the points connected by a link revolves continuously, while the other has a reciprocating motion, it is often desirable to determine the *greatest* value of the ratio borne by the velocity of the recipro-

cating point to that of the revolving point. The general principle upon which that greatest ratio depends is shown in fig. 139, in which T′ represents the reciprocating point, and T the revolving point; T T′, the line of connection; and C T, the crank-arm. Let C A be perpendicular to the direction of motion of the reciprocating point T′, and let A be the point where the line of connection cuts C A; then, as has been already shown in Article 184, page, 196,

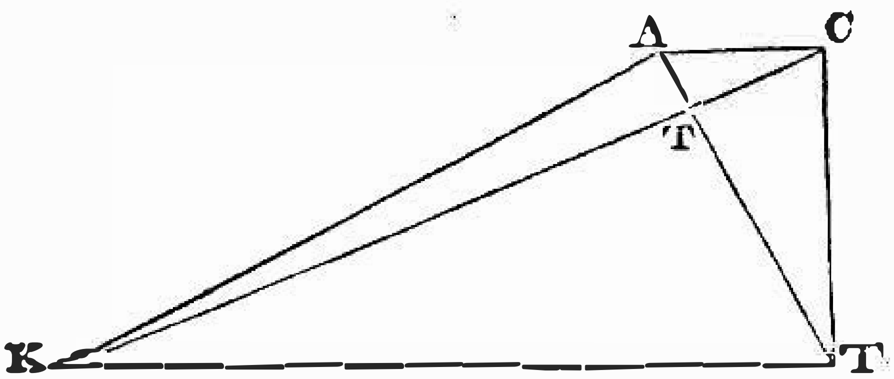

Fig. 139.

$$\frac{\text{Velocity of T}'}{\text{Velocity of T}} = \frac{\text{C A}}{\text{C T}};$$

and at the instant when that ratio is greatest, A is at its greatest distance from C; therefore, at that instant the direction of motion of the point A in the line of connection is along that line itself. Draw T′ K parallel to C A, produce C T till it cuts T′ K in K, the instantaneous centre of motion of the link, and join K A; then the direction of motion of the point A in the line of connection at any instant is perpendicular to A K; and therefore, at the instant when C A is greatest, A K is perpendicular to A T′. Upon this proposition depends the determination of the greatest value of the ratio $\frac{\text{C A}}{\text{C T}}$; but that determination cannot be completed by geometry alone; for it requires the solution of a cubic equation, as stated in the footnote.*

* In fig. 139, let the crank-arm C T $= a$; let the line of connection T T′ $= b$; these two quantities being given; and when the ratio of the velocity of T′ to that of T is a maximum, let the angle C T′ T $= \theta$, and the angle A C T $= \phi$.

Solve the following cubic equation:—

$$\sin^6\theta - \sin^4\theta - \sin^2\theta + \frac{a^2}{b^2} = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

so as to determine the value of $\sin^2\theta$, which is the only root of that equation that is positive and less than 1. Next, calculate the value of the angle ϕ, or those of its trigonometrical functions, by the help of one or more of the following equations (each of which implies the others):—

$$\left.\begin{aligned} \tan\phi &= \cos\theta\sin\theta = \frac{\sin 2\theta}{2}; \\ \sin\phi &= \sqrt{\left\{\frac{\sin^2\theta - \sin^4\theta}{1+\sin^2\theta - \sin^4\theta};\right\}} \\ \cos\phi &= \frac{1}{\sqrt{(1+\sin^2\theta - \sin^4\theta)}}; \end{aligned}\right\} \ldots\ldots\text{a}\ldots\ldots(2.)$$

An *approximate solution* of this question may, however, be obtained by plane geometry, when the line of connection, T T', is not less than about twice the crank-arm, C T. It consists in treating the angle at T as if it were a right angle (from which it differs by the angle A K T); and thus we obtain

$$\frac{\text{C A}}{\text{C T}} = (\text{nearly}) \frac{\text{C T}'}{\text{T T}'}, = \frac{\sqrt{(\text{C T}^2 + \text{T T}'^2)}}{\text{T T}'}.$$

When T T' is great as compared with C T, the error of this solution is inappreciable, or nearly so; when T T' = 2 C T, the approximate solution is too small by about one per cent., and is therefore near enough for practical purposes; when T T' becomes less than 2 C T, the error rapidly increases, so as to make the approximate solution inapplicable; but cases of this last kind are very uncommon in practice.

189. **Doubling of Oscillations by Linkwork.**—When two reciprocating pieces are connected by means of a link, the follower may be made to perform two oscillations or strokes for one of the driver, in the following manner:—In fig. 140, let the driver be an arm or lever, A B; A its axis of motion, and B its connected point.

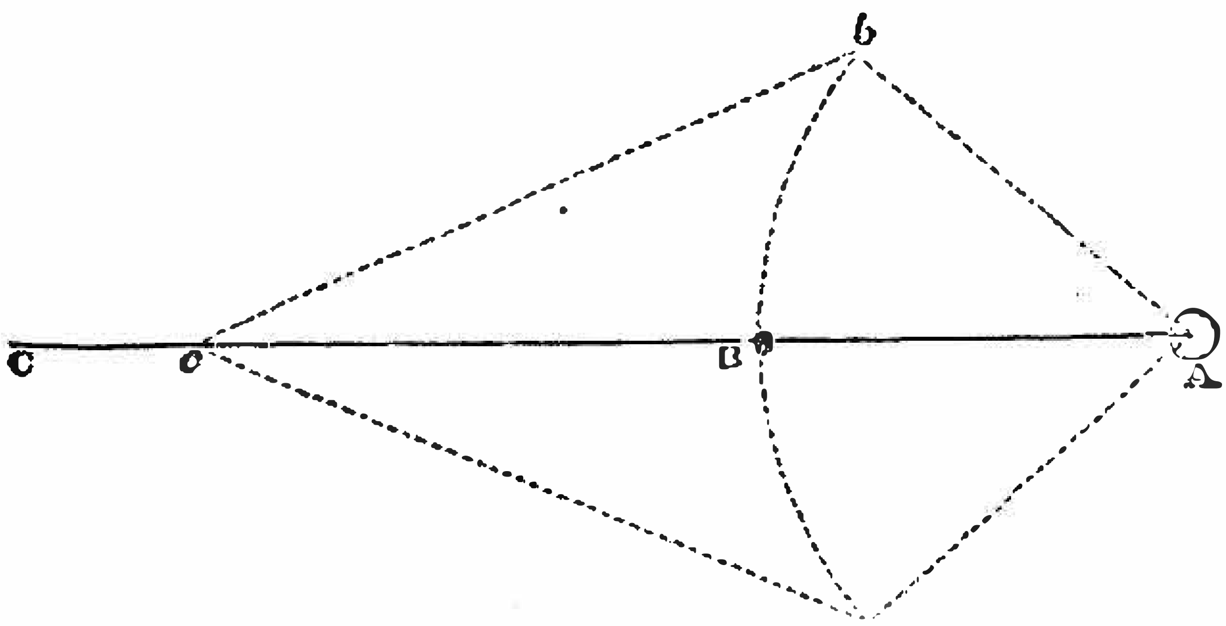

Fig. 140.

Let C be the connected point of the follower, and B C the link. Then the parts of the combination are to be so arranged that the straight line C c, which traverses the two ends of the stroke of the point C, shall traverse also the axis A, and shall bisect the arc of

and finally, calculate the required greatest velocity-ratio by the following formula:—

$$\frac{\text{C A}}{\text{C T}} = \frac{\cos(\theta - \phi)}{\cos\theta} \quad \text{(3.)}$$

In the two extreme cases the values of that ratio are as follows:—When b is immeasurably longer than a, C A ÷ C T sensibly = 1; when $b = a$, C A ÷ C T = 2.

motion, b B b', of the connected point B. The result will be, that while the point B performs a single stroke, from b to b', the point C will perform a double stroke, from c to C and back again.

If C is a point in a second lever, that second lever may, by means of a similar arrangement, be made to drive a third lever, so as again to double the frequency of the strokes; and thus, by a train of linkwork, the last follower may have the frequency of its strokes increased, as compared with those of the first driver, in a ratio expressed by any required power of 2.

190. **Slow Motion by Linkwork.**—As has been already explained in Article 180, page 193, when the connected point in the driver of an elementary combination by linkwork is at a dead point, the velocity of the connected point of the follower is nothing; and when the connected point of the driver is near a dead point, the motion of the connected point of the follower is comparatively very slow, and gradually increases as the connected point of the driver moves away from the dead point. When, therefore, it is desired that the motion of a follower shall, at and near a particular position of the combination, be very slow as compared with that of the driver, or as compared with that of the follower itself when in other positions, arrangements may be used of the class which is exemplified in fig. 141 and fig. 141 A.

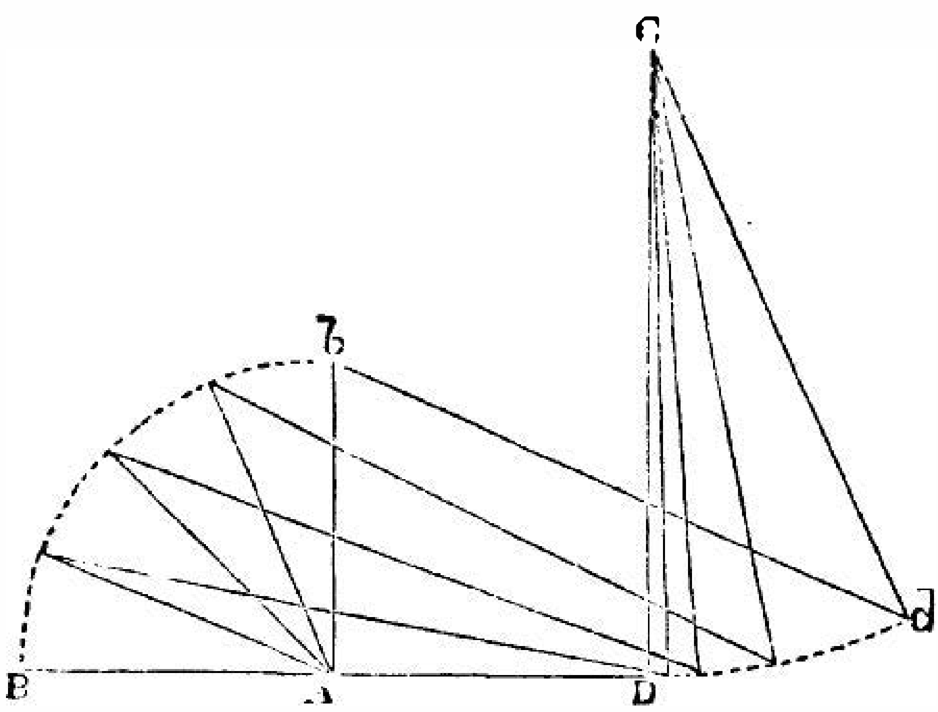

Fig. 141.

In fig. 141 the lever A B, turning about an axis at A, drives, by means of the link B D, the lever C D, which turns about an axis at C. When the driving lever is in the position marked A B, it is in one straight line with the link B D; so that B is a dead point, and the velocity of the follower is null. As the connected point of the driver advances from B towards b, the connected point of the follower advances from D towards d, with a comparative velocity which is at first very small, and goes on increasing by degrees. When the motion is reversed, the comparative velocity of the latter point gradually diminishes as it returns from d towards D, and finally vanishes at the last-named point. Motions of this kind are useful in the opening and closing of steam-valves, in order to prevent shocks.

Fig. 141 A shows a train of two elementary combinations of the same kind with that just described; the effect being to make the motion of a third connected point, E, quite insensible during a certain part of the motion of the first connected point, B.

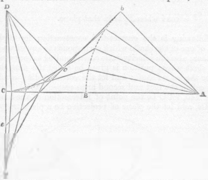

Fig. 141 A.

191. (*A. M.*, 491.) **Hooke's Coupling, or Universal Joint** (fig. 142), is a contrivance for coupling shafts whose axes intersect each other in a point.

Let O be the point of intersection of the axes O C_1, O C_2, and i their angle of inclination to each other. The pair of shafts C_1, C_2, terminate in a pair of forks, F_1, F_2, in bearings at the extremities of which turn the pivots at the ends of the arms of a rectangular cross having its centre at O. This cross is the link; the connected points are the centres of the bearings F_1, F_2. At each instant each of those points moves at right angles to the central plane of its shaft and fork; therefore the line of intersection of the central planes of the two forks, at any instant, is the instantaneous axis of the cross; and the *velocity-ratio* of the points F_1, F_2 (which, as the forks are equal, is also the *angular velocity-ratio* of the shafts), is equal to the ratio of the distances of those points from that instantaneous axis. The *mean* value of that velocity-ratio is that of equality; for each successive *quarter turn* is made by both shafts in the same time; but its instantaneous value fluctuates between the limits,

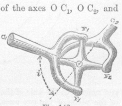

Fig. 142.

$$\left.\begin{array}{l}\dfrac{a_2}{a_1} = \dfrac{1}{\cos i}\text{ when } F_1 \text{ is in the plane of the axes;}\\[2ex] \dfrac{a_2}{a_1} = \cos i \text{ when } F_2 \text{ is in that plane.}\end{array}\right\}\ldots(1.)$$

The following is the geometrical construction for finding the position of one of the shafts which corresponds to any given position of the other; also the velocity-ratio corresponding to that position:— Let the shaft whose position is given be called the *first* shaft, and the other the *second* shaft; and let the corresponding arms of the cross be called the first and second arms respectively.

In fig. 143, let O be the point of intersection of the axes of the two shafts, and let the plane of projection be a plane traversing O,

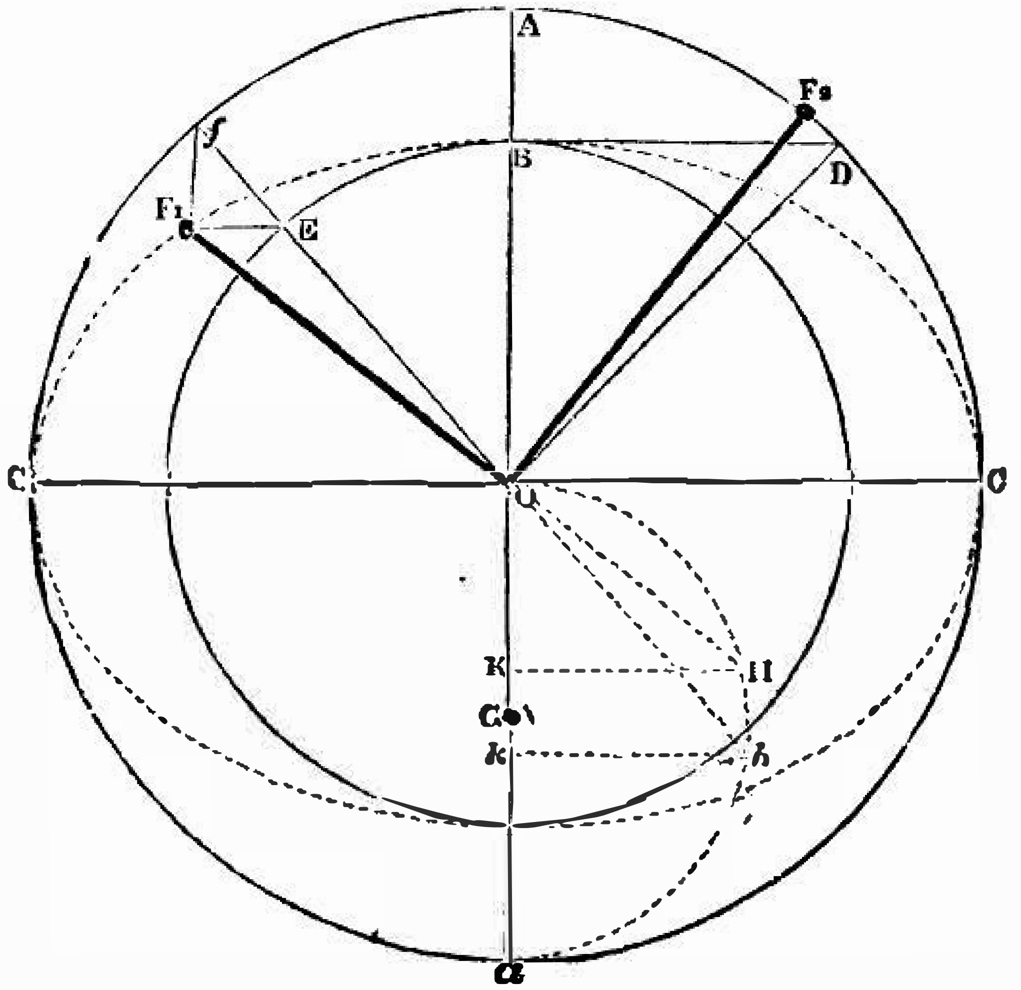

Fig. 143.

and normal to the axis of the *second* shaft. Let A O a be the trace of the plane of the two axes, and C O C, perpendicular to A O a, a normal to that plane. With any convenient radius, O A, describe a circle about O. Lay off the angle A O D equal to the angle i, which the axes of the shafts make with each other. Through D, parallel to C C, draw D B, cutting O A in B; then $\frac{\text{O B}}{\text{O A}} = \cos i$ is the velocity-ratio of the second to the first axis, when the first arm coincides with O C and the second with O A; and $\frac{\text{O A}}{\text{O B}} = \frac{1}{\cos i} = \sec i$ is the velocity-ratio, when the first arm coincides with O A, and the second with O C.

About O, with the radius O B, describe a circle. Draw the radius O E f, cutting the two circles in E and f respectively, and making the angle A O f = the given angle which the *first arm* makes with the plane of the axes:—in other words, let O f be the *rabatment* of the first arm, made by rabatting a plane normal to the first axis upon the plane of projection. Through E, parallel to O C, draw E F_1; and through f, parallel to O A, draw f F_1; the point F_1 will be the projection of the point whose rabatment is f. Draw the straight line O F_1; this will be the *projection of the first arm* on a plane normal to the second axis. Then perpendicular to O F_1 draw O F_2; this will be *the required position of the second arm.*

The projection of the path of the point F_1 is the ellipse C B C.

To find the angular velocity-ratio corresponding to the given position of the arms; about any convenient point, G, in A O a, describe a circle through O, cutting F_1 O and f O (produced if required) in H and h respectively; from which points draw H K and h k parallel to O C, and cutting A O a in K and k respectively. Then we have

$$\frac{a_2}{a_1} = \frac{\mathrm{K\,H}}{k\,h}. \text{*} \qquad (2.)$$

The particular form of universal joint shown in fig. 142 is chosen in order to exhibit all the parts distinctly. In practice, the joint is often made much more compact, the forks not having more space between them and the cross than is necessary in order to admit of the required extent of motion of the cross-arms, and the cross being sometimes made in the form of a circular disc, or of a ring, or of a ball, with four pivots projecting from its circumference. Where the angle of obliquity of the two shafts (i) is small, each of the forks is often made in the form of a round disc on the end of the shaft, having a pair of projecting horns or lugs to carry the bearings of the pivots.

The universal joint belongs to Willis's Class B. When used as a coupling, it is liable to the objection, that although the mean velocity-ratio is uniform, being that of equality, the velocity-ratio at each instant fluctuates, and thus gives rise to vibratory and unsteady motion.

192. (*A. M.*, 492.) The **Double Hooke's Joint** (fig. 144) is used to obviate the vibratory and unsteady motion caused by the fluctuation

* In algebraical symbols, let ϕ_1 = A O f, and ϕ_2 = A O F_2, be the angles made by the first and second arm respectively at a given instant with the plane of the axes of the shafts; then

$$\tan\phi_1 \cdot \tan\phi_2 = \cos i; \text{ and}$$

$$\frac{a_2}{a_1} = -\frac{d\,\phi_2}{d\,\phi_1} = \frac{\sin 2\,\phi_2}{\sin 2\,\phi_1} = \frac{\tan\phi_1 + \operatorname{cotan}\phi_1}{\tan\phi_2 + \operatorname{cotan}\phi_2}.$$

of the velocity-ratio which has already been mentioned. Between the two shafts to be connected, C_1, C_3, there is introduced a short intermediate shaft, C_2, making equal angles with C_1 and C_3, connected with each of them by a Hooke's joint, and having both its own forks in the same plane. The effect of this combination is, that the angular velocities of the *first* and *third* shafts are equal to each other at every instant; and that the planes of the first and third forks make, at every instant, equal angles with the plane of the three axes. Hence, as regards the comparative motion of the first and third shafts, the double Hooke's joint belongs to Class A; but as regards the motion of the second or intermediate shaft, it belongs to Class B.*

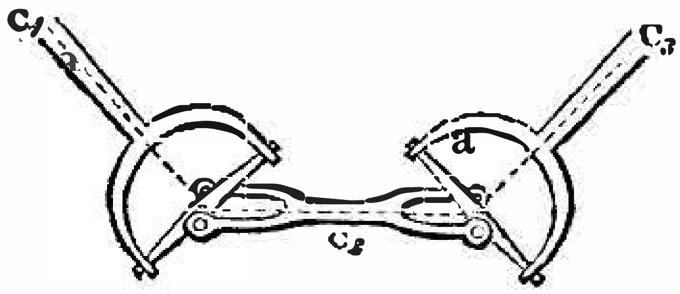

Fig. 144.

The double Hooke's joint works correctly when the third shaft is *parallel* to the first, as well as in the position shown in the figure.

193. **Hooke-and-Oldham Coupling.**—This name may be given to an universal joint in which the pivots of the cross are capable of sliding lengthwise as well as of turning in their bearings in the horns of the forks. It combines the properties of Hooke's coupling with that of Oldham's coupling, formerly described (Article 158, page 166); that is to say, it is capable of transmitting motion between shafts whose axes are neither parallel nor intersecting. It acts by sliding contact and linkwork combined: when single, it belongs to Class B; and when double, with the axes of the three shafts in parallel planes, and the first and third making equal angles with the intermediate axis, to Class A.

194. **Intermittent Linkwork—Click and Ratchet.**—A *click* or *catch*, being a reciprocating bar (such as B C in figs. 145 and 146) acting upon a ratchet wheel or rack, which it pushes or pulls through a certain arc at each forward stroke, and leaves at rest at each backward stroke, is an example of intermittent linkwork. During the forward stroke, the action of the click is governed by the principles of linkwork; during the backward stroke, that action ceases. A *fixed catch*, or *pall*, or *detent* (such as *b c* in fig. 145), turning on a fixed axis, prevents the ratchet wheel or rack from reversing its motion.

* Let i be the angle of inclination of C_1 and C_2, and also that of C_2 and C_3. Let ϕ_1, ϕ_2, ϕ_3, be the angles made at a given instant by the planes of the forks of the three shafts with the plane of their axes, and let a_1, a_2, a_3, be their angular velocities. Then

$$\tan \phi_2 \cdot \tan \phi_3 = \cos i = \tan \phi_1 \cdot \tan \phi_2;$$

whence

$$\tan \phi_3 = \tan \phi_1; \text{ and } a_3 = a_1.$$

The *effective stroke*, being the space through which the ratchet is driven by each forward stroke of the click, is necessarily once, or a

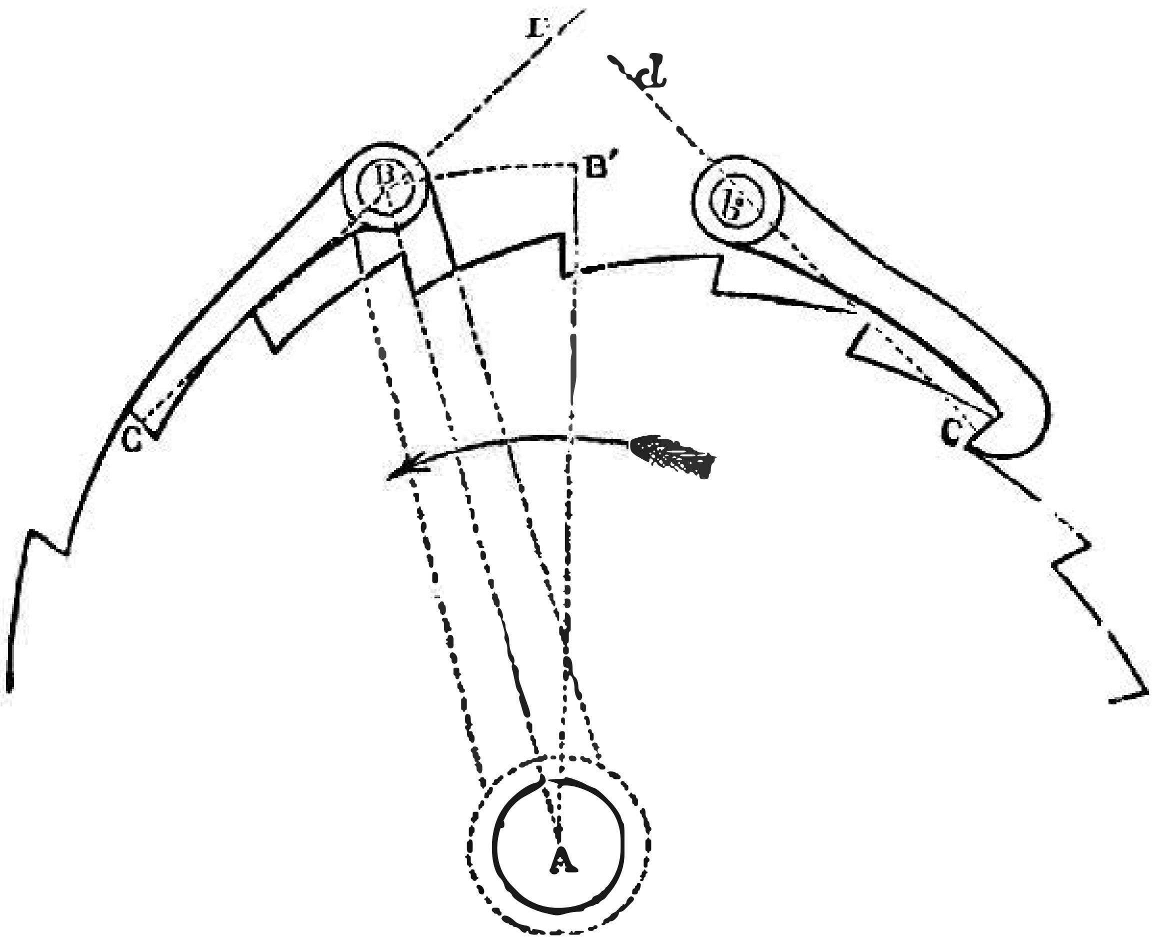

Fig. 145.

whole number of times, the pitch of the teeth of the ratchet; and it is obvious that the length of the total stroke of the click must

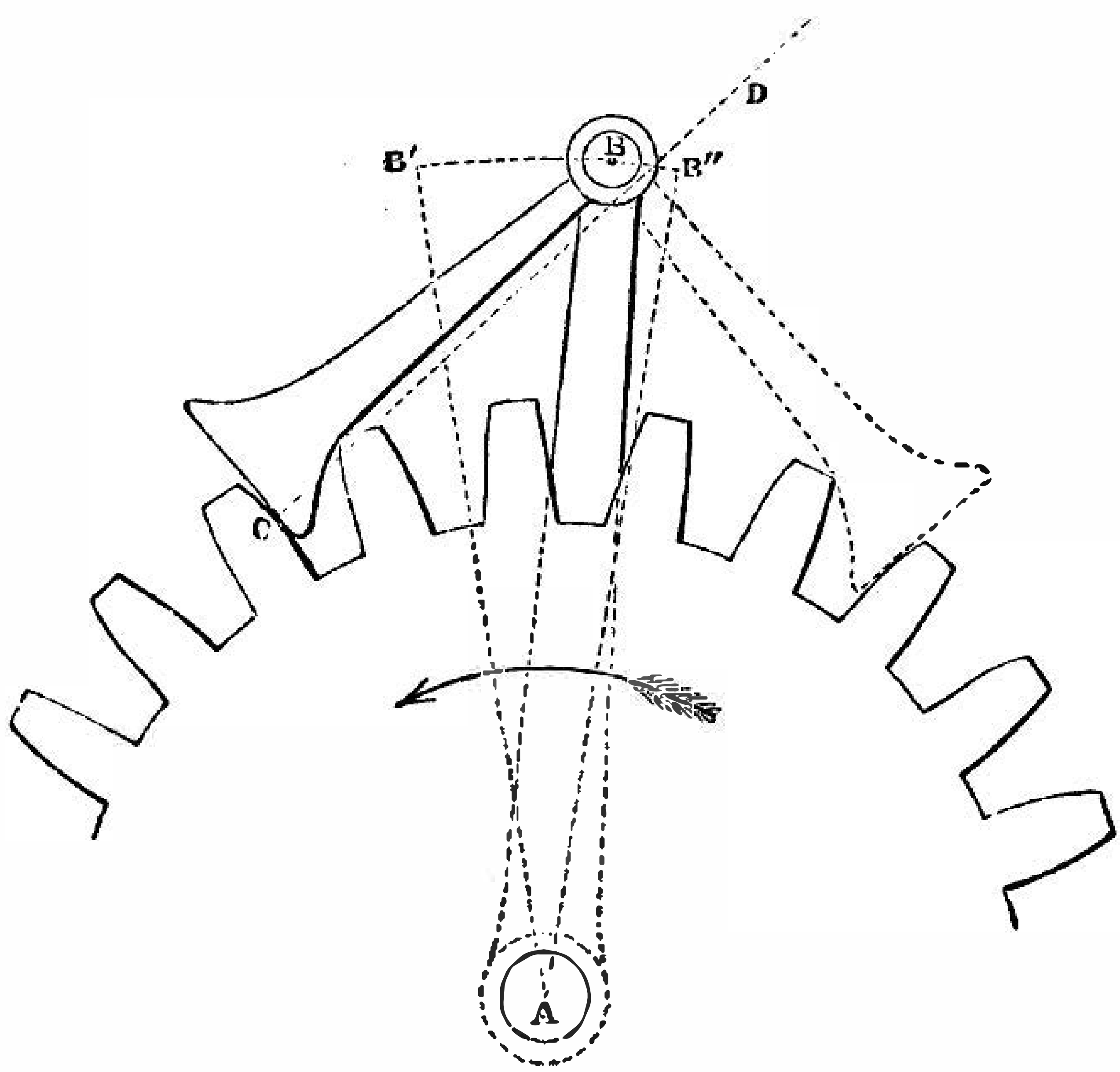

Fig. 146.

be greater than the effective stroke, and less than the next greater whole number of times the pitch. It is advisable, when practicable, to make the excess of the total above the effective stroke no greater than is just sufficient to ensure that the click shall clear each successive tooth of the ratchet. In figs. 145 and 146 the effective stroke is once the pitch of the ratchet; in fig. 147, twice the pitch.

A catch may be made to drop into its place in front of each successive tooth either by gravity or by the pressure of a spring, according to the circumstances of the case.

Some clicks act by thrusting, as B C in fig. 145, and B C in fig. 146; others by pulling, as *b c* in fig. 145.

The direction of the pressure between a click and a tooth is nearly a normal to the acting surfaces of the click and tooth at the centre of their area of contact; for example, in fig. 145, the dotted lines marked C D, *c d*, and in fig. 146, the dotted line marked C D. In order that a click may be certain not to lose its hold of the tooth, that normal *ought to pass inside the axis of motion of a thrusting click, and outside the axis of motion of a pulling click.* For example, in fig. 145, C D passes inside the axis B, and *c d* passes outside the axis *b*; the words "inside" and "outside" being used to denote respectively nearer to and further from the ratchet.

It is convenient, though not essential, that a click for driving a wheel should be carried by an arm concentric with the wheel; such as the arms A B in fig. 145, and A B in fig. 146. In such cases the *total angular stroke* of the click-arm (marked B A B′ in fig. 145, and B′ A B″ in fig. 146) must be a little greater than the effective angular stroke, which is once, or a whole number of times, the pitch-angle of the teeth of the wheel. The axis of motion of the click-arm may, however, be placed elsewhere if necessary, provided care is taken that in all positions of the arm the line of pressure passes to the proper side of the axis of motion of the click. (See figs. 148, 149, further on.)

Fig. 146 represents a *tumbling* or *reversible click*, shaped so as to act upon the teeth of an ordinary toothed wheel. In its present position it drives the wheel in the direction pointed out by the arrow: by throwing it over into the position marked with dotted lines, it is made, when required, to drive the wheel the contrary way.

It is easy to see that the acting surfaces of clicks, and the teeth of ratchets on which they act, may be shaped in a variety of ways besides those exemplified in the figures.

195. **Silent Click.**—This is a contrivance for avoiding the noise and the tear and wear which arise from the sudden dropping of the common click into the space between the teeth of the ratchet.

wheel. The wheel is like an ordinary toothed wheel. B C is the click, which, in the example, is made to push the teeth. It is carried by one branch, A B, of a bell-crank lever, which has a rocking motion about the same axis with the wheel. The other

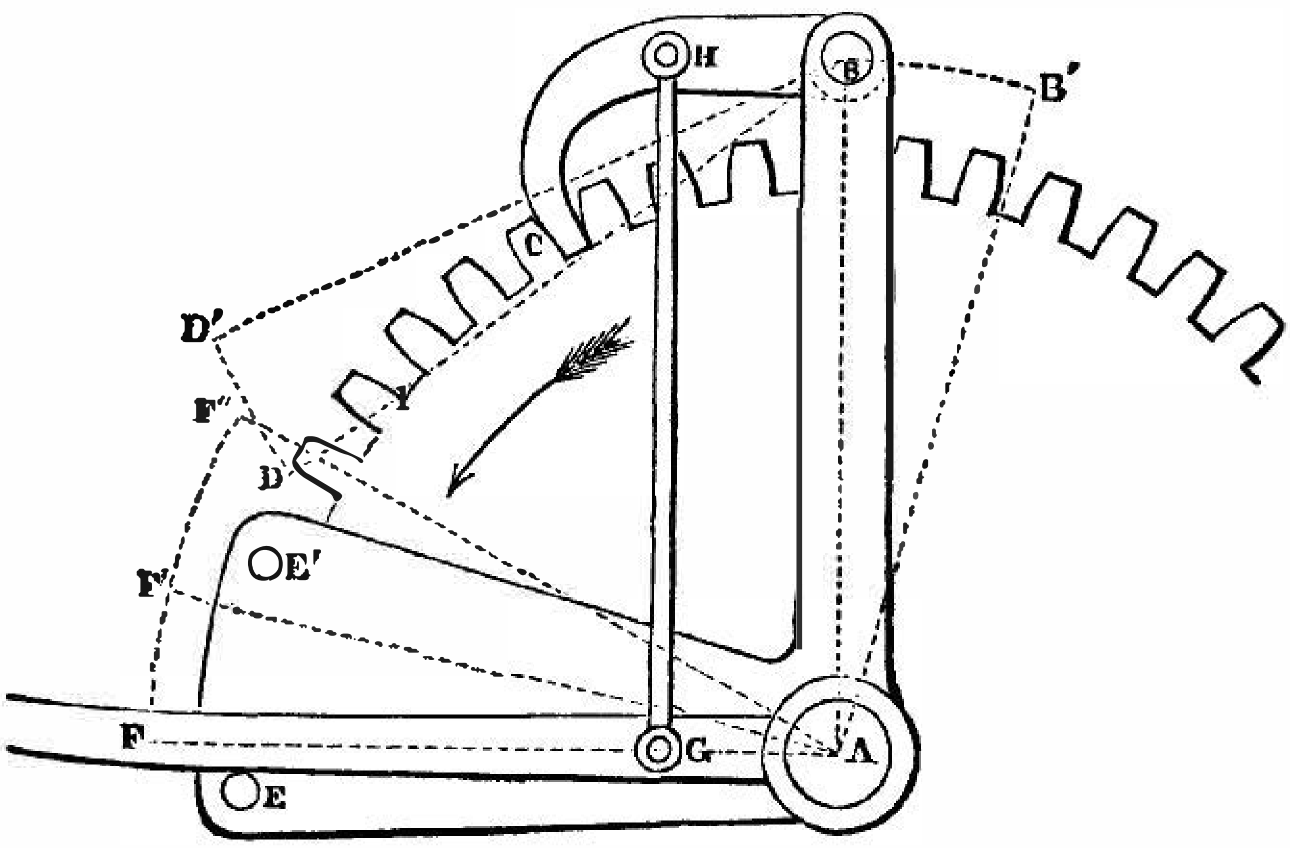

Fig. 147.

branch of the bell-crank lever has two studs or pins in it, E and E′. Between these pins is the driving arm, A F, which has a reciprocating motion about the same axis, and is connected by a link, G H, with the click.

B A B′ is the total angular stroke of the bell-crank lever; D B D′ is the angle through which the click must be moved in order to lift it clear of the teeth. The sum of these angles, B A B′ + D B D′, is = F A F‴, the angular stroke of the driving arm. The positions of the studs, E and E′, are so adjusted, that the driving arm in passing from the one to the other moves through the angle F A F′ = D B D′; being the angular motion that lifts the click clear of the teeth before the return stroke, or makes it take hold before the forward stroke. During those parts of the motion of the driving arm and click, the bell-crank lever stands still: its forward and return strokes are made by the driving arm pressing against the studs E and E′ respectively.

196. **Double-Acting Click.**—This is the contrivance sometimes called, from its inventor, "the lever of La Garousse." It consists of two clicks making alternate strokes, so as to produce a nearly continuous motion of the ratchet which they drive; that motion being intermitted for an instant only at each reversal of the direction of movement of the clicks. In fig. 148 the clicks act by pushing; in fig. 149, by pulling. The former arrangement is on the whole the best adapted to cases in which the mechanism

requires considerable strength; such as windlasses on board ship. Each single stroke of the click-arms advances the ratchet through one-half of its pitch.

Corresponding points in the two figures are marked with the same letters; and as fig. 148 contains some parts which do not

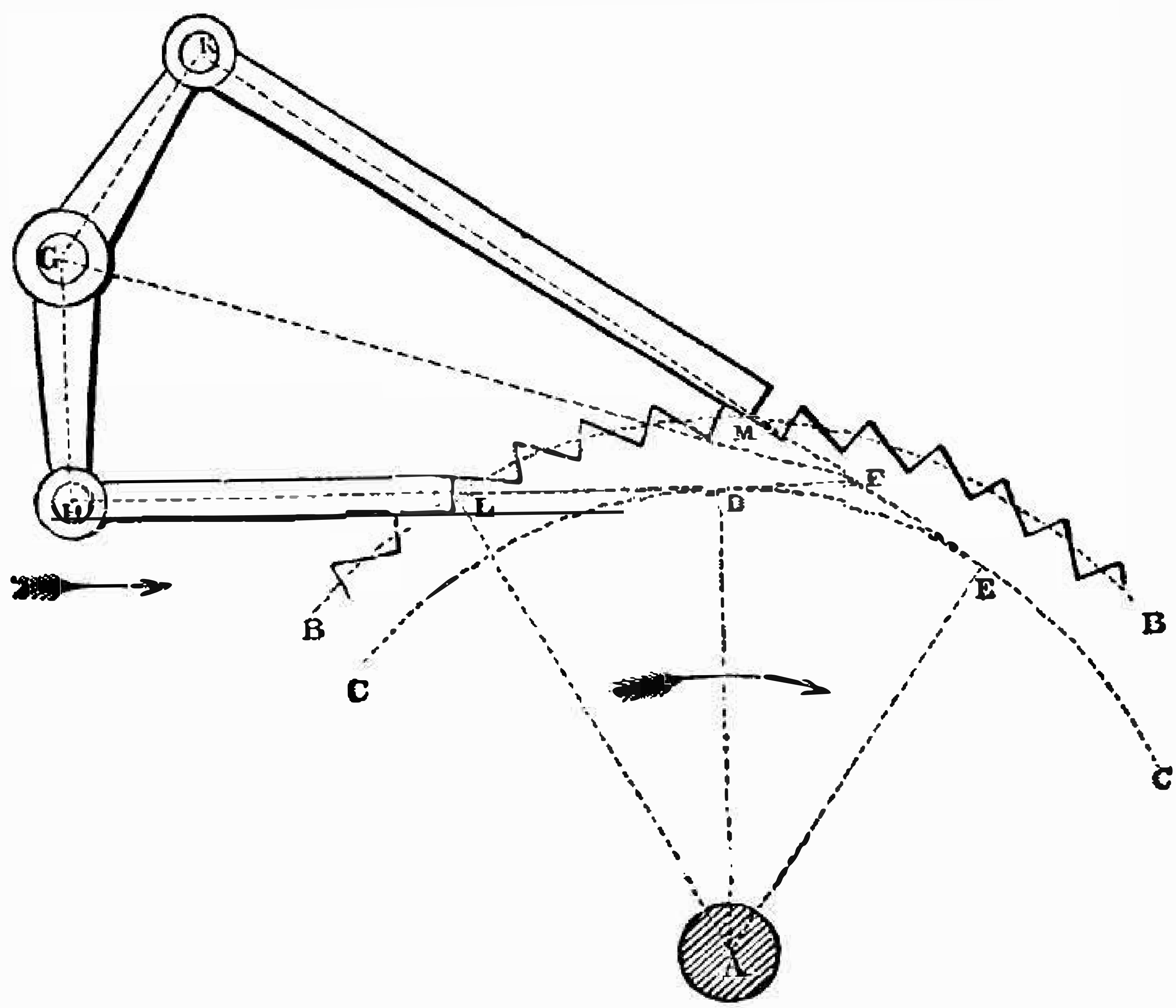

Fig. 148.

occur in fig. 149, the former will, in the first place, be referred to in explaining the principles to be followed in designing such combinations.

Let the figure and dimensions of the ratchet-wheel be given, and let A be its axis, and B B its pitch-circle; that is, a circle midway between the points and roots of the teeth.

Having fixed the mean obliquity of the action of the clicks—that is, the angle which their lines of action, at mid-stroke, are to make with tangents to the pitch-circle—draw any convenient radius of the pitch-circle, as L A, and from it lay off the angle L A D, equal to that obliquity. On A D let fall the perpendicular L D, and with the radius A D describe the circle C C; this will be the *base-circle*, to which the lines of action of the clicks are to be tangents. (As to base-circles, see also Article 131, page 121.) Lay off the angle D A E equal to *an odd number of times half the pitch angle;* then through the points D and E in the base-circle draw two tangents, cutting each other in F. Draw F G, bisecting the angle at F, and take any convenient point in it, G, for the trace of the axis of motion of the rocking-shaft which carries the click-arms. From G let fall G H and G K perpendicular to the tangents F D H and E F K; then H and K will be the positions of the centres of motion of the two clicks at mid-stroke; and G H and F K will represent the click-arms. Let L and M be the points where D H and E K respectively cut the pitch-circle; then H L and K M will be the lengths of the two clicks. The *effective stroke* of each click will be equal to half the pitch, *as measured on the base-circle* C C; and the total stroke must be as much greater as is necessary in order to make the clicks clear the teeth.

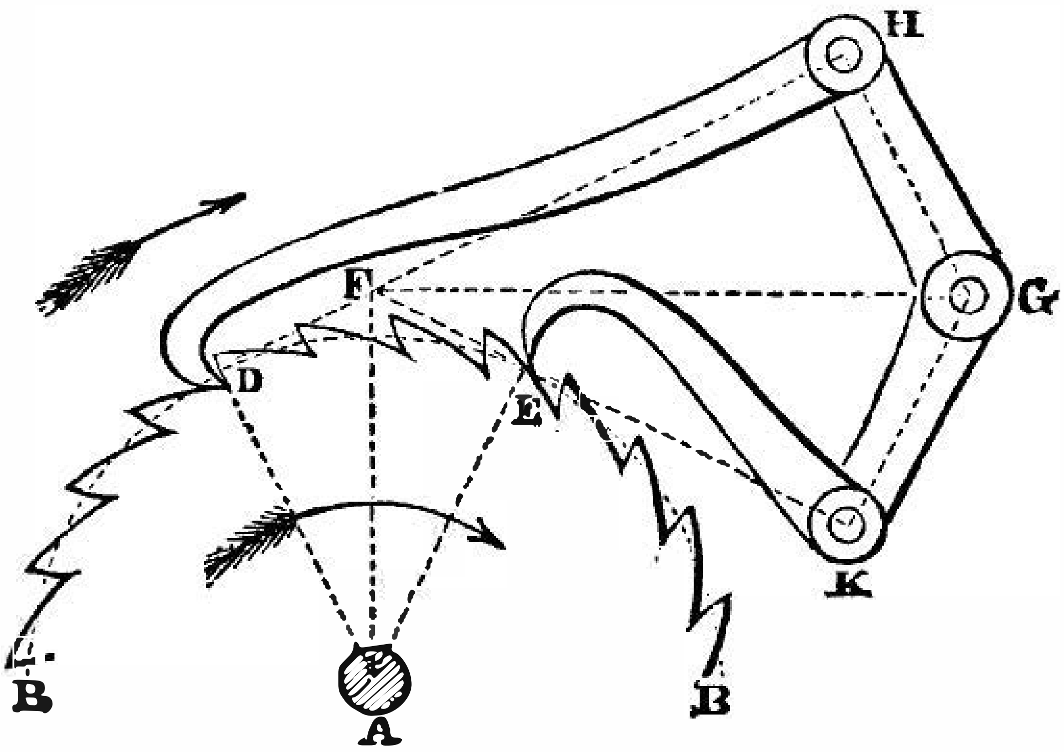

Fig. 149.

In fig. 149, where the clicks pull instead of pushing, the obliquity is nothing; and the consequence is that the base-circle, C C, coincides with the pitch-circle, B B, and that the points L and M coincide respectively with D and E.

197. **Frictional Catch.**—The frictional catch (called sometimes the "silent feed-motion") is a sort of intermittent linkwork, founded on the dynamical principle, that two surfaces will not slide on each other so long as the angle which the direction of the pressure exerted between them makes with their common normal at the place where they touch each other is less than a certain angle called the *angle of repose*, which depends on the nature of the surfaces, and their state of roughness or smoothness, and of lubrication. The smoother and the better lubricated the surfaces, the smaller is the angle of repose.

In trigonometrical language, the angle of repose is the angle whose *tangent is equal to the co-efficient of friction:* that is, to the ratio which the friction between two surfaces, being the force which resists sliding, bears to the normal pressure; or, what is the same thing, it is the angle whose *sine is equal to the ratio that the friction bears to the resultant pressure* when sliding takes place. The subject of friction, and of the angle of repose, properly belong to the dynamical part of this treatise, and will be mentioned in greater detail further on. For the present purpose it is sufficient to state that the sine of the angle of repose for metallic surfaces in a moderately smooth state, and not lubricated, as deduced from the experiments of Morin, ranges from 0·15 to 0·2, or thereabouts; so that an angle whose sine is *one-seventh* of radius may be considered to be less than the angle of repose of any pair of metallic surfaces which are in the above-mentioned condition.

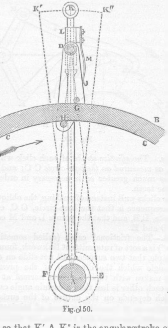

Fig. 150.

The frictional catch, though always depending on the principle just stated, is capable of great variety in detail. The arrangement represented in fig. 150 is constructed in the following manner:—

The shaft and rim of the wheel to be acted upon are shown in section. A K is the catch-arm, having a rocking motion about the axis A of the wheel; the link by which it is driven is supposed to be jointed to it at K; and K′ K″ represents the stroke, or arc of motion, of the point K; so that K′ A K″ is the angular stroke of the catch-arm. L is a socket, capable of sliding longitudinally on the catch-arm to a small extent; a shoulder for limiting the extent of that sliding motion is marked by dotted lines. The socket and the part of the arm on which it slides should be square, and not round, to prevent the socket from turning. From the side of the socket there projects a pin at D, from which the catch D G H hangs. M is a spring pressing against the forward side of the catch. G and H are two studs on the catch, which grip and carry forward the rim, B B C C, of the wheel during the forward stroke, by means of friction, but let it go during the return stroke.

A similar frictional catch, not shown in the figure, hanging from a socket on a fixed instead of a moveable arm, at any convenient part of the rim of the wheel, serves for a detent, to hold the wheel still during the return stroke of the moveable catch-arm.

The following is the graphic construction for determining the proper position of the studs G and H:—Multiply the radii of the outer and inner surfaces, B B and C C, of the rim of the wheel by a co-efficient a little less than the sine of the angle of repose—say $\frac{1}{7}$—and with the lengths so found as radii describe two circular arcs about A; the greater (marked E) lying in the direction of forward motion, and the less (marked F) in the contrary direction. From D, the centre of the pin, draw D E and D F, touching those two arcs. Then G, where D E cuts B B, and H, where D F cuts C C, will be the proper positions for the points of contact of the two studs with the rim of the wheel. For the force by which the catch is driven during the forward stroke acts through D; that force is resolved into two components, acting along the lines D G E and F H D respectively; and those lines make with the normals to the rim of the wheel, at G and H respectively, angles less than the angle of repose of a pair of metallic surfaces that are not lubricated. Should it be thought desirable, the positions of the holding studs, or of one of them, may be made adjustable by means of screws or otherwise.

The stiffness of the spring M ought to be sufficient to bring the catch quickly into the holding position at the end of each return stroke.

The length of stroke of a frictional catch is arbitrary, and may, by suitable contrivances, be altered during the motion. Contrivances for that purpose will be described further on.

A pair of frictional catches may be made double-acting, like the double-acting clicks of the preceding Article.

198. **Slotted Link.**—A slotted link is connected with a pin at one of its ends, not by a round hole fitting the pin closely, but by an oblong opening or slot with semicircular ends. This is an example of intermittent linkwork; the intermission in its action taking place during the middle part of each stroke, while the pin is shifting its position relatively to the link from the one end of the slot to the other. That intermission takes effect by producing a pause in the motion of that piece which is the follower, and which may be either the link or the pin; and the stroke of the follower is shorter than that of the driver by an extent corresponding to the length of the slot, as measured from centre to centre of its two semicircular ends.

199. **Band Links.**—Where tension alone, and not thrust, is to act along a link, it may be flexible, and may consist either

of a single band, or of an endless band passing round a pair of pulleys which turn round axes traversing and moving with the connected points. For example, in fig. 151, A is the axis of a rotating shaft, B that of a crank-pin, C the other connected point, and B C the line of connection; and the connection is effected by means of an endless band, passing round a pulley which is centred upon C, and round the crank-pin itself, which acts as another pulley. The pulleys are of course secondary pieces; and the motion of each of them belongs to the subject of aggregate combinations, being compounded of the motion which they have along with the line of connection, B C, and of their respective rotations relatively to that line as their line of centres; but the motion of the points B and C is the same as if B C were a rigid link, provided that forces act which keep the band always in a state of tension.

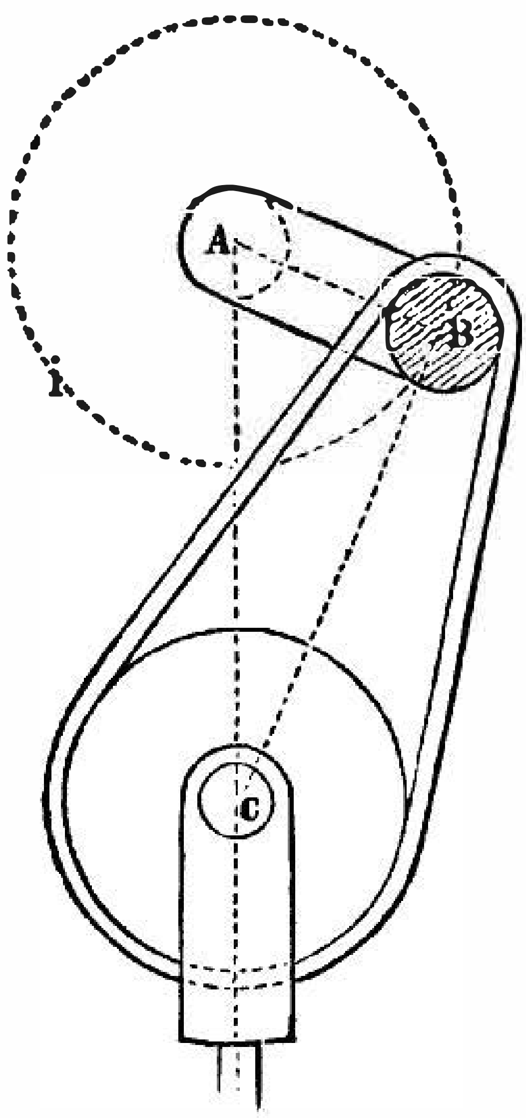

Fig. 151

This combination is used in order to lessen the friction, as compared with that which takes place between a rigid link and a pair of pins; and the band employed is often a leather chain, of the kind already mentioned in Article 176, page 191, because of its flexibility.

SECTION VII.—*Connection by Plies of Cord, or by Reduplication.*

200. **General Explanations.** (*A. M.*, 494.)—The combination of pieces connected by the several plies of a cord, rope, or chain, consists of a pair of cases or frames called *blocks*, each containing one or more pulleys called *sheaves*. One of the blocks (A, figs. 152, 153), called the *fixed block*, or *fall-block*, is fixed; the other, called the *fly-block*, or *running block*, B, is moveable to or from the fall-block, with which it is connected by means of a rope, or *fall*, of which one end is fastened either to a fixed point or to the running block, while the other end, C, called the *hauling part*, is free; and the intermediate portion of the rope passes alternately round the pulleys in the fixed block and running block. The several plies of the rope are called by seamen *parts;* and the part which has its end fastened is called the *standing part*. The whole combination is called a *tackle* or *purchase*. When the hauling part is the driver, and the running block the follower, the two blocks are being drawn

together; when the running block is the driver, and the hauling part the follower, the two blocks are being pulled apart.

201. **Velocity-Ratios.** (*A. M.*, 495, 496.)—The *velocity-ratio* chiefly considered in a purchase is that between the velocities of the running block, B, and of the hauling part, C. That ratio is expressed by the *number of plies* of rope by which the running

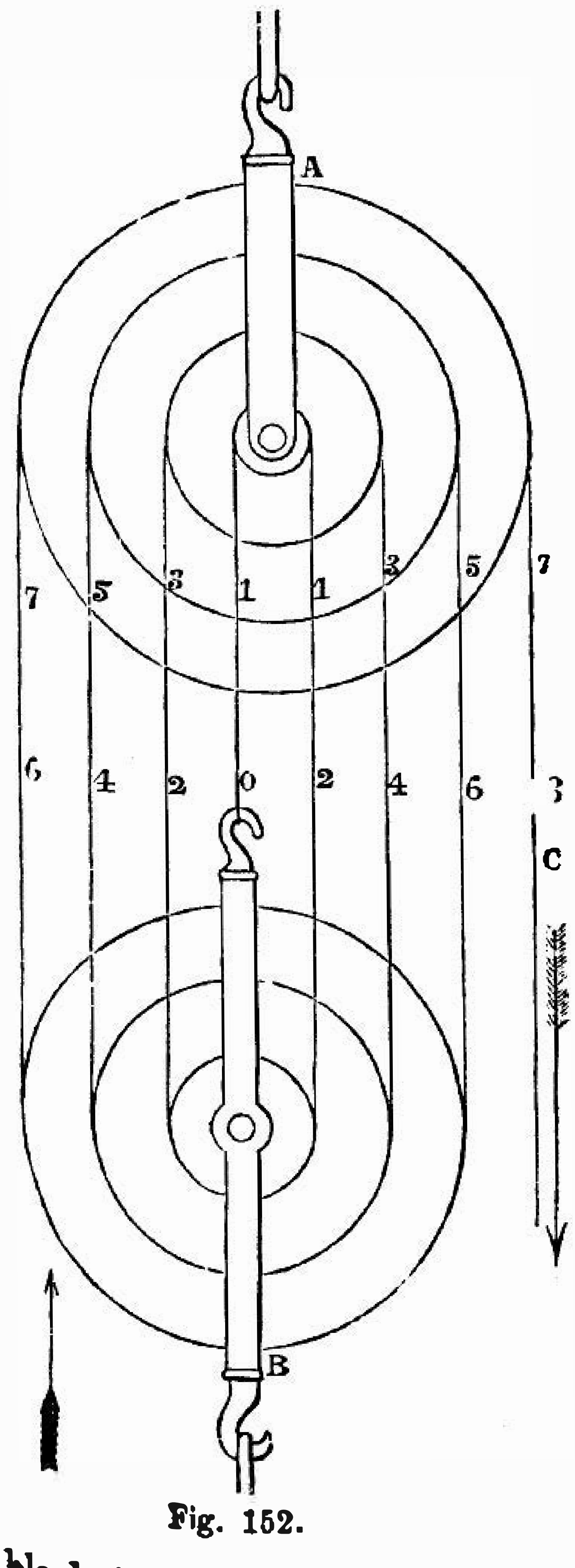

Fig. 152.

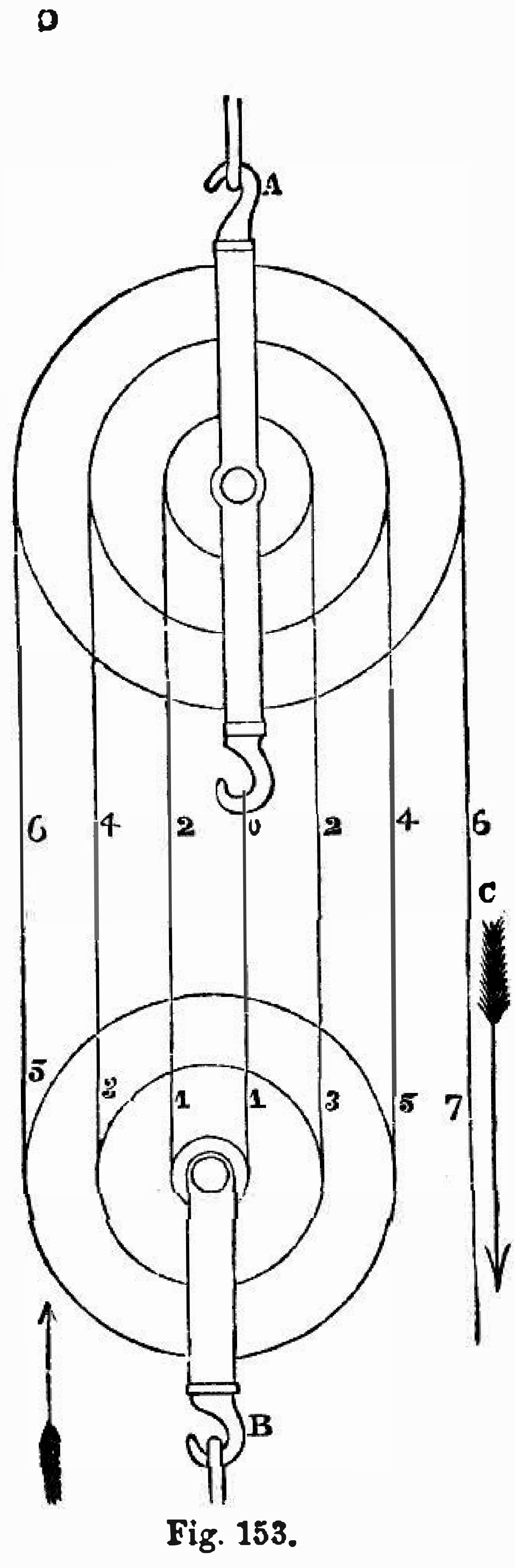

Fig. 153.

block is connected with the fall-block. Thus, in fig. 152, C ÷ B = 7; and in fig. 153, C ÷ B = 6. A tackle is called a *twofold* *purchase*, a *threefold purchase*, and so on, according to the value of the velocity-ratio C ÷ B. For example, fig. 152 is a sevenfold purchase, and fig. 153 a sixfold purchase.

The *velocity of any ply or part* of the rope is found in the following manner:—For a ply on the side of the fall-block, A, next the hauling part, C, it is to be considered what would be the velocity of that ply if it were itself the hauling part: that is to say, the ratio of its velocity to that of the running block is expressed by the number of plies *between* the ply in question and the point of attachment of the standing part. For a ply on the side of the fall-block furthest from the hauling part, the velocity is equal and contrary to that of the next succeeding ply, with which it is directly connected over one of the sheaves of the fall-block. If the standing part is attached to a fixed point, as in fig. 153, its velocity is nothing; if to the running block, as in fig. 152, its velocity is equal to that of the block. The comparative velocities of the several parts of the ropes are expressed by the upper row of figures. The lower row of figures express the velocities of the several parts relatively to the running block.

202. **Ordinary Form of Pulley-Blocks.**—A block, as used on board ship, consists of an oval *shell*, usually of elm or metal, containing one or more pulleys, called *sheaves*, of lignum-vitæ or metal, turning about a cylindrical wrought-iron *pin*. The round hole in the centre of a wooden sheave is lined with a gun-metal tube called the *bushing*. The part of the sheave-hole through which the rope or chain reeves is called the *swallow*. In the bottom and sides of a block is a groove called the *score*, into which fits the *strop* or *strapping* of rope or iron by which the block is hung or secured to its place. Ordinary blocks containing one pin are called *single*, *double*, *treble*, &c., according to the number of sheaves that turn about that pin side by side. Each sheave turns in a separate hole in the shell. Fig. 154 shows examples of the forms of iron pulley-blocks commonly used in machinery on land. A is a treble block; B, a double block. The block B has an eye for the attachment of the standing part of the rope.


A


B

Fig. 154.

203. **White's Pulleys.**—When the sheaves of a block, as in the ordinary form, are all of the same diameter, they all turn with different angular velocities, because of the different velocities of the plies of rope that pass over them. But by making the effective radius of each sheave proportional to the velocity, *relatively to the block*, of the ply of rope which it is to carry, the angular velocities of the sheaves in one block may be rendered equal; so that the

sheaves may be made all in one piece, having two journals which turn in fixed bearings.

These are called "White's Pulleys," from the inventor; and they are represented in figs. 152 and 153, page 215: having been chosen to illustrate the general principles of the action of blocks and tackle, because of the clearness with which they show the positions of all the parts of the rope. They are not, however, much used in practice, because the unequal stretching of different parts of the cord prevents the combination from working with that degree of accuracy which is necessary in order that any advantage may be obtained by means of it over the common construction.

204. **Compound Purchases.** — A compound purchase consists of a *train* of simple purchases; that is to say, the hauling part of one tackle is secured to the running block of another, and so on, for any number of tackles. In practice, however, the number of tackles in a compound purchase is almost always two; and then the rope that has the running block secured to it is usually called the *pendant*, and the rope that is directly hauled upon by hand, the *fall*.

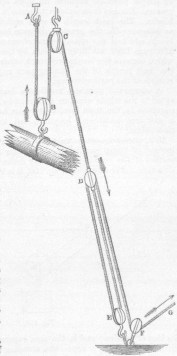

Fig. 155.

The velocity-ratio is, as in other trains of elementary combinations, the product of the velocity-ratios belonging to the elementary or simple tackles of which the compound purchase consists.

For example, in fig. 155, A B C is a twofold purchase; and at D, its pendant is secured to the fly-block of a threefold purchase, D E F, whose hauling part is F G. The velocity-ratio of D to B is 2, and that of G to D is 3; so that the velocity-ratio of G to B is $2 \times 3 = 6$; and the compound purchase is sixfold.

205. **Rope and Space Required for a Purchase.**—An elementary or simple purchase requires no more space to work in than the greatest distance from outside to outside of the fixed and running blocks. The least length of rope sufficient for it may be found as follows:—To the greatest distance between the centres of the blocks add half the *effective* circumference of a sheave (see Article 166, page 180); multiply the sum by the number of plies of rope which connect the blocks with each other; and to the product add the least length of the hauling part required under the circumstances of the particular case.

A compound purchase requires a length of space to work in equal to the whole distance traversed by the fly-block of the last purchase in the train (viz., that whose hauling part is free), with a sufficient additional length added for the blocks and their fastenings.

206. **Obliquely-acting Tackle.**—The parts of the rope of a tackle, instead of being parallel to each other and to the direction of motion of the running block, may make various angles with that direction. For example, in fig. 156, B is the running block, and B *b*

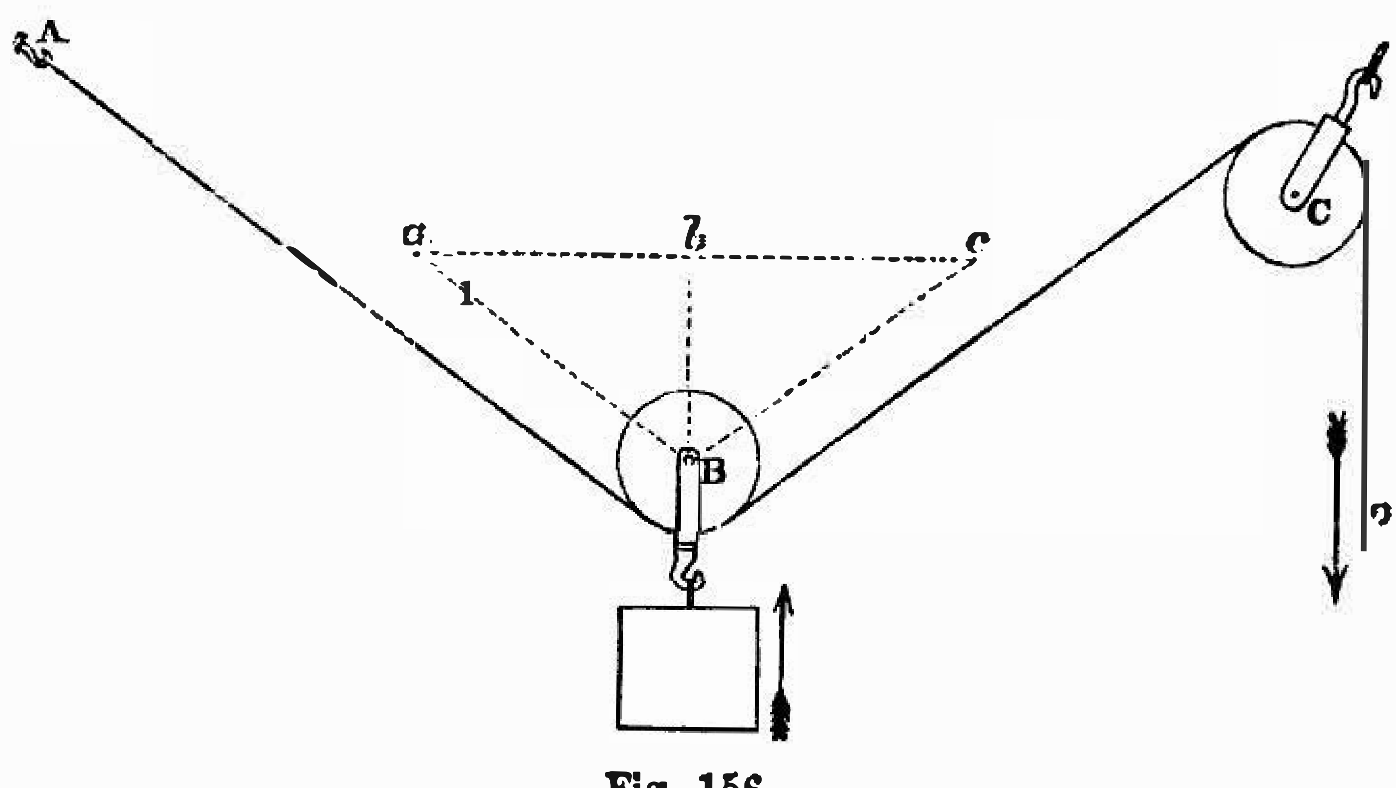

Fig. 156.

its line of motion; and in the case represented, that block hangs from two parts of a rope—the standing part, B A, and another part, B C. To find the velocity-ratio of the hauling part, D, to the running block, B: from the centre, B, of that block, draw straight lines, B *a*, B *c*, parallel to the parts of the rope by which it hangs; at any convenient distance from B, draw the straight line *a b c* perpendicular to B b, and cutting all the straight lines which diverge from B; then,

as B b : is to B a + B c,
: : so is the velocity of B
: to the velocity of D;

and the same rule may be extended to any number of parts, thus:

$$\frac{\text{velocity of D}}{\text{velocity of B}} = \frac{\text{sum of lengths cut off on lines diverging from B}}{\text{B}\,b}.$$

The combination belongs to Class B; because, owing to the continual variation of the obliquity of the parts of the rope, the velocity-ratio is continually changing.

206 A. **Tiller-Ropes.**—The *tiller* of a ship is a horizontal lever projecting from the rudder-head, by means of which the position of

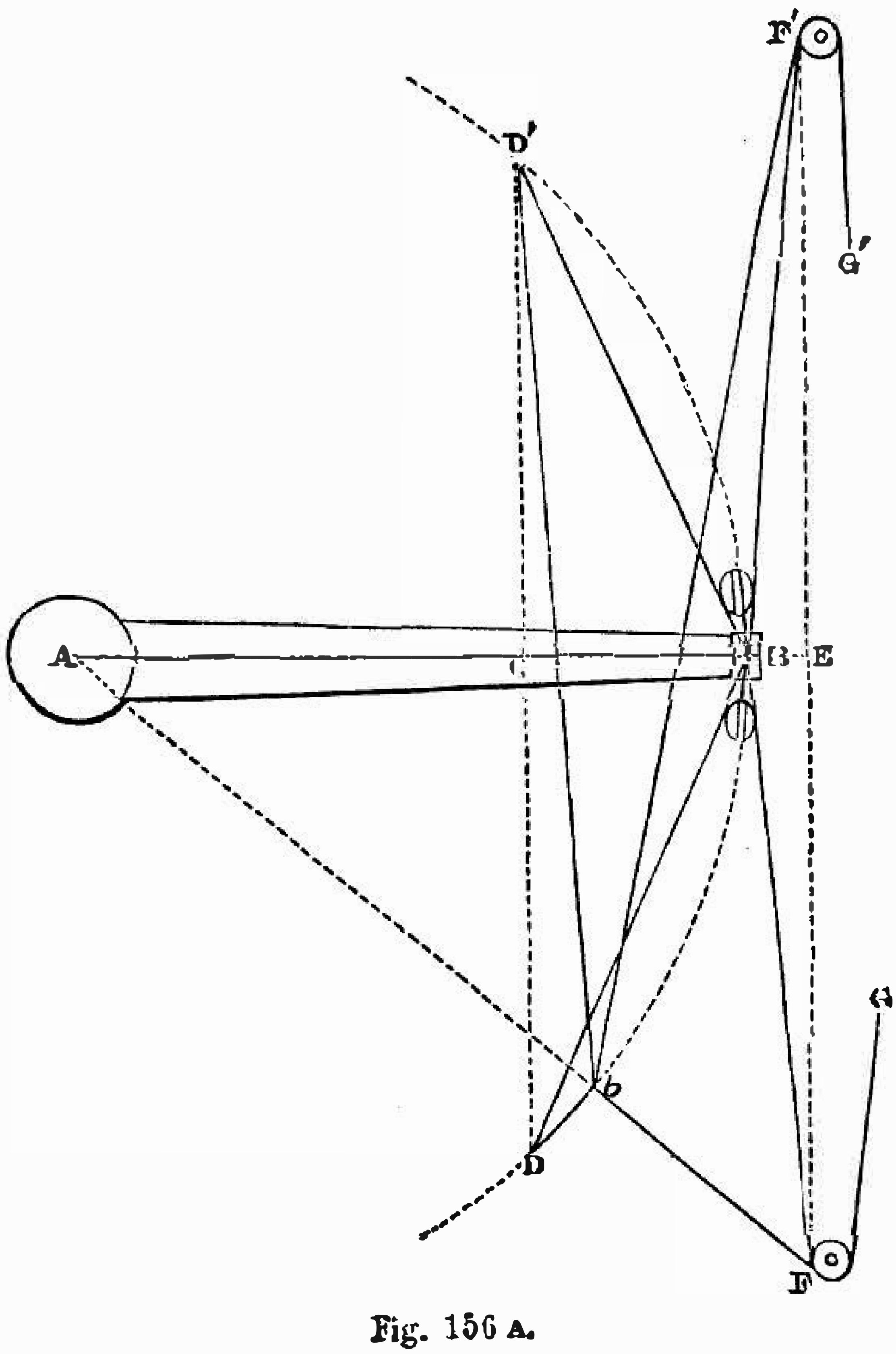

Fig. 156 A.

the rudder is adjusted. It usually points forward; that is, in the contrary direction to the rudder itself. In ships of war the tiller is usually *put over*, or moved to one side or to the other, by means of a pair of obliquely-acting twofold tackles, made of raw hide ropes, which haul it respectively *to starboard* (that is, towards the right) and *to port* (that is, towards the left), when required. The hauling parts of both tackles are guided by fixed pulleys so as to be wound in opposite directions round one barrel, which is turned by means of the steering-wheel.*

Fig. 156 A is a plan of this combination. A is the rudder-head; A B, the tiller, shown as amidships, or pointing right ahead; D B F G is the starboard tiller-rope; D′ B F′ G′, the port tiller-rope. These ropes are made fast to eye-bolts at D and D′; at B they are rove through blocks that are secured to the tiller; at F and F′ they are led round fixed pulleys; and G and G′ are their hauling parts, which are led, by means of pulleys which it is unnecessary to show in the figure, to the barrel of the steering-wheel.

A b is the position of the tiller when put over about 40° to starboard; and the corresponding positions of the tiller-ropes are D b F G and D′ b F′ G′.

In order that the tiller-ropes may never become too slack, it is necessary that the sum of the lengths of their several parts should be nearly constant in all positions of the tiller; that is to say, that we should have, in all positions,

$$\mathrm{D}\,b + b\,\mathrm{F} + \mathrm{D}'\,b + b\,\mathrm{F}' \text{ nearly } = 2\,(\mathrm{D\,B} + \mathrm{B\,F}).$$

That object is attained, with a rough approximation sufficient for practical purposes, by adjusting the positions of the points D, D′, and F, F′, according to the following rule:—

RULE.—About A, with the radius A B, describe a circle. Make $\mathrm{A\,C} = \frac{2}{3}\,\mathrm{A\,B}$; and through C, perpendicular to A B, draw a straight line cutting that circle in D and D′. These will be the points at which the standing parts of the ropes are to be made fast. Then produce A B to E, making $\mathrm{B\,E} = \frac{1}{12}\,\mathrm{A\,B}$; and through E, perpendicular to A B E, draw F′ E F, making $\mathrm{E\,F} = \mathrm{E\,F}' = \frac{5}{8}\,\mathrm{C\,D}$; F and F′ will be the stations for the fixed blocks.

When the angle B A b is about 40°, the sum of the lengths of the parts of the ropes is a little greater than when the tiller is amidships; but the difference (which is about one-50th part of the length expressed in the preceding equation) is not so great as to

* See Peake's *Rudimentary Treatise on Shipbuilding*, second volume, pp. 66, 162; also Watts, Rankine, Napier, and Barnes *On Shipbuilding*, p. 202.

cause any inconvenient increase of tightness. For angles not exceeding 30° the approximation to uniformity of tightness is extremely close.

SECTION VIII.—*Hydraulic Connection.*

207. **General Nature of the Combinations.**—The kind of combinations to which the present section relates are those in which two cylinders fitted with moveable pistons are connected with each other by a passage, and the space between the pistons is entirely filled with a mass of fluid of invariable volume.

Any liquid mass may be treated, in most practical questions respecting the transmission of motion, as if its volume were invariable, because of the smallness of the change of volume produced in a liquid by any possible change of pressure. For example, in the case of water, the compression produced by an increase in the intensity of the pressure to the extent of one atmosphere (or 14·7 lbs. on the square inch), is only one-20,000th part of the whole volume. (See Article 88, page 75.)

The volume, then, of the mass of fluid enclosed in the space between two pistons being invariable, it follows that if one piston (the driver) moves inwards, sweeping through a given volume, the other piston (the follower) must move outwards, sweeping through an exactly equal volume; otherwise the volume of the space contained between the pistons would change; and this is the principle upon which the comparative motion in hydraulic connection depends.

208. **Cylinders, Pistons, and Plungers.**—A piston is a primary piece, sliding in a vessel called a cylinder. The motion of the piston is most commonly straight; and then the bearing surfaces of the piston and cylinder are actually cylindrical, in the mathematical sense of that word.

When the motion of a piston is circular, the bearing surfaces of the piston, and of the vessel in which it slides, are surfaces of revolution described about the axis of rotation of the piston; but that vessel, in common language, is still called a *cylinder*, although its figure may not be cylindrical.

A *plunger* is distinguished from an ordinary piston in the following way:—The bearing surface of a cylinder for a plunger consists merely of a *collar*, of a depth sufficient to prevent the fluid from escaping; and the plunger slides through that collar, and has a bearing surface of a length equal to the depth of the collar added to the length of stroke; so that during the motion different parts of the surface of the plunger come successively into contact with the same surface of the collar. On the other hand, an ordinary piston has a bearing surface of a depth merely sufficient to prevent the fluid from escaping; and the cylinder has a bearing surface of a length equal to the depth of that of the piston added to the length of stroke; so that during the motion the same surface of the piston comes into contact successively with different parts of the surface of the cylinder. For example, in fig. 157, A is a plunger, working through the collar B in the cylinder C; and in fig. 158, A is an

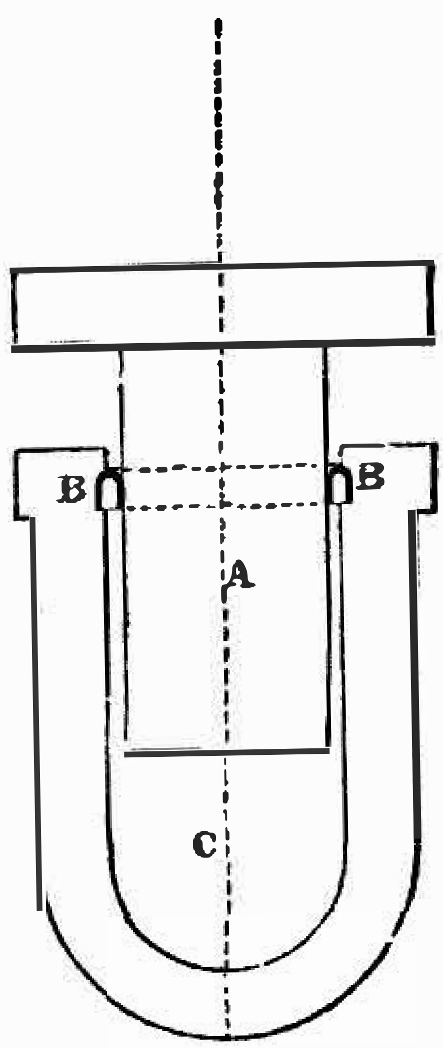

Fig. 157.

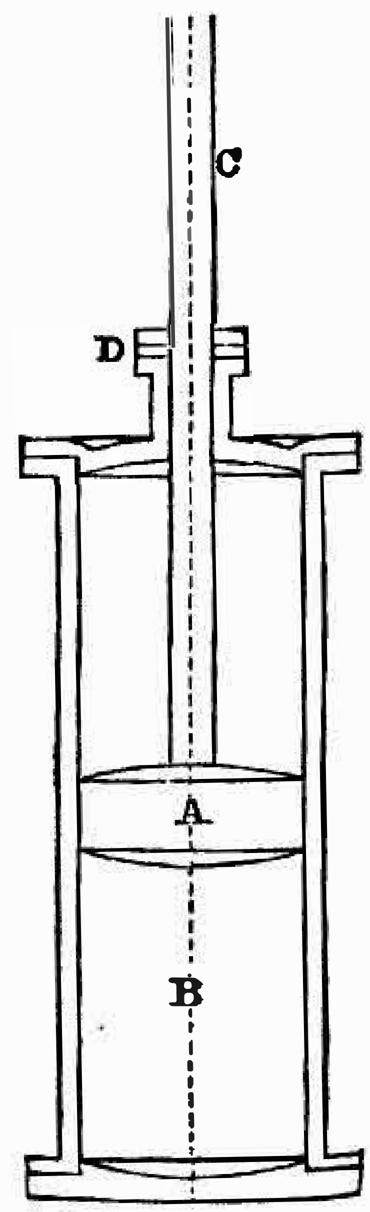

Fig. 158.

ordinary piston, working in the cylinder B. The action of plungers and of ordinary pistons in transmitting motion is exactly the same; and in stating the general principles of that action, the word *piston* is used to include plungers as well as ordinary pistons.

The *volume swept* by a piston in a given time is the product of two factors—transverse area and length. The *transverse area* is that of a plane bounded by the bearing surface of the piston and cylinder, and normal to the direction of motion of the piston, so that it cuts that surface everywhere at right angles. In a straight-sliding piston that plane is normal to the axis of the cylinder; in a piston moving circularly, it traverses the axis of rotation of the piston: in other words, the area is that of a projection of the piston on a plane normal to its direction of motion.

When the motion of the piston is straight, the *length* of the volume swept through is simply the distance moved by each point of the piston. When the motion is circular, that length is the distance moved through by the *centre of the area* of the piston.*

So long as the transverse area and length of the space swept by a piston are the same, it is obvious that the form of the ends of that piston does not affect the volume of that space.

When the space in the cylinder which contains the fluid acted on by a piston is traversed by a *piston-rod*, the effective transverse area is equal to the tranverse area of the piston, with that of the rod subtracted. For example, in fig. 158, the upper division of the cylinder is traversed by the piston-rod C, working through the stuffing-box D; hence the effective transverse area in that division of the cylinder is the difference between the transverse areas of the piston A and rod C. In the lower division of the cylinder, where there is no rod, the whole transverse area of the piston is effective. A *trunk* acts in this respect like a piston-rod of large diameter.

209. **Comparative Velocities of Pistons.**—From the equality of the volumes swept through by a pair of pistons that are connected with each other by means of an intervening fluid mass of invariable volume, it obviously follows that *the velocities of the pistons are inversely as their transverse areas.*

The transverse areas are to be measured, as stated in the preceding Article, on planes normal to the directions of motion of the pistons; and when the motion of a piston is circular, the velocity referred to in the rule is that of the centre of its transverse area.

Let A and A′ denote the transverse areas of the two pistons marked with those letters in fig. 159, page 224, and v and v' their velocities; then their velocity-ratio is $\frac{v'}{v} = \frac{A}{A'}$.

As the velocity-ratio of a given pair of connected pistons is constant, the combination belongs to Willis's Class A.

210. **Comparative Velocities of Fluid Particles.**—It may sometimes be required to find the comparative mean velocities with which

* To find the distance of the centre of a plane area from an axis in the plane of that area: divide the area, by lines parallel to that axis, into a number of narrow bands; let $d\,x$ be the breadth of one of those bands, and y its length; then $y\,d\,x$ is the area of that band; and $\int y\,d\,x$ is the whole area. Let x be the distance from the axis to the centre of the band $y\,d\,x$; then $x\,y\,d\,x$ is the *geometrical moment* of that band, and $\int x\,y\,d\,x$ is the geometrical moment of the whole area relatively to the axis; which moment, being divided by the area, gives the required distance of the centre of the area from the axis, viz.,

$$x_0 = \frac{\int x\,y\,d\,x}{\int y\,d\,x}.$$ (See Article 293, page 334.)

the fluid particles flow through a given section of the passage which connects a pair of pistons; it being understood that the mean velocity of flow through a given section of the passage denotes the mean value of the component velocities, in a direction normal to that section, of all the particles that pass through it. From the fact that in a given time equal volumes of fluid flow through all sectional surfaces that extend completely across the passage, it follows that *the mean velocity of flow through any such section is inversely as its area* (a principle already stated in Article 88, page 76); and this principle applies to all possible sections, transverse and oblique, plane and curved.

For example, in fig. 159, let B denote the area of a transverse section, B B, of the passage which connects the two cylinders, and

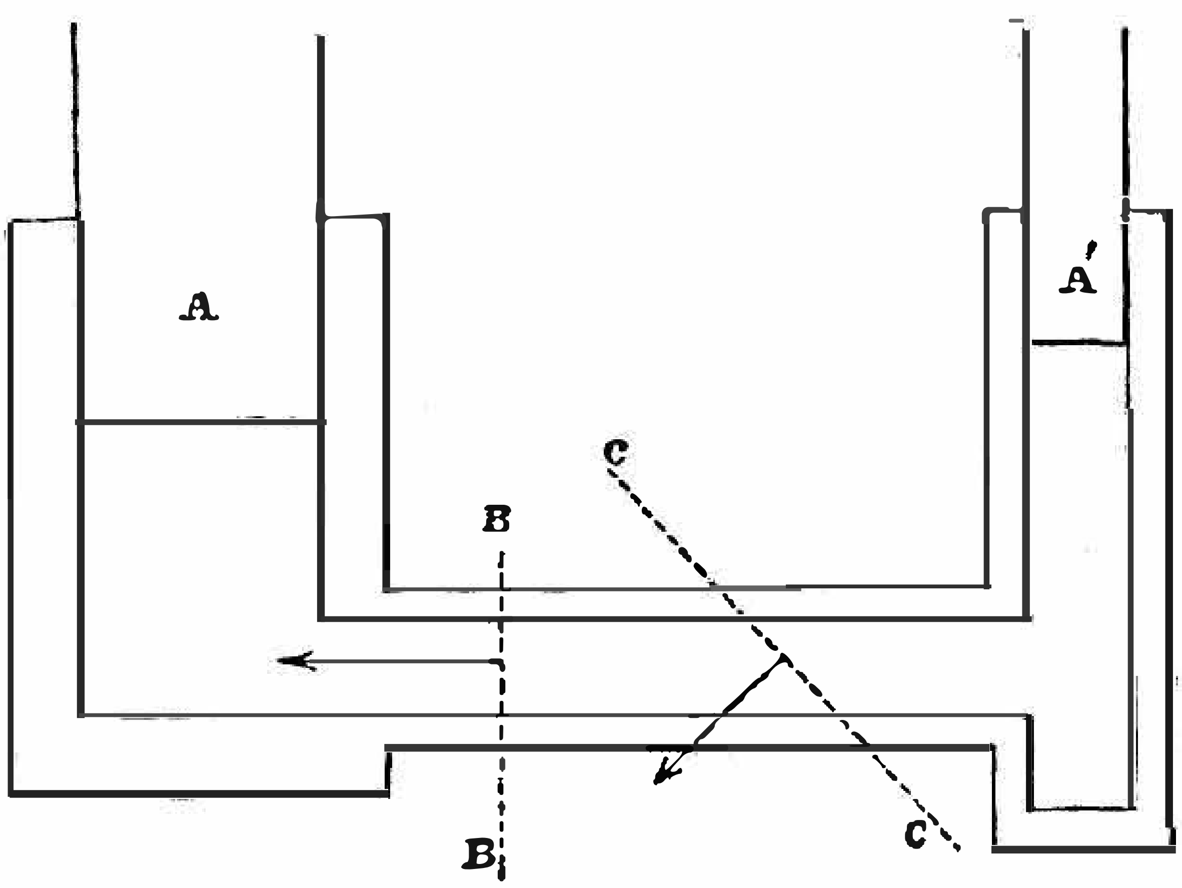

Fig. 159.

u the mean velocity with which the particles of fluid flow through that section; then v, as before, being the velocity of the piston whose transverse area is A, we have

$$\frac{u}{v} = \frac{A}{B}.$$

Also, let C denote the area of an oblique section, C C, of the passage, and w the mean component velocity of the fluid particles in a direction normal to that section; then

$$\frac{w}{v} = \frac{A}{C}; \text{ and } \frac{w}{u} = \frac{B}{C}.$$

211. Use of Valves—Intermittent Hydraulic Connection.—Valves are used to regulate the communication of motion through a fluid by opening and shutting passages through which the fluid flows. For example, a cylinder may be provided with valves which shall cause the fluid to flow in through one passage, and out through another. Of this use of valves two cases may be distinguished.

I. *When the piston drives the fluid*, the valves may be what is called *self-acting;* that is, moved by the fluid. If there be two passages into the cylinder, one provided with a valve opening inwards, and the other with a valve opening outwards, then, during the outward stroke of the piston, the former valve is opened and the latter shut by the inward pressure of the fluid, which flows in through the former passage; and during the inward stroke of the piston the former valve is shut and the latter opened by the outward pressure of the fluid, which flows out through the latter passage. This combination of cylinder, piston, and valves constitutes a *pump.*

II. *When the fluid drives the piston*, the valves must be opened and shut by mechanism, or by hand. In this case the cylinder is a *working cylinder.*

It is by the aid of valves that *intermittent hydraulic connection* between two pistons is effected; and the action produced is analogous to that of the click, ratchet, and detent, in intermittent link-work.

For example, in the Hydraulic Press, the rapid motion of a small plunger in a pump causes the slow motion of a large plunger in a working cylinder; and the connection of the pistons is made intermittent by means of the discharge valve of the pump; being a valve which opens outwards from the pump and inwards as regardse the working cylinder. The pump draws water from a reservoir, and forces it into the working cylinder: during the inward stroke of the pump plunger, the plunger of the working cylinder moves outward with a velocity as much less than that of the pump plunger as its area is greater. At the end of the inward stroke of the pump plunger, the valve between the pump and the working cylinder closes, and prevents any water from returning from the working cylinder into the pump; and it thus answers the purpose of the detent in ratchet-work (see page 206). During the outward stroke of the pump plunger that valve remains shut, and the plunger of the working cylinder stands still, while the pump is again filling itself with water through a valve opening inwards. When the piston of the working cylinder has finished its outward stroke, which may be of any length, and may occupy the time of any number of strokes of the pump, it is permitted to be moved inwards again by opening a valve by hand and allowing the water to escape.

A hydraulic press is often furnished with two, three, or more pumps, making their inward strokes in succession, and so producing a continuous motion of the working plunger. This is analogous to the double-acting click (page 209).

211 A. **Flexible Cylinders and Pistons.**—By an extension of the use of the word "cylinder," it may be made to include vessels made wholly or partly of a flexible material, which answer the purpose of a cylinder with its piston, by altering their shape and internal capacity; such as bellows. Questions as to this class of vessels may be approximately solved according to purely geometrical principles, by assuming the flexible material of which they are made to be inextensible.

In bellows, and pumps constructed on the principle of bellows, the vessel must have at least a pair of rigid ends, which, being moved alternately from and towards each other, answer the purpose of a piston. If those ends are equal and similar, and connected together by sides that may be assumed to be inextensible and perfectly flexible, the volume of fluid alternately drawn in and forced out may be taken as nearly equal to the area of one end multiplied by the distance through which the centre of area of one end moves alternately towards and from the other end.

Another example is furnished by a kind of pump, in which a circular orifice in one of the sides of a box is closed by a rigid flat disc of smaller diameter, and a bag in the form of a conical frustum of leather, or some other suitable material—the inner edge of the leather being made fast to the disc, and the outer edge to the circumference of the orifice. In working, the disc is moved alternately inwards and outwards, so as to draw the conical bag tight in opposite directions alternately. To find the *virtual area* of piston, add together the area of the disc, the area of the orifice, and four times the area of a circle whose diameter is the half-sum of the diameters of the disc and orifice, and divide the sum by six. That virtual area, multiplied by the length of stroke, gives nearly the volume of fluid moved per stroke.

In *Bourdon's pumps and engines* an elastic metal tube, of a flattened form of transverse section, is bent so as to present the figure of a circular arc. The internal capacity of the tube is varied by alternately admitting and expelling fluid; the effect of which is to flatten the curvature of the tube when its capacity is increased, and to sharpen that curvature when that capacity is diminished; so that if one end of the tube is fixed in position and direction, the other end has an oscillating motion.

In fig. 81, page 114, the arcs A D, A D′, A D″ may be taken to represent successive positions of the tube; A being its fixed end, and D its moveable end. The path of the moveable end, D D′ D″, is nearly an arc of a circle of the radius C G $= \frac{3}{4}$ of the length of the tube. The capacities of the tube in its several different positions, A D, A D′, A D″, &c., vary nearly *in the inverse ratio of the* arcs G D, G D′, G D″, &c.; so that if the capacity of the tube, when in a given position, is known, we can calculate its capacity in any other position, and the volume of fluid admitted or expelled in passing from any given position to any other.*

Section IX.—*Miscellaneous Principles respecting Trains.*

212. **Converging Trains.**—The essential principles of a train of mechanism have been stated in Article 93, page 80. Two or more trains may converge into one; that is to say, two or more primary pieces, which are followers in different trains, may all act as drivers to one primary piece. In such cases the comparative motion in each of the elementary combinations formed by the one follower with its several drivers is fixed by the nature of the connection; and thus the comparative motions of all the pieces are determined. As an example of converging trains, we may take a compound steam engine, in which two or more pistons drive one shaft, each by its own connecting-rod and crank.

213. **Diverging Trains.**—One train of mechanism may diverge into two or more; that is to say, one primary piece may act as driver to two or more primary pieces, each of which may be the commencement of a distinct train. In this case, as well as in that of converging trains, the comparative motions of all the pieces are determined.

Examples of diverging trains might be multiplied to any extent. One of the most common cases is that in which a number of different machines in a factory are driven by one prime mover: all those machines are so many diverging trains. In many instances there are diverging trains in one machine; thus in almost every

* Let A D′ be the position for which the capacity of the tube is known, and let V′ be that capacity. Let A D and A D″ be the positions of the tube at the two ends of its stroke; let V and V″ be the corresponding capacities; and let the lengths of the arcs G D, G D′, G D″ be denoted by s, s', s'' respectively. Then we have

$$V\,s = V'\,s' = V''\,s''; \text{ and } \frac{1}{s} : \frac{1}{s'} : \frac{1}{s''} :: V : V' : V'' \text{..........(1.)}$$

The volume of fluid admitted or expelled at each stroke is as follows:—

$$V'' - V = V'\,s'\left(\frac{1}{s''} - \frac{1}{s}\right) = \frac{V'\,s'\,(s - s'')}{s\,s''} \text{.............(2.)}$$

The length of stroke of the point D is $s - s''$; hence the apparatus may be regarded as equivalent to a cylinder and piston of that length of stroke, and of the following transverse area:—

$$\frac{V'' - V}{s - s''} = \frac{V'\,s'}{s\,s''} \text{..................................(3.)}$$

machine tool there are at least two diverging trains—one to produce the cutting motion, and the other the feed motion.

214. **Train for diminishing Fluctuations of Speed.**—The fluctuations in the velocity-ratio, when a revolving and a reciprocating point are connected by means of a link, have been stated in Article 184, pages 196, 197, and in Article 188, pages 199 to 201. In some cases it is desirable that the velocity-ratio of a reciprocating point to a revolving point should be more nearly uniform. For this purpose a train of two combinations may be used,—the first primary piece being a rotating shaft, which may be called A; the second, another rotating shaft, which may be called B; and the third, the reciprocating piece, C. The connection of A with B is by means of a pair of equal and similar two-lobed wheels (see Article 109, page 97); and a crank on B, by means of a connecting-rod, drives C. The two-lobed wheels are to be so placed that the shortest radius of the wheel on B shall be in gearing with the longest radius of the wheel on A at the instants when the crank is passing its dead-points. The result to be aimed at in the arrangement is, that each *quarter-stroke* of C shall be made as nearly as possible in the time of *one-eighth of a revolution of* A; and in order that this may be the case, the following should be the angles moved through by the two shafts respectively in given times:—

Shaft A,..................	0°	45°	90°	135°	180°
Shaft B, commencing at a dead-point of the crank,................	0°	60°	90°	120°	180°

Hence it appears that B is alternately to overtake and to fall behind A by 15°. This angle, then, being given, the rules of Article 109, page 98, are to be applied to the designing of the pitch-lines of the wheels. The greatest and least radii of those wheels are approximately 0·634 and 0·366 of the line of centres respectively.

The following are the comparative velocities, at different instants, of a revolving point in A at a given distance from its axis, of a revolving point in B at the same distance from its axis, and of a point in C connected by a very long link with the point in B*:—

* Mr. Willis, in his *Treatise on Mechanism*, investigates the figures of a pair of wheels on A and B for giving exact uniformity to the ratio C ÷ A. The equations are as follows:—Let c be the line of centres; r, a radius of the wheel on B, making the angle θ with the shortest radius; r', the corresponding radius of the wheel on A, making the angle θ' with the longest radius of this wheel; then we have

$$r = c \cdot \frac{\pi \sin \theta}{\pi \sin \theta + 2}; \; r' = c - r; \text{ and } \theta' = \frac{\pi}{2} \cdot \text{versin } \theta.$$

Mr. Willis points out that the forms of the pitch-lines given by the equations must in practice be slightly modified at the points which gear together when the crank is at its dead-points.

Angles moved through by A,..............h.	0°	45°	90°	135°	180°
Velocity-ratio B ÷ A,	1·732	0·866	0·577	0·866	1·732
Velocity-ratio C ÷ B,	0	0·866	1·000	0 866	0
Velocity-ratio C ÷ A,	0	0·750	0·577	0·750	0

Mean value of each of the velocity-ratios C ÷ B and C ÷ A, 0·637.

A similar adjustment may be made by connecting the shafts A and B by means of an universal joint (Article 191, page 203); the fork on the shaft B being so placed as to have its plane perpendicular to the plane of the axes when the crank is at its dead-points; the angle made by those axes with each other should be that whose cosine is 0·577, viz., 54¾°.

The Double Hooke's Joint (Article 192, page 205) is an example of a train in which the fluctuation of the velocity-ratio is corrected exactly.

Section X.—*References to Combinations arranged in Classes.*

215. **Object of this Section.**—In the preceding sections the various elementary combinations in mechanism have been arranged according to the mode of connection. The object of the present section is to give a list of such combinations, arranged according to Mr. Willis's system—that is, according to the comparative motion—with references to the previous Articles and pages of this treatise, where the several combinations are described. Two deviations from or modifications of Mr. Willis's system are used; first, the addition, at the commencement of each Class, of references to places where the comparative motions of two points in one primary piece are treated of; and secondly, the placing of combinations in which the connection is intermittent, in a class by themselves, entitled Class D.

216. Class A. **Directional-Relation Constant—Velocity-Ratio Constant.**

Combinations.

Velocity-Ratio that of Equality alone.

Any Constant Velocity-Ratio.

	Articles.	Pages.
Pair of Points in one Rotating Primary Piece,	53	31
Pair of Points in one Screw,	60	37
Rolling Contact: Circular Toothless Wheels and Sectors, and Straight Racks,	102 to 106	84 to 92
Rolling Contact: Frictional Gearing,	111	102
Sliding Contact; Circular Toothed Wheels and Sectors, and Straight Racks,	112 to 141, 144 to 151	103 to 139, 143 to 157
Sliding Contact: Screw Gearing,	152 to 157	157 to 166
Bands and Pulleys,	165 to 177	179 to 192
Blocks and Tackle,	200 to 205	214 to 218
Hydraulic Connection: Pistons and Cylinders,	207 to 210	221 to 224

217. Class B. **Directional-Relation Constant; Velocity-Ratio Variable.**

Mean Velocity-Ratio that of Equality alone.

	Articles.	Pages.
Rolling Contact: Smooth Elliptic and Lobed Wheels,	108, 109	95 to 99
Sliding Contact: Toothed Elliptic and Lobed Wheels,	143	141
" " Pin and Slot Coupling,	159	167
Crossed Cord and Elliptic Pulleys,	175	189
Linkwork: Drag-Link,	182	194
" Link for Contrary Rotations,	183	196
" Single Hooke's Joint,	191	203
" Single Hooke-and-Oldham Coupling,	193	206

Any Mean Velocity-Ratio.

	Articles.	Pages.
Rolling Contact: Non-Circular Wheels and Sectors,	107 to 110	92 to 102

Any Mean Velocity-Ratio—Continued.

	Articles.	Pages.
Sliding Contact: Teeth of Non-Circular Wheels and Sectors,	143	141
Bands with Non-Circular Pulleys,	175	188
Linkwork with Rocking Cranks and Levers,....h....	184 190	196 202
Blocks and Tackle, obliquely acting,	206 206 A	218 to 221

218. Class C. **Directional-Relation Variable.**

Sliding Contact: Pin and Slot,...........h............	159	168
,, ,, Cams,	160 to 163	170 to 175
Linkwork: Rotating Cranks and Eccentrics,.....h...	184 to 188	196 to 201
,, Levers for Multiplying Oscillations,	189	201
,, Band-links,	198	213

219. Class D. **Intermittent Connection.**

Sliding Contact: Intermittent Wheel-work,h.........	142	139
,, ,, Wipers and Pallets; Escapements,	164	175
Linkwork: Clicks and Ratchets,...h..................	194 to 196	206 to 211
,, Frictional Catches,	196	211
,, Slotted Link,...h..............h...........	199	213
Hydraulic Connection: Valves, Pumps, Hydraulic Press, Bellows,	211 211 A	224 to 227

Section XI.—*Comparative Motion in the "Mechanical Powers."*

220. **Classification of the Mechanical Powers.**—"Mechanical Powers" is the name given to certain simple or elementary machines, all of which, with the single exception of the pulley, are more simple than even an elementary combination of a driver and follower; for, with that exception, a mechanical power consists essentially of only one primary moving piece; and the comparative motion taken into consideration is simply the velocity-ratio either of a pair of points in that piece, or of two components of the velocity of one point. There are two established classifications of

the mechanical powers; an older classification, which enumerates six; and a newer classification, which ranges the six mechanical powers of the older system under three heads. The following table shows both these classifications:—

Newer Classification.	Older Classification.
The Lever,	The Lever. The Wheel and Axle.
The Inclined Plane, &c.	The Inclined Plane. The Wedge. The Screw.
The Pulley,	The Pulley.

In the present section the comparative motions in the mechanical powers are considered alone. The relations amongst the forces which act in those machines will be treated of in the dynamical division of this Treatise.

221. **Lever—Wheel and Axle.**—In the lever and the wheel and axle of the older classification, which are both comprehended under the lever of the newer classification, the primary moving piece turns about a fixed axis; and the comparative motion taken into consideration is the velocity-ratio of two points in that piece, which may be called respectively the *driving point* and the *following point*. The principle upon which that velocity-ratio depends has already been stated in Article 53, page 31—viz., that the velocity of each point is proportional to the radius of the circular path which it describes; that is, to its perpendicular distance from the axis of motion.

The distinction between the lever and the wheel and axle is this: that in the *lever*, the driving point, D, and the following point, F, are a pair of determinate points in the moving piece, as in figs. 161 to 164; whereas in the *wheel and axle* they may be any pair of points which are situated respectively in a pair of cylindrical pitch-surfaces, D and F, described about the axis A, fig. 160.

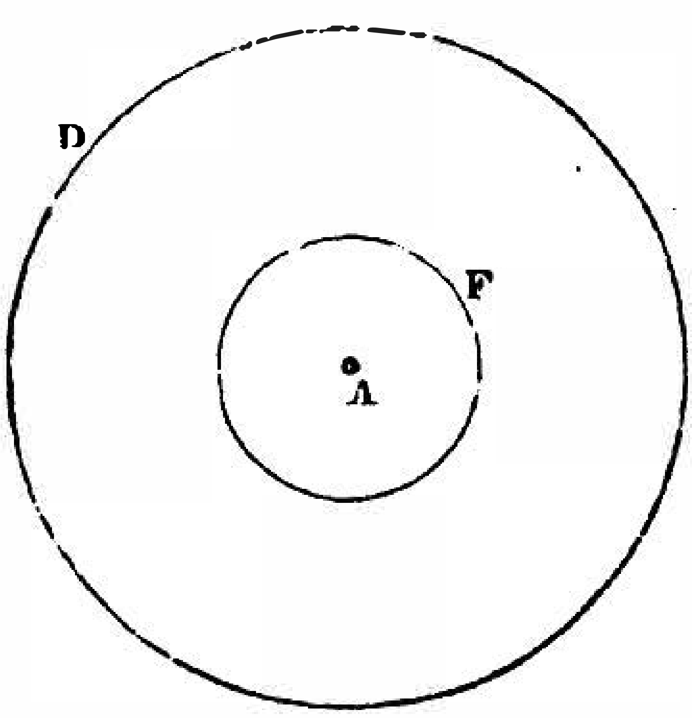

Fig. 160.

In each of these figures the plane of projection is normal to the axis, and A is the trace of the axis. In fig. 160, D and F are the traces of two cylindrical pitch-surfaces. In figs. 161 to 164, D and F are the projections of the driving and following points respectively.

The axis of a lever is often called the *fulcrum.*

A lever is said to be *straight,* when the driving point, D, and following point, F, are in one plane traversing the axis A, as in figs. 161, 162, and 163. In other cases the lever is said to be *bent,* as in fig. 164.

Fig. 161.

Fig 162.

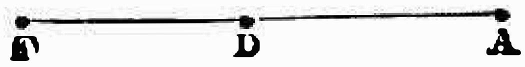

Fig. 163.

The straight lever is said to be of one or other of three kinds, according to the following classification:—

In a *lever of the first kind,* fig. 161, the driving and following points are at opposite sides of the fulcrum A.

In a *lever of the second kind,* fig. 162, the driving and following points are at the same side of the fulcrum, and the driving point is the further from the fulcrum.

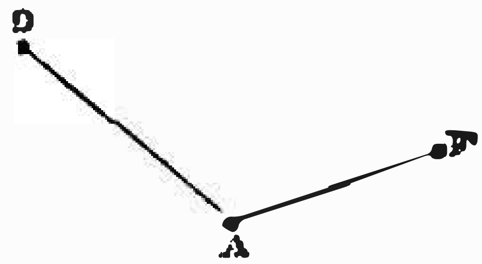

Fig. 164.

In a *lever of the third kind,* fig. 163, the driving and following points are at the same side of the fulcrum, and the following point is the further from the fulcrum.

222. **Inclined Plane—Wedge.**—In the inclined plane, and in the wedge, the comparative motion considered is the velocity-ratio of the entire motion of a straight-sliding primary piece and one of the components of that motion; the principles of which velocity-ratio have been stated in Article 43, pages 22, 23.

In the inclined plane, fig. 165, A A is the trace of a fixed plane; B, a block sliding on that plane in the direction B C; the plane of projection being perpendicular to the plane A A, and parallel to the direction of motion of B. B D is some direction oblique to B C. From any convenient point, C, in B C, let fall C D perpendicular to B D; then B D ÷ B C is the ratio of the component velocity in the direction B D to the entire velocity of B.

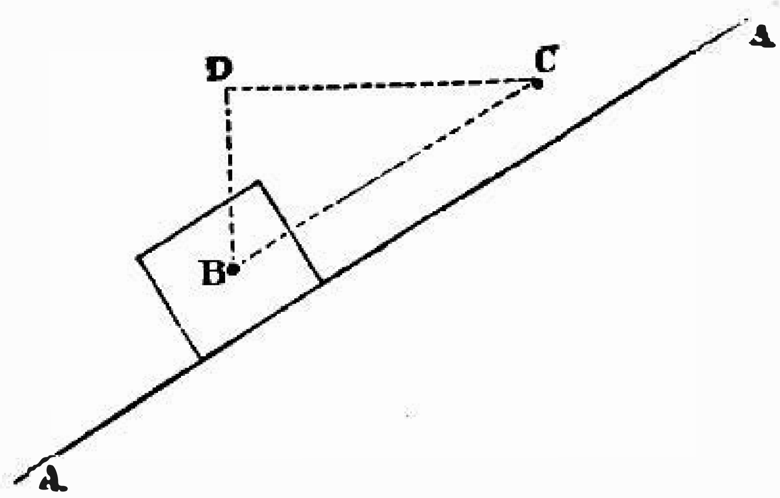

Fig. 165.

In fig. 166, A A is the trace of a fixed plane; B C D, the trace of a wedge which slides on that plane. While the wedge advances through the distance C *c*, its oblique face advances from the position C D to the position *c d*; and if C *e* be drawn normal to the plane C D, the ratio borne by the component velocity of the wedge

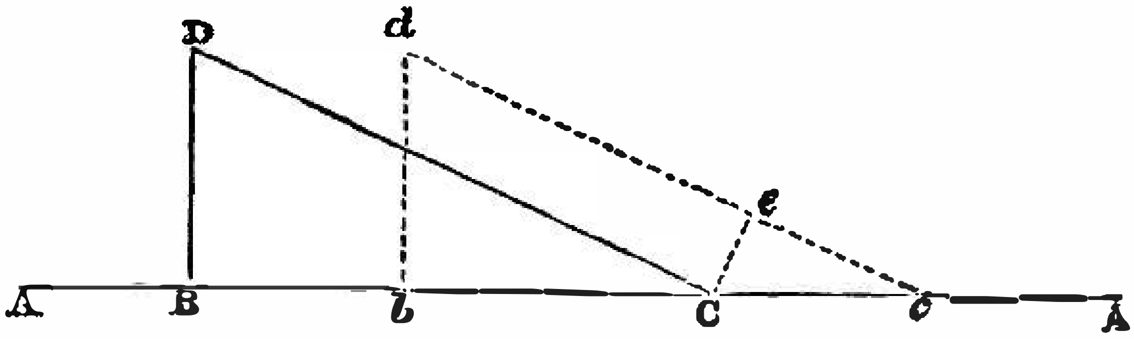

Fig. 166.

in a direction normal to its oblique face to its entire velocity will be expressed by C *e* : C *c*.

223. **Screw.**—In the screw the comparative motion considered is the ratio borne by the entire velocity of some point in, or rigidly connected with, the screw, to the velocity of advance of the screw.

The helical path of motion of a point in, or rigidly attached to, a screw may be developed (as has been already explained in Article 63, page 40) into a straight line: being the hypothenuse of a right angled triangle whose height is equal to the pitch of the screw, and its base to the circumference of a circle whose radius is the distance of the given point from the axis of the screw. Then if B D in fig. 165 be taken to represent the pitch of the screw, and D C, perpendicular to B D, the circumference of the circle described by the point in question about the axis, B C will be the development of one turn of the screw-line described by that point as it revolves and advances along with the screw; and B C ÷ B D will be the ratio of its entire velocity to the velocity of advance; just as in the case of a body sliding on an inclined plane, A A, parallel to B C. This shows why the screw is comprehended under the general head of the inclined plane, in the newer classification of the mechanical powers.

224. **Pulley.**—The term *pulley*, in treating of the mechanical powers, means any purchase or tackle of the class already described in Section VII. of this Chapter, pages 214 to 221.