CHAPTER III.

OF THE MOTIONS OF SECONDARY MOVING PIECES.

67. **General Principles.** (*A. M.*, 383, 384, 503.)—In the present chapter the general principles only of the motions of secondary moving pieces in machines can be given, many of their most important applications being reserved for that chapter which will treat of "Aggregate Combinations in Mechanism," and some for the chapter on "Elementary Combinations." The mechanism for producing the motions of secondary moving pieces belongs wholly to those later chapters.

Secondary moving pieces have already been defined (in Article 37, page 17) as those which are carried by other moving pieces, or which have their motions not wholly guided by their connection with the frame. Their motions, therefore, are not restricted, like those of primary pieces, to translation in a straight line, rotation about a fixed axis, and that combination of those two motions which constitutes the motion of a screw with a fixed axis; they comprehend translations along curved lines of various figures, rotations about shifting axes, and various combinations of translations and rotations. The paths of points, too, in secondary pieces are not restricted to three forms—the straight line, the circle, and the helix; they comprehend a great variety of curved lines, both plane and of double curvature. The comparative motions of any two points in a primary piece are constant. The comparative motions of two points in a secondary piece very often vary from instant to instant as the piece changes its position.

In many cases the motions of secondary pieces are partially guided or restricted. For example, a secondary piece may be so guided that all its movements take place parallel to a fixed plane; in which case its motions are restricted to translations parallel to the fixed plane, and rotations about axes perpendicular to it; and the paths of its points are restricted to lines, straight or curved, in or parallel to that plane; and this restricted case is by far the most common in mechanism. Another kind of restriction on the movements of a secondary piece is when it turns about a ball and socket joint, or some equivalent contrivance, so that one point at the centre of the joint is kept fixed: in this case its motions are restricted to rotations about axes traversing that fixed point; and the motions of points in it are restricted to curves situated in spherical surfaces described about the fixed point. Cases in which the movements of secondary moving pieces are not restricted in one or other of those ways are comparatively rare.

The geometrical problems relating to the motions of secondary moving pieces may be divided into the two following classes:—

I. When the motions, in most cases, of two, and at furthest of three, points in a secondary moving piece are given, and it is required to find the motion of any other point in the piece, or of the piece as a whole. All problems of this class depend for their solution on the principle of Article 54, page 32.

II. When there are two moving pieces or moving points, C and B, the frame of the machine being denoted by A, and two out of the three motions of A, B, and C relatively to each other being given, it is required to find the third of those motions. All problems of this class depend for their solution on the principle (already stated in Article 42, page 21) that the motion of C relatively to A is the resultant of the motions of B relatively to A, and of C relatively to B.

68. **Translation of Secondary Moving Pieces.** (*A. M.*, 369.)—If, in a moving piece whose movements are not restricted, the directions of motion of three points not in the same straight line are parallel to each other and oblique to the plane of the three points; or if, in a moving piece restricted to movements parallel to one plane, the motions of two points are parallel to each other and oblique to the line of connection of the points; then the motion of the whole piece is a translation. All the points in the piece describe equal and similar paths, straight or curved; and all, at a given instant, move with equal velocities in parallel directions. The motion of any pair of points in the moving piece relatively to each other is nothing; and their comparative motion consists in the directional relation of parallelism and the velocity-ratio of equality.

To exemplify the translation of all the points of a moving piece in equal and similar curved paths, we may take the case of a coupling-rod (fig. 32) which connects together a pair of equal

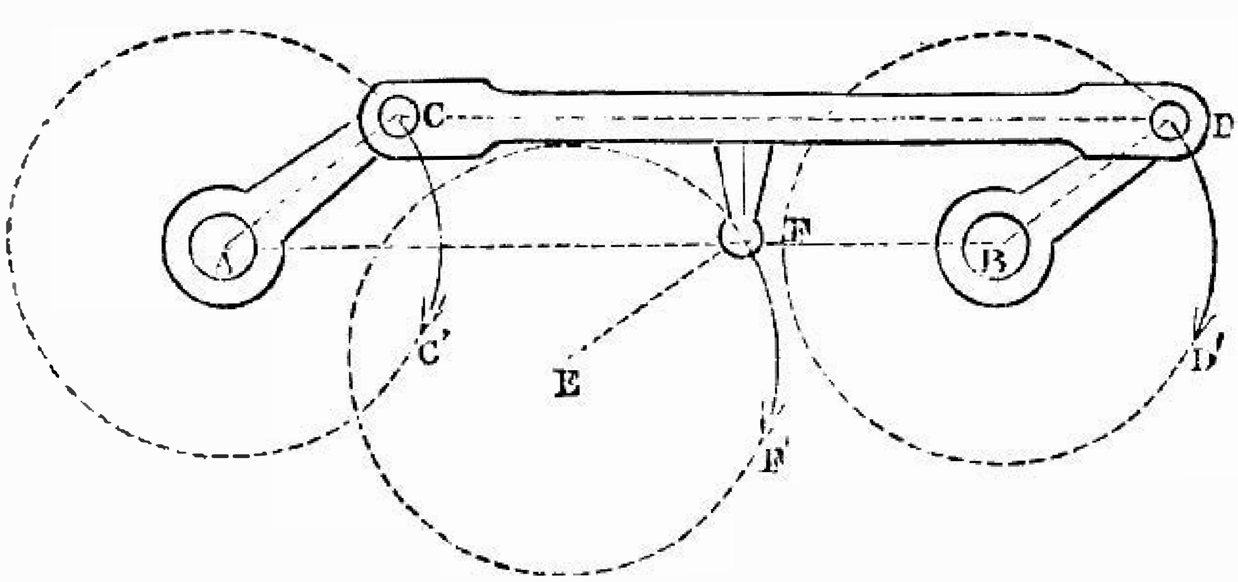

Fig. 32.

cranks, A C, B D, and has its effective length, C D, equal to the perpendicular distance, A B, between the axes of rotation of the two cranks. The motion of that coupling-rod is one of translation, in which all the particles describe with equal speed equal and similar circles of the radius A C = B D, in planes perpendicular to the axes A and B. The same is the case with any particle rigidly attached to the coupling-rod; such as F, which revolves in a circle of the radius E F = A C; so that, for example, the points C, D, and F move simultaneously through the equal and similar arcs C C′, D D′, F F′.

69. **Rotation Parallel to a Fixed Plane — Temporary Axis — Instantaneous Axis.**—The cases next in order as to complexity are those in which all the movements of the piece are parallel to a fixed plane; and the following are the problems which present themselves :—

I. *Given, the paths of two points in a moving piece, the distance between their projections on the plane of motion, and two successive positions of one of them, to find the* temporary axis of motion *of the piece.*

In fig. 33, let the plane of projection and of motion be that of

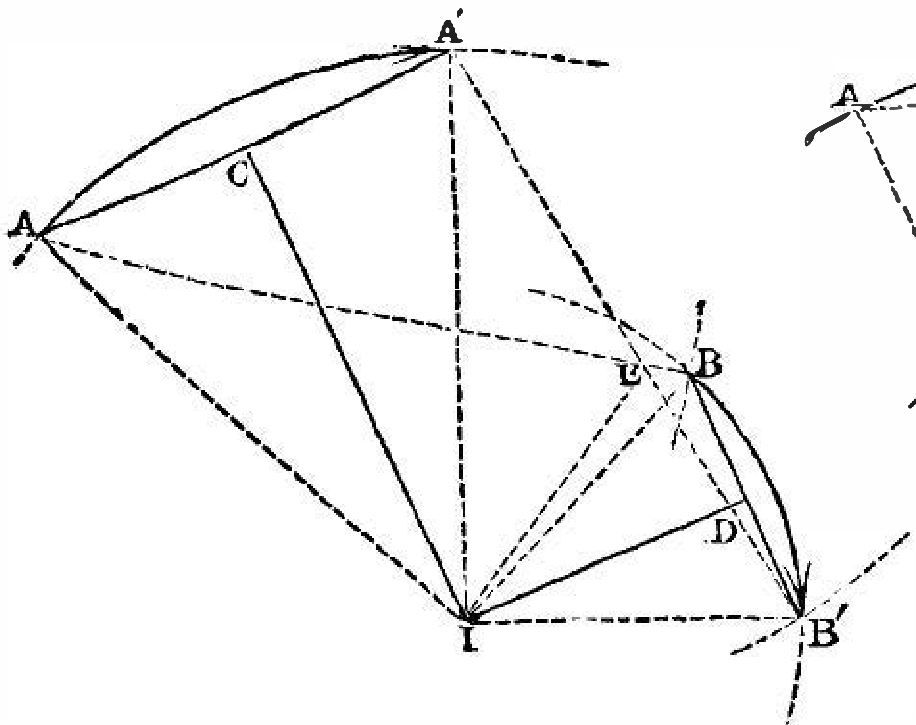

Fig. 33.

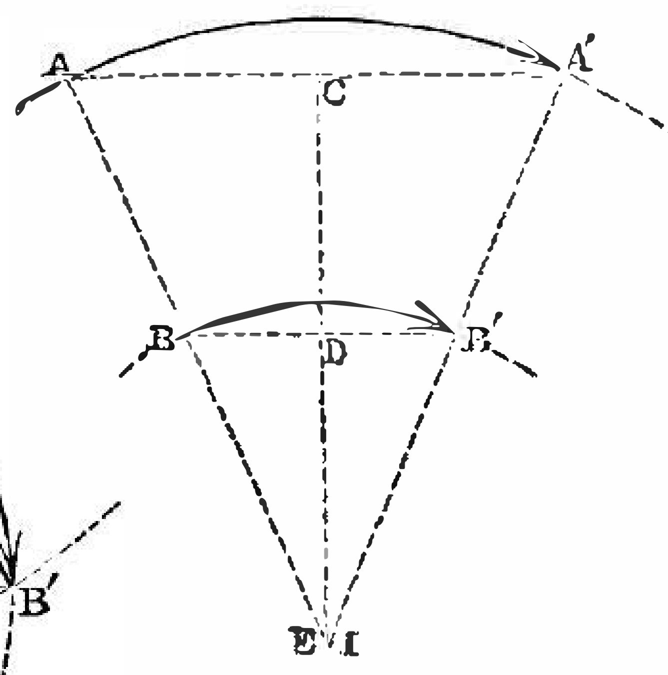

Fig. 34.

the paper, and let the partly dotted lines A A′ and B B′ be the projections of the paths of the two points, which may be straight lines or plane curves of any figure, subject only to the limitation that the distance between the points is invariable. Let A and A′ be the given two successive positions of one of the points. About A and A′ respectively, with the projection of the line of connection as a radius, draw circular arcs cutting the projected path of the other point in B and B′; these will be the projections of the two successive positions of the second point; and the straight lines A B and A′ B′ will be the projections of the line of connection in the two successive positions of the moving piece. Draw the straight lines A A′ and B B′; bisect them in C and D, through which points draw C I perpendicular to A A′ and D I perpendicular to B B′, meeting each other in I. Then, because A′ I = A I and B′ I = B I, I represents the same point in the two positions of the piece; and therefore I is the projection and the trace of a line perpendicular to the plane of motion, whose position is the same after the motion of A to A′ and of B to B′ that it was before. That line may be called the *Temporary Axis of Motion* of the moving piece, because the change of position of the piece is the same as if it had been turned through an angle A I A′ = B I B′ about that line.

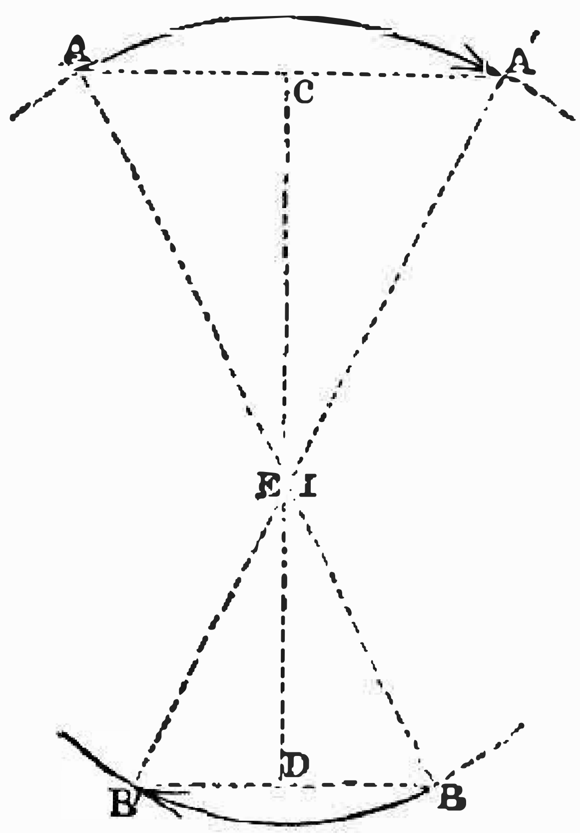

Fig. 35.

Let E be the point of intersection of A B and A′B′. Then the straight line E I bisecting the angle A E B′ traverses the temporary axis I; and this affords a means of finding that axis when C I and D I cut each other at an angle so oblique as to make it difficult to determine precisely their point of intersection.

When B B′ is parallel to A A′, as in figs. 34 and 35, C I and D I become parts of one straight line, and have no intersection; and then the point I is determined by its coinciding with E. In most cases of this kind it is necessary that the two successive positions of B should be given as well as those of A.

II. *Given* (in fig. 36), *the projections* A *and* B, *at a given instant, of two points in a moving piece on the plane of motion, and the simultaneous directions of motion of those points,* A a *and* B b, *to find the* instantaneous axis *of the moving piece; and thence to deduce the comparative motions, at the given instant, of the given points, and of any other points in the moving piece.*

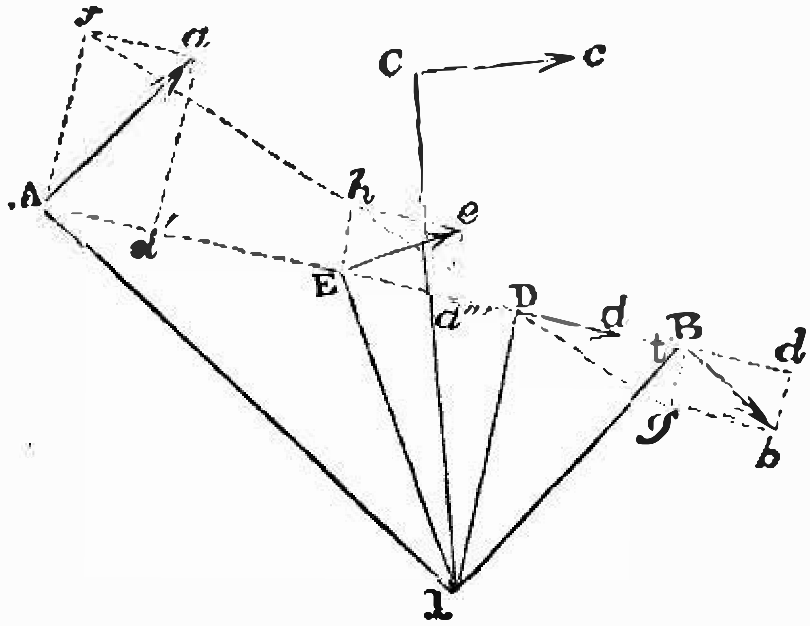

Fig. 36.

If the simultaneous directions of motion of the given points are perpendicular to their line of connection, the problem requires additional data for its solution, which will be stated in Rule III. If those directions are parallel to each other, and not perpendicular to the line of connection, the motion of the piece is one of translation, like that referred to in Article 68, page 44. The present rule comprehends all cases in which the given directions are not parallel to each other.

Through A and B draw A I and B I perpendicular respectively to A a and B b, and cutting each other in I. Then I will be the projection and the trace on the plane of motion of the required INSTANTANEOUS AXIS: that is to say, of a line such that the motion of the piece *at the instant in question* is one of rotation about that axis.

An instantaneous axis is so called because it is an imaginary line which is continually changing its position, both relatively to the frame of the machine and relatively to the secondary piece to which it belongs; so that it occupies any particular position, whether relatively to the frame or relatively to the secondary piece, at a particular instant only.

The comparative motions at the given instant of points in the secondary piece are deduced from the principle that the velocities of those points are proportional in magnitude and perpendicular in direction to the perpendiculars let fall from the points upon the instantaneous axis. For example, let A a, B b, C c, D d, E e, represent the directions and velocities of the points whose projections are A, B, C, D, E; then

$$\mathrm{A}a : \mathrm{B}b : \mathrm{C}c : \mathrm{D}d : \mathrm{E}e$$

are respectively proportional and perpendicular to

$$:: \mathrm{AI} : \mathrm{BI} : \mathrm{CI} : \mathrm{DI} : \mathrm{EI}.$$

From I let fall I D perpendicular to the projection, A B, of the line of connection of the given points. Then all points whose projections are at D are at the given instant in the act of moving parallel to A B; and all points whose projections are in A B, or in A B produced, such as A, B, and E, have for their component velocities along A B velocities equal to the velocity of D; that is to say,

$$\mathrm{D}d = \mathrm{A}d' = \mathrm{B}d'' = \mathrm{E}d''';$$

a consequence which follows also from the principle of Article 53, page 31.

The components perpendicular to A B of the velocities of points whose projections are in that line, such as A, B, and E, are proportional to the distances of those projections from D; that is to say, if A f, B g, and E h represent those transverse component velocities, we have the proportions,

$$\mathrm{DA} : \mathrm{DB} : \mathrm{DE}$$
$$:: \mathrm{A}f : \mathrm{B}g : \mathrm{E}h;$$

and the points f, h, D, g are in one straight line.

Hence, when A I and B I form an angle with each other so oblique as to make it difficult to determine precisely their point of intersection, we may proceed as follows to increase the precision of that determination:—Lay off any convenient equal distances, A d' = B d'', along A B from A and B respectively, to represent the longitudinal component of their velocities. Then complete the rectangular parallelograms A d' a f, B d'' b g; draw the straight line $f g$, cutting A B in D. Then from D perpendicular to A B draw D I; this line will traverse the instantaneous axis, and will increase the precision with which it is determined.

This last way of considering the motion of the piece is equivalent to regarding that motion as compounded of a rotation about an axis at D and a translation of that axis, and of the whole body along with it, with the velocity represented by D d.

III. *Given* (in fig. 37 or fig. 38), *the projections* A *and* B, *at a given instant, of two points in a moving piece on the plane of motion, and the ratio of their velocities, which are both perpendicular to the*

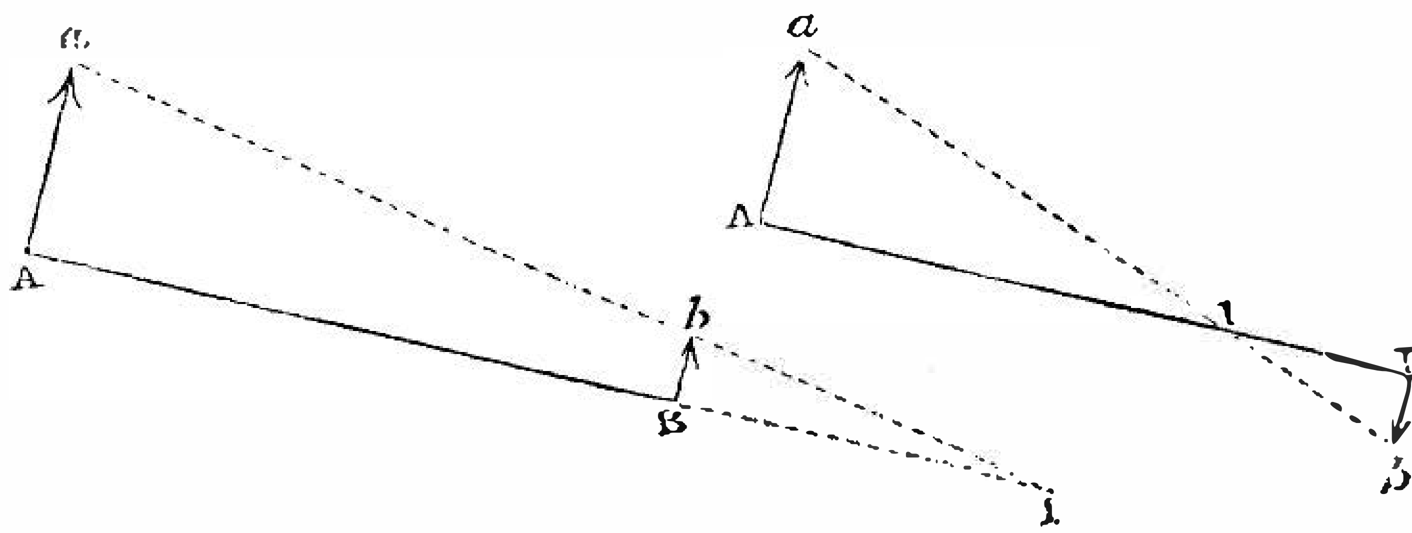

Fig. 37. Fig. 38.

projection, A B, *of their line of connection, to find the* instantaneous axis *of motion of the piece.* Perpendicular to A B draw the straight lines A a, B b, bearing to each other the given proportion of the velocities of the two points: draw the straight line a b; the point of intersection, I, of A B and a b (produced if necessary) will be the projection and trace on the plane of motion of the required instantaneous axis.

That axis may then be used as in the preceding Rule to determine the comparative motions of any set of points in the moving piece.

70. **Rotation about a Fixed Point.**—Every possible motion of a rigid body relatively to a point in the body is reducible to rotation about an axis, permanent, temporary, or instantaneous, as the case may be, which traverses that point. This is proved by showing that the following problem is always capable of solution:—

I. *Given, at any instant, the directions of motion of any two points,* B, C (fig. 39), *in a rigid body relatively to a point,* A, *in the* *body, to find the* instantaneous axis *of the motion of the whole body relatively to* A. In the first place, it is to be observed that when the motions of three points in a rigid body are determined, the

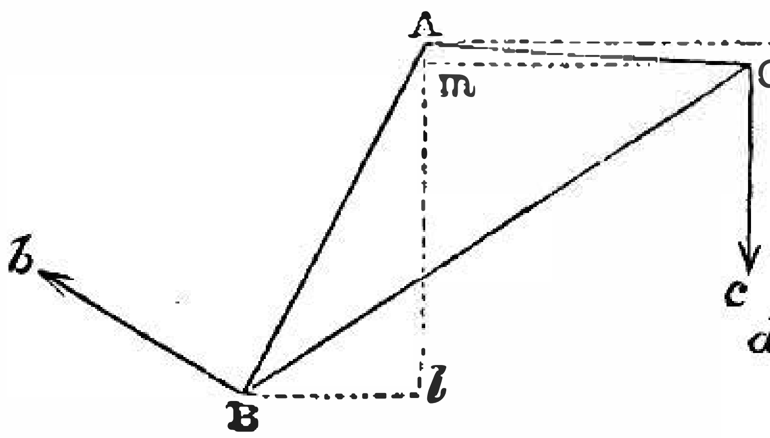

Fig. 40.

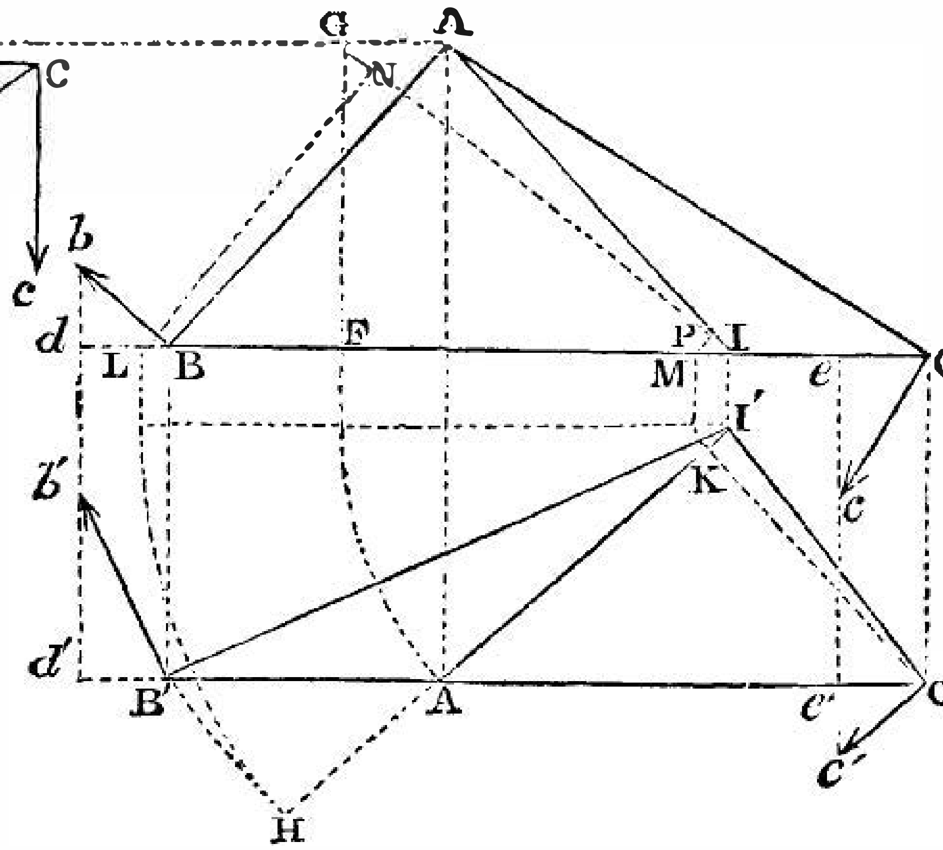

Fig. 39.

motion of the whole body is determined; for the distances of any fourth point in the body from those three points being invariable, the position of that fourth point at every instant is completely determined by the positions of the three points.

In order that the solution may be put in the simplest possible form, let the plane of the three points themselves, or a plane parallel to it, be taken for one plane of projection; and in fig. 39 let A, B, C be the projections of the three points on that plane. For a second plane of projection, take a plane perpendicular to the first plane, and traversing B C, and let A′, B′, and C′ (which are in one straight line) be the projections of the three points on that second plane; so that B′ C′ is parallel to B C, and A A′, B B′, and C C′ are perpendicular to B C.

Because the instantaneous axis must traverse A, it is obvious that A B and A C are the traces on the first plane of projection of two planes traversing the instantaneous axis and the points B and C respectively; and also, that if B *b* and C *c* are the projections on the first plane of projection of the directions of motion of B and C at the given instant, those projections must be perpendicular to A B and A C. Let B′ *b*′ and C′ *c*′ represent the projections of thee directions of motion of B and C on the second plane of projection. Draw B′ I′ and C′ I′ perpendicular respectively to B′ *b*′ and C′ *c*′, and meeting each other in I′; then B′ I′ and C′ I′ are the traces, on the second plane of projection, of two planes perpendicular respectively to the instantaneous directions of motion of B and C; that is to say, of the two planes already mentioned, which traverse the instantaneous axis and the points B and C respectively; and I′ is *the trace of the instantaneous axis on the second plane of projection.* From I′ let fall I′ I perpendicular to B C; then I is the projection of I′ on the first plane of projection. Draw the straight lines A I, A′ I′: those are *the projections of the instantaneous axis.*

II. *To draw the projections of the points* B *and* C *on a plane perpendicular to the instantaneous axis, and to find the comparative motion of those points.* In B C, fig. 39, take I F = I′ A′; draw A G parallel and F G perpendicular to B C, cutting each other in G; join I G: this line will be the rabatment of I A. From B′ and C′ let fall B′ H and C′ K perpendicular to I′ A′ (produced if required). In B C take I L = I′ H, and I M = I′ K; then G, L, and M will represent the respective projections of A, B, and C upon a plane which traverses the instantaneous axis, and is perpendicular to the second plane of projection. From L and M let fall L N and M P perpendicular to I G. Then, in fig. 40, let the paper represent a plane of projection perpendicular to the instantaneous axise let A be the trace and projection of that axis, and A *l* the trace of the plane already mentioned as being perpendicular to the second plane of projection in fig. 39. Make A *l* in fig. 40 = N L in fig. 39, and A *m* in fig. 40 = P M in fig. 39. Draw *l* B in fig. 40 perpendicular to A *l* in fig. 40 and = H B′ in fig. 39; also *m* C in fig. 40 perpendicular to A *l* in fig. 40 and = K C′ in fig. 39. Join A B, A C. Then B and C in fig. 40 will be *the projections required;* and the velocities of B and C relatively to A will be *perpendicular in direction and proportional in magnitude to* A B *and* A C *respectively.*

Another mode of finding the comparative motion of A and B is the following:—According to the principle of Article 54, page 32, the component velocities of B and C along their line of connection, B C, are equal. Therefore, in fig. 39, lay off along B C and B′ C′ the equal distances B *d*, C *e*, B′ *d′*, C′ *e′*, to represent that component; then draw *d′ b′ d b*, *c′ e′ c e* perpendicular to B C, cutting B *b* in *b*, B *b′* in *b′*, C *c* in *c*, and C′ *c′* in *c′*; then B *b* and B′ *b′* will be the projections of the velocity of B relatively to A; and C *e* and C′ *e′* will be the projections of the velocity of C relatively to A. Then, by the rule of Article 19, page 7, find the lengths of the lines of which B *b* and B′ *b′*, C *c* and C′ *c′* are the projections; the ratio of those lengths to each other will be the velocity-ratio of the two points.

71. **Unrestricted Motion of a Rigid Body.**—How complicated soever the motion of a rigid body may be, it may always be regarded as made up of a change of position of the body as a whole—that is, a translation of the body, and a change of position of

the body relatively to some point in it; that is to say, as has been shown in the preceding Article, a rotation about an axis traversing that point, which axis may be either permanent, temporary, or instantaneous. It will be shown further on that a rotation and a translation parallel to the plane of rotation, when compounded, are equivalent to a motion of rotation about an instantaneous axis perpendicular to that plane. Hence it follows, that if a rigid body has any translation combined with any rotation, the translation is to be resolved into two components, one parallel and one normal to the plane of rotation; when the parallel component of the translation, being combined with the rotation, will be equivalent to a rotation about a new instantaneous axis perpendicular to the plane of rotation; and the whole motion of the body will be equivalent to this new rotation combined with the normal component of the translation, the direction of which component is parallel to the axis. In short, how complex soever the motion of a rigid body may be, it is at each instant equivalent to a helical or screw-like motion about an instantaneous axis. The most comprehensive problem that can occur respecting the unrestricted motion of a rigid body at a given instant is this: given the simultaneous velocities and directions of motion of three points (say A, B, and C), to find the motion of the whole body at that instant. This is to be solved by choosing one of the points (say A), and regarding it for the time as fixed, and determining the motions of B and C relatively to A. Then, by means of the rules of Article 70, the position is to be determined of an instantaneous axis traversing A; when the motion of the whole body will be compounded of a rotation about that axis and a translation of the point A, carrying the instantaneous axis and the whole body along with it.

72. **Instantaneous Axis of a Rolling Body.**—The following proposition has many useful applications in mechanism:—*The instantaneous axis of a rigid body, which rolls without slipping upon the surface of another rigid body, passes through all the points in which these bodies touch each other;* for the particles in the rolling body which at any given instant touch the fixed body without slipping must have, at that instant, no velocity, and must therefore be, at that instant, in the instantaneous axis.

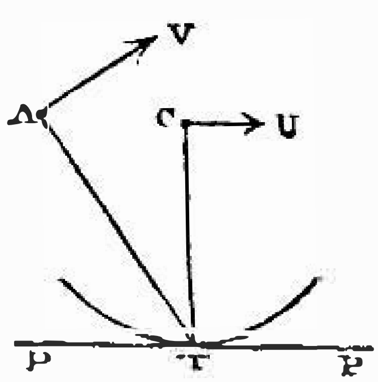

Fig. 41.

Moreover, because an instantaneous axis is a straight line, it follows that, in order that two surfaces which touch each other at more than one point may roll on each other without slipping, their points of contact must all lie in one straight line: a property of plane, cylindrical, and conical surfaces only, the terms "cylindrical" and "conical" being used in the general sense, which comprehends cylinders and cones with bases of any figure, as well as those with circular bases.

In fig. 41, let the plane of the paper represent a plane of projection perpendicular to the straight line in which the fixed and the rolling surfaces touch each other; let T be the projection and trace of that straight line, which is the instantaneous axis of the rolling body. Let A be the projection at a given instant of a point in the rolling body; then at that instant A is moving with a velocity proportional to A T, and in a direction perpendicular to the plane traversing A and the instantaneous axis, of which plane A T is the trace.

It follows that the path traced by a point such as A in a rolling body is *a curve whose normal*, A T, *at any given point*, A, *passes through the corresponding position*, T, *of the instantaneous axis.* Curves of this class are called *rolled curves;* and some of them are useful in mechanism, as will be explained farther on.

73. **Composition of Rotation with Translation.**—From Article 52, page 30, it appears that the single rotation of a body about a fixed axis (such as O, fig. 19, page 26) may be regarded as compounded of a rotation with equal angular velocity about a moving axis parallel to the fixed axis (such as that whose trace is A, fig. 19), and a translation of that moving axis carrying the body along with it in a circle round the fixed axis of the radius O A. A similar resolution of motions may be applied to rotation about an instantaneous axis. For example, the rotation of the rolling body in fig. 41 about the instantaneous axis, T, may be conceived to be made up of a rotation about another axis, C, parallel to the instantaneous axis, and a translation of that axis.

The present Article relates to the converse process, in which there are given a rotation of a secondary piece about an axis occupying a fixed position in the piece, and a translation of that axis relatively to the frame in a direction perpendicular to itself—that is, parallel to the plane of rotation; and it is required to find, at any instant, the instantaneous axis so situated that a rotation about it with the same angular velocity shall express the resultant motion of the piece.

In fig. 41, let the plane of the paper be the plane of motion, and let C be the projection and trace of the moving axis—moving relatively to the frame, but fixed as to its position in the secondary piece. Let C U be the direction of the translation of that axis, carrying the moving piece with it; and let the velocity of translation be so related to the angular velocity of rotation as to be equal to the velocity of revolution about the axis C, of a particle whose distance from that axis is $C\,T = \dfrac{\text{velocity of translation}}{\text{angular velocity}}$. Draw C T of the length so determined, in a direction perpendicular to C U, and pointing towards the right or towards the left of C U, according as the rotation is right-handed or left-handed. Then T will be *the projection and trace of the required instantaneous axis;* so that if A is the projection of any point in the moving piece, the direction of motion of that point is perpendicular to the plane whose trace is A T; and the velocity-ratio of A and C is

$$\frac{A\,V}{C\,U} = \frac{T\,A}{T\,C}.$$

The proof that T is the instantaneous axis is, that any particle whose projection, at a given instant, coincides with T is carried backward relatively to C by the rotation with a speed equal and opposite to that with which C is carried forward by the translation; so that the resultant velocity of every particle at the instant when its projection coincides with T is nothing.

74. **Rolling of a Cylinder on a Plane—Trochoid—Cycloid.** (*A. M.*, 386.)—Every straight line parallel to the moving axis C, in a cylindrical surface described about C with the radius C T, becomes in turn the instantaneous axis. Hence the motion of the body is the same with that produced by the rolling of such a cylindrical surface on a plane, P T P, parallel to C and to C U, at the distance C T.

The path described by any point in the body, such as A, which is not in the moving axis C, is a curve well known by the name of *trochoid.* The particular form of trochoid, called the *cycloid,* is described by each of the points in the rolling cylindrical surface.

75. **Rolling of a Plane on a Cylinder; Involute—Spirals.** (*A. M.*, 387.)—Another mode of representing the combination of rotation with translation in the same plane is as follows:—In fig. 42, let O be the projection and trace on the plane of motion of a fixed axis, about which let the plane whose trace and projection is O C (containing the axis O) rotate (righthandedly, in the figure) with a given angular velocity. Let a secondary piece have *relatively to the rotating plane,* and in a direction perpendicular to it, a translation with a given velocity. In the plane O C, and at right angles to the axis

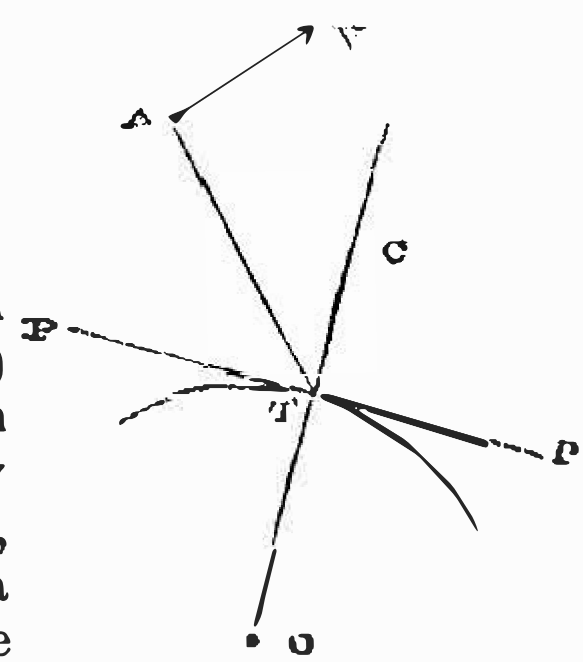

Fig. 42.

O, take $\text{O T} = \dfrac{\text{velocity of translation}}{\text{angular velocity}}$, in such a direction that the velocity which the point T *in the rotating plane* has at a given instant, shall be in the contrary direction to the equal velocity of translation which the secondary piece has relatively to the rotating plane. Then each point in the secondary piece which arrives at the position T, or at any position in a line traversing T parallel to the fixed axis O, is at rest *at the instant* of its occupying that position; therefore the line traversing T parallel to the fixed axis O is *the instantaneous axis;* the motion at a given instant of any point in the secondary piece, such as A, is at right angles to the plane whose trace is A T, perpendicular to the instantaneous axis; and the velocity of that motion bears to the velocity of the translation the ratio of T A to T O.

All the lines in the secondary piece which successively occupy the position of instantaneous axis are situated in a plane of that body, P T P, perpendicular to O C; and all the positions of the instantaneous axis are situated in a cylinder described about O with the radius O T; so that the motion of the secondary piece is such as is produced by the *rolling of the plane whose trace is* P P *on the cylinder whose radius is* O T. Each point in the secondary piece, such as A, describes a plane spiral about the fixed axis O. For each point in the *rolling plane*, P P, that spiral is the *involute of the circle* whose radius is O T, being identical with the curve traced by a pencil tied to a thread while the thread is being unwrapped from the cylinder. For each point whose path of motion traverses the fixed axis O—that is, for each point in a plane of the secondary piece traversing O and parallel to P P—the spiral is Archimedean, having a radius-vector increasing by a length equal to the circumference of the cylinder at each revolution.

76. **Composition of Rotations about Parallel Axes.** (*A. M.*, 388.) —In figs. 43, 44, and 45, let O be the projection and trace on the

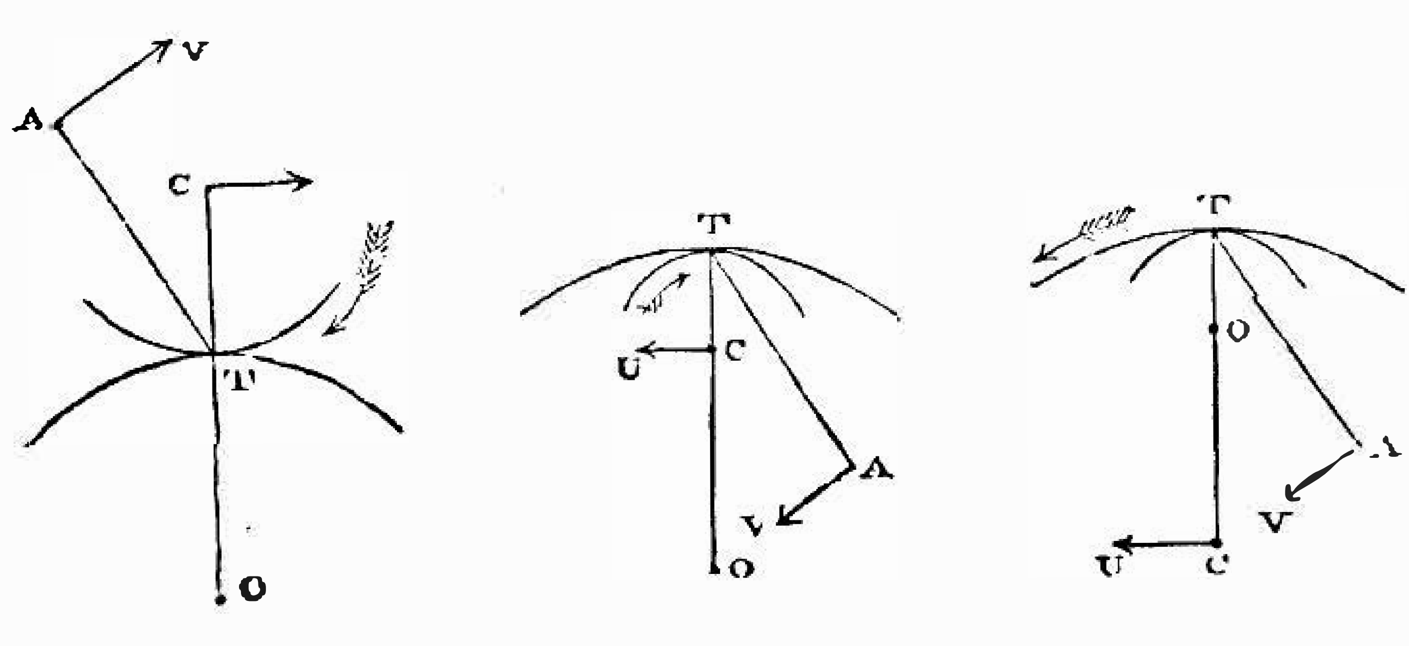

Fig. 43. Fig. 44. Fig. 45.

plane of rotation of a fixed axis, and O C the trace of a plane traversing that axis, and rotating about it with the angular velocity *a*. Let C be the projection and trace of an axis in that plane, parallel to the fixed axis O; and about the moving axis C let a secondary piece rotate with the angular velocity *b relatively to the* *plane* O C; and let the directions of the rotations a and b be distinguished by positive and negative signs. The body is said to have the rotations about the parallel axes O and C *combined* or *compounded*, and it is required to find the result of that combination of parallel rotations.

Fig. 43 represents the case in which a and b are similar in direction: fig. 44, that in which a and b are in opposite directions, and b is the greater; and fig. 45, that in which a and b are in opposite directions, and a is the greater.

Let the trace O C of the rotating plane be intersected in T by a straight line parallel to both axes, in such a manner that the distances of T (the trace of that line) from the fixed and moving axes respectively shall be inversely proportional to the angular velocities of the component rotations about them, as is expressed by the following proportion:—

$$a : b :: \mathrm{CT} : \mathrm{OT} \quad \text{(1.)}$$

When a and b are similar in direction, let T fall between O and C, as in fig. 43; when they are contrary, beyond, as in figs. 44 and 45. Then the velocity of the line T *of the plane* O C is $a \cdot \mathrm{OT}$; and the velocity of the line T *of the secondary piece, relatively to the plane* O C, is $b \cdot \mathrm{CT}$, equal in amount and contrary in direction to the former; therefore each line of the secondary piece which arrives at the position T is at rest at the instant of its occupying that position, and is then *the instantaneous axis.* The *resultant angular velocity* is given by the equation

$$c = a + b; \quad \text{(2.)}$$

regard being had to the directions or signs of a and b; that is to say, if we now take a and b to represent *arithmetical* magnitudes, and affix explicit signs to denote their directions, the direction of c will be the same with that of the greater; the case of fig. 43 will be represented by the equation 2, already given; and those of figs. 44 and 45 respectively by

$$c = b - a; \; c = a - b \quad \text{(2 A.)}$$

The relative proportions of a, b, and c, and of the distances between the fixed, moving, and instantaneous axes, are given by the equation

$$a : b : c :: \mathrm{CT} : \mathrm{OT} : \mathrm{OC} \quad \text{(3.)}$$

The motion of any point, such as A, in the secondary piece, according to the principle of Article 72, is at each instant at right angles to the plane whose trace is A T, drawn from the point A perpendicular to the instantaneous axis; and the velocity of that motion is

$$v = c \cdot \mathrm{A\,T} \quad \text{(4.)}$$

77. **Rolling of a Cylinder on a Cylinder—Epitrochoids—Epicycloids.** (*A. M.*, 389.)—All the lines in the secondary piece which successively occupy the position of instantaneous axis are situated in a cylindrical surface described about C with the radius C T; and all the positions of the instantaneous axis are contained in a cylindrical surface described about O with the radius O T; therefore the resultant motion of the secondary piece is that which is produced by rolling the former cylinder on the latter cylinder.

In fig. 43 a convex cylinder rolls on a convex cylinder; in fig. 44 a smaller convex cylinder rolls in a larger concave cylinder; in fig. 45 a larger concave cylinder rolls on a smaller convex cylinder.

Each point in the rolling rigid body traces, relatively to the fixed axis, a curve of the kind called *epitrochoids.* The epitrochoid traced by a point in the surface of the rolling cylinder is an *epicycloid.*

In certain cases the epitrochoids become curves of a more simple class. For example, each point in the *moving axis* C traces a circle.

When a cylinder rolls, as in fig. 44, within a concave cylinder of *double its radius,* each point in the surface of the rolling cylinder moves backwards and forwards in a straight line, being a diameter of the fixed cylinder; each point in the axis of the rolling cylinder traces a circle of the same radius with that cylinder; and each other point in or attached to the rolling cylinder traces an ellipse of greater or less eccentricity, having its centre in the fixed axis O. This principle has been made available in instruments for drawing and turning ellipses.

There is one case of the composition of rotations about parallel axes in which there is *no instantaneous axis;* and that is when the two component rotations are of equal speed and in contrary directions; for then the resultant is simply a *translation* of the secondary piece along with the moving axis. This may be illustrated by referring to fig. 32, page 44, where the translation of the coupling-rod C D may be looked upon as the resultant of the combination of the rotation of the crank A C about A, with an equal and contrary rotation, *relatively to the crank,* of C D about C.

78. **Curvature of Involutes of Circles, Epicycloids, and Cycloids.**—It is often useful to determine the radius of curvature of a rolled curve at a given point, especially where the fixed curve and rolling curve are circles, and where the tracing point is in the circumference of the rolling curve.

In the case of the *involute of a circle,* the radius of curvature at a given point is simply the length of a normal to the curve at that point measured to the point where that normal touches the circlet; that is to say, it is the length of the straight part of the thread used in drawing the involute.

In the case of a *cycloid* (traced by a point in the circumference of a cylinder which rolls on a plane) the radius of curvature at a given point is twice the length of the normal measured from that point to the corresponding instantaneous axis.

In the case of an epicycloid the construction for finding the radius of curvature is shown in fig. 46; the right-hand division of the figure giving the construction for an *external epicycloid*, I A, traced by a point, A, in the surface of a cylinder, the trace of

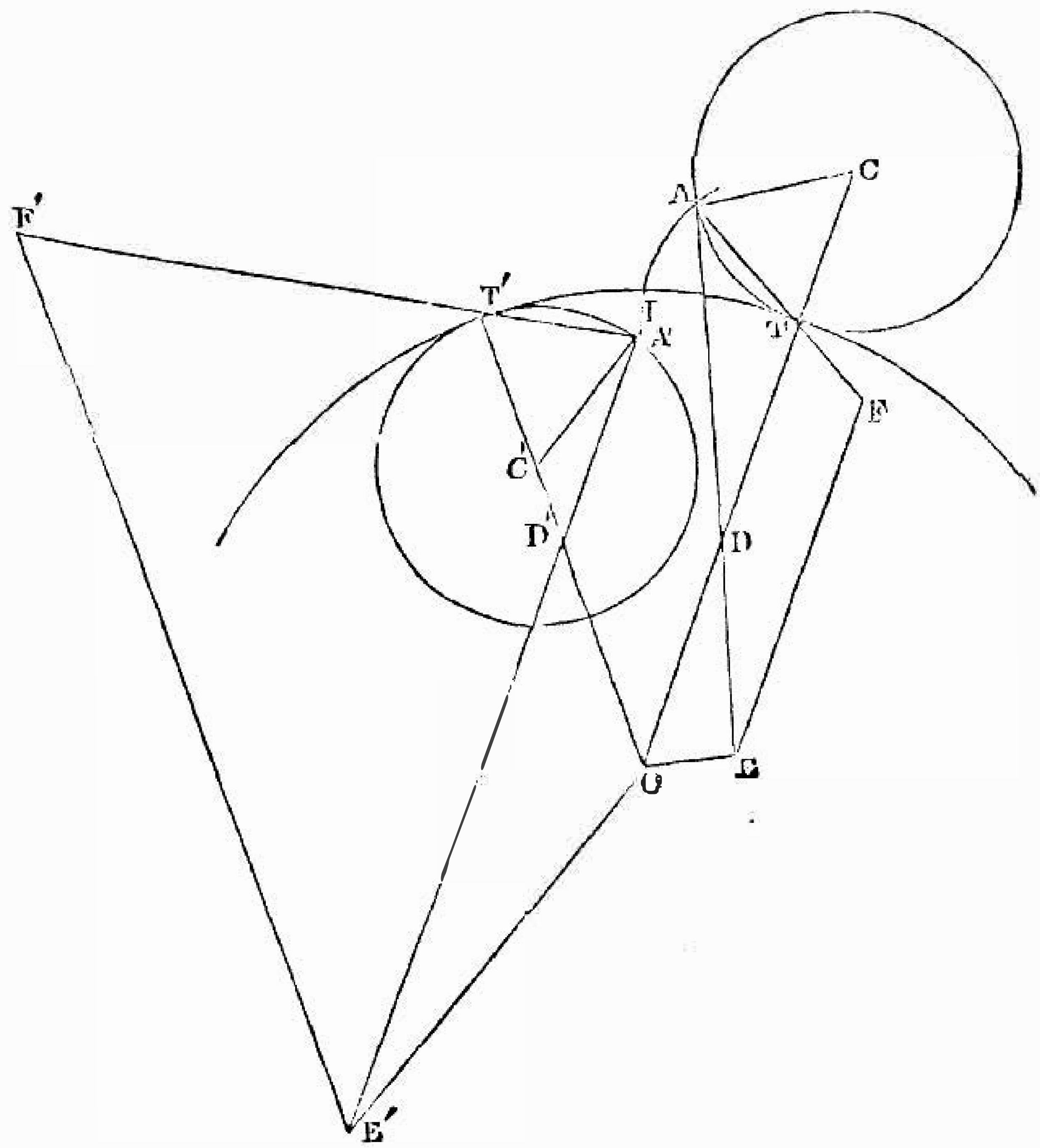

Fig. 46.

whose axis is C, rolling *outside* a fixed cylinder, the trace of whose axis is O; and the left-hand division giving the construction for an *internal epicycloid*, I A′, traced by a point, A′, in the surface of a cylinder, the trace of whose axis is C′, rolling *inside* the same fixed cylinder. The following description applies to both divisions of the figure: it being observed that at the left-hand side the letters are accented:—

Let T be the trace of the instantaneous axis, or line of contact of the cylinders, at the instant when the tracing point is at A; so that A T is the normal to the epicycloid at A, and O T and C T the radii of the fixed and rolling cylinders, being two parts of one straight line. Through O draw O E parallel to A C. Bisect O T in D, and draw the straight line A D E, cutting O E in E. Through E draw E F parallel to O T, and cutting A T (produced as far as required) in F. Then A F will be the radius of curvature of the epicycloid at the point A.

The following formula serves to find A F by calculation;

$$A\,F = \frac{A\,T \cdot O\,C}{C\,D}\text{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

It is sometimes more convenient to calculate the distance, T F, of the *centre of curvature*, F, from the instantaneous axis, T, and that is done by the following formula:

$$T\,F = \frac{A\,T \cdot O\,D}{C\,D} = \frac{A\,T \cdot O\,T}{2\,C\,D}; \quad \ldots\ldots\ldots\ldots\ldots (2.)$$

the use of which, in designing the teeth of wheels by Mr. Willis's method, will appear farther on.

79. **To Draw Rolled Curves.**—A rolled curve may be drawn by actually rolling a disc of the form of the rolling curve, carrying a suitable tracing point, upon the edge of a disc of the form of the fixed curve. But it needs much care to perform that operation with accuracy, except with the aid of machinery specially contrived for the purpose, such as is to be found in certain kinds of turning lathes.

For ordinary purposes in designing machinery, approximate methods of drawing rolled curves are used, such as the following:—

I. *To draw approximately a rolled curve by the help of tangent circles.*—In fig. 47, let A B be the fixed curve, and A D the rolling curve, touching the fixed curve at A, which is also the position of the tracing point at starting. The curve A D rolls from A towards B; and it is required to draw approximately the curve traced by the point A. By Rule III. of Article 51, page 29, lay off on each of the two curves A B and A D a series of equal arcs, A 1, 12, 23, 34, &c. Measure the straight chord from 1 to A on the curve A D, and with 11 = 1A as a radius, and the point 1 on the curve A B as a centre, draw so much of a circle as lies near the probable position of the rolled curve; measure the straight chord from 2 to A on A D, and with 22 = 2 A as a radius, and the point 2 on the curve A B as a centre, draw in like manner part of a circle; and go on, in the same way, drawing a series of

* The proof of this is as follows:—Let the radius of the rolling cylinder, C A = C T = r; let that of the fixed cylinder, O T = R, which is to be

circular arcs with the points 1, 2, 3, 4, &c., in the fixed curve A B, for their centres, and for their radii the lines 11, 22, 33, 44, &c., respectively equal to the distances 1 A, 2 A, 3 A, 4 A, &c., as measured between points on the rolling curve. Then, with the

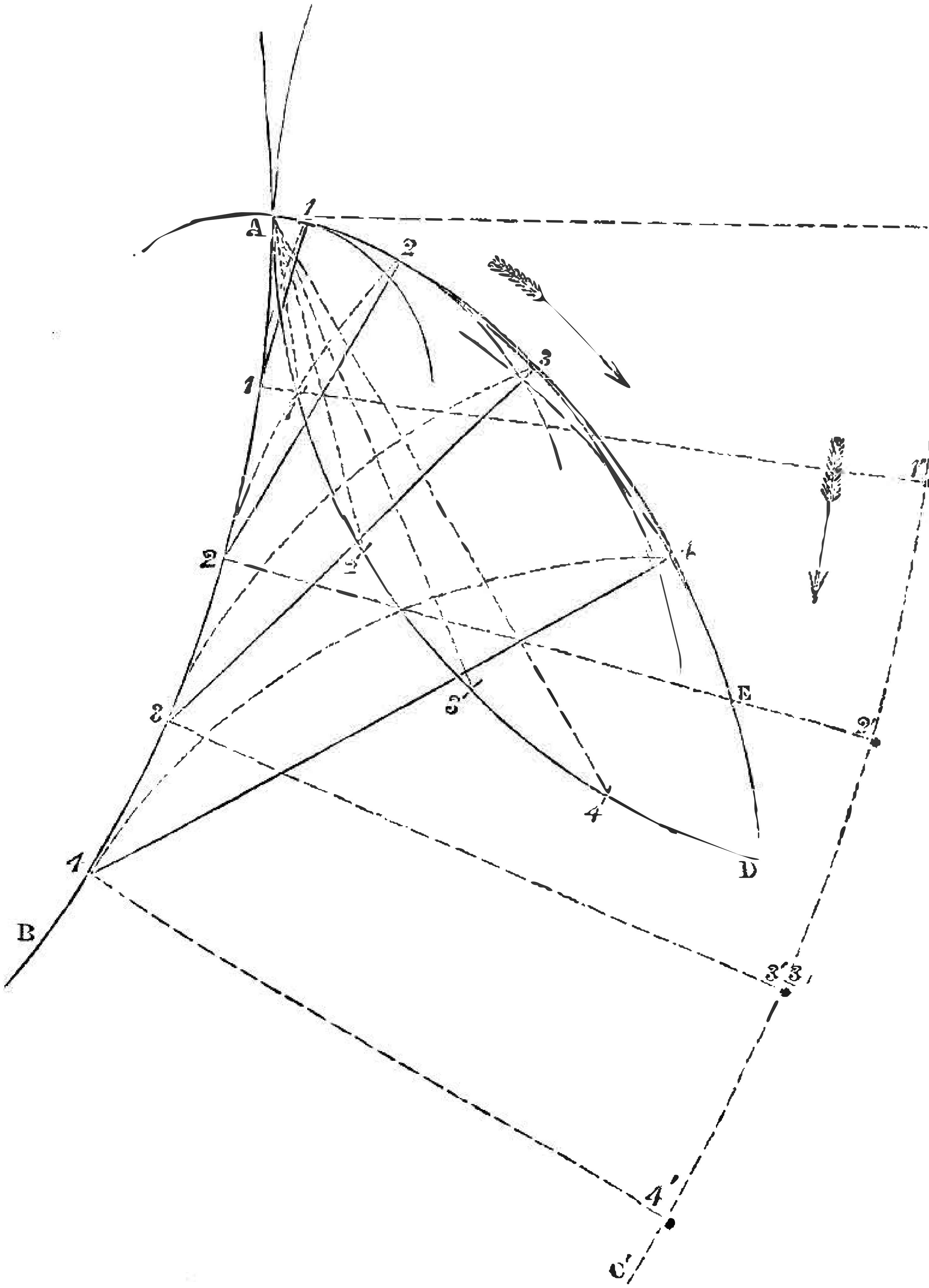

Fig. 47.

regarded as positive or negative according as the rolling cylinder rolls on the outer or inner surface of the fixed cylinder; let the instantaneous radius or free hand, or with the help of a bent spring, draw a curve, A E, so as to touch all those circular arcs; this will be very nearly the rolled curve required.

The curve A E is called the "Envelope" of the series of arcs that it touches.

II. *To find a series of points in a rolled curve.*—Draw a series of tangent circular arcs as in the preceding rule; then draw the several normals, 11, 22, 33, 44, &c., as radii of those arcs; the direction of each normal being determined by the principle, that at the point where it meets the fixed curve A B, it makes an angle with a tangent to that curve equal to the angle which the corre-

normal of the epicycloid, T A $= p$; and let the required radius of curvature, A F $= \rho$.

Let the angular velocity of the rolling cylinder, *relatively to the rotating plane* O C, be denoted by b, and that of the plane O C by a, so that the resultant angular velocity of the rolling cylinder is $a + b$. Then, because the angle C T A is the complement of one-half of the angle T C A, it is evident that the angular velocity of T A is $a + \frac{b}{2}$. But according to Article 76, $a \text{ R} = b\, r$; therefore

$$a + b = b\left(1 + \frac{r}{\text{R}}\right); \; a + \frac{b}{2} = b\left(\frac{1}{2} + \frac{r}{\text{R}}\right).$$

In any indefinitely short time, $d\,t$, the normal sweeps through an angle whose value in circular measure is

$$d\,\theta = \left(a + \frac{b}{2}\right) d\,t = b\left(\frac{1}{2} + \frac{r}{\text{R}}\right) d\,t;$$

and the point A traces an arc of the length

$$d\,s = (a + b)\, p\, d\,t = b\left(1 + \frac{r}{\text{R}}\right) p\, d\,t;$$

therefore the radius of curvature of the epicycloid at the point A is

$$\rho = \frac{d\,s}{d\,\theta} = p \cdot \frac{1 + \frac{r}{\text{R}}}{\frac{1}{2} + \frac{r}{\text{R}}} = \frac{p\left(\text{R} + r\right)}{\frac{1}{2}\text{R} + r} = \frac{\text{A T} \cdot \text{O C}}{\text{C D}}.$$

This formula is made to comprehend the case of a cycloid by making $\text{R} = \infty$, when it becomes $\rho = 2\,p$; and that of the involute of a circle by making $r = \infty$, when we have $\rho = p$. When the epicycloid is internal, and R and r denote arithmetical values of those radii, the sign — is to be substituted for + both in the numerator and in the denominator of the formula. The symbolical expression for equation 2 of the text is

$$\rho - p = \frac{p\,\text{R}}{\text{R} + 2\,r},$$

with the same understanding as to the sign in the denominator. In the case already referred to at the end of Article 77, when a cylinder rolls inside a cylinder of twice its diameter, we have $\text{R} = -2\,r$, and the denominator of the expression for ρ becomes $= 0$; showing that the radius of curvature is infinite; or, in other words, that the epicycloid traced is a straight line, as stated in the text. When the rolling cylinder is concave, r is negative.

sponding chord on the rolling curve A D, makes with a tangent to that curve at the corresponding point. Thus are found a series of points, 1, 2, 3, 4, &c., on the rolled curve A E, at the ends of the normals from the corresponding points on the fixed curve A B.

The two preceding Rules are applicable to fixed and rolling curves of all figures whatsoever. When both curves are circles, the finding of a series of points is facilitated by drawing the circle C C′, which contains the successive positions of the centre of the rolling circle; then marking those successive positions, 1′, 2′, 3′, 4′, &c., on the circle C C′, by drawing radii through the corresponding points 1, 2, 3, 4, &c., on the circle A B; then drawing the rolling circle in its several successive positions (marked with dots in the figure), and laying off the chords 11, 22, 33, 44, &c., of their proper lengths upon those positions of the rolling circle, which chords will be a series of normals to the rolled curve A E.

III. *To approximate to the figure of an epicycloidal arc by means of one circular arc.* By the method of the preceding Rule draw the normal to the epicycloidal arc in question at a point near its middle. For example, if A 3 is the arc of the epicycloid A E, whose figure is to be approximated to by means of one circular arc, draw the normal 22 by Rule II. Then conceive that normal to be represented by A T in fig. 46, page 57; and by the method of Article 78 find the corresponding radius of curvature A F and centre of curvature F. A circular arc described about F, with the radius F A (fig. 46), will be an approximation to the epicycloidal arc.

This is the approximation used in Mr. Willis's method of designing teeth for wheels, to be described farther on. It ensures that the circular arc shall have, at or about the middle of its length, the same position, direction, and curvature with the epicycloidal arc for which it is substituted. Towards the ends of the arcs they gradually deviate from each other.

IV. *To approximate to the figure of an epicycloidal arc by means of two circular arcs.* This method of approximation is closer than the preceding, but more laborious. It substitutes for an epicycloidal arc a curve made up of two circular arcs; and the approximate curve coincides exactly with the true curve at the two ends and at one intermediate point, and has also the same tangents at its two ends

Suppose that A and B (fig. 48) are the two ends of the epicycloidal arc to which an approximation is required, and that A C and B C are normals to the arc at those points: the positions of the ends of the arc and directions of its normals having been determined by Rule II. of this Article. Let C be the point of intersection of the normals. Draw the tangents A D perpendicular to A C, and B D perpendicular to B C, meeting each other in D. Draw the straight line D C, and bisect it in E. About E, with the radius E D = E C, describe a circle, which will pass through the four points, A, D, B, C. Draw the diameter F E G, bisecting the arc A B in F and the arc B C A in G.

Draw the straight line G D, in which take G H = G A = G B. Through H, parallel to F E G, draw the straight line H K L, cutting A C in K and B C in L. Then about K, with the radius K A = K H, draw the circular arc A H; and about L, with the radius L H = L B, draw the circular arc H B: the curve made up of those two circular arcs will be a close approximation to the epicycloidal arc, having the same position and tangents at its two ends, and being very near to the true arc at all intermediate points.

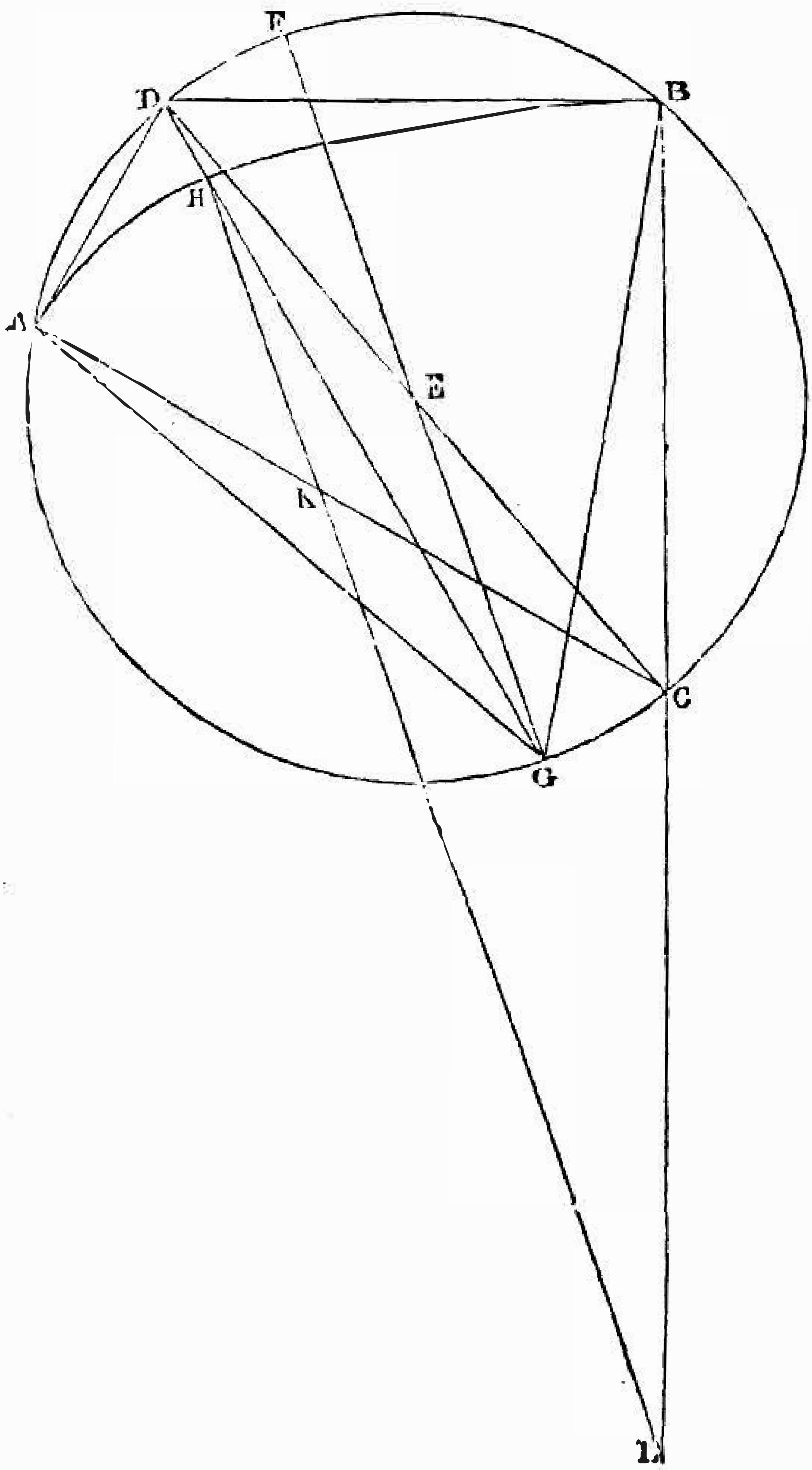

Fig. 48.

It may be remarked that G H = G A = G B = $\sqrt{(HK \cdot HL)}$ approximates very closely to the mean radius of curvature of the epicycloidal arc A B; also that the process described is applicable to the approximate drawing of many curves besides epicycloids; and that the ratio of the two radii, H L : H K, deviates less from equality than that of the radii of any other pair of circular arcs which can be drawn so as to touch A D in A and B D in B, and also to touch each other at an intermediate point.*

* This may be expressed symbolically by stating that $\left(\frac{HL^2 - HK^2}{HK \cdot HL}\right)^2$ is a minimum; or that $\left(\log. \frac{HL}{HK}\right)^2$ is a minimum.

80. **Resolution of Rotation in General.**—The following propositions show how the rotation of a rigid body about a given axis, fixed or instantaneous, may be resolved into two component rotations about any two axes in the same plane with the actual axis.

I. PARALLEL AXES.—*The rotation of a rigid body about a given axis is equivalent to the resultant of two component rotations about two axes parallel to the given axis and in the same plane, the angular velocity of each of the three rotations being proportional to the distance between the axes of the other two rotations.*

In fig. 49, let the plane of the paper be perpendicular to the three parallel axes, and let C be the trace of the axis of the resultant rotation, and A and B the traces of the axes of the component rotations; all three axes being in the same plane, whose trace is A C B. Let the angular velocities about

A, B, C,

be respectively proportional to

B C, C A, A B.

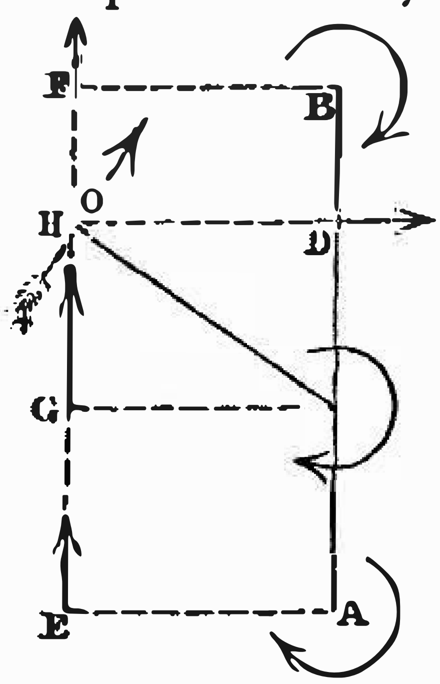

Fig. 49.

As the figure is drawn, all three angular velocities are of the same sign, because A B = B C + C A. If C lay beyond A and B, instead of between them, A B would be the difference of B C and C A, instead of their sum; and the lesser of these two distances and of the corresponding angular velocities would have to be considered as negative.

Let H be the projection of a particle in the rigid body, which particle is moving in a direction perpendicular to H C, with a velocity proportional to C H · A B. Then, *first*, from H let fall H D perpendicular to A B; then, by the principles of Article 55, page 33, the component velocity of H in the direction H D, whether due to rotation about A, B, or C, is the same with that of a particle at D. Now the velocities of a particle at D due to the rotations about

A, B, C

are proportional respectively to

$$+ AD \cdot BC; \ - BD \cdot CA; \ + CD \cdot AB;$$

and $CD \cdot AB = AD \cdot BC - BD \cdot CA$; therefore this component of the velocity of the particle H due to the rotation about C is the resultant of the corresponding components due to the rotations about A and B respectively.

Secondly. Through H draw E G H F parallel to A C B, and on it let fall the perpendiculars A E, B F, C G. Then, by the principles of Article 55, page 33, the component velocities of H along E F due to the rotations about the axes A, B, and C are respectively equal to the velocities of E due to rotation about A, of F due to rotation about B, and of G due to rotation about C; and because A E = B F = C G, these velocities are respectively proportional to

B C, C A, A B;

But A B = B C + C A; therefore the component along E F of the velocity of the particle H due to the rotation about C is the resultant of the corresponding component velocities due to the rotations about A and B respectively. Therefore the whole velocity of the particle H due to rotation about C, with an angular velocity proportional to A B, is the resultant of the velocities of the same particle due respectively to rotations about A, with an angular velocity proportional to B C, and about B, with an angular velocity proportional to C A. And this being true for every particle of the rotating body, is true for the whole body: Q. E. D.

II. Intersecting Axes.—*The rotation of a rigid body about a given axis is equivalent to the resultant of two component rotations about two axes in the same plane with the first axis, and cutting it in one point; the angular velocities of the component and resultant rotations being proportional respectively to the sides and diagonal of a parallelogram, which are parallel respectively to the three axes of rotation.*

In fig. 50 the upper right-hand part of the figure represents a plane perpendicular to the resultant axis of rotation, O″. F″ is the projection of any particle on that plane; and the direction of motion of any particle whose projection is F″ is perpendicular to O″ F″.

O″ Y″ and O″ Z″ are the traces of two planes perpendicular to the first plane of projection and to each other; and D″ and E″ are the projections of F″ on those planes respectively. According to the principle of Article 55, page 33, the component velocity parallel to O″ Y″ of the particle whose projection is F″ is the same with the velocity of a particle at D″; and its component velocity parallel to O″ Z″ is the same with that of a particle at E″.

The upper left-hand part of the figure represents the plane whose trace on the first plane of projection is O″ Z″; O′ X′, on this second plane, is the axis of rotation; O′ Z′ is the trace of the first plane of projection; and D′ is the projection of F″, and is the same point that is marked D″ on the first plane. The lower part of the figure represents the plane whose trace on the first plane of projection is O″ Y″, and on the second plane, O′ X′. On this third plane O X is the axis of rotation, and also the trace of the second plane; O Y is the trace of the first plane; E is the projection of

F′, and is the same point that is marked E″ on the first plane; O is the projection of D″.

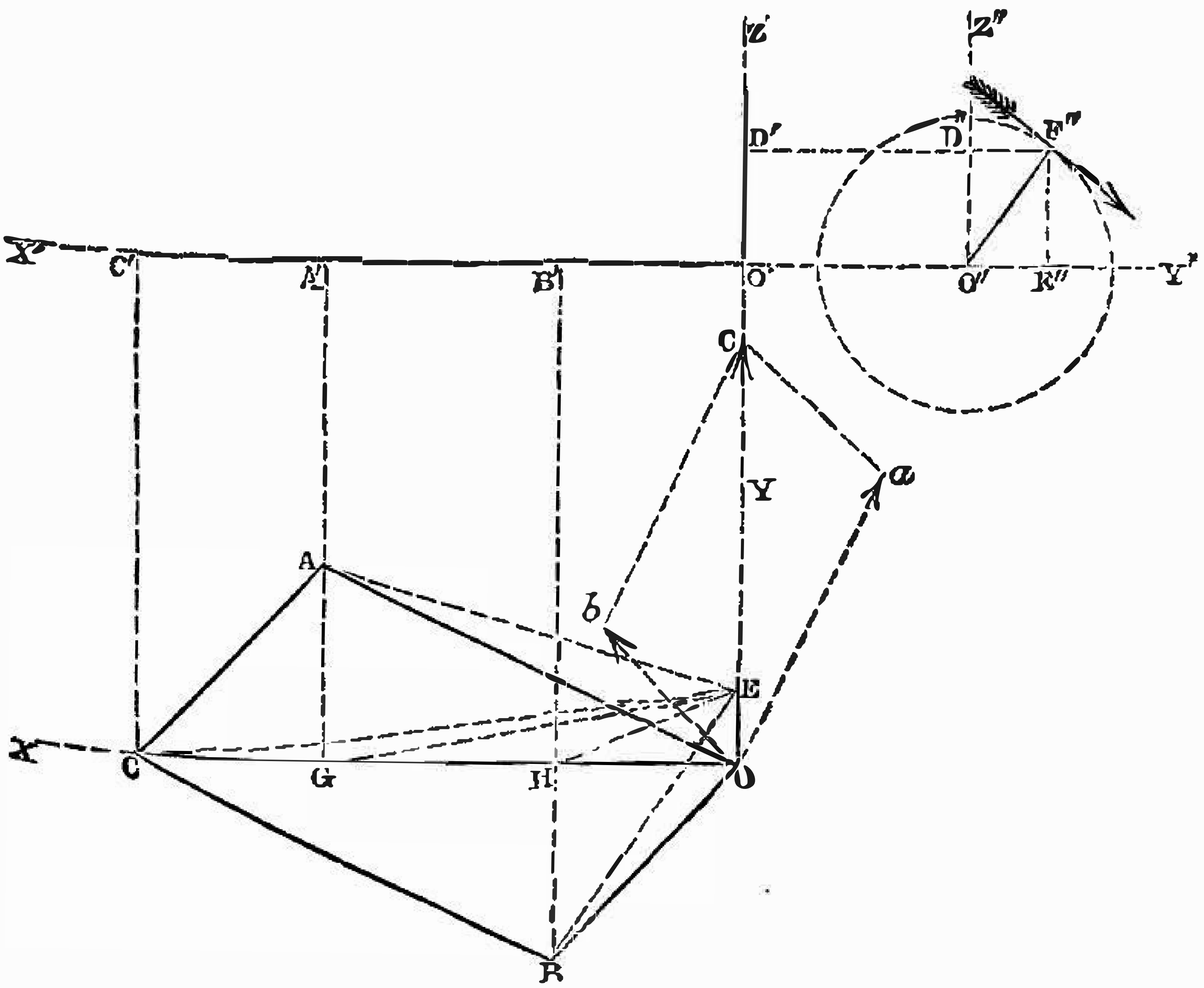

Fig. 50.

The positions of the second and third planes are arbitrary so long as they both traverse the axis of rotation, and are at right angles to each other.

In O X take O C proportional to the angular velocity; and make it point so that to an observer looking from O towards C the rotation shall seem right-handed.

From O draw any two lines, O A and O B, in the third plane; from C draw C B parallel to A O, and C A parallel to B O, so as to complete the parallelogram O B C A. Then the proposition states, that a right-handed rotation about O A, with an angular velocity proportional to O A, and a right-handed rotation about O B, with an angular velocity proportional to O B, being combined, are equivalent to the actual rotation.

To prove this for a particle at E, it is to be considered that the motions impressed on E by the three rotations separately are each of them perpendicular to the third plane; also, that the velocity of E due to any one of the three rotations is proportional to the angular velocity of that rotation multiplied by the perpendicular distance of E from the axis of that rotation, and is therefore proportional to the area of a triangle having for its base the length marked on that axis, to represent that angular velocity, and for its summit the point E; so that the velocities of a particle at E due respectively to the rotations about

$$O\,A, \qquad O\,B, \qquad O\,C$$

are proportional respectively to the areas of the triangles

$$O\,A\,E, \qquad O\,B\,E, \qquad O\,C\,E.$$

Through A and B draw A G and B H perpendicular to O C, and join E G and E H. Then, by plane geometry,

$$OAE = OGE; \text{ and } OBE = OHE = GCE;$$

therefore

$$OCE = OGE + GCE = OAE + OBE.$$

So that the velocity of E due to the actual rotation about O C is the resultant of the velocities due to the rotations about O A and O B; the angular velocities being proportional to the lengths laid off on the axes respectively.

To prove the same proposition for a particle at D″, whose projection on the third plane is O, it is to be considered that the perpendicular distance of this point from the three axes, O A, O B, and O C, is identical, being the line marked O″ D″ and O′ D′ on the first and second planes; so that the velocities of D due to the three rotations are simply proportional to the three angular velocities. To represent, then, those three velocities as projected on the third plane, draw O a, O b, and O c perpendicular and proportional respectively to O A, O B, and O C. It is evident that O a, O b, and O c are the sides and diagonal of a parallelogram similar to O B C A; and therefore that the velocity of D″ due to the actual rotation about O C is the resultant of the velocities due to the rotations about O A and O B, the angular velocities being proportional to the lengths laid off on the axes respectively.

The proposition, therefore, is proved for both components of the velocity of a particle at F″; and it holds for any particle whose projection on a plane perpendicular to the axis O C is F″; that is, for every particle of the body, and therefore for the whole body: Q. E. D.

It appears, then, that rotations, when represented by lengths laid off on their axes proportional to their angular velocities, can be compounded and resolved, like linear velocities, by constructing parallelograms.

81. **Rotations about Intersecting Axes Compounded.** (*A. M.*, 392).

—In fig. 51, let O A be a fixed axis, and about it let the plane A O C rotate with the angular velocity a. Let the plane of projection be that of those two axes at a given instant. Let O C be an axis in the rotating plane; and about that axis let a secondary piece rotate with the angular velocity b relatively to the rotating plane; and let it be required to find the instantaneous axis and the resultant angular velocity of the secondary piece. From the principles of Article 80, Proposition II., page 64, the following rule is deduced:—

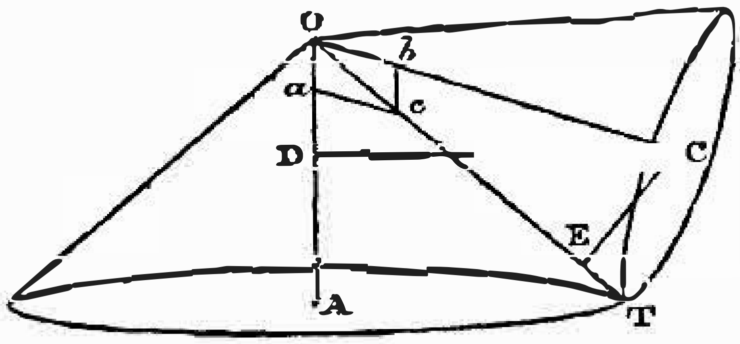

Fig. 51.

On O A take O a proportional to a; and on O C take O b proportional to b. Let those lines be taken in such directions that to an observer looking from O towards their extremities the component rotations shall seem both right-handed. Complete the parallelogram O b c a; the diagonal O c will be the instantaneous axis; and its length will represent *the resultant angular velocity*. Another mode of viewing the question is as follows:—

Because the point O in the secondary piece is fixed, the instantaneous axis must traverse that point. The direction of that axis is determined by considering that each point which arrives at that line must have, in virtue of the rotation about O C, a velocity relatively to the rotating plane, equal and directly opposed to that which the coincident point of the rotating plane has. Hence it followst that the ratio of the perpendicular distances of each point in the instantaneous axis from the fixed and moving axes respectively—that is, the ratio of the sines of the angles which the instantaneous axis makes with the fixed and moving axes—must be the reciprocal of the ratio of the component angular velocities about those axes; or if, in symbols, O T be the instantaneous axis,

$$\sin \mathrm{A\,O\,T} : \sin \mathrm{C\,O\,T} :: b : a \qquad (1.)$$

The resultant angular velocity about this instantaneous axis is found by considering that if C be any point in the moving axis, the linear velocity of that point must be the same whether computed from the angular velocity, a, of the rotating plane about the fixed axis O A, or from the resultant angular velocity, c, of the rigid body about the instantaneous axis. That is to say, let C D, C E be perpendiculars from C upon O A, O T, respectively; then

$$a \cdot \overline{\mathrm{CD}} = c \cdot \overline{\mathrm{CE}};$$

but $\overline{\mathrm{CD}} : \overline{\mathrm{CE}} :: \sin \mathrm{A\,O\,C} : \sin \mathrm{C\,O\,T}$; and therefore

$$\sin \mathrm{C\,O\,T} : \sin \mathrm{A\,O\,C} :: a : c;$$

and, combining this proportion with that given in equation 1, we obtain the following proportional equation :—

$$\left.\begin{array}{ccccccc} & \sin \mathrm{C\,O\,T} & : & \sin \mathrm{A\,O\,T} & : & \sin \mathrm{A\,O\,C} \\ :: & a & : & b & : & c \\ :: & \mathrm{O}a & : & \mathrm{O}b & : & \mathrm{O}c \end{array}\right\} \ldots\ldots(2.)$$

That is to say, *the angular velocities of the component and resultant rotations are each proportional to the sine of the angle between the axes of the other two;* and *the diagonal of the parallelogram* O b c a *represents both the direction of the instantaneous axis and the angular velocity about that axis.*

82. **Rolling Cones.** (*A. M.*, 393.)—All the lines which successively come into the position of instantaneous axis are situated in the surface of a cone described by the revolution of O T about O C; and all the positions of the instantaneous axis lie in the surface of a cone described by the revolution of O T about O A. Therefore the motion of the secondary piece is such as would be produced by the rolling of the former of those cones upon the latter. Circular sections of the two cones are sketched in perspective in fig. 51.

It is to be understood that either of the cones may become a flat disc, or may be hollow, and touched internally by the other. For example, should $\angle$ A O T become a right angle, the fixed cone would become a flat disc; and should $\angle$ A O T become obtuse, that cone would be hollow, and would be touched internally by the rolling cone; and similar changes may be made in the rolling cone.

The path described by a point in or attached to the rolling cone is a *spherical epitrochoid;* and if that point is in the surface of the rolling cone, that curve becomes a *spherical epicycloid.* It will be shown in the next chapter how to draw such curves—not exactly, but with a degree of accuracy sufficient for practical purposes.

83. **Resolution of Helical Motion.**—The resolution of helical or screw-like motion into rotation about an axis and translation along that axis has already been treated of in the last section of the preceding chapter. It remains to be shown how a helical motion may be regarded as compounded of two rotations about two axes which are in different planes.

In fig. 52, let the lower part of the figure represent a plane of projection, and O A and O B the projections upon that plane of two axes which are both parallel to it, but not in the same plane. Let the upper part of the figure represent a second plane of projection perpendicular to the first plane; and let F′ G′ be the projection on that second plane of the *common perpendicular* of those two axes (Article 36, page 14). Let a rigid body have a motion compounded of two rotations about the two axes respectively, with angular velocities represented by O A and O B, these lines being drawn, as before, so that to an observer at O each rotation shall appear right-handed.

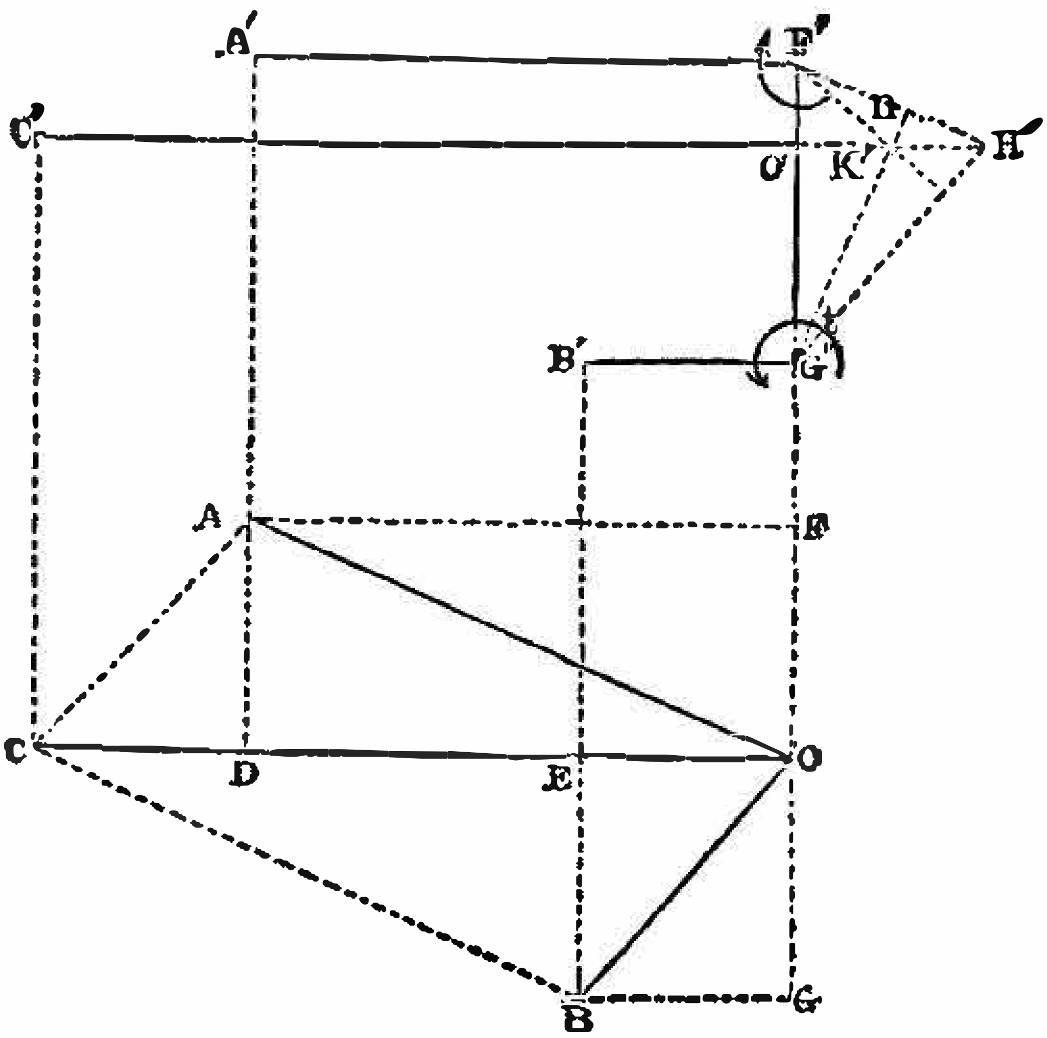

Fig. 52.

Complete the parallelogram O A C B, and draw its diagonal O C. Then, if the axes O A and O B were in the same plane, the rotations about them, being combined, would be equivalent simply to a rotation represented by O C, as in Articles 80 and 81, pages 64 to 66.

Let the second plane of projection be now supposed parallel to O C; and let F′ A′, O′ C′, and G′ B′ be the respective projections of O A, O C, and O B upon it. Draw A D, B E, and F O G perpendicular to O C, and A F and B G parallel to O C. It is obvious that O D = F′ A′, O E = G′ B′, and F O = O G.

According to Article 80, Proposition II., page 64, the rotation represented by O A may now be regarded as compounded of a rotation represented by O D, about an axis of which O D and F′ A′ are the projections, and a rotation represented by O F, about an axis of which O F and the point F′ are the projections; also, the rotation represented by O B may be regarded as compounded of a rotation represented by O E, about an axis of which O E and G′ B′ are the projections, and a rotation represented by O G, about an axis of which O G and the point G′ are the projections.

Then, according to Article 76, page 54, the rotations about the parallel axes F′ A′ and G′ B′, being combined, are equivalent to a rotation about an intermediate axis, O′ C′, in the same plane, with an angular velocity represented by

$$\mathrm{O\ C} = \mathrm{O' C'} = \mathrm{F' A'} + \mathrm{G' B'};$$

and that axis of resultant rotation divides the distance F′ G′ in the following proportion:—

$$\mathrm{O' C'} : \mathrm{F' A'} : \mathrm{G' B'}$$
$$:: \mathrm{F' G'} : \mathrm{O' G'} : \mathrm{O' F'}.$$

To find the point O′ by graphic construction, draw F′ H′, parallel to A O and G′ H′ parallel to B O, cutting each other in H′; then through H′ draw H′ O′ C′ parallel to O C.

Moreover, the component rotations represented by O F and O G, about the axes F′ and G′, are of equal and opposite angular velocities; and therefore, according to Article 76, page 54, they are equivalent to a translation in the direction O C, with a velocity represented by the product O F · F G.

That translation being compounded with the resultant rotation represented by O C, gives finally, for the resultant motion of the body, *a helical motion about the axis whose projections are* O C *and* O C′.

The *pitch* of that helical motion, or advance per turn, is found by multiplying the rate of advance, O F, F′ G′, by the time of one revolution, $\frac{6{\cdot}2832}{\mathrm{O\ C}}$; and is therefore equal to the *circumference of a circle whose radius is* $\frac{\mathrm{O\ F \cdot F' G'}}{\mathrm{O\ C}}$. Draw F′ K′ perpendicular to O B, and G′ K′ perpendicular to O A, cutting each other in K′ (which will be in the straight line H′ O′ C′). Then it is evident that F′ K′ G′ and C A O are similar triangles; and because D A = O F, we have the following proportion:—

$$\mathrm{O\ C} : \mathrm{O\ F} :: \mathrm{F' G'} : \mathrm{O' K'} = \frac{\mathrm{O\ F \cdot F' G'}}{\mathrm{O\ C}};$$

Therefore *the pitch of the resultant helical motion is equal to the circumference of a circle whose radius is* O′ K′; and *the rate of advance may be represented by the product* O C · O′ K′.

84. **Rolling Hyperboloids.**—Conceive the straight line O C to represent an indefinitely long straight edge, rigidly fastened to the arm O′ F′, and sweeping along with that arm round the axis O A; then conceive the same straight line to be rigidly fastened to the arm O′ G′, and to sweep along with this arm round the axis O B. Thus are generated a pair of surfaces called *Rolling Hyperboloïds*,

which touch each other all along the straight line O C. Fig. 53 shows the general appearance of a pair of rollers of that form; and in fig. 54 the projections of their figures are given with greater precision. If one of those bodies is fixed, and the other made to roll upon it, they continue to touch each other in a straight line, which is the instantaneous axis of the rolling body; and the rotation about that instantaneous axis is accompanied by a sliding motion along the same axis, so as to give, as the resultant compound motion, a helical motion about the instantaneous axis, as described in the preceding Article. The following problem sometimes occurs in mechanism:—

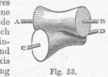

Fig. 53.

Given, the angle between the directions of two axes, and the length of their common perpendicular, to draw the projections of a pair of rolling hyperboloids of which these shall be the axes, and of which one shall roll on the other, so as to have component angular velocities bearing to each other a given ratio.

Let the lower part of fig. 54 (see next page) represent a plane to which the two axes are parallel; and let O a and O b be their projections on that plane, with lengths laid off upon them proportional to the intended component angular velocities. Draw b c parallel to O a, and a c parallel to O b, cutting each other in c; O c will be the projection of the *line of contact*, or instantaneous axis; and the length O c will represent the resultant angular velocity (as in the preceding Article).

Through O, perpendicular to O c, draw O G′ F′, and lay off upon it G′ F′ equal to the given common perpendicular; and let the second plane of projection be perpendicular to the first plane, and parallel to O c and G′ F′. To find the projection of the line of contact upon this second plane, proceed as in the preceding Article; that is, draw F′ H′ and G′ H′ parallel respectively to O a and O b, and H′ O′ parallel to O c; H′ O′ will be the required projection. This projection may also be found, if convenient, by either of the following methods: Draw G′ K′ perpendicular to O a, and F′ K′ perpendicular to O b, cutting each other in K′; and then draw H′ K′ O′ parallel to A c; or otherwise:—Draw g c f perpendicular to O c, and divide F′ G′ in the following proportion:—

$$fg \ : \ cf \ : \ cg$$
$$: : \mathrm{F'\,G'} : \mathrm{O'\,F'} : \mathrm{O'G'}.$$

Draw U O Y perpendicular to O a, making O U = O Y = F′ O′; also draw V O Z perpendicular to O b, making O V = O Z = G′ O′; then U O Y and V O Z will be the projections on the first plane of the *smallest transverse sections*, or what may be called the "*throats*" of the two hyperboloïds; which transverse sections are

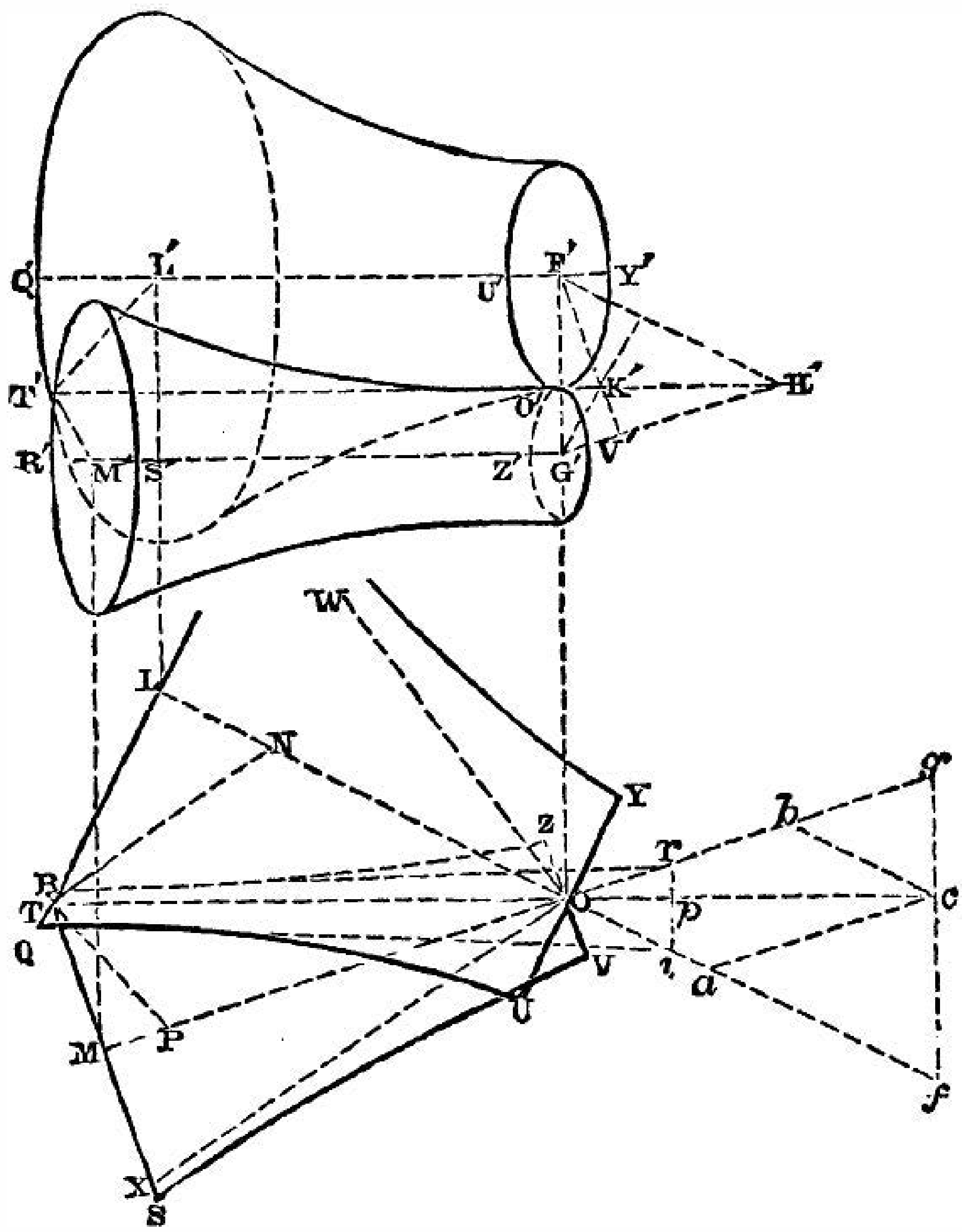

Fig. 54.

circles of the respective radii F′ O′ and G′ O′. The projections of those circles on the second plane of projection are the ellipses U′ O′ Y′ and V′ O′ Z, drawn according to the principles of Article 37, page 15.

To find the projections of a pair of circular transverse sections of the two hyperboloïds, which shall cross each other in any given point of the line of contact, let T and T′ be the projections of that point. Then draw T L perpendicular to a O, and T M perpendicular to b O; L and M will be the projections of the centres of those circular sections on the first plane. Draw F′ L′ and G′ M′ parallel, and L L′ and M′ M′ perpendicular to O c; L′ and M′ will be the projections of those centres on the second plane. In L O take L N = F′ O′, and join N T; then in L T produced, take L Q = N T; this will be the radius of the required section of one hyperboloïd; and Q will be a point in the hyperbola

U Q, which is the longitudinal section or trace of that surface on a plane traversing the axis F L′, and parallel to the first plane of projection. Also, in M O take M P = G′ O′, and join P′ T; then in M T produced take M R = P T; this will be the radius of the required section of the other hyperboloïd; and R will be a point in the hyperbola Z R, which is the longitudinal section or trace of this surface on a plane traversing the axis G′ M′ and parallel to the first plane of projection.

The projection, O T, of the line of contact is an asymptote to both hyperbolas, U Q and Z R; and their other asymptotes are O W, making L O W = L O T, and O X, making M O X = M O T.

The projections on the second plane of projection of the two circular transverse sections which cross each other at the point whose projections are T and T′ are two ellipses, drawn according to the principles of Article 37, page 15.

By the same process may be found the projections of any required number of transverse sections of the two rolling hyperboloïds, and of any required number of points, such as Q and R, in their longitudinal sections.

Additional rules relating to the construction of such figures will be given in the next chapter, in the articles which treat of their application to skew-bevel wheels.

85. **Cylinder Rolling Obliquely.**—The same kind of resultant motion will take place, if for the rolling hyperboloïds there be substituted a pair of cylinders described about the axes whose projections are O A and O B, fig. 52, page 69, with the respective radii O′ F′ and O G; provided the axis of the rolling cylinder is guided so that the point where it is met by the common perpendicular F′ G′ shall revolve in a circle of the radius F′ G′ round the axis of the fixed cylinder, and so that the inclination of those two axes to each other shall remain constant. The general appearance of such a pair of cylinders is shown in fig. 55. They touch each other in a point only, and not along a straight line, as the hyperboloïds do. The uniform transverse sections of such a pair of cylinders are identical with those at the *throats* of the corresponding pair of hyperboloïds. Further explanations as to obliquely-rolling cylinders will be given in the next chapter, under the head of screw-gearing.

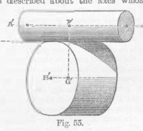

Fig. 55.

86. **Cones Rolling Obliquely.** — The same kind of resultant

motion may be effected also by substituting for the pair of hyperboloïds of fig. 54, page 72, the pair of cones which touch those hyperboloïds in the pair of circles that cross each other in any given point, T, of the instantaneous axis. To draw the projections of those tangent cones, let O′ T′ in fig. 56 (as in fig. 54) be the instantaneous axis, O′ the point where it cuts the common perpendicular, and T′ the intended point of contact of the cones. From O′, perpendicular to O′ T′, draw O′ h = O′ H′ in fig. 56; join T′ h; and perpendicular to T′ h draw $h\,p'$, cutting the instantaneous axis in p'. Then a plane normal to the instantaneous axis at p' passes through the summits of both the required tangent cones. Therefore, in fig. 54,

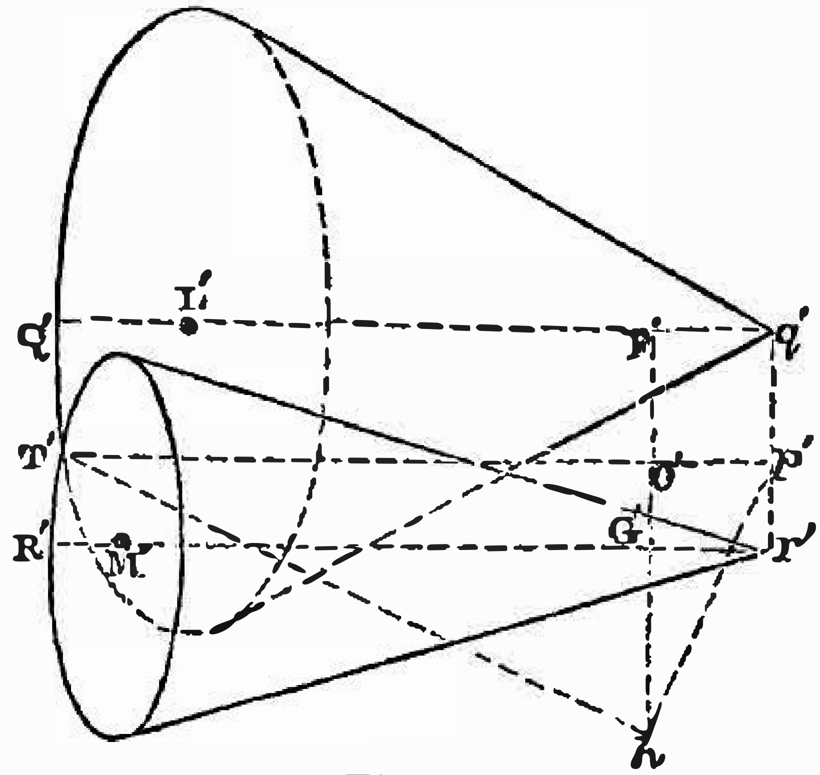

Fig. 56.

on O c lay off O p = O′ p' of fig. 56, and draw q p r perpendicular to O c, cutting O a in q and O b in r; q and r will be *the projections of the summits of the tangent cones on the first plane of projection.* The projections on the same plane of the longitudinal sections or traces of these cones, upon planes traversing their axes parallel to the plane of projection, are q Q and r R. Also, let the plane of fig. 56 be the second plane of projection, and let F′ L′ and G′ M′, as in fig. 54, be the projections of the axes of the hyperboloïds, and F′ O′ G′ that of the common perpendicular. Draw q' p' r', cutting those axes in q' and r'; these points will be *the projections of the summits of the tangent cones on the second plane of projection.* The projections of the bases of these cones on the same plane are the pair of ellipses, with L′ and M′ for their centres, which cross each other at the point T′, as in fig. 54. The cones touch each other in the point T′ only, and not along a straight line, as the hyperboloïds do. Further explanations as to obliquely-rolling cones will be given in the next chapter, under the head of skew-bevel wheels.

87. **Bands or Flexible Secondary Pieces—Cords—Belts—Chains.** (*A. M.*, 400, 401.)—The flexible pieces used in machinery may be classed under three heads:—*Cords*, which approximate to a round form in section; *Belts*, which are flat; and *Chains*, which consist of a series of rigid links so connected together that the chain as a whole is flexible. Mr. Willis gives them all the common name of *wrapping connectors;* and for the sake of brevity in stating principles that apply to them all, they may conveniently be called *bands.*

In treating of questions of pure mechanism, the *centre line* of a band is treated as being of invariable length; for although no substance is absolutely inextensible, and although when a band passes over a curved surface the concave side is shortened and the convex side lengthened, still the variations of length of the centre line of the band are, or ought to be, practically inappreciable.

In order that the figure and motion of a band may be determined from geometrical principles alone, independently of the magnitude and distribution of forces acting on it, its weight must be insensible compared with the tension on it, and it must everywhere be *tight;* and when that is the case, each part of the band which is not straight is maintained in a curved figure by passing over a *convex* surface. When a band is guided by a given actual surface, the centre line of that band may be regarded as guided by an imaginary surface parallel to the actual surface, and at a distance from it equal to half the thickness of the band. The line in which the centre line of a band lies on such guiding surface is the *shortest line* which it is possible to draw on that surface between each pair of points in the course of the band. (It is a well-known principle of the geometry of curved surfaces that the *osculating plane* at each point of such a line is perpendicular to the curved surface.)

Hence it appears that the motions of a tight flexible band, of invariable length along its centre line and insensible weight, are regulated by the following principles:—

I. *The length between each pair of points in the centre line of the band is constant.*

II. *That length is the shortest line which can be drawn between its extremities over the surface by which the centre line of the band is guided.*

The motions of a band are of two kinds—

I. Travelling of a band along a track of invariable form; in which case the velocities of all points of the centre line are equal.

II. Alteration of the figure of the track by the motion of the guiding surfaces.

Those two kinds of motion may be combined.

The most usual problems in practice respecting the motions of bands are those in which bands are the means of transmitting motion between two pieces in a train of mechanism. Such problems will be considered in the next chapter.

88. **Fluid Secondary Pieces.**—A mass of fluid may act as a secondary piece in a machine; and in order that the motion of such a mass may be a subject of pure mechanism, the volume occupied by the mass must be constant; and that not only for the whole mass, but for every part of it, how small soever. In other words, the fluid mass must in every part be of constant *bulkiness;*

this word being used to denote the volume filled by an unit of mass; for example, the number of cubic feet filled by a pound, or the number of cubic metres filled by a kilogramme. Every fluid, whether liquid or gaseous, undergoes variations of bulkiness through variations of pressure and of temperature; but in mechanism such variations of bulkiness may be either so small that they may be disregarded for the practical purpose under consideration (as in the case of most liquids), or, if the fluid employed be gaseous, they may be prevented by keeping the pressure and temperature constant.

Under such conditions the motions of the particles of a fluid mass are regulated by the following principle:—

At a given series of sections of a stream of fluid of constant bulkiness, the mean velocities at each instant of the particles in directions normal to those sections respectively, are inversely proportional to the areas of the sections.